FACULTY
OF MATHEMATICS
AND PHYSICS
Charles University

# DOCTORAL THESIS

Josef Dvořák

# Structural and categorical description of classes of abelian groups

Department of Algebra

Supervisor of the doctoral thesis: doc. Mgr. et Mgr. Jan Žemlička, Ph.D.
Study programme: Mathematics
Study branch: Algebra, number theory, and mathematical logic

Prague 2021

I declare that I carried out this doctoral thesis independently, and only with the cited sources, literature and other professional sources. It has not been used to obtain another or the same degree.

I understand that my work relates to the rights and obligations under the Act No. 121/2000 Sb., the Copyright Act, as amended, in particular the fact that the Charles University has the right to conclude a license agreement on the use of this work as a school work pursuant to Section 60 subsection 1 of the Copyright Act.

In ............. date .............            ......................................

Author's signature

I'd like to express my warmest thanks to my supervisor, Jan Žemlička, for leading me through my whole studies with boundless patience, kindness and enduring encouragement.

This work also would not come to existence without the great support I've been given from my family and especially from my parents and I'm deeply greateful to them.

Title: Structural and categorical description of classes of abelian groups

Author: Josef Dvořák

Department: Department of Algebra

Supervisor: doc. Mgr. et Mgr. Jan Žemlička, Ph.D., Department of Algebra

Abstract: The work presents results concerning the self-smallness and relative smallness properties of products in the category of abelian groups and in *Ab5*-categories. Criteria of relative smallness of abelian groups and a characterization of self-small products of finitely generated abelian groups are also given. A decomposition theory of $UD$-categories and in consequence a unified theory of decomposition in categories of $S$-acts is developed with applications on (auto)compactness properties of $S$-acts. A structural description of compact objects in two categories of $S$-acts is provided. The existence of projective covers in categories $S - \overline{\mathrm{Act}}$ and $S - \mathrm{Act}_0$ and the issue of perfectness of a monoid with zero are discussed.

Keywords: self-small, relatively-small, compact object, autocompact object, perfect monoid

# Contents

# Chapter 1

# Introduction

This dissertation is based on the work presented in the following five articles:

- **i** Dvořák, J.: *On products of self-small abelian groups*, *Stud. Univ. Babeş-Bolyai Math.* 60 (2015), no. 1, 13–17.
- **ii** Dvořák, J., Žemlička, J.: *Self-small products of abelian groups*, accepted for publication in Comment. Math. Univ. Carolin., 2021, arXiv:2102.11443.
- **iii** Dvořák, J., Žemlička, J.: *Autocompact objects of Ab5 categories*, submitted to Theory Appl. Categ., 2021, arXiv:2102.04818.
- **iv** Dvořák, J. ,Žemlička, J.: *Compact objects in categories of $S$-acts*, submitted to Semigroup Forum, 2021, arXiv:2009.12301.
- **v** Dvořák, J., Žemlička, J.: *Perfect monoids with zero and categories of $S$-acts*, submitted to Comm. Algebra, 2021, arXiv:2105.02159.

This first chapter serves then to summarize the main results of the abovementioned papers and to present them in a general context.

The leitmotif connecting papers (**i**) - (**v**) is the notion of *(relative) compactness* in certain categories and its relationship to the structure of corresponding objects, so let us begin with a definition:

Let $\mathcal{F}$ be a family of objects of a locally small concrete category $\mathcal{C}$ such that $\coprod \mathcal{F}$ exists for any family $\mathcal{N}$ of objects of $\mathcal{F}$. Using the covariant functor $\mathrm{Hom}_{\mathcal{C}}(C,-)$ from $\mathcal{C}$ to the category Set (to the category *Ab* of abelian groups for $\mathcal{C} = R{-}\mathrm{Mod}$), we define a natural morphism in the target category

$$\Psi^C_{\mathcal{N}} : \coprod_{N\in\mathcal{N}} \mathrm{Hom}_{\mathcal{C}}(C,N) \to \mathrm{Hom}_{\mathcal{C}}(C, \coprod_{N\in\mathcal{N}} N),$$

which is the unique morphism such that the following triangle is commutative for all $N \in \mathcal{N}$:

$$\begin{tikzcd} & \coprod_{\mathcal{N}} \mathrm{Hom}_{\mathcal{C}}(C,N) \arrow[d, dashed, "\Psi^C_{\mathcal{N}}"] \\ \mathrm{Hom}_{\mathcal{C}}(C,N) \arrow[ur, "\mu_N"] \arrow[r, "\mathrm{Hom}_{\mathcal{C}(C,\nu_N)}"'] & \mathrm{Hom}_{\mathcal{C}}(C, \coprod_{\mathcal{N}} N), \end{tikzcd}$$

where $\mu_N : \mathrm{Hom}_{\mathcal{C}}(C, N) \to \coprod_{\mathcal{N}} \mathrm{Hom}_{\mathcal{C}}(C, N)$ is the coproduct inclusion in Set and $\nu_N : N \to \coprod_{N \in \mathcal{N}}$ is the coproduct injection in $\mathcal{C}$. It is not difficult to see that $\Psi^C_{\mathcal{N}}$ is in the case of module categories a monomorphism, however this fact does not hold in general.

An object $C \in \mathcal{C}$ is then called *$\mathcal{F}$-compact* (*$\mathcal{F}$-small* in the context of module categories), if $\Psi^C_{\mathcal{N}}$ is an epimorphism for any family $\mathcal{N}$ of objects of $\mathcal{F}$. A $\mathcal{C}$-compact object $C$ is called simply *compact* (*small* in module categories) – then the functor $\mathrm{Hom}_{\mathcal{C}}(C, -)$ is said to *preserve coproduts* – and a $\{C\}$-compact object is said to be *autocompact* (*self-small* in module categories).

In this work, the categories we consider, shall be categories of abelian groups, *Ab5*-categories and $S - \overline{\mathrm{Act}}, S - \mathrm{Act}_0$ for a monoid $\mathcal{S} = (S, \cdot, 1)$ with zero (see below).

An infinite coproduct of abelian groups (modules, in general) cannot be self-small, as the authors note immediately in the founding paper of the topic in the branch of abelian groups [1]; on the other hand, the structure of a product is from this point of view in general more complicated and there are several results concerning its self-smallness, begining with [1, Corollary 1.3], which had to be corrected by [12]; a counterexample to the original statement of [1, Corollary 1.3] within the category of abelian groups was then provided in (**i**, Example 3).

The paper (**ii**) investigates on the topic further by building first a general theory of relative smallness in the context of abelian groups. Neither this relitivizing approach to a property nor the notion itself are completely new, since it has been noted in [2] that for an abelian group $A$ of generalized rank 1 and a family $\mathcal{A}$ of $A$-solvable groups, $\bigoplus \mathcal{A}$ is $A$-solvable if and only if $A$ is $\mathcal{A}$-small. The study of other general properties relativized to a given group has also had many forms and applications: $A$-(local)-freeness and $A$-projectivity [1, 11], $A$-reflexive and $A$-cogenerated groups [3] and let us mention the application of this approach in the study of (so–called Warfield) dualities of various subcategories of the category of abelian groups [3].

The article (**ii**) deals first with generalizations of the classical results of [1] via properties of $\omega$-filtrations which provide a negative characterizations of relative smallness; these are then directly applied to prove a set of closure properties criteria of relative smallness and self-smallness (**ii**, Section 2).

Concerning self-smallness then, we obtain a characterization of self-small products via relative smallness (**ii**, Theorem 3.1):

**Theorem.** *Let $\mathcal{M}$ be a family of abelian groups and put $M = \prod \mathcal{M}$ and $S = \bigoplus \mathcal{M}$. Then the following conditions are equivalent:*

1. *$M$ is self-small,*
2. *$M$ is $S$-small,*
3. *$M$ is $\bigoplus \mathcal{C}$-small for each countable family $\mathcal{C} \subseteq \mathcal{M}$,*

and also a complete structural description of self-small products of finitely generated abelian groups given by (**ii**, Theorem 3.10):

**Theorem.** *Let $\mathcal{M}$ be a family of nonzero finitely generated abelian groups and put $M = \prod \mathcal{M}$, $S = \bigoplus \mathcal{M}$ and $Q = S/T_S$. Then the following conditions are equivalent:*

1. *$M$ is self-small,*
2. *either $T_S = 0$, or $S_{(p)}$ is finite for each $p \in \mathbb{P}$ and $Q$ is finitely generated*
3. *either all $A \in \mathcal{M}$ are free, or the family $\{B \in \mathcal{M} \mid Hom(B, A) \neq 0\}$ is finite for each $A \in \mathcal{M}$,*
4. *either $M \cong \mathbb{Z}^\kappa$ for some cardinal $\kappa$, or $M \cong F \oplus \prod_{p\in\mathbb{P}} M_p$ for a finitely generated free group $F$ and finite abelian $p$-groups $M_p$ for each $p \in \mathbb{P}$.*

Let us mention that by the previous theorem, e. g. the well-known Baer–Specker group $\mathbb{Z}^\omega$ is self-small.

A research of relative and auto- compactness from a more general standpoint of abelian category theory is presented in (**iii**), where the case of *Ab5*-categories is considered. Recall that an *Ab5*-category can be characterized as an abelian category such that for any direct system of exact sequences $X_i' \to X_i \to X_i''$, the canonically constructed exact sequence $\varinjlim X_i' \to \varinjlim X_i \to \varinjlim X_i''$ exists ([9, p. 64]). A similar generalizing approach to the compactness property was chosen in [5], where also the behaviour with respect to certain set-theoretical assumptions is considered.

The paper (**iii**) generalizes some of the results of (**ii**) concerning relative compactness, presents a general variant of [1, Proposition 1.1], i. e. the characterization of autocompactness via the endomorphism ring of the object, the characterization of finite autocompact coproducts (recall that an infinite coproduct must fail to be autocompact) and presents sufficient and necessary conditions for a product of objects to be autocompact both in the categorial version of (**ii**, Theorem 3.1), both in a fashion similar to [12, Proposition 1.6] in (**iii**, Theorem 5.8):

**Theorem.** *Let $\mathcal{M}$ be a family of objects of an Ab5-category $\mathcal{A}$ that has a compact projective generator and where the product $M = \prod \mathcal{M}$ exists. Put $M_N = \prod(\mathcal{M} \setminus \{N\})$ and let $\mathcal{A}(M_N, N) = 0$ for each $N$. Then $M$ is autocompact if and only if $N$ is autocompact for each $N \in \mathcal{M}$.*

The articles (**iv**) and (**v**) deal with the algebraic objects called $S$-acts and their categories: unlike abelian groups or $R-$modules in general, $S-$acts together with their homomorphisms form non-additive categories, whose structure, however, may in the case of monoids with zero in certain aspects, e. g. for our topic of (auto)compactness, resemble the behaviour of abelian categories and provide some useful insights.

We shall give here only the basic definitions necessary for the purpose of introduction, any further details and definitions can be found in the corresponding chapter or in [7]: let $\mathcal{S} = (S, \cdot, 1)$ be a monoid and $A$ a nonempty set. If there exists a mapping $- \cdot - : S \times A \to A$ satisfying the following two conditions: $1 \cdot a = a$ and $(s_1 \cdot s_2) \cdot a = s_1 \cdot (s_2 \cdot a)$, then $A$ is said to be a left $S$-act and it is denoted ${}_SA$ (the subscript may, however, be often omitted, if it is clear from the context). A mapping $f : {}_SA \to {}_SB$ is a homomorphism of $S$-acts (an $S$-homomorphism) provided $f(sa) = sf(a)$ holds for all pairs $s \in S, a \in A$.

An $S$-act ${}_SA$ considered as a representation of the monoid $S$ via transformations of the set $A$ can then be thought of as an analogy to an $R$-module as a representation of the ring $R$ via endomorphisms of the underlying abelian group. The theory of $S$-acts (known also under the names "$S$-automata", "$S$-sets", "transition systems" etc.) has many sources of methods and inspirations, some of them being also module theory and homological algebra, as e. g. the monograph [7] shows.

If the monoid $\mathcal{S}$ possesses a zero element (and this is the only class of monoids we shall consider in this work), there are two natural categories of $S$-acts, namely $S - \overline{\mathrm{Act}}$, and $S - \mathrm{Act}_0$ (for definitions see the corresponding chapter). Most of the literature and research in this branch has considered mainly the former category so far, however for our investigation focused on (auto)compactness in abelian categories, the latter will turn out to be very suitable, too.

The article (**iv**) begins with unifying the approach to both of the aforementioned categories from the viewpoint of decompositions into indecomposable objects via the notion of a $UD$-category:

A concrete category $(\mathcal{C}, U)$ over the category Set is a *UD-category*, if it satisfies a set of natural conditions (see the corresponding chapter) which as a consequence ensure the existence of nice decompositions of objects, since we have (**iv**, Theorem 3.10):

**Theorem.** *Every noninitial object of a UD-category has a decomposition into indecomposable objects, which is, up to the order, unique.*

Employing this result, we can prove that the structure of projective objects in a $UD$-category is rather transparent (**iv**, Theorem 4.3):

**Theorem.** *An object of a UD-category is projective if and only if it is isomorphic to a coproduct of indecomposable projective objects.*

Note that similarly to module categories, $S$-acts of the form $Se$ for an idempotent $e \in S$ are projective. In the $S$-act setting also the converse holds true (up to isomorphism) and furthermore an indecomposable projective act is cyclic ([7, Theorem 17. 7]).

Turning the attention to the compactness issue, let us mention the following result (**iv**, Corollary 5.4):

**Theorem.** *An object $C$ of a $UD$-category is compact if and only if for every pair of objects $A_1$ and $A_2$ and each morphism $f \in Mor(C, A_1 \coprod A_2)$ there exists $i \in \{1, 2\}$ such that $f$ factorizes through the structural coproduct morphism $\nu_i$,*

which has a direct consequence on the autocompactness of objects of a $UD$-category. Indecomposability and (auto)compactness of objects are mutually intertwined here, since we have (**iv**, Lemma 5.7):

**Theorem.** *An autocompact object of a $UD$-category is indecomposable.*

Even though the unification of the approach to the two categories of $S$-acts via $UD$-categories proved fruitful regarding decompositions and projectivity, there are differences concerning (auto)compactness, which call for a seperate treatment. Namely, in $S - \overline{\mathrm{Act}}$ we have the following characterization (**iv**, Theorem 6.11):

**Theorem.** *The following conditions are equivalent for an $S$-act $C \in S - \overline{\mathrm{Act}}$:*

1. *$C$ is autocompact,*

*2. $C$ is compact,*
*3. $C$ is indecomposable,*

and let us mention that this transparent behaviour within the category $S-\overline{\mathrm{Act}}$ was employed, e. g. in the study of Morita-type equivalences for monoids in [8].

On the other hand, within the category $S - \mathrm{Act}_0$, the situation is more complicated, as we only have

**Theorem.** *An $S$-act is compact in the category $S - \mathrm{Act}_0$ if and only if it is hollow,*

by (**iv**, Proposition 6.6), where a *hollow* act is such that it is not a union of a pair of proper subacts. A negative characterization of autocompact $S$-acts is then given by (**iv**, Proposition 6.15). Note that in general, unlike for $S - \overline{\mathrm{Act}}$, within the category $S - \mathrm{Act}_0$ the classes of compact, autocompact and indecomposable objects form a strict chain of inclusions.

The paper (**v**) is inspired mainly by the issue concerning the categories $S-\overline{\mathrm{Act}}$ and $S-\mathrm{Act}_0$ noted in (**iv**, Section 5.2), namely their different pullback behavior: call a category $\mathcal{C}$ *extensive*, if it is closed under finite coproducts, it has pullbacks along colimit structural morphisms and for every commutative diagram

$$\begin{tikzcd} X \arrow[r, "f_z"] \arrow[d, "f_a"'] & Z \arrow[d, "\zeta"] & Y \arrow[l, "g_z"'] \arrow[d, "g_B"] \\ A \arrow[r, "\nu_A"] & A\coprod B & B \arrow[l, "\nu_B"'] \end{tikzcd}$$

in $\mathcal{C}$, the top row is a coproduct diagram if and only if the squares are pullbacks. Similarly, a category is said to be *infinitary extensive*, if the condition on the diagram above holds also for infinite coproducts. Now, as Proposition 5.12 and Example 5.13 of (**iv**) show, the category $S - \overline{\mathrm{Act}}$ is infinitary extensive for arbitrary monoid with zero, while the category $S - \mathrm{Act}_0$ may not be. This seemingly negligible difference causes notably different properties regarding (auto)compactness as noted before or at [13, Theorem 3.1], but also poses questions considering further possible differences in behavior of the two categories: that one concerning the projectivity of objects is then answered in (**v**).

Inspired by module categories, call a monoid with zero *left perfect* (*left 0-perfect*), if each $A \in S - \overline{\mathrm{Act}}$ ($A \in S - \mathrm{Act}_0$) has a projective cover, where a *projective cover* $f : P \to A$ is an epimorphism (in the respective category) with $P$ a projective act, such that for any proper subact $P' \subset P$ the restriction $f|_{P'} : P' \to A$ is not an epimorphism. By (**v**, Theorem 18), we have:

**Theorem.** *A monoid with zero is left-perfect if and only if it is left-0-perfect.*

Inspired by module theory again, call a monoid $S$ *left 0-steady* if every compact act in the category $S - \mathrm{Act}_0$ is cyclic. Then (**iv**, Proposition 6.9) gives the following:

**Theorem.** *If a monoid is left 0-perfect, then it is left-0-steady,*

and we also have a characterization of left 0-steady monoids by (**v**, Theorem 22):

**Theorem.** *A monoid $S$ is left 0-steady if and only if it satisfies the ascending chain condition on cyclic subacts.*

Note that by [4, 6] the perfectness of a general monoid (possibly without zero) depends on two structural conditions, one of which is the a.c.c. from the previous theorem.

# Bibliography for the Introduction

[1] D.M. Arnold, C.E. Murley, Abelian groups, A, such that $\mathrm{Hom}(A, -)$ preserves direct sums of copies of A, Pacific J. Math., 56 (1975), 7–20

[2] Albrecht, U., Abelian Groups, A, such that the category of A-solvable groups is preabelian. Contemporary Mathematics 87 (1989):117–131.

[3] Breaz, S., Warfield dualities induced by self-small mixed groups. Journal of Group Theory (2010), 13(3)

[4] J. Isbell, Perfect monoids. Semigroup Forum (1971) 2, 95–118.

[5] Kálnai, P., Zemlička, J., Compactness in abelian categories, *J. Algebra*, 534 (2019), 273–288

[6] M. Kilp, Perfect monoids revisited. Semigroup Forum (1996) 53, 225–229.

[7] M. Kilp, U. Knauer, A.V. Mikhalev, Monoids, acts and categories, de Gruyter, Expositions in Mathematics 29, Walter de Gruyter, Berlin 2000.

[8] C.G. Modoi, Localizations, colocalizations and non additive *-objects, *Semigroup Forum* 81(2010), No. 3, 510-523.

[9] Popescu N., Abelian categories with applications to rings and modules (1973), Boston, Academic Press.

[10] Vinsonhaler, C., Wickless, W., Dualities for torsion-free abelian groups of finite rank. Journal of Algebra (1990), 128(2), 474–487.

[11] Warfield, R. B., Homomorphisms and duality for torsion-free groups. Mathematische Zeitschrift (1968), 107(3), 189–200.

[12] Žemlička, J., When products of self-small modules are self-small. *Commun. Algebra* **36** (2008), No. 7, 2570–2576.

[13] Connected object, on The nLab, URL: https://ncatlab.org/nlab/show/connected+object, downloaded on May 14, 2021.

# Chapter 2

# On products of self-small abelian groups

The notion of self-small module as a generalization of the finitely generated module appears as a useful tool in the study of splitting properties [1], groups of homomorphisms of graded modules [10] or representable equivalences between subcategories of module categories [8].

The paper [4] in which the topic of self-small modules is introduced contains a mistake in the proof of [4, Corollary 1.3], which states when the product of (infinite) system $(A_i \mid i \in I)$ of self-small modules is self-small. A counterexample and correct version of the hypothesis were presented in [12] for a system of modules over a non-steady abelian regular ring. In the present paper an elementary counterexample in the category of $\mathbb{Z}$-modules, i.e. abelian groups, is constructed and as a consequence, an elementary example of two self-small abelian groups such that their product is not self-small is presented.

Throughout the paper a *module* means a right module over an associative ring with unit. If $A$ and $B$ are two modules over a ring $R$, $\mathrm{Hom}_R(A, B)$ denotes the abelian group of all $R$-homomorphisms $A \to B$. The set of all prime numbers is denoted by $\mathbb{P}$, for given $p \in \mathbb{P}$, $\mathbb{Z}_p$ means the cyclic group of order $p$ and $\mathbb{Q}$ is the group of rational numbers. $E(A)$ denotes the injective envelope of the module $A$. Recall that injective $\mathbb{Z}$-modules, i.e. abelian groups, are precisely the divisible ones. For non-explained terminology we refer to [9].

**Definition.** An $R$-module $A$ is *self-small*, if for arbitrary index set $I$ and each homomorphism $f \in \mathrm{Hom}_R(A, \bigoplus_{i\in I} A_i)$, where $A_i \cong A$, there exists a finite $I' \subseteq I$ such that $f(A) \subseteq \bigoplus_{i\in I'} A_i$.

Properties of self-small modules and mainly of self-small groups are thoroughly investigated in [2], [3], [4], [5] and [6] revealing several characterizations of self-small groups and discussing the properties of the category of self-small groups and modules.

For our purpose the following notation will be of use:

**Definition.** For an $R$-module $A$ and $B \subseteq A$ we define the annihilator of $B$

$$B^* := \{f \mid f \in \mathrm{End}_R(A), f(a) = 0 \text{ for each } a \in B\}\ .$$

The first (negative) characterization of self-small modules is given in [4] and it describes non-self-small modules via annihilators and chains of submodules:

**Theorem 2.1.** *[4, Proposition 1.1] For an $R$-module $A$ the following conditions are equivalent:*

1. *$A$ is not self-small*
2. *there exists a chain $A_1 \subseteq A_2 \subseteq \dots \subseteq A_n \subseteq \dots \subsetneq A$ of proper submodules in $A$ such that $\bigcup_{n=1}^{\infty} A_n = A$ and for each $n \in \mathbb{N}$ we have $A_n^* \neq \{0\}$.*

## 2.1 Examples

The key tool for constructions of this paper is the following well-known lemma:

**Lemma 2.2.** $\prod_{p\in\mathbb{P}} \mathbb{Z}_p / \bigoplus_{p\in\mathbb{P}} \mathbb{Z}_p \cong \mathbb{Q}^{(2^\omega)}$.

*Proof.* Since $\bigoplus_{p\in\mathbb{P}} \mathbb{Z}_p$ is the torsion part of $\prod_{p\in\mathbb{P}} \mathbb{Z}_p$, the group $\prod_{p\in\mathbb{P}} \mathbb{Z}_p / \bigoplus_{p\in\mathbb{P}} \mathbb{Z}_p$ is torsion-free. Now the assertion follows from [7, Exercises S 2.5 and S 2.7]. □

Recall that $\mathbb{Q}$ is torsion-free of rank 1 and each nontrivial factor of $\mathbb{Q}$ is a torsion group, hence there is no nonzero non-injective endomorphism of $\mathbb{Q}$, which by Theorem 2.1 implies well-known fact that $\mathbb{Q}$ is self-small.

Using this observation, a counterexample to [4, Corollary 1.3] can be constructed:

**Example 2.3.** Since $\mathbb{Z}_p$ is finite for every $p \in \mathbb{P}$, it is a self-small group. Now, all homomorphisms between $\mathbb{Z}_p$'s for different $p \in \mathbb{P}$, or $\mathbb{Q}$ are trivial:

$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_p, \mathbb{Q}) = \{0\}$, since $\mathbb{Z}_p$ is a torsion group, whereas $\mathbb{Q}$ is torsion-free. $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}_p) = \{0\}$, since every factor of $\mathbb{Q}$ is divisible and 0 is the only divisible subgroup of $\mathbb{Z}_p$. Obviously, $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_q, \mathbb{Z}_p) = \{0\}$.

Let $A = \mathbb{Q} \times \prod_{p\in\mathbb{P}} \mathbb{Z}_p$ and $B = A / \left(\mathbb{Q} \times \bigoplus_{p\in\mathbb{P}} \mathbb{Z}_p\right) \cong \prod_{p\in\mathbb{P}} \mathbb{Z}_p / \bigoplus_{p\in\mathbb{P}} \mathbb{Z}_p$. Then by Lemma 2.2 there exists a countable chain of subgroups $B_i \subseteq B_{i+1}$ of $B$, $i \in \mathbb{N}$, such that $B = \bigcup_{i\in\mathbb{N}} B_i$ and $\mathrm{Hom}_{\mathbb{Z}}(B/B_n, \mathbb{Q}) \neq 0$ for each $n$, where $\mathbb{Q}$ may be viewed as a subgroup of $A$. Now put $A_n$ to be the preimage of $B_n$ in $A$ under factorization by $\mathbb{Q} \times \bigoplus_{p\in\mathbb{P}} \mathbb{Z}_p$. Then the subgroups $A_n$, $n \in \mathbb{N}$ form a chain of subgroups and $A = \bigcup_{n\in\mathbb{N}} A_n$. At the same time, given an $n \in \mathbb{N}$, composing the factorization of $A$ by $\mathbb{Q} \times \bigoplus_{p\in\mathbb{P}} \mathbb{Z}_p$ with factorization by $B_n$ and finally composing it with a non-vanishing $\nu_n : B/B_n \to \mathbb{Q}$, we get an endomorphism $\varphi_n$ of the group $A$ such that $A_n \subseteq \ker \varphi_n$. Therefore the condition of Theorem 2.1 is satisfied, hence the group $A$ is not self-small.

The previous example shows that for two different primes $p, q$

$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_q, \mathbb{Z}_p) = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}_p) = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_p, \mathbb{Q}) = \{0\},$$

all the groups $\mathbb{Z}_p$, $p \in \mathbb{P}$ and $\mathbb{Q}$ are self-small, but the group $\mathbb{Q} \times \prod_{p\in\mathbb{P}} \mathbb{Z}_p$ is not self-small.

Recall that classes of small modules, i.e. modules over which the covariant Hom-functor commutes with all direct sums, are closed under homomorphic images and extensions [11, Proposition 1.3]. Obviously, self-small modules do not satisfy this closure property and, moreover, although the class of self-small modules is closed under direct summands, the last example illustrates that it is not closed under finite direct sums.

**Proposition 2.4.** *The following conditions are equivalent for a finite system of self-small R-modules $(M_i|\ i \leq k)$:*

1. $\prod_{i\leq k} M_i$ *is not self-small*
2. *there exist $i, j \leq k$ and a chain $N_1 \subseteq N_2 \subseteq ... \subseteq N_n \subseteq ...$ of proper submodules of $M_i$ such that $\bigcup_{n=1}^{\infty} N_n = M_i$ and $Hom_R(M_i/N_n, M_j) \neq 0$ for each $n \in \mathbb{N}$.*

*Proof.* Put $M = \prod_{i\leq k} M_i$.

(1)⇒(2) If $M$ is not self-small, there exists a chain $A_1 \subseteq A_2 \subseteq \ldots \subseteq A_n \subseteq \ldots$ of proper submodules of $M$ for which $\bigcup_{n=1}^{\infty} A_n = M$ and $\mathrm{Hom}_R(M/A_n, M) \neq 0$ for each $n \in \mathbb{N}$. Put $A_n^i = M_i \cap A_n$ for each $i \leq k$ and $n \in \mathbb{N}$. Then $M_i = \bigcup_n A_n^i$ for each $i \leq k$ and there exists at least one index $i$ such that the chain $A_1^i \subseteq A_2^i \subseteq \ldots \subseteq A_n^i \subseteq \ldots$ consists of proper submodules of $M_i$ (or else the condition on the original chain is broken) and further on we consider only such $i'$s.

Since for each $n \in \mathbb{N}$ there exist $0 \neq b_n \in M \setminus A_n$ and $f_n : M/A_n \to M$ such that $f_n(b_n + A_n) \neq 0$, for each $n$ we can find an index $i(n) \leq k$ with $f_n \pi_{A_n} \nu_{i(n)} \pi_{i(n)}(b_n) \neq 0$ (where $\pi_{i(n)}$, resp. $\nu_{i(n)}$ are the natural projection, resp. injection and $\pi_{A_n}$ is is the natural projection $M \to M/A_n$). Now, by pigeonhole principle, there must exist at least one index $i_0$ such that $S := \{n \in \mathbb{N} \,|\, i(n) = i_0\}$ is infinite. By the same principle, there must exist at least one index $j_0$ such that $T := \{n \in S \,|\, \pi_{j_0} f_n \pi_{A_n} \nu_{i_0} \pi_{i_0}(b_n) \neq 0\}$ is infinite. The couple $i_0$, $j_0$ proves the implication.

(2)⇒(1) Put $A_n = \pi_i^{-1}(N_n)$ where $\pi_i : M \to M_i$ is the natural projection, so $\bigcup_n A_n = M$. If $0 \neq f_n \in \mathrm{Hom}_R(M_i, M_j)$ such that $N_n \subseteq \ker f_n$ and $f_n(m_n) \neq 0$ for some suitable $m_n \in M_i$, then $\nu_j f_n \pi_i \in \mathrm{Hom}_R(M, M)$, where $\nu_j : M_j \to M$ is the natural injection, $A_n \subseteq \ker \nu_j f_n \pi_i$ and the nonzero element having $m_n$ on the $i$-th position show that the condition of the Theorem 2.1 holds. □

**Corollary 2.5.** *Let $(M_i|\ i \leq k)$ be a finite system of R-modules. Then $\prod_{i\leq n} M_i$ is self-small if and only if for every $i, j$ and every chain $N_1 \subseteq N_2 \subseteq ... \subseteq N_n \subseteq ...$ of proper submodules of $M_i$ such that $\bigcup_{n=1}^{\infty} N_n = M_i$ there exist $n$ for which $Hom_R(M_i/N_n, M_j) = 0$.*

In consequence we see that the "finite version" of [4, Corollary 1.3] remains true:

**Corollary 2.6.** *Let $(M_i|\ i \leq n)$ be a finite system of self-small modules satisfying the condition $Hom_R(M_j, M_i) = 0$ for each $i \neq j$. Then $\prod_{i\leq n} M_i$ is a self-small module.*

Finally, as a consequence of Example 2.3 an elementary example of two self-small abelian groups such that their product is not self-small may be constructed. It illustrates that the assumption $\mathrm{Hom}_{\mathbb{Z}}(M_j, M_i) = 0$ for each $i \neq j$ cannot be omitted even in the category of $\mathbb{Z}$-modules.

**Example 2.7.** By [12, Example 2.7] the group $\prod_{p\in\mathbb{P}} \mathbb{Z}_p$ is self-small as well as the group $\mathbb{Q}$. Moreover, $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \prod_{p\in\mathbb{P}} \mathbb{Z}_p) = \prod_{p\in\mathbb{P}} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}_p) = 0$. Nevertheless, the product $\mathbb{Q} \times \prod_{p\in\mathbb{P}} \mathbb{Z}_p$ is not self-small by Example 2.3. Note that it is not surprising in view of Corollary 2.6 that the structure of $\mathrm{Hom}_{\mathbb{Z}}(\prod_{p\in\mathbb{P}} \mathbb{Z}_p, \mathbb{Q})$ is quite rich as shown in Lemma 2.2.

# Bibliography for chapter 2


[1] Albrecht, U., Breaz, S., Wickless, W., The finite quasi-Baer property, J. Algebra, **293** (2005), 1–16.

[2] Albrecht, U., Breaz, S., Wickless, W., Self-small Abelian groups. *Bull. Aust. Math. Soc.* **80** (2009), No. 2, 205–216.

[3] Albrecht, U., Breaz, S., Wickless, W., Purity and self-small groups. Commun. Algebra **35** (2007), No. 11, 3789-3807.

[4] Arnold, D.M., Murley, C.E., Abelian groups, $A$, such that $\mathrm{Hom}\,(A,-)$ preserves direct sums of copies of $A$. *Pacific Journal of Mathematics*, Vol. **56** (1975), No.1, 7–20.

[5] Breaz, S., Schultz, P., Dualities for Self-small Groups *Proc. A.M.S.*, **140** (2012), No. 1, 69–82 .

[6] Breaz, S., Žemlička, J. When every self-small module is finitely generated. *J. Algebra* **315** (2007), 885–893.

[7] Călugăreanu, G., Breaz, S., Modoi, C., Pelea, C., Vălcan, D, *Exercises in abelian group theory* Kluwer Texts in the Mathematical Sciences, Kluwer, Dordrecht, 2003.

[8] Colpi, R., Menini, C., On the structure of $\star$-modules, J. Algebra **158** (1993), 400–419.

[9] Fuchs, L., Infinite Abelian Groups, Vol.I, Academic press, New York and London 1970

[10] Gómez Pardo, J. L. , Militaru, G., Năstăsescu, C., When is $\mathrm{HOM}(M,-)$ equal to $\mathrm{Hom}(M,-)$ in the category $R-gr$?, Comm. Algebra, **22** (1994), 3171–3181.

[11] J. Žemlička: *Classes of dually slender modules*, Proceedings of the Algebra Symposium, Cluj, 2005, Editura Efes, Cluj-Napoca, 2006, 129–137.

[12] Žemlička, J., When products of self-small modules are self-small. *Commun. Algebra* **36** (2008), No. 7, 2570–2576.

# Chapter 3

# Self-small products of abelian groups

The study of modules whose covariant functor $\mathrm{Hom}(M, -)$ commutes with all direct sums, which is a condition providing a categorial generalization the notion of finitely generated module, started in 60's by the work of Hyman Bass [6, p.54] and Rudolf Rentschler [20]. Such modules have appeared as a useful tool in diverse contexts and under various names (small, $\Sigma$-compact, U-compact, dually slender) in ring theory, module theory and in the study of abelian groups. In 1974, David M. Arnold and Charles E. Murley published their influential paper [5] dedicated to a weaker variant of the studied condition, namely commuting of the functor $\mathrm{Hom}(A, -)$ with direct sums of the tested module itself. Groups and modules satisfying this restricted condition are usually called self-small in literature. Many interesting results concerning self-small modules over unital rings in general have appeared later [1, 10, 11, 16, 18], self-small abelian groups proving to be a particularly successful tool [2, 3, 4, 7, 8, 9].

The aim of this paper is to deepen the present knowledge about structure of self-small groups and about possibilities of testing abelian groups for self-smallness by adopting some ideas of the papers [2, 13, 17] and extending several results of [12, 21]. Namely, we deal with the notion of a relatively small abelian group (defined in [2, 16], cf. also relatively compact objects in [17]) which serves as a tool for characterization of those products of groups that are self-small.

Throughout the paper *module* means a right module over an associative ring with unit and an abelian group is a module over the ring of integers. Note that we will use the term *group* instead of abelian group frequently, as non-abelian groups are not considered here. If $A$ and $B$ are two abelian groups, then $\mathrm{Hom}(A, B)$ denotes the abelian group of homomorphisms $A \to B$. A *family of groups* means a discrete diagram in the category of abelian groups, so a family may contain more that one copy of a group. The set of all prime numbers is denoted by $\mathbb{P}$ and we identify cardinals with least ordinals of given cardinality.

For non-explained terminology we refer to [14, 15].

