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Classical operators of harmonic analysis and Sobolev embeddings on rearrangement-invariant function spaces

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Abstract: Boundedness properties of some classical operators of harmonic analysis (namely the Hilbert and Riesz transforms, the Riesz potentials and (fractional and nonfractional) maximal operators) as well as certain Sobolev-type embeddings on the entire space are studied. The compactness of Sobolev trace embeddings is also investigated. The focus is on the optimality of the results within the class of rearrangement-invariant function spaces. The aforementioned questions are reduced to equivalent problems concerning appropriate Hardy-type operators acting on functions of a single variable. The behavior of the Hardy-type operators on rearrangement-invariant function spaces is investigated first. The results concerning the Hardy-type operators are used as the building blocks from which together with known results from the literature the other results are obtained. To illustrate possible applications, general results are accompanied by particular examples. The results presented in this thesis are based on some of the papers authored or coauthored by the author.

Keywords: integral operators, Sobolev embeddings, traces, rearrangement-invariant function spaces, optimal targets, optimal domains, reduction principle, compactness

Contents

Introduction

Important as Lebesgue spaces are, there certainly are situations in which the Lebesgue scale of function spaces is simply not fine enough. We shall illustrate that by an oft-quoted example, which also adumbrates a common theme of this thesis. Let $W^{1,n}(\Omega)$ be the Sobolev space of weakly differentiable functions that together with their derivatives belong to the Lebesgue space $L^n(\Omega)$, where Ω is a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$. The standard Sobolev embedding theorem tells us that $W^{1,n}(\Omega)$ is (continuously) embedded in $L^q(\Omega)$ for all $q \in [1, \infty)$, a fact that we denote by $W^{1,n}(\Omega) \hookrightarrow L^q(\Omega)$. It is also well known that $W^{1,n}(\Omega)$ is not embedded in $L^\infty(\Omega)$, that is, functions from $W^{1,n}(\Omega)$ need not be (essentially) bounded (consider, for example, the function $u(x) = \log\big(\log\big(1 + \frac{1}{|x-x_0|}\big)\big)$, $x \in \Omega$, where $x_0 \in \Omega$ is a fixed point). At this point, one might wonder whether functions from $W^{1,n}(\Omega)$, which in turn also belong to “the half-open interval of Lebesgue spaces“, have some better integrability properties that cannot be described/captured by Lebesgue spaces. It turns out that this is the case. Due to the result of N.S. Trudinger [94] (see also [68, 91]), we know that functions from $W^{1,n}(\Omega)$ actually exhibit certain exponential integrability. In terms of function spaces, Trudinger's result can be stated as $W^{1,n}(\Omega) \hookrightarrow \exp L^{\frac{n}{n-1}}(\Omega)$, where $\exp L^{\frac{n}{n-1}}(\Omega)$ is the Orlicz space defined by the Young function $(0, \infty) \ni t \mapsto \exp(t^{\frac{n}{n-1}}) - 1$. We have that

$$L^\infty(\Omega) \subsetneq \exp L^{\frac{n}{n-1}}(\Omega) \subsetneq L^q(\Omega) \quad \text{for every } q \in [1, \infty).$$

In other words, there is a better (smaller) function space than $L^q(\Omega)$, $q \in [1, \infty)$, in which $W^{1,n}(\Omega)$ is embedded. However, functions from $W^{1,n}(\Omega)$ actually have even better integrability properties than those revealed by Trudinger. H. Brézis and S. Wainger showed in [13] that

$$W^{1,n}(\Omega) \hookrightarrow L^{\infty,n;-1}(\Omega), \tag{1}$$

where $L^{\infty,n;-1}(\Omega)$ is a certain function space (more precisely, a Lorentz–Zygmund space) such that

$$L^\infty(\Omega) \subsetneq L^{\infty,n;-1}(\Omega) \subsetneq \exp L^{\frac{n}{n-1}}(\Omega) \subsetneq L^q(\Omega) \quad \text{for every } q \in [1, \infty).$$

That means that there is a function space even better (smaller) than $\exp L^{\frac{n}{n-1}}(\Omega)$ in which $W^{1,n}(\Omega)$ is embedded. An imminent question is, are we done? Is the embedding (1) optimal? Or is it possible to improve (1) even further?

If we are to answer such questions, we need to make more precise what we mean by “a function space” first. All of the function spaces above in which $W^{1,n}(\Omega)$ is embedded are particular instances of so-called rearrangement-invariant Banach function spaces (we shall simply say rearrangement-invariant function spaces instead). The class of rearrangement-invariant function spaces contains a large number of function spaces measuring in some sense integrability of functions, such as Lebesgue spaces, two-parameter Lorentz spaces or Orlicz spaces. If we restrict ourselves to the class of rearrangement-invariant function spaces, which is a fairly reasonable restriction inasmuch as we are interested in integrability of functions (admittedly, it would be a much less reasonable restriction if, for example, we wanted to measure smoothness of functions), then the embedding (1) is indeed optimal and cannot be improved. By that we mean that there is no rearrangement-invariant function space $Y(\Omega)$ strictly smaller than $L^{\infty,n;-1}(\Omega)$ such that (1) is valid with $L^{\infty,n;-1}(\Omega)$ replaced by $Y(\Omega)$. However, how do we know that there is no such rearrangement-invariant function space $Y(\Omega)$? The class of rearrangement-invariant function spaces is quite general and going through all rearrangement-invariant function spaces one by one is of course out of the question. It turns out that a possible way to solve the question of whether (1) can be improved is to *reduce* it to an equivalent question that is easier to manage within the framework of rearrangement-invariant function spaces.

R. Kerman and L. Pick showed in [52] (cf. [39] and references therein) that the validity of the Sobolev-type embedding

$$W^m X(\Omega) \hookrightarrow Y(\Omega), \tag{2}$$

where $X(\Omega)$ and $Y(\Omega)$ are rearrangement-invariant function spaces and $W^m X(\Omega)$ is the m-th order ($m \in \mathbb{N}$, $m < n$) Sobolev-type space built upon $X(\Omega)$ ($W^m L^p(\Omega) = W^{m,p}(\Omega)$), (is equivalent to)/(can be reduced to) the validity of a certain one-dimensional Hardy-type inequality, namely

$$\left\| \int_t^1 f(s) s^{\frac{m}{n}-1} \, \mathrm{d}s \right\|_{Y(0,1)} \leq C \|f\|_{X(0,1)} \tag{3}$$

for every nonnegative, measurable function f on $(0,1)$, where C is a positive constant independent of f and where $X(0,1)$ and $Y(0,1)$ are the representation spaces of $X(\Omega)$ and $Y(\Omega)$, respectively (roughly speaking, there is not much difference between rearrangement-invariant function spaces over Ω and rearrangement-invariant function spaces over $(0,1)$). The equivalence of the validity of (2) to the validity of (3) is an example of what is now often called a *reduction principle*. By means of that reduction principle, the authors of [52] were able to describe, among other things, what the *optimal rearrangement-invariant function spaces* in (2) are, that is, if $X(\Omega)$ is a given rearrangement-invariant function space, what the optimal target rearrangement-invariant function space (i.e., the smallest one) $Y(\Omega)$ in (2) is as well as, if $Y(\Omega)$ is a given rearrangement-invariant function space, what the optimal domain rearrangement-invariant function space (i.e., the largest one) $X(\Omega)$ in (2) is. By combining that with certain Hardy-type inequalities, one can show that the optimal target rearrangement-invariant function space in (2) for $X(\Omega) = L^n(\Omega)$ and $m = 1$ is indeed the space $L^{\infty,n;-1}(\Omega)$. The message of this narrative is that the reduction of the complicated Sobolev-type embedding (2) to the one-dimensional Hardy-type inequality (3) transforms the original problem to another one, which is equivalent and more manageable, and the reduced problem is more suitable for pursuing the question of optimality within the framework of rearrangement-invariant function spaces as well as of fine properties such as compactness (cf. [53]).

Since the aforementioned reduction principle was established, a great deal of work on reduction principles and optimality within the framework of rearrangement-invariant function spaces in various settings has been done. Those settings range from all kinds of Sobolev-type embeddings (e.g. [29, 24, 30, 26, 5, 31]) and their compactness (e.g. [53, 85]) to harmonic analysis (e.g. [14]). In this thesis, we shall pursue the question of reduction, optimality and fine properties such as compactness in some new settings. The results presented in this thesis are based on results from some of the papers authored or coauthored by the author (namely [20, 40, 66, 67, 19, 59]). We aim to obtain the results in a unified, coherent way. In order to achieve that, we shall first establish a common theoretical background providing us not only with the building blocks from which our results in particular settings will be built but also with general theorems that might be of use in other settings not considered in this thesis. A lot of results from the literature are of course also used and they are cited at the places where they are used. The thesis is structured as follows.

- In Chapter 1, we introduce some notation and recall some background material that is oft-used throughout the thesis. In particular, we recall basic elements of the theory of rearrangement-invariant function spaces.

- Chapter 2 contains the principal building blocks of the thesis. We investigate behavior and properties of certain one-dimensional operators (namely those defined by (2.1), (2.2), (2.17) and (2.19)) acting on rearrangement-invariant function spaces. Chapter 2 is the most technical and, at the same time, longest chapter of this thesis. One could say that Chapter 2 is the place where the heavy lifting is done. A lot of results in that chapter are stated in a more general way than actually needed in later chapters so that they might be potentially useful in other settings. With that being said, it should be noted that generality is not pursued at all costs and there is definitely room for further generalizations. For example, it would be of interest (and of importance, cf. [30, 85]) to allow for kernels in the definitions of the Hardy-type operators defined by (2.1) and (2.2). However, such a generalization, if approached in a comprehensive way, could be the subject of a short thesis on its own.

- In Chapter 3, we investigate the optimal behavior of certain classical operators of harmonic analysis on rearrangement-invariant function spaces over $\mathbb{R}^n$. This topic was investigated in [40].

- Chapter 4 deals with two instances of Sobolev-type inequalities on $\mathbb{R}^n$ (namely (4.2) and (4.49)), in which the norm of a m-times weakly differentiable function is bounded from above by the norm of the vector of its all m-th order derivatives. This topic was investigated in [66] and [67].

- The final chapter of this thesis, that is, Chapter 5, is devoted to studying the compactness of Sobolev trace embeddings in which target spaces are endowed with upper Ahlfors measures. This question (save for the content of Section 5.3) was researched in [19], which generalizes [20]. Section 5.3 together with Section 2.3.2 is based on results from [59].

Useful as general results are, it is often not obvious how to apply them to particular situations. Therefore, we shall put some effort into illustrating our general results with particular examples. To this end, we will make use of the class of Lorentz–Zygmund spaces, which encompasses many customary function spaces (e.g., Lebesgue spaces, two-parameter Lorentz spaces or some Orlicz spaces).

If the author were to advise the reader on in what order they should read the thesis, the author would recommend that the reader skim over Chapter 1 first and then start reading from Chapter 3 so that they would know what applications we have in our minds before burying themselves in Chapter 2.

1. Preliminaries

Throughout the entire thesis $L \in \{1, \infty\}$.

We write $P \lesssim Q$, where P, Q are nonnegative quantities, when there is a positive constant c independent of all appropriate quantities appearing in the expressions P and Q such that $P \leq c \cdot Q$. If not stated explicitly, what "the appropriate quantities appearing in the expressions P and Q" are should be obvious from the context. At the few places where it is not obvious, we will explicitly specify what the appropriate quantities are. We also write $P \gtrsim Q$ with the obvious meaning. Furthermore, we write $P \approx Q$ when $P \lesssim Q$ and $P \gtrsim Q$ simultaneously.

Throughout the thesis, we adhere to the convention that $\frac{1}{\infty} = 0 \cdot \infty = 0$.

1.1 Rerrangement-invariant function norms and spaces

1.1.1 Basic definitions and properties

Throughout this section, (R, μ) denotes a σ-finite, nonatomic measure space. If $(R, \mu) = (G, \lambda_n)$, where $G \subseteq \mathbb{R}^n$, $n \in \mathbb{N}$, is a Lebesgue-measurable set and λ_n is the n-dimensional Lebesgue measure on G, we simply write G instead of (R, μ). We set

$$\begin{aligned}\mathfrak{M}(R, \mu) &= \{f\colon f \text{ is a } \mu\text{-measurable function on } R \text{ with values in } [-\infty, \infty]\},\\ \mathfrak{M}_0(R, \mu) &= \{f \in \mathfrak{M}(R, \mu)\colon f \text{ is finite } \mu\text{-a.e. on } R\},\end{aligned}$$

and

$$\mathfrak{M}^+(R, \mu) = \{f \in \mathfrak{M}(R, \mu)\colon f \geq 0 \ \mu\text{-a.e.}\}.$$

The *nonincreasing rearrangement* $f^*\colon (0, \infty) \to [0, \infty]$ of a function $f \in \mathfrak{M}(R, \mu)$ is defined as

$$f^*(t) = \inf\{\lambda \in (0, \infty)\colon \mu\left(\{x \in R\colon |f(x)| > \lambda\}\right) \leq t\},\ t \in (0, \infty).$$

Note that $f^*(t) = 0$ for every $t \in [\mu(R), \infty)$. When (R, μ) and (S, ν) are two (possibly different) (nonatomic, σ-finite) measure spaces, we say that functions $f \in \mathfrak{M}(R, \mu)$ and $g \in \mathfrak{M}(S, \nu)$ are *equimeasurable*, and we write $f \sim g$, if $\mu\left(\{x \in R\colon |f(x)| > \lambda\}\right) = \nu\left(\{x \in S\colon |g(x)| > \lambda\}\right)$ for every $\lambda \in (0, \infty)$. We always have that $f \sim f^*$. The relation $\sim$ is transitive.

The *maximal nonincreasing rearrangement* $f^{**}\colon (0, \infty) \to [0, \infty]$ of a function $f \in \mathfrak{M}(R, \mu)$ is defined as

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\,\mathrm{d}s,\ t \in (0, \infty).$$

If there is any possibility of misinterpretation, we use the more explicit notations f^*_μ and f^{**}_μ instead of f^* and f^{**}, respectively, to stress what measure the rearrangements are taken with respect to. The mapping $f \mapsto f^*$ is monotone in the sense that, for every $f, g \in \mathfrak{M}(R, \mu)$,

$$|f| \leq |g| \quad \mu\text{-a.e. on } R \quad \Longrightarrow \quad f^* \leq g^* \quad \text{on } (0,\infty);$$

consequently, the same implication remains true if * is replaced with **. We have that

$$f^* \leq f^{**} \quad \text{for every } f \in \mathfrak{M}(R,\mu). \tag{1.1}$$

The operation $f \mapsto f^*$ is neither subadditive nor multiplicative. The lack of subadditivity of the operation of taking the nonincreasing rearrangement is, up to some extent, compensated by the following fact ([8, Chapter 2, (3.10)]):

$$\int_0^t (f+g)^*(s)\,\mathrm{d}s \leq \int_0^t f^*(s)\,\mathrm{d}s + \int_0^t g^*(s)\,\mathrm{d}s \tag{1.2}$$

for every $t \in (0,\infty)$, $f, g \in \mathfrak{M}_0(R,\mu)$. In other words, the operation $f \mapsto f^{**}$ is subadditive.

There are a large number of inequalities concerning rearrangements (see, e.g., [48]). We state two fundamental instances of them, which shall prove particularly useful for us. The *Hardy-Littlewood inequality* ([8, Chapter 2, Theorem 2.2]) tells us that, for every $f, g \in \mathfrak{M}(R,\mu)$,

$$\int_R |f||g|\,\mathrm{d}\mu \leq \int_0^{\mu(R)} f^*(t)g^*(t)\,\mathrm{d}t. \tag{1.3}$$

In particular, by taking $g = \chi_E$ in (1.3), one obtain that

$$\int_E |f|\,\mathrm{d}\mu \leq \int_0^{|E|} f^*(t)\,\mathrm{d}t \tag{1.4}$$

for every measurable $E \subseteq R$. The *Hardy lemma* ([8, Chapter 2, Proposition 3.6]) states that, for every $f, g \in \mathfrak{M}^+(0,\infty)$ and every nonincreasing $h \in \mathfrak{M}^+(0,\infty)$,

$$\begin{aligned} &\text{if} \quad \int_0^t f(s)\,\mathrm{d}s \leq \int_0^t g(s)\,\mathrm{d}s \quad \text{for all } t \in (0,\infty),\\ &\text{then} \quad \int_0^\infty f(t)h(t)\,\mathrm{d}t \leq \int_0^\infty g(t)h(t)\,\mathrm{d}t. \end{aligned} \tag{1.5}$$

A functional $\varrho\colon \mathfrak{M}^+(0,L) \to [0,\infty]$ is called a *function norm* (on $(0,L)$) if, for all f, g and $\{f_k\}_{k\in\mathbb{N}}$ in $\mathfrak{M}^+(0,L)$, and every $\lambda \in [0,\infty)$:

(P1) $\|f\|_{X(0,L)} = 0$ if and only if $f = 0$ a.e. on $(0,L)$; $\|\lambda f\|_{X(0,L)} = \lambda\|f\|_{X(0,L)}$; $\|f+g\|_{X(0,L)} \leq \|f\|_{X(0,L)} + \|g\|_{X(0,L)}$;

(P2) $\|f\|_{X(0,L)} \leq \|g\|_{X(0,L)}$ if $f \leq g$ a.e. on $(0,L)$;

(P3) $\|f_k\|_{X(0,L)} \nearrow \|f\|_{X(0,L)}$ if $f_k \nearrow f$ a.e. on $(0,L)$;

(P4) $\|\chi_E\|_{X(0,L)} < \infty$ for every measurable $E \subseteq (0,L)$ of finite measure;

(P5) for every measurable $E \subseteq (0,L)$ of finite measure, there is a positive, finite constant $C_{E,X}$, possibly depending on E and $\|\cdot\|_{X(0,L)}$ but not on f, such that $\int_E f(t)\,\mathrm{d}t \leq C_{E,X}\|f\|_{X(0,L)}$.

If, in addition,

(P6) $\|f\|_{X(0,L)} = \|g\|_{X(0,L)}$ whenever $f \sim g$,

then $\|\cdot\|_{X(0,L)}$ is called a *rearrangement-invariant function norm* (on $(0,L)$).

The *Hardy–Littlewood–Pólya principle* ([8, Chapter 2, Theorem 4.6]) asserts that, for every $f, g \in \mathfrak{M}(0,L)$ and every rearrangement-invariant function norm $\|\cdot\|_{X(0,L)}$,

$$\text{if } \int_0^t f^*(s)\,\mathrm{d}s \le \int_0^t g^*(s)\,\mathrm{d}s \text{ for all } t \in (0,L), \text{ then } \|f\|_{X(0,L)} \le \|g\|_{X(0,L)}. \tag{1.6}$$

With every rearrangement-invariant function norm $\|\cdot\|_{X(0,L)}$, we associate another functional $\|\cdot\|_{X'(0,L)}$ defined as

$$\|f\|_{X'(0,L)} = \sup_{\substack{g \in \mathfrak{M}^+(0,L) \\ \|g\|_{X(0,L)} \le 1}} \int_0^L f(t)g(t)\,\mathrm{d}t, \ f \in \mathfrak{M}^+(0,L). \tag{1.7}$$

The functional $\|\cdot\|_{X'(0,L)}$ is also a rearrangement-invariant function norm ([8, Chapter 2, Proposition 4.2]), and it is called the *associate function norm* of $\|\cdot\|_{X(0,L)}$. Furthermore, we always have that ([8, Chapter 1, Theorem 2.7])

$$\|f\|_{X(0,L)} = \sup_{\substack{g \in \mathfrak{M}^+(0,L) \\ \|g\|_{X'(0,L)} \le 1}} \int_0^L f(t)g(t)\,\mathrm{d}t \quad \text{for every } f \in \mathfrak{M}^+(0,L), \tag{1.8}$$

that is,

$$\|\cdot\|_{(X')'(0,L)} = \|\cdot\|_{X(0,L)}. \tag{1.9}$$

Consequently, statements like "Let $\|\cdot\|_{X(0,L)}$ be *the* rearrangement-invariant function norm whose associate function norm is …" are well-justified. The supremum in (1.8) does not change when the functions involved are replaced with their nonincreasing rearrangements ([8, Chapter 2, Proposition 4.2]), that is,

$$\|f\|_{X(0,L)} = \sup_{\substack{g \in \mathfrak{M}^+(0,L) \\ \|g\|_{X'(0,L)} \le 1}} \int_0^L f^*(t)g^*(t)\,\mathrm{d}t \quad \text{for every } f \in \mathfrak{M}^+(0,L). \tag{1.10}$$

If (R,μ) is a nonatomic, σ-finite measure space and $\|\cdot\|_{X(0,L)}$ is a rearrangement-invariant function norm, where

$$L = \begin{cases} 1 & \text{if } \mu(R) < \infty, \\ \infty & \text{if } \mu(R) = \infty, \end{cases}$$

we define the functional $\|\cdot\|_{X(R,\mu)}$ as

$$\|f\|_{X(R,\mu)} = \begin{cases} \|f^*_\mu(\mu(R)\cdot)\|_{X(0,1)} & \text{if } \mu(R) < \infty, \\ \|f^*_\mu\|_{X(0,\infty)} & \text{if } \mu(R) = \infty, \end{cases} \tag{1.11}$$

for every $f \in \mathfrak{M}(R,\mu)$. Note that $\|f\|_{X(R,\mu)} = \||f|\|_{X(R,\mu)}$. When $(R,\mu) = (0,L)$, definition (1.11) extends the given rearrangement-invariant function norm to all $f \in \mathfrak{M}(0,L)$. The functional $\|\cdot\|_{X(R,\mu)}$ restricted to the linear set $X(R,\mu)$ defined as

$$X(R,\mu) = \{f \in \mathfrak{M}(R,\mu)\colon \|f\|_{X(R,\mu)} < \infty\}$$

is a norm (provided that we identify any two functions from $\mathfrak{M}(R,\mu)$ coinciding μ-a.e. on R, as usual). In fact, $X(R,\mu)$ endowed with the norm $\|\cdot\|_{X(R,\mu)}$ is a Banach space ([8, Chapter 1, Theorem 1.6]). We say that $X(R,\mu)$ is a *rearrangement-invariant function space*. Note that $f \in \mathfrak{M}(R,\mu)$ belongs to $X(R,\mu)$ if and only if $\|f\|_{X(R,\mu)} < \infty$. The rearrangement-invariant function space $X(0,L)$ is called the *representation space* of $X(R,\mu)$. We always have that

$$S(R,\mu) \subseteq X(R,\mu) \subseteq \mathfrak{M}_0(R,\mu), \tag{1.12}$$

where $S(R,\mu)$ denotes the set of all simple functions on (R,μ) (by a simple function, we mean a (finite) linear combination of characteristic functions of measurable sets having finite measure). Moreover, the second inclusion is continuous if the linear set $\mathfrak{M}_0(R,\mu)$ is endowed with the (metrizable) topology of convergence in measure on sets of finite measure ([8, Chapter 1, Theorem 1.4]). In particular, every sequence converging in $X(R,\mu)$ to a function $f \in X(R,\mu)$ contains a subseqence converging pointwise μ-a.e. on R to f.

The *dilation operator* is bounded on every representation space $X(0,L)$. More precisely, we have that ([8, Chapter 3, Proposition 5.11])

$$\|D_a f\|_{X(0,L)} \le \max\{1,a\}\|f\|_{X(0,L)} \quad \text{for every } f \in \mathfrak{M}(0,L), \tag{1.13}$$

where

$$D_a f(t) = f\Big(\frac{t}{a}\Big),\ t \in (0,\infty), \quad \text{if } L = \infty,$$
$$D_a f(t) = \begin{cases} f(\frac{t}{a}),\ 0 < \frac{t}{a} < 1, \\ 0,\ \frac{t}{a} \ge 1, \end{cases} \quad \text{if } L = 1.$$

The rearrangement-invariant function space $X'(R,\mu)$ built upon the associate function norm $\|\cdot\|_{X'(0,L)}$ of a rearrangement-invariant function norm $\|\cdot\|_{X(0,L)}$ is called the *associate function space* of $X(R,\mu)$. Thanks to (1.9), we have that $(X')'(R,\mu) = X(R,\mu)$. Furthermore, one has that

$$\int_R |f||g| \,\mathrm{d}\mu \le C_R \|f\|_{X(R,\mu)} \|g\|_{X'(R,\mu)} \quad \text{for every } f,g \in \mathfrak{M}(R,\mu), \tag{1.14}$$

where

$$C_R = \begin{cases} \mu(R) & \text{if } \mu(R) < \infty, \\ 1 & \text{if } \mu(R) = \infty. \end{cases}$$

Inequality (1.14) is a Hölder-type inequality, and we shall refer to it as the Hölder inequality.

Let $X(R,\mu)$ and $Y(R,\mu)$ be rearrangement-invariant function spaces over the same measure space. We say that $X(R,\mu)$ is *embedded in* $Y(R,\mu)$, and we write $X(R,\mu) \hookrightarrow Y(R,\mu)$, if there is a positive constant C such that $\|f\|_{Y(R,\mu)} \le$

$C\|f\|_{X(R,\mu)}$ for every $f \in \mathfrak{M}(R,\mu)$. If $X(R,\mu) \hookrightarrow Y(R,\mu)$ and $Y(R,\mu) \hookrightarrow X(R,\mu)$ simultaneously, we write that $X(R,\mu) = Y(R,\mu)$. We have that ([8, Chapter 1, Theorem 1.8])

$$X(R,\mu) \hookrightarrow Y(R,\mu) \quad \text{if and only if} \quad X(R,\mu) \subseteq Y(R,\mu).$$

Furthermore,

$$X(R,\mu) \hookrightarrow Y(R,\mu) \quad \text{if and only if} \quad Y'(R,\mu) \hookrightarrow X'(R,\mu) \tag{1.15}$$

with the same embedding constants. Note that

$$X(R,\mu) \hookrightarrow Y(R,\mu) \quad \text{if and only if} \quad X(0,L) \hookrightarrow Y(0,L).$$

We always have that ([8, Chapter 2, Theorem 6.6])

$$L^1(R,\mu) \cap L^\infty(R,\mu) \hookrightarrow X(R,\mu) \hookrightarrow L^1(R,\mu) + L^\infty(R,\mu), \tag{1.16}$$

where $L^1(R,\mu)$ and $L^\infty(R,\mu)$ denote the rearrangement-invariant function spaces built upon the standard $L^1(0,L)$ and $L^\infty(0,L)$ norms (see Section 1.1.4), respectively. In particular,

$$L^\infty(R,\mu) \hookrightarrow X(R,\mu) \hookrightarrow L^1(R,\mu) \tag{1.17}$$

provided that $\mu(R) < \infty$.

We conclude this section by noting that the way we define rearrangement-invariant function spaces in this thesis is actually different from the way used in [8]. We start with rearrangement-invariant function norms defined on measurable functions on intervals and then we define rearrangement-invariant function spaces on general measure spaces by means of those functionals. The way used in [8] is the other way around. Rearrangement-invariant function spaces are defined there by means of rearrangement-invariant function norms defined on measurable functions on general measure spaces, but the Luxemburg representation theorem ([8, Chapter 2, Theorem 4.10]) tells us that their approach is actually equivalent to ours (at least as long as we consider nonatomic, σ-finite measure spaces). However, if $\mu(R) < \infty$ (and $\mu(R) \neq 1$), the approach that we use here gives rearrangement-invariant function norms that are equivalent to (but possibly different from) more customary norms. In a way, we actually already encountered that in the Hölder inequality (1.14). The reason for this is that, if $\mu(R) < \infty$, we use rearrangement-invariant function norms defined on the interval $(0,1)$ instead of $(0,\mu(R))$. An advantage of our approach is that it simplifies notation and some calculations, which shall prove particularly useful in more technical parts of this thesis.

Statements like "let $X(R,\mu)$ be a rearrangement-invariant function space" are to be interpreted as "let $\|\cdot\|_{X(0,L)}$ be a rearrangement-invariant function norm and let $X(R,\mu)$ be the corresponding rearrangement-invariant function space".

1.1.2 Fundamental function and endpoint spaces

When $\|\cdot\|_{X(0,L)}$ is a rearrangement-invariant function norm, we define its *fundamental function* $\varphi_{X(0,L)}$ as

$$\varphi_{X(0,L)}(t) = \|\chi_E\|_{X(0,L)},\ t \in [0,L),$$

where E is any measurable subset of $(0, L)$ such that $|E| = t$. The fundamental function is well defined thanks to property (P6) of rearrangement-invariant function norms and is *quasiconcave*, that is, $\varphi_{X(0,L)}(t) = 0$ if and only if $t = 0$, φ is nondecreasing and the function $(0, L) \ni t \mapsto \frac{\varphi(t)}{t}$ is nonincreasing.

The fundamental functions of $\|\cdot\|_{X(0,L)}$ and $\|\cdot\|_{X'(0,L)}$ satisfy ([8, Chapter 2, Theorem 5.2]) that

$$\varphi_{X(0,L)}(t)\varphi_{X'(0,L)}(t) = t \quad \text{for every } t \in [0, L). \tag{1.18}$$

When $\varphi\colon [0, L) \to [0, \infty)$ is a quasiconcave function, there are always the strongest and the weakest rearrangement-invariant function norms whose fundamental functions are (equivalent to) φ, namely $\|\cdot\|_{\Lambda_\varphi(0,L)}$ and $\|\cdot\|_{M_\varphi(0,L)}$, respectively. These rearrangement-invariant function norms are defined as

$$\begin{aligned}
\|f\|_{\Lambda_\varphi(0,L)} &= \int_{[0,L)} f^*(s)\,\mathrm{d}\widetilde{\varphi}(s),\ f \in \mathfrak{M}^+(0, L),\\
\|f\|_{M_\varphi(0,L)} &= \sup_{t\in(0,L)} f^{**}(t)\varphi(t),\ f \in \mathfrak{M}^+(0, L),
\end{aligned}$$

where $\widetilde{\varphi}$ is the least concave majorant of φ, which satisfies ([8, Chapter 2, Proposition 5.10]) that $\frac{1}{2}\widetilde{\varphi} \le \varphi \le \widetilde{\varphi}$, and the integral $\int_{[0,L)} f^*(s)\,\mathrm{d}\widetilde{\varphi}(s)$ is to be interpreted as the Lebesgue-Stieltjes integral. We have that $\varphi_{\Lambda_\varphi(0,L)} = \widetilde{\varphi}$, $\varphi_{M_\varphi(0,L)} = \varphi$ and

$$\Lambda_\varphi(0, L) \hookrightarrow X(0, L) \hookrightarrow M_\varphi(0, L) \tag{1.19}$$

whenever the fundamental function of $\|\cdot\|_{X(0,L)}$ is equivalent to φ. The corresponding rearrangement-invariant function spaces $\Lambda_\varphi(0, L)$ and $M_\varphi(0, L)$ are sometimes called *Lorentz endpoint spaces* and *Marcinkiewicz endpoint spaces*, respectively.

If $L = 1$, we have the following important characterizations of the endpoint spaces $L^\infty(0, 1)$ and $L^1(0, 1)$ ([84, Theorems 5.2 and 5.3], cf. [8, Chapter 2, Theorem 5.5]):

$$\lim_{t\to0^+} \varphi_{X(0,1)}(t) = 0 \quad \text{if and only if} \quad X(0, 1) \ne L^\infty(0, 1) \tag{1.20}$$

and

$$\lim_{t\to0^+} \frac{t}{\varphi_{X(0,1)}(t)} = 0 \quad \text{if and only if} \quad X(0, 1) \ne L^1(0, 1). \tag{1.21}$$

1.1.3 Almost-compact embeddings

Throughout this section, (R, μ) is a finite, nonatomic measure space.

We say that a function $f \in X(R, \mu)$ has *absolutely continuous norm in* $X(R, \mu)$ if

$$\lim_{k\to\infty} \|f\chi_{E_k}\|_{X(R,\mu)} = 0$$

for every sequence of measurable sets $E_k \subseteq R$ such that $\chi_{E_k} \to 0$, $k \to \infty$, μ-a.e. This is equivalent to the fact that

$$\lim_{a\to0^+} \|f^*\chi_{(0,a)}(t)\|_{X(0,1)} = 0.$$

If every function $f \in X(R,\mu)$ has absolutely continuous norm in $X(R,\mu)$, we say that $X(R,\mu)$ has absolutely continuous norm.

We say that a rearrangement-invariant function space $X(R,\mu)$ is *almost-compactly embedded* in a rearrangement-invariant function space $Y(R,\mu)$, and we write $X(R,\mu) \overset{*}{\hookrightarrow} Y(R,\mu)$, if

$$\lim_{k\to\infty} \sup_{\|f\|_{X(R,\mu)}\le 1} \|f\chi_{E_k}\|_{Y(R,\mu)} = 0$$

for every sequence of measurable sets $E_k \subseteq R$ such that $\chi_{E_k} \to 0$, $k \to \infty$, μ-a.e. In other words, the closed unit ball of $X(R,\mu)$ has uniformly absolutely continuous norm in $Y(R,\mu)$. We have that

$$X(R,\mu) \overset{*}{\hookrightarrow} Y(R,\mu) \quad \text{if and only if} \quad X(0,1) \overset{*}{\hookrightarrow} Y(0,1)$$

and

$$X(0,1) \overset{*}{\hookrightarrow} Y(0,1) \quad \text{if and only if} \quad \lim_{a\to 0^+} \sup_{\|f\|_{X(0,1)}\le 1} \|f^*\chi_{(0,a)}\|_{Y(0,1)} = 0. \tag{1.22}$$

Furthermore, we have that

$$X(0,1) \overset{*}{\hookrightarrow} Y(0,1) \quad \text{if and only if} \quad Y'(0,1) \overset{*}{\hookrightarrow} X'(0,1). \tag{1.23}$$

The relation $\overset{*}{\hookrightarrow}$ is stronger than $\hookrightarrow$, that is,

$$X(0,1) \overset{*}{\hookrightarrow} Y(0,1) \quad \Longrightarrow \quad X(0,1) \hookrightarrow Y(0,1).$$

There is this necessary condition for an embedding to be almost compact ([42, (3.1)]):

$$X(0,1) \overset{*}{\hookrightarrow} Y(0,1) \quad \Longrightarrow \quad \lim_{t\to 0^+} \frac{\varphi_{Y(0,1)}(t)}{\varphi_{X(0,1)}(t)} = 0. \tag{1.24}$$

We have the following important characterizations of the endpoint spaces $L^\infty(0,1)$ and $L^1(0,1)$ ([84, Theorems 5.2 and 5.3]):

$$L^\infty(0,1) \overset{*}{\hookrightarrow} X(0,1) \quad \text{if and only if} \quad X(0,1) \ne L^\infty(0,1) \tag{1.25}$$

and

$$X(0,1) \overset{*}{\hookrightarrow} L^1(0,1) \quad \text{if and only if} \quad X(0,1) \ne L^1(0,1). \tag{1.26}$$

For more information on almost-compact embeddings, which are sometimes called *absolutely continuous embeddings* in the literature, the interested reader is referred to [42, 84].

1.1.4 Some customary rearrangement-invariant function norms

Textbook examples of rearrangement-invariant function norms are the standard *Lebesgue function norms.* For $p \in (0,\infty]$, we define the functional $\|\cdot\|_{L^p(0,L)}$ as

$$\|f\|_{L^p(0,L)} = \begin{cases} \left(\int_0^L f(t)^p \,\mathrm{d}t\right)^{\frac{1}{p}} & \text{if } p \in (0,\infty), \\ \operatorname*{ess\,sup}_{t\in(0,L)} f(t) & \text{if } p = \infty, \end{cases}$$

for $f \in \mathfrak{M}^+(0,L)$. The functional $\|\cdot\|_{L^p(0,L)}$ is a rearrangement-invariant function norm if and only if $1 \le p \le \infty$. An important generalization of the Lebesgue function norms is constituted by the two-parameter *Lorentz function norms.* For $0 < p, q \le \infty$, we define the functional $\|\cdot\|_{L^{p,q}(0,L)}$ as

$$\|f\|_{L^{p,q}(0,L)} = \left\| t^{\frac{1}{p}-\frac{1}{q}} f^*(t) \right\|_{L^q(0,L)}, \quad f \in \mathfrak{M}^+(0,L).$$

The functional $\|\cdot\|_{L^{p,q}(0,L)}$ is equivalent to a rearrangement-invariant function norm if and only if $1 < p < \infty$ and $1 \le q \le \infty$, or $p = q = 1$, or $p = q = \infty$ ([79, Corollary 8.2.4]). That means that there are a rearrangement-invariant function norm ϱ over $(0,L)$ and a positive constant C, depending only on p and q, such that

$$C^{-1}\varrho(f) \le \|f\|_{L^{p,q}(0,L)} \le C\varrho(f) \quad \text{for every } f \in \mathfrak{M}^+(0,L).$$

If this is the case, we call the corresponding rearrangement-invariant function space a *Lorentz space.* Note that ([8, Chapter 2, Proposition 1.8])

$$L^{p,p}(0,L) = L^p(0,L).$$

Furthermore, if $1 < p < \infty$ and $1 \le q \le \infty$, or $p = q = 1$, or $p = q = \infty$, we have that ([8, Chapter 4, Theorem 4.7])

$$(L^{p,q})'(0,L) = L^{p',q'}(0,L), \tag{1.27}$$

where p' and q' stand for the standard Hölder conjugates of p and q, respectively, that is, $\frac{1}{p} + \frac{1}{p'} = 1$.

We define the functions ℓ and $\ell\ell$ as $\ell(t) = 1 + |\log t|$ and $\ell\ell(t) = \ell(\ell(t))$, $t \in (0,\infty)$. These functions are called *broken logarithmic functions.* Let $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$. We denote by $\ell^{\mathbb{A}}$ and $\ell\ell^{\mathbb{A}}$ the functions

$$\ell^{\mathbb{A}}(t) = \begin{cases} \ell^{\alpha_0}(t), & t \in (0,1), \\ \ell^{\alpha_\infty}(t), & t \in [1,\infty), \end{cases}$$

and

$$\ell\ell^{\mathbb{A}}(t) = \begin{cases} \ell\ell^{\alpha_0}(t), & t \in (0,1), \\ \ell\ell^{\alpha_\infty}(t), & t \in [1,\infty). \end{cases}$$

We will sometimes need broken logarithmic functions with more than two layers of logarithms. Such functions are defined in the obvious way. For $\mathbb{A} = (\alpha_0, \alpha_\infty), \mathbb{B} = (\beta_0, \beta_\infty) \in \mathbb{R}^2$ and $0 < p, q \le \infty$, we define the functional $\|\cdot\|_{L^{p,q;\mathbb{A},\mathbb{B}}(0,L)}$ as

$$\|f\|_{L^{p,q;\mathbb{A},\mathbb{B}}(0,L)} = \left\| t^{\frac{1}{p}-\frac{1}{q}} \ell^{\mathbb{A}}(t) \ell\ell^{\mathbb{B}}(t) f^*(t) \right\|_{L^q(0,L)}, \quad f \in \mathfrak{M}^+(0,L).$$

Note that the values of α_∞ and β_∞ are immaterial if $L = 1$. If $L = 1$, we simply write $\|f\|_{L^{p,q;\alpha,\beta}(0,1)}$, ℓ^α and $\ell\ell^\beta$, where $\alpha = \alpha_0 \in \mathbb{R}$ and $\beta = \beta_0 \in \mathbb{R}$. If $\mathbb{B} = (0,0)$ or $\beta = 0$, we sometimes omit $\mathbb{B}$ or β, respectively, in the notation. If $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ and $c \in \mathbb{R}$, we denote by $c\mathbb{A}$ and $\mathbb{A} + c$ the vectors $(c\alpha_0, c\alpha_\infty)$ and $(\alpha_0 + c, \alpha_\infty + c)$, respectively. The functional $\|\cdot\|_{L^{p,q;\mathbb{A},\mathbb{B}}(0,\infty)}$ is equivalent to a rearrangement-invariant function norm if and only if ([75, Theorem 7.1])

$$\begin{cases} 1 < p < \infty, 1 \leq q \leq \infty \text{ or} \\ p = q = 1, \alpha_0 > 0, \alpha_\infty < 0 \text{ or} \\ p = q = 1, \alpha_0 > 0, \alpha_\infty = 0, \beta_\infty \leq 0 \text{ or} \\ p = q = 1, \alpha_0 = 0, \beta_0 \geq 0, \alpha_\infty < 0 \text{ or} \\ p = q = 1, \alpha_0 = 0, \beta_0 \geq 0, \alpha_\infty = 0, \beta_\infty \leq 0 \text{ or} \\ p = \infty, 1 \leq q \leq \infty, \alpha_0 + \frac{1}{q} < 0 \text{ or} \\ p = \infty, 1 \leq q \leq \infty, \alpha_0 + \frac{1}{q} = 0, \beta_0 + \frac{1}{q} < 0 \text{ or} \\ p = q = \infty, \alpha_0 = \beta_0 = 0. \end{cases} \tag{1.28}$$

The functional $\|\cdot\|_{L^{p,q;\alpha,\beta}(0,1)}$ is equivalent to a rearrangement-invariant function norm if and only if ([75, Theorem 7.4])

$$\begin{cases} 1 < p < \infty, 1 \leq q \leq \infty \text{ or} \\ p = q = 1, \alpha > 0 \text{ or} \\ p = q = 1, \alpha = 0, \beta \geq 0 \text{ or} \\ p = \infty, 1 \leq q \leq \infty, \alpha + \frac{1}{q} < 0 \text{ or} \\ p = \infty, 1 \leq q \leq \infty, \alpha + \frac{1}{q} = 0, \beta + \frac{1}{q} < 0 \text{ or} \\ p = q = \infty, \alpha = \beta = 0. \end{cases} \tag{1.29}$$

The corresponding rearrangement-invariant function space is called a *Lorentz–Zygmund space.* If either $p \in [1, \infty)$, $q \in [1, \infty]$, or $p = q = \infty$, $\alpha_0 = \alpha \leq 0$, $\alpha_\infty \geq 0$, we have that ([75, Theorems 6.2, 6.6, 6.11])

$$\begin{gathered} (L^{p,q;\mathbb{A}})'(0,\infty) = L^{p',q';-\mathbb{A}}(0,\infty), \\ (L^{p,q;\alpha})'(0,1) = L^{p',q';-\alpha}(0,1). \end{gathered} \tag{1.30}$$

We will occasionally need other Lorentz–Zygmund-type spaces, namely $L^{(p,q;\mathbb{A},\mathbb{B})}(0,\infty)$ and $L^{(p,q;\alpha,\beta)}(0,1)$, which are closely related to the spaces $L^{p,q;\mathbb{A},\mathbb{B}}(0,\infty)$ and $L^{p,q;\alpha,\beta}(0,1)$, respectively. They are defined by means of the same functional but with f^* replaced by f^{**}. If $p \in (1, \infty]$ and one of conditions (1.28) or (1.29) is satisfied, we actually have that

$$\begin{gathered} L^{(p,q;\mathbb{A},\mathbb{B})}(0,\infty) = L^{p,q;\mathbb{A},\mathbb{B}}(0,\infty), \\ L^{(p,q;\alpha,\beta)}(0,1) = L^{p,q;\alpha,\beta}(0,1), \end{gathered} \tag{1.31}$$

respectively. Both types of Lorentz–Zygmund spaces were exhaustively studied in [75].

A large number of rearrangement-invariant function norms (including the Lorentz-Zygmund ones) are (equivalent to) instances of the *Lorentz* Λ*-functionals* $\|\cdot\|_{\Lambda^q(v)}$ for suitable choices of v. Let $v\colon (0, L) \to (0, \infty)$ be a measurable function such that $V(t) < \infty$ for every $t \in (0, L)$, where $V(t) = \int_0^t v(s)\,\mathrm{d}s$, $t \in (0, L)$. We define the functional $\|\cdot\|_{\Lambda^q(v)}$ as

$$\|f\|_{\Lambda^q(v)} = \begin{cases} \left(\int_0^L f^*(t)^q v(t)\,\mathrm{d}t\right)^{\frac{1}{q}} & \text{if } q \in (0, \infty), \\ \operatorname*{ess\,sup}_{t\in(0,L)} f^*(t)v(t) & \text{if } q = \infty, \end{cases}$$

for $f \in \mathfrak{M}^+(0, L)$. If $q \in [1, \infty]$, the functional $\|\cdot\|_{\Lambda^q(v)}$ is equivalent to a rearrangement-invariant function norm if and only if

$$\begin{cases} \frac{1}{t}V(t) \lesssim \frac{1}{s}V(s) \quad \text{for every } 0 < s < t < L & \text{if } q = 1, \\ V^{q'-1}(t)\int_0^t s^{q'}v(s)V^{-q'}(s)\,\mathrm{d}s \lesssim t^{q'} \quad \text{for every } 0 < t < L & \text{if } q \in (1, \infty), \\ \tilde{v} \text{ is a finite function} \quad \text{and} \quad \sup_{t\in(0,L)} \tilde{v}(t)\frac{1}{t}\int_0^t \frac{1}{\tilde{v}(s)}\,\mathrm{d}s < \infty & \text{if } q = \infty, \end{cases} \tag{1.32}$$

where $\tilde{v}(t) = \operatorname*{ess\,sup}_{s\in(0,t)} v(s)$, $t \in (0, L)$, that is, $\tilde{v}$ is the least nondecreasing (essential) majorant of v. We refer the reader to [82] for $q \in (1, \infty)$, to [17] for $q = 1$ (see also [90] with regard to local embedding of $\Lambda^1(v)$ in L^1) and to [45] for $q \in (1, \infty]$. The multiplicative constants in (1.32) may depend only on q and v.

1.2 Sobolev spaces built upon rearrangement-invariant function spaces

1.2.1 Basic definitions and properties

Let $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, be an open set. If $m \in \mathbb{N}$ and u is a m-times weakly differentiable function on Ω, we denote by $\nabla^k u$, $k \in \{0, 1, \dots, m\}$, the vector of all k-th order weak derivatives of u on Ω, where $\nabla^0 u = u$. We denote by $D^m u$ the vector of all weak derivatives of u on Ω up to order m. We shall simply write ∇u and Du instead of $\nabla^1 u$ and $D^1 u$, respectively. We define the m-th order *Sobolev-type space $W^m X(\Omega)$ built upon a rearrangement-invariant function space $X(\Omega)$* as

$$W^m X(\Omega) = \{u \in L^1_{\mathrm{loc}}(\Omega) \colon u \text{ is } m\text{-times weakly differentiable on } \Omega \text{ and } |D^m u| \in X(\Omega)\},$$

where $|D^m u|$ stands for the ℓ^1-norm of the vector. The Sobolev-type space $W^m X(\Omega)$ endowed with the norm

$$\|u\|_{W^m X(\Omega)} = \sum_{k=0}^{m} \||\nabla^k u|\|_{X(\Omega)}, \ u \in W^m X(\Omega),$$

is a Banach space. We have that

$$\|u\|_{W^m X(\Omega)} \approx \||D^m u|\|_{X(\Omega)} \quad \text{for every } u \in W^m X(\Omega). \tag{1.33}$$

We shall simply write $\|\nabla^k u\|_{X(\Omega)}$ and $\|D^m u\|_{X(\Omega)}$ instead of $\||\nabla^k u|\|_{X(\Omega)}$ and $\||D^m u|\|_{X(\Omega)}$, respectively. When $X(\Omega) = L^p(\Omega)$, $p \in [1, \infty]$, we have that $W^m L^p(\Omega) = W^{m,p}(\Omega)$, the standard Sobolev space of m-th order, possibly up to equivalent norms. We denote the subspace of $W^m(\Omega)$ consisting of those functions that together with their derivatives up to order $m-1$ vanish on $\partial\Omega$ (in a suitable sense) by $W_0^m X(\Omega)$, that is,

$$W_0^m X(\Omega) = \{u \in W^m(\Omega)\colon \text{the continuation of } u \text{ by 0 outside } \Omega \text{ is } m\text{-times} \\ \text{weakly differentiable on } \mathbb{R}^n\}.$$

The space $W_0^m X(\Omega)$ is a closed subspace of $W^m X(\Omega)$. In particular, $W_0^m X(\Omega)$ is a Banach space. The closure of $\mathcal{C}_0^\infty(\Omega)$, the space of all infinitely differentiable functions having a compact support in Ω, in the norm of $W^m X(\Omega)$ is contained in $W_0^m X(\Omega)$, but the inclusion may be strict.

We denote by $V^m X(\Omega)$ the linear set of all m-times weakly differentiable functions on Ω whose m-th order gradients belong to $X(\Omega)$, that is,

$$V^m X(\Omega) = \{u \in L^1_{\text{loc}}(\Omega)\colon u \text{ is } m\text{-times weakly differentiable on } \Omega \text{ and } |\nabla^m u| \in X(\Omega)\}.$$

Note that, for a function from $V^m X(\Omega)$, only its m-th order derivatives are required to be elements of $X(\Omega)$, whereas there are no assumptions imposed on its derivatives of lower orders. The derivatives of lower orders are not required to have any extra regularity apart from their existence in the weak sense, that is, as locally integrable functions (cf. [64, 1.1.2]). Nevertheless, we have that ([64, 5.2.3])

$$V^m X(B) \subseteq W^m L^1(B) \quad \text{for every open ball } B \subseteq \mathbb{R}^n. \tag{1.34}$$

In particular, functions from $V^m X(B)$ are integrable over B.

We denote by $V_0^m X(\Omega)$ the linear subset of $V^m X(\Omega)$ consisting of those functions that together with their derivatives up to order $m-1$ have finite measure of all their level sets, that is,

$$V_0^m X(\Omega) = \{u \in V^m X(\Omega)\colon |\{x \in \Omega\colon |\nabla^k u(x)| > \lambda\}| < \infty \text{ for all } k = 0, \dots, m-1 \\ \text{and every } \lambda > 0\}.$$

If $|\Omega| < \infty$, then we plainly have that $V^m X(\Omega) = V_0^m X(\Omega)$.

When Ω is a domain, that is, Ω is not only open but also connected, we say that Ω has *the cone property* if there is a finite cone $C \subseteq \mathbb{R}^n$ such that every $x \in \Omega$ is the vertex of a finite cone C_x that is contained in Ω and congruent to C. When Ω is a bounded domain, we say that Ω is a *bounded Lipschitz domain* if for every $x \in \partial\Omega$ there is a neighborhood U_x of x such that $\Omega \cap U_x$ is the subgraph of a Lipschitz continuous function of $n-1$ variables. Bounded Lipschitz domains as defined here are sometimes called bounded domains having the strong Lipschitz property or $\mathcal{C}^{0,1}$-domains (cf. [64, 1.1.9]). We follow the definitions given in [4, Chapter 4].

1.2.2 Interpolation of Sobolev spaces

The *K-functional* associated to a pair (X_0, X_1) of Banach spaces of functions from $\mathfrak{M}_0(R, \mu)$ is defined as (e.g., [8, 10])

$$K(t, u; X_0, X_1) = \inf_{u = u_1 + u_2} \left(\|u_1\|_{X_0} + t\|u_2\|_{X_1}\right), \; u \in X_0 + X_1, \; t \in (0, \infty),$$

where the infimum is taken over all possible decompositions $u = u_1 + u_2$, where $u_i \in X_i$, $i = 0, 1$.

Assume that Ω is a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$. By [36], the reiteration theorem [8, Chapter 5, Theorem 2.4] and Holmstedt's formulas for K-functionals of pairs of Lorentz spaces [50, Theorem 4.2], if $p_0 = q_0 = 1$, or $1 < p_0 < p_1 < \infty$ and $1 \leq q_0, q_1 < \infty$, then

$$\begin{aligned} K(t, u; W^m L^{p_0,q_0}(\Omega), W^m L^{p_1,q_1}(\Omega)) \approx & \left(\int_0^{t^\alpha} \left[s^{\frac{1}{p_0} - \frac{1}{q_0}} |D^m u|^*(s) \right]^{q_0} \mathrm{d}s \right)^{\frac{1}{q_0}} \\ & + t \left(\int_{t^\alpha}^{\infty} \left[s^{\frac{1}{p_1} - \frac{1}{q_1}} |D^m u|^*(s) \right]^{q_1} \mathrm{d}s \right)^{\frac{1}{q_1}} \end{aligned} \tag{1.35}$$

for every $t \in (0, \infty)$, where $\frac{1}{\alpha} = \frac{1}{p_0} - \frac{1}{p_1}$. Furthermore, we have that

$$K(t, u; W^m L^{p_0,q_0}(\Omega), W^m L^{\infty}(\Omega)) \approx \left(\int_0^{t^{p_0}} \left[s^{\frac{1}{p_0} - \frac{1}{q_0}} |D^m u|^*(s) \right]^{q_0} \mathrm{d}s \right)^{\frac{1}{q_0}} \tag{1.36}$$

for every $t \in (0, \infty)$. The multiplicative constants in (1.35) and (1.36) depend only on Ω, p_0, p_1, q_0, and q_1. Moreover, both equivalences are also valid for any open set $\Omega \subseteq \mathbb{R}^n$ such that $|\Omega| < \infty$ provided that W^m is replaced by W^m_0.

2. One-dimensional operators on rearrangement-invariant function spaces

2.1 Operator-induced rearrangement-invariant function norms

2.1.1 Hardy-type operators

Let $u, v\colon (0,L) \to (0,\infty)$ be measurable functions. Let $\nu \in (0,\infty)$. We define the Hardy-type operators $R_{u,v,\nu}$ and $H_{u,v,\nu}$ as

$$R_{u,v,\nu}f(t) = v(t)\int_0^{t^\nu} |f(s)|u(s)\,\mathrm{d}s,\ t \in (0,L),\ f \in \mathfrak{M}(0,L), \tag{2.1}$$

and

$$H_{u,v,\nu}f(t) = u(t)\int_{t^{\frac{1}{\nu}}}^L |f(s)|v(s)\,\mathrm{d}s,\ t \in (0,L),\ f \in \mathfrak{M}(0,L). \tag{2.2}$$

If $u \equiv 1$, we shall simply write $R_{v,\nu}$ and $H_{v,\nu}$ instead of $R_{u,v,\nu}$ and $H_{u,v,\nu}$, respectively. The operators $R_{u,v,\nu}$ and $H_{u,v,\nu}$ are in a sense dual to each other. More precisely, by using the Fubini theorem, one can easily verify that

$$\int_0^L f(t)R_{u,v,\nu}g(t)\,\mathrm{d}t = \int_0^L g(t)H_{u,v,\nu}f(t)\,\mathrm{d}t \quad \text{for every } f,g \in \mathfrak{M}^+(0,L). \tag{2.3}$$

The following two propositions characterize when certain functionals induced by $R_{u,v,\nu}$ and $H_{u,v,\nu}$ are rearrangement-invariant function norms.

Proposition 2.1.1. *Let $\|\cdot\|_{X(0,L)}$ be a rearrangement-invariant function norm. Let $u\colon (0,L) \to (0,\infty)$ be a nonincreasing function such that $0 < U(t) < \infty$ for every $t \in (0,L)$, where $U(t) = \int_0^t u(s)\,\mathrm{d}s$. If $L = 1$, we assume that $u(1^-) > 0$. Let $v\colon (0,L) \to (0,\infty)$ be measurable. Let $\nu \in (0,\infty)$. Set*

$$\varrho(f) = \left\| v(t)\int_0^{t^\nu} f^*(s)u(s)\,\mathrm{d}s \right\|_{X(0,L)},\ f \in \mathfrak{M}^+(0,L).$$

If $L = 1$, the functional ϱ is a rearrangement-invariant function norm if and only if $v(t)U(t^\nu) \in X(0,1)$.

If $L = \infty$, the functional ϱ is a rearrangement-invariant function norm if and only if $v(t)U(t^\nu)\chi_{(0,1)}(t) + v(t)\chi_{(1,\infty)}(t) \in X(0,\infty)$.

Proof. Property (P1). The positive homogeneity and positive definiteness of ϱ can be readily verified. As for the subadditivity of ϱ, it follows from (1.2) combined with Hardy's lemma (1.5) that

$$\int_0^L (f+g)^*(s)u(s)\chi_{(0,t^\nu)}(s)\,\mathrm{d}s \leq \int_0^L f^*(s)u(s)\chi_{(0,t^\nu)}(s)\,\mathrm{d}s + \int_0^L g^*(s)u(s)\chi_{(0,t^\nu)}(s)\,\mathrm{d}s$$

for every $f, g \in \mathfrak{M}^+(0,L)$ and $t \in (0,L)$ thanks to the fact that u is nonincreasing. Since $\| \cdot \|_{X(0,L)}$ is subadditive, it follows that

$$\varrho(f+g) \leq \varrho(f) + \varrho(g) \quad \text{for every } f, g \in \mathfrak{M}^+(0,L).$$

Properties (P2) *and* (P3). Since $\| \cdot \|_{X(0,L)}$ has these properties, it can be readily verified that ϱ, too, has them.

Property (P4). First, assume that $L = 1$. Clearly, $\varrho(\chi_{(0,1)}) < \infty$ if and only if $v(t)U(t^\nu) \in X(0,1)$. Since ϱ has property (P2), ϱ has property (P4) if and only if $v(t)U(t^\nu) \in X(0,1)$. Second, assume that $L = \infty$. Let $E \subseteq (0,\infty)$ be a set of finite, positive measure. Clearly, $\varrho(\chi_E) < \infty$ if and only if $v(t)U(t^\nu)\chi_{(0,|E|)}(t) + v(t)\chi_{(|E|,\infty)}(t) \in X(0,\infty)$. If $|E| \leq 1$, then

$$\begin{aligned}
&\|v(t)U(t^\nu)\chi_{(0,|E|)}(t) + v(t)\chi_{(|E|,\infty)}(t)\|_{X(0,\infty)} \\
&\leq \|v(t)U(t^\nu)\chi_{(0,1)}(t)\|_{X(0,\infty)} + \|v(t)\chi_{(|E|,1)}(t)\|_{X(0,\infty)} + \|v(t)\chi_{(1,\infty)}(t)\|_{X(0,\infty)} \\
&\leq \|v(t)U(t^\nu)\chi_{(0,1)}(t)\|_{X(0,\infty)} + \frac{1}{U(|E|^\nu)}\|U(t^\nu)v(t)\chi_{(|E|,1)}(t)\|_{X(0,\infty)} \\
&\quad + \|v(t)\chi_{(1,\infty)}(t)\|_{X(0,\infty)} \\
&\leq \Big(1 + \frac{1}{U(|E|^\nu)}\Big)\|v(t)U(t^\nu)\chi_{(0,1)}(t)\|_{X(0,\infty)} + \|v(t)\chi_{(1,\infty)}(t)\|_{X(0,\infty)}.
\end{aligned}$$

If $E \geq 1$, we can similarly obtain that

$$\begin{aligned}
&\|v(t)U(t^\nu)\chi_{(0,|E|)}(t) + v(t)\chi_{(|E|,\infty)}(t)\|_{X(0,\infty)} \\
&\leq \|v(t)U(t^\nu)\chi_{(0,1)}(t)\|_{X(0,\infty)} + (1 + U(|E|^\nu))\|v(t)\chi_{(1,\infty)}(t)\|_{X(0,\infty)}.
\end{aligned}$$

Either way, we have that $\varrho(\chi_E) < \infty$ if and only if

$$v(t)U(t^\nu)\chi_{(0,1)}(t) + v(t)\chi_{(1,\infty)}(t) \in X(0,\infty).$$

Property (P5). Let $E \subseteq (0,L)$ be a set of finite, positive measure. Let $f \in \mathfrak{M}^+(0,L)$. Note that the function $(0,L) \ni t \mapsto \frac{1}{U(t^\nu)}\int_0^{t^\nu} f^*(s)u(s)\,\mathrm{d}s$ is nonincreasing because it is the integral mean of a nonincreasing function over the interval $(0,t^\nu)$ with respect to the measure $u(s)\,\mathrm{d}s$. Thanks to that and the monotonicity of u, we obtain that

$$\begin{aligned}
\Big\|v(t)\int_0^{t^\nu} f^*(s)u(s)\,\mathrm{d}s\Big\|_{X(0,L)} &\geq \Big\|v(t)\chi_{(0,|E|^{\frac{1}{\nu}})}(t)\int_0^{t^\nu} f^*(s)u(s)\,\mathrm{d}s\Big\|_{X(0,L)} \\
&\geq \Big\|v(t)U(t^\nu)\chi_{(0,|E|^{\frac{1}{\nu}})}(t)\Big\|_{X(0,L)}\frac{1}{U(|E|)}\int_0^{|E|} f^*(s)u(s)\,\mathrm{d}s \\
&\geq \Big\|v(t)U(t^\nu)\chi_{(0,|E|^{\frac{1}{\nu}})}(t)\Big\|_{X(0,L)}\frac{u(|E|^-)}{U(|E|)}\int_0^{|E|} f^*(s)\,\mathrm{d}s \\
&\geq \Big\|v(t)U(t^\nu)\chi_{(0,|E|^{\frac{1}{\nu}})}(t)\Big\|_{X(0,L)}\frac{u(|E|^-)}{U(|E|)}\int_E f(s)\,\mathrm{d}s,
\end{aligned}$$

where we used (1.4) in the last inequality.

Property (P6). Since $f^* = g^*$ when $f, g \in \mathfrak{M}^+(0,L)$ are equimeasurable, this is obvious.

We conclude the proof by noting that the necessity of $v(t)U(t^\nu) \in X(0,1)$ or $v(t)U(t^\nu)\chi_{(0,1)}(t) + v(t)\chi_{(1,\infty)}(t) \in X(0,\infty)$ if $L = 1$ or $L = \infty$, respectively, was proved in the paragraph devoted to property (P4). ☐

The situation is more complicated with $H_{u,v,\nu}$.

Proposition 2.1.2. *Let $\|\cdot\|_{X(0,L)}$ be a rearrangement-invariant function norm. Let $u, v\colon (0,L) \to (0,\infty)$ be nonincreasing. If $L = 1$, we assume that $v(1^-) > 0$. Let $\nu \in (0,\infty)$. Set*

$$\varrho(f) = \sup_{h\sim f} \left\| u(t) \int_{t^{\frac{1}{\nu}}}^{L} h(s)v(s)\,\mathrm{d}s \right\|_{X(0,L)}, \ f \in \mathfrak{M}^+(0,L), \tag{2.4}$$

where the supremum is taken over all $h \in \mathfrak{M}^+(0,L)$ equimeasurable with f.

If $L = 1$, the functional ϱ is a rearrangement-invariant function norm if and only if

$$\left\| u(t) \int_{t^{\frac{1}{\nu}}}^{1} v(s)\,\mathrm{d}s \right\|_{X(0,1)} < \infty. \tag{2.5}$$

If $L = \infty$, the functional ϱ is a rearrangement-invariant function norm if and only if

$$\left\| u(t)\chi_{(0,1)}(t) \int_{t^{\frac{1}{\nu}}}^{1} v(s)\,\mathrm{d}s \right\|_{X(0,\infty)} < \infty \tag{2.6}$$

and

$$\sup_{b\in[1,\infty)} v(b)\|u\chi_{(0,b^\nu)}\|_{X(0,\infty)} < \infty. \tag{2.7}$$

Proof. Property (P2). Let $f, g \in \mathfrak{M}^+(0,L)$ be such that $f \leq g$ a.e. Consequently, $f^* \leq g^*$. Suppose that $\varrho(f) > \varrho(g)$. That implies that there is $\widetilde{f} \in \mathfrak{M}^+(0,L)$, $\widetilde{f} \sim f$, such that

$$\sup_{h\sim g} \left\| u(t) \int_{t^{\frac{1}{\nu}}}^{L} h(s)v(s)\,\mathrm{d}s \right\|_{X(0,L)} < \left\| u(t) \int_{t^{\frac{1}{\nu}}}^{L} \widetilde{f}(s)v(s)\,\mathrm{d}s \right\|_{X(0,L)}. \tag{2.8}$$

When $L = \infty$, we may assume that $\lim_{t\to\infty} (\widetilde{f})^*(t) = \lim_{t\to\infty} f^*(t) = 0$, for we would otherwise approximate $\widetilde{f}$ by functions $f_n = \widetilde{f}\chi_{(0,n)}$, $n \in \mathbb{N}$ (the monotone convergence theorem and property (P3) of $\|\cdot\|_{X(0,L)}$ would guarantee that the inequality above holds with $\widetilde{f}$ replaced by f_n for n large enough). Thanks to [8, Chapter 2, Corollary 7.6], there is a measure-preserving transformation (in the sense of [8, Chapter 2, Definition 7.1]) $\sigma\colon (0,L) \to (0,L)$ such that $\widetilde{f} = f^* \circ \sigma$. Since σ is measure preserving, we have that $(g^* \circ \sigma) \sim g^* \sim g$ ([8, Chapter 2, Proposition 7.2]). Consequently,

$$\begin{aligned} \sup_{h\sim g} \left\| u(t) \int_{t^{\frac{1}{\nu}}}^{L} h(s)v(s)\,\mathrm{d}s \right\|_{X(0,L)} &\geq \left\| u(t) \int_{t^{\frac{1}{\nu}}}^{L} g^*(\sigma(s))v(s)\,\mathrm{d}s \right\|_{X(0,L)} \\ &\geq \left\| u(t) \int_{t^{\frac{1}{\nu}}}^{L} f^*(\sigma(s))v(s)\,\mathrm{d}s \right\|_{X(0,L)} \\ &= \left\| u(t) \int_{t^{\frac{1}{\nu}}}^{L} \widetilde{f}(s)v(s)\,\mathrm{d}s \right\|_{X(0,L)}. \end{aligned} \tag{2.9}$$

By combining (2.8) and (2.9), we reach a contradiction. Hence $\varrho(f) \leq \varrho(g)$.

Property (P3). Let $f, f_k \in \mathfrak{M}^+(0,L)$, $k \in \mathbb{N}$, be such that $f_k \nearrow f$ a.e. Thanks to property (P2) of ϱ, the limit $\lim\limits_{k\to\infty} \varrho(f_k)$ exists and we clearly have that $\lim\limits_{k\to\infty} \varrho(f_k) \leq \varrho(f)$. The fact that $\lim\limits_{k\to\infty} \varrho(f_k) = \varrho(f)$ can be proved by contradiction in a similar way to the proof of (P2).

Property (P1). The positive homogeneity and positive definiteness of ϱ can be readily verified. As for the subadditivity of ϱ, let $f, g \in \mathfrak{M}^+(0,L)$ be simple functions. Let $h \in \mathfrak{M}^+(0,L)$ be such that $h \sim f + g$. Being equimeasurable with $f+g$, h is a simple function having the same range as $f+g$. Furthermore, it is easy to see that h can be decomposed as $h = h_1 + h_2$, where $h_1, h_2 \in \mathfrak{M}^+(0,L)$ are simple functions such that $h_1 \sim f$ and $h_2 \sim g$. Using the subadditivity of $\|\cdot\|_{X(0,L)}$, we obtain that

$$\begin{aligned}\Big\|u(t)\int_{t^{\frac{1}{\nu}}}^{L} h(s)v(s)\,\mathrm{d}s\Big\|_{X(0,L)} &\leq \Big\|u(t)\int_{t^{\frac{1}{\nu}}}^{L} h_1(s)v(s)\,\mathrm{d}s\Big\|_{X(0,L)} + \Big\|u(t)\int_{t^{\frac{1}{\nu}}}^{L} h_2(s)v(s)\,\mathrm{d}s\Big\|_{X(0,L)}\\ &\leq \varrho(f) + \varrho(g).\end{aligned}$$

Hence $\varrho(f+g) \leq \varrho(f) + \varrho(g)$. When $f, g \in \mathfrak{M}^+(0,L)$ are general functions, we approximate each of them by a nondecreasing sequence of nonnegative, simple functions and use property (P3) of ϱ to get $\varrho(f+g) \leq \varrho(f) + \varrho(g)$.

Property (P4). When $L = 1$, ϱ has property (P4) if and only if $\varrho(\chi_{(0,1)}) < \infty$ (note that here we use the fact that ϱ has property (P2)). If $h \in \mathfrak{M}^+(0,1)$ is equimeasurable with $\chi_{(0,1)}$, then $h = 1$ a.e.; therefore,

$$\varrho(\chi_{(0,1)}) = \Big\|u(t)\int_{t^{\frac{1}{\nu}}}^{1} v(s)\,\mathrm{d}s\Big\|_{X(0,1)}.$$

Hence ϱ has property (P4) (when $L=1$) if and only if $u(t)\int_{t^{\frac{1}{\nu}}}^{1} v(s)\,\mathrm{d}s \in X(0,1)$. Assume now that $L = \infty$. Let $E \subseteq (0,\infty)$ be of finite measure. Set $b = \max\Big\{1, \frac{2|E|}{2^{\frac{1}{\nu}}-1}\Big\}$. Let $h \in \mathfrak{M}^+(0,\infty)$ be equimeasurable with χ_E. It is easy to see that $h = \chi_F$ for some measurable $F \subseteq (0,\infty)$ such that $|F| = |E|$. Thanks to the (outer) regularity of the Lebesgue measure, there is an open set $G \supseteq F$ such that $|G| \leq 2|F|$. Being an open set on the real line, $G \cap (b,\infty)$ can be expressed as $G \cap (b,\infty) = \bigcup\limits_k (a_k, b_k)$, where $\{(a_k,b_k)\}_k$ is a countable system of mutually disjoint, open intervals. We plainly have that $F \subseteq (0,b] \cup \big(G \cap (b,\infty)\big)$, $b_k - a_k \leq 2|F|$ and $a_k > b$. Furthermore, we have that $b_k - a_k \leq 2|F| \leq (2^{\frac{1}{\nu}}-1)b \leq (2^{\frac{1}{\nu}}-1)a_k$, whence

$$b_k^\nu - a_k^\nu < a_k^\nu. \tag{2.10}$$

We have that

$$\begin{aligned}\Big\|u(t)\int_{t^{\frac{1}{\nu}}}^{\infty} \chi_F(s)v(s)\,\mathrm{d}s\Big\|_{X(0,\infty)} &\leq \Big\|u(t)\int_{t^{\frac{1}{\nu}}}^{\infty} \big(\chi_{(0,b]}(s) + \sum_k \chi_{(a_k,b_k)}(s)\big)v(s)\,\mathrm{d}s\Big\|_{X(0,\infty)}\\ &\leq \Big\|u(t)\chi_{(0,b^\nu)}(t)\int_{t^{\frac{1}{\nu}}}^{b} v(s)\,\mathrm{d}s\Big\|_{X(0,\infty)}\\ &\quad+ \sum_k \Big\|u(t)\chi_{(0,a_k^\nu)}(t)\int_{a_k}^{b_k} v(s)\,\mathrm{d}s\Big\|_{X(0,\infty)}\\ &\quad+ \sum_k \Big\|u(t)\chi_{(a_k^\nu,b_k^\nu)}(t)\int_{t^{\frac{1}{\nu}}}^{b_k} v(s)\,\mathrm{d}s\Big\|_{X(0,\infty)}.\end{aligned} \tag{2.11}$$

Note that (2.6) together with the monotonicity of u and v implies that

$$\|u\chi_{(0,a)}\|_{X(0,\infty)} < \infty \quad \text{for every } a \in (0,\infty).$$

Since u is nonincreasing, it is sufficient to show that $\|u\chi_{(0,\frac{1}{2})}\|_{X(0,\infty)} < \infty$, which follows from

$$\begin{aligned}\infty > \Big\|u(t)\chi_{(0,1)}(t)\int_{t^{\frac{1}{\nu}}}^{1} v(s)\,\mathrm{d}s\Big\|_{X(0,\infty)} &\geq \Big\|u(t)\chi_{(0,\frac{1}{2})}(t)\int_{t^{\frac{1}{\nu}}}^{1} v(s)\,\mathrm{d}s\Big\|_{X(0,\infty)}\\ &\geq v(1)(1-2^{-\frac{1}{\nu}})\Big\|u\chi_{(0,\frac{1}{2})}\Big\|_{X(0,\infty)}.\end{aligned}$$

Now, as for the first term on the right-hand side of (2.11), we have that

$$\begin{aligned}\Big\|u(t)\chi_{(0,b^\nu)}(t)\int_{t^{\frac{1}{\nu}}}^{b} v(s)\,\mathrm{d}s\Big\|_{X(0,\infty)} &\leq \Big\|u(t)\chi_{(0,1)}(t)\int_{t^{\frac{1}{\nu}}}^{1} v(s)\,\mathrm{d}s\Big\|_{X(0,\infty)}\\ &\quad+\Big\|u(t)\chi_{(0,1)}(t)\int_{1}^{b} v(s)\,\mathrm{d}s\Big\|_{X(0,\infty)}\\ &\quad+\Big\|u(t)\chi_{(1,b^\nu)}(t)\int_{t^{\frac{1}{\nu}}}^{b} v(s)\,\mathrm{d}s\Big\|_{X(0,\infty)}\\ &\leq A < \infty,\end{aligned}\tag{2.12}$$

where

$$\begin{aligned}A = \Big\|u(t)\chi_{(0,1)}(t)\int_{t^{\frac{1}{\nu}}}^{1} v(s)\,\mathrm{d}s\Big\|_{X(0,\infty)} &+ v(1)(b-1)\|u\chi_{(0,1)}\|_{X(0,\infty)}\\ + v(1)(b-1)\|u\chi_{(0,b^\nu-1)}\|_{X(0,\infty)}&.\end{aligned}$$

As for the second term on the right-hand side of (2.11), we have that

$$\begin{aligned}\Big\|u(t)\chi_{(0,a_k^\nu)}(t)\int_{a_k}^{b_k} v(s)\,\mathrm{d}s\Big\|_{X(0,\infty)} &\leq v(a_k)(b_k-a_k)\|u\chi_{(0,a_k^\nu)}\|_{X(0,\infty)}\\ &\leq B(b_k-a_k),\end{aligned}\tag{2.13}$$

where B is the supremum in (2.7). In particular, B is independent of k. Next,

$$\begin{aligned}\Big\|u(t)\chi_{(a_k^\nu,b_k^\nu)}(t)\int_{t^{\frac{1}{\nu}}}^{b_k} v(s)\,\mathrm{d}s\Big\|_{X(0,\infty)} &\leq \int_{a_k}^{b_k} v(s)\,\mathrm{d}s\|u\chi_{(a_k^\nu,b_k^\nu)}\|_{X(0,\infty)}\\ &\leq v(a_k)(b_k-a_k)\|u\chi_{(0,b_k^\nu-a_k^\nu)}\|_{X(0,\infty)}\\ &\leq v(a_k)(b_k-a_k)\|u\chi_{(0,a_k^\nu)}\|_{X(0,\infty)}\\ &\leq B(b_k-a_k),\end{aligned}\tag{2.14}$$

where we used the monotonicity of u and v in the second inequality and (2.10) in the third one. By combining (2.11) with (2.12), (2.13) and (2.14), we obtain that

$$\Big\|u(t)\int_{t^{\frac{1}{\nu}}}^{\infty} h(s)v(s)\,\mathrm{d}s\Big\|_{X(0,\infty)} \leq A + 2B\sum_k (b_k-a_k) \leq A + 4B|E| < \infty.\tag{2.15}$$

Hence $\varrho(\chi_E) < \infty$ provided that (2.6) and (2.7) are satisfied. We now turn our attention to the necessity of (2.6) and (2.7). The necessity of (2.6) is obvious because we have that

$$\Big\|u(t)\chi_{(0,1)}(t)\int_{t^{\frac{1}{\nu}}}^{1} v(s)\,\mathrm{d}s\Big\|_{X(0,\infty)} \leq \varrho(\chi_{(0,1)}).$$

As for the necessity of (2.7), suppose that $\sup\limits_{b\in[1,\infty)} v(b)\big\|u\chi_{(0,b^\nu)}\big\|_{X(0,\infty)} = \infty$. It follows that there is a sequence $b_k \nearrow \infty$, $k\to\infty$, such that

$$\lim_{k\to\infty} v(b_k)\|u\chi_{(0,b_k^\nu)}\|_{X(0,\infty)} = \infty.$$

We may clearly assume that $b_k \geq \frac{2^{\frac{1}{\nu}}}{2^{\frac{1}{\nu}}-1}$; hence $b_k^\nu - (b_k-1)^\nu \leq (b_k-1)^\nu$. Using the fact that u is nonincreasing, we obtain that

$$\begin{aligned}\|u\chi_{(0,b_k^\nu)}\|_{X(0,\infty)} &\leq \|u\chi_{(0,(b_k-1)^\nu)}\|_{X(0,\infty)} + \|u\chi_{((b_k-1)^\nu,b_k^\nu)}\|_{X(0,\infty)}\\ &\leq \|u\chi_{(0,(b_k-1)^\nu)}\|_{X(0,\infty)} + \|u\chi_{(0,b_k^\nu-(b_k-1)^\nu)}\|_{X(0,\infty)}\\ &\leq \|u\chi_{(0,(b_k-1)^\nu)}\|_{X(0,\infty)} + \|u\chi_{(0,(b_k-1)^\nu)}\|_{X(0,\infty)}\\ &= 2\|u\chi_{(0,(b_k-1)^\nu)}\|_{X(0,\infty)}.\end{aligned}$$

Therefore,

$$\begin{aligned}\varrho(\chi_{(0,1)}) &\geq \left\|u(t)\chi_{(0,(b_k-1)^\nu)}(t)\int_{t^{\frac{1}{\nu}}}^\infty \chi_{(b_k-1,b_k)}(s)v(s)\,\mathrm{d}s\right\|_{X(0,\infty)}\\ &\geq v(b_k)\|u\chi_{(0,(b_k-1)^\nu)}\|_{X(0,\infty)} \geq \frac{1}{2}v(b_k)\|u\chi_{(0,b_k^\nu)}\|_{X(0,\infty)},\end{aligned}$$

which tends to ∞ as $k\to\infty$. Hence $\varrho(\chi_{(0,1)}) = \infty$, and so ϱ does not have property (P4).

Property (P5). Assume that $L=1$. Note that (2.5) together with $v(1^-)>0$ implies that $\|u\|_{X(0,1)}<\infty$. Let $f\in\mathfrak{M}^+(0,1)$. Since f^* is nonincreasing, we have that $\int_0^1 f^*(s)\,\mathrm{d}s \leq 2\int_0^{\frac{1}{2}} f^*(s)\,\mathrm{d}s$. Since the function $(0,1)\ni t\mapsto f^*(1-t)$ is equimeasurable with f, we have that

$$\begin{aligned}\varrho(f) &\geq \left\|u(t)\int_{t^{\frac{1}{\nu}}}^1 f^*(1-s)v(s)\,\mathrm{d}s\right\|_{X(0,1)}\\ &\geq v(1^-)\left\|u(t)\chi_{(0,2^{-\nu})}(t)\int_{t^{\frac{1}{\nu}}}^1 f^*(1-s)\,\mathrm{d}s\right\|_{X(0,1)}\\ &= v(1^-)\left\|u(t)\chi_{(0,2^{-\nu})}(t)\int_0^{1-t^{\frac{1}{\nu}}} f^*(s)\,\mathrm{d}s\right\|_{X(0,1)}\\ &\geq v(1^-)\|u\chi_{(0,2^{-\nu})}\|_{X(0,1)}\int_0^{\frac{1}{2}} f^*(s)\,\mathrm{d}s\\ &\geq \frac{v(1^-)}{2}\|u\chi_{(0,2^{-\nu})}\|_{X(0,1)}\int_0^1 f^*(s)\,\mathrm{d}s\\ &\geq \frac{v(1^-)}{2}\|u\chi_{(0,2^{-\nu})}\|_{X(0,1)}\int_0^1 f(s)\,\mathrm{d}s,\end{aligned}$$

where we used (1.4) in the last inequality. Since $\frac{v(1^-)}{2}\|u\chi_{(0,2^{-\nu})}\|_{X(0,1)}\in(0,\infty)$ does not depend on f, property (P5) follows. Assume now that $L=\infty$. As was shown in the proof of property (P4), $\|u\chi_{(0,a)}\|_{X(0,\infty)}<\infty$ for every $a\in(0,\infty)$ provided that (2.6) is satisfied. Let $f\in\mathfrak{M}^+(0,\infty)$ and $E\subseteq(0,\infty)$ be of finite measure. The function $(0,\infty)\ni t\mapsto f^*(t-|E|)\chi_{(|E|,\infty)}(t)$ is equimeasurable with f. By arguing similarly to the case $L=1$, we obtain that

$$\varrho(f) \geq v(2|E|)\|u\chi_{(0,|E|^\nu)}\|_{X(0,\infty)}\int_E f(s)\,\mathrm{d}s, \tag{2.16}$$

whence property (P5) follows.

Property (P6). Since the relation $\sim$ is transitive, it plainly follows that ϱ has property (P6).

We conclude the proof by noting that the necessity of (2.5) or of (2.6) and (2.7) if $L=1$ or $L=\infty$, respectively, was proved in the paragraph devoted to property (P4). $\square$

The function norm defined by (2.4) is quite complicated. The next proposition tells us that the function norm can be considerably simplified provided that a certain supremum operator is bounded on the associate space of $X(0,L)$. We define the operator T_φ, where $\varphi\colon (0,L)\to(0,\infty)$ is a fixed function, as

$$T_\varphi f(t)=\frac{1}{\varphi(t)}\sup_{s\in[t,L)}\varphi(s)f^*(s),\ t\in(0,L),\ f\in\mathfrak{M}(0,L). \tag{2.17}$$

Proposition 2.1.3. *Let $\|\cdot\|_{X(0,L)}$ be a rearrangement-invariant function norm. Let $\nu\in(0,\infty)$. Let $u\colon(0,L)\to(0,\infty)$ be nonincreasing. Let $v\colon(0,L)\to(0,\infty)$ be a nonincreasing function such that $\frac{1}{v(t)}=\int_0^{t^\nu}\xi(s)\,\mathrm{d}s$ for every $t\in(0,L)$, where $\xi\colon(0,L)\to(0,\infty)$ is a continuous function. If $L=1$, we assume that $v(1^-)>0$. Assume that*

$$\left\|u(t)\chi_{(0,1)}(t)\int_{t^{\frac{1}{\nu}}}^1 v(s)\,\mathrm{d}s\right\|_{X(0,L)}<\infty$$

and that the operator T_φ, defined by (2.17)*, is bounded on $X'(0,L)$, where $\varphi=\frac{u}{\xi}$. Let ϱ be the functional defined by* (2.4) *and set*

$$\widetilde{\varrho}(f)=\sup_{\substack{g\in\mathfrak{M}^+(0,L)\\ \|g\|_{X'(0,L)}\le 1}}\int_0^L f^*(s)v(t)\int_0^{t^\nu}T_\varphi g(s)u(s)\,\mathrm{d}s\,\mathrm{d}t,\ f\in\mathfrak{M}^+(0,L).$$

The functionals ϱ and $\widetilde{\varrho}$ are rearrangement-invariant function norms. Furthermore, we have that

$$\begin{aligned}\|H_{u,v,\nu}(f^*)\|_{X(0,L)}\le\sup_{h\sim f}\left\|u(t)\int_{t^{\frac{1}{\nu}}}^L h(s)v(s)\,\mathrm{d}s\right\|_{X(0,L)}&\le\widetilde{\varrho}(f)\\ &\le\|T_\varphi\|_{X'(0,L)}\|H_{u,v,\nu}(f^*)\|_{X(0,L)}\end{aligned} \tag{2.18}$$

for every $f\in\mathfrak{M}^+(0,L)$, where $\|T_\varphi\|_{X'(0,L)}$ stands for the operator norm of T_φ on $X'(0,L)$. In particular, the rearrangement-invariant function norms ϱ and $\widetilde{\varrho}$ are equivalent.

Proof. Since $f\sim f^*$ for every $f\in\mathfrak{M}^+(0,L)$, the first inequality in (2.18) plainly holds. As for the second inequality, note that the function $(0,L)\ni t\mapsto R_{u,v,\nu}(T_\varphi g)(t)$ is nonincreasing for every $g\in\mathfrak{M}^+(0,L)$ because it is the integral mean of the nonincreasing function $(0,L)\ni s\mapsto\sup_{\tau\in[s,L)}\varphi(\tau)g^*(\tau)$ over the interval $(0,t^\nu)$ with respect to the measure $\xi(s)\,\mathrm{d}s$. Consequently, for every

$f \in \mathfrak{M}^+(0,L)$ and every $h \in \mathfrak{M}^+(0,L)$ equimeasurable with f, we have that

$$\begin{aligned}
\|H_{u,v,\nu}h\|_{X(0,L)} &= \sup_{\substack{g\in\mathfrak{M}^+(0,L)\\ \|g\|_{X'(0,L)}\leq 1}} \int_0^L h(t)R_{u,v,\nu}(g^*)(t)\,\mathrm{d}t\\
&\leq \sup_{\substack{g\in\mathfrak{M}^+(0,L)\\ \|g\|_{X'(0,L)}\leq 1}} \int_0^L h(t)R_{u,v,\nu}(T_\varphi g)(t)\,\mathrm{d}t\\
&\leq \sup_{\substack{g\in\mathfrak{M}^+(0,L)\\ \|g\|_{X'(0,L)}\leq 1}} \int_0^L h^*(t)R_{u,v,\nu}(T_\varphi g)(t)\,\mathrm{d}t\\
&= \sup_{\substack{g\in\mathfrak{M}^+(0,L)\\ \|g\|_{X'(0,L)}\leq 1}} \int_0^L f^*(t)R_{u,v,\nu}(T_\varphi g)(t)\\
&= \widetilde{\varrho}(f),
\end{aligned}$$

where we used (1.10) (note that the function $H_{u,v,\nu}h$ is nonincreasing for every $h \in \mathfrak{M}^+(0,L)$) together with (2.3) in the first equality, the pointwise estimate $g^* \leq T_\varphi g$ in the first inequality, the Hardy–Littlewood inequality (1.3) in the second inequality, and the equimeasurability of f and h in the last inequality. Hence the second inequality in (2.18) follows. As for the third inequality in (2.18), we have that

$$\begin{aligned}
\sup_{\substack{g\in\mathfrak{M}^+(0,L)\\ \|g\|_{X'(0,L)}\leq 1}} \int_0^L f^*(t)R_{u,v,\nu}(T_\varphi g)(t)\,\mathrm{d}t &= \sup_{\substack{g\in\mathfrak{M}^+(0,L)\\ \|g\|_{X'(0,L)}\leq 1}} \int_0^L T_\varphi g(t)H_{u,v,\nu}(f^*)(t)\,\mathrm{d}t\\
&\leq \|H_{u,v,\nu}(f^*)\|_{X(0,L)} \sup_{\substack{g\in\mathfrak{M}^+(0,L)\\ \|g\|_{X'(0,L)}\leq 1}} \|T_\varphi g\|_{X'(0,L)}\\
&= \|T_\varphi\|_{X'(0,L)}\|H_{u,v,\nu}(f^*)\|_{X(0,L)}
\end{aligned}$$

for every $f \in \mathfrak{M}^+(0,L)$ thanks to (2.3) and the Hölder inequality (1.14).

Second, we shall prove that the functional ϱ, defined by (2.4), is a rearrangement-invariant function norm. If $L = 1$, this follows immediately from Proposition 2.1.2. If $L = \infty$, owing to Proposition 2.1.2 again, we only need to verify that (2.7) is satisfied. It follows from its proof that, if (2.7) did not hold, then we would have

$$\sup_{h\sim\chi_{(0,1)}} \left\| u(t)\int_{t^{\frac{1}{\nu}}}^\infty h(s)v(s)\,\mathrm{d}s \right\|_{X(0,\infty)} = \infty.$$

However, thanks to (2.18), we have that

$$\begin{aligned}
\sup_{h\sim\chi_{(0,1)}} \left\| u(t)\int_{t^{\frac{1}{\nu}}}^L h(s)v(s)\,\mathrm{d}s \right\|_{X(0,\infty)} &\approx \|H_{u,v,\nu}\chi_{(0,1)}\|_{X(0,L)}\\
&= \left\| u(t)\chi_{(0,1)}(t)\int_{t^{\frac{1}{\nu}}}^1 v(s)\,\mathrm{d}s \right\|_{X(0,\infty)} < \infty.
\end{aligned}$$

Therefore, (2.7) is satisfied.

Finally, now that we know that the functionals ϱ and $\widetilde{\varrho}$ are equivalent and the former is a rearrangement-invariant function norm, it readily follows that $\widetilde{\varrho}$, too, is a rearangement-invariant function norm once we observe that $\widetilde{\varrho}$ is subadditive. The subadditivity follows from

$$\begin{aligned}
\widetilde{\varrho}(f+g) &= \sup_{\substack{h\in\mathfrak{M}^+(0,L)\\ \|h\|_{X'(0,L)}\leq 1}} \int_0^L (f+g)^*(t) R_{u,v,\nu}(T_\varphi h)(t)\,\mathrm{d}t \\
&\leq \sup_{\substack{h\in\mathfrak{M}^+(0,L)\\ \|h\|_{X'(0,L)}\leq 1}} \int_0^L f^*(t) R_{u,v,\nu}(T_\varphi h)(t)\,\mathrm{d}t \\
&\quad + \sup_{\substack{h\in\mathfrak{M}^+(0,L)\\ \|h\|_{X'(0,L)}\leq 1}} \int_0^L g^*(t) R_{u,v,\nu}(T_\varphi h)(t)\,\mathrm{d}t \\
&= \widetilde{\varrho}(f) + \widetilde{\varrho}(g) \quad \text{for every } f,g\in\mathfrak{M}^+(0,L),
\end{aligned}$$

where we used (1.2) together with the Hardy lemma (1.5) (recall that the function $R_{u,v,\nu}(T_\varphi h)$ is nonincreasing for every $h\in\mathfrak{M}^+(0,L)$) together with (1.2). ☐

2.1.2 Calderón-type operators

Throughout this subsection, we assume that $L=\infty$.

Let $\sigma=(\alpha,\beta,\gamma,\delta)\in[0,1]^4$ be such that $0\leq\gamma<\alpha\leq 1$ and $0\leq\delta<\beta\leq 1$. We define the operator S_σ as

$$S_\sigma f(t) = \int_0^\infty |f(s)| \min\left\{\frac{s^\alpha}{t^\beta}, \frac{s^\gamma}{t^\delta}\right\} \frac{\mathrm{d}s}{s},\ t\in(0,\infty),\ f\in\mathfrak{M}(0,\infty). \tag{2.19}$$

The operator S_σ is sometimes called the *Calderón operator* corresponding to the interpolation segment $[(\alpha,\beta),(\gamma,\delta)]$ and is closely related to the interpolation of operators of (joint-)weak type (e.g. [6, Chapter IV], [8, Chapter 3, Section 5], [15]).

The function $S_\sigma f$ is plainly nonincreasing on $(0,\infty)$ for every $f\in\mathfrak{M}(0,\infty)$. Furthermore, since the function $(0,\infty)\ni s\mapsto \min\left\{\frac{s^{\alpha-1}}{t^\beta}, \frac{s^{\gamma-1}}{t^\delta}\right\}$ is nonincreasing for every $t\in(0,\infty)$, the Hardy–Littlewood inequality (1.3) implies that

$$S_\sigma f \leq S_\sigma(f^*) \quad \text{for every } f\in\mathfrak{M}(0,\infty). \tag{2.20}$$

Note that, for every $t\in(0,\infty)$, $f\in\mathfrak{M}(0,\infty)$,

$$S_\sigma f(t) = t^{-\beta}\int_0^{t^m} |f(s)| s^{\alpha-1}\,\mathrm{d}s + t^{-\delta}\int_{t^m}^\infty |f(s)| s^{\gamma-1}\,\mathrm{d}s, \tag{2.21}$$

where

$$m = \frac{\beta-\delta}{\alpha-\gamma}. \tag{2.22}$$

Furthermore, by Fubini's theorem,

$$S_\sigma f(t) = (\alpha-\gamma) t^{-\delta}\int_{t^m}^\infty \int_0^s |f(\tau)|\tau^{\alpha-1}\,\mathrm{d}\tau\, s^{\gamma-\alpha-1}\,\mathrm{d}s \tag{2.23}$$

for every $t \in (0,\infty)$ and $f \in \mathfrak{M}(0,\infty)$. If σ is as above, we define σ' as

$$\sigma' = (1-\delta, 1-\gamma, 1-\beta, 1-\alpha). \tag{2.24}$$

We have that $(\sigma')' = \sigma$ and, by Fubini's theorem,

$$\int_0^\infty S_\sigma f(t) g(t)\,\mathrm{d}t = \int_0^\infty f(t) S_{\sigma'} g(t)\,\mathrm{d}t \quad \text{for every } f,g \in \mathfrak{M}^+(0,\infty). \tag{2.25}$$

The next proposition characterizes when a certain functional induced by the operator S_σ is a rearrangement-invariant function norm.

Proposition 2.1.4. *Let* $\|\cdot\|_{X(0,\infty)}$ *be a rearrangement-invariant function norm. Let* $\sigma = (\alpha,\beta,\gamma,\delta) \in [0,1]^4$ *be such that* $0 \le \gamma < \alpha \le 1$ *and* $0 \le \delta < \beta \le 1$. *Set*

$$\varrho(f) = \|S_\sigma(f^*)\|_{X(0,\infty)},\ f \in \mathfrak{M}^+(0,\infty).$$

Set

$$\xi(t) = \begin{cases} t^{-\delta}\log(\frac{e}{t})\chi_{(0,1)}(t) + t^{-\beta}\chi_{(1,\infty)}(t) & \text{if } \gamma = 0,\\ t^{-\delta}\chi_{(0,1)}(t) + t^{-\beta}\chi_{(1,\infty)}(t) & \text{if } \gamma > 0.\end{cases} \tag{2.26}$$

The functional ϱ *is a rearrangement-invariant function norm if and only if* $\xi \in X(0,\infty)$.

Proof. Property (P1). The positive homogeneity and positive definiteness of ϱ can be readily verified. Since the function $(0,\infty) \ni s \mapsto \min\left\{\frac{s^{\alpha-1}}{t^\beta}, \frac{s^{\gamma-1}}{t^\delta}\right\}$ is nonincreasing for every $t \in (0,\infty)$, it follows from (1.2) and Hardy's lemma (1.5) that

$$S_\sigma((f+g)^*)(t) \le S_\sigma(f^*)(t) + S_\sigma(g^*)(t) \quad \text{for every } t \in (0,\infty),\ f,g \in \mathfrak{M}^+(0,\infty),$$

whence the subadditivity of ϱ follows.

Properties (P2) *and* (P3). These properties can be readily verified.

Property (P4). Let $E \subseteq (0,\infty)$ be a measurable set of finite measure. Thanks to the boundedness of the dilation operator (1.13) and (2.19), it is easy to see that $\|S_\sigma(\chi_E^*)\|_{X(0,\infty)} < \infty$ if and only if $\|S_\sigma\chi_{(0,1)}\|_{X(0,\infty)} < \infty$. It follows from (2.21) that

$$\begin{aligned}\|S_\sigma\chi_{(0,1)}\|_{X(0,\infty)} \approx \|t^{m\alpha-\beta}\chi_{(0,1)}(t)\|_{X(0,\infty)} + \left\|t^{-\delta}\chi_{(0,1)}(t)\int_{t^m}^1 s^{\gamma-1}\,\mathrm{d}s\right\|_{X(0,\infty)}\\ + \|t^{-\beta}\chi_{(1,\infty)}(t)\|_{X(0,\infty)}.\end{aligned} \tag{2.27}$$

Assume that $\gamma = 0$. Note that $m\alpha - \beta = -\delta$. Since

$$\lim_{t\to0^+} \frac{t^{-\delta}\int_{t^m}^1 s^{-1}\,\mathrm{d}s}{t^{-\delta}\log(\frac{e}{t})} \in (0,\infty),$$

and

$$\lim_{t\to0^+} \frac{t^{m\alpha-\beta}}{t^{-\delta}\log(\frac{e}{t})} = 0,$$

it follows that

$$\|t^{m\alpha-\beta}\chi_{(0,1)}(t)\|_{X(0,\infty)}+\left\|t^{-\delta}\chi_{(0,1)}(t)\int_{t^m}^1 s^{-1}\,\mathrm{d}s\right\|_{X(0,\infty)}\approx\left\|t^{-\delta}\chi_{(0,1)}(t)\log\left(\frac{e}{t}\right)\right\|_{X(0,\infty)}.$$

Assume now that $\gamma>0$. Note that $m\alpha-\beta>-\delta$. Since

$$\lim_{t\to 0^+}\frac{t^{-\delta}\int_{t^m}^1 s^{\gamma-1}\,\mathrm{d}s}{t^{-\delta}}\in(0,\infty),$$

and

$$\lim_{t\to 0^+}\frac{t^{m\alpha-\beta}}{t^{-\delta}}=0,$$

it follows that

$$\|t^{m\alpha-\beta}\chi_{(0,1)}(t)\|_{X(0,\infty)}+\left\|t^{-\delta}\chi_{(0,1)}(t)\int_{t^m}^1 s^{\gamma-1}\,\mathrm{d}s\right\|_{X(0,\infty)}\approx\|t^{-\delta}\chi_{(0,1)}(t)\|_{X(0,\infty)}.$$

Hence

$$\|t^{m\alpha-\beta}\chi_{(0,1)}(t)\|_{X(0,\infty)}+\left\|t^{-\delta}\chi_{(0,1)}(t)\int_{t^m}^1 s^{\gamma-1}\,\mathrm{d}s\right\|_{X(0,\infty)}\approx\|\xi\|_{X(0,\infty)} \tag{2.28}$$

whether $\gamma=0$ or $\gamma\neq 0$. By combining (2.27) and (2.28), we obtain that ϱ has property (P4) if and only if $\xi\in X(0,\infty)$.