## 3.1 Relatively small groups

Let $A$ be an abelian group and $\mathcal{N}$ a family of abelian groups. It is well-known (and easy to see) that the functor $\mathrm{Hom}(A, -)$ induces an injective homomorphism of abelian groups

$$\Psi_{\mathcal{N}} : \bigoplus_{N\in\mathcal{N}} \mathrm{Hom}(A, N) \to \mathrm{Hom}(A, \bigoplus \mathcal{N})$$

given by the rule $\Psi_{\mathcal{N}}((f_N)_N) = \sum_N f_N$ (cf. e.g. [17, Lemma 1.3]), where $\sum_N f_N \in \mathrm{Hom}(A, \bigoplus \mathcal{N})$ is defined by the rule $a \to \sum_N f_N(a)$ for $f_N$ viewed as a homomorphism into $\bigoplus \mathcal{N}$. Suppose, then, that $\mathcal{C}$ is a class of groups. We say that $A$ is *$\mathcal{C}$-small* if $\Psi_{\mathcal{N}}$ is an isomorphism for any family $\mathcal{N}$ of groups from the class $\mathcal{C}$. If $B$ is an abelian group, $A$ is said to be *$B$-small* provided it is a $\{B\}$-small group (cf. [2, 13, 16, 17]). It is clear that $A$-small abelian groups $A$ are exactly *self-small* ones as defined in [5].

**Example 3.1.** (1) Every finitely generated abelian group is small, so $B$-small for every group $B$. In, particular each finite group is self-small.

(2) Let $A$ and $B$ be two abelian groups such that $\mathrm{Hom}(A, B) = 0$. Then it is easy to see that $A$ is $B$-small.

In particular, if $p, q \in \mathbb{P}$ are different primes, $A_p$ is an abelian $p$-group and $A_q$ is an abelian $q$-group, then $A_p$ is $A_q$-small and $\mathbb{Z}$-small.

**Example 3.2.** It is clear, $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are $\mathbb{Q}$-small groups but neither $\mathbb{Q}$ nor $\mathbb{Q}/\mathbb{Z}$ is $\mathbb{Q}/\mathbb{Z}$-small. Furthermore, $\mathbb{Q}$-small groups are precisely groups of finite torsion-free rank by [2, Corollary 4.3.].

We start with an elementary observation which translates the definition of a relative small group to an easily tested condition (cf. [20, Section 1], [17, Lemma 1.4(2)] and [13, Theorem 1.6(2)]):

**Lemma 3.3.** *Let $A$ be an abelian group and $\mathcal{C}$ a class of abelian groups. Then $A$ is $\mathcal{C}$-small if and only if for each family $\mathcal{N}$ of groups contained in the class $\mathcal{C}$ and every $f \in \mathrm{Hom}(A, \bigoplus \mathcal{N})$ there exists a finite family $\mathcal{F} \subseteq \mathcal{N}$ such that $f(A) \subseteq \bigoplus \mathcal{F}$. In particular, for a group $B$, the group $A$ is $B$-small if and only if for every index set $I$ and every $f \in \mathrm{Hom}(A, B^{(I)})$ there exists a finite subset $F \subseteq I$ such that $f(A) \subseteq B^{(F)}$.*

*Proof.* The argument of the proof is well known; if $\Psi_{\mathcal{N}}$ is onto and we take an $f \in \mathrm{Hom}(A, \bigoplus \mathcal{N})$, then $f$ is of the form $\sum_{N_i} f_{N_i}$ for a finite family $(N_i \mid i = 1, \dots n) \subseteq \mathcal{N}$ and $f_{N_i} \in \mathrm{Hom}(A, N_i)$, hence $f(A) = \sum_{N_i} f_{N_i}(A) \subseteq \bigoplus_{i=1}^n N_i$. On the other hand, if $f(A) \subseteq \bigoplus_{i=1}^n N_i \subseteq \bigoplus \mathcal{N}$, then $\Psi_{\mathcal{N}}((\pi_{N_i} f)_{N_i}) = f$, where $\pi_{N_i}$ denotes the projection onto the corresponding component. $\square$

The observation that the concept of relatively small groups is general enough if we consider relative smallness over a set of groups (cf. general [13, Lemma 2.1]) presents a first application of the previous lemma. To that end, for a class of groups define

$$\mathrm{Add}(\mathcal{C}) = \{A \mid A \text{ is a direct sumand of } \bigoplus_{\alpha<\kappa} C_\alpha \text{ for some cardinal } \kappa \text{ and } C_\alpha \in \mathcal{C}\}$$

and by $\mathrm{Add}(A)$ denote $\mathrm{Add}(\{A\})$.

**Lemma 3.4.** *Let $A$ be an abelian group and $\mathcal{C}$ be a set of abelian groups. Then the following conditions are equivalent:*

1. *$A$ is $\bigoplus \mathcal{C}$-small,*

2. *$A$ is $\mathcal{C}$-small,*

3. *$A$ is $\mathrm{Add}(\bigoplus \mathcal{C})$-small.*

*Proof.* (1)⇒(3) Put $B = \bigoplus \mathcal{C}$, let $\mathcal{N}$ be a family of groups contained in $\mathrm{Add}(B)$, and $f \in \mathrm{Hom}(A, \bigoplus \mathcal{N})$. Then for each $N \in \mathcal{N}$ there exists a cardinal $\kappa_N$ for which $N \subseteq B^{(\kappa_N)}$ ($N$ is also a direct summand of $B^{(\kappa_N)}$), and so $f(A) \subseteq \bigoplus \mathcal{N} \subseteq \bigoplus_{N \in \mathcal{N}} B^{(\kappa_N)}$. Since $A$ is $B$-compact, there exists finite family $\mathcal{F} \subseteq \mathcal{N}$ such that $f(A) \subseteq \bigoplus_{N \in \mathcal{F}} B^{(\kappa_N)}$ which implies that $f(A) \subseteq \bigoplus \mathcal{F}$.

(3)⇒(2) It is obvious since $\mathcal{C} \subseteq \mathrm{Add}(\bigoplus \mathcal{C})$.

(2)⇒(1) As any group $B \in \mathcal{C}$ is a direct summand of $\bigoplus \mathcal{C}$, the same argument as in the implication (1)⇒(3) proves the assertion. □

Since $\mathrm{Add}(B) = \mathrm{Add}(B^{(\kappa)})$ for an arbitrary group $B$ and a nonzero cardinal $\kappa$, we obtain the following useful criterion:

**Corollary 3.5.** *Let $A$ and $B$ be abelian groups and $\kappa$ a nonzero cardinal. Then $A$ is $B$-small if and only if $A$ is $B^{(\kappa)}$-small.*

As a consequence, we can formulate a well-known closure property of the class of all self-small groups.

**Corollary 3.6.** *Let $\kappa$ be a cardinal and $A$ an abelian group. Then $A^{(\kappa)}$ is self-small if and only if $A$ is self-small and $\kappa$ is finite.*

Let us formulate a variant of the assertion [2, Theorem 4.1.], which generalize the classical criterion of self-small groups [5, Proposition 1.1] for the case of relatively small groups (cf. [13, Lemma 3.3]). Recall that the family $(A_i \mid i < \omega)$ is said to be *$\omega$-filtration* of a group $A$, if it is a chain of subgroups of $A$, i.e. $A_i \subseteq A_{i+1}$ for each $i < \omega$, with $A = \bigcup_{n<\omega} A_n$.

**Proposition 3.7.** *The following conditions are equivalent for abelian groups $A$ and $B$:*

1. *$A$ is not $B$-small,*

2. *there exists a homomorphism $f \in Hom(A, B^{(\omega)})$ such that $f(A) \nsubseteq B^{(n)}$ for all $n < \omega$,*

3. *there exists an $\omega$-filtration $(A_i \mid i < \omega)$ of $A$ such that for each $n < \omega$ there exists a nonzero $f_n \in Hom(A, B)$ satisfying $f_n(A_n) = 0$,*

4. *there exists an $\omega$-filtration $(A_i \mid i < \omega)$ of $A$ such that $Hom(A/A_n, B) \neq 0$ for each $n < \omega$.*

*Proof.* The proof works using similar arguments as in [5, Proposition 1.1].

(1)⇒(2) By Lemma 3.3 there exists a set $I$ and $g \in \mathrm{Hom}(A, B^{(I)})$ such that $g(A) \nsubseteq B^{(F)}$ for any finite $F \subset I$. Then we can construct by induction a sequence of finite sets $I_n \subset I$ such that $I_0 = \emptyset$, $|I_n \setminus I_{n-1}| = 1$ and $\ker \pi_{I_{n-1}} g \supsetneq \ker \pi_{I_n} g$ for all $n < \omega$ where $\pi_{I_n} \in \mathrm{Hom}(B^{(I)}, B^{(I_n)})$ denotes the natural projection. If we put $I_\omega = \bigcup_{i<\omega} I_i$, then $\pi_{I_\omega} g \in \mathrm{Hom}(A, B^{(I_\omega)})$ represents the desired homomorphism.

(2)⇒(3) Let $f \in \mathrm{Hom}(A, B^{(\omega)})$ satisfy the condition (2) and define $A_n = f^{-1}(B^{(n,\omega)})$ where $B^{(n,\omega)} = \{b \in B^{(\omega)} \mid \pi_i(b) = 0\ \forall i \leq n\}$ for natural projections $\pi_i : B^{(\omega)} \to B$ onto the $i$-th coordinate. Then $A = \bigcup_{i<\omega} A_i$ and for each $i < \omega$ there exist $n_i > i$ such that $f_i = \pi_{n_i} f \neq 0$ with $f_i(A_i) = 0$.

(3)⇒(4) It is enough to observe that any nonzero $f_n \in \mathrm{Hom}(A, B)$ that satisfies $f_n(A_n) = 0$ can be factorized through the natural projection $A \to A/A_n$, so there exists a nonzero $\tilde{f}_n \in \mathrm{Hom}(A/A_n, B)$.

(4)⇒(1) Let $f_i \in \mathrm{Hom}(A/A_i, B)$ denote a nonzero homomorphism and define a homomorphism $f \in \mathrm{Hom}(A, B^\omega)$ by the rule $\pi_i(f(a)) = f_i(a + A_i)$ for each $a \in A$ and $i < \omega$. Then $f(A) \subseteq B^{(\omega)}$ since for each $a \in A$ there exists $n$ such that $a \in A_i$ for all $i \geq n$, hence $f \in \mathrm{Hom}(A, B^{(\omega)})$. On the other hand, $f(A) \nsubseteq B^{(n)}$ for any $n < \omega$ as $\pi_n f \neq 0$, $i < \omega$. Thus $A$ is not $B$-small by Lemma 3.3. □

The previous assertion applied on $A = B$ allows us to reformulate [12, Proposition 9].

**Corollary 3.8.** *The following conditions are equivalent for an abelian group $A$:*

1. *$A$ is not self-small,*
2. *there exists an $\omega$-filtration $(A_i \mid i < \omega)$ of $A$ such that $Hom(A/A_n, A) \neq 0$ for each $n < \omega$,*
3. *there exists an $\omega$-filtration $(A_i \mid i < \omega)$ of $A$ such that for each $n < \omega$ there exists a nonzero $\varphi_n \in End(A)$ satisfying $\varphi_n(A_n) = 0$.*

**Example 3.9.** Put $P = \prod_{p \in \mathbb{P}} \mathbb{Z}_p$. Then $\mathrm{Hom}(\mathbb{Q}, P) = 0$ by [12, Example 4], hence $\mathbb{Q}$ is $P$-small. On the other hand, if we put $B = P/\bigoplus_{p\in\mathbb{P}} \mathbb{Z}_p$, then there exists exists an $\omega$-filtration $(B_i \mid i < \omega)$ of $B$ such that $\mathrm{Hom}(B/B_n, \mathbb{Q}) \neq 0$ for each $n$ by [12, Example 3]. If we take preimages $A_n$ of all $B_n$ in canonical projection $P \to P/\bigoplus_{p\in\mathbb{P}} \mathbb{Z}_p$, then $(A_i \mid i < \omega)$ forms an $\omega$-filtration of $A$ satisfying $\mathrm{Hom}(A/A_n, \mathbb{Q}) \cong \mathrm{Hom}(B/B_n, \mathbb{Q}) \neq 0$, hence $P$ is not $\mathbb{Q}$-small by Proposition 3.3 (equivalently, we could use [2, Corollary 4.3.]).

## 3.2 Closure properties of relative smallness

First, let us formulate several elementary relations between classes of relatively small groups.

**Lemma 3.10.** *Let $A$, $B$ and $C$ be abelian groups and $I$ be a set. Suppose that $A$ is $B$-small.*

1. *If $C$ is a subgroup of $A$, then $A/C$ is $B$-small.*
2. *If $C$ is embeddable into $B^I$, then $A$ is $C$-small.*

*Proof.* (1) Proving indirectly, we assume that $\overline{A} = A/C$ is not $B$-small. Then there exists an $\omega$-filtration $(\overline{A}_i \mid i < \omega)$ of $\overline{A}$ for which $\mathrm{Hom}(\overline{A}/\overline{A}_n, B) \neq 0$ for all $n < \omega$ by Proposition 3.7. If we lift all the groups of the $\omega$-filtration of $\overline{A}$ to the $\omega$-filtration $(A_i \mid i < \omega)$ of $A$ satisfying the conditions $C \leq A_n$ and $A_n/C = \overline{A}_n$ for each $n$, then $\mathrm{Hom}(A/A_n, B) \cong \mathrm{Hom}(\overline{A}/\overline{A}_n, B) \neq 0$, hence $A$ is not $B$-small by Proposition 3.7.

(2) We may suppose w.l.o.g. that $C \leq B^I$. Assume $A$ is not $C$-small and consider the $\omega$-filtration $(A_i \mid i < \omega)$ of $A$ for which $\mathrm{Hom}(A/A_n, C) \neq 0$ provided by Proposition 3.7. Then we have $\mathrm{Hom}(A/A_n, B^I) \neq 0$ for each $n < \omega$ and since for each nonzero $f_n \in \mathrm{Hom}(A/A_n, B^I)$ there exists $i \in I$ such that $\pi_i f_n \neq 0$, we conclude that $\mathrm{Hom}(A/A_n, B) \neq 0$ for every $n < \omega$, a contradiction. □

**Proposition 3.11.** *Let $A$ be a self-small abelian group.*

1. *If $f \in Hom(A, A^I)$ for an index set $I$, then $f(A)$ is self-small.*
2. *If $I \subseteq End(A)$, then $A/\bigcap\{\ker \iota \mid \iota \in I\}$ is self-small.*

*Proof.* (1) Since $A$ is $A$-small, $f(A)$ is $A$-small by Lemma 3.10(1). Thus $f(A)$ is $f(A)$-small by Lemma 3.10(2).

(2) If $\varphi : A \to A^I$ is defined by the rule $\pi_\iota \varphi = \iota$ for each $\iota \in I$, then $\ker \varphi = \bigcap\{\ker \iota \mid \iota \in I\}$, hence $A/\bigcap\{\ker \iota \mid \iota \in I\} \cong f(A)$ is self-small by (1) (cf. also [13, Example 2.10]). □

The next assertion describes closure properties concerning extensions.

**Proposition 3.12.** *Let $A$ and $C$ be abelian groups and $B \leq C$.*

1. *If both $B$ and $C/B$ are $A$-small, then $C$ is $A$-small.*
2. *If $A$ is $B$-small and $C/B$-small, then $A$ is $C$-small.*

*Proof.* Similarly as in Lemma 3.10, we will use throughout the whole proof the correspondence of relative nonsmallness and properties of $\omega$-filtrations given by Proposition 3.7. Let us denote by $\pi_B : C \to C/B$ the natural projection and by $\iota_B : B \to C$ the inclusion homomorphism.

(1) Suppose that $(C_n \mid n < \omega)$ is an $\omega$-filtration of $C$. Then $(C_n \cap B \mid n < \omega)$ is an $\omega$-filtration of $B$ and $(C_n + B/B \mid n < \omega)$ is an $\omega$-filtration of $C/B$. Since $B$ and $C/B$ are $A$-small, there exists $n$ such that $f(B) = 0$ whenever $f \in \mathrm{Hom}(B, A)$ satisfies $f(B \cap C_n) = 0$, and $\tilde{f}(C/B) = 0$ whenever $\tilde{f} \in \mathrm{Hom}(C/B, A)$ satisfies $\tilde{f}(C_n + B/B) = 0$.

Let $f \in \mathrm{Hom}(C, A)$ be such that $f(C_n) = 0$, then $f(B) = 0$ as $f(C_n \cap B) = 0$ and there exists $\tilde{f} \in \mathrm{Hom}(C/B, A)$ for which $\tilde{f}\pi_B = f$, where $\pi_B : C \to C/B$ denotes the natural projection. Now, $\tilde{f}(C/B) = 0$ since $\tilde{f}(C_n + B/B) = 0$, hence $f = \tilde{f}\pi_B = 0$. We have proved that $C$ is an $A$-small group.

(2) Let $(A_n \mid n < \omega)$ be an $\omega$-filtration of $A$. Since $A$ is $B$-small, there exists $n$ for which both $\mathrm{Hom}(A/A_n, B)$ and $\mathrm{Hom}(A/A_n, C/B)$ vanish. If $f \in \mathrm{Hom}(A/A_n, C)$ is nonzero, then, we have $\pi_B f = 0$ since $\pi_B f \in \mathrm{Hom}(A/A_n, C/B) = 0$. Hence $f$ factorizes through $B$ as $f = \iota_B \tilde{f}$ for some $\tilde{f} \in \mathrm{Hom}(A/A_n, B)$. By the assumption on $\mathrm{Hom}(A/A_n, B)$ we have $\tilde{f} = 0$ and therefore $f = 0$. Thus $\mathrm{Hom}(A/A_n, C) = 0$ and so $A$ is $C$-small. □

**Example 3.13.** The implication of the previous claim cannot be reversed:

(1) $\prod_{p \in \mathbb{P}} \mathbb{Z}_p$ is self-small by [21, Theorem 2.5 and Example 2.7], but $\bigoplus_{p \in \mathbb{P}} \mathbb{Z}_p$ is not $\prod_{p \in \mathbb{P}} \mathbb{Z}_p$-small.

(2) Since $\mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}) = 0$, the group $\mathbb{Q}/\mathbb{Z}$ is $\mathbb{Q}$-small, but $\mathbb{Q}/\mathbb{Z}$ is not $\mathbb{Q}/\mathbb{Z}$-small.

**Lemma 3.14.** *Let $A$ be an abelian group and $\mathcal{M}$ a finite family of abelian groups.*

1. *If $N$ is $A$-small for each $N \in \mathcal{M}$, then $\bigoplus \mathcal{M}$ is $A$-small.*

2. *If $A$ is $N$-small for each $N \in \mathcal{M}$, then $A$ is $\bigoplus \mathcal{M}$-small.*

*Proof.* Put $M = \bigoplus \mathcal{M}$. Both of the proofs proceed by induction on the cardinality of $\mathcal{M}$.

(1) If $|\mathcal{M}| \leq 1$, there is nothing to prove. Let the assertion hold true for $|\mathcal{M}| - 1$ and put $M_N = \bigoplus \mathcal{M} \setminus \{N\}$ for arbitrary $N \in \mathcal{M}$. Since $M_N$ is $A$-small by the induction hypothesis, $N$ is $A$-small by the hypothesis and $M/N \cong M_N$, we get that $M$ is $A$-small by Proposition 3.12(1).

(2) The same induction argument as in (1) shows that $A$ is $M$-small by Lemma 3.12(2), since $A$ is $N$-small by the hypothesis and it is $M_N$-small for each $N \in \mathcal{M}$ by the induction hypothesis. □

As the main result of the section we describe which finite sums of relatively small abelian groups are again relatively small.

**Proposition 3.15.** *Let $\mathcal{M}$ and $\mathcal{N}$ be finite families of abelian groups. The following conditions are equivalent:*

1. *$\bigoplus \mathcal{M}$ is $\bigoplus \mathcal{N}$-small,*

2. *$M$ is $\bigoplus \mathcal{N}$-small for each $M \in \mathcal{M}$,*

3. *$\bigoplus \mathcal{M}$ is $N$-small for each $N \in \mathcal{N}$,*

4. *$M$ is $N$-small for each $M \in \mathcal{M}$ and $N \in \mathcal{N}$,*

5. *for each $M \in \mathcal{M}$, $N \in \mathcal{N}$, and $\omega$-filtration $(M_i \mid i < \omega)$ of $M$, there exist $i < \omega$ with $Hom(M/M_i, N) = 0$.*

*Proof.* (1)⇒(2) Put $F_M := \bigoplus(\mathcal{M} \setminus \{M\}) \leq \bigoplus \mathcal{M}$ and since $(\bigoplus \mathcal{M})/F_M \cong M$, the claim follows from Lemma 3.10(1).

(1)⇒(3) Since $N \leq \bigoplus \mathcal{N}$ the assertion is clear by Lemma 3.10(2).

(2)⇒(4), (3)⇒(4) It follows from Lemma 3.10 again.

The implication (4)⇒(3) is a consequence of Lemma 3.14(1), while the implication (3)⇒(1) is shown in Lemma 3.14(2).

(4)⇔(5) It is an immediate consequence of Proposition 3.7. □

As a consequence we reformulate [12, Proposition 5]:

**Corollary 3.16.** *The following conditions are equivalent for a finite family of abelian groups $\mathcal{M}$ and $M = \bigoplus \mathcal{M}$:*

1. *$M$ is self-small,*

2. *$N_1$ is $N_2$-small for each $N_1, N_2 \in \mathcal{M}$,*

3. *for every $N_1, N_2 \in \mathcal{M}$ and $\omega$-filtration $(M_i \mid i < \omega)$ of $N_1$ there exist $i < \omega$ with $Hom(N_1/M_i, N_2) = 0$.*

**Example 3.17.** Since $\mathrm{Hom}(\mathbb{Q}, \mathbb{Z}) = 0$ and $\mathbb{Q}$ is self-small and $\mathbb{Z}$ is small so $\mathbb{Z}$-small and $\mathbb{Q}$-small, the group $\mathbb{Z} \oplus \mathbb{Q}$ is self-small by Corollary 3.16.

## 3.3 Self-small products

We start the section by a criterion of self-smallness of a general product (cf. [13, Theorem 5.4]).

**Theorem 3.18.** *Let $\mathcal{M}$ be a family of abelian groups and put $M = \prod \mathcal{M}$ and $S = \bigoplus \mathcal{M}$. Then the following conditions are equivalent:*

1. *$M$ is self-small,*
2. *$M$ is $S$-small,*
3. *$M$ is $\bigoplus \mathcal{C}$-small for each countable family $\mathcal{C} \subseteq \mathcal{M}$.*

*Proof.* The implications (1)⇒(2)⇒(3) follow from Lemma 3.10(2), since $S$ is embeddable into $M$ and $\bigoplus \mathcal{C}$ is embeddable into $S$.

(3)⇒(1) Proving indirectly, assume that $M$ is not self-small. Then there exists an $\omega$-filtration $(M_i \mid i < \omega)$ of $M$ for which $\operatorname{Hom}(M/M_n, M) \neq 0$ for all $n < \omega$ by Proposition 3.7. Note that if $0 \neq f : M/M_n \to M = \prod \mathcal{M}$, then there must exist an $N \in \mathcal{M}$ with $0 \neq \pi_N f : M/M_n \to N$, where $\pi_N : M \to N$ denotes the canonical projection of the product. Hence for each $n < \omega$ there exists $A_n \in \mathcal{M}$ such that $\operatorname{Hom}(M/M_n, A_n) \neq 0$. If we put $\mathcal{C} = \{A_i \mid i < \omega\}$, then all $A_i$'s are embeddable into $\bigoplus \mathcal{C}$, hence $\operatorname{Hom}(M/M_n, \bigoplus \mathcal{C}) \neq 0$ for each $n < \omega$, which implies that $M$ is not $\bigoplus \mathcal{C}$-small by Proposition 3.7. □

As $A^\kappa$ is $A^{(\kappa)}$-small if and only if it is $A$-small by Corollary 3.5 we obtain the following consequence of Theorem 3.18.

**Corollary 3.19.** *Let $A$ be an abelian group and $I$ a set. Then $A^I$ is self-small if and only if it is $A$-small.*

**Example 3.20.** (1) $\mathbb{Q}^\omega$ is not self-small, since it is an infinitely generated $\mathbb{Q}$-vector space, hence it is not $\mathbb{Q}$-small.

(2) We have recalled in Example 3.13 that $\prod_{p\in\mathbb{P}} \mathbb{Z}_p$ is self-small, so it is $\bigoplus_{p\in\mathbb{P}} \mathbb{Z}_p$-small group by Theorem 3.18.

Let us denote by $T_A = \bigoplus_{p\in\mathbb{P}} A_{(p)}$ the torsion part of an abelian group $A$ where $A_{(p)}$ denotes the $p$-component of the torsion part.

**Lemma 3.21.** *Let $p \in \mathbb{P}$, $P$ be a nonzero $p$-group, $R$ a nonzero torsion group, $\mathcal{T}$ a family of finite torsion groups, and $\kappa$ be a cardinal. Then:*

1. *$\mathbb{Z}_p^\kappa$ is $P$-small if and only if $\kappa$ is finite,*
2. *$\mathbb{Z}^\kappa$ is $R$-small if and only if $\kappa$ is finite,*
3. *if $\prod \mathcal{T}$ is $P$-small, then $\{T \in \mathcal{T} \mid T_{(p)} \neq 0\}$ is finite,*
4. *if $\prod \mathcal{T}$ is $R$-small, then $\{T \in \mathcal{T} \mid T_{(p)} \neq 0\}$ is finite for each $p \in \mathbb{P}$ satisfying $R_{(p)} \neq 0$.*

*Proof.* (1) If $\kappa$ is finite, then $\mathbb{Z}_p^\kappa$ is finite, and so $P$-small (it is, in fact, small). If $\kappa$ is infinite, then $\mathbb{Z}_p^\kappa$ is an infinitely generated vector space over $\mathbb{Z}_p$. Hence infinite direct sum of groups $\mathbb{Z}_p$, which is not $\mathbb{Z}_p$-small, so it is not $P$-small by Lemma 3.10(2), since there exists $Q \leq P$ with $Q \simeq \mathbb{Z}_p$.

(2) It is enough to prove the direct implication. Suppose that $\kappa$ is infinite. Since there exists $p \in \mathbb{P}$ such that $R_{(p)} \neq 0$ and $\mathbb{Z}^\kappa/(p\mathbb{Z}^\kappa) \cong \mathbb{Z}_p^\kappa$ is not $R_{(p)}$-small by (1). Then $\mathbb{Z}^\kappa$ is not $R$-small by Lemma 3.10(1).

(3) Put $\mathcal{T}_p = \{T_{(p)} \mid T \in \mathcal{T}, T_{(p)} \neq 0\}$ and $\mathcal{S} = \{pS \mid S \in \mathcal{T}_p\}$ and suppose that $\kappa = |\mathcal{T}_p| = |\{T \in \mathcal{T} \mid T_{(p)} \neq 0\}|$ is infinite. Then $(\prod \mathcal{T}_p)/\prod \mathcal{S} \cong \mathbb{Z}_p^\kappa$ which is not $P$-small by (1), and so $\prod \mathcal{T}_p$ is not $P$-small by Lemma 3.10(1). Now $\prod \mathcal{T}$ is not $P$-small by Lemma 3.10(1) again, as $\prod \mathcal{T}_p$ is a direct summand of $\prod \mathcal{T}$.

(4) It follows from (3) and Lemma 3.10(2). □

**Lemma 3.22.** *Let $A_p$ be a finite $p$-group for each $p \in \mathbb{P}$. Then $\prod_{p\in\mathbb{P}} A_p$ is self-small.*

*Proof.* From [14, Section 20, Exercise 5] we have that $A = \prod_{p\in\mathbb{P}} A_p/\bigoplus_{p\in\mathbb{P}} A_p$ is divisible, since $pA_q = A_q$ for all $q \neq p$. Repeating the argument of [21, Lemma 1.7] (cf. also Example 3.20(2)) we get that if $f \in \mathrm{Hom}(\prod_{p\neq q} A_p, A_q)$ where $q \in \mathbb{P}$, then $\bigoplus_{p\neq q} A_p \subseteq \ker f$, hence $\mathrm{Im} f$ is isomorphic to some factor of the divisible group $A$. Therefore $\mathrm{Im} f$ is divisible and at the same time a subgroup of a finite group, hence $\mathrm{Im} f = 0$. In consequence, $\mathrm{Hom}(\prod_{p\neq q} A_p, A_q) = 0$ and the fact that $A_q$ is self-small for each $q \in \mathbb{P}$ implies that $\prod_{p\in\mathbb{P}} A_p$ is self-small by applying [21, Proposition 1.6]. □

Now we are ready to describe self-small products of finitely generated groups.

**Theorem 3.23.** *Let $\mathcal{M}$ be a family of nonzero finitely generated abelian groups such that at least one $N \in \mathcal{M}$ has nonzero torsion part and put $M = \prod \mathcal{M}$, $S = \bigoplus \mathcal{M}$ and $Q = S/T_S$. Then the following conditions are equivalent:*

1. *$M$ is self-small,*
2. *$S$ is $\mathbb{Z}$-small and $S_{(p)}$-small for all $p \in \mathbb{P}$,*
3. *$S_{(p)}$ is finite for each $p \in \mathbb{P}$ and $Q$ is finitely generated*
4. *there are only finitely many $A \in \mathcal{M}$ which are infinite and for each $p \in \mathbb{P}$ there are only finitely many $A \in \mathcal{M}$ with $A_{(p)} \neq 0$,*
5. *the family $\{B \in \mathcal{M} \mid Hom(B, A) \neq 0\}$ is finite for each $A \in \mathcal{M}$,*
6. *there are only finitely many $A \in \mathcal{M}$ which are infinite and the family $\{B \in \mathcal{M} \mid Hom(C, B) \neq 0\}$ is finite for each finite $C \in \mathcal{M}$,*
7. *$M \cong F \oplus \prod_{p\in\mathbb{P}} M_p$ for a finitely generated free group $F$ and finite abelian $p$-groups $M_p$ for each $p \in \mathbb{P}$.*

*Proof.* Any finitely generated group $A$ is by [14, Theorem 15.5] isomorphic to a direct sum of a finite number of cyclic groups, so let $A \cong \bigoplus_{i=1}^n C_i$, where $C_i$ is cyclic. Put $F_A$ be the direct sum of those $C_i$'s that are infinite and similarly $T_A$ a direct sum of those that are finite. Then $F_A$ is free, $T_A$ is the (finite) torsion part of $A$ and $A \cong F_A \oplus T_A$. Put $F = \bigoplus_{A\in\mathcal{M}} F_A$ and $T = \bigoplus_{A\in\mathcal{M}} T_A$ and note that $S \cong F \oplus T$ where $F$ is a free abelian group and $T$ is the torsion part of $S$. Furthermore $M \cong \prod_{A\in\mathcal{M}} F_A \oplus \prod_{A\in\mathcal{M}} T_A$.

(2)⇔(3)⇔(4) Note that $\mathrm{Hom}(T,\mathbb{Z}) = 0$, hence $S$ is $\mathbb{Z}$-small if and only if $F$ is $\mathbb{Z}$-small. Thus $S$ is $\mathbb{Z}$-small if and only if $Q \cong F$ is finitely generated which holds true if and only if there are only finitely many $A \in \mathcal{M}$ with nonzero $F_A$, i.e. which are infinite. Furthermore, it is easy to see that $S$ is $S_{(p)}$-small if and only if $S_{(p)}$ is finite if and only if there exists only finitely many $A \in \mathcal{M}$ such that $A_{(p)} \neq 0$.

(4)⇒(5) Take $A \in \mathcal{M}$. Then $\mathrm{Hom}(B, A) \neq 0$ if and only if $F_B \neq 0$ or there exists $p \in \mathbb{P}$ satisfying $(T_A)_{(p)} \neq 0 \neq (T_B)_{(p)}$. Since $A$ is finitely generated, there exist $p_i \in \mathbb{P}$, $i = 1, \dots, k$ such that $T_A = \bigoplus_{i=1}^{k} (T_A)_{(p_i)}$. In total, we get

$$\{B \in \mathcal{M} \mid \mathrm{Hom}(B, A) \neq 0\} \subseteq \{B \mid F_B \neq 0\} \cup \bigcup_{i=1}^{k} \{B \mid (T_B)_{(p_i)} \neq 0\},$$

where both sets on the right-hand side are finite.

(5)⇒(4) Let $A \in \mathcal{M}$ be infinite, i.e $F_A \neq 0$. If $B$ is infinite, then $\mathrm{Hom}(B, A) \neq 0$, hence there exist only finitely many infinite groups $B \in \mathcal{M}$. Similarly, if $A, B \in \mathcal{M}$ are such that $A_{(p)} \neq 0 \neq B_{(p)}$, then $\mathrm{Hom}(B, A) \neq 0$, so for each $p \in \mathbb{P}$ there are only finitely many $B \in \mathcal{M}$ such that $B_{(p)} \neq 0$.

(1)⇒(4) Since $M$ is self-small, it is $S$-small by Theorem 3.18. Furthermore, $\prod_{A\in\mathcal{M}} F_A$ being a direct summand, hence a factor of $M$, it is $M$-small and in consequence $T$-small by Lemma 3.10(2), so $\prod_{A\in\mathcal{M}} F_A$ is finitely generated by Lemma 3.21(2). Therefore there exist only finitely many $A$ with $F_A \neq 0$. Similarly, since $\prod_{A\in\mathcal{M}} T_A$ is $T$-small, there exist only finitely many $A \in \mathcal{M}$ such that $A_{(p)} = (T_A)_{(p)} \neq 0$ for each $p \in \mathbb{P}$ by Lemma 3.21(4).

(3)⇒(7) Note that by (3) $F = \bigoplus_{A\in\mathcal{M}} F_A$ is finitely generated. Moreover,

$$\prod_{A\in\mathcal{M}} T_A = \prod_{A\in\mathcal{M}} \bigoplus_{p\in\mathbb{P}} (T_A)_{(p)} \cong \prod_{A\in\mathcal{M}} \prod_{p\in\mathbb{P}} (T_A)_{(p)} \cong \prod_{p\in\mathbb{P}} \prod_{A\in\mathcal{M}} (T_A)_{(p)} \cong \prod_{p\in\mathbb{P}} \bigoplus_{A\in\mathcal{M}} (T_A)_{(p)},$$

because $T_A$ is finite for all $A \in \mathcal{M}$ and for each $p \in \mathbb{P}$ there exist only finitely many $A$ with $(T_A)_{(p)} \neq 0$. Then $M_p = \bigoplus_{A\in\mathcal{M}} (T_A)_{(p)}$ is a finite $p$-group for all $p \in \mathbb{P}$ and $M \cong F \oplus \prod_{A\in\mathcal{M}} T_A \cong F \oplus \prod_{p\in\mathbb{P}} M_p$

(7)⇒(1) By Theorem 3.18 it is enough to prove that $M$ is $F \oplus \bigoplus_{p\in\mathbb{P}} M_p$-small. Since $F$ is finitely generated, it is $F \oplus \bigoplus_{p\in\mathbb{P}} M_p$-small. As $\mathrm{Hom}(\prod_{p\in\mathbb{P}} M_p, F) = 0$, it remains to show that $\prod_{p\in\mathbb{P}} M_p$ is $\bigoplus_{p\in\mathbb{P}} M_p$-small by Proposition 3.15, which holds true by Lemma 3.22 and Theorem 3.18.