Property (P5). Let $E\subseteq(0,\infty)$ be a measurable set of finite measure. Since the function $(0,\infty)\ni t\mapsto S_\sigma(f^*)(t)$ is nonincreasing for every $f\in\mathfrak{M}^+(0,\infty)$, we have that

$$\begin{aligned}\|S_\sigma(f^*)\|_{X(0,\infty)}&\geq\|S_\sigma(f^*)(t)\chi_{(0,|E|)}(t)\|_{X(0,\infty)}\geq S_\sigma(f^*)(|E|)\|\chi_{(0,|E|)}\|_{X(0,\infty)}\\&\geq\|\chi_{(0,|E|)}\|_{X(0,\infty)}\int_0^{|E|}f^*(s)\min\left\{\frac{s^\alpha}{|E|^\beta},\frac{s^\gamma}{|E|^\delta}\right\}\frac{\mathrm{d}s}{s}\\&\geq\min\left\{\frac{|E|^{\alpha-1}}{|E|^\beta},\frac{|E|^{\gamma-1}}{|E|^\delta}\right\}\|\chi_{(0,|E|)}\|_{X(0,\infty)}\int_0^{|E|}f^*(s)\,\mathrm{d}s\\&\geq\min\left\{\frac{|E|^{\alpha-1}}{|E|^\beta},\frac{|E|^{\gamma-1}}{|E|^\delta}\right\}\|\chi_{(0,|E|)}\|_{X(0,\infty)}\int_E f(s)\,\mathrm{d}s\end{aligned}$$

for every $f\in\mathfrak{M}^+(0,\infty)$, where we used (1.4) in the last inequality. Hence ϱ has property (P5).

Property (P6). Since $f^*=g^*$ when $f,g\in\mathfrak{M}^+(0,\infty)$ are equimeasurable, ϱ plainly has property (P6). □

2.2 Optimal function norms

In this section, we shall investigate optimal mapping properties of the operators $H_{u,v,\nu}$, $R_{u,v,\nu}$ and S_σ, introduced in the preceding section. Let T be one of those operators. We say that a rearrangement-invariant function space $Y(0,L)$ is *the optimal target space* for the operator T and a rearrangement-invariant function space $X(0,L)$ if $T\colon X(0,L)\to Y(0,L)$ is bounded and $Y(0,L)\hookrightarrow Z(0,L)$ whenever $Z(0,L)$ is a rearrangement-invariant function space such that

$T\colon X(0,L) \to Z(0,L)$ is bounded (in other words, $\|\cdot\|_{Y(0,L)}$ is the strongest target rearrangement-invariant function norm for T and $\|\cdot\|_{X(0,L)}$). We say that a rearrangement-invariant function space $X(0,L)$ is *the optimal domain space* for the operator T and a rearrangement-invariant function space $Y(0,L)$ if $T\colon X(0,L) \to Y(0,L)$ is bounded and $Z(0,L) \hookrightarrow X(0,L)$ whenever $Z(0,L)$ is a rearrangement-invariant function space such that $T\colon Z(0,L) \to Y(0,L)$ is bounded (in other words, $\|\cdot\|_{X(0,L)}$ is the weakest domain rearrangement-invariant function norm for T and $\|\cdot\|_{Y(0,L)}$).

2.2.1 Hardy-type operators and supremum operators

In this subsection, we study the operators $H_{u,v,\nu}$, $R_{u,v,\nu}$ and the supremum operator T_φ, which is closely related to them. First, we describe the optimal domain space for the operator $R_{u,v,\nu}$ and a given rearrangement-invariant function space.

Proposition 2.2.1. *Let $\|\cdot\|_{Y(0,L)}$ be a rearrangement-invariant function norm. Let $u\colon (0,L) \to (0,\infty)$ be a nonincreasing function such that $0 < U(t) < \infty$ for every $t \in (0,L)$, where $U(t) = \int_0^t u(s)\,\mathrm{d}s$. If $L = 1$, we assume that $u(1^-) > 0$. Let $v\colon (0,L) \to (0,\infty)$ be measurable. Let $\nu \in (0,\infty)$. Assume that $\xi \in Y(0,L)$, where ξ is defined as*

$$\xi(t) = \begin{cases} v(t)U(t^\nu),\ t \in (0,1), & \text{if } L = 1,\\ v(t)U(t^\nu)\chi_{(0,1)}(t) + v(t)\chi_{(1,\infty)}(t),\ t \in (0,\infty), & \text{if } L = \infty. \end{cases} \tag{2.29}$$

Let $\|\cdot\|_{X(0,L)}$ be the function norm defined as

$$\|f\|_{X(0,L)} = \left\| v(t)\int_0^{t^\nu} f^*(s)u(s)\,\mathrm{d}s \right\|_{Y(0,L)},\quad f \in \mathfrak{M}^+(0,L). \tag{2.30}$$

The rearrangement-invariant function space $X(0,L)$ is the optimal domain space for the operator $R_{u,v,\nu}$ and the space $Y(0,L)$.

Moreover, if $\xi \notin Y(0,L)$, then there is no rearrangement-invariant function space $Z(0,L)$ such that $R_{u,v,\nu}\colon Z(0,L) \to Y(0,L)$ is bounded.

Proof. First, note that the functional defined by (2.30) is indeed a rearrangement-invariant function norm thanks to Proposition 2.1.1, and so we are entitled to denote the corresponding rearrangement-invariant function space by $X(0,L)$. Second, thanks to the Hardy–Littlewood inequality (1.3) and the monotonicity of u, we have that

$$\|R_{u,v,\nu}f\|_{Y(0,L)} \le \|R_{u,v,\nu}(f^*)\|_{Y(0,L)} = \|f\|_{X(0,L)} \quad \text{for every } f \in \mathfrak{M}^+(0,L).$$

Hence $R_{u,v,\nu}\colon X(0,L) \to Y(0,L)$ is bounded. Next, if $Z(0,L)$ is a rearrangement-invariant function space such that $R_{u,v,\nu}\colon Z(0,L) \to Y(0,L)$ is bounded, then we have that

$$\|f\|_{X(0,L)} = \|R_{u,v,\nu}(f^*)\|_{Y(0,L)} \lesssim \|f^*\|_{Z(0,L)} = \|f\|_{Z(0,L)} \quad \text{for every } f \in \mathfrak{M}^+(0,L),$$

and so $Z(0,L) \hookrightarrow X(0,L)$. Finally, note that, if $R_{u,v,\nu}\colon Z(0,L) \to Y(0,L)$ is bounded, then

$$\|\xi\|_{Y(0,L)} \approx \|R_{u,v,\nu}(\chi_{(0,1)})\|_{Y(0,L)} \lesssim \|\chi_{(0,1)}\|_{X(0,L)} < \infty.$$

□

The following unsurprising proposition, which we state for future reference, reflects the fact that the operators $R_{u,v,\nu}$ and $H_{u,v,\nu}$ are in a sense dual to each other (see (2.3)).

Proposition 2.2.2. *Let $\|\cdot\|_{X(0,L)}$, $\|\cdot\|_{Y(0,L)}$ be rearrangement-invariant function norms. Let $u,v\colon (0,L)\to(0,\infty)$ be measurable. Let $\nu\in(0,\infty)$. We have that*

$$\sup_{\|f\|_{X(0,L)}\leq 1}\|R_{u,v,\nu}f\|_{Y(0,L)}=\sup_{\|g\|_{Y'(0,L)}\leq 1}\|H_{u,v,\nu}g\|_{X'(0,L)}. \tag{2.31}$$

In particular,

$$\begin{aligned} R_{u,v,\nu}\colon X(0,L)\to Y(0,L) &\quad \textit{is bounded if and only if}\\ H_{u,v,\nu}\colon Y'(0,L)\to X'(0,L) &\quad \textit{is bounded.}\end{aligned} \tag{2.32}$$

Proof. We have that

$$\begin{aligned}\sup_{\|f\|_{X(0,L)}\leq 1}\|R_{u,v,\nu}f\|_{Y(0,L)} &= \sup_{\|f\|_{X(0,L)}\leq 1}\sup_{\|g\|_{Y'(0,L)}\leq 1}\int_0^L R_{u,v,\nu}f(t)|g(t)|\,\mathrm{d}t\\ &= \sup_{\|f\|_{X(0,L)}\leq 1}\sup_{\|g\|_{Y'(0,L)}\leq 1}\int_0^L |f(t)|H_{u,v,\nu}g(t)\,\mathrm{d}t\\ &= \sup_{\|g\|_{Y'(0,L)}\leq 1}\|H_{u,v,\nu}g\|_{X'(0,L)}\end{aligned}$$

thanks to (1.8), (2.3) and (1.7). □

Next, we describe the optimal target space for the operator $R_{u,v,\nu}$ and a given rearrangement-invariant function space.

Proposition 2.2.3. *Let $\|\cdot\|_{X(0,L)}$ be a rearrangement-invariant function norm. Let $u,v\colon (0,L)\to(0,\infty)$ be nonincreasing. If $L=1$, we assume that $v(1^-)>0$. Assume that*

(2.33)

$$\begin{cases} u(t)\int_{t^{\frac{1}{\nu}}}^1 v(s)\,\mathrm{d}s\in X'(0,1) & \textit{if } L=1,\\ u(t)\chi_{(0,1)}(t)\int_{t^{\frac{1}{\nu}}}^1 v(s)\,\mathrm{d}s\in X'(0,\infty) \textit{ and } \sup\limits_{b\in[1,\infty)} v(b^{\frac{1}{\nu}})\|u\chi_{(0,b)}\|_{X'(0,\infty)}<\infty & \textit{if } L=\infty.\end{cases}$$

Let $\|\cdot\|_{Y(0,L)}$ be the rearrangement-invariant function norm whose associate function norm $\|\cdot\|_{Y'(0,L)}$ is defined as

$$\|f\|_{Y'(0,L)}=\sup_{h\sim f}\left\|u(t)\int_{t^{\frac{1}{\nu}}}^L h(s)v(s)\,\mathrm{d}s\right\|_{X'(0,L)},\quad f\in\mathfrak{M}^+(0,L), \tag{2.34}$$

where the supremum is taken over all $h\in\mathfrak{M}^+(0,L)$ equimeasurable with f. The rearrangement-invariant function space $Y(0,L)$ is the optimal target space for the operator $R_{u,v,\nu}$ and the space $X(0,L)$.

Moreover, if (2.33) is not satisfied, then there is no rearrangement-invariant function space $Z(0,L)$ such that $R_{u,v,\nu}\colon X(0,L)\to Z(0,L)$ is bounded.

Proof. First, note that the functional defined by (2.34) is indeed a rearrangement-invariant function norm thanks to Proposition 2.1.2. Therefore, owing to (1.9), we are entitled to denote by $\|\cdot\|_{Y(0,L)}$ the rearrangement-invariant function norm whose associate function norm is defined by (2.34). Second, since we plainly have that

$$\|H_{u,v,\nu}f\|_{X'(0,L)} \leq \|f\|_{Y'(0,L)} \quad \text{for every } f \in \mathfrak{M}^+(0,L),$$

it follows that $R_{u,v,\nu}\colon X(0,L) \to Y(0,L)$ is bounded thanks to (2.32). Next, let $Z(0,L)$ be a rearrangement-invariant function space such that $R_{u,v,\nu}\colon X(0,L) \to Z(0,L)$ is bounded. Owing to (2.32) again, this is equivalent to the fact that $H_{u,v,\nu}\colon Z'(0,L) \to X'(0,L)$ is bounded. For every $f \in \mathfrak{M}^+(0,\infty)$ and each $h \in \mathfrak{M}^+(0,L)$ equimeasurable with f, we have that

$$\|H_{u,v,\nu}h\|_{X'(0,L)} \lesssim \|h\|_{Z'(0,L)} = \|f\|_{Z'(0,L)}.$$

Therefore,

$$\|f\|_{Y'(0,L)} \lesssim \|f\|_{Z'(0,L)} \quad \text{for every } f \in \mathfrak{M}^+(0,L).$$

Hence $Z'(0,L) \hookrightarrow Y'(0,L)$, which is equivalent to $Y(0,L) \hookrightarrow Z(0,L)$ owing to (1.15). Finally, we claim that, if there is any rearrangement-invariant function space $Z(0,L)$ such that $H_{u,v,\nu}\colon Z'(0,L) \to X'(0,L)$ is bounded, then (2.33) needs to be satisfied. If $L = 1$, we plainly have that

$$\left\|u(t)\int_{t^{\frac{1}{\nu}}}^{1} v(s)\,\mathrm{d}s\right\|_{X'(0,1)} = \|H_{u,v,\nu}\chi_{(0,1)}\|_{X'(0,1)} \lesssim \|\chi_{(0,1)}\|_{Z'(0,1)} < \infty.$$

If $L = \infty$, we can argue as in the proof of Proposition 2.1.2 to show that, if either of assumptions (2.33) is not satisfied, then

$$\sup_{h\sim\chi_{(0,1)}} \|H_{u,v,\nu}h\|_{X'(0,\infty)} = \infty,$$

whence, thanks to the boundedness of $H_{u,v,\nu}\colon Z'(0,\infty) \to X'(0,\infty)$,

$$\infty = \sup_{h\sim\chi_{(0,1)}} \|H_{u,v,\nu}h\|_{X'(0,\infty)} \lesssim \|\chi_{(0,1)}\|_{Z'(0,\infty)} < \infty,$$

which would be a contradiction. □

Thanks to (2.32) and (1.9), $Y(0,L)$ is the optimal target space for the operator $H_{u,v,\nu}$ and $X(0,L)$ if and only if $Y'(0,L)$ is the optimal domain space for the operator $R_{u,v,\nu}$ and $X'(0,L)$. Similarly, $X(0,L)$ is the optimal domain space for the operator $H_{u,v,\nu}$ and $Y(0,L)$ if and only if $X'(0,L)$ is the optimal target space for the operator $R_{u,v,\nu}$ and $Y'(0,L)$. Therefore, Proposition 2.2.1 and Proposition 2.2.3 actually also characterize optimal rearrangement-invariant function spaces for the operator $H_{u,v,\nu}$.

The next proposition describes the optimal domain space for $H_{u,v,\nu}$ and a given rearrangement-invariant function space.

Proposition 2.2.4. *Let $\|\cdot\|_{Y(0,L)}$ be a rearrangement-invariant function norm. Let $u, v\colon (0,L) \to (0,\infty)$ be nonincreasing. If $L = 1$, we assume that $v(1^-) > 0$. Assume that*

(2.35)

$$\begin{cases} u(t)\int_{t^{\frac{1}{\nu}}}^{1} v(s)\,\mathrm{d}s \in Y(0,1) & \textit{if } L = 1, \\ u(t)\chi_{(0,1)}(t)\int_{t^{\frac{1}{\nu}}}^{1} v(s)\,\mathrm{d}s \in Y(0,\infty) \textit{ and } \sup\limits_{b\in[1,\infty)} v(b^{\frac{1}{\nu}})\|u\chi_{(0,b)}\|_{Y(0,\infty)} < \infty & \textit{if } L = \infty. \end{cases}$$

Let $\| \cdot \|_{X(0,L)}$ be the rearrangement-invariant function norm defined as

$$\|f\|_{X(0,L)} = \sup_{h\sim f} \left\| u(t) \int_{t^{\frac{1}{\nu}}}^{L} h(s)v(s)\,\mathrm{d}s \right\|_{Y(0,L)}, \ f \in \mathfrak{M}^+(0,L), \tag{2.36}$$

where the supremum is taken over all $h \in \mathfrak{M}^+(0,L)$ equimeasurable with f. The rearrangement-invariant function space $X(0,L)$ is the optimal domain space for the operator $H_{u,v,\nu}$ and the space $Y(0,L)$.

Moreover, if (2.35) is not satisfied, then there is no rearrangement-invariant function space $Z(0,L)$ such that $H_{u,v,\nu}\colon Z(0,L) \to Y(0,L)$ is bounded.

The optimal target space for $H_{u,v,\nu}$ and a given rearrangement-invariant function space is described in the following proposition.

Proposition 2.2.5. *Let $\| \cdot \|_{X(0,L)}$ be a rearrangement-invariant function norm. Let $u\colon (0,L) \to (0,\infty)$ be a nonincreasing function such that $0 < U(t) < \infty$ for every $t \in (0,L)$, where $U(t) = \int_0^t u(s)\,\mathrm{d}s$. If $L = 1$, we assume that $u(1^-) > 0$. Let $v\colon (0,L) \to (0,\infty)$ be measurable. Let $\nu \in (0,\infty)$. Assume that $\xi \in X'(0,L)$, where ξ is defined by (2.29). Let $\| \cdot \|_{Y(0,L)}$ be the rearrangement-invariant function norm whose associate function norm $\| \cdot \|_{Y'(0,L)}$ is defined as*

$$\|f\|_{Y'(0,L)} = \left\| v(t) \int_0^{t^\nu} f^*(s)u(s)\,\mathrm{d}s \right\|_{X'(0,L)}, \ f \in \mathfrak{M}^+(0,L).$$

The rearrangement-invariant function space $Y(0,L)$ is the optimal target space for the operator $H_{u,v,\nu}$ and the space $X(0,L)$.

Moreover, if $\xi \notin X'(0,L)$, then there is no rearrangement-invariant function space $Z(0,L)$ such that $H_{u,v,\nu}\colon X(0,L) \to Z(0,L)$ is bounded.

By combining Proposition 2.1.3 and Proposition 2.2.3, we obtain the following theorem, which tells us that the optimal target space for the operator $R_{u,v,\nu}$ and a rearrangement-invariant function space $X(0,L)$ has a much more manageable description than that given by Proposition 2.2.3 provided that the supremum operator T_φ defined by (2.17) with an appropriate function φ is bounded on $X(0,L)$.

Theorem 2.2.6. *Let $\| \cdot \|_{X(0,L)}$ be a rearrangement-invariant function norm. Let $\nu \in (0,\infty)$. Let $u\colon (0,L) \to (0,\infty)$ be nonincreasing. Let $v\colon (0,L) \to (0,\infty)$ be a nonincreasing function such that $\frac{1}{v(t)} = \int_0^{t^\nu} \xi(s)\,\mathrm{d}s$ for every $t \in (0,L)$, where $\xi\colon (0,L) \to (0,\infty)$ is a continuous function. If $L = 1$, we assume that $v(1^-) > 0$. Furthermore, assume that $u(t)\chi_{(0,1)}(t) \int_{t^{\frac{1}{\nu}}}^1 v(s)\,\mathrm{d}s \in X'(0,L)$. Finally, assume that the operator T_φ is bounded on $X(0,L)$, where $\varphi = \frac{u}{\xi}$. Let $\| \cdot \|_{Y(0,L)}$ be the rearrangement-invariant function norm whose associate function norm $\| \cdot \|_{Y'(0,L)}$ is defined as*

$$\|f\|_{Y'(0,L)} = \sup_{\substack{g \in \mathfrak{M}^+(0,L) \\ \|g\|_{X(0,L)} \leq 1}} \int_0^L f^*(s)v(t) \int_0^{t^\nu} T_\varphi g(s)u(s)\,\mathrm{d}s\,\mathrm{d}t, \ f \in \mathfrak{M}^+(0,L).$$

The rearrangement-invariant function space $Y(0,L)$ is the optimal target space for the operator $R_{u,v,\nu}$ and the space $X(0,L)$. Moreover,

$$\|H_{u,v,\nu}(f^*)\|_{X'(0,L)} \leq \|f\|_{Y'(0,L)} \leq \|T_\varphi\|_{X(0,L)} \|H_{u,v,\nu}(f^*)\|_{X'(0,L)} \tag{2.37}$$

for every $f \in \mathfrak{M}^+(0,L)$, where $\|T_\varphi\|_{X(0,L)}$ stands for the operator norm of T_φ on $X(0,L)$.

Remark 2.2.7. Owing to the remark above Proposition 2.2.4, Theorem 2.2.6 can also be used to get a simpler description of optimal domain spaces for the operator $H_{u,v,\nu}$.

We shall see that the supremum operator T_φ and optimal spaces for the operators $R_{v,\nu}$ and $H_{v,\nu}$ are actually far more closely related to each other than it might appear at first sight. Before we reveal how close their connection actually is, we need to prepare ourselves for the task by proving a few technical propositions of independent interest.

Proposition 2.2.8. *Let $\nu \in (0,\infty)$. Let $\varphi\colon (0,L) \to (0,\infty)$ be quasiconcave. Let $f \in \mathfrak{M}^+(0,L)$ be nonincreasing. We have that*

$$\int_0^t \sup_{\tau\in[s,L)} \varphi(\tau)f(\tau)\,\mathrm{d}s \le 6\int_0^t (\varphi f)^*(s)\,\mathrm{d}s \quad \textit{for every } t \in (0,L). \tag{2.38}$$

Consequently,

$$\Big\| \sup_{\tau\in[t,L)} \varphi(\tau)f(\tau) \Big\|_{X(0,L)} \le 6\|\varphi f\|_{X(0,L)} \tag{2.39}$$

for every rearrangement-invariant function norm $\|\cdot\|_{X(0,L)}$.

Proof. Since φ is quasiconcave, it is, in particular, locally absolutely continuous on $(0,L)$ ([56, Chapter II, Lemma 1.1]) and we have that

$$\varphi(s_2) - \varphi(s_1) \le \int_{s_1}^{s_2} \frac{\varphi(u)}{u}\,\mathrm{d}u \quad \text{for every } 0 < s_1 \le s_2 < L. \tag{2.40}$$

Note that

$$\begin{aligned}\int_0^t \sup_{\tau\in[s,L)} \varphi(\tau)f(\tau)\,\mathrm{d}s &\le \int_0^t \sup_{\tau\in[s,t)} \varphi(\tau)f(\tau)\,\mathrm{d}s + t\sup_{\tau\in[t,L)}\varphi(\tau)f(\tau)\\ &\le \int_0^t \sup_{\tau\in[s,t)} (\varphi(\tau)-\varphi(s))f(\tau)\,\mathrm{d}s + \int_0^t \varphi(s)\sup_{\tau\in[s,t)} f(\tau)\,\mathrm{d}s\\ &\quad + t\sup_{\tau\in[t,L)}\varphi(\tau)f(\tau).\end{aligned} \tag{2.41}$$

Since f is nonincreasing, we plainly have that

$$\int_0^t \varphi(s)\sup_{\tau\in[s,t)} f(\tau)\,\mathrm{d}s = \int_0^t \varphi(s)f(s)\,\mathrm{d}s. \tag{2.42}$$

Using (2.40), the monotonicity of f and Fubini's theorem, we have that

$$\begin{aligned}\int_0^t \sup_{\tau\in[s,t)} (\varphi(\tau)-\varphi(s))f(\tau)\,\mathrm{d}s &\le \int_0^t \sup_{\tau\in[s,t)} \int_s^\tau \frac{\varphi(u)}{u}\,\mathrm{d}u\, f(\tau)\,\mathrm{d}s\\ &\le \int_0^t \sup_{\tau\in[s,t)} \int_s^\tau \frac{\varphi(u)f(u)}{u}\,\mathrm{d}u\,\mathrm{d}s\\ &= \int_0^t \int_s^t \frac{\varphi(u)f(u)}{u}\,\mathrm{d}u\,\mathrm{d}s\\ &= \int_0^t \varphi(u)f(u)\,\mathrm{d}u.\end{aligned} \tag{2.43}$$

Using the fact that φ is quasiconcave, it follows that

$$\varphi(s)-\varphi\Big(\frac{s}{2}\Big)\leq\int_{\frac{s}{2}}^{s}\frac{\varphi(u)}{u}\,\mathrm{d}u\leq\frac{2\varphi(\frac{s}{2})}{s}\frac{s}{2}=\varphi\Big(\frac{s}{2}\Big)\quad\text{for every } s\in(0,L),$$

whence

$$\varphi(s)\leq 2\varphi\Big(\frac{s}{2}\Big)=\frac{4}{s}\int_{\frac{s}{2}}^{s}\varphi\Big(\frac{s}{2}\Big)\,\mathrm{d}u\leq 4\frac{1}{s}\int_0^s\varphi(u)\,\mathrm{d}u\quad\text{for every } s\in(0,L). \tag{2.44}$$

Using (2.44), we have that

$$\begin{aligned}t\sup_{\tau\in[t,L)}\varphi(\tau)f(\tau)&\leq 4t\sup_{\tau\in[t,L)}\frac{1}{\tau}\int_0^\tau\varphi(u)\,\mathrm{d}u\,f(\tau)\\&\leq 4t\sup_{\tau\in[t,L)}\frac{1}{\tau}\int_0^\tau\varphi(u)f(u)\,\mathrm{d}u\\&\leq 4t\sup_{\tau\in[t,L)}\frac{1}{\tau}\int_0^\tau(\varphi f)^*(u)\,\mathrm{d}u\\&=4\int_0^t(\varphi f)^*(u)\,\mathrm{d}u,\end{aligned} \tag{2.45}$$

where the third inequality follows from the Hardy–Littlewood inequality (1.3) and the last one from the fact that $(\varphi f)^{**}$ is nonincreasing. By combining (2.41) with (2.42), (2.43), (2.45) and the Hardy–Littlewood inequality (1.3), we obtain (2.38).

Finally, if $\|\cdot\|_{X(0,L)}$ is any rearrangement-invariant function norm, then the Hardy–Littlewood–Pólya principle (1.6) asserts that (2.38) implies (2.39). ☐

The next proposition tells us what a rearrangement-invariant function norm of $H_{u,v,\nu}$ applied to a positive, nonincreasing simple function is equivalent to.

Proposition 2.2.9. *Let $\|\cdot\|_{X(0,L)}$ be a rearrangement-invariant function norm. Let $\nu\in(0,\infty)$. Let $u,v\colon(0,L)\to(0,\infty)$ be nonincreasing. Assume that there is a positive constant C such that*

$$\int_0^{\frac{t}{2}}v(s)\,\mathrm{d}s\leq C\int_{\frac{t}{2}}^t v(s)\,\mathrm{d}s\quad\text{for every } t\in(0,L). \tag{2.46}$$

Set $f=\sum\limits_{i=1}^N c_i\chi_{(0,a_i)}$, where $c_i\in(0,\infty)$, $i=1,\dots,N$, and $0<a_1<\dots<a_N\leq L$. We have that

$$\Big\|u(t)\int_{t^{\frac{1}{\nu}}}^L f(s)v(s)\,\mathrm{d}s\Big\|_{X(0,L)}\approx\Big\|u(t)\sum_{i=1}^N c_i\chi_{(0,a_i^\nu)}(t)v\Big(\frac{a_i}{2}\Big)a_i\Big\|_{X(0,L)}, \tag{2.47}$$

where the multiplicative constants depend only on ν and C.

Proof. On the one hand, we have that

$$
\begin{aligned}
\left\| u(t) \int_{t^{\frac{1}{\nu}}}^{L} f(s) v(s) \, \mathrm{d}s \right\|_{X(0,L)} &= \left\| u(t) \sum_{i=1}^{N} c_i \chi_{(0,a_i^\nu)}(t) \int_{t^{\frac{1}{\nu}}}^{a_i} v(s) \, \mathrm{d}s \right\|_{X(0,L)} \\
&\geq \left\| u(t) \sum_{i=1}^{N} c_i \chi_{(0,4^{-\nu}a_i^\nu)}(t) \int_{t^{\frac{1}{\nu}}}^{\frac{a_i}{2}} v(s) \, \mathrm{d}s \right\|_{X(0,L)} \\
&\geq \frac{1}{4} \left\| u(t) \sum_{i=1}^{N} c_i \chi_{(0,4^{-\nu}a_i^\nu)}(t) v\Big(\frac{a_i}{2}\Big) a_i \right\|_{X(0,L)} \\
&\geq \frac{1}{4^{1+\nu}} \left\| u(4^{-\nu}t) \sum_{i=1}^{N} c_i \chi_{(0,a_i^\nu)}(t) v\Big(\frac{a_i}{2}\Big) a_i \right\|_{X(0,L)} \\
&\geq \frac{1}{4^{1+\nu}} \left\| u(t) \sum_{i=1}^{N} c_i \chi_{(0,a_i^\nu)}(t) v\Big(\frac{a_i}{2}\Big) a_i \right\|_{X(0,L)}
\end{aligned}
\tag{2.48}
$$

thanks to the fact that u and v are nonincreasing and (1.13) (the boundedness of the dilation operator D_{4^ν}).

On the other hand, using (2.46) and the monotonicity of v, we obtain that

$$
\begin{aligned}
\left\| u(t) \int_{t^{\frac{1}{\nu}}}^{L} f(s) v(s) \, \mathrm{d}s \right\|_{X(0,L)} &= \left\| u(t) \sum_{i=1}^{N} c_i \chi_{(0,a_i^\nu)}(t) \int_{t^{\frac{1}{\nu}}}^{a_i} v(s) \, \mathrm{d}s \right\|_{X(0,L)} \\
&\leq \left\| u(t) \sum_{i=1}^{N} c_i \chi_{(0,a_i^\nu)}(t) \int_{0}^{a_i} v(s) \, \mathrm{d}s \right\|_{X(0,L)} \\
&\leq 2 \left\| u(t) \sum_{i=1}^{N} c_i \chi_{(0,a_i^\nu)}(t) \int_{0}^{\frac{a_i}{2}} v(s) \, \mathrm{d}s \right\|_{X(0,L)} \\
&\leq 2C \left\| u(t) \sum_{i=1}^{N} c_i \chi_{(0,a_i^\nu)}(t) \int_{\frac{a_i}{2}}^{a_i} v(s) \, \mathrm{d}s \right\|_{X(0,L)} \\
&\leq C \left\| u(t) \sum_{i=1}^{N} c_i \chi_{(0,a_i^\nu)}(t) v\Big(\frac{a_i}{2}\Big) a_i \right\|_{X(0,L)}.
\end{aligned}
\tag{2.49}
$$

By combining (2.49) and (2.48), we obtain (2.47). ☐

By combining the preceding proposition with the fact that every nonnegative, nonincreasing function on $(0, L)$ is the pointwise limit of a nondecreasing sequence of nonnegative, nonincreasing simple functions, we obtain the following important corollary.

Corollary 2.2.10. *Let $\nu \in (0, \infty)$. Let $v\colon (0, L) \to (0, \infty)$ be a nonincreasing function satisfying (2.46) with some positive constant C. Let $f \in \mathfrak{M}^+(0, L)$ be nonincreasing. There is a nondecreasing sequence $\{f_k\}_{k=1}^\infty$ of nonnegative, nonincreasing simple functions on $(0, L)$ such that, for every rearrangement-invariant function norm $\|\cdot\|_{X(0,L)}$,*

$$\lim_{k\to\infty} \|H_{v,\nu}(f_k)\|_{X(0,L)} \approx \|f\|_{X(0,L)},$$

where the multiplicative constants depend only on C and ν.

The following proposition is a modification of the results of Cianchi, Pick and Slavíková [30, Theorem 9.5] and Peša [78, Theorem 3.10], which is tailored to our purpose because it allows $\nu \neq 1$ and $u \not\equiv 1$. In [19], we needed that, and, while we knew that their proofs would carry over, we still had to carefully check them because there is a plenty of fine analysis involved. It turns out that their proofs carry over nearly verbatim to our setting. The proof in our setting is actually simpler because our operator $H_{u,v,\nu}$ does not have a kernel.

Proposition 2.2.11. *Let $\|\cdot\|_{X(0,L)}$ and $\|\cdot\|_{Y(0,L)}$ be rearrangement-invariant function norms. Let $u, v\colon (0,L) \to (0,\infty)$ be nonincreasing. Let $\nu \in (0,\infty)$. The following two statements are equivalent:*

(i) there is a positive constant C such that

$$\Big\| u(t) \int_{t^{\frac{1}{\nu}}}^{L} f(s) v(s) \,\mathrm{d}s \Big\|_{Y(0,L)} \leq C \|f\|_{X(0,L)} \tag{2.50}$$

for every $f \in \mathfrak{M}^+(0,L)$;

(ii) there is a positive constant C such that

$$\Big\| u(t) \int_{t^{\frac{1}{\nu}}}^{L} f(s) v(s) \,\mathrm{d}s \Big\|_{Y(0,L)} \leq C \|f\|_{X(0,L)} \tag{2.51}$$

for every nonincreasing $f \in \mathfrak{M}^+(0,L)$.

Moreover, if (ii) holds with C, then (i) holds with $2^{1+\nu}C$.

Proof. Since (i) plainly implies (ii), we only need to prove that (ii) implies (i). Since the quantities in (2.50) and (2.51) do not change when the function v is redefined on a countable set, we may assume that v is left continuous. Set $R = R_{u,v,\nu}$ and $H = H_{u,v,\nu}$. Note that Hf is nonincreasing for every $f \in \mathfrak{M}^+(0,L)$. Hence, thanks to (1.10) and (2.3), in order to prove that (ii) implies (i), we need to show that

$$\sup_{\substack{f \in \mathfrak{M}^+(0,L) \\ \|f\|_{X(0,L)} \leq 1}} \sup_{\substack{g \in \mathfrak{M}^+(0,L) \\ \|g\|_{Y'(0,L)} \leq 1}} \int_0^L f(s) R(g^*)(s) \,\mathrm{d}s \lesssim \sup_{\substack{f \in \mathfrak{M}^+(0,L) \\ \|f\|_{X(0,L)} \leq 1}} \sup_{\substack{g \in \mathfrak{M}^+(0,L) \\ \|g\|_{Y'(0,L)} \leq 1}} \int_0^L f^*(s) R(g^*)(s) \,\mathrm{d}s. \tag{2.52}$$

We define the operator G as

$$Gg(t) = \sup_{\tau \in [t,L)} R(g^*)(\tau),\ t \in (0,L),$$

for every $g \in \mathfrak{M}^+(0,L)$. Note that Gg is nonincreasing for every $g \in \mathfrak{M}^+(0,L)$. Fix $g \in \mathfrak{M}^+(0,L)$ such that $|\{t \in (0,L)\colon g(t) > 0\}| < \infty$, and set

$$E = \big\{ t \in (0,L)\colon R(g^*)(t) < Gg(t) \big\}.$$

It can be shown that there is a countable system $\{(a_k, b_k)\}_{k \in \mathcal{I}}$ of mutually disjoint, bounded intervals in $(0,L)$ such that

$$E = \bigcup_{k \in \mathcal{I}} (a_k, b_k); \tag{2.53}$$

$$Gg(t) = R(g^*)(t) \quad \text{if } t \in (0,L) \setminus E; \tag{2.54}$$

$$Gg(t) = R(g^*)(b_k) \quad \text{if } t \in (a_k, b_k) \text{ for } k \in \mathcal{I}. \tag{2.55}$$

This was proved in [30, Proposition 9.3] for $L = 1$ and in [78, Lemma 3.9] for $L = \infty$. Their proofs are for $u \equiv 1$ and $\nu = 1$, but the fact that g^*u is nonincreasing and $R(g^*)$ is upper semicontinuous remains valid in our situation, and so it can be readily seen that their proofs carry over verbatim to our setting.

Thanks to the fact that v and g^*u are nonincreasing, we have, for every $k \in \mathcal{I}$, that

$$\begin{aligned}(b_k - a_k)R(g^*)(b_k) &= 2\int_{\frac{a_k+b_k}{2}}^{b_k} R(g^*)(b_k)\,\mathrm{d}t = 2\int_{\frac{a_k+b_k}{2}}^{b_k} \frac{v(b_k)}{b_k^\nu} b_k^\nu \int_0^{b_k^\nu} g^*(s)u(s)\,\mathrm{d}s\ \mathrm{d}t \\ &\le 2\int_{\frac{a_k+b_k}{2}}^{b_k} \frac{v(t)}{t^\nu} b_k^\nu \int_0^{t^\nu} g^*(s)u(s)\,\mathrm{d}s\ \mathrm{d}t \le 2^{1+\nu}\int_{\frac{a_k+b_k}{2}}^{b_k} R(g^*)(t)\,\mathrm{d}t \\ &\le 2^{1+\nu}\int_{a_k}^{b_k} R(g^*)(t)\,\mathrm{d}t,\end{aligned} \tag{2.56}$$

where we used the fact that v and $(0, L) \ni t \mapsto \frac{1}{t^\nu}\int_0^{t^\nu} g^*(s)u(s)\,\mathrm{d}s$ are nonincreasing, the latter being the integral mean of a nonincreasing function, in the first inequality, and we used the fact that $b_k \le 2t$ for every $t \in (\frac{a_k+b_k}{2}, b_k)$ in the second inequality.

Consider the averaging operator A defined as

$$Af = f^*\chi_{(0,L)\setminus E} + \sum_{k\in\mathcal{I}} \Big(\frac{1}{b_k - a_k}\int_{a_k}^{b_k} f^*(s)\,\mathrm{d}s\Big)\chi_{(a_k,b_k)},\ \ f \in \mathfrak{M}^+(0, L).$$

Note that Af is a nonincreasing function for every $f \in \mathfrak{M}^+(0, L)$. Furthermore, it is known ([8, Chapter 2, Theorem 4.8]) that

$$\|Af\|_{X(0,L)} \le \|f\|_{X(0,L)} \quad \text{for every } f \in \mathfrak{M}^+(0, L). \tag{2.57}$$

We have, for every $f \in \mathfrak{M}^+(0, L)$, that

$$\begin{aligned}\int_0^L f(t)R(g^*)(t)\,\mathrm{d}t &\le \int_0^L f(t)Gg(t)\,\mathrm{d}t \le \int_0^L f^*(t)Gg(t)\,\mathrm{d}t \\ &= \int_{(0,L)\setminus E} f^*(t)R(g^*)(t)\,\mathrm{d}t + \sum_{k\in\mathcal{I}}\int_{a_k}^{b_k} f^*(t)R(g^*)(b_k)\,\mathrm{d}t \\ &\le \int_0^L f^*(t)\chi_{(0,L)\setminus E}(t)R(g^*)(t)\,\mathrm{d}t \\ &\quad + 2^{1+\nu}\sum_{k\in\mathcal{I}}\Big(\frac{1}{b_k - a_k}\int_{a_k}^{b_k} f^*(t)\,\mathrm{d}t\Big)\Big(\int_{a_k}^{b_k} R(g^*)(t)\,\mathrm{d}t\Big) \\ &\le 2^{1+\nu}\int_0^L Af(t)R(g^*)(t)\,\mathrm{d}t,\end{aligned} \tag{2.58}$$

owing to the Hardy–Littlewood inequality (1.3), (2.53), (2.54), (2.55), and (2.56). If $L = \infty$ and $g \in \mathfrak{M}^+(0, \infty)$ is positive on a set of infinite measure, we consider $g\chi_{(0,n)} \nearrow g$, $n \to \infty$, and obtain (2.58) even for such functions g thanks to the monotone convergence theorem; hence we have proved that

$$\int_0^L f(t)R(g^*)(t)\,\mathrm{d}t \le 2^{1+\nu}\int_0^L Af(t)R(g^*)(t)\,\mathrm{d}t \quad \text{for every } f, g \in \mathfrak{M}^+(0, L). \tag{2.59}$$

By combining (2.57) and (2.59), we obtain that

$$\int_0^L f(t)R(g^*)(t)\,\mathrm{d}t \le 2^{1+\nu} \sup_{\substack{h\in\mathfrak{M}^+(0,L)\\ \|h\|_{X(0,L)}\le 1}} \int_0^L h^*(t)R(g^*)(t)\,\mathrm{d}t$$

for every $f \in \mathfrak{M}^+(0,L)$, $\|f\|_{X(0,L)} \le 1$, and $g \in \mathfrak{M}^+(0,L)$. Note that here we used the fact that Af is nonincreasing for every $f \in \mathfrak{M}^+(0,L)$. By taking the supremum over all $f, g \in \mathfrak{M}^+(0,L)$ from the closed unit balls of $X(0,L)$ and $Y'(0,L)$, respectively, we obtain (2.52) with the multiplicative constant equal to $2^{1+\nu}$. ☐

Proposition 2.2.11 together with Proposition 2.2.2 has the following important corollary, in which the first equality is just a consequence of the Hardy–Littlewood inequality (1.3) combined with the obvious inequality

$$\sup_{\|f\|_{X(0,L)}\le 1} \|R_{u,v,\nu}(f^*)\|_{Y(0,L)} \le \sup_{\|f\|_{X(0,L)}\le 1} \|R_{u,v,\nu}f\|_{Y(0,L)}.$$

Corollary 2.2.12. *Let $\|\cdot\|_{X(0,L)}$ and $\|\cdot\|_{Y(0,L)}$ be rearrangement-invariant function norms. Let $u, v\colon (0,L)\to(0,\infty)$ be nonincreasing. Let $\nu\in(0,\infty)$. We have that*

$$\begin{aligned} \sup_{\|f\|_{X(0,L)}\le 1} \|R_{u,v,\nu}(f^*)\|_{Y(0,L)} &= \sup_{\|f\|_{X(0,L)}\le 1} \|R_{u,v,\nu}f\|_{Y(0,L)} \\ &= \sup_{\|g\|_{Y'(0,L)}\le 1} \|H_{u,v,\nu}g\|_{X'(0,L)} \\ &\approx \sup_{\|g\|_{Y'(0,L)}\le 1} \|H_{u,v,\nu}(g^*)\|_{X'(0,L)}. \end{aligned} \tag{2.60}$$

We are now finally in a position to reveal the connection between the supremum operator T_φ and optimal spaces for the operators $R_{v,\nu}$ and $H_{v,\nu}$.

Theorem 2.2.13. *Let $\|\cdot\|_{X(0,L)}$ be a rearrangement-invariant function norm. Let $\nu\in(0,\infty)$. Let $\xi\colon(0,L)\to(0,\infty)$ be a continuous function such that*

$$\frac{1}{t}\int_0^t \xi(s)\,\mathrm{d}s \lesssim \xi(t) \quad \textit{for every } t\in(0,L). \tag{2.61}$$

Set

$$v(t) = \frac{1}{\int_0^{t^\nu} \xi(s)\,\mathrm{d}s},\ t\in(0,L).$$

Assume that v satisfies (2.46) with some positive constant. Set $\varphi = \frac{1}{\xi}$. Assume that the function $(0,L)\ni t\mapsto\varphi(t^\nu)$ is equivalent to a quasiconcave function. Let ϱ be the functional defined by (2.4) with $u\equiv 1$. The following five statements are equivalent:

(i) the operator T_φ is bounded on $X'(0,L)$;

(ii) there is a positive constant C such that

$$\sup_{h\sim f} \|H_{v,\nu}h\|_{X(0,L)} \le C\|H_{v,\nu}(f^*)\|_{X(0,L)} \quad \textit{for every } f\in\mathfrak{M}^+(0,L); \tag{2.62}$$

(iii) there are positive constants C_1 *and* C_2 *such that*

$$\begin{aligned}(2.63)\qquad & C_1 \sup_{\substack{g\in\mathfrak{M}^+(0,L)\\ \varrho(g)\leq 1}} \int_0^L g(t)R_{v,\nu}(f^*)(t)\,\mathrm{d}t \leq \|f\|_{X'(0,L)} \\ & \leq C_2 \sup_{\substack{g\in\mathfrak{M}^+(0,L)\\ \varrho(g)\leq 1}} \int_0^L g(t)R_{v,\nu}(f^*)(t)\,\mathrm{d}t \quad \textit{for every } f\in\mathfrak{M}^+(0,L);\end{aligned}$$

(iv) the space $X'(0,L)$ *is the optimal domain space for the operator* $R_{v,\nu}$ *and some rearrangement-invariant function space* $Y(0,L)$*;*

(v) the space $X(0,L)$ *is the optimal target space for the operator* $H_{v,\nu}$ *and some rearrangement-invariant function space* $Y(0,L)$*.*

Proof. We start off by observing that any of the five statements implies that the functional ϱ is actually a rearrangement-invariant function norm. To this end, note that $\|\chi_{(0,1)}(t)\int_{t^{\frac{1}{\nu}}}^1 v(s)\,\mathrm{d}s\|_{X(0,L)} < \infty$ because

$$\begin{aligned}\Big\|\chi_{(0,1)}(t)\int_{t^{\frac{1}{\nu}}}^1 v(s)\,\mathrm{d}s\Big\|_{X(0,L)} &\leq \int_0^1 v(s)\,\mathrm{d}s\|\chi_{(0,1)}\|_{X(0,L)} \lesssim \int_{\frac{1}{2}}^1 v(s)\,\mathrm{d}s\|\chi_{(0,1)}\|_{X(0,L)} \\ &\lesssim v\Big(\frac{1}{2}\Big)\|\chi_{(0,1)}\|_{X(0,L)} < \infty,\end{aligned}$$

where we used the fact that v satisfies (2.46) and the monotonicity of v. If we assume (i), then the fact that ϱ is a rearrangement-invariant function norm follows from Proposition 2.1.3. If (ii) is assumed, then we have that

$$\sup_{h\sim\chi_{(0,1)}} \|H_{v,\nu}h\|_{X(0,L)} \lesssim \Big\|\chi_{(0,1)}(t)\int_{t^{\frac{1}{\nu}}}^1 v(s)\,\mathrm{d}s\Big\|_{X(0,L)} < \infty.$$

It follows from Proposition 2.1.2 and its proof that this actually guarantees that ϱ is a rearrangement-invariant function norm. If we assume (iii), then, in particular, the set $\{g\in\mathfrak{M}^+(0,L)\colon \varrho(g)\leq 1\}$ needs to contain a function $g\in\mathfrak{M}^+(0,L)$ not equal to 0 a.e. It follows from Proposition 2.1.2 and its proof that ϱ fails to be a rearrangement-invariant function norm only if $\varrho(g)=\infty$ for every $g\in\mathfrak{M}^+(0,L)$ not equal to 0 a.e. Hence ϱ is a rearrangement-invariant function norm if (iii) is assumed. If we assume (iv) or (v), ϱ is a rearrangement-invariant function norm thanks to Proposition 2.2.3 or Proposition 2.2.4, respectively. Therefore, in all of the cases, we are entitled to denote the corresponding rearrangement-invariant function space over $(0,L)$ by $Z(0,L)$. Note that (2.63) actually reads as

$$(2.64)\qquad \|f\|_{X'(0,L)} \approx \|R_{v,\nu}(f^*)\|_{Z'(0,L)} \quad \text{for every } f\in\mathfrak{M}^+(0,L)$$

by (1.7).

First, the implication $(i)\Rightarrow(ii)$ follows immediately from Proposition 2.1.3.

Second, we shall prove the implication $(ii)\Rightarrow(iii)$. Note that $H_{v,\nu}\colon Z(0,L)\to X(0,L)$ is bounded thanks to Proposition 2.2.11 combined with the inequality $\|H_{v,\nu}(f^*)\|_{X(0,L)} \leq \|f\|_{Z(0,L)}$ for every $f\in\mathfrak{M}^+(0,L)$, which plainly holds. Let $Y(0,L)$ be the optimal target space for the operator $H_{v,\nu}$ and the space $Z(0,L)$.

Its existence is guaranteed by Proposition 2.2.5. In particular, $Y(0,L) \hookrightarrow X(0,L)$. Furthermore,

$$\|f\|_{Y'(0,L)} = \|R_{v,\nu}(f^*)\|_{Z'(0,L)} \quad \text{for every } f \in \mathfrak{M}^+(0,L). \tag{2.65}$$

Using the optimality of $Y(0,L)$ and (2.62), we obtain that

$$\begin{aligned} \|H_{v,\nu}(f^*)\|_{X(0,L)} \lesssim \|H_{v,\nu}(f^*)\|_{Y(0,L)} \lesssim \|f^*\|_{Z(0,L)} = \|f\|_{Z(0,L)} \\ \lesssim \|H_{v,\nu}(f^*)\|_{X(0,L)} \quad \text{for every } f \in \mathfrak{M}^+(0,L). \end{aligned}$$

Hence

$$\|H_{v,\nu}(f^*)\|_{X(0,L)} \approx \|H_{v,\nu}(f^*)\|_{Y(0,L)} \quad \text{for every } f \in \mathfrak{M}^+(0,L).$$

In particular, we have that

$$\|H_{v,\nu}h\|_{X(0,L)} \approx \|H_{v,\nu}h\|_{Y(0,L)} \tag{2.66}$$

for every nonincreasing simple function $h \in \mathfrak{M}^+(0,L)$. Combining (2.66) with Corollary 2.2.10, we obtain that

$$\|f^*\|_{X(0,L)} \approx \|f^*\|_{Y(0,L)} \quad \text{for every } f \in \mathfrak{M}^+(0,L).$$

Owing to the rearrangement invariance of both function norms, it follows that $X(0,L) = Y(0,L)$. Hence (2.63) follows from (2.65) combined with (1.7).

Next, note that (*iii*) implies (*iv*). Indeed, (2.64) coupled with Proposition 2.2.1 tells us that $X'(0,L)$ is the optimal domain space for the operator $R_{v,\nu}$ and the space $Z'(0,L)$. Statements (*iv*) and (*v*) are clearly equivalent to each other owing to the remark above Proposition 2.2.4.

Next, the proof of the fact that (*iv*) implies (*iii*) is based on the following important observation. If $X'(0,L)$ is the optimal domain space for the operator $R_{v,\nu}$ and some rearrangement-invariant function space $Y(0,L)$, then, in particular, $R_{v,\nu}\colon X'(0,L) \to Y(0,L)$ is bounded. Consequently, by virtue of Proposition 2.2.3, the rearragement-invariant function space whose associate function norm is given by (2.34) with $u \equiv 1$ is the optimal target space for the operator $R_{v,\nu}$ and the space $X'(0,L)$. By (1.9), that optimal target space is actually the space $Z'(0,L)$. Owing to Proposition 2.2.1, there is the optimal domain space for the operator $R_{v,\nu}$ and the space $Z'(0,L)$, which we denote by $W(0,L)$. Moreover,

$$\|f\|_{W(0,L)} \approx \|R_{v,\nu}(f^*)\|_{Z'(0,L)} \quad \text{for every } f \in \mathfrak{M}^+(0,L). \tag{2.67}$$

The crucial observation is that we have, in fact, that $X'(0,L) = W(0,L)$. The embedding $X'(0,L) \hookrightarrow W(0,L)$ is valid because $R_{v,\nu}\colon X'(0,L) \to Z'(0,L)$ is bounded and $W(0,L)$ is the optimal domain space for the operator $R_{v,\nu}$ and the space $Z'(0,L)$. The validity of the opposite embedding is slightly more complicated. Since $R_{v,\nu}\colon X'(0,L) \to Y(0,L)$ is bounded and $Z'(0,L)$ is the optimal target space for the operator $R_{v,\nu}$ and the space $X'(0,L)$, we have that $Z'(0,L) \hookrightarrow Y(0,L)$. Consequently, since $R_{v,\nu}\colon W(0,L) \to Z'(0,L)$ is bounded, so is $R_{v,\nu}\colon W(0,L) \to Y(0,L)$. Using the fact that $X'(0,L)$ is the optimal domain space for the operator $R_{v,\nu}$ and the space $Y(0,L)$, we obtain that $W(0,L) \hookrightarrow X'(0,L)$. Now that we know that $X'(0,L) = W(0,L)$, (2.64) follows from (2.67).

It only remains to prove that (*iii*) implies (*i*). Using (2.64) combined with the fact that $T_\varphi f$ is equivalent to a nonincreasing function for every $f \in \mathfrak{M}^+(0,L)$, we have that

$$\begin{aligned}
\|T_\varphi f\|_{X'(0,L)} &\approx \|R_{v,\nu}((T_\varphi f)^*)\|_{Z'(0,L)} \approx \|R_{v,\nu}(T_\varphi f)\|_{Z'(0,L)} \\
&= \left\| v(t) \int_0^{t^\nu} \frac{1}{\varphi(s)} \sup_{\tau\in[s,L)} \varphi(\tau) f^*(\tau)\,\mathrm{d}s \right\|_{Z'(0,L)} \\
&\le \left\| v(t) \int_0^{t^\nu} \frac{1}{\varphi(s)} \sup_{\tau\in[s,t^\nu)} \varphi(\tau) f^*(\tau)\,\mathrm{d}s \right\|_{Z'(0,L)} \\
&\quad + \left\| v(t) \sup_{\tau\in[t^\nu,L)} \varphi(\tau) f^*(\tau) \int_0^{t^\nu} \xi(s)\,\mathrm{d}s \right\|_{Z'(0,L)} \\
&= \left\| v(t) \int_0^{t^\nu} \frac{1}{\varphi(s)} \sup_{\tau\in[s,t^\nu)} \varphi(\tau) f^*(\tau)\,\mathrm{d}s \right\|_{Z'(0,L)} \\
&\quad + \left\| \sup_{\tau\in[t^\nu,L)} \varphi(\tau) f^*(\tau) \right\|_{Z'(0,L)}.
\end{aligned} \tag{2.68}$$

Since ξ satisfies (2.61), it follows from [46, Theorem 3.2] that

$$\int_0^{t^\nu} \frac{1}{\varphi(s)} \sup_{\tau\in[s,t^\nu)} \varphi(\tau) f^*(\tau)\,\mathrm{d}s \lesssim \int_0^{t^\nu} f^*(s)\,\mathrm{d}s \quad \text{for every } t \in (0,L);$$

hence

$$\left\| v(t) \int_0^{t^\nu} \frac{1}{\varphi(s)} \sup_{\tau\in[s,t^\nu)} \varphi(\tau) f^*(\tau)\,\mathrm{d}s \right\|_{Z'(0,L)} \lesssim \|R_{v,\nu}(f^*)\|_{Z'(0,L)}. \tag{2.69}$$

Since the function $(0,L) \ni t \mapsto \varphi(t^\nu)$ is equivalent to a quasiconcave function, it follows from Proposition 2.2.8 that

$$\left\| \sup_{\tau\in[t^\nu,L)} \varphi(\tau) f^*(\tau) \right\|_{Z'(0,L)} \lesssim \|\varphi(t^\nu) f^*(t^\nu)\|_{Z'(0,L)}.$$

Furthermore, we have that

$$\begin{aligned}
\|\varphi(t^\nu) f^*(t^\nu)\|_{Z'(0,L)} \le \|\varphi(t^\nu) f^{**}(t^\nu)\|_{Z'(0,L)} &\lesssim \left\| \frac{t^\nu}{\int_0^{t^\nu} \xi(s)\,\mathrm{d}s} f^{**}(t^\nu) \right\|_{Z'(0,L)} \\
&= \|R_{v,\nu}(f^*)\|_{Z'(0,L)},
\end{aligned} \tag{2.70}$$

where we used (1.1) in the first inequality and (2.61) in the second one. By combining (2.68) with (2.69) and (2.70) and using (2.64), we obtain that

$$\|T_\varphi f\|_{X'(0,L)} \lesssim \|R_{v,\nu}(f^*)\|_{Z'(0,L)} \approx \|f\|_{X'(0,L)} \quad \text{for every } f \in \mathfrak{M}^+(0,L);$$

hence T_φ is bounded on $X'(0,L)$. □

Remarks 2.2.14.

(i) If $X'(0,L)$ is the optimal domain space for $R_{v,\nu}$ and some rearrangement-invariant function space $Y(0,L)$, then $X'(0,L)$ is actually the optimal domain space for $R_{v,\nu}$ and its own optimal target space. This follows from the following. Thanks to Proposition 2.2.3 and Proposition 2.2.1, we are entitled to denote by $Z(0,L)$ the optimal target space for $R_{v,\nu}$ and $X'(0,L)$ and by $W(0,L)$ the optimal domain space for $R_{v,\nu}$ and $Z(0,L)$. We need to show that $X'(0,L) = W(0,L)$. On the one hand, since $R_{v,\nu}\colon X'(0,L) \to Z(0,L)$ is bounded and $W(0,L)$ is the optimal domain space for $R_{v,\nu}$ and $Z(0,L)$, we have that

$$X'(0,L) \hookrightarrow W(0,L).$$

On the other hand, since $R_{v,\nu}\colon X'(0,L) \to Y(0,L)$ is bounded and $Z(0,L)$ is the optimal target space for $R_{v,\nu}$ and $X'(0,L)$, we have that $Z(0,L) \hookrightarrow Y(0,L)$; consequently, $R_{v,\nu}\colon W(0,L) \to Y(0,L)$ is bounded. Finally, since $X'(0,L)$ is the optimal domain space for $R_{v,\nu}$ and $Y(0,L)$, we obtain that

$$W(0,L) \hookrightarrow X'(0,L).$$

Furthermore, by combining this observation with the remark above Proposition 2.2.4, we also obtain that, if $X(0,L)$ is the optimal target space for $H_{v,\nu}$ and some rearrangement-invariant function space $Y(0,L)$, then $X(0,L)$ is actually the optimal target space for $H_{v,\nu}$ and its own optimal domain space.

(ii) The assumptions of Theorem 2.2.13 are satisfied, for example, when $v(t) = t^{-1+\gamma}$ (and $\xi(t) = \frac{1-\gamma}{\nu} t^{\frac{1-\gamma}{\nu}-1}$), $t \in (0,L)$, provided that $\gamma \in (0,1)$ and $1 \leq \nu + \gamma \leq 2$.

We devote the rest of this subsection to so-called *sharp interation principles* for the operators $R_{v,\nu}$ and $H_{v,\nu}$.

Proposition 2.2.15. *Let* $\|\cdot\|_{X(0,L)}$ *be a rearrangement-invariant function norm. Let* $\nu_1, \nu_2 \in (0,\infty)$. *Let* $v_1\colon (0,L) \to (0,\infty)$ *be measurable. Let* $v_2\colon (0,L) \to (0,\infty)$ *be a nonincreasing function satisfying* (2.46) *with some constant* $C > 0$. *We have that*

$$\left\| v_1(t) v_2(t^{\nu_1}) t^{\nu_1} \int_0^{t^{\nu_1 \nu_2}} f^*(s)\,\mathrm{d}s \right\|_{X(0,L)} \lesssim \|R_{v_1,\nu_1}((R_{v_2,\nu_2}(f^*))^*)\|_{X(0,L)}$$

for every $f \in \mathfrak{M}^+(0,L)$, *where the multiplicative constant depends only on* C *and* ν_2.

Proof. Note that, for every $f \in \mathfrak{M}^+(0,L)$, we have that

$$\int_0^t f^*(s)\,\mathrm{d}s \leq 2^{\lceil \nu_2 \rceil} \int_0^{\frac{t}{2^{\lceil \nu_2 \rceil}}} f^*(s)\,\mathrm{d}s \leq 2^{\lceil \nu_2 \rceil} \int_0^{\frac{t}{2^{\nu_2}}} f^*(s)\,\mathrm{d}s \tag{2.71}$$

for every $t \in (0,L)$ owing to the fact that f^* is nonincreasing. Thanks to the monotonicity of v_2, the fact that v_2 satisfies (2.46) and the inequality (2.71), we have that

$$\begin{aligned}
\left\|v_1(t)v_2(t^{\nu_1})t^{\nu_1}\int_0^{t^{\nu_1\nu_2}} f^*(s)\,\mathrm{d}s\right\|_{X(0,L)} &\lesssim \left\|v_1(t)\int_0^{\frac{t^{\nu_1}}{2}} v_2(s)\,\mathrm{d}s\int_0^{t^{\nu_1\nu_2}} f^*(s)\,\mathrm{d}s\right\|_{X(0,L)}\\
&\lesssim \left\|v_1(t)\int_{\frac{t^{\nu_1}}{2}}^{t^{\nu_1}} v_2(s)\,\mathrm{d}s\int_0^{t^{\nu_1\nu_2}} f^*(s)\,\mathrm{d}s\right\|_{X(0,L)}\\
&\lesssim \left\|v_1(t)\int_{\frac{t^{\nu_1}}{2}}^{t^{\nu_1}} v_2(s)\,\mathrm{d}s\int_0^{\frac{t^{\nu_1\nu_2}}{2^{\nu_2}}} f^*(s)\,\mathrm{d}s\right\|_{X(0,L)}\\
&\le \left\|v_1(t)\int_{\frac{t^{\nu_1}}{2}}^{t^{\nu_1}} v_2(s)\int_0^{s^{\nu_2}} f^*(\tau)\,\mathrm{d}\tau\,\mathrm{d}s\right\|_{X(0,L)}\\
&\le \left\|v_1(t)\int_0^{t^{\nu_1}} v_2(s)\int_0^{s^{\nu_2}} f^*(\tau)\,\mathrm{d}\tau\,\mathrm{d}s\right\|_{X(0,L)}\\
&\le \left\|R_{v_1,\nu_1}((R_{v_2,\nu_2}(f^*))^*)\right\|_{X(0,L)}
\end{aligned}$$

for every $f \in \mathfrak{M}^+(0,L)$, where we used the Hardy–Littlewood inequality (1.3) in the last inequality. $\square$

The following theorem generalizes [26, Theorem 3.4], which is limited only to certain power weights and $L = 1$.

Theorem 2.2.16. *Let* $\|\cdot\|_{X(0,L)}$ *be a rearrangement-invariant function norm. Let* $\nu_1, \nu_2 \in (0,\infty)$. *Let* $v_1\colon (0,L) \to (0,\infty)$ *be measurable. Let* $\xi\colon (0,L) \to (0,\infty)$ *be a measurable function such that* $0 < \int_0^t \xi(s)\,\mathrm{d}s < \infty$ *for every* $t \in (0,L)$. *Set*

$$v_2(t) = \frac{1}{\int_0^{t^{\nu_2}} \xi(s)\,\mathrm{d}s},\ t \in (0,L).$$

Assume that v_2 *satisfies* (2.61) *with some multiplicative constant* $C_1 > 0$. *Set*

$$v(t) = t^{\nu_1}v_1(t)v_2(t^{\nu_1}),\ t \in (0,L),$$

and $\nu = \nu_1\nu_2$. *Assume that*

$$\begin{cases} v(t)t^{\nu} \in X(0,1) & \textit{if } L = 1,\\ v(t)t^{\nu}\chi_{(0,1)}(t) + v(t)\chi_{(1,\infty)}(t) \in X(0,\infty) & \textit{if } L = \infty. \end{cases} \tag{2.72}$$

Furthermore, set

$$\eta(t) = \frac{1}{tv(t^{\frac{1}{\nu}})},\ t \in (0,L).$$

Assume that η *and* $\frac{\eta}{\xi}$ *are nonincreasing. Furthermore, assume that* η *satisfies* (2.61) *with some multiplicative constant* $C_2 > 0$ *and that*

$$\frac{1}{t}\int_0^t s^{\nu}v(s)\,\mathrm{d}s \gtrsim t^{\nu}v(t) \quad \textit{for every } t \in (0,L). \tag{2.73}$$

We have that

$$\|R_{v_1,\nu_1}((R_{v_2,\nu_2}(f^*))^*)\|_{X(0,L)} \approx \|R_{v,\nu}(f^*)\|_{X(0,L)} \tag{2.74}$$

for every $f \in \mathfrak{M}^+(0,L)$, *where the multiplicative constants depend only on* ν_1, ν_2, C_1, C_2 *and the multiplicative constant in* (2.73).

Proof. First, note that the fact that v_2 is nonincreasing and satisfies (2.61) ensures that v_2 also satisfies (2.46). Hence

$$\|R_{v_1,\nu_1}((R_{v_2,\nu_2}(f^*))^*)\|_{X(0,L)} \gtrsim \|R_{v,\nu}(f^*)\|_{X(0,L)} \quad \text{for every } f \in \mathfrak{M}^+(0,L)$$

thanks to Proposition 2.2.15. Therefore, we only need to prove the opposite inequality.

Since we assume (2.72), Proposition 2.2.1 with $u \equiv 1$ guarantees that there is a rearrangement-invariant function space $Z(0,L)$ such that

$$\|f\|_{Z(0,L)} = \|R_{v,\nu}(f^*)\|_{X(0,L)} \quad \text{for every } f \in \mathfrak{M}^+(0,L).$$

Furthermore, by (2.31) and the Hardy–Littlewood inequality (1.3), we have that

$$\sup_{\|g\|_{X'(0,L)}\leq 1} \|H_{v,\nu}g\|_{Z'(0,L)} = 1. \tag{2.75}$$

Note that, for every $f \in \mathfrak{M}^+(0,L)$, the function

$$(0,L) \ni t \mapsto v_2(t)\int_0^{t^{\nu_2}} \xi(s) \sup_{\tau\in[s,L)} \frac{1}{\xi(\tau)} f^*(\tau)\,\mathrm{d}s$$

is nonincreasing because it is the integral mean of the nonincreasing function $(0,L) \ni s \mapsto \sup\limits_{\tau\in[s,L)} \frac{1}{\xi(\tau)} f^*(\tau)$ over the interval $(0,t^{\nu_2})$ with respect to the measure $\xi(s)\,\mathrm{d}s$. In particular, it coincides with its nonincreasing rearrangement. By (1.8) and (2.3), we have that

$$\begin{aligned}
&\|R_{v_1,\nu_1}((R_{v_2,\nu_2}(f^*))^*)\|_{X(0,L)} = \sup_{\|g\|_{X'(0,L)}\leq 1} \int_0^L (R_{v_2,\nu_2}(f^*))^*(t) H_{v_1,\nu_1}g(t)\,\mathrm{d}t\\
&= \sup_{\|g\|_{X'(0,L)}\leq 1} \int_0^L \left[v_2(s)\int_0^{s^{\nu_2}} f^*(\tau)\,\mathrm{d}\tau\right]^*(t) H_{v_1,\nu_1}g(t)\,\mathrm{d}t\\
&\leq \sup_{\|g\|_{X'(0,L)}\leq 1} \int_0^L \left[v_2(s)\int_0^{s^{\nu_2}} \xi(\tau) \sup_{x\in[\tau,L)} \frac{1}{\xi(x)} f^*(x)\,\mathrm{d}\tau\right]^*(t) H_{v_1,\nu_1}g(t)\,\mathrm{d}t\\
&= \sup_{\|g\|_{X'(0,L)}\leq 1} \int_0^L v_2(t)\int_0^{t^{\nu_2}} \xi(s) \sup_{\tau\in[s,L)} \frac{1}{\xi(\tau)} f^*(\tau)\,\mathrm{d}s\, H_{v_1,\nu_1}g(t)\,\mathrm{d}t\\
&= \sup_{\|g\|_{X'(0,L)}\leq 1} \int_0^L \Big(\xi(s) \sup_{\tau\in[s,L)} \frac{1}{\xi(\tau)} f^*(\tau)\Big)\Big(\int_{s^{\frac{1}{\nu_2}}}^L v_2(t)\int_{t^{\frac{1}{\nu_1}}}^L g(x)v_1(x)\,\mathrm{d}x\,\mathrm{d}t\Big)\,\mathrm{d}s\\
&\leq \Big\|\xi(t) \sup_{s\in[t,L)} \frac{1}{\xi(s)} f^*(s)\Big\|_{Z(0,L)} \sup_{\|g\|_{X'(0,L)}\leq 1} \Big\|\int_{t^{\frac{1}{\nu_2}}}^L v_2(s)\int_{s^{\frac{1}{\nu_1}}}^L g(\tau)v_1(\tau)\,\mathrm{d}\tau\,\mathrm{d}s\Big\|_{Z'(0,L)}\\
&= \Big\|\xi(t) \sup_{s\in[t,L)} \frac{1}{\xi(s)} f^*(s)\Big\|_{Z(0,L)} \sup_{\|g\|_{X'(0,L)}\leq 1} \Big\|\int_{t^{\frac{1}{\nu}}}^L g(\tau)v_1(\tau)\int_{t^{\frac{1}{\nu_2}}}^{\tau^{\nu_1}} v_2(s)\,\mathrm{d}s\,\mathrm{d}\tau\Big\|_{Z'(0,L)}\\
&\lesssim \Big\|\xi(t) \sup_{s\in[t,L)} \frac{1}{\xi(s)} f^*(s)\Big\|_{Z(0,L)} \sup_{\|g\|_{X'(0,L)}\leq 1} \Big\|\int_{t^{\frac{1}{\nu}}}^L g(\tau)v_1(\tau)\tau^{\nu_1}v_2(\tau^{\nu_1})\,\mathrm{d}\tau\Big\|_{Z'(0,L)}\\
&= \Big\|\xi(t) \sup_{s\in[t,L)} \frac{1}{\xi(s)} f^*(s)\Big\|_{Z(0,L)} \sup_{\|g\|_{X'(0,L)}\leq 1} \|H_{v,\nu}g\|_{Z'(0,L)}\\
&= \Big\|\xi(t) \sup_{s\in[t,L)} \frac{1}{\xi(s)} f^*(s)\Big\|_{Z(0,L)},
\end{aligned}$$

for every $f \in \mathfrak{M}^+(0,L)$, where we used Fubini's theorem in the fourth and fifth equalities, the Hölder inequality (1.14) in the second inequality, the fact that v_2 satisfies (2.61) in the last inequality, and (2.75) in the last equality. Therefore, the proof will be finished once we show that

$$\Big\|\xi(t) \sup_{s\in[t,L)} \frac{1}{\xi(s)} f^*(s)\Big\|_{Z(0,L)} \lesssim \|R_{v,\nu}(f^*)\|_{X(0,L)} \quad \text{for every } f \in \mathfrak{M}^+(0,L).$$

Since the function $\frac{\eta}{\xi}$ is nonincreasing, we have that

$$\Big\|\xi(t) \sup_{s\in[t,L)} \frac{1}{\xi(s)} f^*(s)\Big\|_{Z(0,L)} \leq \Big\|\eta(t) \sup_{s\in[t,L)} \frac{1}{\eta(s)} f^*(s)\Big\|_{Z(0,L)}$$

for every $f \in \mathfrak{M}^+(0,L)$. Hence it is sufficient to show that

$$\Big\|\eta(t) \sup_{s\in[t,L)} \frac{1}{\eta(s)} f^*(s)\Big\|_{Z(0,L)} \lesssim \|R_{v,\nu}(f^*)\|_{X(0,L)} \quad \text{for every } f \in \mathfrak{M}^+(0,L). \tag{2.76}$$

Being the product of two nonincreasing functions, the function $(0,L) \ni t \mapsto \eta(t)\sup_{s\in[t,L)} \frac{1}{\eta(s)} f^*(s)$ is nonincreasing for every $f \in \mathfrak{M}^+(0,L)$. Thanks to that fact, we have that

$$\begin{aligned} \Big\|\eta(t) \sup_{s\in[t,L)} \frac{1}{\eta(s)} f^*(s)\Big\|_{Z(0,L)} &= \Big\|v(t)\int_0^{t^\nu} \eta(s) \sup_{\tau\in[s,L)} \frac{1}{\eta(\tau)} f^*(\tau)\,\mathrm{d}s\Big\|_{X(0,L)} \\ &\leq \Big\|v(t)\int_0^{t^\nu} \eta(s) \sup_{\tau\in[s,t^\nu)} \frac{1}{\eta(\tau)} f^*(\tau)\,\mathrm{d}s\Big\|_{X(0,L)} \\ &\quad + \Big\|v(t) \sup_{\tau\in[t^\nu,L)} \frac{1}{\eta(\tau)} f^*(\tau) \int_0^{t^\nu} \eta(s)\,\mathrm{d}s\Big\|_{X(0,L)} \end{aligned} \tag{2.77}$$

for every $f \in \mathfrak{M}^+(0,L)$. Since the function η is nonincreasing and satisfies (2.61), [46, Theorem 3.2] guarantees that

$$\int_0^{t^\nu} \eta(s) \sup_{\tau\in[s,t^\nu)} \frac{1}{\eta(\tau)} f^*(\tau)\,\mathrm{d}s \lesssim \int_0^{t^\nu} f^*(s)\,\mathrm{d}s$$

for every $t \in (0,L)$ and every $f \in \mathfrak{M}^+(0,L)$, whence

$$\begin{aligned} \Big\|v(t)\int_0^{t^\nu} \eta(s) \sup_{\tau\in[s,t^\nu)} \frac{1}{\eta(\tau)} f^*(\tau)\,\mathrm{d}s\Big\|_{X(0,L)} &\lesssim \Big\|v(t)\int_0^{t^\nu} f^*(s)\,\mathrm{d}s\Big\|_{X(0,L)} \\ &= \|R_{v,\nu}(f^*)\|_{X(0,L)} \end{aligned} \tag{2.78}$$

for every $f \in \mathfrak{M}^+(0,L)$. Furthermore, thanks to the fact that η satisfies (2.61) again, we have that

$$\begin{aligned} \Big\|v(t) \sup_{\tau\in[t^\nu,L)} \frac{1}{\eta(\tau)} f^*(\tau) \int_0^{t^\nu} \eta(s)\,\mathrm{d}s\Big\|_{X(0,L)} &\lesssim \Big\|v(t)t^\nu\eta(t^\nu) \sup_{\tau\in[t^\nu,L)} \frac{1}{\eta(\tau)} f^*(\tau)\Big\|_{X(0,L)} \\ &= \Big\| \sup_{\tau\in[t^\nu,L)} \frac{1}{\eta(\tau)} f^*(\tau)\Big\|_{X(0,L)} \\ &= \Big\| \sup_{\tau\in[t,L)} \frac{1}{\eta(\tau^\nu)} f^*(\tau^\nu)\Big\|_{X(0,L)} \end{aligned} \tag{2.79}$$

for every $f \in \mathfrak{M}^+(0,L)$. We claim that

$$\Big\| \sup_{\tau\in[t,L)} \frac{1}{\eta(\tau^\nu)} f^*(\tau^\nu) \Big\|_{X(0,L)} \lesssim \|R_{v,\nu}(f^*)\|_{X(0,L)}. \tag{2.80}$$

Thanks to the Hardy–Littlewood–Pólya principle (1.6), it is sufficient to show that

$$\int_0^t \sup_{\tau\in[s,L)} \frac{1}{\eta(\tau^\nu)} f^*(\tau^\nu)\,\mathrm{d}s \lesssim \int_0^t (R_{v,\nu}(f^*))^*(s)\,\mathrm{d}s \quad \text{for every } t\in(0,L). \tag{2.81}$$

To this end, we have that

$$\begin{aligned} \int_0^t \sup_{\tau\in[s,t)} \frac{1}{\eta(\tau^\nu)} f^*(\tau^\nu)\,\mathrm{d}s &\lesssim \int_0^t \frac{1}{\eta(s^\nu)} f^*(s^\nu)\,\mathrm{d}s = \int_0^t s^\nu v(s) f^*(s^\nu)\,\mathrm{d}s \\ &\le \int_0^t R_{v,\nu}(f^*)(s)\,\mathrm{d}s \le \int_0^t (R_{v,\nu}(f^*))^*(s)\,\mathrm{d}s \end{aligned} \tag{2.82}$$

for every $t \in (0,L)$, where the first inequality follows from [46, Theorem 3.2] (the fact that the function $(0,L) \ni s \mapsto \frac{1}{\eta(s^\nu)} = s^\nu v(s)$ is nondecreasing and satisfies (2.73) was used here), the second inequality follows from (1.1), and the last one follows from the Hardy–Littlewood inequality (1.3). Furthermore, owing to (2.73) again, we have that

$$\begin{aligned} \sup_{\tau\in[t,L)} \frac{1}{\eta(\tau^\nu)} f^*(\tau^\nu) &= \sup_{\tau\in[t,L)} \tau^\nu v(\tau) f^*(\tau^\nu) \lesssim \sup_{\tau\in[t,L)} \frac{1}{\tau}\int_0^\tau s^\nu v(s)\,\mathrm{d}s\, f^*(\tau^\nu) \\ &\le \sup_{\tau\in[t,L)} \frac{1}{\tau}\int_0^\tau s^\nu v(s) f^*(s^\nu)\,\mathrm{d}s \le \sup_{\tau\in[t,L)} \frac{1}{\tau}\int_0^\tau R_{v,\nu}(f^*)(s)\,\mathrm{d}s \\ &\le \sup_{\tau\in[t,L)} \frac{1}{\tau}\int_0^\tau (R_{v,\nu}(f^*))^*(s)\,\mathrm{d}s = \frac{1}{t}\int_0^t (R_{v,\nu}(f^*))^*(s)\,\mathrm{d}s. \end{aligned} \tag{2.83}$$

Inequality (2.81) now follows from (2.82) and (2.83) inasmuch as

$$\int_0^t \sup_{\tau\in[s,L)} \frac{1}{\eta(\tau^\nu)} f^*(\tau^\nu)\,\mathrm{d}s \le \int_0^t \sup_{\tau\in[s,t)} \frac{1}{\eta(\tau^\nu)} f^*(\tau^\nu)\,\mathrm{d}s + t \sup_{\tau\in[t,L)} \frac{1}{\eta(\tau^\nu)} f^*(\tau^\nu)$$

for every $t \in (0,L)$.

Finally, by combining (2.77) with (2.78), (2.79) and (2.80), we obtain (2.76). □

Remarks 2.2.17.

(i) Since Theorem 2.2.16 has several assumptions, it is instructive to provide concrete, important examples. Let $\nu_1, \nu_2, \gamma_1, \gamma_2 \in (0,\infty)$. Set $v_j(t) = t^{\gamma_j - 1}$, $t \in (0,L)$, $j = 1,2$. If

$$\gamma_1 + \nu_1 \ge 1,\ \gamma_1 + \gamma_2\nu_1 < 1,\ \gamma_1 + \nu_1(\nu_2 + \gamma_2) \ge 1 \tag{2.84}$$

(note that, in turn, $\gamma_j < 1$, $j = 1,2$) and

$$t^{\gamma_1+\gamma_2\nu_1}\chi_{(1,\infty)}(t) \in X(0,\infty) \quad \text{if } L = \infty, \tag{2.85}$$

then all assumptions of Theorem 2.2.16 are satisfied (with $\xi(t) = \frac{\nu_2}{1-\gamma_2} t^{\frac{1-\gamma_2}{\nu_2}-1}$) and we have that

$$\Big\| t^{\gamma_1 - 1} \int_0^{t^{\nu_1}} \big[\tau^{\gamma_2 - 1} \int_0^{\tau^{\nu_2}} f^*(\sigma) \, \mathrm{d}\sigma\big]^*(s) \, \mathrm{d}s \Big\|_{X(0,L)} \approx \Big\| t^{\gamma_1 + \gamma_2 \nu_1} \int_0^{t^{\nu_1 \nu_2}} f^*(s) \, \mathrm{d}s \Big\|_{X(0,L)}$$

for every $f \in \mathfrak{M}^+(0,L)$. Moreover, the multiplicative constants depend only on ν_1, ν_2, γ_1 and γ_2. Furthermore, assumptions (2.84) and (2.85) are actually also necessary for this choice of functions v_j, $j = 1, 2$. Note that assumptions (2.84) are satisfied provided that

$$\gamma_1 + \nu_1 \geq 1, \ \gamma_1 + \gamma_2 \nu_1 < 1, \ \nu_2 + \gamma_2 \geq 1. \tag{2.86}$$

With $L = 1$ and the parameters $\nu_1, \nu_2, \gamma_1, \gamma_2 \in (0, \infty)$ satisfying (2.86), Theorem 2.2.16 recovers [26, Theorem 3.4].

(ii) When the functions η and $\frac{\eta}{\xi}$ are merely equivalent to nonincreasing functions, (2.74) remains valid but with different multiplicative constants.

We conclude this subsection with a $H_{v,\nu}$ counterpart to Theorem 2.2.16, whose proof is substantially more simpler than that of the theorem.

Proposition 2.2.18. *Let* $\|\cdot\|_{X(0,L)}$ *be a rearrangement-invariant function norm. Let* $\nu_1, \nu_2 \in (0,\infty)$*. Let* $v_1 \colon (0,L) \to (0,\infty)$ *be a nonincreasing function satisfying* (2.61) *with some multiplicative constant* $C > 0$*. Let* $v_2 \colon (0,L) \to (0,\infty)$ *be measurable. Set*

$$v(t) = t^{\nu_2} v_1(t^{\nu_2}) v_2(t), \ t \in (0,L),$$

and $\nu = \nu_1 \nu_2$*. We have that*

$$\|H_{v_1,\nu_1}(H_{v_2,\nu_2} f)\|_{X(0,L)} \approx \|H_{v,\nu} f\|_{X(0,L)} \quad \text{for every } f \in \mathfrak{M}^+(0,L),$$

where the multiplicative constants depend only on ν_1 *and* C*.*

Proof. On the one hand, we have that

$$\begin{aligned}
\|H_{v_1,\nu_1}(H_{v_2,\nu_2} f)\|_{X(0,L)} &= \Big\| \int_{t^{\frac{1}{\nu_1}}}^L \int_{s^{\frac{1}{\nu_2}}}^L f(\tau) v_2(\tau) \, \mathrm{d}\tau \, v_1(s) \, \mathrm{d}s \Big\|_{X(0,L)} \\
&= \Big\| \int_{t^{\frac{1}{\nu}}}^L f(\tau) v_2(\tau) \int_{t^{\frac{1}{\nu_1}}}^{\tau^{\nu_2}} v_1(s) \, \mathrm{d}s \, \mathrm{d}\tau \Big\|_{X(0,L)} \\
&\leq \Big\| \int_{t^{\frac{1}{\nu}}}^L f(\tau) v_2(\tau) \int_0^{\tau^{\nu_2}} v_1(s) \, \mathrm{d}s \, \mathrm{d}\tau \Big\|_{X(0,L)} \\
&\lesssim \Big\| \int_{t^{\frac{1}{\nu}}}^L f(\tau) v_2(\tau) \tau^{\nu_2} v_1(\tau^{\nu_2}) \, \mathrm{d}\tau \Big\|_{X(0,L)}
\end{aligned}$$

for every $f \in \mathfrak{M}^+(0,L)$ thanks to the fact that v_1 satisfies (2.61). On the other hand, we have that

$$
\begin{aligned}
\|H_{v_1,\nu_1}(H_{v_2,\nu_2}f)\|_{X(0,L)} &= \Big\| \int_{t^{\frac{1}{\nu}}}^{L} f(\tau)v_2(\tau) \int_{t^{\frac{1}{\nu_1}}}^{\tau^{\nu_2}} v_1(s)\,\mathrm{d}s\,\mathrm{d}\tau \Big\|_{X(0,L)} \\
&\geq \Big\| \chi_{(0,2^{-\nu_1}L)}(t) \int_{2^{\frac{1}{\nu_2}} t^{\frac{1}{\nu}}}^{L} f(\tau)v_2(\tau) \int_{t^{\frac{1}{\nu_1}}}^{\tau^{\nu_2}} v_1(s)\,\mathrm{d}s\,\mathrm{d}\tau \Big\|_{X(0,L)} \\
&\geq \Big\| \chi_{(0,2^{-\nu_1}L)}(t) \int_{2^{\frac{1}{\nu_2}} t^{\frac{1}{\nu}}}^{L} f(\tau)v_2(\tau)v_1(\tau^{\nu_2})(\tau^{\nu_2} - t^{\frac{1}{\nu_1}})\,\mathrm{d}\tau \Big\|_{X(0,L)} \\
&\geq \frac{1}{2}\Big\| \chi_{(0,2^{-\nu_1}L)}(t) \int_{2^{\frac{1}{\nu_2}} t^{\frac{1}{\nu}}}^{L} f(\tau)v_2(\tau)v_1(\tau^{\nu_2})\tau^{\nu_2}\,\mathrm{d}\tau \Big\|_{X(0,L)} \\
&\geq \frac{1}{2^{1+\nu_1}}\Big\| \int_{t^{\frac{1}{\nu}}}^{L} f(\tau)v_2(\tau)v_1(\tau^{\nu_2})\tau^{\nu_2}\,\mathrm{d}\tau \Big\|_{X(0,L)}
\end{aligned}
$$

for every $f \in \mathfrak{M}^+(0,L)$, where we used the monotonicity of v_1 and the boundedness of the dilation operator $D_{2^{\nu_1}}$ (1.13). ☐

2.2.2 Calderón-type operators

Throughout this subsection, we assume that $L = \infty$.

Proposition 2.2.19. *Let* $\|\cdot\|_{Y(0,\infty)}$ *be a rearrangement-invariant function norm. Let* $\sigma = (\alpha,\beta,\gamma,\delta) \in [0,1]^4$ *be such that* $0 \leq \gamma < \alpha \leq 1$ *and* $0 \leq \delta < \beta \leq 1$. *Assume that* $\xi \in Y(0,\infty)$, *where* ξ *is defined by* (2.26). *Let* $\|\cdot\|_{X(0,\infty)}$ *be the rearrangement-invariant function norm defined as*

$$
\|f\|_{X(0,\infty)} = \|S_\sigma(f^*)\|_{Y(0,\infty)},\ f \in \mathfrak{M}^+(0,\infty). \tag{2.87}
$$

The rearrangement-invariant function space $X(0,\infty)$ *is the optimal domain space for the operator* S_σ *and the space* $Y(0,\infty)$.

Moreover, if $\xi \notin Y(0,\infty)$, *then there is no rearrangement-invariant function space* $Z(0,\infty)$ *such that* $S_\sigma\colon Z(0,\infty) \to Y(0,\infty)$ *is bounded.*

Proof. First, note that the functional defined by (2.87) is indeed a rearrangement-invariant function norm thanks to Proposition 2.1.4, and so we are entitled to denote the corresponding rearrangement-invariant function space by $X(0,\infty)$. Second, owing to (2.20), we have that

$$
\|S_\sigma f\|_{Y(0,\infty)} \leq \|S_\sigma(f^*)\|_{Y(0,\infty)} = \|f\|_{X(0,\infty)} \quad \text{for every } f \in \mathfrak{M}^+(0,\infty).
$$

Hence $S_\sigma\colon X(0,\infty) \to Y(0,\infty)$ is bounded. Next, if $Z(0,\infty)$ is a rearrangement-invariant function space such that $S_\sigma\colon Z(0,\infty) \to Y(0,\infty)$ is bounded, then we have that

$$
\|f\|_{X(0,\infty)} = \|S_\sigma(f^*)\|_{Y(0,\infty)} \lesssim \|f^*\|_{Z(0,\infty)} = \|f\|_{Z(0,\infty)} \quad \text{for every } f \in \mathfrak{M}^+(0,\infty).
$$

Hence $Z(0,\infty) \hookrightarrow X(0,\infty)$. Finally, note that, if $S_\sigma\colon Z(0,\infty) \to Y(0,\infty)$ is bounded, then

$$
\|\xi\|_{Y(0,\infty)} \approx \|S_\sigma(\chi_{(0,1)})\|_{Y(0,\infty)} \lesssim \|\chi_{(0,1)}\|_{Z(0,\infty)} < \infty,
$$

where the equivalence was established in the proof of Proposition 2.1.4. ☐

The next two propositions are not surprising in light of (2.25).

Proposition 2.2.20. *Let* $\|\cdot\|_{X(0,\infty)}$ *and* $\|\cdot\|_{Y(0,\infty)}$ *be rearrangement-invariant function norms. Let* $\sigma=(\alpha,\beta,\gamma,\delta)\in[0,1]^4$ *be such that* $0\leq\gamma<\alpha\leq 1$ *and* $0\leq\delta<\beta\leq 1$*. We have that*

$$\sup_{\|f\|_{X(0,\infty)}\leq 1}\|S_\sigma f\|_{Y(0,\infty)}=\sup_{\|g\|_{Y'(0,\infty)}\leq 1}\|S_{\sigma'}g\|_{X'(0,\infty)},$$

where σ' *is defined by* (2.24)*. In particular,*

$$\begin{aligned}&S_\sigma\colon X(0,\infty)\to Y(0,\infty) &&\text{is bounded if and only if}\\ &S_{\sigma'}\colon Y'(0,\infty)\to X'(0,\infty) &&\text{is bounded.}\end{aligned}\tag{2.88}$$

Proof. We have that

$$\begin{aligned}\sup_{\|f\|_{X(0,\infty)}\leq 1}\|S_\sigma f\|_{Y(0,\infty)}&=\sup_{\|f\|_{X(0,\infty)}\leq 1}\sup_{\|g\|_{Y'(0,\infty)}\leq 1}\int_0^L S_\sigma f(t)|g(t)|\,\mathrm{d}t\\ &=\sup_{\|f\|_{X(0,\infty)}\leq 1}\sup_{\|g\|_{Y'(0,\infty)}\leq 1}\int_0^\infty |f(t)|S_{\sigma'}g(t)\,\mathrm{d}t\\ &=\sup_{\|g\|_{Y'(0,\infty)}\leq 1}\|S_{\sigma'}g\|_{X'(0,\infty)}\end{aligned}$$

thanks to (1.8), (2.25) and (1.7). □

The following proposition describes the optimal target space for S_σ and a given rearrangement-invariant function space.

Proposition 2.2.21. *Let* $\|\cdot\|_{X(0,\infty)}$ *be a rearrangement-invariant function norm. Let* $\sigma=(\alpha,\beta,\gamma,\delta)\in[0,1]^4$ *be such that* $0\leq\gamma<\alpha\leq 1$ *and* $0\leq\delta<\beta\leq 1$*. Assume that* $\xi\in X'(0,\infty)$*, where* ξ *is defined by* (2.26) *with parameters* $(\tilde{\alpha},\tilde{\beta},\tilde{\gamma},\tilde{\delta})=\sigma'$*, where the parameters with tildes correspond to those used in* (2.26)*.*

Let $\|\cdot\|_{Y(0,\infty)}$ *be the rearrangement-invariant function norm whose associate function norm* $\|\cdot\|_{Y'(0,\infty)}$ *is defined as*

$$\|f\|_{Y'(0,\infty)}=\|S_{\sigma'}(f^*)\|_{X'(0,\infty)},\ f\in\mathfrak{M}^+(0,\infty).\tag{2.89}$$

The rearrangement-invariant function space $Y(0,\infty)$ *is the optimal target space for the operator* S_σ *and the space* $X(0,\infty)$*.*

Moreover, if $\xi\notin X'(0,\infty)$*, then there is no rearrangement-invariant function space* $Z(0,\infty)$ *such that* $S_\sigma\colon X(0,\infty)\to Z(0,\infty)$ *is bounded.*

Proof. First, note that the functional defined by (2.89) is indeed a rearrangement-invariant function norm thanks to Proposition 2.1.4. Therefore, owing to (1.9), we are entitled to denote by $\|\cdot\|_{Y(0,\infty)}$ the rearrangement-invariant function norm whose associate function norm is defined by (2.89).

Second, since we plainly have that

$$\|S_{\sigma'}(f^*)\|_{X'(0,\infty)}\leq\|f\|_{Y'(0,\infty)}\quad\text{for every } f\in\mathfrak{M}^+(0,\infty),$$

it follows that $S_\sigma\colon X(0,\infty)\to Y(0,\infty)$ is bounded thanks to (2.88). Next, let $Z(0,\infty)$ be a rearrangement-invariant function space such that $S_\sigma\colon X(0,\infty)\to Z(0,\infty)$ is bounded. Owing to (2.88) again, this is equivalent to the fact that $S_{\sigma'}\colon Z'(0,\infty)\to X'(0,\infty)$ is bounded; consequently,

$$\|f\|_{Y'(0,\infty)}\lesssim\|f\|_{Z'(0,\infty)}\quad\text{for every } f\in\mathfrak{M}^+(0,\infty).$$

Hence $Z'(0,\infty)\hookrightarrow Y'(0,\infty)$, which is equivalent to $Y(0,\infty)\hookrightarrow Z(0,\infty)$ by (1.15).

Finally, if there is any rearrangement-invariant function space $Z(0,\infty)$ such that $S_{\sigma'}\colon Z'(0,\infty)\to X'(0,\infty)$ is bounded, then ξ has to belong to $X'(0,\infty)$. Indeed, we have that

$$\|\xi\|_{X'(0,\infty)}\approx\|S_{\sigma'}(\chi_{(0,1)})\|_{X'(0,\infty)}\lesssim\|\chi_{(0,1)}\|_{Z'(0,\infty)}<\infty,$$

where the equivalence was established in the proof of Proposition 2.1.4. □

2.2.3 Particular examples of optimal function norms

In this section, we provide some particular examples of optimal function norms for the operators studied in the previous two sections and Lorentz–Zygmund spaces.

2.2.3.1 Optimal spaces for $R_{v,\nu}$

Throughout the following two subsections, we denote by v the function $v(t)=t^{-1+\gamma}$, $t\in(0,L)$, where $\gamma\in[0,1)$. For the sake of readability, the results of these two subsections are split based on whether L is finite or infinite and whether γ is positive or zero.

First, we provide optimal domain spaces for the operator $R_{v,\nu}$ and Lorentz–Zygmund spaces.

Proposition 2.2.22. *Let $\gamma\in(0,1)$ and $\nu\in(0,\infty)$. Assume that $L^{p,q;\mathbb{A}}(0,\infty)$ is equivalent to a rearrangement-invariant function space, that is, the parameters p,q and $\mathbb{A}=(\alpha_0,\alpha_\infty)$ satisfy one of conditions (1.28) (with $\beta_0=\beta_\infty=0$). There is a rearrangement-invariant function space $Z(0,\infty)$ such that the operator $R_{v,\nu}\colon Z(0,\infty)\to L^{p,q;\mathbb{A}}(0,\infty)$ is bounded if and only if one of the following conditions holds:*

- *$p=\frac{1}{1-\gamma}$ and*
    - ⋆ *$q<\infty$, $\alpha_\infty+\frac{1}{q}<0$ or*
    - ⋆ *$q=\infty$, $\alpha_\infty\le 0$;*
- *$\gamma+\nu\ge 1$ and $p>\frac{1}{1-\gamma}$;*
- *$\gamma+\nu<1$ and $p\in(\frac{1}{1-\gamma},\frac{1}{1-\gamma-\nu})$;*
- *$\gamma+\nu<1$, $p=\frac{1}{1-\gamma-\nu}$ and*
    - ⋆ *$q<\infty$, $\alpha_0+\frac{1}{q}<0$ or*
    - ⋆ *$q=\infty$, $\alpha_0\le 0$.*

Furthermore, if this is the case and we denote by $X(0,\infty)$ the optimal domain space for the operator $R_{v,\nu}$ and the space $L^{p,q;\mathbb{A}}(0,\infty)$, then

$$X(0,\infty)=\begin{cases} L^{1,1;(0,\alpha_\infty+1)}(0,\infty), & p=\frac{1}{1-\gamma},\ q=1,\ \alpha_0+1<0,\ \alpha_\infty+1<0;\\ L^{1,1;\mathbb{A}+1}(0,\infty), & p=\frac{1}{1-\gamma},\ q=1,\ \alpha_0+1>0,\ \alpha_\infty+1<0;\\ L^{1,1;(0,\alpha_\infty+1),(1,0)}(0,\infty), & p=\frac{1}{1-\gamma},\ q=1,\ \alpha_0+1=0,\ \alpha_\infty+1<0;\\ L^{(1,q;\mathbb{A})}(0,\infty), & p=\frac{1}{1-\gamma},\ q\in(1,\infty),\ \alpha_\infty+\frac{1}{q}<0 \textit{ or}\\ & p=\frac{1}{1-\gamma},\ q=\infty,\ \alpha_\infty\le 0;\\ L^{\frac{\nu p}{1+p(\gamma+\nu-1)},q;\mathbb{A}}(0,\infty), & p\in(\frac{1}{1-\gamma},\frac{1}{1-\gamma-\nu}),\ \gamma+\nu<1 \textit{ or}\\ & p\in(\frac{1}{1-\gamma},\infty),\ \gamma+\nu\ge 1;\\ L^{\infty,q;\mathbb{A}}(0,\infty), & p=\frac{1}{1-\gamma-\nu},\ \alpha_0+\frac{1}{q}<0,\ \gamma+\nu\le 1 \textit{ or}\\ & p=\frac{1}{1-\gamma-\nu},\ q=\infty,\ \alpha_0\le 0,\ \gamma+\nu\le 1;\\ L^{\frac{\nu}{\gamma+\nu-1},\infty;\mathbb{A}}(0,\infty), & p=q=\infty,\ \alpha_0\le 0,\ \alpha_\infty\ge 0,\ \gamma+\nu>1. \end{cases}$$

Proof. Thanks to Proposition 2.2.1, there is a rearrangement-invariant function space $Z(0,\infty)$ such that $R_{v,\nu}\colon Z(0,\infty)\to L^{p,q;\mathbb{A}}(0,\infty)$ is bounded if and only if

$$\|t^{-1+\gamma+\nu}\chi_{(0,1)}(t)\|_{L^{p,q;\mathbb{A}}(0,\infty)}<\infty \quad\text{and}\quad \|t^{-1+\gamma}\chi_{(1,\infty)}(t)\|_{L^{p,q;\mathbb{A}}(0,\infty)}<\infty. \tag{2.90}$$

If $\gamma+\nu\ge 1$, then we plainly have that

$$\|t^{-1+\gamma+\nu}\chi_{(0,1)}(t)\|_{L^{p,q;\mathbb{A}}(0,\infty)}\le\|\chi_{(0,1)}\|_{L^{p,q;\mathbb{A}}(0,\infty)}<\infty.$$

If $\gamma+\nu<1$ and $q<\infty$, then we have that

$$\begin{aligned}\|t^{-1+\gamma+\nu}\chi_{(0,1)}(t)\|^q_{L^{p,q;\mathbb{A}}(0,\infty)}&=\|t^{\frac{1}{p}-\frac{1}{q}+\gamma+\nu-1}\ell^{\mathbb{A}}(t)\chi_{(0,1)}(t)\|^q_{L^q(0,\infty)}\\&=\int_0^1 t^{q(\frac{1}{p}+\gamma+\nu-1)-1}\ell^{q\alpha_0}(t)\,\mathrm{d}t,\end{aligned}$$

which is finite if and only if $\frac{1}{p}>1-\gamma-\nu$, or $\frac{1}{p}=1-\gamma-\nu$ and $q\alpha_0<-1$. If $\gamma+\nu<1$ and $q=\infty$, then we have that

$$\|t^{-1+\gamma+\nu}\chi_{(0,1)}(t)\|_{L^{p,\infty;\mathbb{A}}(0,\infty)}=\sup_{t\in(0,1)}t^{\frac{1}{p}+\gamma+\nu-1}\ell^{\alpha_0}(t),$$

which is finite if and only if $\frac{1}{p}>1-\gamma-\nu$, or $\frac{1}{p}=1-\gamma-\nu$ and $\alpha_0\le 0$. As for the second term in (2.90), whether $\gamma+\nu\ge 1$ or $\gamma+\nu<1$, we have that, for $q<\infty$,

$$\begin{aligned}\|t^{-1+\gamma}\chi_{(1,\infty)}(t)\|^q_{L^{p,q;\mathbb{A}}(0,\infty)}&\approx\|t^{\frac{1}{p}-\frac{1}{q}+\gamma-1}\ell^{\mathbb{A}}(t)\chi_{(1,\infty)}(t)\|^q_{L^q(0,\infty)}\\&=\int_1^\infty t^{q(\frac{1}{p}+\gamma-1)-1}\ell^{q\alpha_\infty}(t)\,\mathrm{d}t,\end{aligned}$$

which is finite if and only if $\frac{1}{p}<1-\gamma$, or $\frac{1}{p}=1-\gamma$ and $q\alpha_\infty<-1$. If $q=\infty$, then

$$\|t^{-1+\gamma}\chi_{(1,\infty)}(t)\|_{L^{p,\infty;\mathbb{A}}(0,\infty)}\approx\sup_{t\in(1,\infty)}t^{\frac{1}{p}+\gamma-1}\ell^{\alpha_\infty}(t)$$

is finite if and only if $\frac{1}{p}<1-\gamma$, or $\frac{1}{p}=1-\gamma$ and $\alpha_\infty\le 0$.