(5)⇔(6) The assertion concerning infinite groups follows from the equivalence of (4) and (5). The rest is a consequence of the fact that $\mathrm{Hom}(C, B) \neq 0$ if and only if $\mathrm{Hom}(B, C) \neq 0$ for each pair of finitely generated torsion abelian groups $B, C$. □

An uncountable cardinal $\kappa$ is *measurable* if it admits a $\kappa$-additive measure $\mu : \kappa \to \{0; 1\}$ such that $\mu(\kappa) = 1$ and $\mu(x) = 0$ for $x \in \kappa$. A group $G$ is called *slender*, if for any homomorphism $f : \mathbb{Z}^\omega \to G$, $f(\mathbf{e}_i) = 0$ for almost all $i \in \omega$, where $\mathbf{e}_i$ denotes the element of $\mathbb{Z}^\omega$ with $\pi_j(\mathbf{e}_i) = \delta_{i,j}$. Recall that $\mathbb{Z}$ is slender by [15, Theorem 94.2] and that for a nonmeasurable cardinal $\kappa$ we have $\mathrm{Hom}(\mathbb{Z}^\kappa, \mathbb{Z}) \cong \mathbb{Z}^{(\kappa)}$ by [15, Corollary 94.5] (cf. also [2, Theorem 3.6]).

**Lemma 3.24.** *$\mathbb{Z}^\kappa$ is $\mathbb{Z}$-small for each cardinal $\kappa$.*

*Proof.* For finite $\kappa$ there is nothing to prove, so let us suppose that $\kappa$ is infinite and assume that $\mathbb{Z}^\kappa$ is not $\mathbb{Z}$-small. Then there exists a homomorphism $g \in \mathrm{Hom}(\mathbb{Z}^\kappa, \mathbb{Z}^{(\omega)})$ such that $\mathrm{Im}\, g$ is infinitely generated by Proposition 3.7, hence $\mathrm{Im}\, g$ is a free abelian group of infinite rank. Since $\mathrm{Im}\, g \cong \mathbb{Z}^{(\omega)}$ is projective, $\mathbb{Z}^{(\omega)}$ is a direct summand of $\mathbb{Z}^\kappa$, i.e. there exists a group $A$ for which $\mathbb{Z}^\kappa \cong \mathbb{Z}^{(\omega)} \oplus A$.

First, assume that $\kappa = \omega$. Then $\mathrm{Hom}(\mathbb{Z}^\omega, \mathbb{Z}) \cong \mathbb{Z}^{(\omega)}$ by [15, Corollary 94.5] as $\mathbb{Z}$ is slender by [15, Theorem 94.2]. Hence

$$\begin{aligned}\mathbb{Z}^{(\omega)} \cong \mathrm{Hom}(\mathbb{Z}^\omega, \mathbb{Z}) \cong &\mathrm{Hom}(\mathbb{Z}^{(\omega)} \oplus A, \mathbb{Z}) \cong \\ \cong &\mathrm{Hom}(\mathbb{Z}^{(\omega)}, \mathbb{Z}) \oplus \mathrm{Hom}(A, \mathbb{Z}) \cong \mathbb{Z}^\omega \oplus \mathrm{Hom}(A, \mathbb{Z})\end{aligned}$$

which is impossible for cardinality reasons (i.e. $|\mathbb{Z}^{(\omega)}| < |\mathbb{Z}^\omega|$).

We have proved that $\mathbb{Z}^\omega$ is $\mathbb{Z}$-small, so $\kappa > \omega$. Let $\lambda \geq \kappa$ be a nonmeasurable cardinal (it exists, as for instance each singular cardinal is nonmeasurable). Then $\mathrm{Hom}(Z^\lambda, \mathbb{Z}) \cong Z^{(\lambda)}$ by [15, Corollary 94.5] and $\mathbb{Z}^\lambda \cong \mathbb{Z}^\lambda \oplus \mathbb{Z}^\kappa$ as $\lambda + \kappa = \lambda$, hence $\mathbb{Z}^\lambda \cong \mathbb{Z}^{(\omega)} \oplus B$ for $B = \mathbb{Z}^\lambda \oplus A$. We get

$$\mathbb{Z}^{(\lambda)} \cong \mathrm{Hom}(\mathbb{Z}^\lambda, \mathbb{Z}) \cong \mathrm{Hom}(\mathbb{Z}^{(\omega)} \oplus B, \mathbb{Z}) \cong \mathbb{Z}^\omega \oplus \mathrm{Hom}(A, \mathbb{Z}),$$

which implies that $\mathbb{Z}^\omega$ is embeddable into $\mathbb{Z}^{(\lambda)}$, so it is an infinitely generated free group. This contradicts the fact that $\mathbb{Z}^\omega$ is $\mathbb{Z}$-small. □

**Example 3.25.** Expressing Proposition 3.12(1) via the language of short exact sequences, we can say that relative smallness is transferred from the outer members to the middle one. The other direction, however, is more complicated: while Lemma 3.10(1) implies the transfer from the middle member to the right, the previous example shows that the transfer to the left does not occur generally: we have $\mathbb{Z}^{(\omega)} \hookrightarrow \mathbb{Z}^\omega$, but $\mathbb{Z}^{(\omega)}$ is not $\mathbb{Z}$-small.

Using Corollary 3.19 we can formulate an important consequence:

**Corollary 3.26.** *$\mathbb{Z}^\kappa$ is self-small for each cardinal $\kappa$.*

We finish the paper by a general criterion of self-small products of finitely generated groups.

**Theorem 3.27.** *Let $\mathcal{M}$ be a family of nonzero finitely generated abelian groups and put $M = \prod \mathcal{M}$, $S = \bigoplus \mathcal{M}$ and $Q = S/T_S$. Then the following conditions are equivalent:*

1. *$M$ is self-small,*
2. *either $T_S = 0$, or $S_{(p)}$ is finite for each $p \in \mathbb{P}$ and $Q$ is finitely generated*
3. *either all $A \in \mathcal{M}$ are free, or the family $\{B \in \mathcal{M} \mid Hom(B, A) \neq 0\}$ is finite for each $A \in \mathcal{M}$,*
4. *either $M \cong \mathbb{Z}^\kappa$ for some cardinal $\kappa$, or $M \cong F \oplus \prod_{p \in \mathbb{P}} M_p$ for a finitely generated free group $F$ and finite abelian $p$-groups $M_p$ for each $p \in \mathbb{P}$.*

*Proof.* The torsion part of $M$ is zero if and only all groups $A \in \mathcal{M}$ are free which means that $M \cong \mathbb{Z}^\kappa$ for some cardinal $\kappa$ is self-small by Corollary 3.26. The case when the torsion part of $M$ is nonzero follows directly from Theorem 3.23. $\square$

**Example 3.28.** The assumption in condition (4) of the previous theorem that $F$ is finitely generated cannot be omitted without additional conditions, since, e.g., the group $\mathbb{Q} \times \prod_{p\in\mathbb{P}} \mathbb{Z}_p$ is not self-small by [12, Example 3].

# Bibliography for chapter 3


[1] Albrecht U., Breaz, S., A note on self-small modules over RM-domains, *J. Algebra Appl.* 13(1) (2014), 8 pages.

[2] Albrecht, U., Breaz, S., Schultz, P., Functorial properties of Hom and Ext. In: Groups and model theory. In honor of Rüdiger Göbel's 70th birthday. Contemporary Mathematics 576, 1–15, Providence, 2012.

[3] Albrecht, U., Breaz, S., Wickless, W., Self-small Abelian groups. *Bull. Aust. Math. Soc.* **80** (2009), No. 2, 205–216.

[4] Albrecht, U., Breaz, S., Wickless, W., Purity and self-small groups. Commun. Algebra **35** (2007), No. 11, 3789-3807.

[5] Arnold, D.M., Murley, C.E., Abelian groups, $A$, such that Hom $(A, -)$ preserves direct sums of copies of $A$. *Pacific Journal of Mathematics*, Vol. **56** (1975), No.1, 7–20.

[6] Bass, H., *Algebraic K-theory*, Mathematics Lecture Note Series, New York-Amsterdam: W.A. Benjamin, 1968.

[7] Breaz S., Self-small abelian groups as modules over their endomorphism rings, *Comm. Algebra* 31 (2003), no. 10, 4911–4924.

[8] Breaz S., *A mixed version for a Fuchs' Lemma.* Rend. Sem. Mat. Univ. Padova 144 (2020), 61-71.

[9] Breaz, S., Schultz, P., Dualities for Self-small Groups *Proc. A.M.S.*, **140** (2012), No. 1, 69–82.

[10] Breaz, S., Žemlička, J. When every self-small module is finitely generated. *J. Algebra* **315** (2007), 885–893.

[11] Colpi, R., Menini, C., On the structure of $\star$-modules, J. Algebra **158** (1993), 400–419.

[12] Dvořák, J., *On products of self-small abelian groups*, *Stud. Univ. Babeş-Bolyai Math.* 60 (2015), no. 1, 13–17.

[13] Dvořák, J., Žemlička, J., *Autocompact objects of Ab5 categories*, submitted, 2021, arXiv:2102.04818.

[14] Fuchs, L., Infinite Abelian Groups, Vol.I, Academic press, New York and London 1970

[15] Fuchs, L., Infinite Abelian groups. Volume II. New York: Academic Press, 1973.

[16] Gómez Pardo, J. L. , Militaru, G., Năstăsescu, C., When is HOM$(M,-)$ equal to Hom$(M,-)$ in the category $R-gr$?, Comm. Algebra, **22** (1994), 3171–3181.

[17] Kálnai, P., Zemlička, J., *Compactness in abelian categories*, J. Algebra, 534 (2019), 273–288

[18] Modoi, C.G., *Constructing large self-small modules*, *Stud. Univ. Babeş-Bolyai Math.* 64(2019), No. 1, 3-10.

[19] Popescu N., *Abelian categories with applications to rings and modules*, 1973, Boston, Academic Press.

[20] Rentschler, R., *Sur les modules $M$ tels que Hom$(M,-)$ commute avec les sommes directes*, C.R. Acad. Sci. Paris, **268** (1969), 930–933.

[21] Žemlička, J., When products of self-small modules are self-small. *Commun. Algebra* **36** (2008), No. 7, 2570–2576.

# Chapter 4

# Autocompact objects of Ab5 categories

An object $C$ of an abelian category $\mathcal{A}$ closed under coproducts is said to be *autocompact*, if the corresponding covariant hom-functor $\mathcal{A}(C, -)$ with target category being the category of abelian groups commutes with coproducts $C^{(\kappa)}$ for all cardinals $\kappa$, i.e. there is a canonical abelian group isomorphism between objects $\mathcal{A}(C, C^{(\kappa)})$ and $\mathcal{A}(C, C)^{(\kappa)}$. Note that it generalizes the profoundly treated notion of compact objects whose covariant hom-functors commute with arbitrary coproducts.

A systematic study of compact objects in categories of modules began in late 60's with Hyman Bass remarking in [4, p.54] that the class of compact modules extends the class of finitely generated ones. This observation was elaborated in the work of Rudolf Rentschler [19], where he presented basic constructions and conditions of existence of infinitely generated compact modules. The attention to autocompact objects within the category of abelian groups was then attracted by the work [3]. The later research was motivated mainly by progress in the structural theory of abelian groups [2, 5, 6] and modules [1, 7, 17]. Although the notions of compactness and autocompactness were in fact studied in various algebraic contexts and with heterogeneous motivation (structure of modules [14, 12], graded rings [13], representable equivalences of module categories [8], the structure of almost free modules [20]), their overall categorial nature was omitted for a long time. Nevertheless, there have been several recent papers dedicated to the description of compactness in both non-abelian [16, 10] and abelian [15] categories published.

The present paper follows the undertaking begun with [15] and its main goal is not only to survey results concerning self-small abelian groups and modules from the standpoint of abelian categories, but it tries to deepen and extend some of them in a way that they could be applied back in the algebraic context. We initiate with an investigation of the more general concept of relative compactness. The second section summarizes some basic tools developed in [15], which allows for the description of structure and closure properties of relative compactness, in particular, Proposition 4.18 shows that $\bigoplus \mathcal{M}$ is $\bigoplus \mathcal{N}$-compact for finite families of objects $\mathcal{M}$ and $\mathcal{N}$ of an Ab5-category if and only if $M$ is $N$-compact for all $M \in \mathcal{M}$ and $N \in \mathcal{N}$. The third section presents a general criterion of an object to be autocompact via the structure of its endomorphism ring (Theorem 4.22) and, as a consequence, a description of autocompact coproducts (Proposition 4.23). The main result of the paper presented in Theorem 4.30 which proves that $\prod \mathcal{M}$ is an autocompact object if and only if it is $\bigoplus \mathcal{M}$-compact.

## 4.1 Preliminaries

A category with a zero object is called *abelian* if the following four conditions are satisfied:

1. for each discrete diagram the product and coproduct exist and they are canonically isomorphic,
2. each Hom-set has a structure of an abelian group such that the composition of morphisms is bilinear,
3. with each morphism it contains its kernel and cokernel,
4. monomorphisms are kernels of suitable morphisms, while epimorphisms are cokernels of suitable morphisms.

A category is said to be *complete* (*cocomplete*) if it contains limits (colimits) of all small diagrams; a cocomplete abelian category where all filtered colimits of exact sequences preserve exactness is then called an *Ab5* category.

Any small discrete diagram is said to be a *family*. Let $\mathcal{M}$ be a family of objects from $\mathcal{A}$; then the corresponding coproduct (product) is denoted $(\bigoplus \mathcal{M}, (\nu_M \mid M \in \mathcal{M}))$ $((\prod \mathcal{M}, (\pi_M \mid M \in \mathcal{M})))$ and $\nu_M$ $(\pi_M)$ are called *structural morphisms* of the coproduct (of the product). In case $\mathcal{M} = \{M_i \mid i \in K\}$ with $M_i = M$ for all $i \in K$, where $M$ is an object of $\mathcal{A}$, we shall write $M^{(K)}$ $(M^K)$ instead of $\bigoplus \mathcal{M}$ $(\prod \mathcal{M})$ and the corresponding structural morphisms shall be denoted by $\nu_i := \nu_{M_i}$ $(\pi_i := \pi_{M_i}$ resp.) for each $i \in K$.

Let $\mathcal{N}$ be a subfamily of $\mathcal{M}$. Following the terminology set in [15] the coproduct $(\bigoplus \mathcal{N}, (\overline{\nu}_N \mid N \in \mathcal{N}))$ in $\mathcal{A}$ is called a *subcoproduct* and dually the product $(\prod \mathcal{N}, (\overline{\pi}_N \mid N \in \mathcal{N}))$ is said to be a *subproduct*. Recall there exists a unique canonical morphism $\nu_{\mathcal{N}} \in \mathcal{A}(\bigoplus \mathcal{N}, \bigoplus \mathcal{M})$ $(\pi_{\mathcal{N}} \in \mathcal{A}(\prod \mathcal{M}, \prod \mathcal{N}))$ given by the universal property of $\bigoplus \mathcal{N}$ $(\prod \mathcal{N})$ satisfying $\nu_N = \nu_{\mathcal{N}} \circ \overline{\nu}_N$ $(\pi_N = \overline{\pi}_N \circ \pi_{\mathcal{N}})$ for each $N \in \mathcal{N}$, to which we shall refer as to the *structural morphism* of the subcoproduct (the subproduct) over a subfamily $\mathcal{N}$ of $\mathcal{M}$. If $\mathcal{M} = \{M_i \mid i \in K\}$ and $\mathcal{N} = \{M_i \mid i \in L\}$ where $M_i = M$ for an object $M$ and for $i$ from index sets $L \subseteq K$, the corresponding structural morphisms are denoted by $\nu_L$ and $\pi_L$ respectively. The symbol $1_M$ denotes the identity morphism of an object $M$ and the phrase *the universal property of a limit (colimit)* refers to the existence of unique morphism into the limit (from the colimit).

For basic properties of introduced notions and unspecified terminology we refer to [18].

Throughout the whole paper we assume that $\mathcal{A}$ is an *Ab5* category.

## 4.2 $\mathcal{C}$-compact objects

In order to capture in detail the idea of relative compactness, which is the central notion of this paper, let us suppose that $M$ is an object of the category $\mathcal{A}$ and

$\mathcal{N}$ is a family of objects of $\mathcal{A}$. Note that the functor $\mathcal{A}(M,-)$ on any additive category maps into Hom-sets with a structure of abelian groups, which allows for a definition of the mapping

$$\Psi_{\mathcal{N}} : \bigoplus (\mathcal{A}(M,N) \mid N \in \mathcal{N}\ ) \to \mathcal{A}(M, \bigoplus \mathcal{N})$$

by the rule

$$\Psi_{\mathcal{N}}(\varphi) = \nu_{\mathcal{F}} \circ \nu^{-1} \circ \pi_{\mathcal{F}} \circ \tau$$

where the symbol $\mathcal{F}$ for the element $\varphi = (\varphi_N \mid N \in \mathcal{N})$ of the abelian group $\bigoplus(\mathcal{A}(M,N) \mid N \in \mathcal{N})$ denotes the finite family $\{N \in \mathcal{N} \mid \varphi_N \neq 0\}$, the morphism $\nu \in \mathcal{A}(\bigoplus \mathcal{F}, \prod \mathcal{F})$ is the canonical isomorphism and $\tau \in \mathcal{A}(M, \prod \mathcal{N})$ is the unique morphism given by the universal property of the product $(\prod \mathcal{N}, (\pi_N \mid N \in \mathcal{F}))$ applied on the cone $(M, (\varphi_N \mid N \in \mathcal{N}))$, i.e. $\pi_N \circ \tau = \varphi_N$ for each $N \in \mathcal{N}$:

$$\begin{tikzcd}
M \arrow[r, squiggly, "\tau"] \arrow[dr, "\varphi_N"'] & \prod \mathcal{N} \arrow[r, "\pi_{\mathcal{F}}"] \arrow[d, "\pi_N"] & \prod \mathcal{F} \arrow[r, "\nu^{-1}"] & \bigoplus \mathcal{F} \arrow[r, "\nu_{\mathcal{F}}"] & \bigoplus \mathcal{N} \\
& N & & &
\end{tikzcd}$$

Recall a key observation regarding the algebraic concept of compactness:

**Lemma 4.1.** *[15, Lemma 1.3] For each family of objects $\mathcal{N} \subseteq \mathcal{A}$, the mapping $\Psi_{\mathcal{N}}$ is a monomorphism in the category of abelian groups.*

Let $M$ be an object and $\mathcal{C}$ a class of objects of the category $\mathcal{A}$. In accordance with [15], $M$ is called $\mathcal{C}$-*compact* if $\Psi_{\mathcal{N}}$ is an isomorphism for each family $\mathcal{N} \subseteq \mathcal{C}$. For objects $M, N \in Ac$ we say that $M$ is $N$-*compact* (or relatively compact over $N$) if it is an $\{N\}$-compact object and $M$ is said to be *autocompact* whenever it is $M$-compact.

**Example 4.2.** *(1) If $M$ and $N$ are objects such that $\mathcal{A}(N,M) = 0$, then $N$ is $M$-compact object, in particular $\mathbb{Q}$ is a $\mathbb{Z}$-compact abelian group.*

*(2) Self-small right modules over a unital associative ring, in particular finitely generated ones, are autocompact objects in the category of all right modules.*

Let us formulate an elementary but useful observation:

**Lemma 4.3.** *Let $M$ be an object and let $\mathcal{B} \subseteq \mathcal{C}$ be families of objects of the category $\mathcal{A}$. If $M$ is $\mathcal{C}$-compact, then it is $\mathcal{B}$-compact.*

We shall need two basic structural observations concerning the category $\mathcal{A}$ formulated in [15], which express relationship between coproducts and products using their structural morphisms. For the convenience of the reader we quote both of the results, the first one is formulated for the special case of products coproducts of copies of $M$, while the second one is kept in the original form.

**Lemma 4.4.** *[15, Lemma 1.1] Let $M$ be an object of $\mathcal{A}$ and $L \subseteq K$ be sets. If $\mathcal{A}$ contains products $(M^L, (\overline{\pi}_i \mid i \in L))$ and $(M^K, (\pi_i \mid i \in K))$, then*

1. *There exist unique morphisms $\rho_L \in \mathcal{A}(M^{(K)}, M^{(L)})$ and $\mu_L \in \mathcal{A}(M^L, M^K)$ such that $\rho_L \circ \nu_i = \overline{\nu}_i$, $\pi_i \circ \mu_L = \overline{\pi}_i$ for $i \in L$, and $\rho_L \circ \nu_i = 0$, $\pi_i \circ \mu_L = 0$ for $i \notin L$.*

2. *For each $i \in K$ there exist unique morphisms $\rho_i \in \mathcal{A}(M^{(K)}, M)$ and $\mu_i \in \mathcal{A}(M, M^K)$ such that $\rho_i \circ \nu_i = 1_M$, $\pi_i \circ \mu_i = 1_M$ and $\rho_i \circ \nu_j = 0$, $\pi_j \circ \mu_i = 0$ whenever $i \neq j$. Denoting by $\overline{\rho}_i$ and $\overline{\mu}_i$ the corresponding morphisms for $i \in L$, we have $\mu_L \circ \overline{\mu}_j = \mu_j$ and $\rho_L \circ \overline{\rho}_j = \rho_j$ for all $j \in L$.*

3. *There exists a unique morphism $t \in \mathcal{A}(M^{(K)}, M^K)$ such that $\pi_i \circ t = \rho_i$ and $t \circ \nu_i = \mu_i$ for each $i \in K$.*

**Lemma 4.5.** *[15, Lemma 1.1(i) and 1.2] Let $\mathcal{N} \subseteq \mathcal{M}$ be families of objects of $\mathcal{A}$ and let there exist products $(\prod \mathcal{N}, (\overline{\pi}_N \mid N \in \mathcal{N}))$ and $(\prod \mathcal{M}, (\pi_N \mid N \in \mathcal{M}))$ in $\mathcal{A}$.*

1. *There exist unique morphisms $\rho_{\mathcal{N}} \in \mathcal{A}(\bigoplus \mathcal{M}, \bigoplus \mathcal{N})$ and $\mu_{\mathcal{N}} \in \mathcal{A}(\prod \mathcal{N}, \prod \mathcal{M})$ such that $\rho_{\mathcal{N}} \circ \nu_{\mathcal{N}} = 1_{\bigoplus \mathcal{N}}$, $\pi_{\mathcal{N}} \circ \mu_{\mathcal{N}} = 1_{\prod \mathcal{N}}$ and $\rho_{\mathcal{N}} \circ \nu_M = 0$, $\pi_M \circ \mu_{\mathcal{N}} = 0$ for each $M \notin \mathcal{N}$.*

2. *There exist unique morphisms $\overline{t} \in \mathcal{A}(\bigoplus \mathcal{N}, \prod \mathcal{N})$ and $t \in \mathcal{A}(\bigoplus \mathcal{M}, \prod \mathcal{M})$ such that $\pi_N \circ t = \rho_N$ and $t \circ \nu_N = \mu_N$ for each $N \in \mathcal{M}$, $\overline{\pi}_N \circ \overline{t} = \overline{\rho}_N$ and $\overline{t} \circ \overline{\nu}_N = \overline{\mu}_N$ for each $N \in \mathcal{N}$. Furthermore, the diagram*

$$\begin{tikzcd}
\bigoplus \mathcal{N} \arrow[r, "\nu_{\mathcal{N}}"] \arrow[d, "\overline{t}"'] & \bigoplus \mathcal{M} \arrow[r, "\rho_{\mathcal{N}}"] \arrow[d, "t"] & \bigoplus \mathcal{N} \arrow[d, "\overline{t}"] \\
\prod \mathcal{N} \arrow[r, "\mu_{\mathcal{N}}"] & \prod \mathcal{M} \arrow[r, "\pi_{\mathcal{N}}"] & \prod \mathcal{N}
\end{tikzcd}$$

*commutes.*

3. *Let $\kappa$ be an ordinal and let $(\mathcal{N}_\alpha \mid \alpha < \kappa)$ be a disjoint partition of $\mathcal{M}$. For $\alpha < \kappa$ set $S_\alpha := \bigoplus \mathcal{N}_\alpha$, $P_\alpha := \prod \mathcal{N}_\alpha$ and denote families of the corresponding limits and colimits as $\mathcal{S} := (S_\alpha \mid \alpha < \kappa)$, $\mathcal{P} := (P_\alpha \mid \alpha < \kappa)$. Then $\bigoplus \mathcal{M} \simeq \bigoplus \mathcal{S}$ and $\prod \mathcal{M} \simeq \prod \mathcal{P}$ where both isomorphisms are canonical, i.e. for each object $M \in \mathcal{M}$ the following diagrams commute:*

$$\begin{tikzcd}
M \arrow[r, "\nu_M^{(\alpha)}"] \arrow[d, "\nu_M"'] & S_\alpha \arrow[d, "\nu_{S_\alpha}"] \\
\bigoplus \mathcal{M} \arrow[r, squiggly, "\simeq"] & \bigoplus \mathcal{S}
\end{tikzcd}
\qquad
\begin{tikzcd}
\prod \mathcal{P} \arrow[r, squiggly, "\simeq"] \arrow[d, "\pi_{P_\alpha}"'] & \prod \mathcal{M} \arrow[d, "\pi_M"] \\
P_\alpha \arrow[r, "\pi_M^{(\alpha)}"] & M
\end{tikzcd}$$

Morphisms $\rho_L$, $\rho_{\mathcal{N}}$, ($\mu_L$, $\mu_{\mathcal{N}}$) from Lemma 4.4(1) and Lemma 4.5(1) are called the *associated morphisms* to the structural morphisms $\nu_L$, $\nu_{\mathcal{N}}$ ($\pi_L$, $\pi_{\mathcal{N}}$) over the subcoproduct (the subproduct) of $M$. The unique morphism $t$ from Lemma 4.5(2) is said to be the *compatible coproduct-to-product* morphism. Note that in an Ab5-category $t$ is a monomorphism by [18, Chapter 2, Corollary 8.10] and if $K$ is finite, it is by definition an isomorphism.

We translate now a general criteria [15, Lemma 1.4, Theorem 1.5] of categorial $\mathcal{C}$-compactness to the description of $N$-compactness for an arbitrary object $N$:

**Theorem 4.6.** *The following conditions are equivalent for objects $M$ and $N$ of the category $\mathcal{A}$:*

1. *$M$ is $N$-compact,*

2. *for each cardinal $\kappa$ and $f \in \mathcal{A}(M, N^{(\kappa)})$ there exists a finite set $F \subset \kappa$ and a morphism $f' \in \mathcal{A}(M, N^{(F)})$ such that $f = \nu_F \circ f'$.*

3. *for each cardinal $\kappa$ and $f \in \mathcal{A}(M, N^{(\kappa)})$ there exists a finite set $F \subset \kappa$ such that $f = \sum_{\alpha \in F} \nu_\alpha \circ \rho_\alpha \circ f$,*

4. *for each morphism $\varphi \in \mathcal{A}(M, N^{(\omega)})$ there exists $\alpha < \omega$ such that $\rho_\alpha \circ \varphi = 0$.*

5. *there exists a family $\mathcal{G}$ consisting of $N$-compact objects and an epimorphism $e \in \mathcal{A}(\bigoplus \mathcal{G}, M)$ such that for each countable family $\mathcal{G}_\omega \subseteq \mathcal{G}$ there exists a non-$N$-compact object $F$ and morphism $f \in \mathcal{A}(F, M)$ such that $f^c \circ e \circ \nu_{\mathcal{G}_\omega} = 0$ for the cokernel $f^c$ of $f$.*

*Proof.* Equivalences (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) follow immediately from [15, Lemma 1.4], while (1) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) are consequences of [15, Theorem 1.5] □

## 4.3 Correspondences of compact objects

As the base step of our research we describe $C$-compact objects for a single object $C$ of an Ab5 category $\mathcal{A}$. Let us begin with the observation that we can study $C$-compactness of a suitable object instead of the compactness over a set of objects.

Let us denote the class

$$\mathrm{Add}(\mathcal{C}) = \{A \mid \exists B, \exists \kappa, \forall \alpha < \kappa, \exists C_\alpha \in \mathcal{C} : A \oplus B \cong \bigoplus_{\alpha < \kappa} C_\alpha\}$$

for every family $\mathcal{C}$ of objects of $\mathcal{A}$ and put $\mathrm{Add}(C) := \mathrm{Add}(\{C\})$.

**Lemma 4.7.** *The following conditions are equivalent for an object $M$ and a set of objects $\mathcal{C}$ of the category $\mathcal{A}$:*

1. *$M$ is $\bigoplus \mathcal{C}$-compact,*

2. *$M$ is $\mathcal{C}$-compact,*

3. *$M$ is $\mathrm{Add}(\bigoplus \mathcal{C})$-compact,*

4. *$M$ is $\mathrm{Add}(\mathcal{C})$-compact.*

*Proof.* Since $\mathrm{Add}(\bigoplus \mathcal{C}) = \mathrm{Add}(\mathcal{C})$, the equivalence (3)$\Leftrightarrow$(4) is obvious. Implications (3)$\Rightarrow$(1) and (4)$\Rightarrow$(2) are clear from Lemma 4.3.

(2)$\Rightarrow$(4) Let $\varphi \in \mathcal{A}(M, \bigoplus \mathcal{D})$ for a family $\mathcal{D}$ of objects of $\mathrm{Add}(\bigoplus \mathcal{C})$. For each $D \in \mathcal{D}$ there exists a family $\mathcal{C}_D$ of objects of $\mathcal{C}$ and a monomorphism $\nu_D : D \to \bigoplus \mathcal{C}_D$, hence there exists a monomorphism $\nu : \bigoplus \mathcal{D} \to \bigoplus_{D \in \mathcal{D}} \bigoplus \mathcal{C}_D$. Since $M$ is $\mathcal{C}$-compact, the morphism $\nu\varphi$ factorizes through a finite subcoproduct by [15, Lemma 1.4], hence $\varphi$ factorizes through a finite subcoproduct, so $M$ is $\mathrm{Add}(\mathcal{C})$-compact by [15, Lemma 1.4] again.

(1)$\Rightarrow$(3) Follows from the implication (2)$\Rightarrow$(4) where we take $\{\bigoplus \mathcal{C}\}$ instead of $\mathcal{C}$. □

**Corollary 4.8.** *If $\mathcal{N} \subseteq \mathcal{M}$ are families of objects such that $\mathcal{N}$ contains infinitely many nonzero objects, then $\bigoplus \mathcal{M}$ is not $\mathcal{N}$-compact, so it is not $\bigoplus \mathcal{N}$-compact.*

Since $\mathrm{Add}(M) = \mathrm{Add}(M^{(n)})$ for any integer $n$, we have the following consequence:

**Corollary 4.9.** *Let $\kappa$ be a cardinal and $M$ an autocompact object. Then $M^{(\kappa)}$ is autocompact if and only if $\kappa$ is finite.*

The next result shows the correspondence between classes of compact objects over different pairs of objects.

**Lemma 4.10.** *Let $A, B, M$ be objects of $\mathcal{A}$ and let there exist a cardinal $\lambda$ and a monomorphism $\mu \in \mathcal{A}(A, B^\lambda)$. If $M$ is $B$-compact, then $M$ is $A$-compact.*

*Proof.* Denote by $\nu_\alpha$ and $\tilde{\nu}_\alpha$ the corresponding structural morphisms of coproducts $A^{(\omega)}$ and $B^{(\omega)}$, and by $\rho_\alpha$ and $\tilde{\rho}_\alpha$ their associated morphisms, respectively.

Suppose that $M$ is not $A$-compact. Then there exists $\varphi \in \mathcal{A}(M, A^{(\omega)})$ such that $\rho_\alpha\varphi \neq 0$ for all $\alpha < \omega$ by Theorem 4.6. Since $\mu$ is a monomorphism by assumption, we get that $\mu\rho_\alpha\varphi \neq 0$, which implies that there exists $\beta_\alpha < \lambda$ such that $\pi_{\beta_\alpha}\mu\rho_\alpha\varphi \neq 0$ for each $\alpha < \omega$ by the universal property of the product $B^\lambda$. Put $\mu_\alpha = \pi_{\beta_\alpha}\mu \in \mathcal{A}(A, B)$ and note we have proved that $\mu_\alpha\rho_\alpha\varphi$ is a nonzero morphism $M \to B$ for each $\alpha < \omega$.

The universal property of the coproduct $A^{(\omega)}$ implies that there exists a uniquely determined morphism $\psi \in \mathcal{A}(A^{(\omega)}, B^{(\omega)})$ for which the diagram

$$\begin{tikzcd} A \arrow[r, "\mu_\alpha"] \arrow[d, "\nu_\alpha"'] & B \arrow[d, "\tilde{\nu}_\alpha"] & \\ A^{(\omega)} \arrow[r, squiggly, "\psi"] & B^{(\omega)} \arrow[r, "\tilde{\rho}_\gamma"] & B \end{tikzcd}$$

commutes, i.e. we have equalities $\psi\nu_\alpha = \tilde{\nu}_\alpha\mu_\alpha$ and $\tilde{\rho}_\gamma\psi\nu_\alpha = \tilde{\rho}_\gamma\tilde{\nu}_\alpha\mu_\alpha$ for each $\alpha, \gamma < \omega$. Hence for every $\alpha < \omega$ we get $\tilde{\rho}_\alpha\psi\nu_\alpha = \mu_\alpha$ and $\tilde{\rho}_\gamma\psi\nu_\alpha = 0$ whenever $\gamma \neq \alpha$ by Lemma 4.4(2). Note that it means that $\tilde{\rho}_\gamma\psi\nu_\alpha = \tilde{\rho}_\gamma\psi\nu_\gamma\rho_\gamma\nu_\alpha$ for all $\alpha, \gamma < \omega$

By applying Theorem 4.6 again we need to show that $\tilde{\rho}_\gamma\psi\varphi \neq 0$ for all $\gamma < \omega$. The universal property of the coproduct $A^{(\omega)}$ implies that for every $\gamma < \omega$ there exists a unique morphism $\tau_\gamma \in \mathcal{A}(A^{(\omega)}, B)$ such that the diagram

$$\begin{tikzcd} A \arrow[r, "\nu_\alpha"] \arrow[d, "\nu_\alpha"'] & A^{(\omega)} \arrow[d, "\tilde{\rho}_\gamma\psi"] \\ A^{(\omega)} \arrow[r, squiggly, "\tau_\gamma"] & B \end{tikzcd}$$

commutes for each $\alpha < \omega$. Since $\tilde{\rho}_\gamma\psi\nu_\alpha = \tilde{\rho}_\gamma\psi\nu_\gamma\rho_\gamma\nu_\alpha$ for all $\alpha, \gamma < \omega$, we get the equality $\tilde{\rho}_\gamma\psi = \tau_\gamma = \tilde{\rho}_\gamma\psi\nu_\gamma\rho_\gamma$ by the universal property of the coproduct $A^{(\omega)}$. Now, it remains to compute for every $\gamma < \omega$

$$\tilde{\rho}_\gamma\psi\varphi = \tau_\gamma\varphi = \tilde{\rho}_\gamma\psi\nu_\gamma\rho_\gamma\varphi = \mu_\gamma\rho_\gamma\varphi \neq 0,$$

so $M$ is not $B$-compact by Theorem 4.6. □

**Corollary 4.11.** *Let $M$ and $N$ be objects such that there exists a cardinal $\lambda$ and a monomorphism $\mu \in \mathcal{A}(M, N^\lambda)$ and $M$ is $N$-compact, then $M$ is autocompact.*

As another consequence of Lemma 4.10 we can observe that general compactness can be tested by a single object.