Assume now that the parameters p, q and $\mathbb{A}$ are such that (2.90) is true. Owing to Proposition 2.2.1 again, we have that

$$\|f\|_{X(0,\infty)} \approx \left\| t^{-1+\gamma} \int_0^{t^\nu} f^*(s)\,\mathrm{d}s \right\|_{L^{p,q;\mathbb{A}}(0,\infty)} \quad \text{for every } f \in \mathfrak{M}(0,\infty),$$

where $X(0,\infty)$ is the optimal domain space for the operator $R_{v,\nu}$ and the space $L^{p,q;\mathbb{A}}(0,\infty)$. First, assume that $\gamma + \nu \le 1$. Since the function $(0,\infty) \ni t \mapsto t^{-1+\gamma+\nu} f^{**}(t^\nu)$ is nonincreasing for every $f \in \mathfrak{M}(0,\infty)$, being the product of two nonincreasing functions, we have that

$$\begin{aligned}
\left\| t^{-1+\gamma} \int_0^{t^\nu} f^*(s)\,\mathrm{d}s \right\|_{L^{p,q;\mathbb{A}}(0,\infty)} &= \| t^{-1+\gamma+\nu} f^{**}(t^\nu) \|_{L^{p,q;\mathbb{A}}(0,\infty)} \\
&= \| t^{\frac{1}{p}+\gamma+\nu-1-\frac{1}{q}} \ell^{\mathbb{A}}(t) f^{**}(t^\nu) \|_{L^q(0,\infty)} \\
&\approx \| t^{\frac{1}{\nu}(\frac{1}{p}+\gamma+\nu-1)-\frac{1}{q}} \ell^{\mathbb{A}}(t) f^{**}(t) \|_{L^q(0,\infty)} \\
&= \|f\|_{L^{(\frac{\nu p}{1+p(\gamma+\nu-1)},q;\mathbb{A})}(0,\infty)}
\end{aligned}$$

for every $f \in \mathfrak{M}(0,\infty)$. Hence $X(0,\infty) = L^{(\frac{\nu p}{1+p(\gamma+\nu-1)},q;\mathbb{A})}(0,\infty)$, where $\frac{\nu p}{1+p(\gamma+\nu-1)}$ is to be interpreted as ∞ if $p = \frac{1}{1-\gamma-\nu}$. Second, assume that $\gamma + \nu > 1$. On the one hand, we have that

$$\begin{aligned}
\left\| t^{-1+\gamma} \int_0^{t^\nu} f^*(s)\,\mathrm{d}s \right\|_{L^{p,q;\mathbb{A}}(0,\infty)} &= \left\| t^{\frac{1}{p}-\frac{1}{q}} \ell^{\mathbb{A}}(t) \Big[s^{-1+\gamma+\nu} f^{**}(s^\nu) \Big]^*(t) \right\|_{L^q(0,\infty)} \\
&\le \left\| t^{\frac{1}{p}-\frac{1}{q}} \ell^{\mathbb{A}}(t) \sup_{s\in[t,\infty)} s^{-1+\gamma+\nu} f^{**}(s^\nu) \right\|_{L^q(0,\infty)}
\end{aligned} \tag{2.91}$$

for every $f \in \mathfrak{M}(0,\infty)$. Furthermore, we have that

$$\left\| t^{\frac{1}{p}-\frac{1}{q}} \ell^{\mathbb{A}}(t) \sup_{s\in[t,\infty)} s^{-1+\gamma+\nu} f^{**}(s^\nu) \right\|_{L^q(0,\infty)} \lesssim \left\| t^{\frac{1}{p}-\frac{1}{q}} \ell^{\mathbb{A}}(t) t^{-1+\gamma+\nu} f^{**}(t^\nu) \right\|_{L^q(0,\infty)} \tag{2.92}$$

for every $f \in \mathfrak{M}(0,\infty)$. Indeed, if $p, q < \infty$, the inequality follows from [46, Theorem 3.2]. If $p = q = \infty$, $\alpha_0 \le 0$, $\alpha_\infty \ge 0$, or $p < \infty$ and $q = \infty$, the inequality follows from

$$\begin{aligned}
\left\| t^{\frac{1}{p}} \ell^{\mathbb{A}}(t) \sup_{s\in[t,\infty)} s^{-1+\gamma+\nu} f^{**}(s^\nu) \right\|_{L^\infty(0,\infty)} &= \sup_{t\in(0,\infty)} t^{\frac{1}{p}} \ell^{\mathbb{A}}(t) \sup_{s\in[t,\infty)} s^{-1+\gamma+\nu} f^{**}(s^\nu) \\
&= \sup_{s\in(0,\infty)} s^{-1+\gamma+\nu} f^{**}(s^\nu) \sup_{t\in(0,s]} t^{\frac{1}{p}} \ell^{\mathbb{A}}(t) \\
&\approx \sup_{s\in(0,\infty)} s^{-1+\gamma+\nu} f^{**}(s^\nu) s^{\frac{1}{p}} \ell^{\mathbb{A}}(s),
\end{aligned}$$

where we used the fact that the function $(0,\infty) \ni t \mapsto t^{\frac{1}{p}} \ell^{\mathbb{A}}(t)$ is equivalent to a nondecreasing function. By combining (2.91) and (2.92), we obtain that

$$\begin{aligned}
\left\| t^{-1+\gamma} \int_0^{t^\nu} f^*(s)\,\mathrm{d}s \right\|_{L^{p,q;\mathbb{A}}(0,\infty)} &\lesssim \left\| t^{\frac{1}{p}+\gamma+\nu-1-\frac{1}{q}} \ell^{\mathbb{A}}(t) f^{**}(t^\nu) \right\|_{L^q(0,\infty)} \\
&\approx \left\| t^{\frac{1}{\nu}(\frac{1}{p}+\gamma+\nu-1)-\frac{1}{q}} \ell^{\mathbb{A}}(t) f^{**}(t) \right\|_{L^q(0,\infty)}
\end{aligned} \tag{2.93}$$

for every $f \in \mathfrak{M}(0,\infty)$. On the other hand, we have that

$$\begin{aligned}
\left\|t^{\frac{1}{\nu}(\frac{1}{p}+\gamma+\nu-1)-\frac{1}{q}}\ell^{\mathbb{A}}(t)f^{**}(t)\right\|_{L^q(0,\infty)} &\approx \left\|t^{\frac{1}{p}+\gamma+\nu-1-\frac{1}{q}}\ell^{\mathbb{A}}(t)f^{**}(t^\nu)\right\|_{L^q(0,\infty)} \\
&\leq \left\|t^{\frac{1}{p}-\frac{1}{q}}\ell^{\mathbb{A}}(t)\sup_{s\in[t,\infty)} s^{\gamma+\nu-1}f^{**}(s^\nu)\right\|_{L^q(0,\infty)} \\
&\approx \left\|t^{\frac{1}{p}-\frac{1}{q}}\ell^{\mathbb{A}}(t)\sup_{s\in[t,\infty)} \frac{1}{s}\int_0^s \tau^{\gamma+\nu-1}\,\mathrm{d}\tau f^{**}(s^\nu)\right\|_{L^q(0,\infty)} \\
&\leq \left\|t^{\frac{1}{p}-\frac{1}{q}}\ell^{\mathbb{A}}(t)\sup_{s\in[t,\infty)} \frac{1}{s}\int_0^s \tau^{\gamma+\nu-1}f^{**}(\tau^\nu)\,\mathrm{d}\tau\right\|_{L^q(0,\infty)} \\
&\leq \left\|t^{\frac{1}{p}-\frac{1}{q}}\ell^{\mathbb{A}}(t)\sup_{s\in[t,\infty)} [\tau^{\gamma+\nu-1}f^{**}(\tau^\nu)]^{**}(s)\right\|_{L^q(0,\infty)} \\
&= \left\|t^{\frac{1}{p}-\frac{1}{q}}\ell^{\mathbb{A}}(t)[\tau^{\gamma+\nu-1}f^{**}(\tau^\nu)]^{**}(t)\right\|_{L^q(0,\infty)} \\
&= \|t^{\gamma+\nu-1}f^{**}(t^\nu)\|_{L^{(p,q;\mathbb{A})}(0,\infty)} \\
&\approx \|t^{\gamma+\nu-1}f^{**}(t^\nu)\|_{L^{p,q;\mathbb{A}}(0,\infty)}
\end{aligned}\tag{2.94}$$

for every $f \in \mathfrak{M}(0,\infty)$, where we used the fact that the maximal nonincreasing rearrangement of a measurable function is nonincreasing in the second inequality and the first equality, the Hardy–Littlewood inequality (1.3) in the last inequality, and (1.31) in the last equivalence (recall that $p \geq \frac{1}{1-\gamma} > 1$).

By combining (2.93) and (2.94), we obtain that $X(0,\infty) = L^{(\frac{\nu p}{1+p(\gamma+\nu-1)},q;\mathbb{A})}(0,\infty)$ also when $\gamma+\nu > 1$, where $\frac{\nu p}{1+p(\gamma+\nu-1)}$ is to be interpreted as $\frac{\nu}{\gamma+\nu-1}$ if $p=\infty$.

Finally, note that $\frac{\nu p}{1+p(\gamma+\nu-1)} = 1$ if $p = \frac{1}{1-\gamma}$ and $\frac{\nu p}{1+p(\gamma+\nu-1)} > 1$ if $p > \frac{1}{1-\gamma}$; hence we have that

$$L^{(\frac{\nu p}{1+p(\gamma+\nu-1)},q;\mathbb{A})}(0,\infty) = L^{\frac{\nu p}{1+p(\gamma+\nu-1)},q;\mathbb{A}}(0,\infty) \quad \text{provided that } p > \frac{1}{1-\gamma}$$

thanks to (1.31). We finish the proof by noting that the space $L^{(1,1;\mathbb{A})}(0,\infty)$ is described by [75, Theorem 3.8].

For future reference, we note that the only point where we used the fact that $\gamma > 0$ was the last equivalence in (2.94) when $p = \frac{1}{1-\gamma}$. $\square$

The following proposition is an analogue of Proposition 2.2.22 in the case where $\gamma = 0$. As was pointed out at the end of the proof of Proposition 2.2.22, the only point where the proof fails when $\gamma = 0$ is (2.94). However, the inequality

$$\left\|t^{\frac{1}{\nu}(\frac{1}{p}+\gamma+\nu-1)-\frac{1}{q}}\ell^{\mathbb{A}}(t)f^{**}(t)\right\|_{L^q(0,\infty)} \lesssim \|t^{\gamma+\nu-1}f^{**}(t^\nu)\|_{L^{p,q;\mathbb{A}}(0,\infty)}$$

is still valid for every $f \in \mathfrak{M}(0,\infty)$ even when $\gamma = 0$, $p = \frac{1}{1-\gamma} = 1$, $\alpha_\infty < -1$ and, consequently, $q = 1$, $\alpha_0 \geq 0$ (recall (1.28)). Indeed, we have that

$$\begin{aligned}
\left\|t^{\frac{1}{\nu}(1+0+\nu-1)-1}\ell^{\mathbb{A}}(t)f^{**}(t)\right\|_{L^1(0,\infty)} &= \|t^{1-1}\ell^{\mathbb{A}}(t)f^{**}(t)\|_{L^1(0,\infty)} \\
&\approx \|t^{\nu-1}\ell^{\mathbb{A}}(t)f^{**}(t^\nu)\|_{L^1(0,\infty)} \\
&\leq \|\ell^{\mathbb{A}}(t)[s^{\nu-1}f^{**}(s^\nu)]^{*}(t)\|_{L^1(0,\infty)} \\
&= \|t^{\nu-1}f^{**}(t^\nu)\|_{L^{1,1;\mathbb{A}}(0,\infty)},
\end{aligned}$$

where we used the Hardy–Littlewood inequality (1.3) together with the fact that the function $\ell^{\mathbb{A}}$ is nonincreasing. Hence we have the following result when $\gamma = 0$.

Proposition 2.2.23. *Let $\gamma = 0$ and $\nu \in (0,\infty)$. Assume that $L^{p,q;\mathbb{A}}(0,\infty)$ is equivalent to a rearrangement-invariant function space, that is, the parameters p, q and $\mathbb{A} = (\alpha_0, \alpha_\infty)$ satisfy one of conditions* (1.28) *(with $\beta_0 = \beta_\infty = 0$). There is a rearrangement-invariant function space $Z(0,\infty)$ such that the operator $R_{v,\nu}\colon Z(0,\infty) \to L^{p,q;\alpha}(0,\infty)$ is bounded if and only if one of the following conditions holds:*

- $p = q = 1$, $\alpha_0 \geq 0$ *and* $\alpha_\infty < -1$;
- $\nu \geq 1$ *and* $p > 1$;
- $\nu < 1$ *and* $p \in (1, \frac{1}{1-\nu})$;
- $\nu < 1$, $p = \frac{1}{1-\nu}$ *and*
    - ⋆ $q < \infty$, $\alpha_0 + \frac{1}{q} < 0$ *or*
    - ⋆ $q = \infty$, $\alpha_0 \leq 0$.

Furthermore, if this is the case and we denote by $X(0,\infty)$ the optimal domain space for the operator $R_{v,\nu}$ and the space $L^{p,q;\mathbb{A}}(0,\infty)$, then

$$X(0,\infty) = \begin{cases} L^{1,1;\mathbb{A}+1}(0,\infty), & p = 1,\ q = 1,\ \alpha_0 \geq 0,\ \alpha_\infty + 1 < 0; \\ L^{\frac{\nu p}{1+p(\nu-1)},q;\mathbb{A}}(0,\infty), & p \in (1, \frac{1}{1-\nu}),\ \nu < 1 \text{ or} \\ & p \in (1,\infty),\ \nu \geq 1; \\ L^{\infty,q;\mathbb{A}}(0,\infty), & p = \frac{1}{1-\nu},\ \alpha_0 + \frac{1}{q} < 0,\ \nu \leq 1 \text{ or} \\ & p = \frac{1}{1-\nu},\ q = \infty,\ \alpha_0 \leq 0,\ \nu \leq 1; \\ L^{\frac{\nu}{\nu-1},\infty;\mathbb{A}}(0,\infty), & p = q = \infty,\ \alpha_0 \leq 0,\ \alpha_\infty \geq 0,\ \nu > 1. \end{cases}$$

Remark 2.2.24. Proposition 2.2.22 and Proposition 2.2.23 together with Proposition 2.2.1 tell us that the rearrangement-invariant function space $X(0,\infty)$ whose norm satisfies

$$\|f\|_{X(0,\infty)} \approx \|t^{-1+\gamma+\nu} f^{**}(t^\nu)\|_{L^{\infty,q;\mathbb{A}}(0,\infty)} \quad \text{for every } f \in \mathfrak{M}(0,\infty)$$

is the optimal domain space for the operator $R_{v,\nu}$, $\gamma \in [0,1)$, and the space $L^{\infty,q;\mathbb{A}}(0,\infty)$ when $\gamma + \nu > 1$ and either $q \in [1,\infty)$, $\alpha_0 + \frac{1}{q} < 0$ or $q = \infty$, $\alpha_0 \leq 0$, $\alpha_\infty < \infty$, but they do not provide us with a description of $X(0,\infty)$ in terms of Lorentz–Zygmund spaces in those cases.

The next proposition is an analogue of Proposition 2.2.22 in the case where $L = 1$.

Proposition 2.2.25. *Let $\gamma \in (0,1)$ and $\nu \in (0,\infty)$. Assume that $L^{p,q;\alpha}(0,1)$ is equivalent to a rearrangement-invariant function space, that is, the parameters p, q and α satisfy one of conditions* (1.29) *(with $\beta = 0$). There is a rearrangement-invariant function space $Z(0,1)$ such that the operator $R_{v,\nu}\colon Z(0,1) \to L^{p,q;\alpha}(0,1)$ is bounded if and only if one of the following conditions holds:*

- $\gamma + \nu \geq 1$;
- $\gamma + \nu < 1$ *and* $p < \frac{1}{1-\gamma-\nu}$;

- *$\gamma+\nu<1$, $p=\frac{1}{1-\gamma-\nu}$ and*
  - ⋆ *$q<\infty$, $\alpha+\frac{1}{q}<0$ or*
  - ⋆ *$q=\infty$, $\alpha\le 0$.*

Furthermore, if this is the case and we denote by $X(0,1)$ the optimal domain space for the operator $R_{v,\nu}$ and the space $L^{p,q;\alpha}(0,1)$, then

$$X(0,1)=\begin{cases} L^1(0,1), & p\in[1,\frac{1}{1-\gamma}) \text{ or}\\ & p=\frac{1}{1-\gamma},\ \alpha+\frac{1}{q}<0 \text{ or}\\ & p=\frac{1}{1-\gamma},\ q=\infty,\ \alpha\le 0;\\ L^{1,1;\alpha+1}(0,1), & p=\frac{1}{1-\gamma},\ q=1,\ \alpha+1>0;\\ L^{1,1;0,1}(0,1), & p=\frac{1}{1-\gamma},\ q=1,\ \alpha+1=0;\\ L^{(1,q;\alpha)}(0,1), & p=\frac{1}{1-\gamma},\ q\in(1,\infty),\ \alpha+\frac{1}{q}\ge 0 \text{ or}\\ & p=\frac{1}{1-\gamma},\ q=\infty,\ \alpha>0;\\ L^{\frac{\nu p}{1+p(\gamma+\nu-1)},q;\alpha}(0,1), & p\in(\frac{1}{1-\gamma},\frac{1}{1-\gamma-\nu}),\ \gamma+\nu<1 \text{ or}\\ & p\in(\frac{1}{1-\gamma},\infty),\ \gamma+\nu\ge 1;\\ L^{\infty,q;\alpha}(0,1), & p=\frac{1}{1-\gamma-\nu},\ \alpha+\frac{1}{q}<0,\ \gamma+\nu\le 1 \text{ or}\\ & p=\frac{1}{1-\gamma-\nu},\ q=\infty,\ \alpha\le 0,\ \gamma+\nu\le 1;\\ L^{\frac{\nu}{\gamma+\nu-1},\infty;\alpha}(0,1), & p=q=\infty,\ \alpha\le 0,\ \gamma+\nu>1. \end{cases}$$

Proof. Thanks to Proposition 2.2.1, there is a rearrangement-invariant function space $Z(0,1)$ such that $R_{v,\nu}\colon Z(0,1)\to L^{p,q;\alpha}(0,1)$ is bounded if and only if $\|t^{-1+\gamma+\nu}\|_{L^{p,q;\alpha}(0,1)}<\infty$. It is easy to see (cf. the proof of Proposition 2.2.22) that $\|t^{-1+\gamma+\nu}\|_{L^{p,q;\alpha}(0,1)}<\infty$ if and only if $\gamma+\nu\ge 1$, or $\gamma+\nu<1$ and $p<\frac{1}{1-\gamma-\nu}$, or $\gamma+\nu<1$, $p=\frac{1}{1-\gamma-\nu}$, $q<\infty$ and $\alpha+\frac{1}{q}<0$, or $\gamma+\nu<1$, $p=\frac{1}{1-\gamma-\nu}$, $q=\infty$ and $\alpha\le 0$. Assume that the parameters p,q and α are such that this is the case. Owing to Proposition 2.2.1 again, we have that

$$\|f\|_{X(0,1)}\approx\Big\|t^{-1+\gamma}\int_0^{t^\nu}f^*(s)\,\mathrm{d}s\Big\|_{L^{p,q;\alpha}(0,1)}\quad\text{for every } f\in\mathfrak{M}(0,1),$$

where $X(0,1)$ is the optimal domain space for the operator $R_{v,\nu}$ and the space $L^{p,q;\alpha}(0,1)$.

If $p\ge\frac{1}{1-\gamma}$, we can proceed in the same way as in the proof of Proposition 2.2.22. We just note that the space $L^{(1,q;\alpha)}(0,1)$ is described in [75, Lemma 3.15, Theorem 3.16]. If $p\in[1,\frac{1}{1-\gamma})$, we need to show that $X(0,1)=L^1(0,1)$. On the one hand, since $X(0,1)$ is a rearrangement-invariant function space, we have that

$$X(0,1)\hookrightarrow L^1(0,1)\tag{2.95}$$

thanks to (1.17). On the other hand, we have that

$$\begin{aligned}\Big\|t^{-1+\gamma}\int_0^{t^\nu}f^*(s)\,\mathrm{d}s\Big\|_{L^{p,q;\alpha}(0,1)}&\le\int_0^1 f^*(s)\,\mathrm{d}s\|t^{-1+\gamma}\|_{L^{p,q;\alpha}(0,1)}\\&=\|f\|_{L^1(0,1)}\|t^{\frac{1}{p}+\gamma-1-\frac{1}{q}}\ell^\alpha(t)\|_{L^q(0,1)}\end{aligned}\tag{2.96}$$

for every $f \in \mathfrak{M}(0,1)$. Note that

$$\|t^{\frac{1}{p}+\gamma-1-\frac{1}{q}}\ell^{\alpha}(t)\|_{L^q(0,1)} < \infty$$

thanks to the fact that $\frac{1}{p}+\gamma-1>0$. Hence (2.96) implies that

$$L^1(0,1) \hookrightarrow X(0,1).$$

By combining that with (2.95), we obtain that $X(0,1) = L^1(0,1)$. □

The analogue of Proposition 2.2.23 in the case where $L = 1$ is the following. Its proof is similar to that of Proposition 2.2.25 and is omitted.

Proposition 2.2.26. *Let* $\gamma = 0$ *and* $\nu \in (0,\infty)$. *Assume that* $L^{p,q;\alpha}(0,1)$ *is equivalent to a rearrangement-invariant function space, that is, the parameters* p, q *and* α *satisfy one of conditions* (1.29) *(with* $\beta = 0$*). There is a rearrangement-invariant function space* $Z(0,1)$ *such that the operator* $R_{v,\nu}\colon Z(0,1) \to L^{p,q;\alpha}(0,1)$ *is bounded if and only if one of the following conditions holds:*

- $\gamma + \nu \geq 1$;
- $\gamma + \nu < 1$ *and* $p < \frac{1}{1-\nu}$;
- $\gamma + \nu < 1$, $p = \frac{1}{1-\nu}$ *and*
  - ⋆ $q < \infty$, $\alpha + \frac{1}{q} < 0$ *or*
  - ⋆ $q = \infty$, $\alpha \leq 0$.

Furthermore, if this is the case and we denote by $X(0,1)$ *the optimal domain space for the operator* $R_{v,\nu}$ *and the space* $L^{p,q;\alpha}(0,1)$*, then*

$$X(0,1) = \begin{cases} L^{1,1;\alpha+1}(0,1), & p = q = 1,\ \alpha \geq 0; \\ L^{\frac{\nu p}{1+p(\nu-1)},q;\alpha}(0,1), & p \in (1, \frac{1}{1-\nu}),\ \nu < 1 \text{ or} \\ & p \in (1,\infty),\ \nu \geq 1; \\ L^{\infty,q;\alpha}(0,1), & p = \frac{1}{1-\nu},\ \alpha + \frac{1}{q} < 0,\ \nu \leq 1 \text{ or} \\ & p = \frac{1}{1-\nu},\ q = \infty,\ \alpha \leq 0,\ \nu \leq 1; \\ L^{\frac{\nu}{\nu-1},\infty;\alpha}(0,1), & p = q = \infty,\ \alpha \leq 0,\ \gamma + \nu > 1. \end{cases}$$

Remark 2.2.27. Proposition 2.2.25 and Proposition 2.2.26 together with Proposition 2.2.1 tell us that the rearrangement-invariant function space $X(0,1)$ whose norm satisfies

$$\|f\|_{X(0,1)} \approx \|t^{-1+\gamma+\nu} f^{**}(t^{\nu})\|_{L^{\infty,q;\alpha}(0,1)} \quad \text{for every } f \in \mathfrak{M}(0,1)$$

is the optimal domain space for the operator $R_{v,\nu}$, $\gamma \in [0,1)$, and the space $L^{\infty,q;\alpha}(0,1)$ when $\gamma + \nu > 1$, $q \in [1,\infty)$ and $\alpha_0 + \frac{1}{q} < 0$, but they do not provide us with a description of $X(0,1)$ in terms of Lorentz–Zygmund spaces in that case.

Next, we describe optimal target spaces for the operator $R_{v,\nu}$ and Lorentz–Zygmund spaces.

Proposition 2.2.28. *Let* $\gamma \in (0,1)$ *and* $\nu \in (0,\infty)$. *Assume that* $L^{p,q;\mathbb{A}}(0,\infty)$ *is equivalent to a rearrangement-invariant function space, that is, the parameters* p, q *and* $\mathbb{A} = (\alpha_0, \alpha_\infty)$ *satisfy one of conditions* (1.28) *(with* $\beta_0 = \beta_\infty = 0$*). There is a rearrangement-invariant function space* $Z(0,\infty)$ *such that the operator* $R_{v,\nu}\colon L^{p,q;\mathbb{A}}(0,\infty) \to Z(0,\infty)$ *is bounded if and only if one of the following conditions holds:*

- $\gamma + \nu \leq 1$;
- $\gamma + \nu > 1$ *and*
    - ⋆ $p < \frac{\nu}{\gamma+\nu-1}$ *or*
    - ⋆ $p = \frac{\nu}{\gamma+\nu-1}$, $\alpha_\infty \geq 0$.

Furthermore, if this is the case and we denote by $Y(0,\infty)$ *the optimal target space for the operator* $R_{v,\nu}$ *and the space* $L^{p,q;\mathbb{A}}(0,\infty)$*, then*

$$Y(0,\infty) = \begin{cases} L^{\frac{1}{1-\gamma},\infty}(0,\infty), & p = q = 1,\ \alpha_0 = \alpha_\infty = 0; \\ L^{\frac{p}{p(1-\gamma-\nu)+\nu},q;\mathbb{A}}(0,\infty), & p \in (1,\infty),\ \gamma+\nu \leq 1 \text{ or} \\ & p \in (1, \frac{\nu}{\gamma+\nu-1}),\ \gamma+\nu > 1; \\ L^{\frac{1}{1-\gamma-\nu},\infty;\mathbb{A}}(0,\infty), & p = q = \infty,\ \alpha_0 \leq 0,\ \alpha_\infty \geq 0,\ \gamma+\nu \leq 1; \\ L^{\infty,\infty;\mathbb{A}}(0,\infty), & p = \frac{\nu}{\gamma+\nu-1},\ q = \infty,\ \alpha_0 \leq 0,\ \alpha_\infty \geq 0,\ \gamma+\nu > 1; \\ L^{\infty}(0,\infty), & p = \frac{\nu}{\gamma+\nu-1},\ q < \infty,\ \alpha_0 = \alpha_\infty = 0,\ \gamma+\nu > 1. \end{cases}$$

Proof. Thanks to Proposition 2.2.3, there is a rearrangement-invariant function space $Z(0,\infty)$ such that the operator $R_{v,\nu}\colon L^{p,q;\mathbb{A}}(0,\infty) \to Z(0,\infty)$ is bounded if and only if

$$\left\| \chi_{(0,1)}(t) \int_{t^{\frac{1}{\nu}}}^{1} s^{-1+\gamma}\,\mathrm{d}s \right\|_{(L^{p,q;\mathbb{A}})'(0,\infty)} < \infty \tag{2.97}$$

and

$$\sup_{b\in[1,\infty)} \frac{b^{\frac{\gamma-1}{\nu}+1}}{\varphi_{L^{p,q;\mathbb{A}}(0,\infty)}(b)} < \infty, \tag{2.98}$$

where we also used (1.18). Since $\gamma > 0$, (2.97) is plainly satisfied. If $\gamma + \nu \leq 1$, then (2.98) is also plainly satisfied because we have that

$$\sup_{b\in[1,\infty)} \frac{b^{\frac{\gamma-1}{\nu}+1}}{\varphi_{L^{p,q;\mathbb{A}}(0,\infty)}(b)} \leq \frac{1}{\varphi_{L^{p,q;\mathbb{A}}(0,\infty)}(1)}.$$

If $\gamma + \nu > 1$, we use the fact that ([75, Lemma 3.7])

$$\varphi_{L^{p,q;\mathbb{A}}(0,\infty)}(t) \approx \begin{cases} t^{\frac{1}{p}}\ell^{\alpha_\infty}(t), & p < \infty; \\ 1, & p = \infty,\ \alpha_\infty + \frac{1}{q} < 0 \text{ or} \\ & p = q = \infty,\ \alpha_\infty \leq 0; \\ \ell^{\alpha_\infty+\frac{1}{q}}(t), & p = \infty,\ \alpha_\infty + \frac{1}{q} > 0; \\ \ell\ell^{\frac{1}{q}}(t), & p = \infty,\ q < \infty,\ \alpha_\infty + \frac{1}{q} = 0, \end{cases} \tag{2.99}$$

for every $t \in (1,\infty)$, whence it follows that we need to have that either

$$p < \frac{\nu}{\gamma+\nu-1}$$

or

$$p = \frac{\nu}{\gamma+\nu-1} \quad \text{and} \quad \alpha_\infty \geq 0.$$

From now on, assume that the parameters p, q and $\mathbb{A}$ are such that both (2.97) and (2.98) are satisfied. We denote by $Y(0,\infty)$ the optimal target space for the operator $R_{v,\nu}$ and the space $L^{p,q;\mathbb{A}}(0,\infty)$, whose existence is guaranteed by Proposition 2.2.3. We define the function $\varphi\colon (0,\infty) \to (0,\infty)$ as

$$\varphi(t) = t^{1-\frac{1-\gamma}{\nu}},\ t \in (0,\infty). \tag{2.100}$$

We denote by T the operator T_φ defined by (2.17). Assume that $\gamma + \nu \leq 1$. We plainly have that $Tf = f^*$ for every $f \in \mathfrak{M}(0,\infty)$ because the function φf^* is nonincreasing. Hence T is bounded on any rearrangement-invariant function space over $(0,\infty)$ (and its operator norm is equal to 1). In particular, it is bounded on $L^{p,q;\mathbb{A}}(0,\infty)$. Assume now that $\gamma + \nu > 1$ and $p \leq \frac{\nu}{\gamma+\nu-1}$. If $q < \infty$, then

$$\|Tf\|_{L^{p,q;\mathbb{A}}(0,\infty)} = \Big\|t^{\frac{1}{p}-\frac{1}{q}+\frac{1-\gamma}{\nu}-1}\ell^{\mathbb{A}}(t) \sup_{s\in[t,\infty)} s^{1-\frac{1-\gamma}{\nu}} f^*(s)\Big\|_{L^q(0,\infty)} \lesssim \|f\|_{L^{p,q;\mathbb{A}}(0,\infty)}$$

for every $f \in \mathfrak{M}(0,\infty)$ if and only if

$$t^{q(1-\frac{1-\gamma}{\nu})} \int_0^t s^{q(\frac{1}{p}+\frac{1-\gamma}{\nu}-1)-1} \ell^{q\mathbb{A}}(s)\,\mathrm{d}s \lesssim \int_0^t s^{\frac{q}{p}-1}\ell^{q\mathbb{A}}(s)\,\mathrm{d}s \quad \text{for every } t \in (0,\infty)$$

owing to [46, Theorem 3.2]. It is easy to see that this is the case if and only if $p < \frac{\nu}{\gamma+\nu-1}$. If $q = \infty$, then we have that

$$\sup_{t\in(0,\infty)} t^{\frac{1}{p}+\frac{1-\gamma}{\nu}-1}\ell^{\mathbb{A}}(t) \sup_{s\in[t,\infty)} s^{1-\frac{1-\gamma}{\nu}} f^*(s) = \sup_{t\in[0,\infty)} t^{1-\frac{1-\gamma}{\nu}} f^*(t) \sup_{s\in(0,t]} s^{\frac{1}{p}+\frac{1-\gamma}{\nu}-1}\ell^{\mathbb{A}}(s),$$

whence it follows that T is bounded on $L^{p,\infty;\mathbb{A}}(0,\infty)$ if and only if $p < \frac{\nu}{\gamma+\nu-1}$, or $p = \frac{\nu}{\gamma+\nu-1}$, $\alpha_0 \leq 0$, $\alpha_\infty \geq 0$. Hence we have shown that T is bounded on $L^{p,q;\mathbb{A}}(0,\infty)$ if and only if one of the following conditions holds:

- $\gamma + \nu \leq 1$;
- $\gamma + \nu > 1$ and
  - ⋆ $p < \frac{\nu}{\gamma+\nu-1}$ or
  - ⋆ $p = \frac{\nu}{\gamma+\nu-1}$, $q = \infty$, $\alpha_0 \leq 0$, $\alpha_\infty \geq 0$.

In these cases Theorem 2.2.6 tells us that

$$\|f\|_{Y'(0,\infty)} \approx \Big\| \int_{t^{\frac{1}{\nu}}}^\infty f^*(s)s^{-1+\gamma}\,\mathrm{d}s \Big\|_{(L^{p,q;\mathbb{A}})'(0,\infty)} \quad \text{for every } f \in \mathfrak{M}(0,\infty). \tag{2.101}$$

Recall that $(L^{p,q;\mathbb{A}})'(0,\infty) = L^{p',q';-\mathbb{A}}(0,\infty)$ provided that $p < \infty$ or that $p = q = \infty$, $\alpha_0 \leq 0$, $\alpha_\infty \geq 0$ (see (1.30)).

Let $p=q=1$ and $\alpha_0=\alpha_\infty=0$. We have that

$$\|f\|_{Y'(0,\infty)} \approx \Big\| \int_{t^{\frac{1}{\nu}}}^\infty f^*(s)s^{-1+\gamma}\,\mathrm{d}s \Big\|_{L^\infty(0,\infty)} = \int_0^\infty f^*(s)s^{-1+\gamma}\,\mathrm{d}s = \|f\|_{L^{\frac{1}{\gamma},1}(0,\infty)},$$

whence $Y(0,\infty)=L^{\frac{1}{1-\gamma},\infty}(0,\infty)$.

Let $1<p<\infty$ and $\gamma+\nu\leq 1$, or $p=q=\infty$, $\alpha_0\leq 0$, $\alpha_\infty\geq 0$ and $\gamma+\nu\leq 1$, or $1<p<\frac{\nu}{\gamma+\nu-1}$ and $\gamma+\nu>1$, or $p=\frac{\nu}{\gamma+\nu-1}$, $q=\infty$, $\alpha_0\leq 0$, $\alpha_\infty\geq 0$ and $\gamma+\nu>1$. On the one hand, we have that

$$\begin{aligned}\Big\|t^{\frac{1}{p'}-\frac{1}{q'}}\ell^{-\mathbb{A}}(t)\int_{t^{\frac{1}{\nu}}}^\infty f^*(s)s^{-1+\gamma}\,\mathrm{d}s\Big\|_{L^{q'}(0,\infty)} &\approx \Big\|t^{\frac{\nu}{p'}-\frac{1}{q'}}\ell^{-\mathbb{A}}(t)\int_t^\infty f^*(s)s^{-1+\gamma}\,\mathrm{d}s\Big\|_{L^{q'}(0,\infty)}\\ &\lesssim \|t^{\frac{\nu}{p'}+\frac{1}{q}}\ell^{-\mathbb{A}}(t)f^*(t)t^{-1+\gamma}\|_{L^{q'}(0,\infty)}\\ &= \|t^{\frac{\nu}{p'}+\gamma-\frac{1}{q'}}\ell^{-\mathbb{A}}(t)f^*(t)\|_{L^{q'}(0,\infty)},\end{aligned} \tag{2.102}$$

where we used the Hardy inequality [69, Theorem 2] (see also [12, Theorem 2], cf. [57]). On the other hand, we have that

$$\begin{aligned}\Big\|t^{\frac{1}{p'}-\frac{1}{q'}}\ell^{-\mathbb{A}}(t)\int_{t^{\frac{1}{\nu}}}^\infty f^*(s)s^{-1+\gamma}\,\mathrm{d}s\Big\|_{L^{q'}(0,\infty)} &\approx \Big\|t^{\frac{\nu}{p'}-\frac{1}{q'}}\ell^{-\mathbb{A}}(t)\int_t^\infty f^*(s)s^{-1+\gamma}\,\mathrm{d}s\Big\|_{L^{q'}(0,\infty)}\\ &\geq \Big\|t^{\frac{\nu}{p'}-\frac{1}{q'}}\ell^{-\mathbb{A}}(t)\int_t^{2t} f^*(s)s^{-1+\gamma}\,\mathrm{d}s\Big\|_{L^{q'}(0,\infty)}\\ &\gtrsim \|t^{\frac{\nu}{p'}-\frac{1}{q'}}\ell^{-\mathbb{A}}(t)f^*(2t)t^{\gamma}\|_{L^{q'}(0,\infty)}\\ &\approx \|t^{\frac{\nu}{p'}+\gamma-\frac{1}{q'}}\ell^{-\mathbb{A}}(t)f^*(t)\|_{L^{q'}(0,\infty)}.\end{aligned} \tag{2.103}$$

Note that $\frac{\nu}{p'}+\gamma\in(0,1]$ thanks to the restriction imposed on p. By combining (2.102) and (2.103), we obtain that $Y'(0,\infty)=L^{\frac{p'}{\nu+p'\gamma},q';-\mathbb{A}}(0,\infty)$, whence it follows that

$$Y(0,\infty)=L^{\frac{p}{p(1-\gamma-\nu)+\nu},q;\mathbb{A}}(0,\infty),$$

where $\frac{p}{p(1-\gamma-\nu)+\nu}$ is to be interpreted as ∞ if $p=\frac{\nu}{\gamma+\nu-1}$ (and $\gamma+\nu>1$) and as $\frac{1}{1-\gamma-\nu}$ if $p=\infty$ (and $\gamma+\nu\leq 1$).

Finally, let $p=\frac{\nu}{\gamma+\nu-1}$, $q<\infty$, $\alpha_0=\alpha_\infty=0$ and $\gamma+\nu>1$. Owing to (2.34), we have that

$$\|f\|_{Y'(0,\infty)} \approx \sup_{h\sim f}\Big\|\int_{t^{\frac{1}{\nu}}}^\infty h(s)s^{-1+\gamma}\,\mathrm{d}s\Big\|_{L^{\frac{\nu}{1-\gamma},q'}(0,\infty)} \tag{2.104}$$

for every $f\in\mathfrak{M}(0,\infty)$, where the supremum is taken over all $h\in\mathfrak{M}^+(0,\infty)$ equimeasurable with f. We would like to show that

$$Y'(0,\infty)=L^1(0,\infty). \tag{2.105}$$

To this end, note that $Z(0,\infty)=L^1(0,\infty)$ whenever $Z(0,\infty)$ is a rearrangement-invariant function space whose fundamental function satisfies $\varphi_{Z(0,\infty)}(t)\approx t$ on $(0,\infty)$. Indeed, this follows from (1.19) combined with the fact that $\Lambda_\psi(0,\infty)=M_\psi(0,\infty)=L^1(0,\infty)$, where $\psi(t)=t$, $t\in(0,\infty)$. Therefore, in order to prove

(2.105), we only need to show that $\varphi_{Y'(0,\infty)}(t) \approx t$ on $(0,\infty)$. As for the lower bound, it follows from (2.16) with $f = \chi_{(0,t)}$, $t \in (0,\infty)$, that

$$\varphi_{Y'(0,\infty)}(t) \gtrsim t^{-1+\gamma}\varphi_{L^{\frac{\nu}{1-\gamma},q'}(0,\infty)}(t^{\nu})t \approx t^{\gamma}t^{1-\gamma} = t \tag{2.106}$$

for every $t \in (0,\infty)$. As for the upper bound, observe that we have that

$$\sup_{t\in(0,\infty)} \frac{t^{\frac{\gamma-1}{\nu}+1}}{\varphi_{L^{\frac{\nu}{\gamma+\nu-1},q}(0,\infty)}(t)} < \infty; \tag{2.107}$$

consequently, we can argue in a similar way to the proof of (2.15) to obtain that

$$\varphi_{Y'(0,\infty)}(a) \leq \left\| \chi_{(0,b^{\nu})}(t) \int_{t^{\frac{1}{\nu}}}^{b} s^{-1+\gamma}\,\mathrm{d}s \right\|_{L^{\frac{\nu}{1-\gamma},q'}(0,\infty)} + 4Ba \quad \text{for every } a \in (0,\infty),$$

where $b = \frac{2a}{2^{\frac{1}{\nu}}-1}$ and B is the supremum in (2.107), whence it follows that

$$\varphi_{Y'(0,\infty)}(a) \lesssim b^{\gamma}\varphi_{L^{\frac{\nu}{1-\gamma},q'}(0,\infty)}(b^{\nu}) + a \approx b^{\gamma}b^{1-\gamma} + a \approx a \tag{2.108}$$

for every $a \in (0,\infty)$. Therefore, (2.105) is true owing to (2.106) and (2.108); hence $Y(0,\infty) = L^{\infty}(0,\infty)$ owing to (1.27). ☐

Remark 2.2.29. When $p = q = 1$, $\alpha_0 \geq 0$, $\alpha_\infty \leq 0$, $|\alpha_0| + |\alpha_\infty| > 0$, or $p = q = \infty$, $\alpha_0 \leq 0$, $\alpha_\infty < 0$, $\gamma + \nu \leq 1$, or $p = \infty$, $q \in [1,\infty)$, $\alpha_0 + \frac{1}{q} < 0$, $\gamma+\nu \leq 1$, Proposition 2.2.28 together with its proof tells us that the rearrangement-invariant function space $Y(0,\infty)$ whose associate function norm satisfies (2.101) is the optimal target space for the operator $R_{v,\nu}$ and the space $L^{p,q;\mathbb{A}}(0,\infty)$, but it does not provide us with a description of $Y(0,\infty)$ in terms of Lorentz–Zygmund spaces in those cases.

When $p = \frac{\nu}{\gamma+\nu-1}$, $q \in [1,\infty)$, $\alpha_\infty \geq 0$, $|\alpha_0| + |\alpha_\infty| > 0$, $\gamma+\nu > 1$, or $p = \frac{\nu}{\gamma+\nu-1}$, $q = \infty$, $\alpha_0 > 0$, $\alpha_\infty \geq 0$, $\gamma + \nu > 1$, Proposition 2.2.28 together with its proof tells us that the rearrangement-invariant function space $Y(0,\infty)$ whose associate function norm satisfies (2.104) is the optimal target space for the operator $R_{v,\nu}$ and the space $L^{p,q;\mathbb{A}}(0,\infty)$, but it does not provide us with a description of $Y(0,\infty)$ in terms of Lorentz–Zygmund spaces in those cases (see also Theorem 2.2.13).

The next proposition is an analogue of Proposition 2.2.28 in the case where $\gamma = 0$.

Proposition 2.2.30. *Let $\gamma = 0$ and $\nu \in (0,\infty)$. Assume that $L^{p,q;\mathbb{A}}(0,\infty)$ is equivalent to a rearrangement-invariant function space, that is, the parameters p, q and $\mathbb{A} = (\alpha_0, \alpha_\infty)$ satisfy one of conditions* (1.28) *(with $\beta_0 = \beta_\infty = 0$). There is a rearrangement-invariant function space $Z(0,\infty)$ such that the operator $R_{v,\nu}\colon L^{p,q;\mathbb{A}}(0,\infty) \to Z(0,\infty)$ is bounded if and only if one of the following conditions holds:*

- *$p = q = 1$, $\alpha_0 \geq 1$, $\alpha_\infty \leq 0$;*
- *$\nu \leq 1$ and $p > 1$;*
- *$\nu > 1$ and*

$\star$ $p \in (1, \frac{\nu}{\nu-1})$ *or*

$\star$ $p = \frac{\nu}{\nu-1}$, $\alpha_\infty \geq 0$.

Furthermore, if this is the case and we denote by $Y(0,\infty)$ *the optimal target space for the operator* $R_{v,\nu}$ *and the space* $L^{p,q;\mathbb{A}}(0,\infty)$*, then*

$$Y(0,\infty) = \begin{cases} L^{1,1;\mathbb{A}-1}(0,\infty), & p = q = 1,\ \alpha_0 \geq 1,\ \alpha_\infty < 0; \\ L^{p,q;\mathbb{A}}(0,\infty), & p \in (1,\infty),\ \nu = 1 \text{ or} \\ & p = \infty,\ q \in [1,\infty),\ \alpha_0 + \frac{1}{q} < 0,\ \nu = 1 \text{ or} \\ & p = q = \infty,\ \alpha_0 \leq 0,\ \nu = 1; \\ L^{\frac{p}{p(1-\nu)+\nu},q;\mathbb{A}}(0,\infty), & p \in (1,\infty),\ \nu < 1 \text{ or} \\ & p \in (1, \frac{\nu}{\nu-1}),\ \nu > 1; \\ L^{\frac{1}{1-\nu},\infty;\mathbb{A}}(0,\infty), & p = q = \infty,\ \alpha_0 \leq 0,\ \alpha_\infty \geq 0,\ \nu < 1; \\ L^{\infty,\infty;\mathbb{A}}(0,\infty), & p = \frac{\nu}{\nu-1},\ q = \infty,\ \alpha_0 \leq 0,\ \alpha_\infty \geq 0,\ \nu > 1. \end{cases}$$

Proof. Thanks to Proposition 2.2.3, there is a rearrangement-invariant function space $Z(0,\infty)$ such that the operator $R_{v,\nu}\colon L^{p,q;\mathbb{A}}(0,\infty) \to Z(0,\infty)$ is bounded if and only if

$$\left\| \log\left(\frac{1}{t}\right)\chi_{(0,1)}(t) \right\|_{(L^{p,q;\mathbb{A}})'(0,\infty)} < \infty \tag{2.109}$$

and

$$\sup_{b\in[1,\infty)} \frac{b^{1-\frac{1}{\nu}}}{\varphi_{L^{p,q;\mathbb{A}}(0,\infty)}(b)} < \infty, \tag{2.110}$$

where we also used (1.18). If $\nu \leq 1$, then (2.110) is plainly satisfied because we have that

$$\sup_{b\in[1,\infty)} \frac{b^{1-\frac{1}{\nu}}}{\varphi_{L^{p,q;\mathbb{A}}(0,\infty)}(b)} \leq \frac{1}{\varphi_{L^{p,q;\mathbb{A}}(0,\infty)}(1)}.$$

If $\nu > 1$, it follows from (2.99) that (2.110) is satisfied if and only if

$$p < \frac{\nu}{\nu-1}$$

or

$$p = \frac{\nu}{\nu-1} \quad \text{and} \quad \alpha_\infty \geq 0.$$

As for (2.109), it is easy to see that (2.109) is satisfied when $p > 1$ (see (1.30) when $p < \infty$, and [75, Theorems 6.2 and 6.6] when $p = \infty$). When $p = 1$ (and, consequently, $q = 1$, $\alpha_0 \geq 0$, $\alpha_\infty \leq 0$), we have that

$$\left\| \log\left(\frac{1}{t}\right)\chi_{(0,1)}(t) \right\|_{L^{\infty,\infty;-\mathbb{A}}(0,\infty)} \approx \sup_{t\in(0,1)} \ell^{-\alpha_0+1}(t),$$

which is finite if and only if $\alpha_0 \geq 1$.

From now on, assume that the parameters p, q and $\mathbb{A}$ are such that both (2.109) and (2.110) are satisfied. We denote by $Y(0,\infty)$ the optimal target space for the operator $R_{v,\nu}$ and the space $L^{p,q;\mathbb{A}}(0,\infty)$, whose existence is guaranteed by Proposition 2.2.3. By arguing in the same way as in the proof of Proposition 2.2.28, we can show that the operator T_φ, where φ is defined by (2.100) with $\gamma = 0$, is bounded on $L^{p,q;\mathbb{A}}(0,\infty)$ if and only if one of the following conditions holds:

- $\nu \le 1$;
- $\nu > 1$ and
  - ⋆ $p < \frac{\nu}{\nu-1}$ or
  - ⋆ $p = \frac{\nu}{\nu-1}$, $q = \infty$, $\alpha_0 \le 0$, $\alpha_\infty \ge 0$.

In these cases, Theorem 2.2.6 tells us that

$$\|f\|_{Y'(0,\infty)} \approx \left\| \int_{t^{\frac{1}{\nu}}}^{\infty} f^*(s)s^{-1}\,\mathrm{d}s \right\|_{(L^{p,q;\mathbb{A}})'(0,\infty)} \quad \text{for every } f \in \mathfrak{M}(0,\infty). \tag{2.111}$$

Recall that $(L^{p,q;\mathbb{A}})'(0,\infty) = L^{p',q';-\mathbb{A}}(0,\infty)$ provided that $p < \infty$ or that $p = q = \infty$, $\alpha_0 \le 0$, $\alpha_\infty \ge 0$ (see (1.30)).

Let $p = q = 1$, $\alpha_0 \ge 1$, $\alpha_\infty < 0$. On the one hand, we have that

$$\begin{aligned}
\|f\|_{Y'(0,\infty)} &\approx \left\| \ell^{-\mathbb{A}}(t) \int_{t^{\frac{1}{\nu}}}^{\infty} f^*(s)s^{-1}\,\mathrm{d}s \right\|_{L^\infty(0,\infty)} \\
&\approx \left\| \ell^{-\mathbb{A}}(t) \int_{t}^{\infty} f^*(s)s^{-1}\,\mathrm{d}s \right\|_{L^\infty(0,\infty)} \\
&= \left\| \ell^{-\mathbb{A}}(t) \int_{t}^{\infty} f^*(s)\ell^{-\mathbb{A}+1}(s)\ell^{\mathbb{A}-1}(s)s^{-1}\,\mathrm{d}s \right\|_{L^\infty(0,\infty)} \\
&\le \left\| \ell^{-\mathbb{A}}(t) \int_{t}^{\infty} \ell^{\mathbb{A}-1}(s)s^{-1}\,\mathrm{d}s \right\|_{L^\infty(0,\infty)} \sup_{t\in(0,\infty)} f^*(t)\ell^{-\mathbb{A}+1}(t) \\
&\approx \sup_{t\in(0,\infty)} f^*(t)\ell^{-\mathbb{A}+1}(t) \\
&= \|f\|_{L^{\infty,\infty;-\mathbb{A}+1}(0,\infty)}.
\end{aligned}$$

On the other hand, we have that

$$\begin{aligned}
\|f\|_{Y'(0,\infty)} &\approx \left\| \ell^{-\mathbb{A}}(t) \int_{t}^{\infty} f^*(s)s^{-1}\,\mathrm{d}s \right\|_{L^\infty(0,\infty)} \\
&= \max\Big\{ \sup_{t\in(0,1]} \ell^{-\alpha_0}(t) \int_t^\infty f^*(s)s^{-1}\,\mathrm{d}s, \sup_{t\in[1,\infty)} \ell^{-\alpha_\infty}(t) \int_t^\infty f^*(s)s^{-1}\,\mathrm{d}s \Big\} \\
&\ge \max\Big\{ \sup_{t\in(0,1]} \ell^{-\alpha_0}(t) \int_t^{2\sqrt{t}} f^*(s)s^{-1}\,\mathrm{d}s, \sup_{t\in[1,\infty)} \ell^{-\alpha_\infty}(t) \int_t^{2t^2} f^*(s)s^{-1}\,\mathrm{d}s \Big\} \\
&\ge \max\Big\{ \sup_{t\in(0,1]} \ell^{-\alpha_0}(t) f^*\big(2\sqrt{t}\big) \int_t^{2\sqrt{t}} s^{-1}\,\mathrm{d}s, \sup_{t\in[1,\infty)} \ell^{-\alpha_\infty}(t) f^*\big(2t^2\big) \int_t^{2t^2} s^{-1}\,\mathrm{d}s \Big\} \\
&= \max\Big\{ \sup_{t\in(0,1]} \ell^{-\alpha_0}(t) f^*\big(2\sqrt{t}\big) \log\Big(\frac{2}{\sqrt{t}}\Big), \sup_{t\in[1,\infty)} \ell^{-\alpha_\infty}(t) f^*\big(2t^2\big) \log(2t) \Big\} \\
&\approx \max\Big\{ \sup_{t\in(0,1]} \ell^{-\alpha_0+1}(t) f^*(t), \sup_{t\in[1,\infty)} \ell^{-\alpha_\infty+1}(t) f^*(t) \Big\} \\
&= \|f\|_{L^{\infty,\infty;-\mathbb{A}+1}(0,\infty)}.
\end{aligned}$$

Hence $Y(0,\infty) = L^{1,1;\mathbb{A}-1}(0,\infty)$.

Let $1 < p < \infty$ and $\nu < 1$, or $p = q = \infty$, $\alpha_0 \le 0$, $\alpha_\infty \ge 0$ and $\nu < 1$, or $1 < p < \frac{\nu}{\nu-1}$ and $\nu > 1$, or $p = \frac{\nu}{\nu-1}$, $q = \infty$, $\alpha_0 \le 0$, $\alpha_\infty \ge 0$ and $\nu > 1$. On the one hand, by arguing in the same way as in (2.102), we have that

$$\left\| t^{\frac{1}{p'}-\frac{1}{q'}} \ell^{-\mathbb{A}}(t) \int_{t^{\frac{1}{\nu}}}^{\infty} f^*(s)s^{-1}\,\mathrm{d}s \right\|_{L^{q'}(0,\infty)} \lesssim \|t^{\frac{\nu}{p'}-\frac{1}{q'}} \ell^{-\mathbb{A}}(t) f^*(t)\|_{L^{q'}(0,\infty)}. \tag{2.112}$$

On the other hand, we have that

$$\begin{aligned}\Big\|t^{\frac{1}{p'}-\frac{1}{q'}}\ell^{-\mathbb{A}}(t)\int_{t^{\frac{1}{\nu}}}^{\infty} f^*(s)s^{-1}\,\mathrm{d}s\Big\|_{L^{q'}(0,\infty)} &\approx \Big\|t^{\frac{\nu}{p'}-\frac{1}{q'}}\ell^{-\mathbb{A}}(t)\int_{t}^{\infty} f^*(s)s^{-1}\,\mathrm{d}s\Big\|_{L^{q'}(0,\infty)}\\ &\geq \Big\|t^{\frac{\nu}{p'}-\frac{1}{q'}}\ell^{-\mathbb{A}}(t)\int_{t}^{2t} f^*(s)s^{-1}\,\mathrm{d}s\Big\|_{L^{q'}(0,\infty)}\\ &\geq \Big\|t^{\frac{\nu}{p'}-\frac{1}{q'}}\ell^{-\mathbb{A}}(t)f^*(2t)\int_{t}^{2t} s^{-1}\,\mathrm{d}s\Big\|_{L^{q'}(0,\infty)}\\ &\approx \|t^{\frac{\nu}{p'}-\frac{1}{q'}}\ell^{-\mathbb{A}}(t)f^*(t)\|_{L^{q'}(0,\infty)}.\end{aligned} \tag{2.113}$$

Note that $\frac{\nu}{p'} \in (0,1]$ thanks to the restriction imposed on p. By combining (2.112) and (2.113), we obtain that $Y'(0,\infty) = \big(L^{\frac{p'}{\nu},q';-\mathbb{A}}\big)'(0,\infty)$, whence it follows that

$$Y(0,\infty) = L^{\frac{p}{p(1-\nu)+\nu},q;\mathbb{A}}(0,\infty),$$

where $\frac{p}{p(1-\nu)+\nu}$ is to be interpreted as ∞ if $p = \frac{\nu}{\nu-1}$ (and $\nu > 1$) and as $\frac{1}{1-\nu}$ if $p = \infty$ (and $\nu < 1$).

Finally, let $\nu = 1$ and $p > 1$. We claim that $Y(0,\infty) = L^{p,q;\mathbb{A}}(0,\infty)$. Note that $R_{v,1}(f^*) = f^{**}$ for every $f \in \mathfrak{M}(0,\infty)$. We have that

$$\begin{aligned}\|R_{v,1}f\|_{L^{p,q;\mathbb{A}}(0,\infty)} &\leq \|R_{v,1}(f^*)\|_{L^{p,q;\mathbb{A}}(0,\infty)}\\ &= \Big\|t^{\frac{1}{p}-\frac{1}{q}-1}\ell^{\mathbb{A}}(t)\int_0^t f^*(s)\,\mathrm{d}s\Big\|_{L^q(0,\infty)}\\ &\lesssim \|t^{\frac{1}{p}-1}t^{1-\frac{1}{q}}\ell^{\mathbb{A}}(t)f^*(t)\|_{L^q(0,\infty)}\\ &= \|f\|_{L^{p,q;\mathbb{A}}(0,\infty)}\end{aligned}$$

for every $f \in \mathfrak{M}(0,\infty)$, where we used the Hardy–Littlewood inequality (1.3), the fact that f^{**} is nonincreasing, and the Hardy inequality [69, Theorem 1] (cf. [57]). Hence $R_{v,1}\colon L^{p,q;\mathbb{A}}(0,\infty) \to L^{p,q;\mathbb{A}}(0,\infty)$ is bounded. It only remains to show that $L^{p,q;\mathbb{A}}(0,\infty)$ is the optimal target space, that is, we need to show that $L^{p,q;\mathbb{A}}(0,\infty) \hookrightarrow Z(0,\infty)$ whenever $Z(0,\infty)$ is a rearrangement-invariant function space such that $R_{v,1}\colon L^{p,q;\mathbb{A}}(0,\infty) \to Z(0,\infty)$ is bounded. This, however, follows immediately from (1.1) because we have that

$$\|f\|_{Z(0,\infty)} \leq \|f^{**}\|_{Z(0,\infty)} \lesssim \|f^*\|_{L^{p,q;\mathbb{A}}(0,\infty)} = \|f\|_{L^{p,q;\mathbb{A}}(0,\infty)}$$

for every $f \in \mathfrak{M}(0,\infty)$. $\square$

Remark 2.2.31. When $p = q = 1$, $\alpha_0 \geq 1$, $\alpha_\infty = 0$, or $p = q = \infty$, $\alpha_0 \leq 0$, $\alpha_\infty < 0$, $\nu < 1$, or $p = \infty$, $q \in [1,\infty)$, $\alpha_0 + \frac{1}{q} < 0$, $\nu < 1$, Proposition 2.2.30 together with its proof tells us that the rearrangement-invariant function space $Y(0,\infty)$ whose associate function norm satisfies (2.111) is the optimal target space for the operator $R_{v,\nu}$ and the space $L^{p,q;\mathbb{A}}(0,\infty)$, but it does not provide us with a description of $Y(0,\infty)$ in terms of Lorentz–Zygmund spaces in those cases.

When $p = \frac{\nu}{\nu-1}$, $q \in [1,\infty)$, $\alpha_\infty \geq 0$, $\nu > 1$, or $p = \frac{\nu}{\nu-1}$, $q = \infty$, $\alpha_0 > 0$, $\alpha_\infty \geq 0$, $\nu > 1$, Proposition 2.2.30 together with its proof tells us that the rearrangement-invariant function space $Y(0,\infty)$ whose associate function norm satisfies (2.104) with $\gamma = 0$ is the optimal target space for the operator $R_{v,\nu}$ and the space $L^{p,q;\mathbb{A}}(0,\infty)$, but it does not provide us with a description of $Y(0,\infty)$ in terms of Lorentz–Zygmund spaces in those cases.

The next proposition is an analogue of Proposition 2.2.28 in the case where $L = 1$.

Proposition 2.2.32. *Let $\gamma \in (0,1)$ and $\nu \in (0,\infty)$. Assume that $L^{p,q;\alpha}(0,1)$ is equivalent to a rearrangement-invariant function space, that is, the parameters p, q and α satisfy one of conditions* (1.29) *(with $\beta = 0$). There is an optimal target space $Y(0,1)$ for the operator $R_{v,\nu}$ and the space $L^{p,q;\alpha}(0,1)$. Furthermore, we have that*

$$Y(0,1) = \begin{cases} L^{\frac{1}{1-\gamma},\infty}(0,1), & p = q = 1,\ \alpha = 0; \\ L^{\frac{p}{p(1-\gamma-\nu)+\nu},q;\alpha}(0,1), & p \in (1,\infty),\ \gamma+\nu \le 1 \text{ or} \\ & p \in (1, \frac{\nu}{\gamma+\nu-1}),\ \gamma+\nu > 1; \\ L^{\frac{1}{1-\gamma-\nu},\infty;\alpha}(0,1), & p = q = \infty,\ \alpha \le 0,\ \gamma+\nu \le 1; \\ L^{\infty,\infty;\alpha}(0,1), & p = \frac{\nu}{\gamma+\nu-1},\ q = \infty,\ \alpha \le 0,\ \gamma+\nu > 1; \\ L^{\infty}(0,1), & p = \frac{\nu}{\gamma+\nu-1},\ q < \infty,\ \alpha = 0,\ \gamma+\nu > 1 \text{ or} \\ & p > \frac{\nu}{\gamma+\nu-1},\ \gamma+\nu > 1. \end{cases}$$

Proof. If $\gamma + \nu \le 1$, or $\gamma + \nu > 1$ and $p \le \frac{\nu}{\gamma+\nu-1}$, the proof is similar to that of Proposition 2.2.28 and we omit it. Let $p > \frac{\nu}{\gamma+\nu-1}$ and $\gamma + \nu > 1$. It can be easily verified that $L^{p,q;\alpha}(0,1) \hookrightarrow L^{\frac{\nu}{\gamma+\nu-1},\infty}(0,1)$; consequently, the optimal target space for the operator $R_{v,\nu}$ and the space $L^{p,q;\alpha}(0,1)$ is embedded in the optimal target space for the operator $R_{v,\nu}$ and the space $L^{\frac{\nu}{\gamma+\nu-1},\infty}(0,1)$, that is,

$$Y(0,1) \hookrightarrow L^{\infty}(0,1).$$

By combining that with (1.17), we obtain $Y(0,1) = L^{\infty}(0,1)$. □

Remark 2.2.33. When $p = q = 1$, $\alpha > 0$, or $p = \infty$, $q \in [1,\infty)$, $\alpha + \frac{1}{q} < 0$, $\gamma + \nu \le 1$, the associate function norm of the optimal target space $Y(0,1)$ satisfies

$$\|f\|_{Y'(0,1)} \approx \begin{cases} \left\| \int_{t^{\frac{1}{\nu}}}^{1} f^*(s) s^{-1+\gamma}\, \mathrm{d}s \right\|_{L^{\infty,\infty;-\alpha}(0,1)}, & p = q = 1,\ \alpha > 0; \\ \left\| \int_{t^{\frac{1}{\nu}}}^{1} f^*(s) s^{-1+\gamma}\, \mathrm{d}s \right\|_{L^{(1,q';-\alpha-1)}(0,1)}, & p = \infty,\ q \in [1,\infty),\ \alpha + \frac{1}{q} < 0 \\ & \text{and } \gamma+\nu \le 1, \end{cases}$$

for every $f \in \mathfrak{M}(0,1)$, but Proposition 2.2.32 does not provide us with a description of $Y(0,1)$ in terms of Lorentz–Zygmund spaces in those cases.

When $p = \frac{\nu}{\gamma+\nu-1}$, $q \in [1,\infty)$, $\alpha \ne 0$, $\gamma + \nu > 1$, or $p = \frac{\nu}{\gamma+\nu-1}$, $q = \infty$, $\alpha > 0$, $\gamma + \nu > 1$, the associate function space of the optimal target space $Y(0,1)$ satisfies

$$\|f\|_{Y'(0,1)} \approx \sup_{g \sim f} \left\| \int_{t^{\frac{1}{\nu}}}^{1} g(s) s^{-1+\gamma}\, \mathrm{d}s \right\|_{L^{\frac{\nu}{1-\gamma},q';-\alpha}(0,1)}$$

for every $f \in \mathfrak{M}(0,1)$, where the supremum is taken over all $g \in \mathfrak{M}^+(0,1)$ equimeasurable with f, but Proposition 2.2.32 does not provide us with a description of $Y(0,1)$ in terms of Lorentz–Zygmund spaces in those cases (see also Theorem 2.2.13).

When $L = 1$, Proposition 2.2.30 transforms into the following. The proof is similar to that of Proposition 2.2.30 and we omit it.

Proposition 2.2.34. *Let $\gamma = 0$ and $\nu \in (0, \infty)$. Assume that $L^{p,q;\alpha}(0,1)$ is equivalent to a rearrangement-invariant function space, that is, the parameters p, q and α satisfy one of conditions* (1.29) *(with $\beta = 0$). There is a rearrangement-invariant function space $Z(0,1)$ such that the operator $R_{v,\nu}\colon L^{p,q;\alpha}(0,1) \to Z(0,1)$ is bounded if and only if one of the following conditions holds:*

- *$p = q = 1$ and $\alpha \geq 1$;*
- *$p > 1$.*

Furthermore, if this is the case and we denote by $Y(0,1)$ the optimal target space for the operator $R_{v,\nu}$ and the space $L^{p,q;\alpha}(0,1)$, then

$$Y(0,1) = \begin{cases} L^{1,1;\alpha-1}(0,1), & p = q = 1,\ \alpha \geq 1; \\ L^{p,q;\alpha}(0,1), & p \in (1,\infty),\ \nu = 1 \text{ or} \\ & p = \infty,\ q \in [1,\infty),\ \alpha + \frac{1}{q} < 0,\ \nu = 1 \text{ or} \\ & p = q = \infty,\ \alpha \leq 0,\ \nu = 1; \\ L^{\frac{p}{p(1-\nu)+\nu},q;\alpha}(0,1), & p \in (1,\infty),\ \nu < 1 \text{ or} \\ & p \in (1, \frac{\nu}{\nu-1}),\ \nu > 1; \\ L^{\frac{1}{1-\nu},\infty;\alpha}(0,1), & p = q = \infty,\ \alpha \leq 0,\ \nu < 1; \\ L^{\infty,\infty;\alpha}(0,1), & p = \frac{\nu}{\nu-1},\ q = \infty,\ \alpha \leq 0,\ \nu > 1; \\ L^{\infty}(0,1), & p > \frac{\nu}{\nu-1},\ \nu > 1. \end{cases}$$

Remark 2.2.35. When $p = \infty$, $q \in [1,\infty)$, $\alpha + \frac{1}{q} < 0$, $\nu < 1$, the associate function norm of the optimal target space $Y(0,1)$ satisfies

$$\|f\|_{Y'(0,1)} \approx \left\| \int_{t^{\frac{1}{\nu}}}^{1} f^*(s)s^{-1}\,\mathrm{d}s \right\|_{L^{(1,q';-\alpha-1)}(0,1)}$$

for every $f \in \mathfrak{M}(0,1)$, but Proposition 2.2.34 does not provide us with a description of $Y(0,1)$ in terms of Lorentz–Zygmund spaces in that case.

When $p = \frac{\nu}{\nu-1}$, $q \in [1,\infty)$, $\nu > 1$, or $p = \frac{\nu}{\nu-1}$, $q = \infty$, $\alpha > 0$, $\nu > 1$, the associate function space of the optimal target space $Y(0,1)$ satisfies

$$\|f\|_{Y'(0,1)} \approx \sup_{g \sim f} \left\| \int_{t^{\frac{1}{\nu}}}^{1} g(s)s^{-1}\,\mathrm{d}s \right\|_{L^{\nu,q';-\alpha}(0,1)}$$

for every $f \in \mathfrak{M}(0,1)$, where the supremum is taken over all $g \in \mathfrak{M}^+(0,1)$ equimeasurable with f, but Proposition 2.2.34 does not provide us with a description of $Y(0,1)$ in terms of Lorentz–Zygmund spaces in those cases.

2.2.3.2 Optimal spaces for $H_{v,\nu}$

Taking into account the remark above Proposition 2.2.4, we can obtain optimal spaces for the operator $H_{v,\nu}$ and Lorentz–Zygmund spaces by combining the results that we obtained in the previous subsection with the characterizations of the associate spaces of Lorentz–Zygmund spaces provided in [75, Section 6]. We omit their lengthy but straightforward proofs.

First, we describe optimal domain spaces for $H_{v,\nu}$ and Lorentz–Zygmund spaces. Their descriptions follow from the corresponding results concerning optimal target spaces for the operator $R_{v,\nu}$, Proposition 2.2.4 and [75, Section 6].

Proposition 2.2.36. *Let $\gamma \in (0,1)$ and $\nu \in (0,\infty)$. Assume that $L^{p,q;\mathbb{A}}(0,\infty)$ is equivalent to a rearrangement-invariant function space, that is, the parameters p, q and $\mathbb{A}$ satisfy one of conditions* (1.28) *(with $\beta_0 = \beta_\infty = 0$). There is a rearrangement-invariant function space $Z(0,\infty)$ such that the operator $H_{v,\nu}\colon Z(0,\infty) \to L^{p,q;\mathbb{A}}(0,\infty)$ is bounded if and only if one of the following conditions holds:*

- $\gamma + \nu \leq 1$;
- $\gamma + \nu > 1$ *and* $p > \frac{\nu}{1-\gamma}$;
- $\gamma + \nu > 1$, $p = \frac{\nu}{1-\gamma}$ *and* $\alpha_\infty \leq 0$.

Furthermore, if this is the case and we denote by $X(0,\infty)$ the optimal domain space for the operator $H_{v,\nu}$ and the space $L^{p,q;\mathbb{A}}(0,\infty)$, then

$$X(0,\infty) = \begin{cases} L^{\frac{1}{\gamma+\nu},1;\mathbb{A}}(0,\infty), & p = q = 1,\ \alpha_0 \geq 0,\ \alpha_\infty \leq 0,\ \gamma + \nu \leq 1; \\ L^{1,1;\mathbb{A}}(0,\infty), & p = \frac{\nu}{1-\gamma},\ q = 1,\ \alpha_0 \geq 0,\ \alpha_\infty \leq 0,\ \gamma + \nu > 1; \\ L^{1}(0,\infty), & p = \frac{\nu}{1-\gamma},\ \alpha_0 = \alpha_\infty = 0,\ \gamma + \nu > 1; \\ X_1(0,\infty), & p = \frac{\nu}{1-\gamma},\ q = 1,\ \alpha_0 < 0,\ \alpha_\infty \leq 0,\ \gamma + \nu > 1 \text{ or} \\ & p = \frac{\nu}{1-\gamma},\ q \in (1,\infty],\ |\alpha_0| + |\alpha_\infty| > 0,\ \alpha_\infty \leq 0,\ \gamma + \nu > 1; \\ L^{\frac{p}{p\gamma+\nu},q;\mathbb{A}}(0,\infty), & p \in (1,\infty),\ \gamma + \nu \leq 1 \text{ or} \\ & p \in (\frac{\nu}{1-\gamma},\infty),\ \gamma + \nu > 1; \\ L^{\frac{1}{\gamma},1}(0,\infty), & p = q = \infty,\ \alpha_0 = \alpha_\infty = 0; \\ X_2(0,\infty), & p = q = \infty,\ \alpha_0 \leq 0,\ |\alpha_0| + |\alpha_\infty| > 0 \text{ or} \\ & p = \infty,\ q \in [1,\infty),\ \alpha_0 + \frac{1}{q} < 0, \end{cases}$$

where $X_1(0,\infty)$ and $X_2(0,\infty)$ are rearrangement-invariant function spaces such that

$$\begin{aligned} \|f\|_{X_1(0,\infty)} &\approx \sup_{g \sim f} \left\| t^{\frac{1-\gamma}{\nu} - \frac{1}{q}} \ell^{\mathbb{A}}(t) \int_{t^{\frac{1}{\nu}}}^{\infty} g(s) s^{-1+\gamma} \,\mathrm{d}s \right\|_{L^q(0,\infty)}, \\ \|f\|_{X_2(0,\infty)} &\approx \left\| t^{-\frac{1}{q}} \ell^{\mathbb{A}}(t) \int_{t^{\frac{1}{\nu}}}^{\infty} f^*(s) s^{-1+\gamma} \,\mathrm{d}s \right\|_{L^q(0,\infty)}, \end{aligned}$$

for every $f \in \mathfrak{M}(0,\infty)$, where the supremum is taken over all $g \in \mathfrak{M}^+(0,\infty)$ equimeasurable with f.

When $\gamma = 0$, Proposition 2.2.36 transforms into the following.

Proposition 2.2.37. *Let $\gamma = 0$ and $\nu \in (0,\infty)$. Assume that $L^{p,q;\mathbb{A}}(0,\infty)$ is equivalent to a rearrangement-invariant function space, that is, the parameters p, q and $\mathbb{A}$ satisfy one of conditions* (1.28) *(with $\beta_0 = \beta_\infty = 0$). There is a rearrangement-invariant function space $Z(0,\infty)$ such that the operator $H_{v,\nu}\colon Z(0,\infty) \to L^{p,q;\mathbb{A}}(0,\infty)$ is bounded if and only if one of the following conditions holds:*

- $\nu \leq 1$ *and* $p < \infty$;
- $\nu > 1$ *and* $p \in (\nu,\infty)$;
- $\nu > 1$, $p = \nu$ *and* $\alpha_\infty \leq 0$;

- *$p = \infty$, $q \in [1, \infty)$ and $\alpha_0 + 1 + \frac{1}{q} < 0$;*
- *$p = q = \infty$ and $\alpha_0 + 1 \leq 0$.*

Furthermore, if this is the case and we denote by $X(0, \infty)$ the optimal domain space for the operator $H_{v,\nu}$ and the space $L^{p,q;\alpha}(0, \infty)$, then

$$X(0,\infty) = \begin{cases} L^{\frac{1}{\nu},1;\mathbb{A}}(0,\infty), & p = q = 1,\ \alpha_0 \geq 0,\ \alpha_\infty \leq 0,\ \nu \leq 1; \\ L^{1,1;\mathbb{A}}(0,\infty), & p = \nu,\ q = 1,\ \alpha_0 \geq 0,\ \alpha_\infty \leq 0,\ \nu > 1; \\ X_1(0,\infty), & p = \nu,\ q = 1,\ \alpha_0 < 0,\ \alpha_\infty \leq 0,\ \nu > 1 \text{ or} \\ & p = \nu,\ q \in (1,\infty],\ \alpha_\infty \leq 0,\ \nu > 1; \\ L^{\frac{p}{\nu},q;\mathbb{A}}(0,\infty), & p \in (1,\infty),\ \nu \leq 1 \text{ or} \\ & p \in (\nu,\infty),\ \nu > 1; \\ L^{\infty,\infty;\mathbb{A}+1}(0,\infty), & p = q = \infty,\ \alpha_0 + 1 \leq 0,\ \alpha_\infty > 0; \\ X_2(0,\infty), & p = q = \infty,\ \alpha_0 + 1 \leq 0,\ \alpha_\infty \leq 0 \text{ or} \\ & p = \infty,\ q \in [1,\infty),\ \alpha_0 + 1 + \frac{1}{q} < 0, \end{cases}$$

where $X_1(0, \infty)$ and $X_2(0, \infty)$ are rearrangement-invariant function spaces such that

$$\|f\|_{X_1(0,\infty)} \approx \sup_{g \sim f} \left\| t^{\frac{1}{\nu} - \frac{1}{q}} \ell^{\mathbb{A}}(t) \int_{t^{\frac{1}{\nu}}}^{\infty} g(s) s^{-1} \, \mathrm{d}s \right\|_{L^q(0,\infty)},$$
$$\|f\|_{X_2(0,\infty)} \approx \left\| t^{-\frac{1}{q}} \ell^{\mathbb{A}}(t) \int_{t^{\frac{1}{\nu}}}^{\infty} f^*(s) s^{-1} \, \mathrm{d}s \right\|_{L^q(0,\infty)},$$

for every $f \in \mathfrak{M}(0, \infty)$, where the supremum is taken over all $g \in \mathfrak{M}^+(0, \infty)$ equimeasurable with f.

We now consider the case where $L = 1$.

Proposition 2.2.38. *Let $\gamma \in (0, 1)$ and $\nu \in (0, \infty)$. Assume that $L^{p,q;\alpha}(0, 1)$ is equivalent to a rearrangement-invariant function space, that is, the parameters p, q and α satisfy one of conditions* (1.29) *(with $\beta = 0$). There is an optimal domain space $X(0, 1)$ for the operator $H_{v,\nu}$ and the space $L^{p,q;\alpha}(0, 1)$. Furthermore, we have that*

$$X(0,1) = \begin{cases} L^{\frac{1}{\gamma+\nu},1;\alpha}(0,1), & p = q = 1,\ \alpha \geq 0,\ \gamma + \nu \leq 1; \\ L^1(0,1), & p \in [1, \frac{\nu}{1-\gamma}),\ \gamma + \nu > 1 \text{ or} \\ & p = \frac{\nu}{1-\gamma},\ \alpha = 0,\ \gamma + \nu > 1; \\ L^{1,1;\alpha}(0,1), & p = \frac{\nu}{1-\gamma},\ q = 1,\ \alpha \geq 0,\ \gamma + \nu > 1; \\ X_1(0,1), & p = \frac{\nu}{1-\gamma},\ q = 1,\ \alpha < 0,\ \gamma + \nu > 1 \text{ or} \\ & p = \frac{\nu}{1-\gamma},\ q \in (1,\infty],\ \alpha \neq 0,\ \gamma + \nu > 1; \\ L^{\frac{p}{p\gamma+\nu},q;\alpha}(0,1), & p \in (1,\infty),\ \gamma + \nu \leq 1 \text{ or} \\ & p \in (\frac{\nu}{1-\gamma},\infty),\ \gamma + \nu > 1; \\ L^{\frac{1}{\gamma},1}(0,1), & p = q = \infty,\ \alpha = 0; \\ X_2(0,1), & p = q = \infty,\ \alpha < 0 \text{ or} \\ & p = \infty,\ q \in [1,\infty),\ \alpha + \frac{1}{q} < 0, \end{cases}$$

where $X_1(0,1)$ and $X_2(0,1)$ are rearrangement-invariant function spaces such that

$$\|f\|_{X_1(0,1)} \approx \sup_{g\sim f} \Big\| t^{\frac{1-\gamma}{\nu}-\frac{1}{q}} \ell^\alpha(t) \int_{t^{\frac{1}{\nu}}}^1 g(s)s^{-1+\gamma}\,\mathrm{d}s \Big\|_{L^q(0,1)},$$

$$\|f\|_{X_2(0,1)} \approx \Big\| t^{-\frac{1}{q}} \ell^\alpha(t) \int_{t^{\frac{1}{\nu}}}^1 f^*(s)s^{-1+\gamma}\,\mathrm{d}s \Big\|_{L^q(0,1)},$$

for every $f \in \mathfrak{M}(0,1)$, where the supremum is taken over all $g \in \mathfrak{M}^+(0,1)$ equimeasurable with f.

The next proposition is an analogue of Proposition 2.2.38 in the case where $\gamma = 0$.

Proposition 2.2.39. *Let $\gamma = 0$ and $\nu \in (0,\infty)$. Assume that $L^{p,q;\alpha}(0,1)$ is equivalent to a rearrangement-invariant function space, that is, the parameters p, q and α satisfy one of conditions* (1.29) *(with $\beta = 0$). There is a rearrangement-invariant function space $Z(0,1)$ such that the operator $H_{v,\nu}\colon Z(0,1) \to L^{p,q;\alpha}(0,1)$ is bounded if and only if one of the following conditions holds:*

- $p < \infty$;
- $p = \infty$, $q \in [1,\infty)$ *and* $\alpha + 1 + \frac{1}{q} < 0$;
- $p = q = \infty$ *and* $\alpha + 1 \le 0$.

Furthermore, if this is the case and we denote by $X(0,1)$ the optimal domain space for the operator $H_{v,\nu}$ and the space $L^{p,q;\alpha}(0,1)$, then

$$X(0,1) = \begin{cases} L^{\frac{1}{\nu},1;\alpha}(0,1), & p = q = 1,\ \alpha \ge 0,\ \nu \le 1; \\ L^1(0,1), & p \in [1,\nu),\ \nu > 1; \\ L^{1,1;\alpha}(0,1), & p = \nu,\ q = 1,\ \alpha \ge 0,\ \nu > 1; \\ X_1(0,1), & p = \nu,\ q = 1,\ \alpha < 0,\ \nu > 1 \text{ or} \\ & p = \nu,\ q \in (1,\infty],\ \nu > 1; \\ L^{\frac{p}{\nu},q;\alpha}(0,1), & p \in (1,\infty),\ \nu \le 1 \text{ or} \\ & p \in (\nu,\infty),\ \nu > 1; \\ L^{\infty,\infty;\alpha+1}(0,1), & p = q = \infty,\ \alpha + 1 \le 0; \\ X_2(0,1), & p = \infty,\ q \in [1,\infty),\ \alpha + 1 + \frac{1}{q} < 0, \end{cases}$$

where $X_1(0,1)$ and $X_2(0,1)$ are rearrangement-invariant function spaces such that

$$\|f\|_{X_1(0,1)} \approx \sup_{g\sim f} \Big\| t^{\frac{1}{\nu}-\frac{1}{q}} \ell^\alpha(t) \int_{t^{\frac{1}{\nu}}}^1 g(s)s^{-1}\,\mathrm{d}s \Big\|_{L^q(0,1)},$$

$$\|f\|_{X_2(0,1)} \approx \Big\| t^{-\frac{1}{q}} \ell^\alpha(t) \int_{t^{\frac{1}{\nu}}}^1 f^*(s)s^{-1}\,\mathrm{d}s \Big\|_{L^q(0,1)},$$

for every $f \in \mathfrak{M}(0,1)$, where the supremum is taken over all $g \in \mathfrak{M}^+(0,1)$ equimeasurable with f.

Next, we describe optimal target spaces for $H_{v,\nu}$ and Lorentz–Zygmund spaces. Their descriptions follow from the corresponding results concerning optimal domain spaces for the operator $R_{v,\nu}$, Proposition 2.2.5 and [75, Section 6].

Proposition 2.2.40. *Let $\gamma \in (0,1)$ and $\nu \in (0,\infty)$. Assume that $L^{p,q;\mathbb{A}}(0,\infty)$ is equivalent to a rearrangement-invariant function space, that is, the parameters p, q and $\mathbb{A}$ satisfy one of conditions (1.28) (with $\beta_0 = \beta_\infty = 0$). There is a rearrangement-invariant function space $Z(0,\infty)$ such that the operator $H_{v,\nu}\colon L^{p,q;\mathbb{A}}(0,\infty) \to Z(0,\infty)$ is bounded if and only if one of the following conditions holds:*

- $\gamma + \nu \geq 1$ *and* $p < \frac{1}{\gamma}$;
- $\gamma + \nu < 1$ *and* $p \in (\frac{1}{\gamma+\nu}, \frac{1}{\gamma})$;
- $\gamma + \nu < 1$, $p = \frac{1}{\gamma+\nu}$ *and*
  - ⋆ $q = 1$, $\alpha_0 \geq 0$ *or*
  - ⋆ $q > 1$, $\alpha_0 > 1 - \frac{1}{q}$;
- $p = \frac{1}{\gamma}$ *and*
  - ⋆ $q = 1$, $\alpha_\infty \geq 0$ *or*
  - ⋆ $q > 1$, $\alpha_\infty > 1 - \frac{1}{q}$.

Furthermore, if this is the case and we denote by $Y(0,\infty)$ the optimal target space for the operator $H_{v,\nu}$ and the space $L^{p,q;\mathbb{A}}(0,\infty)$, then

$$Y(0,\infty) = \begin{cases} L^{\frac{p\nu}{1-p\gamma},q;\mathbb{A}}(0,\infty), & p = q = 1,\ \alpha_0 \geq 0,\ \alpha_\infty \leq 0,\ \gamma+\nu \geq 1 \ \textit{or} \\ & p \in (1, \frac{1}{\gamma}),\ \gamma+\nu \geq 1 \ \textit{or} \\ & p \in (\frac{1}{\gamma+\nu}, \frac{1}{\gamma}),\ \gamma+\nu < 1; \\ L^{1,1;\mathbb{A}}(0,\infty), & p = \frac{1}{\gamma+\nu},\ q = 1,\ \alpha_0 \geq 0,\ \alpha_\infty \leq 0,\ \gamma+\nu < 1; \\ L^{1,1;(\alpha_0,0)}(0,\infty), & p = \frac{1}{\gamma+\nu},\ q = 1,\ \alpha_0 \geq 0,\ \alpha_\infty > 0,\ \gamma+\nu < 1; \\ L^{(1,q;\mathbb{A}-1)}(0,\infty), & p = \frac{1}{\gamma+\nu},\ q \in (1,\infty],\ \alpha_0 > 1-\frac{1}{q},\ \alpha_\infty < 1-\frac{1}{q} \\ & \textit{and}\ \gamma+\nu < 1; \\ L^{(1,q;(\alpha_0-1,-\frac{1}{q}),(0,-1))}(0,\infty), & p = \frac{1}{\gamma+\nu},\ q \in (1,\infty],\ \alpha_0 > 1-\frac{1}{q},\ \alpha_\infty = 1-\frac{1}{q} \\ & \textit{and}\ \gamma+\nu < 1; \\ Y_1(0,\infty), & p = \frac{1}{\gamma+\nu},\ q \in (1,\infty],\ \alpha_0 > 1-\frac{1}{q},\ \alpha_\infty > 1-\frac{1}{q} \\ & \textit{and}\ \gamma+\nu < 1; \\ L^{\infty,q;\mathbb{A}-1}(0,\infty), & p = \frac{1}{\gamma},\ \alpha_0 < 1-\frac{1}{q},\ \alpha_\infty > 1-\frac{1}{q}; \\ Y_2(0,\infty), & p = \frac{1}{\gamma},\ q \in [1,\infty),\ \alpha_0 > 1-\frac{1}{q},\ \alpha_\infty > 1-\frac{1}{q}; \\ L^{\infty,\infty;(0,\alpha_\infty-1)}(0,\infty), & p = \frac{1}{\gamma},\ q = \infty,\ \alpha_0 > 1,\ \alpha_\infty > 1; \\ L^{\infty,1;(-1,\alpha_\infty-1),(-1,0),(-1,0)}(0,\infty), & p = \frac{1}{\gamma},\ q = 1,\ \alpha_0 = 0,\ \alpha_\infty > 0; \\ L^{\infty,q;(-\frac{1}{q},\alpha_\infty-1),(-1,0)}(0,\infty), & p = \frac{1}{\gamma},\ q \in (1,\infty],\ \alpha_0 = 1-\frac{1}{q},\ \alpha_\infty > 1-\frac{1}{q}; \\ Y_3(0,\infty), & p = \frac{1}{\gamma},\ q = 1,\ \alpha_0 < 0,\ \alpha_\infty = 0; \\ L^\infty(0,\infty), & p = \frac{1}{\gamma},\ q = 1,\ \alpha_0 \geq 0,\ \alpha_\infty = 0, \end{cases}$$

where $Y_1(0,\infty)$, $Y_2(0,\infty)$ and $Y_3(0,\infty)$ are rearrangement-invariant function *spaces such that*

$$\begin{aligned}\|f\|_{Y_1(0,\infty)} &\approx \|t^{1-\frac{1}{q}}\ell^{\alpha_0-1}(t)f^{**}(t)\|_{L^1(0,1)}+\|f\|_{L^1(0,\infty)},\\ \|f\|_{Y_2(0,\infty)} &\approx \|t^{-\frac{1}{q}}\ell^{\alpha_\infty-1}(t)f^{*}(t)\chi_{(1,\infty)}(t)\|_{L^q(0,\infty)}+\|f\|_{L^\infty(0,\infty)},\\ \|f\|_{Y_3(0,\infty)} &\approx \|t^{-1}\ell^{\alpha_0-1}(t)f^{*}(t)\|_{L^1(0,1)},\end{aligned}\tag{2.114}$$

for every $f\in\mathfrak{M}(0,\infty)$.

When $\gamma=0$, Proposition 2.2.40 transforms into the following.

Proposition 2.2.41. *Let* $\gamma=0$ *and* $\nu\in(0,\infty)$. *Assume that* $L^{p,q;\mathbb{A}}(0,\infty)$ *is equivalent to a rearrangement-invariant function space, that is, the parameters* p,q *and* $\mathbb{A}$ *satisfy one of conditions* (1.28) *(with* $\beta_0=\beta_\infty=0$*). There is a rearrangement-invariant function space* $Z(0,\infty)$ *such that the operator* $H_{\nu,\nu}\colon L^{p,q;\mathbb{A}}(0,\infty)\to Z(0,\infty)$ *is bounded if and only if one of the following conditions holds:*

- $\nu\geq 1$ *and* $p<\infty$*;*
- $\nu<1$ *and* $p\in(\frac{1}{\nu},\infty)$*;*
- $\nu<1$, $p=\frac{1}{\nu}$ *and*
  - ⋆ $q=1$, $\alpha_0\geq 0$ *or*
  - ⋆ $q>1$, $\alpha_0>1-\frac{1}{q}$*;*
- $p=\infty$ *and*
  - ⋆ $q=1$, $\alpha_\infty\geq 0$ *or*
  - ⋆ $q>1$, $\alpha_\infty>1-\frac{1}{q}$.

Furthermore, if this is the case and we denote by $Y(0,\infty)$ *the optimal target space for the operator* $H_{v,\nu}$ *and the space* $L^{p,q;\mathbb{A}}(0,\infty)$*, then*

$$Y(0,\infty)=\begin{cases} L^{p\nu,q;\mathbb{A}}(0,\infty), & p=q=1,\ \alpha_0\geq 0,\ \alpha_\infty\leq 0,\ \nu\geq 1\ \textit{or}\\ & p\in(1,\infty),\ \nu\geq 1\ \textit{or}\\ & p\in(\frac{1}{\nu},\infty),\ \nu<1;\\ L^{1,1;\mathbb{A}}(0,\infty), & p=\frac{1}{\nu},\ q=1,\ \alpha_0\geq 0,\ \alpha_\infty\leq 0,\ \nu<1;\\ L^{1,1;(\alpha_0,0)}(0,\infty), & p=\frac{1}{\nu},\ q=1,\ \alpha_0\geq 0,\ \alpha_\infty> 0,\ \nu<1;\\ L^{(1,q;\mathbb{A}-1)}(0,\infty), & p=\frac{1}{\nu},\ q\in(1,\infty],\ \alpha_0>1-\frac{1}{q},\ \alpha_\infty<1-\frac{1}{q}\\ & \textit{and}\ \nu<1;\\ L^{(1,q;(\alpha_0-1,-\frac{1}{q}),(0,-1))}(0,\infty), & p=\frac{1}{\nu},\ q\in(1,\infty],\ \alpha_0>1-\frac{1}{q},\ \alpha_\infty=1-\frac{1}{q}\\ & \textit{and}\ \nu<1;\\ Y_1(0,\infty), & p=\frac{1}{\nu},\ q\in(1,\infty],\ \alpha_0>1-\frac{1}{q},\ \alpha_\infty>1-\frac{1}{q}\\ & \textit{and}\ \nu<1;\\ L^{\infty,\infty;\mathbb{A}-1}(0,\infty), & p=q=\infty,\ \alpha_0\leq 0,\ \alpha_\infty>1, \end{cases}$$

where $Y_1(0,\infty)$ *is the rearrangement-invariant function space whose norm satisfies* (2.114).

Remark 2.2.42. Proposition 2.2.41 together with Proposition 2.2.5 tells us that the rearrangement-invariant function space $Y(0,\infty)$ whose associate function norm satisfies

$$\|f\|_{Y'(0,\infty)} \approx \|t^{-1+\nu} f^{**}(t^\nu)\|_{L^{(1,q';-\mathbb{A}-1)}(0,\infty)} \quad \text{for every } f \in \mathfrak{M}(0,\infty)$$

is the optimal target space for the operator $H_{v,\nu}$, where $v(t) = t^{-1}$, $t \in (0,\infty)$, and the space $L^{\infty,q;\mathbb{A}}(0,\infty)$ when either $q = 1$, $\alpha_0 + 1 < 0$, $\alpha_\infty \geq 0$, or $q \in (1,\infty)$, $\alpha_0 + \frac{1}{q} < 0$, $\alpha_\infty > 1 - \frac{1}{q}$, but Proposition 2.2.41 does not provide us with a description of $Y(0,\infty)$ in terms of Lorentz–Zygmund spaces in those cases.

Finally, we conclude this subsection with descriptions of optimal target spaces for $H_{v,\nu}$ and Lorentz–Zygmund spaces when $L = 1$.

Proposition 2.2.43. *Let $\gamma \in (0,1)$ and $\nu \in (0,\infty)$. Assume that $L^{p,q;\alpha}(0,1)$ is equivalent to a rearrangement-invariant function space, that is, the parameters p, q and α satisfy one of conditions* (1.29) *(with $\beta = 0$). There is a rearrangement-invariant function space $Z(0,1)$ such that the operator $H_{v,\nu}\colon L^{p,q;\alpha}(0,1) \to Z(0,1)$ is bounded if and only if one of the following conditions holds:*

- $\gamma + \nu \geq 1$*;*
- $\gamma + \nu < 1$ *and* $p > \frac{1}{\gamma+\nu}$*;*
- $\gamma + \nu < 1$*,* $p = \frac{1}{\gamma+\nu}$ *and*
    - ⋆ $q > 1$*,* $\alpha > 1 - \frac{1}{q}$ *or*
    - ⋆ $q = 1$*,* $\alpha \geq 0$*.*

Furthermore, if this is the case and we denote by $Y(0,1)$ the optimal target space for the operator $H_{v,\nu}$ and the space $L^{p,q;\alpha}(0,1)$, then

$$Y(0,1) = \begin{cases} L^{\frac{p\nu}{1-p\gamma},q;\alpha}(0,1), & p = q = 1,\ \alpha \geq 0,\ \gamma+\nu \geq 1 \textit{ or} \\ & p \in (1,\frac{1}{\gamma}),\ \gamma+\nu \geq 1 \textit{ or} \\ & p \in (\frac{1}{\gamma+\nu},\frac{1}{\gamma}),\ \gamma+\nu < 1; \\ L^{1,1;\alpha}(0,1), & p = \frac{1}{\gamma+\nu},\ q = 1,\ \alpha \geq 0,\ \gamma+\nu < 1; \\ L^{(1,q;\alpha-1)}(0,1), & p = \frac{1}{\gamma+\nu},\ q \in (1,\infty],\ \alpha > 1-\frac{1}{q},\ \gamma+\nu < 1; \\ L^{\infty,q;\alpha-1}(0,1), & p = \frac{1}{\gamma},\ \alpha < 1-\frac{1}{q}; \\ L^{\infty,q;-\frac{1}{q},-1}(0,1), & p = \frac{1}{\gamma},\ q \in (1,\infty],\ \alpha = 1-\frac{1}{q}; \\ L^{\infty}(0,1), & p = \frac{1}{\gamma},\ \alpha > 1-\frac{1}{q} \textit{ or} \\ & p = \frac{1}{\gamma},\ q = 1,\ \alpha \geq 0 \textit{ or} \\ & p > \frac{1}{\gamma}. \end{cases}$$

The next proposition is an analogue of Proposition 2.2.43 in the case where $\gamma = 0$.