**Proposition 4.12.** *Let $E$ be an injective cogenerator and $M$ be an object of $\mathcal{A}$. Then $M$ is $E$-compact if and only if it is compact.*

*Proof.* Clearly, it is enough to prove the direct implication. Let $M$ be $E$-compact and $\mathcal{M}$ be a family of objects. Since $E$ is an injective cogenerator , there exists a cardinal $\lambda$ and a monomorphism $\mu \in \mathcal{A}(\bigoplus \mathcal{M}, E^\lambda)$. Then $M$ is $\bigoplus \mathcal{M}$-compact by Lemma 4.10 and so $\mathcal{M}$-compact by Lemma 4.7. □

The rest of this section is dedicated to description of relative compactness over finite coproducts of finite coproducts of objects.

**Lemma 4.13.** *Let $A$ be an object and $\mathcal{M}$ a finite family of objects.*

1. *If $N$ is $A$-compact for each $N \in \mathcal{M}$, then $\bigoplus \mathcal{M}$ is $A$-compact.*
2. *If $A$ is $N$-compact for each $N \in \mathcal{M}$, then $A$ is $\bigoplus \mathcal{M}$-compact.*

*Proof.* (1) Assume that $\bigoplus \mathcal{M}$ is not $A$-compact. Then by Theorem 4.6 there exists a morphism $\varphi \in \mathcal{A}(\bigoplus \mathcal{M}, A^{(\omega)})$ with $\rho_n \varphi \neq 0$ for all associated morphisms $\rho_n$ of $A^{(\omega)}$.

Note that for each $n < \omega$ there exists some $N \in \mathcal{M}$ such that $\rho_n \varphi \nu_N \neq 0$ by the universal property of the coproduct $\bigoplus \mathcal{M}$ , where $\nu_N$ are the corresponding structural morphisms of $\bigoplus \mathcal{M}$. Therefore there exists $N \in \mathcal{N}$ for which the set

$$I = \{n < \omega \mid \rho_n \varphi \nu_N \neq 0\}$$

is infinite and the morphism $\tilde{\varphi} = \rho_I \varphi \nu_N$ ensured by Lemma 4.4 satisfies $\rho_n \tilde{\varphi} = \rho_n \rho_I \varphi \nu_N \neq 0$ for each $n \in I$. Now, Theorem 4.6 implies that $N$ is not $A$-compact.

(2) Put $M = \bigoplus \mathcal{M}$ and denote by $\rho_i$, $\tilde{\rho}_i$ and $\rho_N$ the corresponding associate morphisms of coproducts $M^{(\omega)}$ and $N^{(\omega)}$ for each $N \in \mathcal{M}$ and $i < \omega$. Denote furthermore by $\rho_{N^{(\omega)}} \in \mathcal{A}(M^{(\omega)}, N^{(\omega)})$ the morphism given by Lemma 4.5 which satisfies $\tilde{\rho}_i \rho_{N^{(\omega)}} = \rho_N \rho_i$ for each $N \in \mathcal{M}$ and $i < \omega$. Assume that $A$ is not $\bigoplus \mathcal{M}$-compact: there exists a morphism $\varphi \in \mathcal{A}(A, M^{(\omega)})$ such that $\rho_n \varphi \neq 0$ for infinitely many $n$ by Theorem 4.6 and now using the same argument as in the proof of (1) we can find $N \in \mathcal{M}$ such that the set

$$J = \{i < \omega \mid \rho_N \rho_i \varphi \neq 0\}$$

is infinite. Since $\tilde{\rho}_i \rho_{N^{(\omega)}} \varphi = \rho_N \rho_i \varphi \neq 0$ for every $i \in J$, the object $A$ is not $N$-compact. □

**Lemma 4.14.** *Let $M$, $N$ and $A$ be objects of $\mathcal{A}$ and $n$ be a natural number. If there exists an epimorphism $M^{(n)} \to N$ and $M$ is $A$-compact, then $N$ is $A$-compact.*

*Proof.* Assume that $N$ is not $A$-compact. Then there exists a morphism $\varphi \in \mathcal{A}(N, A^{(\omega)})$ such that $\rho_\alpha \varphi \neq 0$ for all associated morphisms $\rho_\alpha$ of $A^{(\omega)}$ by Theorem 4.6. If $\mu \in \mathcal{A}(M^{(n)}, N)$ is an epimorphism, $\rho_i \varphi \mu \neq 0$ for each $i < \omega$, hence $M^{(n)}$ is not $A$-compact. Then $M$ is not $A$-compact by Lemma 4.13(1). □

We can summarize the obtained necessary condition of autocompactness.

**Proposition 4.15.** *Let $M$ and $N$ be objects of $\mathcal{A}$ such that there exists an epimorphism $M^{(n)} \to N$ for an integer $n$ and a monomorphism $N \to M^{\lambda}$ for a cardinal $\lambda$. If $M$ is is autocompact, then $N$ is autocompact as well.*

*Proof.* $N$ is $M$-compact, as follows from Lemma 4.14. Hence it is $N$-compact, so autocompact by Corollary 4.11. □

The next consequence presents a categorial version of the classical fact that an endomorphic image of a self-small module is self-small.

**Corollary 4.16.** *If $M$ is an autocompact object, such that there exist an epimorphism $\epsilon \in \mathcal{A}(M, N)$ and a monomorphism $\mu \in \mathcal{A}(N, M)$, then $N$ is autocompact.*

**Example 4.17.** *Let $A$ be a self-small right modules over a ring, i.e. autocompact object in the category of right modules. Denote $\mathcal{K} = \{\ker f \mid f \in End(A)\}$ and let $\mathcal{L} \subset \mathcal{K}$. Then $A/\bigcap \mathcal{L}$ is a self-small module by Proposition 4.15 since there exist monomorphisms $A/\bigcap \mathcal{L} \hookrightarrow \prod_{L\in\mathcal{L}} A/L \hookrightarrow \prod_{L\in\mathcal{L}} A$*

We conclude the section mentioning closure properties of relatively compact objects.

**Proposition 4.18.** *Let $\mathcal{M}$ and $\mathcal{N}$ be finite families of objects. Then $\bigoplus \mathcal{M}$ is $\bigoplus \mathcal{N}$-compact if and only if $M$ is $N$-compact for all $M \in \mathcal{M}$ and $N \in \mathcal{N}$.*

*Proof.* ($\Rightarrow$) Since the associate morphism $\rho_N \in \mathcal{A}(\bigoplus \mathcal{M}, M)$ is an epimorphism for each $M \in \mathcal{M}$ and $\bigoplus \mathcal{M}$ is $\bigoplus \mathcal{N}$-compact, each object $M$ is $\bigoplus \mathcal{N}$-compact by Lemma 4.14. As $\nu_N \in \mathcal{A}(N, \bigoplus \mathcal{N})$ is a monomorphism for each $N \in \mathcal{N}$, any object $M \in \mathcal{M}$ is $N$-compact by Lemma 4.10.

($\Leftarrow$) Lemma 4.13(1) implies that $\bigoplus \mathcal{M}$ is $N$-compact for each $N \in \mathcal{N}$ and then Lemma 4.13(2) implies that $\bigoplus \mathcal{M}$ is $\bigoplus \mathcal{N}$-compact. □

## 4.4 Description of autocompact objects

This section is dedicated mainly to the generalization of a classical autocompactness criteria [3] to an Ab5 category $\mathcal{A}$.

Assume $M$ is an object such that the category $\mathcal{A}$ is closed under products $M^{\lambda}$ for all $\lambda \leq |\mathrm{End}_{\mathcal{A}}(M)|$ and take $I \subseteq \mathrm{End}_{\mathcal{A}}(M) = \mathcal{A}(M, M)$. Then there exists a unique morphism $\tau_I \in \mathcal{A}(M, M^I)$ satisfying $\pi_\iota \tau_I = \iota$ for each $\iota \in I$ by the universal property of the product $M^I$. Let us denote by $\mathcal{K}(I) = (K_I, \nu_I)$ the kernel of the morphism $\tau_I$ and note that $\mathcal{K}(I)$ is defined uniquely up to isomorphism.

For an object $K$ of $\mathcal{A}$ consider a morphism $\nu \in \mathcal{A}(K, M)$. We will then set

$$\mathcal{I}(K, \nu) = \{\iota \in \mathrm{End}_{\mathcal{A}}(M) \mid \iota\nu = 0\}.$$

It is easy to see that the set $\mathcal{I}(K, \nu)$ forms a left ideal of the endomorphism ring $\mathrm{End}_{\mathcal{A}}(M)$. We say that a left ideal $I$ of $\mathrm{End}_{\mathcal{A}}(M)$ is an *annihilator ideal* if $\mathcal{I}(\mathcal{K}(I)) = I$.

**Lemma 4.19.** *Let $\mathcal{A}$ be closed under products $M^\lambda$ for all $\lambda \le |End_{\mathcal{A}}(M)|$. Then $\mathcal{I}(K,\nu)$ is an annihilator ideal of $End(M)$ for all $\nu \in \mathcal{A}(K,M)$.*

*Proof.* Put $I = \mathcal{I}(K,\nu)$, $(K_I,\nu_I) = \mathcal{K}(I)$ and $\tilde{I} = \mathcal{IK}(I) = \mathcal{I}(K_I,\nu_I)$. Furthermore, denote by $\tau_I \in \mathcal{A}(K, M^I)$ the morphism satisfying $\pi_\iota\tau_I = \iota$ for each $\iota \in I$, i.e. $(K_I,\nu_I)$ is the kernel of $\tau_I$. Since $\tau_I\nu_I = 0$, we can easily compute that $\iota\nu_I = \pi_\iota\tau_I\nu_I = 0$ for every $\iota \in I$, which implies $I \subseteq \tilde{I}$.

To prove the reverse inclusion $\tilde{I} \subseteq I$, let us note that by the universal property of the kernel $\nu_I$ there exists a unique morphism $\alpha \in \mathcal{A}(K,K_I)$ such that all squares in the diagram

$$\begin{tikzcd} K \arrow[r, "\nu"] \arrow[d, squiggly, "\alpha"'] & M \arrow[r, "\tau_I"] \arrow[d, equal] & M^I \arrow[d, "\pi_\iota"] \\ K_I \arrow[r, "\nu_I"] & M \arrow[r, "\iota"] & M \end{tikzcd}$$

commute for each $\iota \in I$. Consider a morphism $\gamma \in \mathrm{End}(M)$ such that $\gamma \notin I$. Then $\gamma\nu_I\alpha = \gamma\nu \neq 0$ by the definition of the ideal $I$. Hence $\gamma\nu_I \neq 0$ and so $\gamma \notin \tilde{I}$. $\square$

Recall the concepts of exactness and inverse limits in Ab5 categories.

The diagram $A_0 \xrightarrow{\alpha_1} A_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} A_{n-1} \xrightarrow{\alpha_n} A_n$ is said to be an *exact sequence* provided for each $i = 1,\dots,n-1$ the equality $\alpha_{i+1}\alpha_i = 0$ holds and there exist an object $K_i$ together with morphisms $\xi_i \in \mathcal{A}(A_i,K_i)$ and $\theta_i \in \mathcal{A}(K_i,A_i)$ such that $(K_i,\theta_i)$ is a kernel of $\alpha_{i+1}$, $(K_i,\xi_i)$ is a cokernel of $\alpha_i$ and $\xi_i\theta_i = 1_{K_i}$. In particular, the diagram $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ is a *short exact sequence* provided $\alpha$ is a kernel of $\beta$ and $\beta$ is a cokernel of $\alpha$, hence $\alpha$ is a monomorphism and $\beta$ is an epimorphism. Recall that any monomorphism (epimorphism) can be expressed as the first (second) morphism of some short exact sequence in an Ab5-category.

A diagram $\mathcal{D} = (\{M_i\}_{i<\omega}, \{\nu_{i,j}\}_{i<j<\omega})$ is called an *$\omega$-spectrum of $M$*, if $\nu_{i,j} \in \mathcal{A}(M_i,M_j)$, $\nu_{j,k}\nu_{i,j} = \nu_{i,k}$ for each $i<j<k<\omega$, and there exist morphisms $\nu_i \in \mathcal{A}(M_i,M)$ for all $i<\omega$ such that $(M,\{\nu_i\}_{i<\omega})$ is a colimit of the diagram $\mathcal{D}$ (i.e. it is a direct limit of the spectrum $\mathcal{D}$).

**Lemma 4.20.** *Let $M$ be an object and $M^{(\omega)}$ be a coproduct with structural morphisms $\nu_i$ and associated morphisms $\rho_i$, $i<\omega$. Put*

$$n = \{0,\dots,n-1\}, \quad [n,\omega) = \omega \setminus n = \{i<\omega \mid i \ge n\},$$

*let $M^{(n)}$ and $M^{([n,\omega))}$ be subcoproducts of the coproduct $M^{(\omega)}$. Denote by $\nu_{(n,m)} \in \mathcal{A}(M^{(n)},M^{(m)})$, $\nu_{<n} \in \mathcal{A}(M^{(n)},M^{(\omega)})$ the structural morphisms and simlarly denote by $\rho_{(n,m)} \in \mathcal{A}(M^{([n,\omega))},M^{([m,\omega))})$, $\rho_{\ge n} \in \mathcal{A}(M^{(\omega)},M^{([n,\omega))})$ the associated morphisms given by Lemma 4.5 for all $n<m<\omega$. Then*

1. *for each $n<m<\omega$ all squares in the diagram with exact rows*

$$\begin{tikzcd} \mathfrak{M}_n : & 0 \arrow[r] & M^{(n)} \arrow[r, "\nu_{<n}"] \arrow[d, "\nu_{(n,m)}"'] & M^{(\omega)} \arrow[r, "\rho_{\ge n}"] \arrow[d, equal] & M^{([n,\omega))} \arrow[r] \arrow[d, "\rho_{(n,m)}"] & 0 \\ \mathfrak{M}_m : & 0 \arrow[r] & M^{(m)} \arrow[r, "\nu_{<m}"] & M^{(\omega)} \arrow[r, "\rho_{\ge m}"] & M^{([m,\omega))} \arrow[r] & 0 \end{tikzcd}$$

*commute,*

2. *the short exact sequence*

$$0 \longrightarrow M^{(\omega)} \xrightarrow{id} M^{(\omega)} \xrightarrow{0} 0 \longrightarrow 0$$

*with morphisms* $(\nu_{<n}, id, 0)$ *forms a colimit of the* $\omega$*-spectrum* $(\{\mathfrak{M}_n\}_n, \{(\nu_{(n,m)}, id, \rho_{(n,m)})\}_{n<m})$ *in the category of complexes,*

3. $\rho_i \nu_{<n} = \rho_i$ *if* $i < n$ *and* $\rho_i \nu_{<n} = 0$ *otherwise.*

*Proof.* An easy exercise of application of Lemma 4.5 in a Ab5-category. $\square$

Before we formulate the categorial version of [3, Proposition 1.1] we prove a more general result:

**Lemma 4.21.** *Let for* $M \in \mathcal{A}$ *the category contain the products* $M^\omega$. *The following conditions are equivalent for an object* $N \in \mathcal{A}$*:*

1. $N$ *is not* $M$*-compact,*
2. *there exists an* $\omega$*-spectrum* $(\{N_i\}_{i<\omega}, \{\mu_{i,j}\}_{i<j<\omega})$ *of* $N$ *with colimit* $(N, \{\mu_i\}_{i<\omega})$ *such that all* $\mu_i$ *and* $\mu_{i,j}$ *for all* $i < j < \omega$ *are monomorphisms and for each* $i < \omega$ *there exists a nonzero morphism* $\varphi_i \in \mathcal{A}(N, M)$ *satisfying* $\varphi_i \mu_i = 0$*,*
3. *there exists an* $\omega$*-spectrum with colimit* $(N, \{\mu_i\}_{i<\omega})$ *such that for each* $i < \omega$ *there exists a nonzero morphism* $\varphi_i \in \mathcal{A}(N, M)$ *satisfying* $\varphi_i \mu_i = 0$.

*Proof.* We will use the notation of Lemma 4.20 throughout the whole proof.

(1)$\Rightarrow$(2) Let $\varphi \in \mathcal{A}(N, M^{(\omega)})$ satisfying $\rho_i \varphi \neq 0$ for all $i < \omega$, which is ensured by (1) and Theorem 4.6. Furthermore, let us denote $\varphi_{\geq n} = \rho_{\geq n} \varphi$. Then $\rho_i \varphi = \rho_i \varphi_{\geq n}$ for all $i \geq n$. Now, for each $n < \omega$ denote by $(N_n, \mu_n)$ the kernel of the morphism $\varphi_{\geq n}$ and note that by the universal property of the kernel there exists a morphism $\mu_{n,n+1} \in \mathcal{A}(N_n, N_{n+1})$ such that all squares in the diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & N_n \arrow[r, "\mu_n"] \arrow[d, "\mu_{n,n+1}"'] & N \arrow[r, "\varphi_{\geq n}"] \arrow[d, equal] & M^{([n,\omega))} \arrow[d, "\rho_{(n,n+1)}"] \\
0 \arrow[r] & N_{n+1} \arrow[r, "\mu_{n+1}"] & N \arrow[r, "\varphi_{\geq n+1}"] & M^{([n+1,\omega))}
\end{tikzcd}$$

commute. Now let us define inductively for each $n < m < \omega$ morphisms

$$\mu_{n,m} := \mu_{m-1,m} \mu_{m-2,m-1} \dots \mu_{n+1,n+2} \mu_{n,n+1} \in \mathcal{A}(N_n, N_m).$$

Denoting by $(X, \{\xi_i\}_{i<\omega})$ the colimit of the $\omega$-spectrum $\mathcal{N} = (\{N_i\}_{i<\omega}, \{\mu_{i,j}\}_{i<j<\omega})$ then from Lemma 4.20 we obtain the following commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & N_n \arrow[r, "\mu_n"] \arrow[d, "\mu_{n,m}"'] & N \arrow[r, "\varphi_{\geq n}"] \arrow[d, equal] & M^{([n,\omega))} \arrow[d, "\rho_{(n,m)}"] \\
0 \arrow[r] & N_m \arrow[r, "\mu_m"] \arrow[d, "\xi_n"'] & N \arrow[r, "\varphi_{\geq m}"] \arrow[d, equal] & M^{([m,\omega))} \arrow[d, "0"] \\
0 \arrow[r] & X \arrow[r, "\xi"] & N \arrow[r] & 0
\end{tikzcd}$$

because $\mathcal{A}$ is an Ab5-category. Thus $\xi$ is an isomorphism, which implies that $(N, \{\mu_i\}_{i<\omega})$ is a colimit of the $\omega$-spectrum $\mathcal{N}$. Since all $\mu_n$'s are kernel morphisms, they are monomorphisms. Furthermore, $\mu_{n,m}$ are monomorphisms, because $\mu_n = \mu_m\mu_{n,m}$ for all $n < m < \omega$. Finally, put $\varphi_i := \rho_i\varphi$, which is nonzero by the hypothesis, and compute $\varphi_i\mu_i = \rho_i\varphi\mu_i = \rho_i\varphi_{\geq i}\mu_i = 0$.

(2)⇒(3) This is clear.

(3)⇒(1) Let us denote by $\tau \in \mathcal{A}(N, M^\omega)$ the morphism satisfying $\pi_i\tau = \varphi_i$, which is (uniquely) given by the universal property of the product $M^\omega$. Recall that for each $n < \omega$ we have denoted by $\pi_n \in \mathcal{A}(M^\omega, M^n)$ the corresponding structural morphism and we may identify objects $M^n$ and $M^{(n)}$ so we shall consider $\pi_n$ as a morphism in $\mathcal{A}(M^\omega, M^{(n)})$.

Put $\tau_n := \pi_n\tau\mu_n$. Since $\rho_i\tau_n = \rho_i\pi_n\tau\mu_n = \pi_i\pi_n\tau\mu_n$ we obtain that $\rho_i\tau_n = \varphi_i\mu_n \neq 0$ for each $i < n$ and $\rho_i\tau_n = 0$ for each $i \geq n$. Then the diagram

$$\begin{array}{ccc} N_n & \xrightarrow{\tau_n} & M^{(n)} \\ {\scriptstyle \mu_{n,m}}\downarrow & & \downarrow{\scriptstyle \nu_{(n,m)}} \\ N_m & \xrightarrow{\tau_m} & M^{(m)} \end{array}$$

commutes for every $n < m < \omega$. Hence there exists $\varphi \in \mathcal{A}(N, M^{(\omega)})$ such that the diagram

$$\begin{array}{ccc} N_n & \xrightarrow{\tau_n} & M^{(n)} \\ {\scriptstyle \mu_{n}}\downarrow & & \downarrow{\scriptstyle \nu_{<n}} \\ N & \xrightarrow{\varphi} & M^{(\omega)} \end{array}$$

commutes for each $n < \omega$ by Lemma 4.20, as $(N, \{\mu_i\}_{i<\omega})$ is the colimit of the $\omega$-spectrum $(\{N_i\}_{i<\omega}, \{\mu_{i,j}\}_{i<j<\omega})$ and $(\{M^{(\omega)}\}_{i<\omega}, \{\mu_{(i)}\}_{i<\omega})$ is the colimit of the $\omega$-spectrum $(\{M^{(i)}\}_{i<\omega}, \{\nu_{(i,j)}\}_{i<j<\omega})$ in the Ab5-category $\mathcal{A}$.

Applying Theorem 4.6, it is enough to prove that $\rho_i\varphi \neq 0$ for each $i < \omega$. We have shown that $\rho_i\tau_n \neq 0$ for each $i < n$, hence

$$\rho_i\varphi\mu_n = \rho_i\nu_{<n}\tau_n = \rho_i\tau_n \neq 0$$

for each $i < n$, which implies $\rho_i\varphi \neq 0$. □

We are now ready to formulate a basic characterization of autocompact objects which generalizes the classical result [3, Proposition 1.1].

**Theorem 4.22.** *Let $M$ be an object such that $\mathcal{A}$ is closed under products $M^\lambda$ for all $\lambda \leq \max(|End_{\mathcal{A}}(M)|, \omega)$. Then the following conditions are equivalent:*

1. *$M$ is not autocompact,*
2. *there exists an $\omega$-spectrum with colimit $(M, \{\mu_i\}_{i<\omega})$ such that for each $i < \omega$ there exists a nonzero morphism $\varphi_i \in End_{\mathcal{A}}(M)$ satisfying $\varphi_i\mu_i = 0$,*
3. *there exists an $\omega$-spectrum $(\{M_i\}_{i<\omega}, \{\mu_{i,j}\}_{i<j<\omega})$ with colimit $(M, \{\mu_i\}_{i<\omega})$ such that $\{\mathcal{I}(M_i, \mu_i)\}_{i<\omega}$ forms a strictly increasing chain of nonzero ideals of the ring $End_{\mathcal{A}}(M)$ with $\bigcap_{i<\omega} \mathcal{I}(M_i, \mu_i) = 0$.*

*Proof.* (1)⇔(2) follows form Lemma 4.21 for $N = M$.

(2)⇒(3) If $(M, \{\mu_i\}_{i<\omega})$ is the colimit which exists by (2), then $\{\mathcal{I}(M_i, \mu_i)\}_{i<\omega}$ is an increasing chain of nonzero ideals of $\mathrm{End}_{\mathcal{A}}(M)$. Suppose that $\gamma\mu_i = 0$ for all $i < \omega$. Then $\gamma = 0$, since there exists unique such morphism by the universal property of the colimit $(M, \{\nu_i\}_{i<\omega})$. Thus $\bigcap_{i<\omega} \mathcal{I}(M_i, \mu_i) = 0$ and $\mathcal{I}(M_i, \mu_i) \neq 0$ for each $i$. If we put $J = \{j < \omega \mid \mathcal{I}(M_j, \mu_j) \neq \mathcal{I}(M_{j+1}, \mu_{j+1})\}$, then it is easy to see that $(M, \{\mu_j\}_{j\in J})$ is the colimit of the $\omega$-spectrum $(\{M_j\}_{j\in J}, \{\mu_{i,j}\}_{i<j\in J})$ with a strictly increasing chain of nonzero ideals $\{\mathcal{I}(M_j, \mu_j)\}_{j\in J}$.

(3)⇒(2) It is enough to choose $\varphi_i \in \mathcal{I}(M_i, \mu_i) \setminus \mathcal{I}(M_{i+1}, \mu_{i+1})$. □

The following criterion of autocompactness of finite coproducts generalizes results [9, Proposition 5, Corollary 6] formulated in categories of modules.

**Proposition 4.23.** *The following conditions are equivalent for a finite family of objects $\mathcal{M}$ and $M = \bigoplus \mathcal{M}$:*

1. *$M$ is autocompact,*
2. *$N$ is $M$-compact for each $N \in \mathcal{M}$,*
3. *$M$ is $N$-compact for each $N \in \mathcal{M}$,*
4. *$N_1$ is $N_2$-compact for each $N_1, N_2 \in \mathcal{M}$,*
5. *for each $N_1, N_2 \in \mathcal{M}$ and any $\omega$-spectrum $(\{K_i\}_{i<\omega}, \{\mu_{i,j}\}_{i<j<\omega})$ of $N_1$ with colimit $(N_1, \{\mu_i\}_{i<\omega})$ and for each $i < \omega$ and nonzero $\varphi \in \mathcal{A}(N_1, N_2)$, the morphism $\varphi\mu_i$ is nonzero.*

*Proof.* (1)⇔(4) This is proved in Proposition 4.18

(2)⇔(3)⇔(4) These equivalences follow from Proposition 4.18 again, when applied on pairs of families $\{M\}, \mathcal{M}$ and $\mathcal{M}, \{M\}$.

(4)⇔(5) This is an immediate consequence of Lemma 4.21. □

As a consequence, we can formulate the assertion of Corollary 4.9 more precisely.

**Corollary 4.24.** *Let $\mathcal{M}$ be a family of nonzero objects. Then $\bigoplus \mathcal{M}$ is autocompact if and only if $\mathcal{M}$ is finite and $N_1$ is $N_2$-compact for each $N_1, N_2 \in \mathcal{M}$.*

The last direct consequence of Proposition 4.23 presents a categorial variant of [9, Corollary 7].

**Corollary 4.25.** *Let $\mathcal{M}$ be a finite family of autocompact objects satisfying the condition $\mathcal{A}(N_1, N_2) = 0$ whenever $N_1 \neq N_2$. Then $\bigoplus \mathcal{M}$ is autocompact.*

If $\mathcal{M}$ is a finite family of objects, then $\bigoplus \mathcal{M}$ and $\prod \mathcal{M}$ are canonically isomorphic (cf. Lemma 4.5), so the Proposition 4.23 holds true in case we replace any $\bigoplus$ by $\prod$ there. Although there is no autocompact coproduct of infinitely many nonzero objects by Corollary 4.8, the natural question that arises is, under which conditions the products of infinite families of objects are autocompact. The following example shows that the straightforward generalization of the claim does not hold true in general.

**Example 4.26.** *Denote by $\mathbb{P}$ the set of all prime numbers and consider the full subcategory $\mathcal{T}$ of the category of abelian groups Ab consisting of all torsion abelian groups. If $A$ is a torsion abelian group and $A_p$ denotes its p-component for each $p \in \mathbb{P}$, then the decomposition $\bigoplus_{p\in\mathbb{P}} A_p$ forms both the coproduct and product of the family $\mathcal{A} = \{A_p \mid p \in \mathbb{P}\}$. Indeed, if $B$ is a torsion abelian group and $\tau_p \in Ab(B, A_p)$ for $p \in \mathbb{P}$, then for every $b \in B$ there exist only finitely many $p \in P$ for which $\tau_p(b) \neq 0$, hence the image of the homomorphism $f \in Ab(B, \prod_p A_p)$ given by the universal property of the product $\prod_p A_p$ is contained in $\bigoplus_{p\in\mathbb{P}} A_p$, hence $\bigoplus_{p\in\mathbb{P}} A_p$ is the product of $\mathcal{A}$ in the category $\mathcal{T}$.*

*Thus, e.g. $\bigoplus_{p\in\mathbb{P}} \mathbb{Z}_p$ is the product of the family $\{\mathbb{Z}_p \mid p \in \mathbb{P}\}$ in $\mathcal{T}$, which is not autocompact in $\mathcal{T}$ by Corollary 4.24, however $\mathbb{Z}_p$ is $\mathbb{Z}_q$-compact for every $p, q \in \mathbb{P}$.*

## 4.5 Which products are autocompact?

Although the final section tries to answer the question formulated in its title, we start with one more closure property.

**Lemma 4.27.** *If $0 \to A \to B \to C \to 0$ is a short exact sequence such that an object $M$ is $A$-compact and $C$-compact, then it is $B$-compact.*

*Proof.* Proving indirectly, assume that $M$ is not $B$-compact. Then by Lemma 4.21 there exists a colimit $(M, \{\mu_i\}_{i<\omega})$ of some $\omega$-spectrum $(\{M_i\}_{i<\omega}, \{\mu_{i,j}\}_{i<j<\omega})$ and nonzero morphisms $\varphi_i \in \mathcal{A}(M, B)$ such that $\varphi_i\mu_i = 0$, $i < \omega$. If we suppose that $M$ is $C$-compact and consider the short exact sequence

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0,$$

then $\beta\varphi_i\mu_i = 0$ for each $i \in \omega$, hence there exists $n$ such that $\beta\varphi_i = 0$ for all $i \geq n$ by Lemma 4.21. By the universal property of the kernel $\alpha$ of (the cokernel) $\beta$ there exist $\psi_i$ satisfying $\alpha\psi_i = \varphi_i \neq 0$ for each $i \geq n$. As $\alpha$ is a monomorphism, $\psi_i \neq 0$ for each $i \geq n$, hence $M$ is not $A$-compact by Lemma 4.21 again, a contradiction. □

**Corollary 4.28.** *If $0 \to A \to B \to C \to 0$ is a short exact sequence such that the object $B$ is $A$-compact and $C$-compact, then $B$ is autocompact.*

As the next example shows, the previous assertion cannot be reversed.

**Example 4.29.** *If we consider the short exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ in the category of abelian groups, then $\mathbb{Q}$ is self-small, i.e. autocompact abelian group and $\mathbb{Z}$-compact, but it is not $\mathbb{Q}/\mathbb{Z}$-compact.*

Now, we can formulate a criterion for autocompact objects which generalizes [11, Theorem 3.1].

**Theorem 4.30.** *Let $\mathcal{M}$ be a family of autocompact objects such that the product $M = \prod \mathcal{M}$ exists in $\mathcal{A}$ and put $S = \bigoplus \mathcal{M}$. Then the following conditions are equivalent:*

1. *$M$ is autocompact,*

2. $M$ *is* $S$*-compact,*

3. $M$ *is* $\bigoplus\mathcal{C}$*-compact for each countable family* $\mathcal{C} \subseteq \mathcal{M}$.

*Proof.* (1)⇒(2) Since $M$ is $\mathrm{Add}(M)$-compact by Lemma 4.7 and $S = \bigoplus\mathcal{M} \in \mathrm{Add}(M)$, it is $S$-compact by Lemma 4.3.

(2)⇒(3) This is an easy consequence of Proposition 4.18.

(3)⇒(1) Assume on contrary that the object $M$ is not autocompact. Then by Lemma 4.21 there exists an $\omega$-spectrum $(\{M_i\}_{i<\omega}, \{\mu_{i,j}\}_{i<j<\omega})$ of $M$ with the colimit $(M, \{\mu_i\}_{i<\omega})$ such that $\mu_i$ is a monomorphism for all $i < \omega$ and for each $i < \omega$ there exists a nonzero morphism $\varphi_i \in \mathcal{A}(M, M)$ with $\varphi_i\mu_i = 0$. Then for each $i < \omega$ there exists $N_i \in \mathcal{M}$ such that $\pi_{N_i}\varphi_i \neq 0$. Put $\mathcal{C} = \{N_i \mid i < \omega\}$ and denote by $\tilde{\nu}_{N_i}$ the structural morphisms of the coproduct $\bigoplus\mathcal{C}$. Since $\tilde{\nu}_{N_i}\pi_{N_i}\varphi_i \in \mathcal{A}(M, \bigoplus\mathcal{C})$ such that $\tilde{\nu}_{N_i}\pi_{N_i}\varphi_i\mu_i = 0$, there exists $n$ for which $\tilde{\nu}_{N_n}\pi_{N_n}\varphi_n = 0$ by Lemma 4.21, which contradicts the hypothesis $\pi_{N_i}\varphi_i \neq 0$ for each $i < \omega$. □

**Corollary 4.31.** *Let* $M$ *be an object and* $I$ *be a set. Then* $M^I$ *is autocompact if and only if* $M^I$ *is* $M$*-compact.*

Let us make a categorial observation about transfer of $\omega$-spectra via morphisms.

**Lemma 4.32.** *Let* $G$ *and* $M$ *be a pair of objects of* $\mathcal{A}$ *and let* $\alpha \in \mathcal{A}(G, M)$. *If* $(\{M_i\}_{i<\omega}, \{\mu_{i,j}\}_{i<j<\omega})$ *is an* $\omega$*-spectrum of* $M$ *with the colimit* $(M, \{\mu_i\}_{i<\omega})$ *such that all* $\mu_i$*'s are monomorphisms,*

1. *then there exists an* $\omega$*-spectrum* $(\{G_i\}_{i<\omega}, \{\gamma_{i,j}\}_{i<j<\omega})$ *of* $G$ *with the colimit* $(G, \{\gamma_i\}_{i<\omega})$ *where* $\gamma_i$ *are monomorphisms for all* $i$ *and there exist morphisms* $\alpha_i \in \mathcal{A}(G_i, M_i)$ *such that the diagram*

$$\begin{tikzcd} G_i \arrow[r, "\gamma_i"] \arrow[d, "\alpha_i"'] & G \arrow[d, "\alpha"] \\ M_i \arrow[r, "\mu_i"] & M \end{tikzcd}$$

*commutes for each* $i < \omega$.

2. *If* $G$ *is* $A$*-compact for an object* $A$ *and* $t_i \in \mathcal{A}(M, A)$ *are morphisms satisfying* $t_i\mu_i = 0$ *for each* $i < \omega$, *then there exists* $n$ *such that* $t_i\alpha = 0$ *for each* $i \geq n$.

*Proof.* (1) If we denote by $c_i \in \mathcal{A}(M, T_i)$ the cokernel of $\mu_i$ and $\gamma_i \in \mathcal{A}(G_i, G)$ the kernel of $c_i\alpha$ for every $i < j < \omega$, then $\mu_i$ is the kernel of $c_i$ and by the universal property of the kernel, there exists a morphism $\alpha_i \in \mathcal{A}(G_i, M_i)$ such that the diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & G_i \arrow[r, "\gamma_i"] \arrow[d, "\alpha_i"'] & G \arrow[r, "c_i\alpha"] \arrow[d, "\alpha"] & T_i \arrow[d, equal] & \\ 0 \arrow[r] & M_i \arrow[r, "\mu_i"] & M \arrow[r, "c_i"] & T_i \arrow[r] & 0 \end{tikzcd}$$

commutes for each $i < \omega$. Furthermore, if we construct morphisms $\gamma_{i,j}$, $i < j < \omega$ using the universal property of the kernels as in the proof of Lemma 4.21 (1)⇒(2), then we get the following commutative diagram

$$\begin{tikzcd} G_i \arrow[r, "\gamma_i"] \arrow[d, "\gamma_{i,j}"'] & G \arrow[d, equal] \\ G_j \arrow[r, "\gamma_j"] & G \end{tikzcd}$$

and checking that $(G, \{\gamma_i\}_{i<\omega})$ is a colimit of the $\omega$-spectrum $(\{G_i\}_{i<\omega}, \{\gamma_{i,j}\}_{i<j<\omega})$ of $G$ is easy.