Proposition 2.2.44. *Let $\gamma = 0$ and $\nu \in (0,\infty)$. Assume that $L^{p,q;\alpha}(0,1)$ is equivalent to a rearrangement-invariant function space, that is, the parameters p, q and α satisfy one of conditions* (1.29) *(with $\beta = 0$). There is a rearrangement-invariant function space $Z(0,1)$ such that the operator $H_{v,\nu}\colon L^{p,q;\alpha}(0,1) \to Z(0,1)$ is bounded if and only if one of the following conditions holds:*

- *$\nu \geq 1$;*
- *$\nu < 1$ and $p > \frac{1}{\nu}$;*
- *$\nu < 1$, $p = \frac{1}{\nu}$ and*
    - ⋆ *$q > 1$, $\alpha > 1 - \frac{1}{q}$ or*
    - ⋆ *$q = 1$, $\alpha \geq 0$.*

Furthermore, if this is the case and we denote by $Y(0,1)$ the optimal target space for the operator $H_{v,\nu}$ and the space $L^{p,q;\alpha}(0,1)$, then

$$Y(0,1) = \begin{cases} L^{p\nu,q;\alpha}(0,1), & p = q = 1,\ \alpha \geq 0,\ \nu \geq 1 \ \textit{or} \\ & p \in (1,\infty),\ \nu \geq 1 \ \textit{or} \\ & p \in (\frac{1}{\nu},\infty),\ \nu < 1; \\ L^{1,1;\alpha}(0,1), & p = \frac{1}{\nu},\ q = 1,\ \alpha \geq 0,\ \nu < 1; \\ L^{(1,q;\alpha-1)}(0,1), & p = \frac{1}{\nu},\ q \in (1,\infty],\ \alpha > 1 - \frac{1}{q},\ \nu < 1; \\ L^{\infty,\infty;\alpha-1}(0,1), & p = q = \infty,\ \alpha \leq 0. \end{cases}$$

Remark 2.2.45. Proposition 2.2.44 together with Proposition 2.2.5 tells us that the rearrangement-invariant function space $Y(0,1)$ whose associate function norm satisfies

$$\|f\|_{Y'(0,1)} \approx \|t^{-1+\nu} f^{**}(t^\nu)\|_{L^{(1,q';-\alpha-1)}(0,1)} \quad \text{for every } f \in \mathfrak{M}(0,1)$$

is the optimal target space for the operator $H_{v,\nu}$, where $v(t) = t^{-1}$, $t \in (0,1)$, and the space $L^{\infty,q;\alpha}(0,1)$ when $q \in [1,\infty)$ and $\alpha + \frac{1}{q} < 0$, but Proposition 2.2.44 does not provide us with a description of $Y(0,1)$ in terms of Lorentz–Zygmund spaces in that case.

2.2.3.3 Optimal spaces for S_σ

In this subsection, we provide optimal function spaces for the Calderón-type operator S_σ with $\sigma = (1,\beta,1-\beta,0)$, where $\beta \in (0,1]$. Note that we have that $m = 1$, where m is defined by (2.22), and $\sigma = \sigma'$, where σ' is defined by (2.24). Furthermore, thanks to (2.23), we have that

$$S_\sigma(f^*) = \beta H_{v,1}(f^{**}) \quad \text{for every } f \in \mathfrak{M}(0,\infty), \tag{2.115}$$

where

$$v(t) = t^{-\beta},\ t \in (0,\infty). \tag{2.116}$$

For the sake of readability, we split the results of this subsection based on whether $\beta < 1$ or $\beta = 1$.

We start off with optimal domain spaces.

Proposition 2.2.46. *Let $\beta \in (0,1)$. Set $\sigma = (1,\beta,1-\beta,0)$. Assume that $L^{p,q;\mathbb{A}}(0,\infty)$ is equivalent to a rearrangement-invariant function space, that is, the parameters p,q and $\mathbb{A} = (\alpha_0,\alpha_\infty)$ satisfy one of conditions* (1.28) *(with $\beta_0 = \beta_\infty = 0$). There is a rearrangement-invariant function space $Z(0,\infty)$ such that the operator $S_\sigma\colon Z(0,\infty) \to L^{p,q;\mathbb{A}}(0,\infty)$ is bounded if and only if one of the following conditions holds:*

- *$p = \frac{1}{\beta}$, $q < \infty$ and $\alpha_\infty + \frac{1}{q} < 0$;*
- *$p = \frac{1}{\beta}$, $q = \infty$ and $\alpha_\infty \le 0$;*
- *$p > \frac{1}{\beta}$;*

Furthermore, if this is the case and we denote by $X(0,\infty)$ the optimal domain space for the operator S_σ and the space $L^{p,q;\mathbb{A}}(0,\infty)$, then

$$X(0,\infty) = \begin{cases} L^{1,1;(0,\alpha_\infty+1)}(0,\infty), & p = \frac{1}{\beta},\ q = 1,\ \alpha_0 + 1 < 0,\ \alpha_\infty + 1 < 0; \\ L^{1,1;\mathbb{A}+1}(0,\infty), & p = \frac{1}{\beta},\ q = 1,\ \alpha_0 + 1 > 0,\ \alpha_\infty + 1 < 0; \\ L^{1,1;(0,\alpha_\infty+1),(1,0)}(0,\infty), & p = \frac{1}{\beta},\ q = 1,\ \alpha_0 + 1 = 0,\ \alpha_\infty + 1 < 0; \\ L^{(1,q;\mathbb{A})}(0,\infty), & p = \frac{1}{\beta},\ q \in (1,\infty),\ \alpha_\infty + \frac{1}{q} < 0 \text{ or} \\ & p = \frac{1}{\beta},\ q = \infty,\ \alpha_\infty \le 0; \\ L^{\frac{p}{1+p(1-\beta)},q;\mathbb{A}}(0,\infty), & p \in (\frac{1}{\beta},\infty); \\ L^{\frac{1}{1-\beta},1}(0,\infty), & p = q = \infty,\ \alpha_0 = \alpha_\infty = 0. \end{cases}$$

Proof. First, assume that $p < \infty$. Thanks to (1.30), we have that $(L^{p,q;\mathbb{A}})'(0,\infty) = L^{p',q';-\mathbb{A}}(0,\infty)$ and $p' > 1$. We claim that $S_\sigma\colon Z(0,\infty) \to L^{p,q;\mathbb{A}}(0,\infty)$ is bounded if and only if $R_{v,1}\colon Z(0,\infty) \to L^{p,q;\mathbb{A}}(0,\infty)$, where v is defined by (2.116), is bounded. Once we prove it, we can use Proposition 2.2.22 with $\gamma = 1-\beta$ and $\nu = 1$ to obtain the description of $X(0,\infty)$ provided that p is finite. Owing to Proposition 2.2.20 combined with the fact that $\sigma = \sigma'$, we have that $S_\sigma\colon Z(0,\infty) \to L^{p,q;\mathbb{A}}(0,\infty)$ is bounded if and only if $S_\sigma\colon L^{p',q';-\mathbb{A}}(0,\infty) \to Z'(0,\infty)$ is bounded. We now observe that $S_\sigma\colon L^{p',q';-\mathbb{A}}(0,\infty) \to Z'(0,\infty)$ is bounded if and only if $R_{v,1}\colon Z(0,\infty) \to L^{p,q;\mathbb{A}}(0,\infty)$ is bounded. To this end, note that

$$\sup_{\|f\|_{L^{p',q';-\mathbb{A}}(0,\infty)} \le 1} \|H_{v,1}(f^{**})\|_{Z'(0,\infty)} \approx \sup_{\|f\|_{L^{p',q';-\mathbb{A}}(0,\infty)} \le 1} \|H_{v,1}(f^{*})\|_{Z'(0,\infty)}. \tag{2.117}$$

Indeed, "$\ge$" follows directly from (1.1), while "$\lesssim$" follows from the fact that $\|f^{**}\|_{L^{p',q';-\mathbb{A}}(0,\infty)} \lesssim \|f\|_{L^{p',q';-\mathbb{A}}(0,\infty)}$ for every $f \in \mathfrak{M}^+(0,\infty)$ inasmuch as $p' > 1$. Consequently, we have that

$$\begin{aligned} \sup_{\|f\|_{L^{p',q';-\mathbb{A}}(0,\infty)} \le 1} \|S_\sigma f\|_{Z'(0,\infty)} &= \sup_{\|f\|_{L^{p',q';-\mathbb{A}}(0,\infty)} \le 1} \|S_\sigma(f^*)\|_{Z'(0,\infty)} \\ &\approx \sup_{\|f\|_{L^{p',q';-\mathbb{A}}(0,\infty)} \le 1} \|H_{v,1}(f^{**})\|_{Z'(0,\infty)} \\ &\approx \sup_{\|f\|_{L^{p',q';-\mathbb{A}}(0,\infty)} \le 1} \|H_{v,1}(f^{*})\|_{Z'(0,\infty)} \\ &\approx \sup_{\|g\|_{Z(0,\infty)} \le 1} \|R_{v,1} g\|_{L^{p,q;\mathbb{A}}(0,\infty)}, \end{aligned}$$

where we used the Hardy–Littlewood inequality (1.3) in the equality, (2.115) in the first equivalence, and (2.60) in the last equivalence.

Second, assume that $p = \infty$. It can be readily verified that the assumptions of Proposition 2.2.19 are satisfied; hence the rearrangement-invariant function space $X(0,\infty)$ whose function norm satisfies

$$\|f\|_{X(0,\infty)} \approx \|S_\sigma(f^*)\|_{L^{\infty,q;\mathbb{A}}(0,\infty)} \quad \text{for every } f \in \mathfrak{M}^+(0,\infty)$$

is the optimal domain space for S_σ and $L^{\infty,q;\mathbb{A}}(0,\infty)$. If $q=\infty$, $\alpha_0=\alpha_\infty=0$, then we have that

$$\begin{aligned}\|S_\sigma(f^*)\|_{L^\infty(0,\infty)} \approx \|H_{v,1}(f^{**})\|_{L^\infty(0,\infty)} &= \int_0^\infty f^{**}(s)s^{-\beta}\,\mathrm{d}s\\ &= \|f\|_{L^{(\frac{1}{1-\beta},1)}(0,\infty)} \approx \|f\|_{L^{\frac{1}{1-\beta},1}(0,\infty)}\end{aligned}$$

for every $f\in\mathfrak{M}(0,\infty)$ thanks to (2.115) and (1.31). ☐

Remark 2.2.47. If $p=\infty$, $q\in[1,\infty)$, $\alpha_0+\frac{1}{q}<0$, or $p=q=\infty$, $\alpha_0\le 0$, $|\alpha_0|+|\alpha_\infty|>0$, then the rearrangement-invariant function space $X(0,\infty)$ whose function norm satisfies

$$\|f\|_{X(0,\infty)} \approx \left\|t^{-\frac{1}{q}}\ell^{\mathbb{A}}(t)\int_t^\infty f^{**}(s)s^{-\beta}\,\mathrm{d}s\right\|_{L^q(0,\infty)} \quad \text{for every } f\in\mathfrak{M}^+(0,\infty)$$

is the optimal domain space for S_σ and $L^{\infty,q;\mathbb{A}}(0,\infty)$, where $\sigma=(1,\beta,1-\beta,0)$, $\beta\in(0,1)$, but Proposition 2.2.46 does not provide us with a description of $X(0,\infty)$ in terms of Lorentz–Zygmund spaces in those cases.

When $\beta=1$, Proposition 2.2.46 transforms into the following.

Proposition 2.2.48. *Set $\sigma=(1,1,0,0)$. Assume that $L^{p,q;\mathbb{A}}(0,\infty)$ is equivalent to a rearrangement-invariant function space, that is, the parameters p,q and $\mathbb{A}=(\alpha_0,\alpha_\infty)$ satisfy one of conditions* (1.28) *(with $\beta_0=\beta_\infty=0$). There is a rearrangement-invariant function space $Z(0,\infty)$ such that the operator $S_\sigma\colon Z(0,\infty)\to L^{p,q;\mathbb{A}}(0,\infty)$ is bounded if and only if one of the following conditions holds:*

- $p=q=1$, $\alpha_0\ge 0$, $\alpha_\infty<-1$;
- $p\in(1,\infty)$;
- $p=\infty$, $q<\infty$, $\alpha_0+1+\frac{1}{q}<0$;
- $p=q=\infty$, $\alpha_0+1\le 0$;

Furthermore, if this is the case and we denote by $X(0,\infty)$ the optimal domain space for the operator S_σ and the space $L^{p,q;\mathbb{A}}(0,\infty)$, then

$$X(0,\infty)=\begin{cases} L^{1,1;\mathbb{A}+1}(0,\infty), & p=1,\ q=1,\ \alpha_0\ge 0,\ \alpha_\infty+1<0;\\ L^{p,q;\mathbb{A}}(0,\infty), & p\in(1,\infty);\\ L^{\infty,q;\mathbb{A}+1}(0,\infty), & p=\infty,\ q\in[1,\infty),\ \alpha_0+1+\frac{1}{q}<0,\ \alpha_\infty+\frac{1}{q}>0 \text{ or}\\ & p=q=\infty,\ \alpha_0+1\le 0,\ \alpha_\infty>0. \end{cases}$$

Proof. If $p<\infty$, we can proceed in the same way as in the proof of Proposition 2.2.46 (we just make use of Proposition 2.2.23 instead of Proposition 2.2.22).

Assume that $p=\infty$. It can be readily verified that the assumptions of Proposition 2.2.19 are satisfied if and only if $q<\infty$ and $\alpha_0+1+\frac{1}{q}<0$, or $q=\infty$ and $\alpha_0+1\le 0$; hence, in these cases, the rearrangement-invariant function space $X(0,\infty)$ whose function norm satisfies

$$\|f\|_{X(0,\infty)} \approx \|S_\sigma(f^*)\|_{L^{\infty,q;\mathbb{A}}(0,\infty)} \quad \text{for every } f\in\mathfrak{M}^+(0,\infty)$$

is the optimal domain space for S_σ and $L^{\infty,q;\mathbb{A}}(0,\infty)$. Assume now, in addition, that $\alpha_\infty + \frac{1}{q} > 0$. On the one hand, we have that

$$\begin{aligned}\|S_\sigma(f^*)\|_{L^{\infty,q;\mathbb{A}}(0,\infty)} &= \Big\| t^{-\frac{1}{q}} \ell^{\mathbb{A}}(t) \int_t^\infty f^{**}(s) s^{-1} \,\mathrm{d}s \Big\|_{L^q(0,\infty)} \\ &\lesssim \|t^{1-\frac{1}{q}} \ell^{\mathbb{A}+1}(t) f^{**}(t) t^{-1}\|_{L^q(0,\infty)} = \|f\|_{L^{(\infty,q;\mathbb{A}+1)}(0,\infty)} \\ &\approx \|f\|_{L^{\infty,q;\mathbb{A}+1}(0,\infty)}\end{aligned} \tag{2.118}$$

for every $f \in \mathfrak{M}(0,\infty)$, where we used (2.115), the Hardy inequality [69, Theorem 2] (its validity is the point where the extra assumption $\alpha_\infty + \frac{1}{q} > 0$ was used) and (1.31). On the other hand, we have that

$$\begin{aligned}\|S_\sigma(f^*)\|_{L^{\infty,q;\mathbb{A}}(0,\infty)} &= \Big\| t^{-\frac{1}{q}} \ell^{\mathbb{A}}(t) \int_t^\infty f^{**}(s) s^{-1} \,\mathrm{d}s \Big\|_{L^q(0,\infty)} \\ &\approx \Big\| t^{-\frac{1}{q}} \ell^{\mathbb{A}}(t) \chi_{(0,1)}(t) \int_t^\infty f^{**}(s) s^{-1} \,\mathrm{d}s \Big\|_{L^q(0,\infty)} \\ &\quad + \Big\| t^{-\frac{1}{q}} \ell^{\mathbb{A}}(t) \chi_{(1,\infty)}(t) \int_t^\infty f^{**}(s) s^{-1} \,\mathrm{d}s \Big\|_{L^q(0,\infty)} \\ &\geq \Big\| t^{-\frac{1}{q}} \ell^{\alpha_0}(t) \chi_{(0,1)}(t) f^{**}\big(2\sqrt{t}\big) \int_t^{2\sqrt{t}} s^{-1} \,\mathrm{d}s \Big\|_{L^q(0,\infty)} \\ &\quad + \Big\| t^{-\frac{1}{q}} \ell^{\alpha_\infty}(t) \chi_{(1,\infty)}(t) f^{**}\big(2t^2\big) \int_t^{2t^2} s^{-1} \,\mathrm{d}s \Big\|_{L^q(0,\infty)} \\ &\approx \big\| t^{-\frac{1}{q}} \ell^{\alpha_0+1}(t) \chi_{(0,1)}(t) f^{**}\big(2\sqrt{t}\big) \big\|_{L^q(0,\infty)} \\ &\quad + \big\| t^{-\frac{1}{q}} \ell^{\alpha_\infty+1}(t) \chi_{(1,\infty)}(t) f^{**}\big(2t^2\big) \big\|_{L^q(0,\infty)} \\ &\approx \| t^{-\frac{1}{q}} \ell^{\alpha_0+1}(t) \chi_{(0,1)}(t) f^{**}(t) \|_{L^q(0,\infty)} \\ &\quad + \| t^{-\frac{1}{q}} \ell^{\alpha_\infty+1}(t) \chi_{(1,\infty)}(t) f^{**}(t) \|_{L^q(0,\infty)} \\ &\approx \| t^{-\frac{1}{q}} \ell^{\mathbb{A}+1}(t) f^{**}(t) \|_{L^q(0,\infty)} \\ &= \|f\|_{L^{(\infty,q;\mathbb{A}+1)}(0,\infty)} \approx \|f\|_{L^{\infty,q;\mathbb{A}+1}(0,\infty)}\end{aligned} \tag{2.119}$$

for every $f \in \mathfrak{M}(0,\infty)$. ☐

Remark 2.2.49. If $p = \infty$, $q \in [1,\infty)$, $\alpha_0 + 1 + \frac{1}{q} < 0$, $\alpha_\infty + \frac{1}{q} \leq 0$, or $p = q = \infty$, $\alpha_0 + 1 \leq 0$, $\alpha_\infty \leq 0$, then the rearrangement-invariant function space $X(0,\infty)$ whose function norm satisfies

$$\|f\|_{X(0,\infty)} \approx \Big\| t^{-\frac{1}{q}} \ell^{\mathbb{A}}(t) \int_t^\infty f^{**}(s) s^{-1} \,\mathrm{d}s \Big\|_{L^q(0,\infty)} \quad \text{for every } f \in \mathfrak{M}^+(0,\infty)$$

is the optimal domain space for S_σ and $L^{\infty,q;\mathbb{A}}(0,\infty)$, where $\sigma = (1,1,0,0)$, but Proposition 2.2.48 does not provide us with a description of $X(0,\infty)$ in terms of Lorentz–Zygmund spaces in those cases.

Next, we describe optimal target spaces.

Proposition 2.2.50. *Let $\beta \in (0,1)$. Set $\sigma = (1,\beta,1-\beta,0)$. Assume that $L^{p,q;\mathbb{A}}(0,\infty)$ is equivalent to a rearrangement-invariant function space, that is,* *the parameters p, q and $\mathbb{A} = (\alpha_0, \alpha_\infty)$ satisfy one of conditions (1.28) (with $\beta_0 = \beta_\infty = 0$). There is a rearrangement-invariant function space $Z(0,\infty)$ such that the operator $S_\sigma \colon L^{p,q;\mathbb{A}}(0,\infty) \to Z(0,\infty)$ is bounded if and only if one of the following conditions holds:*

- $p < \frac{1}{1-\beta}$;
- $p = \frac{1}{1-\beta}$, $q = 1$ *and* $\alpha_\infty \geq 0$;
- $p = \frac{1}{1-\beta}$, $q \in (1,\infty]$ *and* $\alpha_\infty > 1 - \frac{1}{q}$.

Furthermore, if this is the case and we denote by $Y(0,\infty)$ the optimal target space for the operator S_σ and the space $L^{p,q;\mathbb{A}}(0,\infty)$, then

$$Y(0,\infty) = \begin{cases} L^{\frac{1}{\beta},\infty}(0,\infty), & p = q = 1,\ \alpha_0 = \alpha_\infty = 0; \\ L^{\frac{p}{1-p(1-\beta)},q;\mathbb{A}}(0,\infty), & p \in (1, \frac{1}{1-\beta}); \\ L^{\infty,q;\mathbb{A}-1}(0,\infty), & p = \frac{1}{1-\beta},\ \alpha_0 < 1 - \frac{1}{q},\ \alpha_\infty > 1 - \frac{1}{q}; \\ Y_1(0,\infty), & p = \frac{1}{1-\beta},\ q \in [1,\infty),\ \alpha_0 > 1 - \frac{1}{q} \\ & \text{and } \alpha_\infty > 1 - \frac{1}{q}; \\ L^{\infty,\infty;(0,\alpha_\infty-1)}(0,\infty), & p = \frac{1}{1-\beta},\ q = \infty,\ \alpha_0 > 1,\ \alpha_\infty > 1; \\ L^{\infty,1;(-1,\alpha_\infty-1),(-1,0),(-1,0)}(0,\infty), & p = \frac{1}{1-\beta},\ q = 1,\ \alpha_0 = 0 \\ & \text{and } \alpha_\infty > 0; \\ L^{\infty,q;(-\frac{1}{q},\alpha_\infty-1),(-1,0)}(0,\infty), & p = \frac{1}{1-\beta},\ q \in (1,\infty],\ \alpha_0 = 1 - \frac{1}{q} \\ & \text{and } \alpha_\infty > 1 - \frac{1}{q}; \\ Y_2(0,\infty), & p = \frac{1}{1-\beta},\ q = 1,\ \alpha_0 < 0,\ \alpha_\infty = 0; \\ L^\infty(0,\infty), & p = \frac{1}{1-\beta},\ q = 1,\ \alpha_0 \geq 0,\ \alpha_\infty = 0, \end{cases}$$

where $Y_1(0,\infty)$ and $Y_2(0,\infty)$ are the rearrangement-invariant function spaces whose function norms satisfy

$$\begin{aligned} \|f\|_{Y_1(0,\infty)} &\approx \|t^{-\frac{1}{q}} \ell^{\alpha_\infty - 1}(t) f^*(t) \chi_{(1,\infty)}(t)\|_{L^q(0,\infty)} + \|f\|_{L^\infty(0,\infty)}, \\ \|f\|_{Y_2(0,\infty)} &\approx \|t^{-1} \ell^{\alpha_0 - 1}(t) f^*(t)\|_{L^1(0,1)}, \end{aligned}$$

for every $f \in \mathfrak{M}(0,\infty)$.

Proof. First, assume that $p > 1$. In this case, we obtain the description of $Y(0,\infty)$ from Proposition 2.2.40 with $\gamma = 1 - \beta$ and $\nu = 1$ upon observing that $S_\sigma \colon L^{p,q;\mathbb{A}}(0,\infty) \to Z(0,\infty)$ is bounded if and only if $H_{v,1} \colon L^{p,q;\mathbb{A}}(0,\infty) \to Z(0,\infty)$ is bounded, where v is defined by (2.116). Indeed, that observation follows from

$$\begin{aligned} \sup_{\|f\|_{L^{p,q;\mathbb{A}}(0,\infty)} \leq 1} \|S_\sigma f\|_{Z(0,\infty)} &= \sup_{\|f\|_{L^{p,q;\mathbb{A}}(0,\infty)} \leq 1} \|S_\sigma(f^*)\|_{Z(0,\infty)} \\ &\approx \sup_{\|f\|_{L^{p,q;\mathbb{A}}(0,\infty)} \leq 1} \|H_{v,1}(f^{**})\|_{Z(0,\infty)} \\ &\approx \sup_{\|f\|_{L^{p,q;\mathbb{A}}(0,\infty)} \leq 1} \|H_{v,1}(f^*)\|_{Z(0,\infty)} \\ &\approx \sup_{\|f\|_{L^{p,q;\mathbb{A}}(0,\infty)} \leq 1} \|H_{v,1} f\|_{Z(0,\infty)}, \end{aligned}$$

where we used the Hardy–Littlewood inequality (1.3) in the equality, (2.115) in the first equivalence, (2.117) in the second equivalence and (2.60) in the last equivalence.

Assume now that $p = q = 1$, $\alpha_0 \geq 0$ and $\alpha_\infty \leq 0$. It can be readily verified that the assumptions of Proposition 2.2.21 are satisfied; consequently, owing to (2.115), the rearrangement-invariant function space $Y(0,\infty)$ whose associate function norm satisfies

$$\|f\|_{Y'(0,\infty)} \approx \sup_{t\in(0,\infty)} \ell^{-\mathbb{A}}(t) \int_t^\infty f^{**}(s)s^{-\beta}\,\mathrm{d}s \quad \text{for every } f \in \mathfrak{M}^+(0,\infty) \tag{2.120}$$

is the optimal target space for S_σ and $L^{1,1;\mathbb{A}}(0,\infty)$. If, in addition, $\alpha_0 = \alpha_\infty = 0$, then, thanks to (1.31),

$$\|f\|_{Y'(0,\infty)} \approx \int_0^\infty f^{**}(s)s^{-\beta}\,\mathrm{d}s = \|f\|_{L^{(\frac{1}{1-\beta},1)}(0,\infty)} \approx \|f\|_{L^{\frac{1}{1-\beta},1}(0,\infty)}$$

for every $f \in \mathfrak{M}^+(0,\infty)$, whence $Y(0,\infty) = L^{\frac{1}{\beta},\infty}(0,\infty)$ owing to (1.30). $\square$

Remark 2.2.51. If $p = q = 1$, $\alpha_0 \geq 0$, $\alpha_\infty \leq 0$, $|\alpha_0| + |\alpha_\infty| > 0$, then the rearrangement-invariant function space $Y(0,\infty)$ whose associate function norm satisfies (2.120) is the optimal target space for S_σ and $L^{1,1;\mathbb{A}}(0,\infty)$, where $\sigma = (1,\beta,1-\beta,0)$, $\beta \in (0,1)$, but Proposition 2.2.50 does not provide us with a description of $Y(0,\infty)$ in terms of Lorentz–Zygmund spaces in that case.

The final proposition of this subsection is an analogue of Proposition 2.2.50 in the case where $\beta = 1$.

Proposition 2.2.52. *Set $\sigma = (1,1,0,0)$. Assume that $L^{p,q;\mathbb{A}}(0,\infty)$ is equivalent to a rearrangement-invariant function space, that is, the parameters p, q and $\mathbb{A} = (\alpha_0,\alpha_\infty)$ satisfy one of conditions (1.28) (with $\beta_0 = \beta_\infty = 0$). There is a rearrangement-invariant function space $Z(0,\infty)$ such that the operator $S_\sigma \colon L^{p,q;\mathbb{A}}(0,\infty) \to Z(0,\infty)$ is bounded if and only if one of the following conditions holds:*

- *$p = q = 1$, $\alpha_0 \geq 1$, $\alpha_\infty \leq 0$;*
- *$p \in (1,\infty)$;*
- *$p = \infty$, $q = 1$ and $\alpha_\infty \geq 0$;*
- *$p = \infty$, $q \in (1,\infty]$ and $\alpha_\infty > 1 - \frac{1}{q}$.*

Furthermore, if this is the case and we denote by $Y(0,\infty)$ the optimal target space for the operator S_σ and the space $L^{p,q;\mathbb{A}}(0,\infty)$, then

$$Y(0,\infty) = \begin{cases} L^{1,1;\mathbb{A}-1}(0,\infty), & p = q = 1,\ \alpha_0 \geq 1,\ \alpha_\infty < 0; \\ L^{p,q;\mathbb{A}}(0,\infty), & p \in (1,\infty); \\ L^{\infty,\infty;\mathbb{A}-1}(0,\infty), & p = q = \infty,\ \alpha_0 \leq 0,\ \alpha_\infty > 1. \end{cases}$$

Proof. If $p > 1$, we can proceed in the same way as in the proof of Proposition 2.2.50 (we just make use of Proposition 2.2.41 instead of Proposition 2.2.40).

Assume now that $p = q = 1$, $\alpha_0 \geq 0$ and $\alpha_\infty \leq 0$. Recall that $(L^{1,1;\mathbb{A}})'(0,\infty) = L^{\infty,\infty;-\mathbb{A}}(0,\infty)$ thanks to (1.30). It can be readily verified that the assumptions of Proposition 2.2.21 are satisfied if and only if $\alpha_0 \geq 1$; consequently, if $p = q = 1$, $\alpha_0 \geq 1$ and $\alpha_\infty \leq 0$, owing to (2.115), the rearrangement-invariant function space $Y(0,\infty)$ whose associate function norm satisfies

$$\|f\|_{Y'(0,\infty)} \approx \sup_{t\in(0,\infty)} \ell^{-\mathbb{A}}(t) \int_t^\infty f^{**}(s) s^{-1} \, \mathrm{d}s \quad \text{for every } f \in \mathfrak{M}^+(0,\infty)$$

is the optimal target space for S_σ and $L^{1,1;\mathbb{A}}(0,\infty)$. If $\alpha_\infty < 0$, by repeating the same argument as in (2.118) and (2.119), we obtain that

$$\|f\|_{Y'(0,\infty)} \approx \|f\|_{L^{(\infty,\infty;-\mathbb{A}+1)}(0,\infty)} \approx \|f\|_{L^{\infty,\infty;-\mathbb{A}+1}(0,\infty)}.$$

Hence $Y'(0,\infty) = L^{\infty,\infty;-\mathbb{A}+1}(0,\infty)$, whence $Y(0,\infty) = L^{1,1;\mathbb{A}-1}(0,\infty)$ owing to (1.30).

If $\alpha_\infty = 0$, then

$$\begin{aligned}
\|f\|_{Y'(0,\infty)} &\approx \sup_{t\in(0,1)} \ell^{-\alpha_0}(t) \int_t^\infty f^{**}(s) \frac{\mathrm{d}s}{s} + \int_1^\infty f^{**}(s) \frac{\mathrm{d}s}{s} \\
&\approx \sup_{t\in(0,1)} \ell^{-\alpha_0}(t) \int_t^1 f^{**}(s) \frac{\mathrm{d}s}{s} + \int_1^\infty f^{**}(s) \frac{\mathrm{d}s}{s} \\
&\approx \sup_{t\in(0,1)} \ell^{-\alpha_0+1}(t) f^{**}(t) + \int_1^\infty f^{**}(s) \frac{\mathrm{d}s}{s} \\
&\approx \sup_{t\in(0,1)} \ell^{-\alpha_0+1}(t) f^{*}(t) + \int_1^\infty f^{**}(s) \frac{\mathrm{d}s}{s},
\end{aligned}$$

where we again used the same argument as in (2.118) and (2.119) and the Hardy inequality [69, Theorem 1]. ☐

Remark 2.2.53. Set $\sigma = (1,1,0,0)$. If $p = q = 1$, $\alpha_0 \geq 1$, $\alpha_\infty = 0$, then the rearrangement-invariant function space $Y(0,\infty)$ whose associate function norm satisfies

$$\|f\|_{Y'(0,\infty)} \approx \sup_{t\in(0,1)} \ell^{-\alpha_0+1}(t) f^{*}(t) + \int_1^\infty f^{**}(s) \frac{\mathrm{d}s}{s} \quad \text{for every } f \in \mathfrak{M}^+(0,\infty)$$

is the optimal target space for S_σ and $L^{1,1;\mathbb{A}}(0,\infty)$. If $p = \infty$, $q = 1$, $\alpha_0 + 1 < 0$, $\alpha_\infty \geq 0$, or $p = \infty$, $q \in (1,\infty)$, $\alpha_0 + \frac{1}{q} < 0$, $\alpha_\infty > 1 - \frac{1}{q}$, then the rearrangement-invariant function space $Y(0,\infty)$ whose associate function norm satisfies

$$\|f\|_{Y'(0,\infty)} \approx \left\| t^{-\frac{1}{q'}} \ell^{-\mathbb{A}-1}(t) \int_0^t f^{**}(s) \, \mathrm{d}s \right\|_{L^{q'}(0,\infty)} \quad \text{for every } f \in \mathfrak{M}^+(0,\infty)$$

is the optimal target space for S_σ and $L^{\infty,q;\mathbb{A}}(0,\infty)$. However, Proposition 2.2.52 does not provide us with a description of $Y(0,\infty)$ in terms of Lorentz–Zygmund spaces in those cases.

2.3 Compactness of Hardy-type operators

In this section, we investigate compactness of the operator $H_{u,v,\nu}$ between two rearrangement-invariant function spaces. The results of our effort will come handy when we study the compactness of certain Sobolev-type embeddings in Chapter 5. Throughout this section, we assume that $L = 1$.

Since we have (1.7), (1.9) and (2.3) at our disposal, it is an easy exercise to verify that $H_{u,v,\nu}\colon X(0,1) \to Y(0,1)$ is compact if and only if $R_{u,v,\nu}\colon Y'(0,1) \to X'(0,1)$ is compact, for the standard proof of the Schauder theorem, which states that a linear operator between two Banach spaces is compact if and only if its dual operator (in the classical sense) is compact, can be easily modified to serve the purpose. To this end, it should be noted that the fact that the operators $H_{u,v,\nu}$ and $R_{u,v,\nu}$ are merely sublinear does not cause any trouble (see the proof of Proposition 2.3.2). Therefore, the results that we will obtain for the operator $H_{u,v,\nu}$ could be easily translated into similar results for the operator $R_{u,v,\nu}$. However, we opt not to state them, because we shall not need them at all.

2.3.1 Compactness criteria for $H_{u,v,\nu}$

The following proposition provides a necessary condition for the compactness of $H_{u,v,\nu}$ between two rearrangement-invariant function spaces. The proof proceeds along the same lines as that of Slavíková's result [85, Lemma 4.10], which covers the case where $u \equiv 1$ and $\nu = 1$, but it translates verbatim to our setting.

Proposition 2.3.1. *Let* $\|\cdot\|_{X(0,1)}$ *and* $\|\cdot\|_{Y(0,1)}$ *be rearrangement-invariant function norms and* $\nu \in (0,\infty)$. *Let* $u, v\colon (0,1) \to (0,\infty)$ *be measurable. If* $H_{u,v,\nu}\colon X(0,1) \to Y(0,1)$ *is compact, then*

$$\lim_{r\to 0^+} \sup_{\|f\|_{X(0,1)}\le 1} \|H_{u,v,\nu}(f\chi_{(0,r)})\|_{Y(0,1)} = 0.$$

Proof. First, note that the function $(0,1) \ni r \mapsto \sup_{\|f\|_{X(0,1)}\le 1} \|H_{u,v,\nu}(f\chi_{(0,r)})\|_{Y(0,1)}$ is nonincreasing, and so it is sufficient to show that

$$\lim_{k\to\infty} \sup_{\|f\|_{X(0,1)}\le 1} \|H_{u,v,\nu}(f\chi_{(0,r_k)})\|_{Y(0,1)} = 0 \tag{2.121}$$

for some sequence $\{r_k\}_{k=1}^{\infty} \subseteq (0,1)$ such that $r_k \to 0$ as $k \to \infty$. Since $H_{u,v,\nu}\colon X(0,1) \to Y(0,1)$ is compact, it is also bounded; consequently, for every $k \in \mathbb{N}$, there is a function $f_k \in X(0,1)$, $\|f_k\|_{X(0,1)} \le 1$, such that

$$\sup_{\|f\|_{X(0,1)}\le 1} \|H_{u,v,\nu}(f\chi_{(0,\frac{1}{k})})\|_{Y(0,1)} < \|H_{u,v,\nu}(f_k\chi_{(0,\frac{1}{k})})\|_{Y(0,1)} + \frac{1}{k}. \tag{2.122}$$

Furthermore, owing to the fact that the sequence $\{f_k\chi_{(0,\frac{1}{k})}\}_{k=1}^{\infty}$ is bounded in $X(0,1)$ and $H_{u,v,\nu}\colon X(0,1) \to Y(0,1)$ is compact, there is a subsequence of $\{H_{u,v,\nu}(f_k\chi_{(0,\frac{1}{k})})\}_{k=1}^{\infty}$ converging to a function $h \in Y(0,1)$ in $Y(0,1)$. We may assume that $\lim_{k\to\infty} H_{u,v,\nu}(f_k\chi_{(0,\frac{1}{k})}) = h$ in $Y(0,1)$. In particular,

$$\lim_{k\to\infty} \|H_{u,v,\nu}(f_k\chi_{(0,\frac{1}{k})})\|_{Y(0,1)} = \|h\|_{Y(0,1)}. \tag{2.123}$$

Moreover, we may also assume that $\{H_{u,v,\nu}(f_k\chi_{(0,\frac{1}{k})})\}_{k=1}^{\infty}$ converges pointwise to h a.e. on $(0,1)$ (see (1.12)).

Second, observe that $h=0$ a.e. on $(0,1)$, Indeed, for every $k\in\mathbb{N}$, we plainly have that $H_{u,v,\nu}(f_k\chi_{(0,\frac{1}{k})})=0$ on $[\frac{1}{k^\nu},1)$, and so $h=0$ a.e. on $(0,1)$ thanks to the pointwise convergence a.e. Consequetly, it follows from (2.123) that

$$\lim_{k\to\infty}\|H_{u,v,\nu}(f_k\chi_{(0,\frac{1}{k})})\|_{Y(0,1)}=0. \tag{2.124}$$

Finally, by combining (2.122) and (2.124), we obtain (2.121). ☐

The next proposition characterizes the compactness of $H_{u,v,\nu}$ from $X(0,1)$ to $Y(0,1)$ provided that u has absolutely continuous norm in $Y(0,1)$.

Proposition 2.3.2. *Let $\|\cdot\|_{X(0,1)}$ and $\|\cdot\|_{Y(0,1)}$ be rearrangement-invariant function norms and $\nu\in(0,\infty)$. Let $u,v\colon(0,1)\to(0,\infty)$ be nonincreasing. Assume that u has absolutely continuous norm in $Y(0,1)$ and $0<U(t)<\infty$ for every $t\in(0,1)$. Furthermore, assume that $u(1^-)>0$. The following six statements are equivalent:*

(i) the operator $H_{u,v,\nu}\colon X(0,1)\to Y(0,1)$ is compact;

(ii) $\displaystyle\lim_{r\to0^+}\sup_{\|f\|_{X(0,1)}\le1}\|H_{u,v,\nu}(f\chi_{(0,r)})\|_{Y(0,1)}=0$*;*

(iii) $\displaystyle\lim_{r\to0^+}\sup_{\|f\|_{X(0,1)}\le1}\|H_{u,v,\nu}(f^*\chi_{(0,r)})\|_{Y(0,1)}=0$*;*

(iv) $\displaystyle\lim_{r\to0^+}\sup_{\|f\|_{X(0,1)}\le1}\|\chi_{(0,r)}H_{u,v,\nu}f\|_{Y(0,1)}=0$*;*

(v) $\displaystyle\lim_{r\to0^+}\sup_{\|f\|_{X(0,1)}\le1}\|\chi_{(0,r)}H_{u,v,\nu}(f^*)\|_{Y(0,1)}=0$*;*

(vi) we have that $Y_X(0,1)\overset{*}{\hookrightarrow}Y(0,1)$, where $Y_X(0,1)$ is the optimal target space for the operator $H_{u,v,\nu}$ and the space $X(0,1)$, whose associate function norm satisfies

$$\|f\|_{Y_X'(0,1)}=\|R_{u,v,\nu}(f^*)\|_{X'(0,1)}\quad\text{for every } f\in\mathfrak{M}(0,1).$$

Proof. (*i*) *implies* (*ii*)*.* This follows immediately from Proposition 2.3.1.

(*ii*) *is equivalent to* (*iii*)*.* This follows from (2.60).

(*ii*) *implies* (*iv*)*.* Thanks to the assumption that u has absolutely continuous norm in $Y(0,1)$ we have that

$$\lim_{r\to0^+}\|u\chi_{(0,r)}\|_{Y(0,1)}=0. \tag{2.125}$$

Fix $f\in X(0,1)$ and $r\in(0,1)$. We have that

$$\begin{aligned}\chi_{(0,r)}(t)H_{u,v,\nu}f(t)&\le H_{u,v,\nu}(f\chi_{(0,a)})(t)+\chi_{(0,r)}(t)H_{u,v,\nu}(f\chi_{(a,1)})(t)\\&\le H_{u,v,\nu}(f\chi_{(0,a)})(t)+v(a)\|f\|_{X(0,1)}\varphi_{X'(0,1)}(1)u(t)\chi_{(0,r)}(t)\end{aligned}$$

for every $t\in(0,1)$ and every $a\in(0,1)$, where we used the monotonicity of v and the Hölder inequality (1.14). Consequently,

$$\begin{aligned}\|\chi_{(0,r)}H_{u,v,\nu}f\|_{Y(0,1)}&\le\|H_{u,v,\nu}(f\chi_{(0,a)})\|_{Y(0,1)}\\&\quad+v(a)\|f\|_{X(0,1)}\varphi_{X'(0,1)}(1)\|u\chi_{(0,r)}\|_{Y(0,1)}\end{aligned} \tag{2.126}$$

for every $a \in (0,1)$. By taking the supremum over all $f \in X(0,1)$, $\|f\|_{X(0,1)} \leq 1$, in (2.126) and letting $r \to 0^+$, we obtain that
(2.127)

$$\begin{aligned}\lim_{r\to 0^+} \sup_{\|f\|_{X(0,1)}\leq 1} \|\chi_{(0,r)} H_{u,v,\nu} f\|_{Y(0,1)} &\leq \sup_{\|f\|_{X(0,1)}\leq 1} \|H_{u,v,\nu}(f\chi_{(0,a)})\|_{Y(0,1)} \\ &\quad + v(a)\varphi_{X'(0,1)}(1) \lim_{r\to 0^+} \|u\chi_{(0,r)}\|_{Y(0,1)} \\ &= \sup_{\|f\|_{X(0,1)}\leq 1} \|H_{u,v,\nu}(f\chi_{(0,a)})\|_{Y(0,1)}\end{aligned}$$

for every $a \in (0,1)$, where the equality is true thanks to (2.125). Inequality (2.127) together with (ii) plainly implies (iv).

(iv) is equivalent to (v). This follows from (2.60).

(iv) is equivalent to (vi). Note that $\chi_{(0,r)} H_{u,v,\nu} f$ is a nonincreasing function for every $f \in X(0,1)$ and $r \in (0,1)$. By (1.10), (2.3), and (1.7), we have that

$$\sup_{\|f\|_{X(0,1)}\leq 1} \|\chi_{(0,r)} H_{u,v,\nu} f\|_{Y(0,1)} = \sup_{\|g\|_{Y'(0,1)}\leq 1} \|R_{u,v,\nu}(g^*\chi_{(0,r)})\|_{X'(0,1)} \tag{2.128}$$

for every $r \in (0,1)$. It is easy to observe that (2.128) together with (iv) implies that $R_{u,v,\nu}\chi_{(0,1)} \in X'(0,1)$. Consequently, thanks to Proposition 2.2.5, $Y_X(0,1)$ is indeed the optimal target space for the operator $H_{u,v,\nu}$ and the space $X(0,1)$. Since

$$\sup_{\|g\|_{Y'(0,1)}\leq 1} \|R_{u,v,\nu}(g^*\chi_{(0,r)})\|_{X'(0,1)} = \sup_{\|g\|_{Y'(0,1)}\leq 1} \|g^*\chi_{(0,r)}\|_{Y'_X(0,1)},$$

it follows from (2.128) that (iv) is equivalent $Y'(0,1) \overset{*}{\hookrightarrow} Y'_X(0,1)$, which is equivalent to (vi) by (1.23).

(iv) implies (i). We define an auxiliary linear operator $\widetilde{H}$ as

$$\widetilde{H}f(t) = u(t)\int_{t^{\frac{1}{\nu}}}^1 f(s)v(s)\,\mathrm{d}s,\ t \in (0,1),\ f \in X(0,1). \tag{2.129}$$

Clearly, $\widetilde{H}|f| = H_{u,v,\nu} f$ for every $f \in X(0,1)$. Using the equivalence of (iv) and (vi), we know that $H_{u,v,\nu}\colon X(0,1) \to Y_X(0,1)$ is bounded. In particular, it follows that $\widetilde{H}f(t)$ is well defined for every $t \in (0,1)$ and every $f \in X(0,1)$, and so $\widetilde{H}$ is a well-defined linear operator on $X(0,1)$. It is easy to see that (i) will follow once we prove that $\widetilde{H}\colon X(0,1) \to Y(0,1)$ is compact. In order to prove that, we define, for each $r \in (0,1)$, the linear operator $\widetilde{H}_r$ as

$$\widetilde{H}_r f = \chi_{[r,1)}\widetilde{H}f,\ f \in X(0,1).$$

Note that (iv) implies that

$$\lim_{r\to 0^+} \|\widetilde{H} - \widetilde{H}_r\|_{X(0,1)\to Y(0,1)} = 0, \tag{2.130}$$

where $\|\cdot\|_{X(0,1)\to Y(0,1)}$ stands for the standard operator norm on the Banach space of all bounded linear operators from $X(0,1)$ to $Y(0,1)$. Owing to (2.130), it is sufficient to show that each operator $\widetilde{H}_r$, $r \in (0,1)$, is a compact linear operator from $X(0,1)$ to $Y(0,1)$. The linearity is obvious. We shall show the compactness by proving that, for every $r \in (0,1)$, $\widetilde{H}_r$ is the limit (in the operator norm) of a sequence of (bounded linear) finite-rank operators from $X(0,1)$ to $Y(0,1)$. Fix $r \in (0,1)$. Let $\varepsilon > 0$. Since u has absolutely continuous norm on $Y(0,1)$ and u is nonincreasing, we have that $\lim\limits_{a\to 0^+} \varphi_{Y(0,1)}(a) = 0$ because $\lim\limits_{a\to 0^+} \varphi_{Y(0,1)}(a) \leq \frac{1}{u(1^-)} \lim\limits_{a\to 0^+} \|u\chi_{(0,a)}\|_{Y(0,1)} = 0$. Hence we can find a partition $\{a_k \colon k = 0,1,\dots,m\}$, $m \in \mathbb{N}$, of the interval $[r,1]$, $r = a_0 < a_1 < \dots < a_m = 1$, such that

$$\|\chi_{(r_{k-1},r_k)}\|_{Y(0,1)} \leq \frac{\varepsilon}{u(r)v(r^{\frac{1}{\nu}})\varphi_{X'(0,1)}(1-r^{\frac{1}{\nu}})} \tag{2.131}$$

for every $k \in \{1,\dots,m\}$. Set

$$S_\varepsilon f = \sum_{k=1}^m \widetilde{H}f(a_k)\chi_{(a_{k-1},a_k)},\ f \in X(0,1).$$

Clearly, S_ε is linear and of rank at most m. Furthermore, thanks to the Hölder inequality (1.14) and the monotonicity of v, we have that

$$\widetilde{H}|f|(r) \leq u(r)v(r^{\frac{1}{\nu}})\|f\|_{X(0,1)}\varphi_{X'(0,1)}(1-r^{\frac{1}{\nu}}) \tag{2.132}$$

for every $f \in X(0,1)$, and so $S_\varepsilon \colon X(0,1) \to Y(0,1)$ is bounded because

$$\begin{aligned}\|S_\varepsilon f\|_{Y(0,1)} &\leq \sum_{k=1}^m \widetilde{H}|f|(a_k)\varphi_Y(a_k - a_{k-1}) \leq m\widetilde{H}|f|(r)\varphi_{Y(0,1)}(1)\\ &\leq mu(r)v(r^{\frac{1}{\nu}})\varphi_{X'(0,1)}(1-r^{\frac{1}{\nu}})\varphi_{Y(0,1)}(1)\|f\|_{X(0,1)}\end{aligned}$$

for every $f \in X(0,1)$. Hence S_ε is a (bounded linear) finite-rank operator from $X(0,1)$ to $Y(0,1)$. Finally, we have that

$$\begin{aligned}\|\widetilde{H}_r f - S_\varepsilon f\|_{Y(0,1)} &\leq \sum_{k=1}^m \|(\widetilde{H}f(t) - \widetilde{H}f(a_k))\chi_{(a_{k-1},a_k)}(t)\|_{Y(0,1)}\\ &\leq \sum_{k=1}^m \|(\widetilde{H}(f^+)(t) - \widetilde{H}(f^+)(a_k))\chi_{(a_{k-1},a_k)}(t)\|_{Y(0,1)}\\ &\quad + \sum_{k=1}^m \|(\widetilde{H}(f^-)(t) - \widetilde{H}(f^-)(a_k))\chi_{(a_{k-1},a_k)}(t)\|_{Y(0,1)}\\ &\leq \sum_{k=1}^m (\widetilde{H}(f^+)(a_{k-1}) - \widetilde{H}(f^+)(a_k))\|\chi_{(a_{k-1},a_k)}\|_{Y(0,1)}\\ &\quad + \sum_{k=1}^m (\widetilde{H}(f^-)(a_{k-1}) - \widetilde{H}(f^-)(a_k))\|\chi_{(a_{k-1},a_k)}\|_{Y(0,1)}\\ &\leq \frac{\varepsilon}{u(r)v(r^{\frac{1}{\nu}})\varphi_{X'(0,1)}(1-r^{\frac{1}{\nu}})}(\widetilde{H}(f^+)(r) + \widetilde{H}(f^-)(r))\\ &= \frac{\varepsilon}{u(r)v(r^{\frac{1}{\nu}})\varphi_{X'(0,1)}(1-r^{\frac{1}{\nu}})}\widetilde{H}|f|(r)\\ &\leq \varepsilon\|f\|_{X(0,1)}\end{aligned}$$

for every $f \in X(0,1)$, where we used the fact that $\widetilde{H}|f|$ is nonincreasing for every $f \in X(0,1)$, (2.131) and (2.132). Consequently, $\lim\limits_{\varepsilon\to 0^+} \|\widetilde{H}_r - S_\varepsilon\|_{X(0,1)\to Y(0,1)} = 0$. Hence $\widetilde{H}_r \colon X(0,1) \to Y(0,1)$ is compact. □

Remark 2.3.3. Assume that $u \equiv 1$. The assumptions $0 < U(t) < \infty$ for every $t \in (0,1)$ and $u(1^-) > 0$ are plainly satisfied. The assumption that u has absolutely continuous norm in $Y(0,1)$ is equivalent to the fact that $\lim_{t\to 0^+} \varphi_{Y(0,1)}(t) = 0$. By (1.20), the latter is equivalent to the fact that $\|\cdot\|_{Y(0,1)}$ is not equivalent to $\|\cdot\|_{L^\infty(0,1)}$. Therefore, Proposition 2.3.2 characterizes the compactness of $H_{v,\nu}$ from an arbitrary rearrangement-invariant function space $X(0,1)$ into rearrangement-invariant function spaces different from $L^\infty(0,1)$.