(2) From (1) we have an $\omega$-spectrum $(\{G_i\}_{i<\omega}, \{\gamma_{i,j}\}_{i<j<\omega})$ with the colimit $(G, \{\gamma_i\}_{i<\omega})$ and morphisms $\alpha_i \in \mathcal{A}(G_i, M_i)$ such that the diagram

$$\begin{tikzcd} 0 \arrow[r] & G_i \arrow[r, "\gamma_i"] \arrow[d, "\alpha_i"'] & G \arrow[d, "\alpha"] & \\ 0 \arrow[r] & M_i \arrow[r, "\mu_i"] & M \arrow[r, "t_i"] & A \end{tikzcd}$$

commutes for every $i < \omega$. Since $t_i\alpha\gamma_i = t_i\mu_i\alpha_i = 0$ by the hypothesis, Lemma 4.21 applied on the $A$-compact object $G$ and morphisms $t_i\alpha$, $i < \omega$, give us $n$ such that $t_i\alpha = 0$ for all $i \geq n$. □

**Lemma 4.33.** *Let $A$ and $B$ be objects of $\mathcal{A}$ and $\mathcal{A}(A, B) = 0$. If $\alpha \in \mathcal{A}(A \prod B, B)$ then there exists $\tau \in \mathcal{A}(B, B)$ for which $\alpha = \tau\pi_B$.*

*Proof.* Since $0 \to A \xrightarrow{\nu_A} A \oplus B \xrightarrow{\rho_B} B \to 0$ is a short exact sequence and $\alpha\nu_A = 0$, the claim follows from the universal property of the cokernel $\rho_B$ and by applying the canonical isomorphism $A \oplus B \cong A \prod B$. □

Recall that $G$ is a *projective generator* of $\mathcal{A}$, if for any nonzero object $B$ in $\mathcal{A}$, $\mathcal{A}(G, B) \neq 0$ holds and for each pair of objects $A$, $B$, any epimorphism $\pi \in \mathcal{A}(A, B)$ and any morphism $\varphi \in \mathcal{A}(G, B)$ there exists $\tau \in \mathcal{A}(G, A)$ such that $\varphi = \pi\tau$.

The following assertion is a categorial version of [21, Proposition 1.6] (cf. also [3, Corollary 1.3]). Call an $\mathcal{A}$-compact object briefly *compact* object.

**Theorem 4.34.** *Let $\mathcal{M}$ be a family of objects, $\mathcal{A}$ contain a compact projective generator and the product $M = \prod \mathcal{M}$. Denote $M_N = \prod(\mathcal{M} \setminus \{N\})$ and let $\mathcal{A}(M_N, N) = 0$ for each $N$. Then $M$ is autocompact if and only if $N$ is autocompact for each $N \in \mathcal{M}$.*

*Proof.* (⇒) Since $M \cong N \oplus M_N$ for every $N \in \mathcal{M}$, the assertion follows from Proposition 4.23.

(⇐) First note that $M_N$ is a trivial example of an $N$-compact module (cf. Example 4.2), so $M$ is $N$-compact for every $N \in \mathcal{M}$ by Proposition 4.23.

Assume that $M$ is not $M$-compact, hence by Lemma 4.21 there exists an $\omega$-spectrum $(\{M_i\}_{i<\omega}, \{\mu_{i,j}\}_{i<j<\omega})$ with the colimit $(M, \{\mu_i\}_{i<\omega})$ such that for each $i < \omega$ there exists a nonzero $\tilde{\varphi}_i \in \mathcal{A}(M, M)$ and $N_i \in \mathcal{M}$ for which $\tilde{\varphi}_i\mu_i = 0$ and $\pi_{N_i}\tilde{\varphi}_i \neq 0$. Put $\varphi_i = \pi_{N_i}\tilde{\varphi}_i$ for each $i < \omega$ and $\mathcal{C} = \{N_i \mid i < \omega\}$ and note that there exist $\psi_i \in \mathcal{A}(N_i, N_i)$ satisfying $\psi_i \pi_{N_i} = \varphi_i$ by Lemma 4.33 applied on $M_{N_i} \prod N_i$ for each $i < \omega$.

If $\mathcal{C}$ is finite, then $M$ is $\prod \mathcal{C}$-compact by Proposition 4.18 applied on $\{M\}$ and $\mathcal{C}$, hence there exists $n$ such that $\varphi_n = \pi_{N_n} \pi_{\mathcal{C}} \varphi = \pi_{N_n} \varphi = 0$, which contradicts the fact that $\varphi_i \neq 0$ for all $i < \omega$.

Thus $\mathcal{C}$ is infinite and we may assume w.l.o.g. that $N_i \neq N_j$ whenever $i \neq j$. Denote the cokernel of the composition $\pi_{N_i} \mu_i$ by $\sigma_i \in \mathcal{A}(N_i, T_i)$ for $i < \omega$. Then we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & M_i \arrow[r, "\mu_i"] & M \arrow[r, "\pi_{N_i}"] \arrow[d, "\varphi_i"'] & N_i \arrow[r, "\sigma_i"] \arrow[d, "\psi_i"] & T_i \arrow[r, equal] \arrow[d, "\tilde{\mu}_{T_i}"] & T_i \arrow[r] \arrow[d, equal] & 0 \\
 & & N_i \arrow[r, equal] & N_i & \prod_j T_j \arrow[r, "\tilde{\pi}_{T_i}"] & T_i \arrow[r] & 0
\end{tikzcd}$$

for each $i < \omega$, where $\tilde{\pi}_{T_i}$ and $\tilde{\mu}_{T_i}$ denote the structural and associated morphisms of the product $\prod_j T_j$. Since $\psi_i \pi_{N_i} \mu_i = \varphi_i \mu_i = 0$ and $\psi_i \neq 0$, the morphism $\pi_{N_i} \mu_i$ is not an epimorphism and so $T_i \neq 0$. As $G$ is a projective generator, there exists $\zeta_i \in \mathcal{A}(G, N_i)$ satisfying $\sigma_i \zeta_i \neq 0$ for each $i < \omega$. Then by the universal property of the product $\prod \mathcal{C}$, there is $\zeta \in \mathcal{A}(G, \prod \mathcal{C})$ such that $\hat{\pi}_{N_i} \zeta = \zeta_i$, hence $\sigma_i \hat{\pi}_{N_i} \zeta = \sigma_i \zeta_i \neq 0$ for all $i < \omega$. If we define $t_i = \tilde{\mu}_{T_i} \sigma_i \pi_{N_i}$ and denote by $\mu_{\mathcal{C}} \in \mathcal{A}(\prod \mathcal{C}, M)$ the associated morphism, we can easily compute

$$t_i \mu_i = \tilde{\mu}_{T_i} \sigma_i \pi_{N_i} \mu_i = 0 \quad \text{and} \quad t_i \mu_{\mathcal{C}} \zeta = \tilde{\mu}_{T_i} \sigma_i \pi_{N_i} \mu_{\mathcal{C}} \zeta = \tilde{\mu}_{T_i} \sigma_i \zeta_i \neq 0$$

as $\tilde{\mu}_{T_i}$ for $i < \omega$ is a monomorphism, which contradicts the hypothesis that $G$ is compact by Lemma 4.32(2). $\square$

The following example shows that the existence of the compact projective generator cannot be removed from the assumptions of the last assertion.

**Example 4.35.** *Consider the category of all torsion abelian groups $\mathcal{T}$ from Example 4.26. Then $M = \bigoplus_{q \in \mathbb{P}} \mathbb{Z}_q$ is the product of the family $\{\mathbb{Z}_q \mid q \in \mathbb{P}\}$ and $M_p = \bigoplus_{q \neq p} \mathbb{Z}_p$ is the product of the family $\{\mathbb{Z}_q \mid q \in \mathbb{P} \setminus \{p\}\}$ for all $p \in \mathbb{P}$ in the category $\mathcal{T}$. Although $Hom_{\mathcal{T}}(M_p, \mathbb{Z}_p) = 0$ and $\mathbb{Z}_p$ is autocompact in $\mathcal{T}$ for each $p \in \mathbb{P}$, $M$ is not autocompact. Let us remark that the category $\mathcal{T}$ contains no compact generator. [21, Corollary 1.8]).*

We conclude with a well-known example of an autocompact product.

**Example 4.36.** *Any finitely generated free abelian group is a compact projective generator in the category of abelian groups and the family $\{\mathbb{Z}_q \mid q \in \mathbb{P}\}$ satisfies the hypothesis of Proposition 4.34 by [21, Lemma 1.7], hence $\prod_{q \in \mathbb{P}} \mathbb{Z}_q$ is autocompact (cf.[21, Corollary 1.8]).*

# Bibliography for chapter 4


[1] Albrecht U., Breaz, S., A note on self-small modules over RM-domains, *J. Algebra Appl.* 13(1) (2014), 8 pages.

[2] Albrecht U., Breaz, S., Wickless, W,: Purity and Self-Small Groups, *Communications in Algebra*, 35(2007) No.11, 3789 — 3807.

[3] Arnold, D.M., Murley, C.E., Abelian groups, $A$, such that $\mathrm{Hom}\,(A, -)$ preserves direct sums of copies of $A$, *Pacific Journal of Mathematics*, Vol. 56(1975), No.1, 7–20.

[4] Bass, H., Algebraic K-theory, *Mathematics Lecture Note Series*, New York-Amsterdam: W.A. Benjamin, 1968.

[5] Breaz, S., Self-small abelian groups as modules over their endomorphism rings, *Comm. Algebra* 31 (2003), no. 10, 4911–4924.

[6] Breaz, S., Schultz, P., Dualities for Self-small Groups, *Proc. A.M.S.*, **140** (2012), No. 1, 69–82.

[7] Breaz, S., Žemlička, J., When every self-small module is finitely generated, *J. Algebra* **315** (2007), 885–893.

[8] Colpi, R. and Menini, C., On the structure of $*$-modules, *J. Algebra* **158**, 1993, 400–419.

[9] Dvořák, J., On products of self-small abelian groups, *Stud. Univ. Babeş-Bolyai Math.* 60 (2015), no. 1, 13–17.

[10] Dvořák, J., Žemlička, J., Compact objects in categories of S-acts, submitted, 2020, arXiv:2009.12301.

[11] Dvořák, J., Žemlička, J., Self-small products of abelian groups, accepted for publication, 2021, arXiv:2102.11443.

[12] Eklof, P.C., Goodearl, K.R., Trlifaj, J., Dually slender modules and steady rings, *Forum Math.* **9** (1997), 61–74.

[13] Gómez Pardo J.L., Militaru G., Năstăsescu C., When is $HOM_R(M, -)$ equal to $Hom_R(M, -)$ in the category $R$-gr, *Comm.Alg*, **22/8**, 1994, 3171-3181.

[14] T. Head, Preservation of coproducts by $\mathrm{Hom}_R$(M,-), *Rocky Mt. J. Math.* 2, (1972), 235–237.

[15] Kálnai, P., Zemlička, J., Compactness in abelian categories, *J. Algebra*, 534 (2019), 273–288

[16] Modoi, C.G., Localizations, colocalizations and non additive *-objects, *Semigroup Forum* 81(2010), No. 3, 510-523.

[17] Modoi, C.G., Constructing large self-small modules, *Stud. Univ. Babeş-Bolyai Math.* 64(2019), No. 1, 3-10.

[18] Popescu N., Abelian categories with applications to rings and modules, 1973, Boston, Academic Press.

[19] Rentschler, R., les modules $M$ tels que $\mathrm{Hom}(M, -)$ commute avec les sommes directes,*Sur C.R. Acad. Sci. Paris*, **268** (1969), 930–933.

[20] Trlifaj, J., Strong incompactness for some nonperfect rings, *Proc. Amer. Math. Soc.* 123 (1995), 21–25.

[21] Žemlička, J., When products of self-small modules are self-small. *Commun. Algebra* **36** (2008), No. 7, 2570–2576.

# Chapter 5

# Compact objects in categories of $S$-acts

While the great impact of the category theory on ring and module theory is well known, the analogous concept in the context of theory of monoids and acts over monoids on sets is significantly less studied, however it seems to be promising and fruitful (as it is demonstrated in the monograph [18]).

Recall that an object $c$ of an abelian category closed under coproducts and products is said to be *compact* if the corresponding covariant functor $\mathrm{Hom}(c, -)$ commutes with arbitrary direct sums i.e. there is a canonical isomorphism in the category of abelian groups $\mathrm{Hom}(c, \coprod \mathcal{D}) \cong \coprod \mathrm{Hom}(c, \mathcal{D})$ for every system of objects $\mathcal{D}$, where $\coprod$ denotes a coproduct. In particular, a (right $R$-)module is called *small*, if it is compact in the category of all modules. The aim of the present paper is translating the notion of compactness from abelian categories to a more general context. The constitutive example of such a generalization is provided by the analogy between (abelian) categories of modules over rings and (non-abelian) categories of acts over monoids (cf. also the corresponding description of compactness in Ab5 categories [11]).

The list of works dedicated to the research of compactness in various categorial contexts is long. Let us mention only those related to our conception of linking (auto)compact objects in abelian and non-abelian categories. However, the notion of autocompactness of modules [4, Proposition 1.1] was generalized to Grothendieck categories in [14], the main motivation for the study of compact objects in abelian categories comes from the context of representable equivalences of module categories [7, 8], where the notion of (generalized) $*$-module plays a key role. Analogous problem in non-abelian case, in particular (generalized) $*$-objects and (auto)compact objects, is studied in the paper [22]. Compact objects play also an important role in triangulated categories [23, Section 8], as they are compactly generated, in particular, for the description of Brown representability [20]. The notion of a compact object in non-abelian categories often has different meaning; it is usually defined as an object such that the corresponding covariant hom-functors commutes with filtered colimits. Nevertheless, it can be proved that that this notion is stronger that our definition based on commuting with coproducts (cf. Lemma 5.1 below and [13]). Although locally presentable and accessible categories deal with compactness in the narrow sense [2, 21], the motivation and application of the notion is closed to the abelian case.

Turning the attention towards the categories of modules, as is shown in [5] and in [24, $1^o$], small modules can be structurally described in a natural way by the language of systems of submodules:

**Lemma 5.1.** *[5, 24] The following conditions are equivalent for a module $M$:*

*(1) $M$ is small,*

*(2) if $M = \bigcup_{i<\omega} M_n$ for an increasing chain of submodules $M_n \subseteq M_{n+1} \subseteq M$, then there exists $n$ such that $M = M_n$,*

*(3) if $M = \sum_{i<\omega} M_n$ for a system of submodules $M_n \subseteq M$, $n < \omega$, then there exists $k$ such that $M = \sum_{i<k} M_n$.*

Note that the condition (2) implies immediately that every finitely generated module is small and (3) shows that there are no countably infinitely generated small modules. On the other hand, there are natural constructions of infinitely generated small modules:

**Example 5.2.** (1) A union of a strictly increasing chain of length $\kappa$, for an arbitrary cardinal $\kappa$ of uncountable cofinality, consisting of small (in particular finitely generated) submodules is small.

(2) Every $\omega_1$-generated uniserial module is small.

A ring over which the class of all small right modules coincides with the class of all finitely generated ones is called *right steady*. Note that the class of all right steady rings is closed under factorization [8, Lemma 1.9], finite products [27, Theorem 2.5], and Morita equivalence [12, Lemma 1.7]. However, a ring theoretical characterization of steadiness remains an open problem with partial results concerning right steadiness of certain natural classes of rings including right noetherian [24, $7^0$], right perfect [8, Corollary 1.6], right semiartinian of finite socle length [28, Theorem 1.5] countable commutative [24, $11^0$], and abelian regular rings with countably generated ideals [29, Corollary 7].

The main task of the first half of the paper is presenting two variants of categories of acts over monoids, namely acts enriched by the empty object and acts with zero elements, in joint general categorial language, in particular, the notion of a $UD$-category is introduced. Section 3 deals with the crucial issue of decompositions in $UD$-categories and their necessary basic properties follow, with Theorem 5.17 formulating the existence and uniqueness of indecomposable decomposition of any object in a UD-category. A general composition theory of projective objects in a $UD$-category is built in the next section, where the main result of the section, Theorem 5.20, characterizes projective objects as coproducts of indecomposable projective objects. Section 5 lists general properties of compact objects in a $UD$-category, Theorem 5.24 presents a general criterion for compactness of objects in a UD-category. Furthermore, the more general property of autocompactness is studied, too. As an application of this theory the characterization of (auto)compact objects in categories of $S$-acts is provided (Theorem 5.45, Proposition 5.49].

## 5.1 Axiomatic description of categories of acts

Before we start the study of common categorial properties of classes of acts over monoids, let us recall some necessary terminology and notation.

Let $\mathcal{C}$ be a category. Denote by $\mathrm{Mor}_{\mathcal{C}}(\theta, A)$ the class of all morphisms $A \to B$ in $\mathcal{C}$ for every pair of objects $A, B$ of $\mathcal{C}$; in case $\mathcal{C}$ is clear from the context, the subscript will be omitted. A monomorphism (epimorphism) in $\mathcal{C}$ is a left (right)-cancellable morphism, i.e., a morphism $\mu$ such that $\mu\alpha = \mu\beta$ ($\alpha\mu = \beta\mu$) implies $\alpha = \beta$. A morphism is a bimorphism, if it is both mono- and epimorphism. A category is balanced, if bimorphisms are isomorphisms (the reversed inclusion holds in general). An object $\theta$ is called initial provided $|\mathrm{Mor}(\theta, A)| = 1$ for each object $A$. The category is (co)product complete if the class of objects is closed under all (co)products. Note that any coproduct complete category contains the initial object, which is isomorphic to $\coprod \emptyset$. A pair $(\mathcal{C}, U)$ is said to be a *concrete category* over the category of sets, which is denoted Set in the whole paper, if $\mathcal{C}$ is a category and $U : \mathcal{C} \to \mathrm{Set}$ is a faithful functor. Finally, a *family* of objects means any discrete diagram and the phrase *the universal property of a coproduct* refers to the existence of unique morphism from the coproduct.

Let $\mathcal{S} = (S, \cdot, 1)$ be a monoid and $A$ a nonempty set. If there is a mapping $\mu : S \times A \to A$ satisfying the following two conditions: $\mu(1, a) = a$ and $\mu(s_2, \mu(s_1, a)) = \mu(s_2 \cdot s_1, a)$ then $A$ is said to be a left $S$-act and it is denoted ${}_SA$. For simplicity, $\mu(s, a)$ is often written as $s \cdot a$ or $sa$. A mapping $f : {}_SA \to {}_SB$ is a homomorphism of $S$-acts, or an $S$-homomorphism provided $f(sa) = sf(a)$ holds for any $s \in S, a \in A$. In compliance with [18, Example I.6.5.] we denote by $S - \mathrm{Act}$ the category of all left $S$-acts with homomorphisms of $S$-acts and $S - \overline{\mathrm{Act}}$ the category $S - \mathrm{Act}$ enriched by an initial object ${}_S\emptyset$. If the monoid $\mathcal{S}$ contains a (necessarily unique) zero element 0, then the category of all left $S$-acts with homomorphisms of $S$-acts compatible with zero as morphisms will be denoted by $S - \mathrm{Act}_0$. Observe that $\{0\}$ is the initial object of the category $S - \mathrm{Act}_0$.

Recall that both categories $S - \overline{\mathrm{Act}}$ and $S - \mathrm{Act}_0$ are complete and cocomplete [18, Remarks II.2.11, Remark II.2.22], in particular, the coproduct of a system of objects $(A_i, i \in I)$ is

(i) a disjunct union $\coprod_{i \in I} A_i = \dot{\bigcup} A_i$ in $S$-$\overline{\mathrm{Act}}$ by [18, Proposition II.1.8] and

(ii) $\coprod_{i \in I} A_i = \{(a_i) \in \prod_{i \in I} A_i | \ \exists j : a_i = 0 \forall i \neq j\}$ in $S$-$\mathrm{Act}_0$ by [18, Remark II.1.16].

Furthermore, if we denote the natural forgetful functor from $S - \overline{\mathrm{Act}}$ and $S - \mathrm{Act}_0$ into Set by $U$ (which maps an act to the underlying set of elements and an $S$-homomorphism to the corresponding mapping between sets) both $(S - \mathrm{Act}_0, U)$ and $(S - \overline{\mathrm{Act}}, U)$ are concrete categories over Set.

Let $\mathcal{C}$ be a coproduct complete category with an initial object $\theta \cong \coprod \emptyset$. An object $A \in \mathcal{C}$ is called *indecomposable* if it is not isomorphic to the initial object nor to a coproduct of two non-initial objects. Note that cyclic acts present natural examples of indecomposable objects in both categories $S - \overline{\mathrm{Act}}$ and $S - \mathrm{Act}_0$. Nevertheless, the class of indecomposable acts can be much larger, e. g. the rational numbers form a non-cyclic indecomposable $(\mathbb{Z}, \cdot)$-act.

As we have declared, the main motivation of the present paper is to describe and investigate compactness properties of categories of acts over monoids in the general categorial language. In particular, we focus on the categories $S-\overline{\mathrm{Act}}$ and $S-\mathrm{Act}_0$. The key feature of both of these categories is the existence of a unique decomposition of every object into indecomposable objects, which is proved in [18, Theorem I.5.10] for the case of the category $S-\overline{\mathrm{Act}}$.

First of all, we list several natural categorial properties which ensure an easy handling of the category, the uniqueness of decomposition and provide the existence condition as well. Recall that a pair $(S,\nu)$ is said to be a subobject of an object $A$ if $S$ is an object and $\nu: S\to A$ is a monomorphism.

We say that a concrete category $(\mathcal{C},U)$ over the category Set is a *UD-category* (unique decomposition) if the following conditions hold:

(UD1) $\mathcal{C}$ is a coproduct complete balanced category with an initial object $\theta\cong\coprod\emptyset$, for which each morphism $\theta\to A$ is a monomorphism and there is at most one morphism $A\to\theta$,

(UD2) for any morphism $f\in\mathrm{Mor}(A,B)$ in $\mathcal{C}$, there exists a subobject $(A^f,\iota)$ of $B$ such that $U(A^f)=U(f)(U(A))\subseteq U(B)$ and $U(\iota)\in\mathrm{Mor}(U(A^f),U(B))$ is the subset inclusion map,

(UD3) for each morphism $f\in\mathrm{Mor}(A,B)$ and every subobject $(S,\nu)$ of $B$ such that $U(f)(U(A))\subseteq U(\nu)(U(S))$, there exists a morphism $g\in\mathrm{Mor}(A,S)$ such that $f=\nu g$,

(UD4) for every system $(A_i,\nu_i)_{i\in I}$ of subobjects of an object $A$, there exist subobjects denoted by $(\bigcap_i A_i^{\nu_i},\iota_\cap)$ and $(\bigcup_i A_i^{\nu_i},\iota_\cup)$ such that

$$U(\bigcap_i A_i^{\nu_i})=\bigcap_i U(\nu_i)(U(A_i))=\bigcap_i U(A_i^{\nu_i}),$$
$$U(\bigcup_i A_i^{\nu_i})=\bigcup_i U(\nu_i)(U(A_i))=\bigcup_i U(A_i^{\nu_i})$$

and both $U(\iota_\cap)$, $U(\iota_\cup)$ are the corresponding subset inclusion mappings,

(UD5) if $(A,(\nu_0,\nu_1))$ is a coproduct of a pair of objects $(A_0,A_1)$, then $\nu_0$ and $\nu_1$ are monomorphisms and $\bigcap_{i=1}^2 A_i^{\nu_i}$ is isomorphic to $\theta$,

(UD6) for every object $A$ and every $x\in U(A)$ there exists an indecomposable subobject $(B,\nu)$ of $A$ such that $x\in U(B^\nu)=U(\nu)(U(B))\subseteq U(A)$.

Any monomorphism $\iota: A\to B$ such that $U(\iota)$ is the subset inclusion map is called *inclusion morphism.* Note that we will use the notation $(A^f,\iota)$ from (UD2) and $(\bigcup_i A_i^{\nu_i},\iota_\cup)$ from (UD4) freely without other explanations. Moreover, we will write $A_0^{\nu_0}\cap A_1^{\nu_0}$ ($A_0^{\nu_0}\cup A_1^{\nu_0}$ respectively) instead of $\bigcap_{i=1}^2 A_i^{\nu_i}$ ($\bigcup_{i=1}^2 A_i^{\nu_i}$ respectively).

First we make an elementary but frequently used (sometimes without reference) observation:

**Lemma 5.3.** *Let $\psi$ be a morphism of a UD-category $(\mathcal{C},U)$.*

*(1) If $U(\psi)$ is injective, then $\psi$ is a monomorphism.*

*(2) If $U(\psi)$ is surjective, then $\psi$ is an epimorphism.*

*(3) $\psi$ is an isomorphism if and only if $U(\psi)$ is a bijection.*

*Proof.* Since injective maps are monomorphisms and surjective maps are epimorphisms in Set, (1) and (2) follow immediately from the hypothesis that $U$ is a faithful functor.

(3) If $\psi$ is an isomorphism, then there exists its inverse isomorphism $\psi^{-1}$, hence $\mathrm{id} = U(\mathrm{id}) = U(\psi)U(\psi^{-1}) = U(\psi^{-1})U(\psi)$, and so $U(\psi)$ is a bijection. Since $\mathcal{C}$ is a balanced category, the reverse implication follows from (1) and (2). □

As an easy consequence we obtain a natural property of subobjects in UD-categories:

**Lemma 5.4.** *Let $(B, \nu)$ be a subobject of an object $A$ in a UD-category $(\mathcal{C}, U)$. If $(B^\nu, \iota)$ is a subobject with the inclusion morphism $\iota$ from (UD2) and $\tilde{\nu} \in Mor(B, B^\nu)$ from (UD3) satisfying $\nu = \iota\tilde{\nu}$, then $\tilde{\nu}$ is an isomorphism.*

*Proof.* Since $U(\tilde{\nu})(U(B)) = U(\iota)U(\tilde{\nu})(U(B)) = U(\nu)(U(B)) = U(B^\nu)$ by (UD2), the morphism $\tilde{\nu}$ is an epimorphism by Lemma 5.3(2). As $\tilde{\nu}$ is a monomorphism and $\mathcal{C}$ is a balanced category, $\tilde{\nu}$ is an isomorphism. □

Let us note that both categories of acts treated in this paper satisfy the previous axiomatics:

**Example 5.5.** (1) Let $\mathcal{S} = (S, \cdot, 1)$ be a monoid. We show that all conditions (UD1)–(UD6) are satisfied by $(\mathcal{S} - \overline{\mathrm{Act}}, U)$ for the natural forgetful functor $U : S - \overline{\mathrm{Act}} \to \mathrm{Set}$, hence it is a UD-category.

We have already mentioned that $\mathcal{S} - \overline{\mathrm{Act}}$ is a coproduct complete category and that $(\mathcal{S} - \overline{\mathrm{Act}}, U)$ is a concrete category over Set. Furthermore, the empty act $\emptyset$ with the empty mapping represents an initial object and the empty map is a monomorphism, since there is no morphism of a nonempty act into $\emptyset$. Since monomorphisms are exactly injective morphisms, epimorphisms are surjective morphisms and isomorphisms are bijections, $\mathcal{S} - \overline{\mathrm{Act}}$ is (epi,mono)-structured hence a balanced category (cf. [1, Section 14]), which proves (UD1). Let us put $A^f = f(A)$ for every morphism $f : A \to B$ and note that intersections and unions of subacts forms subacts as well, then the conditions (UD2), (UD3), (UD4) and (UD5) follow either immediately from the definition of an act or from well-known basic properties (cf. [18]), and (UD6) holds true since cyclic acts are indecomposable.

(2) Let $\mathcal{S}_0 = (S_0, \cdot, 1)$ be a monoid with a zero element 0. Then $\mathcal{S}_0 - \mathrm{Act}_0$, similarly as in (1) is also a coproduct complete category and $(\mathcal{S}_0 - \mathrm{Act}_0, U)$ is a concrete category over Set, where $U$ is the forgetful functor. Clearly, the zero object $\{0\}$ with the zero (mono)morphism forms an initial object of the category $\mathcal{S}_0 - \mathrm{Act}_0$. Since there is exactly one (zero) morphism from an arbitrary object to the zero object, (UD1) holds true. A similar argumentation as in (1) shows that $(\mathcal{S}_0 - \mathrm{Act}_0, U)$ satisfies also the conditions (UD2)–(UD6), i.e., it is a UD-category.

**Example 5.6.** The concrete category $(\mathrm{Set}, \mathrm{id}_{\mathrm{Set}})$ is a trivial example of a UD-category: conditions (UD1)–(UD5) are clearly satisfied and for (UD6) note that singletons are indecomposable objects.

**Example 5.7.** Observe that the faithful functor $U$ from the definition of UD-category need not preserve coproducts. While coproducts of the category $\mathcal{S} - \overline{\mathrm{Act}}$ are precisely disjoint unions, which are coproducts also in Set, and so the forgetful functor $U$ preserves coproducts here, the coproducts in $\mathcal{S}_0 - \mathrm{Act}_0$ glue together zero elements, so they do not coincide with coproducts of the category of all sets, hence the forgetful functor $U$ does not preserve coproducts of $\mathcal{S}_0 - \mathrm{Act}_0$.

## 5.2 Decomposition and coproduct

We suppose in the sequel that $(\mathcal{C}, U)$ is a UD-category, i.e., a concrete category over Set satisfying all axioms (UD1)–(UD6) and the notions of objects and morphisms refer to objects and morphisms of the underlying category $\mathcal{C}$. First, we prove a key observation that the description of coproducts in both categories of acts [18, Proposition II.1.8, Remark II.1.16] can be easily unified within the context of UD-categories.

If $\mathcal{A}$ is a family of objects, the corresponding coproduct will be designated $(\coprod \mathcal{A}, (\nu_A)_{A\in\mathcal{A}})$ where $\nu_A$ is said to be the *structural morphisms* of the coproduct for each $A \in \mathcal{A}$.

**Proposition 5.8.** *Let $\mathcal{A} = (A_i)_{i\in I}$ be a set of objects and $(\coprod_j A_j, (\nu_i)_{i\in I})$ be the coproduct of $\mathcal{A}$. Then there exist morphisms $\mu_j \in Mor(A_j, \bigcup_{i\in I} A_i^{\nu_i})$ and an inclusion morphism $\iota_\cup \in Mor(\bigcup_{i\in I} A_i^{\nu_i}, \coprod_{i\in I} A_i)$ satisfying $\iota_\cup \mu_j = \nu_j$ for each $j$. Furthermore, $(\bigcup_i A_i^{\nu_i}, (\mu_i)_i)$ is a coproduct of $\mathcal{A}$ and $U(\coprod_{i\in I} A_i) = \bigcup_{i\in I} U(A_i^{\nu_i})$.*

*Proof.* Note that the inclusion morphism $\iota_\cup \in \mathrm{Mor}(\bigcup_i A_i^{\nu_i}, \coprod A_i)$ exists by (UD4) and morphisms $\mu_j \in \mathrm{Mor}(A_j, \bigcup_i A_i^{\nu_i})$ with $\iota_\cup \mu_j = \nu_j$ exist by (UD3) for each $j$. Using the universal property of the coproduct, we obtain a morphism $\varphi$ such that $\varphi\nu_j = \mu_j$ for every $j$, i.e. the left square of the diagram

$$\begin{tikzcd} A_j \arrow[r, "\nu_j"] \arrow[d, equal] & \coprod A_i \arrow[r, equal] \arrow[d, "\varphi"] & \coprod A_i \\ A_j \arrow[r, "\mu_j"] & \bigcup_i A_i^{\nu_i} \arrow[r, equal] & \bigcup_i A_i^{\nu_i} \arrow[u, "\iota_\cup"'] \end{tikzcd}$$

commutes in $\mathcal{C}$ and we will show that the right square commutes as well.

Since $\iota_\cup\varphi\nu_j = \iota_\cup\mu_j = \nu_j$ for each $j$, we get again by the colimit universal property that $\iota_\cup\varphi = \mathrm{id}_{\coprod A_i}$. Furthermore, $U(\iota_\cup)$ is the inclusion mapping and

$$U(\iota_\cup)U(\varphi) = U(\iota_\cup\varphi) = U(\mathrm{id}_{\coprod A_i}) = \mathrm{id}_{U(\coprod A_i)},$$

which implies that the inclusion $U(\iota_\cup)$ is a bijection. Then $\iota_\cup$ and $\varphi$ are isomorphisms by Lemma 5.3(3), hence $U(\coprod A_i) = U(\iota_\cup)(\bigcup_i A_i^{\nu_i}) = \bigcup_i U(A_i^{\nu_i})$ and $(\bigcup_i A_i^{\nu_i}, (\mu_i)_i) = (\bigcup_i A_i^{\nu_i}, (\varphi\nu_i)_i)$ is a coproduct of the family $\mathcal{A}$. $\square$

Let $A$ be an object, and $(A_j, \iota_j)$ be subobjects such that $\iota_j$ is the inclusion morphism for each $j \in J$. We say that $((A_j, \iota_j), j \in J)$ is a *decomposition* of $A$ if $(A, (\iota_j)_{j\in J})$ is a coproduct of the family $(A_j, j \in J)$. Note that we have defined the decomposition for sets of subobjects with inclusion morphisms mainly for clarity of exposition: had we defined it for general subobjects, we would have got the same result using (UD2) and Lemma 5.4 afterwards.

The following assertion describes a natural decomposition of a coproduct in $\mathcal{C}$.

**Lemma 5.9.** *Let $\mathcal{A} = (A_j, j \in J)$ be a family of objects and $(A, (\nu_j)_j)$ be the coproduct of $\mathcal{A}$. Then for each $j \in J$ there exists an inclusion morphism $\iota_j \in$ Mor$(A_j^{\nu_j}, A)$ such that $((A_j^{\nu_j}, \iota_j), j \in J)$ forms a decomposition of $A$.*

*Proof.* Note that the inclusion morphisms $\iota_j \in \mathrm{Mor}(A_j^{\nu_j}, A)$, $j \in J$, exist by (UD2) and denote by $\tilde{\nu_j} \in \mathrm{Mor}(A_j, A_j^{\nu_j})$ the morphism satisfying $\nu_j = \iota_j \tilde{\nu_j}$ which exists by (UD3) for all $j \in J$. Since the maps $U(\tilde{\nu_j})$ are onto $U(A_j^{\nu_j})$ by (UD2) for all $j$, we get that $\tilde{\nu_j}$ are epimorphisms in $\mathcal{C}$ by Lemma 5.3(2).

In order to prove that $(A, (\iota_j)_{j\in J})$ is a coproduct of $(A^{\nu_j}, j \in J)$, let us suppose that $B$ is an arbitrary object and $\rho_j \in \mathrm{Mor}(A^{\nu_j}, B)$ are arbitrary morphisms for all $j \in J$. Fix $\varphi \in \mathrm{Mor}(A, B)$ satisfying the property $\rho_j \tilde{\nu_j} = \varphi \nu_j$ for each $j \in J$, which exists by the universal property of the coproduct $(A, (\nu_j)_j)$. It remains to show that $\varphi$ is the unique morphism such that $\rho_j = \varphi \iota_j$ for all $j \in J$.

Since $\rho_j \tilde{\nu_j} = \varphi \nu_j = \varphi \iota_j \tilde{\nu_j}$ and $\tilde{\nu_j}$ is an epimorphism, we get the equality $\rho_j = \varphi \iota_j$ for each $j$. Finally, for any morphism $\tilde{\varphi}$ such that $\rho_j = \tilde{\varphi} \iota_j$ for each $j$, the equality $\rho_j \tilde{\nu_j} = \tilde{\varphi} \iota_j \tilde{\nu_j} = \tilde{\varphi} \nu_j$ holds for each $j$, hence $\tilde{\varphi} = \varphi$ by the universal property of the coproduct $(A, (\nu_j)_{j\in J})$. $\square$

It is easy to see that $\theta \coprod A \cong A$ for every object $A$ and there is a canonical isomorphism $\coprod_{i\in I} (\coprod \mathcal{A}_i) \cong \coprod \left(\bigcup_{i\in I} \mathcal{A}_i\right)$ for every family of sets of objects $\mathcal{A}_i$, $i \in I$ in any coproduct-complete category with the initial object $\theta$.

As a straightforward consequence of the previous lemma, we obtain an important property of decompositions in UD-category.

**Lemma 5.10.** *Let $A$ be an object and $(\mathcal{A}_i, i \in I)$ a family of families of subobjects $(C, \iota_C)$ of $A$ such that $\iota_C$ is the inclusion morphism and let $B_i = \bigcup_{(C,\iota_C)\in\mathcal{A}_i} C^{\iota_C}$ and $\iota_i$ be the inclusion morphism ensured by (UD4) for each $i \in I$. The following conditions are equivalent:*

1. *For each $i \in I$, the set $\mathcal{A}_i$ forms a decomposition of the object $B_i$ and the family $((B_i, \iota_i), i \in I)$ is a decomposition of $A$,*
2. *$\dot{\bigcup}_{i\in I} \mathcal{A}_i$ is a decomposition of $A$.*

As the morphism of an initial object to an arbitrary object is a monomorphism by (UD1), for every object $A$, there exists a subobject denoted by $(\theta_A, \vartheta_A)$ with the inclusion morphism $\vartheta_A$ and $\theta_A \cong \theta$.

It is not a priori clear that the existence of different (even though necessarily isomorphic) initial objects will not cause obstacles. However, the following observations, that will also turn out to be useful for further dealing with decompositions of objects in a general UD-category, show that the situation is favourable.

**Lemma 5.11.** *An initial object $\theta$ has no proper subobjects, i.e. $\nu$ is an isomorphism for any subobject $(S, \nu)$ of $\theta$.*

*Proof.* Since $\mathrm{id}_\theta = \nu \vartheta_S$ by the uniqueness of the endomorphism of $\theta$, $\nu$ is an epimorphism. Then the monomorphism $\nu$ is an isomorphism because $\mathcal{C}$ is a balanced category. $\square$

**Lemma 5.12.** *If $A$ is an object and $(S, \iota)$ is a subobject of $A$ with the inclusion morphism $\iota$, then*

*(1)* $U(\theta_S) = U(\theta_A)$,

*(2)* $S \cong \theta$ *if and only if* $U(S) = U(\theta_A)$ *if and only if* $U(S) \subseteq U(\theta_A)$.

*Proof.* (1) Since there exists the unique isomorphism $\tau : \theta_S \to \theta_A$ and both $\iota\vartheta_S\tau = \vartheta_A$ and $\iota\vartheta_S = \vartheta_A\tau^{-1}$ are inclusion morphisms, we get the equality $U(\theta_S) = U(\theta_A)$.

(2) If $S \cong \theta$, then $U(S) = U(\theta_S) = U(\theta_A)$. The implication $U(S) = U(\theta_A) \Rightarrow U(S) \subseteq U(\theta_A)$ is clear and, proving indirectly, suppose that $S$ is not isomorphic to $\theta$. Then $\vartheta_S$ is a monomorphism by (UD1) which is not an epimorphism. Hence $U(\vartheta_S)$ is not surjective by Lemma 5.3(2) and so $U(\theta_A) = U(\theta_S) \subsetneq U(S)$ by (1). We have proved that $U(S) \not\subseteq U(\theta_A)$. $\square$

Now, we formulate a natural description of decompositions of objects and of its subobjects.

**Lemma 5.13.** *Let $A$ be an object and and $(A_j, \iota_j)$ be a subobject of $A$ with the inclusion morphism $\iota_j$ into $A$ for every $j \in J$. Then $((A_j, \iota_j), j \in J)$ is a decomposition of $A$ if and only if $U(A) = \bigcup_j U(A_j)$ and $U(A_i) \cap \bigcup_{j\neq i} U(A_j) = U(\theta_A)$ for each $i \in J$.*

*Proof.* Let $((A_j, \iota_j), j \in J)$ be a decomposition of $A$. By Proposition 5.8 we then have $U(A) = \bigcup_j U(\iota_j)U(A_j) = \bigcup_j U(A_j)$ . Since $((A_i, \iota_i), (\bigcup_{j\neq i} A_j^{\iota_j}, \iota))$, where the morphism $\iota \in \mathrm{Mor}(\bigcup_{j\neq i} A_j^{\iota_j}, A)$ is the inclusion morphism, forms a decomposition of $A$ by Lemma 5.10, (UD3) and (UD4), we get that $A_i^{\iota_i} \cap (\bigcup_{j\neq i} A_j^{\iota_j})^{\iota} \cong \theta$ by (UD5). Thus $U(A_i) \cap \bigcup_{j\neq i} U(A_j) = U(A_i^{\iota_i}) \cap U((\bigcup_{j\neq i} A_j^{\iota_j})^{\iota}) = U(\theta_A)$ by Lemma 5.12.

In order to prove the reverse implication, let us suppose that $U(A) = \bigcup_j U(A_j)$ and $U(A_i)\cap\bigcup_{j\neq i} U(A_j) = U(\theta_A)$ for each $i \in J$ and $(\coprod_j A_j, (\nu_j)_{j\in J})$ is a coproduct of the family $(A_j, j \in J)$. Then there exists a morphism $\varphi \in \mathrm{Mor}(\coprod_j A_j, A)$ such that $\varphi\nu_j = \iota_j$ for all $j$ by the universal property of the coproduct. Since $U(A) = \bigcup_j U(A_j) \subseteq U(\varphi)(U(\coprod_j A_j))$, the mapping $U(\varphi)$ is surjective. Let $U(\varphi)(a) = U(\varphi)(b)$ for elements $a, b \in U(\coprod_j A_j)$. Then there are indexes $j_0, j_1 \in J$ and elements $\tilde{a} \in U(A_{j_0})$, $\tilde{b} \in U(A_{j_1})$ for which $a = U(\nu_{j_0})(\tilde{a})$, $b = U(\nu_{j_1})(\tilde{b})$ by Proposition 5.8, hence

$$\tilde{a} = U(\iota_{j_0})(\tilde{a}) = U(\varphi\nu_{j_0})(\tilde{a}) = U(\varphi)(a) = U(\varphi)(b) = U(\varphi\nu_{j_1})(\tilde{b}) = U(\iota_{j_1})(\tilde{b}) = \tilde{b},$$

which proves that $U(\varphi)$ is an injective map. Since $U(\varphi)$ is a bijection, $\varphi$ is an isomorphism by Lemma 5.3(3). Thus $(A, (\varphi\nu_j)_j)$ is a coproduct of the family $(A_j, j \in J)$ and all $\varphi\nu_j$ are the inclusion morphisms, which means that $(A_j, j \in J)$ is a decomposition of $A$. $\square$

Note that the argument of the reverse implication depends strongly on the fact that $(\mathcal{C}, U)$ is a concrete category over Set.

**Lemma 5.14.** *Let $A$ be an object, $(B, \mu)$ its subobject with inclusion morphism $\mu$ and let $((A_j, \iota_j), j \in J)$ be a decomposition of $A$. Then there exists a decomposition $((B_j, \mu_j), j \in J)$ of $B$ such that $U(B_j) = U(B) \cap U(A_j)$ for each $i \in J$.*

*Proof.* It is enough to define $B_j = A_j^{\iota_j} \cap B^{\mu}$ with a morphism $\mu_j \in \mathrm{Mor}(B_j, B)$ such that $\mu\mu_j$ is the inclusion morphism $B_j \to A$ and $U(B_j) = U(B) \cap U(A_j)$ for each $j \in J$, which exists by (UD4) and (UD3). Since $U(\mu\mu_j) = U(\mu)U(\mu_j)$ and $U(\mu)$ are inclusions, $\mu_j$ is an inclusion morphism for all $j \in J$. Finally, since $\bigcup_j U(B_j) = \bigcup_j U(B) \cap U(A_j) = U(B)$ and

$$U(\theta_A) \subseteq U(B_i) \cap \bigcup_{j \neq i} U(B_j) = U(B) \cap U(A_i) \cap \bigcup_{j \neq i} U(A_j) = U(\theta_A)$$

for each $i \in J$, we obtain that $((B_j, \mu_j), j \in J)$ is a decomposition by Lemma 5.13. □

Recall that an object $A$ in a category with an initial object is indecomposable if for each pair of objects $A_1$, $A_2$ such that $A \cong A_1 \coprod A_2$ either $A_1 \cong \theta$ or $A_2 \cong \theta$ holds, hence an object of a UD-category $(\mathcal{C}, U)$ is *indecomposable* provided is indecomposable object of $\mathcal{C}$. Note that the definition of an indecomposable object in a UD-category reflects the definition of such an object in categories of acts and we do not include initial objects into the class of indecomposable ones. The natural description of indecomposability is formulated in the following assertion:

**Proposition 5.15.** *The following conditions are equivalent for an object $A$:*

*(1) $A$ is indecomposable,*

*(2) for every decomposition $((A_0, \iota_0), (A_1, \iota_1))$ of $A$, either $U(A_0) = U(\theta_A)$ and $U(A_1) = U(A)$, or $U(A_1) = U(\theta_A)$ and $U(A_0) = U(A)$,*

*(3) for every decomposition $((A_j, \iota_j), j \in J)$ of $A$ there exists $i \in J$ such that $U(A_j) = U(\theta_{A_j})$ for each $j \neq i$ and $U(A_i) = U(A)$.*

*Proof.* (1)⇒(2) If $((A_0, \iota_0), (A_1, \iota_1))$ is a decomposition of $A$, then $A \cong A_0 \coprod A_1$, hence by indecomposability of $A$ either $A_0 \cong \theta$ so $U(A_0) = U(\theta_A)$ or $A_1 \cong \theta$ and so $U(A_1) = U(\theta_A)$ by Lemma 5.12. In the first case, since we have $U(A) = U(A_0) \cup U(A_1)$ with $U(A_0) \cap U(A_1) = U(\theta_A)$ by Lemma 5.13, we get $U(A_0) \subseteq U(A_1)$, and in consequence $U(A_1) = U(A)$. The latter case is proved analogously.

(2)⇒(1) Let $A \cong A_0 \coprod A_1$ and $\nu_0$, $\nu_1$ be structural morphisms of the coproduct. We may assume w.l.o.g that $A = A_0 \coprod A_1$. Then for $i = 0, 1$ there exists an object $A_i^{\nu_i}$, an inclusion morphism $\iota_i \in \mathrm{Mor}(A_i^{\nu_i}, A)$ and a monomorphism $\mu_i \in \mathrm{Mor}(A_i, A_i^{\nu_i})$, such that $\nu_i = \iota_i \mu_i$ by (UD2) and (UD3). Then the pair $((A_0^{\nu_0}, \iota_0), (A_1^{\nu_1}, \iota_1))$ forms a decomposition of $A$ by Lemma 5.9, hence there exists $i \in \{0, 1\}$ for which $U(A_i) = U(\theta_A)$ by the hypothesis, hence $A_i^{\nu_i} \cong \theta$ by Lemma 5.12 and $A_i \cong \theta$ by Lemma 5.11 as $(A_i, \mu_i)$ is a subobject of $A_i^{\nu_i}$.

(2)⇔(3) The direct implication follows from Lemma 5.10 and the fact that any coproduct of initial objects is isomorphic to the initial object. The reverse implication is clear. □

**Lemma 5.16.** *Let $((A_i, \nu_i), i \in I)$ be a family of subobjects of an object $A$ such that $A_i$ is indecomposable for each $i \in I$. If $\bigcap_{i \in I} U(A_i^{\nu_i}) \neq U(\theta_A)$, then there exists an inclusion morphism $\iota_\cup \in$ Mor$(\bigcup_{i \in I} A_i^{\nu_i}, A)$ such that $(\bigcup_{i \in I} A_i^{\nu_i}, \iota_\cup)$ is an indecomposable subobject of $A$.*

*Proof.* Put $A' = \bigcup_{i\in I} A_i^{\nu_i}$ and let $\iota_\cup \in \mathrm{Mor}(A', A)$ be the inclusion morphism ensured by (UD4). Since $(A', \iota_\cup)$ is a subobject, we may suppose that $A = A'$. Remark that the proof repeats the argument of the proof of [18, Lemma I.5.9].

Assume that $((B_0, \iota_0), (B_1, \iota_1))$ is a decomposition of $A$ such that $U(B_i) \neq U(\theta_A)$ for both $i = 0, 1$. Since $\bigcap_{i\in I} U(A_i^{\nu_i}) \neq U(\theta_A)$ and $U(B_0) \cup U(B_1) = U(A)$ by Lemma 5.13, there exists $j$ for which $U(B_j) \cap \bigcap_{i\in I} U(A_i^{\nu_i}) \neq U(\theta_A)$, we may w.l.o.g. assume that $j = 0$. Moreover, there exists $i$ such that $U(B_1) \cap U(A_i) \neq U(\theta_A)$. Thus $U(B_j) \cap U(A_i) \neq U(\theta_A)$ for both $j = 0, 1$. Then by Lemma 5.14 there exists a decomposition $((\tilde{B}_0, \tilde{\iota}_0), (\tilde{B}_0, \tilde{\iota}_0))$ of $A_i$ such that $U(\tilde{B}_j) \neq U(\theta_A) = U(\theta_{A_i})$ for both $j = 0, 1$. Hence we obtain by Proposition 5.15 a contradiction with the hypothesis that $A_i$ is indecomposable. $\square$

Now we can formulate a version of [18, Theorem I.5.10] valid in a general UD-category:

**Theorem 5.17.** *Every noninitial object $A$ has a decomposition into indecomposable objects. If $((A_j, \iota_j), j \in J)$ and $((\tilde{A}_j, \tilde{\iota}_j), j \in \tilde{J})$ are two decompositions into indecomposable objects, there exists a bijection $b : J \to \tilde{J}$ such that $U(A_j) = U(\tilde{A}_{b(j)})$ for each $j \in J$.*

*Proof.* For $a \in U(A) \setminus U(\theta_A)$, which exists by Lemma 5.12, consider the set

$$I_a = \{C \mid (C, \nu_C) \text{ is an indecomposable subobject of } A \text{ and } a \in U(C)\}$$

and let $(A_a, \iota_a)$ be a subobject with the inclusion map, where $A_a = \bigcup_{(C,\nu_C)\in I_a} C^{\nu_C}$, which exists by (UD4). Then $(A_a, \iota_a)$ is an indecomposable subobject of $A$ by Lemma 5.16.

Furthermore, if $a \neq b$ then either $U(A_a) = U(A_b)$, or $U(A_a) \cap U(A_b) = U(\theta_A)$. Indeed, let $U(A_a) \cap U(A_b) \neq U(\theta_A)$, take $z \in (U(A_a) \cap U(A_b)) \setminus U(\theta_A)$, which exists by Lemma 5.11 and (UD4) and consider the indecomposable object $A_z$. Since $z \in U(A_a)$, we have $(A_a, \iota_a) \in I_z$, hence $(A_a, \iota_a)$ is a subobject of $A_z$, similarly for $b$ and vice versa. Therefore $U(A_z) = U(A_a) = U(A_b)$.

Note that for each $a \in U(A)$ there exists an indecomposable subobject $(C, \nu_C)$ of $A$ such that $U(C)$ contains $a$ by (UD6), hence $a \in A_a$. Moreover, as $A$ is not isomorphic to $\theta$, we get that $U(A) = \bigcup_{a\in U(A)\setminus U(\theta_A)} U(A_a)$, and we have proved that the representative set of subobjects of the form $(A_x, \iota_x)$ is the desired decomposition.

Let $((A_j, \iota_j), j \in J)$ and $((\tilde{A}_j, \tilde{\iota}_j), j \in \tilde{J})$ be two indecomposable decompositions, then for each $j \in J$ there exists a decomposition $((B_i, \mu_i), i \in \tilde{J})$ of $A_j$ such that $U(B_k) = U(A_j) \cap U(\tilde{A}_k)$ for each $k \in \tilde{J}$ by Lemma 5.14 . Since $A_j$ is indecomposable there exists exactly one $b(j) \in \tilde{J}$ for which $U(B_{b(j)}) = U(A_j)$ and $U(B_k) = U(\theta_A)$ for all $k \neq b(j)$. We have determined an injective mapping $b : J \to \tilde{J}$. The surjectivity follows from the symmetric argument for each $\tilde{A}_j$, $j \in \tilde{J}$. $\square$

## 5.3 Projective objects

Recall that $(\mathcal{C}, U)$ is supposed to be a UD-category. We say that an object $P \in \mathcal{C}$ is *projective*, if for any pair of objects $A, B \in \mathcal{C}$ and any pair of morphisms

$\pi : A \to B$, $\alpha : P \to B$, where $\pi$ is an epimorphism, there exists a morphism $\overline{\alpha} : P \to A$ in $\mathcal{C}$ such that $\alpha = \pi\overline{\alpha}$, i.e. any diagram

$$\begin{array}{ccc} & & P \\ & & \downarrow{\scriptstyle \alpha} \\ A & \xrightarrow{\pi} & B \end{array}$$

in $\mathcal{C}$ with $\pi$ an epimorphism, can be completed into a commutative diagram

$$\begin{array}{ccc} P & = & P \\ {\scriptstyle \overline{\alpha}}\downarrow & & \downarrow{\scriptstyle \alpha} \\ A & \xrightarrow{\pi} & B. \end{array}$$

Note that the notion of projectivity is one of basic tools of category theory and issue of description of projective objects seems to be important task in research of any (concrete) category (see e.g. [1, Chapter 9] or [18, Section III.17]). The main goal of the section is to confirm that the structure of projective objects of the underlying category $\mathcal{C}$ of a UD-category $(C, U)$ can be described as a coproduct of indecomposable projective objects in accordance with the case of categories of acts.

**Lemma 5.18.** *The coproduct of a family $(P_i, i \in I)$ of projective objects is projective.*

*Proof.* Let the projective situation

$$\begin{array}{ccc} & & \coprod P_i \\ & & \downarrow{\scriptstyle \alpha} \\ A & \xrightarrow{\pi} & B \end{array}$$

be given.

For each $i \in I$ consider the structural monomorphism $\nu_i : P_i \to \coprod P_{j \in I}$, which gives

$$\begin{array}{ccc} P_i & = & P_i \\ \downarrow{\scriptstyle \varphi_i} & & \downarrow{\scriptstyle \alpha\nu_i} \\ A & \xrightarrow{\pi} & B, \end{array}$$

where $\varphi_i : P_i \to A$ is obtained from the projectivity of $P_i$. Then the family $(\varphi_i, i \in I)$ induces the unique morphism $\varphi : \coprod P_i \to A$ with $\varphi\nu_i = \varphi_i$. By Proposition 5.8, each element $x \in U(\coprod P_i)$ can be written as $U(\nu_i)(y)$ for some (not necessarily unique) $i \in I$ and $y \in U(P_i)$, hence

$$U(\pi\varphi)(x) = U(\pi\varphi\nu_i)(y) = U(\alpha\nu_i)(y) = U(\alpha)(x).$$

As $U(\pi\varphi) = U(\alpha)$ and since $U$ is a faithful functor,the equality $\pi\varphi = \alpha$ holds, which finishes the proof. □

**Lemma 5.19.** *If the coproduct of objects $P = \coprod_I P_i$ is projective, then each object $P_i$, $i \in I$, is projective.*

*Proof.* As $\coprod_j P_i \cong P_i \coprod(\coprod_{j\neq i} Pj)$, it is enough to prove that for any pair of objects $P_0, P_1$, if $P_0 \coprod P_1$ is projective, then $P_0$ is projective.

Let the projective situation

$$\begin{array}{ccc} & & P_0 \\ & & \downarrow{\scriptstyle\alpha} \\ A & \xrightarrow{\pi} & B \end{array}$$

be given and let $\lambda_X : X \to P_0 \coprod P_1$, for $X \in \{P_0, P_1\}$, $\mu_X : X \to A \coprod P_1$ for $X \in \{A, P_1\}$ and $\nu_X : X \to B \coprod P_1$ for $X \in \{B, P_1\}$ be structural coproduct morphisms, which all are monomorphisms by (UD5). Denote by $\tilde{\alpha} : P_0 \coprod P_1 \to B \coprod P_1$ the coproduct of morphisms $\alpha : P_0 \to B$ and $1_{P_1} : P_1 \to P_1$ which is uniquely determined by the universal property of the coproduct $P_0 \coprod P_1$, i.e. $\tilde{\alpha}\lambda_{P_0} = \nu_B\alpha$ and $\tilde{\alpha}\lambda_{P_1} = \nu_{P_1}$. Similarly, denote by $\tilde{\pi} : A \coprod P_1 \to B \coprod P_1$ the coproduct of morphisms $\pi : A \to B$ and $1_{P_1} : P_1 \to P_1$, which means that $\tilde{\pi}\mu_{P_0} = \nu_B\pi$ and $\tilde{\pi}\mu_{P_1} = \nu_{P_1}$. It is easy to compute applying the universal property of the coproduct $B \coprod P_1$, that $\tilde{\pi}$ is an epimorphism since both $\pi$ and $1_{P_1}$ are epimorphisms.

Hence we obtain another projective situation:

$$\begin{array}{ccc} & & P_0 \coprod P_1 \\ & & \downarrow{\scriptstyle\tilde{\alpha}} \\ A \coprod P_1 & \xrightarrow{\tilde{\pi}} & B \coprod P_1 \end{array}$$

By the assumption, there exists a morphism $\varphi \in \mathrm{Mor}(P_0 \coprod P_1, A \coprod P_1)$ such that $\tilde{\pi}\varphi = \tilde{\alpha}$. Let us show that $U(\varphi\lambda_{P_0})(U(P_0)) \subseteq U(\mu_A)(U(A))$.

By (UD2), (UD3) and (UD4) there exists a subobject $(S, \iota)$ of $A \coprod P_1$ with the inclusion morphism $\iota$ satisfying $U(S) = U(\varphi\lambda_{P_0})(U(P_0)) \cap U(\mu_{P_1})(U(P_1))$. Since $\mu_{P_1}$ is a monomorphism, there exists a monomorphism $\sigma : S \to P_1$, such that $\mu_{P_1}\sigma = \iota$. Then $\tilde{\pi}\iota = \tilde{\pi}\mu_{P_1}\sigma = \nu_{P_1}\sigma$ is a monomorphism. Since

$$\begin{aligned} U(\tilde{\pi}\iota)(U(S)) \subseteq U(\tilde{\pi})U(\varphi\lambda_{P_0})(U(P_0)) = U(\tilde{\alpha}\lambda_{P_0})(U(P_0)) = \\ = U(\nu_B\alpha)(U(P_0)) \subseteq U(\nu_B)(U(B)) \end{aligned}$$

and

$$U(\tilde{\pi}\iota)(U(S)) = U(\nu_{P_1}\sigma)(U(S)) \subseteq U(\nu_{P_1})(U(P_1)),$$

by Lemma 5.13 and Proposition 5.8 $U(\tilde{\pi}\iota)(U(S)) = U(\theta_{B\coprod P_1})$ and so $\tilde{\pi}\iota$ factorizes through the morphism $\vartheta : \theta \to B \coprod P_1$. Thus $S \cong \theta$, which implies that

$$U(\varphi\lambda_{P_0})(U(P_0)) \cap U(\mu_{P_1})(U(P_1)) = U(\theta_{B\coprod P_1}).$$

Since $U(\varphi\lambda_{P_0})(U(P_0)) \subseteq U(\mu_A)(U(A)) \cup U(\mu_{P_1})(U(P_1))$ by Lemma 5.13 and Proposition 5.8, we get that $U(\varphi\lambda_{P_0})(U(P_0)) \subseteq U(\mu_A)(U(A))$. In consequence, by (UD3) there exists a morphism $\tau : P_0 \to A$ such that $\mu_A\tau = \varphi\lambda_{P_0}$; therefore $\tilde{\pi}\varphi\lambda_{P_0} = \tilde{\pi}\mu_A\tau = \nu_B\pi\tau$ and on the other hand $\tilde{\pi}\varphi\lambda_{P_0} = \tilde{\alpha}\lambda_{P_0} = \nu_B\alpha$. Finally, as $\nu_B\pi\tau = \nu_B\alpha$ and the morphism $\nu_B$ is a monomorphism by (UD5), we have $\pi\tau = \alpha$. □

Now we are ready to prove an important property of UD-categories:

**Theorem 5.20.** *An object of a UD-category is projective if and only if it is isomorphic to a coproduct of indecomposable projective objects.*

*Proof.* If an object is projective, it possesses a decomposition by Theorem 5.17, which consists of projective objects by Lemma 5.19. The reverse implication follows immediately from Lemma 5.18. □

Let $A$ and $B$ be a pair of objects. Recall that $B$ is a *retract of* $A$ if there are morphisms $f \in \mathrm{Mor}(A, B)$ and $g \in \mathrm{Mor}(B, A)$ such that $fg = 1_B$. The morphism $f$ is then called *retraction* and $g$ *coretraction*. Note that each retraction is an epimorphism and each coretraction is a monomorphism. An object $G$ of a category is said to be *generator* if for any object $A \in \mathcal{C}$ there exists an index set $I$ and an epimorphism $\pi : \coprod_{i\in I} G_i \to A$ where $G_i \simeq G$.

**Lemma 5.21.** *If $\mathcal{C}$ contains a generator $G$, then every indecomposable projective object is a retract of $G$.*

*Proof.* We generalize arguments of [18, Propositions III.17.4 and III.17.7].

Let $P$ be an indecomposable projective object. Since $G$ is a generator, there are a coproduct $\coprod_i G_i$ of objects $G_i \cong G$ with the corresponding structural morphisms $\nu_i \in \mathrm{Mor}(G_i, \coprod_i G_i)$ and an epimorphism $\pi \in \mathrm{Mor}(\coprod_i G_i, P)$. Moreover, there exists a (mono)morphism $\gamma \in \mathrm{Mor}(P, \coprod_i G_i)$ for which $\pi\gamma = 1_P$ due to the projectivity of $P$. Note that $P \cong P^\gamma$ by Lemma 5.3 and there exists a decomposition $((H_i, \mu_i), i \in I)$ of $P^\gamma$ for which $U(H_i) = U(P^\gamma) \cap U(G_i^{\nu_i}) = U(\gamma)(U(P)) \cap U(\nu_i)(U(G_i))$ for each $i \in I$ by Lemma 5.14. As $P^\gamma$ is indecomposable, there exists an $i \in I$ such that $U(\gamma)(U(P)) = U(P^\gamma) \subseteq U(\nu_i)(U(G_i))$ by Proposition 5.15, hence there exists a morphism $\varphi \in \mathrm{Mor}(P, G_i)$ such that $\nu_i\varphi = \gamma$ by (UD3). Thus $\pi\nu_i\varphi = \pi\gamma = 1_P$ which shows that $\pi\nu_i$ is the desired retraction. □

## 5.4 Compact objects

### 5.4.1 Compactness in UD categories

Now we are ready to translate the concept of compactness to the context of a UD-category $(\mathcal{C}, U)$.

Let $C$ be an object, $\mathcal{A} = (A_i, i \in I)$ a family of objects and $(\coprod_{i\in I} A_i, \{\nu_i\}_{i\in I})$ a coproduct of the family $\mathcal{A}$. Using the covariant functor $\mathrm{Mor}(C, -)$ from $\mathcal{C}$ to Set, we define a natural morphism in the category Set

$$\Psi^C_{\mathcal{A}} : \coprod_{i\in I} \mathrm{Mor}(C, A_i) \to \mathrm{Mor}(C, \coprod_{i\in I} A_i)$$

which is the unique morphism such that the following square is commutative for all $i \in I$

$$\begin{tikzcd} \mathrm{Mor}\,(C, A_i) \arrow[r, "\mu_i"] \arrow[d, "\mathrm{Mor}(C,\nu_i)"'] & \coprod_I \mathrm{Mor}\,(C, A_i) \arrow[d, "\Psi^C_{\mathcal{A}}"'] \\ \mathrm{Mor}\,(C, \coprod_I A_i) \arrow[r, equal] & \mathrm{Mor}\,(C, \coprod_I A_i) \end{tikzcd}$$

where $\mu_i : \mathrm{Mor}\,(C, A_i) \to \coprod_I \mathrm{Mor}\,(C, A_i)$ is the coproduct structural inclusion in Set. Since coproducts of objects in Set are isomorphic to disjoint unions of the corresponding objects, we have $\coprod_I \mathrm{Mor}\,(C, A_i) = \dot{\bigcup} \mathrm{Mor}\,(C, A_i)$ and we can describe $\Psi^C_{\mathcal{A}}$ explicitly as $\Psi^C_{\mathcal{A}}(\alpha) = \nu_i\alpha$ for each index $i$ satisfying $\alpha \in \mathrm{Mor}(C, A_i)$.

It is worth mentioning that it is natural to consider morphisms

$$\tilde{\Psi}^C_{\mathcal{A}} : \coprod_{i\in I} \mathrm{Mor}(U(C), U(A_i)) \to \mathrm{Mor}(U(C), U(\coprod_{i\in I} A_i)) = \mathrm{Mor}(U(C), \bigcup_{i\in I} U(A_i))$$

as we deal with concrete category. Since $U$ is a faithful functor, such a concept is equivalent to original one, however it seems to be technically more difficult.

**Lemma 5.22.** *Let $C$ be an object, $I$ an index set consisting of at least two elements, $\mathcal{A} = \{A_i \mid i \in I\}$ a family of objects. Suppose that $(A, \{\nu_i\}_{i\in I})$ is a coproduct of the family $\mathcal{A}$ and $\alpha, \beta \in \coprod_{i\in I} \mathit{Mor}\,(C, A_i) = \dot{\bigcup}_{i\in I} \mathit{Mor}\,(C, A_i)$.*

*(1) If $\alpha \neq \beta$ and $\Psi^C_{\mathcal{A}}(\alpha) = \Psi^C_{\mathcal{A}}(\beta)$, then there exist indeces $i \neq j$ such that $\alpha \in \mathit{Mor}\,(C, A_i)$, $\beta \in \mathit{Mor}\,(C, A_j)$ and $U(\nu_i\alpha)(U(C)) = U(\theta_A) = U(\nu_j\beta)(U(C))$.*

*(2) If $i, j \in I$ and $\alpha \in \mathit{Mor}\,(C, A_i)$ and $\beta \in \mathit{Mor}\,(C, A_j)$ such that $U(\alpha)(U(C)) = U(\theta_{A_i})$ and $U(\beta)(U(C)) = U(\theta_{A_j})$, then $\Psi^C_{\mathcal{A}}(\alpha) = \Psi^C_{\mathcal{A}}(\beta)$.*

*(3) $\Psi^C_{\mathcal{A}}$ is injective (i.e. it is a monomorphism in the category Set) if and only if $\mathit{Mor}\,(C, \theta) = \emptyset$.*

*Proof.* (1) If there exists an $i$ for which $\alpha, \beta \in \mathrm{Mor}\,(C, A_i)$, then $\nu_i\alpha = \nu_i\beta$, hence $\alpha = \beta$ as $\nu_i$ is a monomorphism by (UD5). In consequence, the hypotheses $\alpha \neq \beta$ and $\Psi^C_{\mathcal{A}}(\alpha) = \Psi^C_{\mathcal{A}}(\beta)$ imply that there exists $i \neq j$ such that $\alpha \in \mathrm{Mor}\,(C, A_i)$, $\beta \in \mathrm{Mor}\,(C, A_j)$, so we get

$$U(\nu_i\alpha)(U(C)) = U(\Psi^C_{\mathcal{A}}(\alpha))(U(C)) = U(\Psi^C_{\mathcal{A}}(\beta))(U(C)) = U(\nu_j\beta)(U(C)).$$

Since $U(\nu_i)(U(A_i)) \cap U(\nu_j)(U(A_j)) = U(\theta_A)$ by Lemmas 5.9 and 5.13 and since

$$U(\nu_i\alpha)(U(C)) = U(\nu_j\beta)(U(C)) \subseteq U(\nu_i)(U(A_i)) \cap U(\nu_j)(U(A_j)) = U(\theta_A),$$

we get that $U(\nu_i)\alpha(U(C)) = U(\nu_j\beta)(U(C)) = U(\theta_A)$ by Lemma 5.12.

(2) Since $U(\nu_i)(U(\theta_{A_i})) = U(\theta_A) = U(\nu_j)(U(\theta_{A_j}))$, we have

$$\begin{aligned} U(\Psi^C_{\mathcal{A}}(\alpha))(U(C)) &= U(\nu_i\alpha)(U(C)) = \\ &= U(\nu_j\beta)(U(C)) = U(\Psi^C_{\mathcal{A}}(\beta))(U(C)) = U(\theta_A) \end{aligned}$$

again by the same argument as in (1) using Lemmas 5.9, 5.11, ZeroSubobj and 5.13. As $U$ is a faithful functor, both morphisms $\Psi^C_{\mathcal{A}}(\alpha), \Psi^C_{\mathcal{A}}(\beta)$ can be viewed as elements of $\mathrm{Mor}(C, \theta_A)$. Since $|\mathrm{Mor}(C, \theta_A)| = |\mathrm{Mor}(C, \theta)| \leq 1$ by (UD1), we get the required equality $\Psi^C_{\mathcal{A}}(\alpha) = \Psi^C_{\mathcal{A}}(\beta)$.

(3) If $\mathrm{Mor}\,(C, \theta) \neq \emptyset$, there exists $\alpha_i \in \mathrm{Mor}(C, A_i)$ such that $U(\alpha_i)(U(C)) = U(\theta_{A_i})$ for all $i \in I$ by Lemma 5.12 again. Thus $\Psi^C_{\mathcal{A}}(\alpha_i) = \Psi^C_{\mathcal{A}}(\alpha_j)$ for all $i, j \in I$ by (2), which implies that $\Psi^C_{\mathcal{A}}$ is not injective.

On the other hand, if $\Psi^C_{\mathcal{A}}$ is not injective, then there exists an index $i$ and $\alpha \in \mathrm{Mor}\,(C, A_i)$ such that $U(\nu_i\alpha)(U(C)) = U(\theta_A)$ by (1). As $\theta_A \cong \theta$, there exists a morphism in $\mathrm{Mor}\,(C, \theta)$ by (UD3). □

Since there is no morphism of a nonempty act $C$ into the empty act $\emptyset$, all mappings $\Psi^C_{\mathcal{A}}$ are injective in the category $\mathcal{S}-\overline{\mathrm{Act}}$ by Lemma 5.22(3), similarly to the case of abelian categories (cf [16, Lemma 1.3]). Applying the same assertion, we can see that it is not the case of the category $\mathcal{S}_0 - \mathrm{Act}_0$.

**Example 5.23.** If $\mathcal{S}_0 = (S_0, \cdot, 1)$ is a monoid with a zero element (for example $(\mathbb{Z}, \cdot, 1)$), $C$ is a right $S_0$-act and $\mathcal{A} = \{A_i \mid i \in I\}$ is a family of $S_0$-acts contained in the category $\mathcal{S}_0 - \mathrm{Act}_0$ satisfying $|\mathcal{A}| \geq 2$, then the mapping $\Psi^C_{\mathcal{A}}$ is not injective by Lemma 5.22(3).

In particular, if we put $C = A_i = \{0\}$ for every $i \in I$, then $|\coprod_I \mathrm{Mor}\,(C, A_i)| = |I|$ and $|\mathrm{Mor}\,(C, \coprod_I A_i)| = 1$, so the mapping $\Psi^C_{\mathcal{A}}$ glues together all morphisms of the arbitrarily large set $\coprod_I \mathrm{Mor}\,(C, A_i)$.

Using the notation of the mapping $\Psi^C_{\mathcal{A}}$ we are ready to generalize abelian-category definition of a compact object to UD-categories.

We say that an object $C$ is $\mathcal{D}$-*compact* (or compact with respect to $\mathcal{D}$), if the morphism $\Psi^C_{\mathcal{A}}$ is surjective for each family $\mathcal{A}$ of objects from the class $\mathcal{D}$ and $C$ is *compact* if it is $O_{\mathcal{C}}$-compact for the class $O_{\mathcal{C}}$ of all objects of the category $\mathcal{C}$. Finally, an object $C$ is called *autocompact*, if it is $\{C\}$-compact. Observe that every compact object is $\mathcal{D}$-compact for an arbitrary class $\mathcal{D}$ of objects, in particular, it is autocompact.

Let $\mathcal{D}$ be a class of objects of the category $\mathcal{C}$ and denote by $\mathcal{D}^{\coprod} = \{\coprod_i D_i \mid D_i \in \mathcal{D}\}$ the class of all coproducts of all families of objects of $\mathcal{D}$.

Let us formulate a non-abelian version of [14, Proposition 2.1] (cf. also [16, Theorem 2.5]):

**Theorem 5.24.** *The following conditions are equivalent for an object $C$ and a class of objects $\mathcal{D}$:*

*(1) $C$ is $\mathcal{D}$-compact,*

*(2) for each pair of objects $A_1 \in \mathcal{D}$ and $A_2 \in \mathcal{D}^{\coprod}$ and each morphism $f \in$ Mor$(C, A_1 \coprod A_2)$ there exists $i \in \{1, 2\}$ such that $f$ factorizes through $\nu_i$,*

*(3) for each pair of objects $A_1 \in \mathcal{D}$ and $A_2 \in \mathcal{D}^{\coprod}$ and each morphism $f \in$ Mor$(C, A_1 \coprod A_2)$ there exists $i \in \{1, 2\}$ with $U(f)(U(C)) \subseteq U(\nu_i)(U(A_i))$,*

*(4) for each family $(A_i, i \in I)$ of objects of the class $\mathcal{D}$ and each morphism $f \in$ Mor$(C, \coprod_{i \in I} A_i)$ there exists $i \in I$ such that $f$ factorizes through $\nu_i$,*

*(5) for each family $(A_i, i \in I)$ of objects of th class $\mathcal{D}$ and each morphism $f \in$ Mor$(C, \coprod_{i \in I} A_i)$ there exists $i \in I$ such that $U(f)(U(C)) \subseteq U(\nu_i)(U(A_i))$,*

*where $\nu_i$ denotes the structural morphism of a corresponding coproduct $A_1 \coprod A_2$ or $\coprod_{i \in I} A_i$.*

*Proof.* (1)$\Rightarrow$(4) Let $\mathcal{A} = \{A_i \mid i \in I\}$ be a family of objects of the class $\mathcal{D}$ and $f \in \mathrm{Mor}\left(C, \coprod_{i \in I} A_i\right)$. Since $C$ is $\mathcal{D}$-compact, the mapping $\Psi^C_{\mathcal{A}}$ is surjective by definition, hence there exists $i$ and $\alpha \in \mathrm{Mor}\,(C, A_i)$ such that $f = \nu_i \alpha$.

(4)$\Rightarrow$(1) Let $f \in \mathrm{Mor}\left(C, \coprod_{i \in I} A_i\right)$ for a family $\mathcal{A} = (A_i, i \in I) \subseteq \mathcal{D}$. Then there exists $i \in I$ and $\tilde{f} \in \mathrm{Mor}\,(C, A_i)$ such that $f = \nu_i \tilde{f}$, hence $\Psi^C_{\mathcal{A}}(\tilde{f}) = f$.

(4)⇒(5) Since there exists $i \in I$ and $\tilde{f} \in \mathrm{Mor}(C, A_i)$ for which $f = \nu_i \tilde{f}$ we get

$$U(f)(U(C)) = U(\nu_i)U(\tilde{f})(U(C)) \subseteq U(\nu_i)(U(A_i)).$$

(5)⇒(4) It is a direct consequence of (UD3).

The equivalence (2)⇔(3) is a special case of (4)⇔(5).

(3)⇒(5) Let $f \in \mathrm{Mor}(C, \coprod_{i\in I} A_i)$ for a family $(A_i, i \in I)$ of objects of $\mathcal{D}$, put $A = \coprod_{i\in I} A_i$ and assume to a contrary that $U(f)U(C) \nsubseteq U(\nu_i)U(A_i)$ for all $i \in I$. Then by (3) $U(f)U(C) \subseteq \bigcup_{i\neq j} U(A_i^{\nu_i})$ for every $j \in I$, hence by Lemma5.13

$$U(f)U(C) \subseteq \bigcap_{j\in I}\bigcup_{i\neq j} U(A_i^{\nu_i}) = U(\theta_A) \cup \bigcap_{j\in I} U(A) \setminus U(A_j^{\nu_j}) = U(\theta_A),$$

a contradiction.

The implication (4)⇒(2) is clear, since $A \in \mathcal{D}^{\coprod}$ if and only if there exists a family $\mathcal{A} = \{A_i \mid i \in I\}$ of objects of $\mathcal{D}$ satisfying $A = \coprod_{i\in I} A_i$. □

Let us reformulate the Theorem 5.24 for the particular (but important) case of compactness:

**Corollary 5.25.** *The following conditions are equivalent for an object $C$:*

*(1) $C$ is compact,*

*(2) for every pair of objects $A_1$ and $A_2$ and each morphism $f \in Mor(C, A_1 \coprod A_2)$ there exists $i \in \{1, 2\}$ such that $f$ factorizes through the structural coproduct morphism $\nu_i$,*

*(3) for every pair of objects $A_1$ and $A_2$ and a morphism $f \in Mor(C, A_1 \coprod A_2)$, either $U(f)(U(C)) \subseteq U(\nu_1)(U(A_1))$ or $U(f)(U(C)) \subseteq U(\nu_2)(U(A_2))$ holds, where $\nu_i$, $i = 1, 2$, is the structural coproduct morphism.*

The following description of autocompactness presents another consequence of Theorem 5.24:

**Corollary 5.26.** *The following conditions are equivalent for an object $C$:*

*(1) $C$ is autocompact,*

*(2) for each morphism $f \in Mor\left(C, \coprod_{i\in I} C_i\right)$, where $C_i \cong C$ for all $i \in I$, there exists an index $i$ such that $U(f)(U(C)) \subseteq U(\nu_i)(U(C_i))$,*

*(3) for each morphism $f \in Mor\left(C, \coprod_{i\in I} C_i\right)$, where $C_i \cong C$ for all $i \in I$, there exists an index $i$ such that $U(f)(U(C)) \cap U(\nu_j)(U(C_j)) = U(\theta_{\coprod C_i})$ for each $j \neq i$,*

*where $\nu_i$ are the structural morphism of the coproduct $\coprod_{i\in I} C_i$.*

In order to obtain useful characterization of compact objects in a general UD category we say that an object $B$ is an *image* of an object $A$ if there is a morphism $\pi \in \mathrm{Mor}(A, B)$ with $U(\pi)$ surjective. Observe that compact objects in the category $\mathcal{C}$ are precisely objects whose every image is indecomposable:

**Proposition 5.27.** *An object $C$ of the category $\mathcal{C}$ is compact if and only if every image of $C$ is indecomposable.*

*Proof.* ($\Rightarrow$) Let $\pi : C \to \overline{C}$ be a morphism such that $U(\pi)$ is surjective and $((A_1, \iota_1), (A_2, , \iota_2))$ is a decomposition of $\overline{C}$ with $A_1 \ncong \theta \ncong A_2$. Then we have $U(\pi)(U(C)) = U(\overline{C}) \nsubseteq U(\iota_i)(U(A_i))$ for both $i = 1, 2$ by Lemma 5.13, hence $C$ is not compact by Corollary 5.25.

($\Leftarrow$) If $C$ is not compact, then there exists a pair of objects $A_1$ and $A_2$ and a morphism $\pi \in \mathrm{Mor}\,(C, A_1 \coprod A_2)$ such that $U(C^\pi) = U(\pi)(U(C)) \nsubseteq U(\nu_i)(U(A_i))$ for $i \in \{1, 2\}$ by Corollary 5.25, where $\nu_1$, $\nu_2$ are the structural coproduct morphisms. Then there exists a morphism $\tilde{\pi} \in \mathrm{Mor}(C, C^\pi)$ with $U(\tilde{\pi})$ surjective and $\iota$ an inclusion morphism satisfying $\pi = \iota\tilde{\pi}$ by (UD2) and (UD3). Furthermore, $(U(\nu_1)(U(A_1), U(\nu_2)(U(A_2))$ induces a nontrivial decomposition of $C^\pi$ by Lemma 5.14. We have proved that the image $C^\pi$ of $C$ is not indecomposable. $\square$

Let us mention, as an easy consequence of the last claim, that every compact object in $\mathcal{C}$ is indecomposable.

The compactness (smallness) property originally studied in the branch of (left $R$-)modules has been defined in a similar fashion and the notion of self-smallness as a generalization of the property of being finitely generated can be transferred via the notion of an autocompact object to $UD$-categories and specially to those of $S$-acts. (see e.g. [4], [10]).

Using a similar argument as in the direct implication of Proposition 5.27 we get a necessary condition of autocompact objects:

**Lemma 5.28.** *Any autocompact object is indecomposable.*

*Proof.* Assume that the autocompact object $C$ has a decomposition into $\mathcal{B} = ((B_1, \iota_1), (B_2, \iota_2))$. Let $\nu \in \mathrm{Mor}\,(C, C \coprod C)$ be the morphism satisfying $\nu\iota_i = \nu_i\iota_i$ for $i = 1, 2$ which exists by the universal property of the coproduct $B_1 \coprod B_2$, where $\nu_1, \nu_2$ are the structural morphisms of the coproduct $C \coprod C$. Then there exists $i$ such that $U(\nu)U(C) \subseteq U(\nu_i)U(C)$ by Corollary 5.26, w.l.o.g. we may suppose $i = 1$. Then

$$U(\nu_2)U(B_2^{\iota_2}) = U(\nu_2\iota_2)U(B_2) \subseteq U(\nu_1)U(C) \cap U(\nu_2)U(C) = U(\theta_{C \coprod C}),$$

which implies that $B_2^{\iota_2} \cong \theta$ by Lemma 5.12. Thus the decomposition $\mathcal{B}$ is trivial, so $C$ is indecomposable by Proposition 5.15. $\square$

**Proposition 5.29.** *For an autocompact object $C \in \mathcal{C}$ and an endomorphism $f \in Mor(C, C)$, the object $C^f$ is autocompact, too.*

*Proof.* Note that there exist morphisms $\tilde{f} \in \mathrm{Mor}\left(C, C^f\right)$ and $\iota \in \mathrm{Mor}\left(C^f, C\right)$ such that $\iota$ is an inclusion morphism and $f = \iota\tilde{f}$ by (UD2) and (UD3). Suppose $C^f$ is not autocompact. Then by Corollary 5.26 there is a morphism $g : C^f \to \coprod_{i \in I} D_i$ such that $D_i \cong C^f$ and $U(g)(U(C^f)) \nsubseteq U(\nu_i)(U(D_i))$ for any $i \in I$, hence $U(g)(U(C^f)) \cap U(\nu_i)(U(D_i)) \neq U(\theta_{D_i})$ for each $i \in I$, where $\nu_i$, $i \in I$ is the coproduct structural morphism. Let $(\coprod_{i \in I} C_i, (\mu_i)_{i \in I})$ be the coproduct of objects $C_i \cong C$. Then it is easy to see that there exists $\nu \in \mathrm{Mor}(\coprod_{i \in I} D_i, \coprod_{i \in I} C_i)$ such that $U(\nu_i)(U(D_i)) \subseteq U(\mu_i)U(C_i)$ for all $i \in I$ by the universal property of coproduct. Since

$$U(g)(U(C^f)) \cap U(\nu_i)(U(D_i)) \subseteq U(g\tilde{f})(U(C)) \cap U(\mu_i)(U(C_i)),$$

which implies that

$$U(g\tilde{f})(U(C)) \cap U(\mu_i)(U(C_i)) \neq U(\theta_{D_i}) \text{ and so } U(g\tilde{f})(U(C)) \nsubseteq U(\mu_i)(U(C_i))$$

for each i. Thus $C$ is not autocompact by Corollary 5.26. □

### 5.4.2 Compactness in extensive categories

Even though our motivation is based on algebraic structures (cf. [3]), it seems to be useful to clarify the concept of compactness in other classes of categories, as it may offer useful interconnections of viewpoints, so let us focus here for a moment on extensive categories: a notion with a wide range of applicability spanning from being a starting point for construction of distributive categories, which seem to be the correct setting for study of acyclic programs in computer science (cf. [6], [9]), to the theory of elementary topoi.

Recall that an *extensive category* can be characterized as a category $\mathcal{B}$ with (finite) coproducts which has pullbacks along colimit structural morphisms and in every commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{f_z} & Z & \xleftarrow{g_z} & Y \\ {\scriptstyle f_a}\downarrow & & \downarrow{\scriptstyle \zeta} & & \downarrow{\scriptstyle g_B} \\ A & \xrightarrow{\nu_A} & A\coprod B & \xleftarrow{\nu_B} & B \end{array}$$

the squares are pullbacks if and only if the top row is a coproduct diagram in $\mathcal{B}$ (see [9, Proposition 2.2]). An extensive category $\mathcal{B}$ is said to be *infinitary extensive* if the coproduct diagram above is considered also for infinite coproducts.

In the rest of the section, by $\mathcal{A}$ we shall always mean an extensive category. The following presentation of properties of extensive categories follows the exposition given at [30] (cf. aslo [9]).

Firstly, a stronger version of the assertion of Lemma 5.27 holds true for extensive categories:

**Lemma 5.30.** *An object $A \in \mathcal{A}$ is compact if and only if it is indecomposable.*

*Proof.* See [30, Theorem 3.3]. □

While compactness in $\mathcal{A}$ naturally implies preservation of binary (finite) coproducts, the reverse implication need not hold true in general. It, however, holds true, if $\mathcal{A}$ is inifinitary extensive, so in that case we have the following characterization, which is analogous to Corollary 5.25:

**Proposition 5.31.** *An object $A$ of infinitary extensive category $\mathcal{A}$ is compact if and only if the corresponding $Mor_{\mathcal{A}}(A, -)$-functor preserves binary coproducts.*

*Proof.* See [30, Theorem 3.1]. □

**Corollary 5.32.** *An object of an infinitary extensive category $\mathcal{A}$ is compact if and only if it is indecomposable.*

**Proposition 5.33.** *The category $S - \overline{\mathrm{Act}}$ is infinitary extensive.*

*Proof.* Let us for $i \in I$ consider the following diagram in $S - \overline{\mathrm{Act}}$ with pullback squares

$$\begin{array}{ccc} A_i & \xrightarrow{\alpha_i} & A \\ {\scriptstyle f_i}\downarrow & & \downarrow{\scriptstyle f} \\ B_i & \xrightarrow{\nu_i} & \coprod_{i\in I} B_i \end{array}$$

and let us prove $A$ is a coproduct of $A_i$'s. Since we can by [18, 2.2, 2.5] consider $A_i = \{(b,a) \in B_i \times A \,|\, \nu_i(b) = f(a)\}$ and $\coprod B_i = \dot{\bigcup}_{i\in I}\nu_i(B_i)$, then for each $a$ there exist $i \in I, b \in B_i$ with $\nu_i(b) = f(a)$, so $\alpha_i((b,a)) = a$. In consequence $A = \bigcup \alpha_i(A_i)$. Assume $a \in \alpha_{i_0}(x_{i_0}) \cap \alpha_{j_0}(x_{j_0})$; then $f(a) = f\alpha_{i_0}(x_{i_0}) \in \nu_{i_0}(B_{i_0})$, hence $f(a) \in \nu_{i_0}(B_{i_0}) \cap \nu_{j_0}(B_{j_0}) = \emptyset$, so $\alpha_i(A_i) \cap \alpha_j(A_j) = \emptyset$ for $j \neq i$ and $A \simeq \coprod_{i\in I} A_i$. □

**Example 5.34.** (1) Consider the monoid of integers $\mathcal{Z} = (\mathbb{Z}, \cdot, 1)$. Then the $\mathcal{Z}$-act $A = 2\mathbb{Z} \cup 3\mathbb{Z}$ shows that the category $S - \mathrm{Act}_0$ is not extensive (and so it is not infinitary extensive). See the following commutative diagram with pullback squares:

$$\begin{array}{ccccc} A_3 & \xrightarrow{\alpha_3} & 2\mathbb{Z} \cup 3\mathbb{Z} & \xleftarrow{\alpha_2} & A_2 \\ {\scriptstyle \tilde{\pi}_{6\mathbb{Z}}}\downarrow & & \downarrow{\scriptstyle \pi_{6\mathbb{Z}}} & & \downarrow{\scriptstyle \overline{\pi}_{6\mathbb{Z}}} \\ \mathbb{Z}_3 & \xrightarrow{\nu_3} & \mathbb{Z}_3 \coprod \mathbb{Z}_2 & \xleftarrow{\nu_2} & \mathbb{Z}_2, \end{array}$$

where $A_2 = \{(0,a) \,|\, a \in 6\mathbb{Z}\} \cup \{(1,a) \,|\, a \in 3 + 6\mathbb{Z}\}$, $A_3 = \{(0,a) \,|\, a \in 6\mathbb{Z}\} \cup \{(1,a) \,|\, a \in 2 + 6\mathbb{Z}\} \cup \{(2,a) \,|\, a \in 4 + 6\mathbb{Z}\}$, $\alpha_i$ denotes projections on the second coordinate, while $\tilde{\pi}, \overline{\pi}$ on the first one.

(2) The category *Top* of topological spaces with continuous maps is extensive, but it is not UD, since it does not satisfy the condition (UD1): the inverse of a continuous bijection need not be continuous, hence *Top* is not balanced. Also note that compactness in *Top* actually means connectedness (the term *connected object* is used in this context rather than *compact object*). A general topological space is not a disjoint union (coproduct in *Top*) of its connected components (this property defines *locally connected topological spaces*), which pushes this extensive category yet further from being a UD-category.

## 5.5 Categories of $S$-acts

Let $\mathcal{S} = (S, \cdot, 1)$ be a monoid (or a monoid with zero 0) through the whole section. Recall that for $\mathcal{S}$ both categories $S - \mathrm{Act}_0$ and $S - \overline{\mathrm{Act}}$ of $S$-acts are UD-categories by Example 5.5. We will use basic properties of these categories summarized in the axiomatics (UD1)–(UD6) freely in the sequel. For standard terminology concerning the theory of acts we refer to the monograph [18].

### 5.5.1 Compact acts

The following consequence of Corollary 5.25 shows that the reverse implication of [18, Lemma I.5.36] holds true.

**Lemma 5.35.** *Compact objects in the category $S - \overline{\mathrm{Act}}$ are precisely indecomposable objects.*

*Proof.* The category $S - \overline{\mathrm{Act}}$ is infinitary extensive by Proposition 5.33, so the assertion follows from Lemma 5.30. Alternatively, the assertion follows from [18, Lemma I.5.36] and Corollary 5.25. $\square$

Recall that a left $S$-act $A$ is called *cyclic* if there exists $a \in A$ for which $Sa = \{sa \mid s \in S\} = A$, and $A$ is called *locally cyclic* if for any pair $a, b \in A$ there exists $c \in A$ such that $a, b \in Sc$. Since cyclic acts are locally cyclic and locally cyclic acts are indecomposable, we obtain an immediate consequence of Lemma 5.35:

**Corollary 5.36.** *Every locally cyclic left act is compact in the category* $S - \overline{\mathrm{Act}}$.

Furthermore, we prove a sufficient condition of compactness for both considered categories of acts.

**Proposition 5.37.** *Every cyclic left act is compact in both categories* $S - \overline{\mathrm{Act}}$ *and* $S - \mathrm{Act}_0$.

*Proof.* By Corollary 5.36 we only need to prove the claim for the category $S - \mathrm{Act}_0$. Since any factor of a cyclic act is cyclic, and so indecomposable, the Proposition 5.27 gives us the result in the category $S - \mathrm{Act}_0$. $\square$

The corresponding variant of Lemma 5.35 as the criterion of compactness in the category $S-\mathrm{Act}_0$ shall deal with all factors of an act, namely, compact objects in the category $S-\mathrm{Act}_0$ are precisely objects whose every image is indecomposable by Proposition 5.27.

The following example shows that in the case of the category $S - \mathrm{Act}_0$ the implication in Proposition 5.37 cannot be inverted in general:

**Example 5.38.** Let $\mathcal{Z} = (\mathbb{Z}, \cdot, 1)$ be a monoid with zero.

(1) Consider again $\mathcal{Z}$-act $A = 2\mathbb{Z} \cup 3\mathbb{Z}$ from Example 5.34. Then $A$ is an indecomposable act which is not compact in the category $S - \mathrm{Act}_0$. Indeed, if we consider the morphism $f_6 : A \to \mathbb{Z}_6$ given by $f_6(a) = a \mod 6$, then the image $f_6(A) = \{0, 2, 4\} \cup \{0, 3\}$ decomposes, hence it is not compact by Proposition 5.27.

(2) Every abelian group is compact in the category $\mathcal{Z} - \overline{\mathrm{Act}}$ since every $\mathcal{Z}$-subact contains 0. More generally, for a monoid $S$ with zero, any $A \in S - \mathrm{Act}_0$ can be recognized as an object of $S - \overline{\mathrm{Act}}$ and it becomes indecomposable in $\mathcal{Z} - \overline{\mathrm{Act}}$ , hence compact by Lemma 5.35.

In compliance with [18, Definition 4.20] recall that for a subact $B$ of an act $A$ the *Rees congruence* $\rho_B$ on $A$ is defined by setting $a_1 \rho a_2$ if $a_1 = a_2$ or $a_1, a_2 \in B$. The corresponding factor act $A/B$ is called *Rees factor of $A$ by $B$* then.

**Lemma 5.39.** *Let $A \in S - \mathrm{Act}_0$ and $A_1$ and $A_2$ be its proper subacts. If $A = A_1 \cup A_2$ and $A_i \setminus (A_1 \cap A_2) \neq \emptyset$ for both $i = 1, 2$, then $A$ is not compact in* $S - \mathrm{Act}_0$.

*Proof.* Consider the projection $\pi$ of $A$ onto the Rees factor $A/(A_1 \cap A_2)$, which is decomposable into $\pi(A_1) \coprod \pi(A_2)$. Now use Corollary 5.25. $\square$

Note that a subact $B$ of an act $A$ can be viewed as an subobject $(B, \iota)$ of $A$ with the inclusion morphism $\iota$ and recall that a subact $B$ of a left $S$-act $A$ (in $S - \overline{\mathrm{Act}}$ or $S - \mathrm{Act}_0$) is called *superfluous* if $B \cup C \neq A$ for any proper subact $C$ of $A$ (see [19, Definition 2.1]). An act is called *hollow* if each of its proper subacts is superfluous (see [19, Definition 3.1]). Note that the situation of Lemma 5.39 is precisely that of non-hollow acts.

**Proposition 5.40.** *An $S$-act $A$ is compact in the category $S - \mathrm{Act}_0$ if and only if it is hollow.*

*Proof.* Suppose $A$ is hollow and it is not compact, i.e., there is a decomposable factor $\pi(A) = A_1 \coprod A_2$ by Proposition 5.27. Then the preimages $\pi^{-1}(A_1)$ and $\pi^{-1}(A_2)$ form subacts of the act $A$ such that $A = \pi^{-1}(A_1) \cup \pi^{-1}(A_2)A$, but neither of $\pi^{-1}(A_i)$ equals $A$. Since the decomposition is proper, we get a contradiction.

On the other hand, if $A$ is not hollow, use the construction of Lemma 5.39. □

### 5.5.2 Steady monoids

In accordance with the definition of steady rings (cf. [8, 12, 28]) we say that a monoid (resp. monoid with zero element) $S$ is *left steady* (resp. *left* 0*-steady*) provided every compact left act in the category $S - \overline{\mathrm{Act}}$ (resp. $S - \mathrm{Act}_0$) is necessarily cyclic. Note that every cyclic act is compact by Proposition 5.37.

**Example 5.41.** (1) If $S$ is a group, then it is easy to see that indecomposable $S$-acts are cyclic. Hence compact $S$-acts are precisely cyclic ones by [18, Theorem I.5.10] (cf. Theorem 5.17), thus groups are (left) steady monoids.

(2) The Prüfer group $\mathbb{Z}_{p^\infty}$ is a compact act over the monoid $(\mathbb{N}, +, 0)$. Clearly, it is not a cyclic $\mathbb{N}$-act, as it is not a cyclic $\mathbb{Z}$-act. Hence $(\mathbb{N}, +, 0)$ is not steady.

The following assertion presents an analogy of the description of compact projective objects in categories of modules.

**Proposition 5.42.** *Let $\mathcal{C}$ be either $S - \overline{\mathrm{Act}}$ or $S - \mathrm{Act}_0$. Then a projective left act is compact in $\mathcal{C}$ if and only if it is cyclic.*

*Proof.* For the direct implication note that, by Theorem 5.20 any projective act has a decomposition into indecomposable projective subacts, since both $S - \overline{\mathrm{Act}}$ and $S - \mathrm{Act}_0$ are UD-categories. As it is compact, it is indecomposable by Proposition 5.27. Now the result follows from Lemma 5.21 since $S$ generates both of the categories $S - \overline{\mathrm{Act}}$ and $S - \mathrm{Act}_0$.

The reverse implication is a consequence of Proposition 5.37. □

A monoid $S$ is called *left perfect* (*left 0-perfect*) if each $A \in S - \overline{\mathrm{Act}}$ ($A \in S - \mathrm{Act}_0$) has a projective cover, i.e., there exists (up to isomorphism unique) a projective $S$-act $P$ and an epimorphism $f : P \to A$ such that for any proper subact $P' \subset P$ the restriction $f|_{P'} : P' \to A$ is not an epimorphism (cf. [15, 17]).

Analogously to the case of perfect rings, which are known to be steady, we prove that 0-perfect monoids are 0-steady.

**Proposition 5.43.** *Let $S$ be a monoid with zero. If $S$ is left* 0*-perfect, then compact objects of $S - \mathrm{Act}_0$ are precisely cyclic acts. Hence $S$ is left* 0*-steady.*

*Proof.* Let $A$ be a compact $S$-act and $\pi \in \mathrm{Mor}(P, A)$ be a projective cover of $A$. Assume that $P$ is not irreducible with a nontrivial decomposition $(P_0, P_1)$. Then neither $\pi(P_0)$ nor $\pi(P_1)$ is not equal to $A$ and $B = \pi(P_0) \cap \pi(P_1)$ is a subact of $A$. Then $(\pi(P_0)/B, \pi(P_1)/B)$ forms a decomposition of Rees factor $A/B$. Note that it is non-trivial, otherwise $\pi(P_0) \subseteq \pi(P_1)$ or $\pi(P_1) \subseteq \pi(P_0)$ which contradicts to the fact that $\pi(P_0) \neq A \neq \pi(P_1)$. Since every factor of $A$ is indecomposable by Lemma 5.35, we obtain a contradiction. □

### 5.5.3 Autocompact acts

Let us formulate a direct consequence of Lemma 5.28 and Proposition 5.29:

**Lemma 5.44.** *Let $C$ be an a autocompact object in either $S-\mathrm{Act}_0$ or $S-\overline{\mathrm{Act}}$ and let $\varphi$ be an endomorphism of $C$. Then $\varphi(C)$ is autocompact and indecomposable, in particular, $C$ is indecomposable.*

Now we can formulate a criterion of autocompactness in $S - \overline{\mathrm{Act}}$ (cf. [22, Lemma 4.1]):

**Theorem 5.45.** *The following conditions are equivalent for an act $C \in S - \overline{\mathrm{Act}}$:*

*(1) $C$ is autocompact,*

*(2) $C$ is compact,*

*(3) $C$ is indecomposable.*

*Proof.* The implication (2)⇒(1) is clear, the implication (1)⇒(3) follows from Lemma 5.44 and the equivalence (2)⇔(3) is proved in Lemma 5.35. □

**Example 5.46.** Consider the monoid $\mathcal{Z} = (\mathbb{Z}, \cdot, 1)$ and the $\mathcal{Z}$-act $A = 2\mathbb{Z} \cup 3\mathbb{Z}$ from Examples 5.38 and 5.34. Then $A$ is autocompact in $S - \mathrm{Act}_0$, since for any morphism $A \to \coprod_{i\in I} A_i$ with $A_i \cong A$, the component in which the image lies is determined by the image of the element 6.

The previous example shows that within the category $S - \mathrm{Act}_0$ the class of autocompact acts is in general strictly larger than the class of compact acts; whereas the following example will show that the class of autocompact acts is in general strictly smaller than that of indecomposable objects, even for left perfect monoids.

**Example 5.47.** Consider the commutative monoid $\mathcal{S} = (\{0, 1, s, s^2\}, \cdot, 1)$ (which could be embedded into the multiplicative monoid of the factor ring $\mathbb{Z}[s]/(s^3)$) with the following multiplication table:

| $\cdot$ | $0$ | $1$ | $s$ | $s^2$ |
|---|---|---|---|---|
| $0$ | $0$ | $0$ | $0$ | $0$ |
| $1$ | $0$ | $1$ | $s$ | $s^2$ |
| $s$ | $0$ | $s$ | $s^2$ | $0$ |
| $s^2$ | $0$ | $s^2$ | $0$ | $0$ |

.

Then consider the $\mathcal{S}$-act $A = \{x, y, z, t, \theta\}$ with the action of $\mathcal{S}$ given as follows:

$$0 \cdot a = \theta, \quad 1 \cdot a = a \text{ for any } a \in A$$

$$s \cdot x = s \cdot y = z, \quad s \cdot z = t, \quad s \cdot t = \theta.$$

Then $A$ is indecomposable, while the Rees factor $A/\langle z\rangle$ decomposes into two isomorphic components (so $A$ is not compact), each of which can be mapped onto $\langle t\rangle \leq A$, hence $A$ is not autocompact.

One can furthermore prove that $\mathcal{S}$ is left perfect using [15, Theorem 1.1].

For $S$-acts $A_1, A_2 \in S - \mathrm{Act}_0$ denote by $\pi_i : A_1 \coprod A_2 \to A_i$, $i = 1, 2$ the canonical projections and note that any canonical projection is a correctly defined morphism in the category $S - \mathrm{Act}_0$.

**Lemma 5.48.** *Let $C, C_1, C_2 \in S - \mathrm{Act}_0$ and $C \cong C_1 \cong C_2$. Then $C$ is autocompact if and only if for each morphism $f : C \to C_1 \coprod C_2$ there exists $i$ such that $\pi_i f(C) = \theta$.*

*Proof.* The direct implication follows immediately from Corollary 5.26.

If $C$ is not autocompact, then by Corollary 5.26 there exists a morphism $f : C \to \coprod_{i\in I} C_i$ where $C_i \cong C$ and there exist $i \neq j$ such that such that $f(C) \nsubseteq \nu_i(C_i)$ and $f(C) \nsubseteq \nu_j(C_j)$. Thus it is enough to compose $f$ with the canonical projection to $C_i \coprod C_j$. □

For a pair $B_1, B_2$ of subacts of a left $S$-act $A$ with inclusions $\iota_i : B_i \to A$ denote by $\rho_{B_1B_2} : B_1 \coprod B_2 \to A$ the unique morphism satisfying $\rho_{B_1B_2}\nu_i = \iota_i$ for $i = 1, 2$, where $\nu_i$ denotes the coproduct structural morphism. We finish the paper by a characterization of non-autocompact $S$-acts in the category $S - \mathrm{Act}_0$, which can by provided by narrowing the class of non-hollow (i. e. non-compact) acts by

**Proposition 5.49.** *The following conditions are equivalent for a triple of isomorphic acts $A, A_1, A_2$ in the category $S - \mathrm{Act}_0$:*

*(1) $A$ is not autocompact in $S - \mathrm{Act}_0$,*

*(2) there exists a pair $B_1, B_2$ of proper subacts of $A$ satisfying $A = B_1 \cup B_2$ and there exists a morphism $f : B_1 \coprod B_2 \to A_1 \coprod A_2$ such that $\pi_i f(B_1 \coprod B_2) \neq \theta_{A_i}$ for $i = 1, 2$ and $ker \rho_{B_1B_2} \subseteq ker\, f$.*

*Proof.* Sufficiency follows from the Homomorphism Theorem [18, Theorem 4.21] which ensures the existence of a morphism $f' : A \to A_1 \coprod A_2$, which turns to be the witnessing morphism for non-autocompactness thanks to the property $\pi_i f(B_1 \coprod B_2) \neq \theta_{A_i}$ for both $i = 1, 2$.

$$\begin{tikzcd}
B_1 \coprod B_2 \arrow[r, "f"] \arrow[d, "\rho_{B_1B_2}"'] & A_1 \coprod A_2 \\
A \arrow[r, equal] & A \arrow[u, "f'"']
\end{tikzcd}$$

Let $g : A \to A_1 \coprod A_2$ be the morphism witnessing non-autocompactness by Lemma 5.48, hence $\pi_i g(A) \neq \theta_{A_i}$ for both $i = 1, 2$. Let $\nu_i : A_i \to A_1 \coprod A_2$ denote the coproduct structural morphism and set $B_i = g^{-1}(g(A) \cap \nu_i(A_i))$; then clearly $A = B_1 \cup B_2$. Set now $f = g\rho_{B_1B_2}$. □

# Bibliography for chapter 5


[1] J. Adámek, H. Herrlich, G.E. Strecker, *Abstract and Concrete Categories*, John Wiley and Sons, Inc, 1990, http://katmat.math.uni-bremen.de/acc/acc.pdf.

[2] J. Adámek, J. Rosický, Locally Presentable and Accessible Categories, Cambridge University Press in the London Mathematical Society Lecture Note Series, 189, 1994, Cambridge.

[3] J. Adámek, J. Rosický, E.M. Vitale, Algebraic Theories: A Categorical Introduction to General Algebra, Cambridge University Press, Cambridge, 2010.

[4] D.M. Arnold, C.E. Murley, Abelian groups, A, such that $\mathrm{Hom}(A, -)$ preserves direct sums of copies of A, Pacific J. Math., 56 (1975), 7–20

[5] H. Bass, *Algebraic K-theory*, New York 1968, Benjamin.

[6] J. R. B. Cockett, *Introduction to distributive categories*. Mathematical Structures in Computer Science, 3(3), 1993, 277–307.

[7] R.Colpi and C.Menini, On the structure of *-modules, J. Algebra 158 (1993), 400–419.

[8] R. Colpi and J. Trlifaj, *Classes of generalized $*$-modules*, Comm. Algebra **22**, 1994, 3985–3995.

[9] A. Carboni, S. Lack, R.F.C.Walters, *Introduction to extensive and distributive categories*, Journal of Pure and Applied Algebra, 1993, **84**, 145–158

[10] J. Dvořák, On products of self-small abelian groups, *Stud. Univ. Babeş-Bolyai Math.* 60 (2015), no. 1, 13–17.

[11] J. Dvořák, J. Žemlička, *Autocompact objects of Ab5 categories*, submitted, 2021, arXiv:2102.04818.

[12] P.C. Eklof, K.R. Goodearl and J. Trlifaj, Dually slender modules and steady rings, Forum Math., 1997, **9**, 61–74.

[13] R. El Bashir, T. Kepka, P. Němec, Modules commuting (via Hom) with some colimits: *Czechoslovak Math. J.* **53** (2003), 891–905.

[14] Gómez Pardo, J. L. , Militaru, G., Năstăsescu, C., When is $\mathrm{HOM}(M, -)$ equal to $\mathrm{Hom}(M, -)$ in the category $R - gr$?, Comm. Algebra, **22** (1994), 3171–3181.

[15] J. Isbell, Perfect monoids. Semigroup Forum (1971) 2, 95–118.

[16] P.Kálnai, J. Žemlička, Compactness in abelian categories, J. Algebra, 534 (2019), 273–288.

[17] M. Kilp, Perfect monoids revisited. Semigroup Forum (1996) 53, 225–229.

[18] M. Kilp, U. Knauer, A.V. Mikhalev, Monoids, acts and categories, de Gruyter, Expositions in Mathematics 29, Walter de Gruyter, Berlin 2000.

[19] R. Khosravi, M. Roueentan, Co-uniform and hollow $S$-acts over monoids, submitted, arXiv: 1908.04559v1

[20] H. Krause, On Neeman's well generated triangulated categories. Doc. Math. 6 (2001), 121-126,

[21] M. Makkai, R. Paré, Accessible categories: The foundations of categorical model theory Contemporary Mathematics 104. American Mathematical Society, 1989, Rhode Island.

[22] C.G. Modoi, Localizations, colocalizations and non additive *-objects, *Semigroup Forum* 81(2010), No. 3, 510-523.

[23] Neeman, A. Triangulated categories, volume 148 of Annals of Mathematics Studies. Princeton University Press, Princeton, NY, 2001.

[24] R. Rentschler, Sur les modules $M$ tels que $\mathrm{Hom}(M, -)$ commute avec les sommes directes, C.R. Acad. Sci. Paris, **268**, 1969, 930–933.

[25] M. Roueentan, M. Sedaghatjoo, On uniform acts over semigroups, Semigroup Forum (2018) 97, 229–243.

[26] M. Sedaghatjoo, A. Khaksari, Monoids over which products of indecomposable acts are indecomposable, Hacet. J. Math. Stat. 46 (2017), No. 2, 229–237.

[27] J. Trlifaj, Strong incompactness for some nonperfect rings, *Proc. Amer. Math. Soc.* 123 (1995), 21–25.

[28] J. Trlifaj, Steady rings may contain large sets of orthogonal idempotents, in *Abelian groups and objects (Padova, 1994)*, Math. Appl., 343, Kluwer Acad. Publ., Dordrecht, 1995, 467–473.

[29] J. Žemlička and J. Trlifaj, Steady ideals and rings, *Rend. Sem. Mat. Univ. Padova*, **98** (1997), 161–172.

[30] Connected object, on The nLab, downloaded on March 5, 2021, URL: https://ncatlab.org/nlab/show/connected+object.

# Chapter 6

# Perfect monoids with zero and categories of $S$-acts

The usefulness of the notion of projective cover within the context of module theory has been confirmed in countless occasions since the publication of the founding works of Bass [2], who coined the term, and Eilenberg [4], who effectively considered the notion for the first time. Together with the idea of projective cover, the closely related notion of a perfect ring, for which projective covers exist in the corresponding module category, appears. The definition of both terms uses a purely categorial language, yet structural and homological characterizations can be given, e.g.,

**Theorem 6.1.** *[2] The following conditions are equivalent for a ring $R$*

1. *$R$ is left perfect*
2. *$R$ satisfies d.c.c. on principal right ideals*
3. *the class of projective $R$-modules coincides with the class of flat $R$-modules.*

The corresponding notion within the branch of monoids and acts turned out to be similarly fruitful with applications to category theory and topological monoids (see [6]). Note that the Theorem 6.1 has its counterpart stated for monoids:

**Theorem 6.2.** *[5, 6, 8] The following conditions are equivalent for a monoid $S$*

1. *$S$ is left perfect*
2. *$R$ satisfies the minimum condition on principal right ideals and each left $S$-act satisfies the a.c.c for cyclic subacts*
3. *the class of projective $S$-acts coincides with the class of strongly flat left $S$-acts.*

The previous result as well as other results have been formulated and considered within the context of the category $S - \overline{\mathrm{Act}}$ (see below), but for a monoid with zero, the monograph [9] introduces another natural category, $S - \mathrm{Act}_0$, which turns out to possess notably different categorial properties regarding e.g. its extensivity or compactness of objects (cf. [3]), hence the question of relationship of these two categories from the viewpoint of perfectness arises naturally and the aim of the present paper is an investigation on this topic.

The question of perfectness appears to be related to the problem over which monoids $S$ (or rings) there exists a non-cyclic act such that the corresponding covariant Hom-functor from a category of $S$-acts (or $S$-modules) commutes with coproducts (such monoid is then called non-steady). It is known that non-steady monoids are necessarily non-perfect in the category $S - \overline{\text{Act}}$ (as well as in the case of modules).

The main tool of the paper is the functor $\mathbf{F} : S - \overline{\text{Act}} \to S - \text{Act}_0$ gluing all zero elements to one using Rees factor. It allows translating the properties of $S - \overline{\text{Act}}$ to the category $S - \text{Act}_0$. Namely, Theorem 6.18 shows that the left perfectness of categories $S - \overline{\text{Act}}$ and $S - \text{Act}_0$ coincide and a monoid with zero is left perfect if and only if it is left 0-perfect and it is left 0-steady if and only if it satisfies the ascending chain condition on cyclic subacts by Theorem 6.22.

## 6.1 Preliminaries

Before we begin the exposition, let us recall some necessary terminology and notations.

Let $\mathcal{S} = (S, \cdot, 1)$ be a monoid and $A$ a nonempty set. If there exists a mapping $- \cdot - : S \times A \to A$ satisfying the following two conditions: $1 \cdot a = a$ and $(s_1 \cdot s_2) \cdot a = s_1 \cdot (s_2 \cdot a)$ then $A$ is said to be a left $S$-act and it is denoted ${}_SA$. A mapping $f : {}_SA \to {}_SB$ is a homomorphism of $S$-acts (an $S$-homomorphism) provided $f(sa) = sf(a)$ holds for all pairs $s \in S, a \in A$. In compliance with [9, Example I.6.5.] we denote by $S - \text{Act}$ the category of all left $S$-acts with homomorphisms of $S$-acts and $S - \overline{\text{Act}}$ the category $S - \text{Act}$ enriched by an initial object ${}_S\emptyset$. Let the monoid $\mathcal{S}$ contain a (necessarily unique) zero element 0, which satisfies $0 \cdot s = s \cdot 0 = 0$ for all $s \in S$. Then the category of all left $S$-acts $A$ with a unique zero element $\theta_A = 0A$ and homomorphisms of $S$-acts compatible with zero as morphisms will be denoted $S - \text{Act}_0$. Observe that $\theta := \{0\}$ is the initial object of the category $S - \text{Act}_0$ (but not of the category $S - \overline{\text{Act}}$).

Recall that both of the categories $S - \overline{\text{Act}}$ and $S - \text{Act}_0$ are complete and cocomplete [9, Remarks II.2.11, Remark II.2.22]. In particular, the coproduct of a system of objects $(A_i, i \in I)$ is

(i) $\coprod_{i\in I} A_i = \dot{\bigcup} A_i$ in $S$-$\overline{\text{Act}}$ by [9, Proposition II.1.8] and

(ii) $\coprod_{i\in I} A_i = \{(a_i) \in \prod_{i\in I} A_i | \ \exists j : a_i = 0 \forall i \neq j\}$ in $S$-$\text{Act}_0$ by [9, Remark II.1.16].

Recall that for a subact $B$ of an act $A$ the *Rees congruence* $\rho_B$ on $A$ is defined by setting $a_1 \rho a_2$ if $a_1 = a_2$ or $a_1, a_2 \in B$ and the corresponding factor act is denoted by $A/B$ (cf. [9, Definition 4.20] )

## 6.2 The functor F between categories of $S$-acts

Throughout the paper, all monoids are considered to **contain the zero element** 0, in particular, $S$ denotes a monoid $(S, \cdot, 1)$ with the zero element 1 and we suppose that $0 \neq 1$.

Let $A$ be a left $S$-act. Since $Sz = 0z = z$ for each $z \in 0A = \{0a \mid\in A\}$ we say that $0A$ is a set of zero elements. Observe that $0A$ can contain more than one element in general and notice that while a morphism $\alpha : C \to D$ in the category $S - \mathrm{Act}_0$ is required to preserve the unique zero, i. e., $\alpha(\theta_C) = \theta_D$, the category $S - \overline{\mathrm{Act}}$ is less restrictive: for a morphism $\beta : A \to B$ the image of a zero element of $A$ from the set $0A$ is some zero element of $B$, in other words $\beta(0A) \subseteq 0B$. This leads to the following idea:

Define the functor $\mathbf{F}$ from the category $S - \mathrm{Act}$ to the category $S - \mathrm{Act}_0$ as follows:

- for an object $A \in S - \mathrm{Act}$, let $\mathbf{F}(A) = A/0A$, i. e. the $S$-act obtained by gluing all zeroes of $A$ together or, in other words the image of the natural projection onto the Rees factor $\pi_{0A} : A \twoheadrightarrow A/0A$
- for a morphism $\alpha : A \to B$ define $\mathbf{F}(\alpha)$ in the natural way so that the following square commutes:

$$\begin{tikzcd}
A \arrow[r, two heads, "\pi_{0A}"] \arrow[d, "\alpha"] & \mathbf{F}(A) \arrow[d, "\mathbf{F}(\alpha)"] \\
B \arrow[r, two heads, "\pi_{0B}"] & \mathbf{F}(B)
\end{tikzcd}$$

The morphism $\mathbf{F}(\alpha)$ can be obtained from the Homomorphism Theorem [9, Theorem 4.21], since $\ker \pi_{0A} \subseteq \ker \pi_{0B}\alpha$. The explicit formula for $\mathbf{F}(\alpha)$ is then:

$$\begin{cases} \mathbf{F}(\alpha)([a]) = [\alpha(a)] \text{ for } a \notin 0A \\ \mathbf{F}(\alpha)(\theta_{\mathbf{F}(A)}) = \theta_{\mathbf{F}(B)} \end{cases} .$$

Now we formulate the key categorial observation on $\mathbf{F}$; for the definition of a reflective subcategory we refer, e.g. to [1, Definition 4.16].

**Proposition 6.3.** *The category* $S - \mathrm{Act}_0$ *is a reflective subcategory of the category* $S - \mathrm{Act}$ *via the reflector* $\mathbf{F}$.

*Proof.* Firstly, we show that $S - \mathrm{Act}_0$ is a full subcategory of $S - \mathrm{Act}$, i.e. we have $\mathrm{Mor}_{S-\mathrm{Act}}(A, B) = \mathrm{Mor}_{S-\mathrm{Act}_0}(A, B)$. For $A, B \in S - \mathrm{Act}_0$ consider an $f \in \mathrm{Mor}_{S-\mathrm{Act}}(A, B)$. Both $A, B$ being objects of $S - \mathrm{Act}_0$ have their respective unique zeros $\theta_A, \theta_B$. Let $f(\theta_A) = b \in B$. Then

$$f(\theta_A) = f(0\theta_A) = 0f(\theta_A) = \theta_B = 0\theta_B$$

so $f$ preserves zero and as a consequence $f \in \mathrm{Mor}_{S-\mathrm{Act}_0}(A, B)$. The reverse inclusion of morphism sets is clear.

Let now be $A \in S - \mathrm{Act}$, $X \in S - \mathrm{Act}_0$ and $f : A \to X$ a morphism in $S - \mathrm{Act}$. We claim that $\mathbf{F}(f)$ is the unique morphism in $\mathrm{Mor}_{S-\mathrm{Act}_0}(\mathbf{F}(A), X)$ that makes the following square commute:

$$\begin{tikzcd}
A \arrow[r, two heads, "\pi_{0A}"] \arrow[d, "f"'] & \mathbf{F}(A) \arrow[d, "\mathbf{F}(f)"] \\
X \arrow[r, equal] & \mathbf{F}(X) = X
\end{tikzcd}$$

Indeed, if $\beta : \mathbf{F}(A) \to X$ satisfies $\beta\pi_{0A} = \mathbf{F}(f)\pi_{0A}$, then $\beta = \mathbf{F}(f)$, since $\pi_{0A}$ is surjective. $\square$

**Example 6.4.** (1) Let $S$ be an arbitrary non-trivial monoid and consider $A_1 = \{\theta\}$ and $A_1 = A_1 \coprod A_1 = \{\theta_1, \theta_2\}$ are two acts in $S-\mathrm{Act}$. Then $\mathbf{F}(A_1) = \mathbf{F}(A_2)$.

Note that the functor $\mathbf{F}$ is not faithful since $|\mathrm{Hom}(A_1, A_2)| = 2$, while by applying $\mathbf{F}$, we get $|\mathrm{Hom}(\mathbf{F}(A_1), \mathbf{F}(A_2))| = |\mathrm{Hom}(A_1, A_1)| = 1$.

(2) The functor $\mathbf{F}$ is not left-exact (i.e. it does not preserve finite limits): consider the monoid $S = (\mathbb{Z}_2, \cdot, 1)$ and the $S$-act $A = \{\theta_A, a\}$ with Cayley graph (omitting unit loops)

$$a \xrightarrow{0} \theta_A.$$

Put $B = A \dot\cup \theta_S$, an object of $S - \mathrm{Act}$ with two zeros. Then $\mathbf{F}(B \prod B)$ has 6 elements, while $\mathbf{F}(B) \prod \mathbf{F}(B) = A \prod A$ is a 4-element act.

The previous examples show that $\mathbf{F}(A)$ cannot be considered in a reasonable way an analogy of localization or completion of $A$.

**Lemma 6.5.** *The functor* $\mathbf{F}$ *preserves coproducts.*

*Proof.* Since $\mathbf{F}$ is a reflector, hence a left adjoint (of the embedding functor $S - \mathrm{Act}_0 \hookrightarrow S - \mathrm{Act}$), it preserves colimits by the dual assertion of [10, Theorem 1, page 114]. □

Recall that an act $P$ is projective, if for any pair of acts $A, B$, a homomorphism $\alpha : P \to B$ and an epimorphism $\pi : A \to B$, there exists a morphism $\overline{\alpha} : P \to A$ in $\mathcal{C}$ such that $\alpha = \pi\overline{\alpha}$.

**Lemma 6.6.** *Let* $P \in S - \mathrm{Act}$ *be projective. Then* $\mathbf{F}(P)$ *is projective in* $S - \mathrm{Act}_0$.

*Proof.* Let the projective situation in $S - \mathrm{Act}_0$ be given:

$$\begin{tikzcd}
\mathbf{F}(P) \arrow[dr, "f"] & \\
& A \\
B \arrow[ur, two heads, "\pi"'] &
\end{tikzcd}$$

Since $S - \mathrm{Act}_0$ is a subcategory of $S - \mathrm{Act}$ and we have $\pi_{0P} : P \twoheadrightarrow \mathbf{F}(P)$, the projectivity of $P$ provides a morphism $\alpha : P \to B$ in $S - \mathrm{Act}$ such that $\pi\alpha = f\pi_{0P}$; furthermore, $\ker \pi_{0P} \subseteq \ker \alpha$, hence $\alpha$ factorizes through $\pi_{0P}$ via some $\alpha' : \mathbf{F}(P) \to B$:

$$\begin{tikzcd}
P \arrow[r, "\pi_{0P}"] \arrow[d, squiggly, "\alpha"'] & \mathbf{F}(P) \arrow[dl, squiggly, "\alpha'"] \arrow[dr, "f"] & \\
B \arrow[r, "\pi_{0B}=id"'] & B \arrow[r, two heads, "\pi"'] & A
\end{tikzcd}$$

In total: $\pi\alpha'\pi_{0P} = \pi\alpha = f\pi_{0P}$ and since $\pi_{0P}$ is an epimorphism, we get $\pi\alpha' = f$. □

Let as observe that the description of projectivity in $S - \mathrm{Act}_0$ works similarly as in $S - \mathrm{Act}$ [9, Theorem III.17.8].

**Lemma 6.7.** *For an indecomposable projective act* $A$ *in* $S - \mathrm{Act}_0$ *there exists an idempotent* $e \in S$ *such that* $A \cong Se$.

*Proof.* We follow the arguments of the proof of [9, Proposition III.17.7].

By [3, Lemma 4.4] there exist a retraction $p : S \to A$ and a coretraction $i : A \to S$ such that $pi = \mathrm{id}_A$. If we put $e = ip(1)$ it is easy to see that $e = ip(1) = ip(e) = e^2$ and $A \cong i(A) = Se$. □

**Proposition 6.8.** *An act $A$ is projective in $S - \mathrm{Act}_0$ if and only if there exist idempotents $e_i$, $i \in I$ such that $A = \coprod_{i \in I} Se_i$.*

*Proof.* We follow the arguments of the proof of [9, Theorem III.17.8].

By [3, Theorem 4.3], $A$ is an projective act if and only if it is isomorphic to a direct sum of indecomposable projective acts. Since every indecomposable projective act is isomorphic to $Se$ for some idempotent $e$ by Lemma 6.7, it remains to observe that for each act $Se$, where $e$ is an idempotent, the inclusion morphism $i : Se \to S$ forms a coretraction and the projection $p : S \to Se$ given by the rule $p(s) = se$ forms a retraction and since $S$ is projective, $Se$ is projective, too. □

Now, we show that locally cyclic acts contains only one zero-element.

**Lemma 6.9.** *Any cyclic $S$-act $A$ contains a unique zero element $\theta_A$.*

*Proof.* Since the act $A$ is cyclic, there exists a $g \in G$ for which $A = Sg$. Let $\theta$ be a zero element of $A$. Then there exists an $s \in S$ such that $\theta = s \cdot a$, and so $\theta = 0\theta = 0sa = 0a$. Thus $0A = \{\theta\}$. □

Recall that an $S$-act is locally cyclic, if for any pair of elements $a_1, a_2 \in A$ there exists a $b \in A$ with $a_i \in Sb$ for $i = 1, 2$.

**Corollary 6.10.** *If $A$ is a locally cyclic $S$-act, then it contains a unique zero element $\theta_A$, the morphism $\pi_{0A}$ is bijective, and we can assume $\mathbf{F}(A) = A$.*

For any act $A \in S - \mathrm{Act}$ we can consider the one-element $S$-act ${}_S\theta$ being adjoined, $A \,\dot\cup\, {}_S\theta \simeq A \coprod_S \theta$. Therefore define a property $\mathcal{P}$ of an $S$-act $A \in S-\mathrm{Act}$ to hold *up to zeros* in the case $A \simeq A' \,\dot\cup\, \dot{\bigcup}_{i \in I} {}_S\theta$, $A'$ cannot be decomposed as $A'' \,\dot\cup\, {}_S\theta$ and it has the property $\mathcal{P}$. Call then $A'$ the *substantial summand* of $A$. Finally, a subact $B$ of $A$ is said to be *superfluous* if $B \cup C \neq A$ for each proper subact $C$ of $A$.

Note that in $S - \mathrm{Act}_0$ the adjunction of ${}_S\theta$ is trivial, since $A \coprod_S \theta \simeq A$, and let us list now some elementary properties of zero elements and substantial summands.

**Lemma 6.11.** *Let $A \in S - \mathrm{Act}$.*

1. *If $\emptyset \neq C \subseteq 0A$, then $C = \dot{\bigcup}_{c \in C} \{c\} \cong \coprod_{c \in C} \theta$ is a subact of $A$.*
2. *If $B$ is a subact of $A$ satisfying $A = B \cup 0A$, then $A \cong B \coprod (0A \setminus B) \cong B \coprod (\coprod_{c \in 0A \setminus B} \theta)$.*
3. *$A$ contains a substantial summand.*
4. *If $A$ is indecomposable, then it is the substantial summand of itself and $0A$ is a superfluous subact of $A$.*

*Proof.* (1) It is clear as $Sc = c = 0c$ for all $c \in 0A$.

(2) Since $A = B \dot\cup (0A \setminus B)$ and $(0A \setminus B) \subseteq 0A$, the claim follows from (1).

(3) By [9, Theorem I.5.10] there exists, up to a permutation, a unique decomposition $A = \dot{\bigcup}_{i \in I} A_i$ of $A$ into indecomposable subacts. If we put $B = \dot{\bigcup}\{A_i \mid A_i \nsubseteq 0A\}$ and $C = \dot{\bigcup}\{A_i \mid A_i \subseteq 0A\}$, then $A = B \dot\cup C \cong B \coprod (\coprod_{c \in C} \theta)$ by

(1) and (2), hence $B$ is the substantial summand of $A$ by the uniqueness of the decomposition.

(4) If $B$ is a subact of an indecomposable act $A$ such that $B \cup 0A = A$, then $A \cong B \coprod (\coprod_{\theta \in 0A \setminus B} \theta)$ by (2), hence we have $0A \subseteq B$ and $B = A$. $\square$

**Lemma 6.12.** *If $A \in S - \overline{\mathrm{Act}}$ such that $\mathbf{F}(A)$ is a nonzero cyclic $S$-act, then $A$ is, up to zeros, cyclic.*

*Proof.* As $\mathbf{F}(A) = \pi_{0A}(A)$ is cyclic, there exists $g \in A$ such that $\mathbf{F}(A) = S\pi_{0A}(g)$, hence $A = Sg \cup 0A$ by the definition of the Rees factor. Since $A \cong Sg \coprod (\coprod_{c \in 0A \setminus Sg} \theta)$ by Lemma 6.11(2), $A$ is, up to zeros, cyclic. $\square$

Note that the image nor the preimage under $\mathbf{F}$ of an indecomposable act may not be indecomposable, as the following examples illustrate:

**Example 6.13.** (1) Consider the monoid $S$ from Example 6.4(2) and the $S$-act $A = \{\theta_A, a, b\}$ with Cayley graph (omitting unit loops)

$$a \xrightarrow{0} \theta_A \xleftarrow{0} b.$$

Then $A$ is indecomposable in $S - \overline{\mathrm{Act}}$, but $\mathbf{F}(A) = A$ is decomposable in $S - \mathrm{Act}_0$.

(2) For any indecomposable $A \in S - \mathrm{Act}_0$ and a nonempty index set $I$, the act $B = A \dot{\bigcup}_{i \in I} (\theta_i) \in S - \overline{\mathrm{Act}}$ is decomposable with $\mathbf{F}(B) \simeq \mathbf{F}(A)$ indecomposable.

Recall that projective objects of both categories $S - \overline{\mathrm{Act}}$ and $S - \mathrm{Act}_0$ are isomorphic to coproducts (in the respective category) $\coprod_{i \in I} Se_i$ of cyclic $S$-acts of the form $Se_i$ with $e_i \in S$ idempotents by [9, Proposition 17.8] and Proposition 6.8.

**Example 6.14.** The functor $\mathbf{F}$ is not bijective on the class of projective objects of $S - \mathrm{Act}$ for any monoid $S$, as there exists a non-projective $A \in S - \mathrm{Act}$ with $\mathbf{F}(A)$ projective: consider the coproduct $A = S_1 \dot{\cup} S_2$, where $S_i \cong S$. Then $\mathbf{F}(S_1 \dot{\cup} S_2)$ is not projective in $S - \mathrm{Act}$ while $\mathbf{F}(\mathbf{F}(S_1 \dot{\cup} S_2)) = \mathbf{F}(S_1 \dot{\cup} S_2)$ is projective in $S - \mathrm{Act}_0$ by Lemma 6.5. In particular, $A = \{(a, b) \in \mathbb{Z}^2 \mid a = 0 \vee b = 0\}$ is not projective in $\mathbb{Z} - \mathrm{Act}$ and $\mathbf{F}(A) \cong A$ is projective in $\mathbb{Z} - \mathrm{Act}_0$.

## 6.3 Perfect monoids

Recall that for an act $A$, a pair $(C, f)$ is a *cover* provided $f : C \to A$ is an epimorphism, and for any proper subact $C' \subset P$ the restriction $f|_{C'} : C' \to A$ is not an epimorphism in the corresponding category. A cover $(P, f)$ is called *projective* in case $P$ is projective (cf. [9, chapter 17]). Note that a projective cover is maximal among all covers.

**Lemma 6.15.** *Let $(P, f)$ be a projective cover of $A$ in the category $S - \mathrm{Act}$. Then $(\mathbf{F}(P), \mathbf{F}(f))$ is a projective cover of $\mathbf{F}(A)$ in the category $S - \mathrm{Act}_0$.*

*Proof.* By Lemma 6.6, $\mathbf{F}(P)$ is projective. Let $Q \subsetneq \mathbf{F}(P)$ be a subact and put $\tilde{Q} = \pi_{0P}^{-1}(Q)$. Then $0P \subseteq \tilde{Q} \subsetneq P$, hence $f(\tilde{Q}) \neq A$ by the hypothesis and $0A = 0f(P) = f(0P) \subseteq f(\tilde{Q})$, as $f$ is surjective. It implies that $\pi_{0A}(\tilde{Q}) \neq \pi_{0P}(A) = \mathbf{F}(A)$, thus

$$\mathbf{F}(f)(Q) = \mathbf{F}(f)(\pi_{0P}(\tilde{Q})) = \pi_{0A} f(\tilde{Q})) \neq \mathbf{F}(A).$$

$\square$

In analogy with module categories, call a monoid *left perfect* (*left 0-perfect*) if each $A \in S - \overline{\mathrm{Act}}$ ($A \in S - \mathrm{Act}_0$) has a projective cover (cf. [3, 6, 8]). Let us recall a characterization of left-perfect monoids:

**Theorem 6.16.** *[6, 1.1] A monoid $S$ is left-perfect if and only if each cyclic $S$-act has a projective cover and every locally cyclic $S$-act is cyclic.*

**Proposition 6.17.** *If a monoid $S$ is left-0-perfect, then it is left perfect.*

*Proof.* Suppose that $S$ is left-0-perfect and let us prove the two conditions from Theorem 6.16.

First suppose that $A \in S - \overline{\mathrm{Act}}$ is a locally cyclic act. Then $A$ contains a unique zero $\theta_A$ and $A \cong \pi_{0A}(A)$ can be considered an act of the category $S - \mathrm{Act}_0$ by Corollary 6.10. Let $f : P \to A$ be a projective cover in $S - \mathrm{Act}_0$. We show that $P$ is indecomposable.

Applying Proposition 6.8 assume to the contrary that $P = P_1 \coprod P_2$ is a non-trivial decomposition, where $P_1 = Se$ is cyclic. Since $f(P) = A = f(P_1) \cup f(P_2)$, there exists $y \in A \setminus f(P_1)$ and there exists $z \in f(P_2)$ such that $f(e), y \in Sz \subseteq f(P_2)$. Hence $f(P_1) \subseteq Sz \subseteq f(P_2)$, and so $A = f(P_1) \cup f(P_2) = f(P_2)$, a contradiction.

Since $P$ is indecomposable, $P \cong Se$ for an idempotent $e \in S$ by Lemma 6.7, which implies that $A$ is cyclic. Furthermore, as $Se$ is a projective act also in the category $S - \overline{\mathrm{Act}}$ by [9, Proposition 17.8], the morphism $f$ constitutes a projective cover in $S - \overline{\mathrm{Act}}$. □

**Theorem 6.18.** *A monoid is left perfect if and only if it is left 0-perfect.*

*Proof.* The direct implication follows from Lemma 6.15 and the reverse one is proven by Proposition 6.17. □

**Example 6.19.** By [6], the examples of monoids which are left perfect (the argument does not require the zero) comprise: monoid of square matrices over a division ring, and finite monoids. By Theorem 6.17, the former is also left-0-perfect, while the latter in case it contains a zero element.

On the other hand, in the case of another class of perfect monoids (without zero) mentioned in [6], groups, the presented result cannot be employed, as adding 0 to a group may in general change the situation notably (see Example 6.23 below).

## 6.4 Steady monoids

An $S$-act $A$ is called *hollow* if each of its proper subacts is superfluous (cf. [7, Definition 3.1]). It is easy to see that hollow acts are indecomposable in both categories $S - \overline{\mathrm{Act}}$ and $S - \mathrm{Act}_0$ (see [7, Theorem 3.4] and [3, Propositions 5.6 and 6.6]).

In compliance with [3] call an act $C \in S - \overline{\mathrm{Act}}$ ($\in S - \mathrm{Act}_0$, resp.) *compact*, if the corresponding covariant Hom-functor commutes with coproducts, i.e. for any family $(A_i, i \in I)$ of $S$-acts in the given category, for the natural functor

$\mathrm{Hom}(C, -) : S - \overline{\mathrm{Act}} \to \mathrm{Set}$ ($S - \mathrm{Act}_0 \to \mathrm{Set}$, resp.) we have a surjective natural morphism

$$\mathrm{Hom}(C, \coprod_{i \in I} A_i) \to \coprod_{i \in I} \mathrm{Hom}(C, A_i) \to 0.$$

Recall that an act in the category $S - \overline{\mathrm{Act}}$ is compact if and only if it is hollow by [3, Proposition 6.6]. It is easy to see that cyclic acts are compact and we say that a monoid $S$ is *left steady* (resp. *left 0-steady*) if every compact act in the category $S - \overline{\mathrm{Act}}$ (resp. $S - \mathrm{Act}_0$) is cyclic (see [3, 6.2]).

**Lemma 6.20.** *Let $A$ be an act in $S - \mathrm{Act}$ such that $0A$ is superfluous in $A$. Then $A$ is hollow in the category $S - \mathrm{Act}$ if and only if $\mathbf{F}(A)$ is hollow in the category $S - \mathrm{Act}_0$.*

*Proof.* Let $A$ be hollow and $\mathbf{F}(A) = B_1 \cup B_2$ for subacts $B_i$, $i = 1, 2$. Then $A = \pi_{0A}^{-1}(B_1) \cup \pi_{0A}^{-1}(B_2)$, hence there exists $i$ such that $A = \pi_{0A}^{-1}(B_i)$ and so $\mathbf{F}(A) = \pi_{0A}(A) = B_i$. Thus $\mathbf{F}(A)$ is hollow.

Conversely, suppose that $A = B_1 \cup B_2$ for subacts $B_i$ of $A$ and $i = 1, 2$. Then $\mathbf{F}(A) = \pi_{0A}(B_1) \cup \pi_{0A}(B_2)$ and so there exists $i$ for which $\mathbf{F}(A) = \pi_{0A}(B_i)$. It implies that $B_i \cup 0A = A$, thus $B_i = A$ since $0A$ is superfluous in $A$. □

Recall a description of the monoid structure via a property of hollow acts, which is employed in the next result:

**Lemma 6.21.** *[7, Lemma 3.8] A monoid $S$ satisfies the ascending chain condition on cyclic subacts of an arbitrary $S$-act if and only if every hollow act in $S - \mathrm{Act}$ is cyclic.*

**Theorem 6.22.** *A monoid $S$ is left 0-steady if and only if it satisfies the ascending chain condition on cyclic subacts.*

*Proof.* By Lemma 6.21 it is enough to prove that $S$ is left 0-steady if and only if every hollow $S$-act in $S - \mathrm{Act}$ is cyclic.

Let $S$ be left 0-steady and let $A$ be a hollow $S$-act in $S - \mathrm{Act}$. Since $A$ is indecomposable, $0A$ is superfluous by Lemma 6.11. Applying Lemma 6.20 we obtain that $\mathbf{F}(A)$ is hollow in the category $S - \mathrm{Act}_0$, which implies that $\mathbf{F}(A)$ is compact in $S - \mathrm{Act}_0$ by [3, Proposition 6.6]. Thus $\mathbf{F}(A) = \pi_{0A}(A)$ is cyclic by the hypothesis and by Lemma 6.12 we get $A = Sa \cup 0A$. Finally, since $0A$ is superfluous, $A$ is cyclic.

Conversely, suppose that $A$ is a compact act in the category $S - \mathrm{Act}_0$. Then it is hollow by [3, Proposition 6.6], and so indecomposable. Now, it follows from Lemmas 6.11 and 6.20 that $A \cong \mathbf{F}(A)$ is hollow in $S - \mathrm{Act}$, hence it is cyclic by the hypothesis. □

We conclude the paper by an example.

**Example 6.23.** Any group $G$ is right steady by [3, Example 6.7(1)], however 0-steadiness of a monoid $G_0$ obtained from $G$ by adding a zero element depends on the structure of subgroups by the last theorem. In particular $\mathbb{Q}^*$ is steady, while $\mathbb{Q}$ is not 0-steady.

# Bibliography for Chapter 6


[1] J. Adámek, H. Herrlich, G. E. Strecker, *Abstract and Concrete Categories*, John Wiley and Sons, Inc, 1990, http://katmat.math.uni-bremen.de/acc/acc.pdf.

[2] H. Bass, Finitistic dimension and a homological generalization of semi-primary rings. Transactions of the American Mathematical Society (1960), 95(3), 466.

[3] J. Dvořák, J. Žemlička, Compact objects in categories of $S$-acts, submitted, 2021, arXiv:2009.12301.

[4] S. Eilenberg, Homological Dimension and Syzygies. The Annals of Mathematics (1956), 64(2), 328–336.

[5] J. Fountain, Perfect semigroups. Proceedings of the Edinburgh Mathematical Society (1976), 20(2), 87–93.

[6] J. Isbell, Perfect monoids. Semigroup Forum (1971) 2, 95–118.

[7] R. Khosravi, M. Roueentan, Co-uniform and hollow $S$-acts over monoids, arXiv: 1908.04559v1

[8] M. Kilp, Perfect monoids revisited. Semigroup Forum (1996) 53, 225–229.

[9] M. Kilp, U. Knauer, A.V. Mikhalev, Monoids, acts and categories, de Gruyter, Expositions in Mathematics 29, Walter de Gruyter, Berlin 2000.

[10] S. Mac Lane, Categories for the Working Mathematician. Springer New York, 1971.