When $u \equiv 1$, we have the following criteria for the compactness of $H_{v,\nu}$ to $L^\infty(0,1)$. Note that the value of ν is completely immaterial.

Proposition 2.3.4. *Let $\|\cdot\|_{X(0,1)}$ be a rearrangement-invariant function norm and $\nu \in (0,\infty)$. Let $v\colon (0,1) \to (0,\infty)$ be a nonincreasing, integrable function. The following five statements are equivalent:*

(i) the operator $H_{v,\nu}\colon X(0,1) \to L^\infty(0,1)$ is compact;

(ii) $\lim_{r\to 0^+} \sup_{\|f\|_{X(0,1)}\le 1} \int_0^r |f(s)|v(s)\,\mathrm{d}s = 0$*;*

(iii) $\lim_{r\to 0^+} \sup_{\|f\|_{X(0,1)}\le 1} \int_0^r f^*(s)v(s)\,\mathrm{d}s = 0$*;*

(iv) $\lim_{r\to 0^+} \|v\chi_{(0,r)}\|_{X'(0,1)} = 0$*, that is, v has absolutely continuous norm in $X'(0,1)$;*

(v) $X(0,1) \overset{*}{\hookrightarrow} \Lambda_V(0,1)$*, where $V(t) = \int_0^t v(s)\,\mathrm{d}s$, $t \in [0,1]$.*

Proof. (i) *implies* (ii). This follows immediately from Proposition 2.3.1.

(ii) *is equivalent to* (iii). Unlike the equivalence of (ii) and (iii) in Proposition 2.3.2, where the integration is carried out over an interval "far from 0", this equivalence is obvious because now the integration is carried out over a right-neighborhood of 0. Indeed, this equivalence follows immediately from the Hardy–Littlewood inequality (1.3) together with the fact that v is nonincreasing.

(ii) *is equivalent to* (iv). We plainly have that, for every $r \in (0,1)$,

$$\sup_{\|f\|_{X(0,1)}\le 1} \int_0^r |f(s)|v(s)\,\mathrm{d}s = \sup_{\|f\|_{X(0,1)}\le 1} \int_0^1 |f(s)|v(s)\chi_{(0,r)}(s)\,\mathrm{d}s = \|v\chi_{(0,r)}\|_{X'(0,1)}$$

thanks to (1.7).

(ii) *implies* (i). For every $a \in (0,1)$, set

$$\widetilde{H}_a f = \widetilde{H}(f\chi_{[a,1)}),\ f \in X(0,1),$$

where the operator $\widetilde{H}$ is defined by (2.129) with $u \equiv 1$. By an argument similar to that at the beginning of the proof of (iv) implies (i) in Proposition 2.3.2, it is sufficient to prove that each linear operator $\widetilde{H}_a$, $a \in (0,1)$, is compact from $X(0,1)$ to $L^\infty(0,1)$. Fix $a \in (0,1)$. We claim that the set $\{\widetilde{H}_a f\colon \|f\|_{X(0,1)} \le 1\}$ is uniformly equicontinuous on $[0,1]$. Let $\varepsilon > 0$ be given. Thanks to (iv), which (as already proved) is equivalent to (ii), there is $\delta > 0$ such that $\|v\chi_{(0,r)}\|_{X'(0,1)} < \varepsilon$ for every $r \in (0,\delta)$. Set

$$\widetilde{\delta} = \min\Big\{\frac{a^\nu}{2}, \nu \min\Big\{\Big(\frac{a}{2^{\frac{1}{\nu}}}\Big)^{\nu-1}, 1\Big\}\delta\Big\}.$$

Let $t_1, t_2 \in [0,1]$, $t_1 < t_2$, be such that $t_2 - t_1 < \widetilde{\delta}$. We claim that

$$\sup_{\|f\|_{X(0,1)} \le 1} |\widetilde{H}_a f(t_2) - \widetilde{H}_a f(t_1)| < \varepsilon.$$

If $t_1 \le \frac{a^\nu}{2}$, then we have that

$$\sup_{\|f\|_{X(0,1)} \le 1} |\widetilde{H}_a f(t_2) - \widetilde{H}_a f(t_1)| \le \sup_{\|f\|_{X(0,1)} \le 1} \int_{t_1^{\frac{1}{\nu}}}^{t_2^{\frac{1}{\nu}}} |f(s)| \chi_{[a,1)}(s) v(s) \, \mathrm{d}s = 0$$

inasmuch as $t_2^{\frac{1}{\nu}} < a$. Assume now that $t_1 > \frac{a^\nu}{2}$. Observe that

$$t_2^{\frac{1}{\nu}} - t_1^{\frac{1}{\nu}} \le \frac{1}{\nu} \max\left\{ t_1^{\frac{1}{\nu}-1}, 1 \right\} (t_2 - t_1) \le \frac{1}{\nu} \max\left\{ \Big(\frac{a}{2^{\frac{1}{\nu}}} \Big)^{1-\nu}, 1 \right\} (t_2 - t_1) < \delta,$$

whence we have that, by the Hölder inequality (1.14),

$$\begin{aligned} \sup_{\|f\|_{X(0,1)} \le 1} |\widetilde{H}_a f(t_2) - \widetilde{H}_a f(t_1)| &\le \sup_{\|f\|_{X(0,1)} \le 1} \int_{t_1^{\frac{1}{\nu}}}^{t_2^{\frac{1}{\nu}}} |f(s)| v(s) \, \mathrm{d}s \\ &\le \|v \chi_{(t_1^{\frac{1}{\nu}}, t_2^{\frac{1}{\nu}})}\|_{X'(0,1)} \le \|v \chi_{(0, t_2^{\frac{1}{\nu}} - t_1^{\frac{1}{\nu}})}\|_{X'(0,1)} < \varepsilon \end{aligned}$$

inasmuch as $t_2^{\frac{1}{\nu}} - t_1^{\frac{1}{\nu}} < \delta$, where we used the monotonicity of v in the last but one inequality. Hence $\{\widetilde{H}_a f \colon \|f\|_{X(0,1)} \le 1\}$ is uniformly equicontinuous on $[0,1]$. Furthermore, the set is bounded in $C([0,1])$ because

$$\sup_{\|f\|_{X(0,1)} \le 1} \sup_{t \in [0,1]} |\widetilde{H}_a f(t)| \le v(a) \varphi_{X'(0,1)}(1)$$

thanks to the monotonicity of v and the Hölder inequality (1.14). Therefore, by virtue of the Arzelà–Ascoli theorem, the set $\{\widetilde{H}_a f \colon \|f\|_{X(0,1)} \le 1\}$ is precompact in $C([0,1])$, and so it is also precompact in $L^\infty(0,1)$. Hence $\widetilde{H}_a \colon X(0,1) \to L^\infty(0,1)$ is compact.

(*iii*) *is equivalent to* (*v*). Note that V is a concave function on $[0,1]$ such that $V(0^+) = 0$ and $V'(t) = v(t)$ for a.e. $t \in (0,1)$. We have that

$$\begin{aligned} \sup_{\|f\|_{X(0,1)} \le 1} \int_0^r f^*(s) v(s) \, \mathrm{d}s &= \sup_{\|f\|_{X(0,1)} \le 1} \int_0^1 f^*(s) \chi_{(0,r)}(s) v(s) \, \mathrm{d}s \\ &= \sup_{\|f\|_{X(0,1)} \le 1} \int_{[0,1)} f^*(s) \chi_{(0,r)}(s) \, \mathrm{d}V(s) \\ &= \sup_{\|f\|_{X(0,1)} \le 1} \|f^* \chi_{(0,r)}\|_{\Lambda_V(0,1)}, \end{aligned}$$

whence the equivalence follows. ☐

Remark 2.3.5. Should $H_{v,\nu}$ be bounded (compact) from a rearrangement-invariant function space to $L^\infty(0,1)$, v has to be integrable because we have that $\|H_{v,\nu} \chi_{(0,1)}\|_{L^\infty(0,1)} = \|v\|_{L^1(0,1)}$. Hence the assumption in Proposition 2.3.4 that v is integrable is not restrictive at all.

2.3.1.1 Particular examples

In this short subsection, we shall address the question of when the operator $H_{v,\nu}$ with $v(t) = t^{-1+\gamma}$, $t \in (0,1)$, $\gamma \in (0,1)$, is compact between two Lorentz–Zygmund spaces. We will make use of the following characterization of almost-compact embeddings between two Lorentz–Zygmund spaces. Assume that $p_1, q_1, p_2, q_2 \in [1,\infty]$, $\alpha_1, \beta_1, \alpha_2, \beta_2 \in \mathbb{R}$ are such that $L^{p_1,q_1;\alpha_1,\beta_1}(0,1)$ and $L^{p_2,q_2;\alpha_2,\beta_2}(0,1)$ are equivalent to rearrangement-invariant function spaces, that is, the parameters satisfy one of conditions (1.29). We have that ([85, Proposition 7.12] and [75, Theorem 4.5], cf. [58, Theorem 3.12])

$$L^{p_1,q_1;\alpha_1,\beta_1}(0,1) \overset{*}{\hookrightarrow} L^{p_2,q_2;\alpha_2,\beta_2}(0,1)$$

if and only if one of the following conditions is true:
(2.133)

$$\begin{cases} p_1 > p_2; \\ p_1 = p_2 < \infty,\ \alpha_2 < \alpha_1 + \min\left\{\frac{1}{q_1} - \frac{1}{q_2}, 0\right\}; \\ p_1 = p_2 < \infty,\ \alpha_2 = \alpha_1 + \min\left\{\frac{1}{q_1} - \frac{1}{q_2}, 0\right\},\ \beta_2 < \beta_1 + \min\left\{\frac{1}{q_1} - \frac{1}{q_2}, 0\right\}; \\ p_1 = p_2 = \infty,\ \alpha_2 + \frac{1}{q_2} < \alpha_1 + \frac{1}{q_1}; \\ p_1 = p_2 = \infty,\ q_1 \le q_2,\ \alpha_1 + \frac{1}{q_1} = \alpha_2 + \frac{1}{q_2} = 0,\ \beta_2 + \frac{1}{q_2} < \beta_1 + \frac{1}{q_1}; \\ p_1 = p_2 = \infty,\ q_1 \le q_2,\ \alpha_1 + \frac{1}{q_1} = \alpha_2 + \frac{1}{q_2} < 0,\ \beta_2 < \beta_1; \\ p_1 = p_2 = \infty,\ q_1 > q_2,\ \alpha_1 + \frac{1}{q_1} = \alpha_2 + \frac{1}{q_2},\ \beta_2 + \frac{1}{q_2} < \beta_1 + \frac{1}{q_1}. \end{cases}$$

In light of Section 2.3.1, it is not surprising that (2.133) will come in useful while we characterize the compactness of $H_{v,\nu}$ between two Lorentz–Zygmund spaces.

Proposition 2.3.6. *Let $\gamma \in (0,1)$ and $\nu \in (0,\infty)$. Assume that $L^{p_1,q_1;\alpha_1}(0,1)$ and $L^{p_2,q_2;\alpha_2,\beta}(0,1)$ are equivalent to rearrangement-invariant function spaces, that is, the parameters satisfy one of conditions (1.29). The operator $H_{v,\nu}\colon L^{p_1,q_1;\alpha_1}(0,1) \to L^{p_2,q_2;\alpha_2,\beta}(0,1)$ is compact if and only if one of the following conditions holds:*

- $p_1 = q_1 = 1$, $\alpha_1 \ge 0$, $\gamma + \nu > 1$ *and*
    - ⋆ $p_2 < \frac{\nu}{1-\gamma}$ *or*
    - ⋆ $p_2 = \frac{\nu}{1-\gamma}$, $\alpha_2 < \alpha_1$ *or*
    - ⋆ $p_2 = \frac{\nu}{1-\gamma}$, $\alpha_2 = \alpha_1$, $\beta < 0$*;*
- $p_1 = \max\{1, \frac{1}{\gamma+\nu}\}$, $q_1 = 1$, $\alpha_1 > 0$, $\gamma + \nu \le 1$ *and*
    - ⋆ $p_2 = q_2 = 1$, $0 < \alpha_2 < \alpha_1$ *or*
    - ⋆ $p_2 = q_2 = 1$, $0 = \alpha_2 < \alpha_1$, $\beta \ge 0$ *or*
    - ⋆ $p_2 = q_2 = 1$, $0 < \alpha_2 = \alpha_1$, $\beta < 0$*;*
- $p_1 = \frac{1}{\gamma+\nu}$, $q \in (1,\infty]$, $\alpha_1 > 1 - \frac{1}{q_1}$, $\gamma + \nu < 1$ *and*
    - ⋆ $p_2 = q_2 = 1$, $0 < \alpha_2 < \alpha_1 - 1 + \frac{1}{q_1}$ *or*
    - ⋆ $p_2 = q_2 = 1$, $\alpha_2 = 0$ *and* $\beta \ge 0$ *or*

  - ⋆ $p_2 = q_2 = 1$, $\alpha_2 = \alpha_1 - 1 + \frac{1}{q_1}$ *and* $\beta < -1 + \frac{1}{q_1}$*;*

- $p \in (\max\{1, \frac{1}{\gamma+\nu}\}, \frac{1}{\gamma})$ *and*
  - ⋆ $p_2 < \frac{p_1\nu}{1-p_1\gamma}$ *or*
  - ⋆ $p_2 = \frac{p_1\nu}{1-p_1\gamma}$, $\alpha_2 < \alpha_1 + \min\{\frac{1}{q_1} - \frac{1}{q_2}, 0\}$ *or*
  - ⋆ $p_2 = \frac{p_1\nu}{1-p_1\gamma}$, $\alpha_2 = \alpha_1 + \min\{\frac{1}{q_1} - \frac{1}{q_2}, 0\}$, $\beta < \min\{\frac{1}{q_1} - \frac{1}{q_2}, 0\}$*;*

- $p_1 = \frac{1}{\gamma}$, $\alpha_1 < 1 - \frac{1}{q_1}$ *and*
  - ⋆ $p_2 < \infty$ *or*
  - ⋆ $p_2 = \infty$, $\alpha_2 + \frac{1}{q_2} < \alpha_1 + \frac{1}{q_1} - 1$ *or*
  - ⋆ $p_2 = \infty$, $\alpha_2 + \frac{1}{q_2} = \alpha_1 + \frac{1}{q_1} - 1$, $\beta < \min\{\frac{1}{q_1} - \frac{1}{q_2}, 0\}$*;*

- $p_1 = \frac{1}{\gamma}$, $\alpha_1 = 1 - \frac{1}{q_1}$ *and*
  - ⋆ $p_2 < \infty$ *or*
  - ⋆ $p_2 = \infty$, $\alpha_2 + \frac{1}{q_2} < 0$ *or*
  - ⋆ $p_2 = \infty$, $\alpha_2 + \frac{1}{q_2} = 0$, $\beta + \frac{1}{q_2} < -1 + \frac{1}{q_1}$*;*

- *either* $p_1 = \frac{1}{\gamma}$, $\alpha_1 > 1 - \frac{1}{q_1}$ *or* $p > \frac{1}{\gamma}$*.*

Proof. Thanks to Proposition 2.2.43, we have that $H_{v,\nu}\colon L^{p_1,q_1;\alpha_1}(0,1) \to Z(0,1)$, where

$$Z(0,1) = \begin{cases} L^{\frac{p_1\nu}{1-p_1\gamma},q_1;\alpha_1}(0,1), & p_1 = q_1 = 1,\ \alpha_1 \geq 0,\ \gamma+\nu \geq 1 \text{ or} \\ & p_1 \in (1, \frac{1}{\gamma}),\ \gamma+\nu \geq 1 \text{ or} \\ & p_1 \in (\frac{1}{\gamma+\nu}, \frac{1}{\gamma}),\ \gamma+\nu < 1; \\ L^{1,1;\alpha_1}(0,1), & p_1 = \frac{1}{\gamma+\nu},\ q_1 = 1,\ \alpha_1 \geq 0,\ \gamma+\nu < 1; \\ L^{(1,q_1;\alpha_1-1)}(0,1), & p_1 = \frac{1}{\gamma+\nu},\ q_1 \in (1,\infty],\ \alpha_1 > 1 - \frac{1}{q_1},\ \gamma+\nu < 1; \\ L^{\infty,q_1;\alpha_1-1}(0,1), & p_1 = \frac{1}{\gamma},\ \alpha_1 < 1 - \frac{1}{q_1}; \\ L^{\infty,q_1;-\frac{1}{q_1},-1}(0,1), & p_1 = \frac{1}{\gamma},\ q_1 \in (1,\infty],\ \alpha_1 = 1 - \frac{1}{q_1}; \\ L^{\infty}(0,1), & p_1 = \frac{1}{\gamma},\ \alpha_1 > 1 - \frac{1}{q_1} \text{ or} \\ & p_1 = \frac{1}{\gamma},\ q_1 = 1,\ \alpha_1 \geq 0 \text{ or} \\ & p_1 > \frac{1}{\gamma}, \end{cases}$$

and the space $Z(0,1)$ is the optimal target space for $H_{v,\nu}$ and $L^{p_1,q_1;\alpha_1}(0,1)$. In the cases where $Z(0,1) \neq L^{\infty}(0,1)$, the assertion follows from Proposition 2.3.2 (see also Remark 2.3.3) combined with (2.133) save for the case where $p_1 = \frac{1}{\gamma+\nu}$, $q_1 \in (1,\infty]$, $\alpha_1 > 1 - \frac{1}{q_1}$, $\gamma+\nu < 1$. In that case, we need to know when $L^{(1,q_1;\alpha_1-1)}(0,1) \overset{*}{\hookrightarrow} L^{p_2,q_2;\alpha_2,\beta}(0,1)$, which can be obtained from [58, Theorem 3.29]. If $p = \frac{1}{\gamma}$, $q_1 = 1$, $\alpha_1 = 0$, then $Z(0,1) = L^{\infty}(0,1)$, but $H_{v,\nu}\colon L^{\frac{1}{\gamma},1}(0,1) \to L^{\infty}(0,1)$ is not compact owing to Proposition 2.3.4 inasmuch as $\|t^{-1+\gamma}\chi_{(0,a)}(t)\|_{L^{\frac{1}{1-\gamma},\infty}(0,1)} = 1$ for every $a \in (0,1)$. If either $p_1 = \frac{1}{\gamma}$, $\alpha_1 > 1 - \frac{1}{q_1}$ or $p_1 > \frac{1}{\gamma}$, then $L^{p_1,q_1;\alpha_1}(0,1) \overset{*}{\hookrightarrow} L^{\frac{1}{\gamma},1}(0,1)$, and so $H_{v,\nu}\colon L^{p_1,q_1;\alpha_1}(0,1) \to L^{\infty}(0,1)$ is compact owing to Proposition 2.3.4. □

2.3.2 Almost-compact embeddings versus fundamental functions

As we have seen in Section 2.3.1, compactness of $H_{u,v,\nu}$ between two rearrangement-invariant function spaces is closely related to the relation of almost compact embeddings. We know that

$$X(0,1) \overset{*}{\hookrightarrow} Y(0,1) \tag{2.134}$$

implies

$$\lim_{t\to 0^+} \frac{\varphi_{Y(0,1)}(t)}{\varphi_{X(0,1)}(t)} = 0. \tag{2.135}$$

When both $X(0,1)$ and $Y(0,1)$ are Lebesgue spaces (or, more generally, two-parametric Lorentz spaces), then (2.135) is also sufficient for (2.134). It might be tempting to think that (2.135) guarantees (2.134) in general. However, it is well known that it is not the case. When we consider general rearrangement-invariant function spaces, we may find $X(0,1)$ and $Y(0,1)$ in such a way that (2.134) is satisfied but $X(0,1)$ is not embedded in $Y(0,1)$ at all. For example, this is the case when $X(0,1) = L^{2,\infty;\frac{1}{4}}(0,1)$ and $Y(0,1) = L^2(0,1)$ ([75, Theorem 4.5]). However, such a (counter)example is not particularly satisfactory, because $X(0,1)$ is not even included in $Y(0,1)$. Nevertheless, it is known that, when $X(0,1)$ and $Y(0,1)$ are Orlicz spaces, (2.135) guarantees that $X(0,1) \hookrightarrow Y(0,1)$, but it does not necessarily guarantee that the embedding is almost compact (see [55, Chapter II, §13], [79, Section 4.12]); therefore, when general rearrangement-invariant function spaces are considered, it may happen that $X(0,1) \hookrightarrow Y(0,1)$ and (2.135) is satisfied, and yet the embedding $X(0,1) \hookrightarrow Y(0,1)$ is not almost compact.

In this subsection, we investigate the question of how to "fundamentally enlarge" a given rearrangement-invariant function space to a bigger rearrangement-invariant function space in such a way that the resulting embedding is still not almost compact. The results that we will obtain in this subsection can be used to produce various (counter)examples related to compactness in the framework of rearrangement-invariant function spaces.

First, we consider the situation where $X(0,1)$ is a Marcinkiewicz space. The following elementary proposition concerning concave functions shows that every Marcinkiewicz space (not equivalent to L^1 or L^∞) can be enlarged to a strictly bigger Marcinkiewicz space without making the resulting embedding almost compact.

Proposition 2.3.7. *Let $\varphi\colon [0,1] \to [0,\infty)$ be a nondecreasing, concave function vanishing only at the origin. Assume that*

$$\lim_{t\to 0^+} \frac{t}{\varphi(t)} = 0 \tag{2.136}$$

and

$$\lim_{t\to 0^+} \varphi(t) = 0. \tag{2.137}$$

There is a nondecreasing, concave function $\psi\colon [0,1] \to [0,\infty)$ vanishing only at the origin such that $\psi \le \varphi$, $\liminf_{t\to 0^+} \frac{\psi(t)}{\varphi(t)} = 0$ and $\limsup_{t\to 0^+} \frac{\psi(t)}{\varphi(t)} = 1$.

Proof. We shall find, by induction, two sequences $\{t_k\}_{k=1}^\infty$, $\{\tau_k\}_{k=1}^\infty$ of positive numbers converging to 0 such that, for each $k \in \mathbb{N}$,

$$t_{k+1} < \tau_k < t_k,$$

$$\frac{\varphi(\tau_k) - \varphi(t_{k+1})}{\tau_k - t_{k+1}} > 2^{k+1} \frac{\varphi(t_k) - \varphi(t_{k+1})}{t_k - t_{k+1}}, \tag{2.138}$$

$$\varphi(t_{k+1}) \le 2^{-k-1}\varphi(\tau_k). \tag{2.139}$$

Set $t_1 = 1$. Assume that we have already found $t_1, \dots, t_k$ and $\tau_1, \dots, \tau_{k-1}$ for some $k \in \mathbb{N}$. By (2.136) there is $\tau_k \in (0, \frac{t_k}{2})$ such that $\frac{\varphi(\tau_k)}{\tau_k} > 2^{k+1}\frac{\varphi(t_k)}{t_k}$. Since $\lim\limits_{t\to 0^+} \frac{\varphi(\tau_k)-\varphi(t)}{\tau_k - t} = \frac{\varphi(\tau_k)}{\tau_k}$ by (2.137), there is $t_{k+1} \in (0, \tau_k)$ such that $\frac{\varphi(\tau_k)-\varphi(t_{k+1})}{\tau_k - t_{k+1}} > 2^{k+1}\frac{\varphi(t_k)}{t_k}$. Moreover, we can find that t_{k+1} in such a way that $\varphi(t_{k+1}) \le 2^{-k-1}\varphi(\tau_k)$ thanks to (2.137) and the fact that $\varphi(\tau_k) \neq 0$.

Clearly $t_{k+1} < \tau_k < t_k$ and $t_{k+1} \le \frac{1}{2^k}$. Since φ is concave, we have that $\frac{\varphi(t_k)}{t_k} \ge \frac{\varphi(t_k)-\varphi(t_{k+1})}{t_k - t_{k+1}}$. Hence $\frac{\varphi(\tau_k)-\varphi(t_{k+1})}{\tau_k - t_{k+1}} > 2^{k+1}\frac{\varphi(t_k)-\varphi(t_{k+1})}{t_k - t_{k+1}}$. This completes the inductive step.

We define the function $\psi\colon [0,1] \to [0,\infty)$ as

$$\psi(t) = \begin{cases} \varphi(t_{k+1}) + \frac{\varphi(t_k)-\varphi(t_{k+1})}{t_k - t_{k+1}}(t - t_{k+1}), & t \in (t_{k+1}, t_k],\ k \in \mathbb{N}, \\ 0, & t = 0. \end{cases}$$

Note that, since φ is concave and nondecreasing, so is ψ. Clearly, $\psi(t_k) = \varphi(t_k)$ for each $k \in \mathbb{N}$. Furthermore, since φ is concave and vanishes only at the origin, we have that $0 < \psi \le \varphi$ on $(0,1]$. Hence $\limsup\limits_{t\to 0^+} \frac{\psi(t)}{\varphi(t)} = 1$.

Finally, for each $k \in \mathbb{N}$,

$$\begin{aligned} \frac{\psi(\tau_k)}{\varphi(\tau_k)} &= \frac{\varphi(t_{k+1}) + \frac{\varphi(t_k)-\varphi(t_{k+1})}{t_k - t_{k+1}}(\tau_k - t_{k+1})}{\varphi(\tau_k)} \\ &\le \frac{\varphi(t_{k+1}) + 2^{-k-1}\frac{\varphi(\tau_k)-\varphi(t_{k+1})}{\tau_k - t_{k+1}}(\tau_k - t_{k+1})}{\varphi(\tau_k)} \\ &= \frac{(1 - 2^{-k-1})\varphi(t_{k+1}) + 2^{-k-1}\varphi(\tau_k)}{\varphi(\tau_k)} \\ &\le \frac{2^{-k}\varphi(\tau_k)}{\varphi(\tau_k)} = 2^{-k}, \end{aligned}$$

where the first and the second inequalities follow from (2.138) and (2.139), respectively. Hence $\liminf\limits_{t\to 0^+} \frac{\psi(t)}{\varphi(t)} = 0$. □

Corollary 2.3.8. *Let* $\varphi\colon [0,1] \to [0,\infty)$ *be a quasiconcave function satisfying* (2.136) *and* (2.137). *There is a nondecreasing, concave function* $\psi\colon [0,1] \to [0,\infty)$ *vanishing only at the origin such that* $M_\varphi(0,1) \subsetneq M_\psi(0,1)$, *but the embedding* $M_\varphi(0,1) \hookrightarrow M_\psi(0,1)$ *is not almost compact.*

Proof. Let $\widetilde{\varphi}$ be the least concave majorant of φ, which satisfies $\frac{1}{2}\widetilde{\varphi} \le \varphi \le \widetilde{\varphi}$ ([8, Chapter 2, Proposition 5.10]). We plainly have that

$$M_\varphi(0,1) = M_{\widetilde{\varphi}}(0,1).$$

Let ψ be the function from Proposition 2.3.7 associated with $\widetilde{\varphi}$. Since $\psi \leq \widetilde{\varphi}$ and $\liminf\limits_{t\to 0^+} \frac{\psi(t)}{\widetilde{\varphi}(t)} = 0$, we have that

$$M_\varphi(0,1) \subsetneq M_\psi(0,1);$$

however the embedding $M_\varphi(0,1) \hookrightarrow M_\psi(0,1)$ is not almost compact due to (1.24), for $\limsup\limits_{t\to 0^+} \frac{\psi(t)}{\widetilde{\varphi}(t)} = 1$. □

Remark 2.3.9. Assumptions (2.136) and (2.137) amount to the fact that $M_\varphi(0,1) \neq L^\infty(0,1)$ and $M_\varphi(0,1) \neq L^1(0,1)$ (recall (1.20) and (1.21)). Therefore, those assumptions are completely natural in light of (1.25) and (1.26).

Although the Marcinkiewicz space $M_\psi(0,1)$ constructed in Corollary 2.3.8 was such that $M_\varphi(0,1) \subsetneq M_\psi(0,1)$ and the embedding was not almost compact, $M_\psi(0,1)$ was not "fundamentally larger" than $M_\varphi(0,1)$ inasmuch as $\limsup\limits_{t\to 0^+} \frac{\psi(t)}{\varphi(t)} > 0$. However, that is hardly surprising, because it can be easily seen that (2.135) implies (2.134) when both spaces in question are Marcinkiewicz spaces. Therefore, if we are to "fundamentally enlarge" $M_\varphi(0,1)$ to a rearrangement-invariant function space $Y(0,1)$ in such a way that the embedding $M_\varphi(0,1) \hookrightarrow Y(0,1)$ is not almost compact, $Y(0,1)$ must not be a Marcinkiewicz space. This brings us to the following theorem, which is inspired by [42, Example 3.4]. Before we state it, we need to introduce the notion of slowly varying functions. We say that a continuous function $b\colon (0,1) \to (0,\infty)$ such that $0 < \liminf\limits_{t\to 1^-} b(t) \leq \limsup\limits_{t\to 1^-} b(t) < \infty$ is *slowly varying near* 0 if for every $\varepsilon > 0$ there is $t_0 \in (0,1)$ such that the functions $t \mapsto t^\varepsilon b(t)$ and $t \mapsto t^{-\varepsilon} b(t)$ are nondecreasing and nonincreasing, respectively, on the interval $(0, t_0)$.

Theorem 2.3.10. *Assume that* $\varphi\colon [0,1] \to [0,\infty)$ *is a quasiconcave function satisfying*

$$\frac{1}{t}\int_0^t \frac{1}{\varphi(s)}\,\mathrm{d}s \lesssim \frac{1}{\varphi(t)} \quad \textit{for every } t \in (0,1). \tag{2.140}$$

Let η *be a positive, measurable function on* $(0,1)$ *such that*

$$\int_0^t \frac{\varphi(s)}{\eta(s)}\,\mathrm{d}s \lesssim \varphi(t) \quad \textit{for every } t \in (0,1), \tag{2.141}$$

$$\int_t^1 \frac{1}{\eta(s)}\,\mathrm{d}s < \infty \quad \textit{for every } t \in (0,1), \tag{2.142}$$

$$\int_0^1 \frac{1}{\eta(s)}\,\mathrm{d}s = \infty. \tag{2.143}$$

Furthermore, assume that the function $b\colon (0,1) \to (0,\infty)$ *defined as*

$$b(t) = 1 + \int_t^1 \frac{1}{\eta(s)}\,\mathrm{d}s,\ t \in (0,1),$$

is slowly varying near 0, *and that*

$$\frac{\eta(t)}{\varphi(t)} \approx \int_0^t \xi(s)\,\mathrm{d}s \quad \textit{for every } t \in (0,1),$$

where ξ is some positive, continuous function on $(0,1)$ that is equivalent to a nonincreasing function. Finally, assume that

$$\frac{1}{\xi(t)}\int_0^t \xi(s)b(s)\,\mathrm{d}s \lesssim \int_0^t b(s)\,\mathrm{d}s \quad \textit{for every } t\in(0,1). \tag{2.144}$$

The functional $\|\cdot\|_{Y(0,1)}$ defined as

$$\|f\|_{Y(0,1)} = \sup_{t\in(0,1)} \frac{1}{b(t)}\int_t^1 f^*(s)\frac{\varphi(s)}{\eta(s)}\,\mathrm{d}s,\ f\in\mathfrak{M}^+(0,1), \tag{2.145}$$

is equivalent to a rearrangement-invariant function norm and we have that $M_\varphi(0,1)\hookrightarrow Y(0,1)$ and $\lim_{t\to0^+}\frac{\varphi_{Y(0,1)}(t)}{\varphi(t)}=0$, but the embedding is not almost compact.

Proof. We start off by observing that the functional defined by (2.145) is indeed equivalent to a rearrangement-invariant function norm, and so we are entitled to denote the corresponding rerrangement-invariant function space by $Y(0,1)$. Set $b^{-1}=\frac{1}{b}$. Note that we have that

$$\|f\|_{Y(0,1)} = \left\| \int_t^1 f^*(s)\frac{\varphi(s)}{\eta(s)}\,\mathrm{d}s \right\|_{L^{\infty,\infty;b^{-1}}(0,1)} \qquad \text{for every } f\in\mathfrak{M}^+(0,1),$$

where the functional $\|\cdot\|_{L^{\infty,\infty;b^{-1}}(0,1)}$ is defined as

$$\|f\|_{L^{\infty,\infty;b^{-1}}(0,1)} = \sup_{t\in(0,1)} b^{-1}(t)f^*(t),\ f\in\mathfrak{M}^+(0,1).$$

The functional $\|\cdot\|_{L^{\infty,\infty;b^{-1}}(0,1)}$ is equivalent to a rearrangement-invariant function norm (see [77, Theorem 3.26]), and so we are entitled to denote by $L^{\infty,\infty;b^{-1}}(0,1)$ the corresponding rearrangement-invariant function space. Furthermore, assumption (2.144) guarantees that the supremum operator T_ξ, defined by (2.17), is bounded on $L^{\infty,\infty;b^{-1}}(0,1)$ owing to [46, Theorem 3.2]. Therefore, the functional $\|\cdot\|_{Y(0,1)}$ is indeed equivalent to a rearrangement-invariant function norm by virtue of Proposition 2.1.3.

We now turn our attention to the relation between the Marcinkiewicz space $M_\varphi(0,1)$ and $Y(0,1)$. Note that $M_\varphi(0,1)\hookrightarrow Y(0,1)$. To this end, note that, since φ satisfies (2.140), we have that

$$\|f\|_{M_\varphi(0,1)} \approx \sup_{t\in(0,1)} \varphi(t)f^*(t) \quad \text{for every } f\in\mathfrak{M}(0,1) \tag{2.146}$$

(see [72, Lemma 2.1]); therefore, it is sufficient to verify that $\frac{1}{\varphi}\in Y(0,1)$, which is easy (recall (2.142) and (2.143)).

Next, we claim that

$$\varphi_{Y(0,1)}(t) \lesssim \frac{\varphi(t)}{b(t)} \quad \text{for every } t\in(0,1).$$

Indeed, thanks to (2.141) and the fact that the function b is nonincreasing, we have that

$$\varphi_{Y(0,1)}(t) \approx \sup_{u\in(0,t)} \frac{1}{b(u)}\int_u^t \frac{\varphi(s)}{\eta(s)}\,\mathrm{d}s \le \frac{1}{b(t)}\int_0^t \frac{\varphi(s)}{\eta(s)}\,\mathrm{d}s \lesssim \frac{\varphi(t)}{b(t)}.$$

Hence $\lim_{t\to 0^+} \frac{\varphi_{Y(0,1)}(t)}{\varphi(t)} = 0$ thanks to (2.143).

Finally, in order to prove that the embedding $M_\varphi(0,1) \hookrightarrow Y(0,1)$ is not almost compact, we consider functions $f_k \in \mathfrak{M}^+(0,1)$, $k \in \mathbb{N}$, such that

$$f_k^*(t) = \frac{1}{\varphi(\frac{1}{k})}\chi_{(0,\frac{1}{k})}(t) + \frac{1}{\varphi(t)}\chi_{[\frac{1}{k},1)}(t),\ t \in (0,1).$$

Note that the set $M = \{f_k \colon k \in \mathbb{N}\}$ is bounded in $M_\varphi(0,1)$, for, thanks to (2.146),

$$\|f_k\|_{M_\varphi(0,1)} \approx \max\left\{\frac{1}{\varphi(\frac{1}{k})}\sup_{t\in(0,\frac{1}{k})}\varphi(t),\ \sup_{t\in(\frac{1}{k},1)}\frac{1}{\varphi(t)}\varphi(t)\right\} = 1,$$

where the equivalence constants are independent of k. However, we claim that M does not have uniformly absolutely continuous norm in $Y(0,1)$. Let $\delta \in (0,1)$. Owing to (2.143) and (2.142), we can find $k \in \mathbb{N}$ large enough that

$$\frac{1}{k} < \delta,$$

$$b\Big(\frac{1}{k}\Big) \le 2\int_{\frac{1}{k}}^1 \frac{1}{\eta(s)}\,\mathrm{d}s, \tag{2.147}$$

$$\int_{\frac{1}{k}}^1 \frac{1}{\eta(s)}\,\mathrm{d}s \ge 2\int_{\delta}^1 \frac{1}{\eta(s)}\,\mathrm{d}s. \tag{2.148}$$

Hence

$$\begin{aligned}\|f_k^*\chi_{(0,\delta)}\|_{Y(0,1)} &\gtrsim \sup_{t\in[\frac{1}{k},\delta]}\frac{1}{b(t)}\int_t^\delta \frac{1}{\eta(s)}\,\mathrm{d}s \ge \frac{1}{b\left(\frac{1}{k}\right)}\int_{\frac{1}{k}}^\delta \frac{1}{\eta(s)}\,\mathrm{d}s\\ &\ge \frac{1}{2}\frac{\int_{\frac{1}{k}}^\delta \frac{1}{\eta(s)}\,\mathrm{d}s}{\int_{\frac{1}{k}}^1 \frac{1}{\eta(s)}\,\mathrm{d}s} = \frac{1}{2}\frac{\int_{\frac{1}{k}}^1 \frac{1}{\eta(s)}\,\mathrm{d}s - \int_{\delta}^1 \frac{1}{\eta(s)}\,\mathrm{d}s}{\int_{\frac{1}{k}}^1 \frac{1}{\eta(s)}\,\mathrm{d}s}\\ &\ge \frac{1}{2}\frac{\int_{\frac{1}{k}}^1 \frac{1}{\eta(s)}\,\mathrm{d}s - \frac{1}{2}\int_{\frac{1}{k}}^1 \frac{1}{\eta(s)}\,\mathrm{d}s}{\int_{\frac{1}{k}}^1 \frac{1}{\eta(s)}\,\mathrm{d}s} = \frac{1}{4},\end{aligned} \tag{2.149}$$

where the last but one inequality is valid thanks to (2.147) and the last one is true owing to (2.148). Since $\delta \in (0,1)$ was arbitrary, (2.149) implies that M does not have uniformly absolutely continuous norm in $Y(0,1)$. □

Remarks 2.3.11.

(i) Since the question of whether a rearrangement-invariant function space is (almost compactly) embedded in another one is invariant with respect to equivalently renorming the spaces, Theorem 2.3.10 can be used even when the function φ is merely equivalent to a quasiconcave function on $[0,1]$.

(ii) Since Theorem 2.3.10 has a large number of assumptions, it is worth noting some concrete, important examples of functions that satisfy them.

- If $\varphi(t) = t^\alpha \ell(t)^\beta$ with $\alpha \in (0,1)$ and $\beta \in \mathbb{R}$, then we can take $\eta(t) = t$, $b(t) = \ell(t)$ and $\xi(t) = t^{-\alpha}\ell(t)^{-\beta}$.

- If $\varphi(t) = \ell(t)^\beta$ with $\beta < 0$, then we can take $\eta(t) = t\ell(t)$, $b(t) = \ell\ell(t)$ and $\xi(t) = \ell(t)^{1-\beta}$.

In each of the two cases, the function φ is equivalent to a quasiconcave function on $[0, 1]$, and φ together with the functions η, b and ξ satisfies the assumptions of Theorem 2.3.10. Moreover, these examples illustrate how to use Theorem 2.3.10 when φ has the form $\varphi(t) = t^\alpha b(t)$, where $\alpha \in [0, 1)$ and b is a slowly-varying function near 0, which is often the case.

The next proposition is a general guideline on how to fundamentally enlarge a rearrangement-invariant function space without making the resulting embedding almost compact.

Proposition 2.3.12. *Let $Z_1(0,1)$ and $Z_2(0,1)$ be rearrangement-invariant function spaces. Assume that $\lim_{t\to 0^+} \frac{\varphi_{Z_2(0,1)}(t)}{\varphi_{Z_1(0,1)}(t)} = 0$ and $Z_1(0,1) \not\subseteq Z_2(0,1)$. Set $Y(0,1) = (Z_1'(0,1) \cap Z_2'(0,1))'$. We have that $\lim_{t\to 0^+} \frac{\varphi_{Y(0,1)}(t)}{\varphi_{Z_1(0,1)}(t)} = 0$ and $Z_1(0,1) \hookrightarrow Y(0,1)$, but the embedding is not almost compact.*

Proof. By combining (1.9) and (1.15), we obtain that $Z_1(0,1) \hookrightarrow Y(0,1)$. Furthermore, since $\varphi_{Y'(0,1)} = \max\{\varphi_{Z_1'(0,1)}, \varphi_{Z_2'(0,1)}\}$, we have that

$$\lim_{t\to 0^+} \frac{\varphi_{Y(0,1)}(t)}{\varphi_{Z_1(0,1)}(t)} = \lim_{t\to 0^+} \frac{\varphi_{Z_1'(0,1)}(t)}{\varphi_{Y'(0,1)}(t)} \le \lim_{t\to 0^+} \frac{\varphi_{Z_1'(0,1)}(t)}{\varphi_{Z_2'(0,1)}(t)} = \lim_{t\to 0^+} \frac{\varphi_{Z_2(0,1)}(t)}{\varphi_{Z_1(0,1)}(t)} = 0,$$

where we used (1.18).

We would like to show that $Z_1(0,1)$ is not almost-compactly embedded in $Y(0,1)$, which is equivalent to showing that $Z_1'(0,1) \cap Z_2'(0,1)$ is not almost-compactly embedded in $Z_1'(0,1)$ thanks to (1.15). Owing to (1.15), we have that $Z_2'(0,1) \not\subseteq Z_1'(0,1)$; hence the embedding $Z_1'(0,1) \cap Z_2'(0,1) \hookrightarrow Z_1'(0,1)$ is not almost compact by virtue of [42, Lemma 3.7]. ☐

Remark 2.3.13. If at least one of the spaces $Z_1'(0,1)$ and $Z_2'(0,1)$ in Proposition 2.3.12 has absolutely continuous norm, then $Y(0,1) = Z_1(0,1) + Z_2(0,1)$ ([8, Chapter 3, Exercise 5]).

A large number of rearrangement-invariant function spaces are particular instances of Lorentz Λ spaces. The following two theorems can be used for fundamentally enlarging a Lorentz Λ space.

Theorem 2.3.14. *Let $q \in (1, \infty)$. Let $v\colon (0,1) \to (0,\infty)$ be a measurable function such that $V(t) < \infty$ for every $t \in (0,1)$, where $V(t) = \int_0^t v(s)\,\mathrm{d}s$, $t \in (0,1)$, that satisfies*

$$V(t)^{q'-1} \int_0^t s^{q'} v(s) V(s)^{-q'}\,\mathrm{d}s \lesssim t^{q'} \quad \text{for every } t \in (0,1).$$

Set $Z(0,1) = \Lambda^q(v)$. Let $r \in [1, q)$. Assume that $w\colon (0,1) \to (0,\infty)$ is a measurable function such that $W(t) < \infty$ for every $t \in (0,1)$ that satisfies (1.32) with v and q replaced by w and r, respectively,

$$\lim_{t\to 0^+} \frac{W(t)^{\frac{1}{r}}}{V(t)^{\frac{1}{q}}} = 0 \tag{2.150}$$

and

$$\int_0^1 \left(\frac{W(t)}{V(t)}\right)^{\frac{q}{q-r}} v(t)\,\mathrm{d}t = \infty. \tag{2.151}$$

Set $Y(0,1) = \Lambda^q(v) + \Lambda^r(w)$. *We have that* $\lim_{t\to 0^+} \frac{\varphi_{Y(0,1)}(t)}{\varphi_{Z(0,1)}(t)} = 0$ *and* $Z(0,1) \hookrightarrow Y(0,1)$, *but the embedding is not almost compact.*

Proof. The assertion will immediately follow from Proposition 2.3.12 with $Z_1(0,1) = \Lambda^q(v)$ and $Z_2(0,1) = \Lambda^r(w)$ once we verify its assumptions. Note that $\Lambda^q(v)$ and $\Lambda^r(w)$ are equivalent to rearrangement-invariant function spaces (see (1.32)). Assumption (2.151) is equivalent to the fact that $Z_1(R,\mu) \not\subseteq Z_2(R,\mu)$ by [90, Proposition 1]. Since $\varphi_{Z_1(0,1)} \approx V^{\frac{1}{q}}$ and $\varphi_{Z_2(0,1)} \approx W^{\frac{1}{r}}$, we plainly have that $\lim_{t\to 0^+} \frac{\varphi_{Z_2(0,1)}(t)}{\varphi_{Z_1(0,1)}(t)} = 0$ thanks to (2.150). It only remains for us to observe that $Z_1'(0,1)$ has absolutely continuous norm. Indeed, owing to [82, Theorem 1] combined with (1.10), we have that

$$\begin{aligned}\|g^*\chi_{(0,a)}\|_{Z_1'(0,1)} \approx \left(\int_0^1 \left(g^*\chi_{(0,a)}\right)^{**}(t)^{q'} t^{-q'} V(t)^{-q'} v(t)\,\mathrm{d}t\right)^{\frac{1}{q'}} \\ + V(1)^{-q} \int_0^a g^*(t)\,\mathrm{d}t\end{aligned}$$

for every $g \in Z_1'(0,1)$ and $a \in (0,1)$, whence the claim follows thanks to the Lebesgue dominated convergence theorem. □

The final theorem of this subsection is an analogue of Theorem 2.3.14 in the case where $q = \infty$.

Theorem 2.3.15. *Let* $v\colon (0,1) \to (0,\infty)$ *be a measurable function such that* $v \in L^\infty(0,1)$ *and*

$$\sup_{t\in(0,1)} \tilde{v}(t)\frac{1}{t}\int_0^t \frac{1}{\tilde{v}(s)}\,\mathrm{d}s < \infty,$$

where $\tilde{v}(t) = \operatorname*{ess\,sup}_{s\in(0,t)} v(s)$, $t \in (0,1)$. *Set* $Z(0,1) = \Lambda^\infty(v)$. *Let* $r \in [1,\infty)$. *Assume that* $w\colon (0,1) \to (0,\infty)$ *is a measurable function such that* $W(t) < \infty$ *for every* $t \in (0,1)$ *that satisfies* (1.32) *with* v *and* q *replaced by* w *and* r, *respectively,*

$$\lim_{t\to 0^+} \frac{W(t)^{\frac{1}{r}}}{\tilde{v}(t)} = 0$$

and

$$\int_0^1 \tilde{v}(t)^{-r} w(t)\,\mathrm{d}t = \infty.$$

Set $Y(0,1) = \Lambda^\infty(v) + \Lambda^r(w)$. *We have that* $\lim_{t\to 0^+} \frac{\varphi_{Y(0,1)}(t)}{\varphi_{Z(0,1)}(t)} = 0$ *and* $Z(0,1) \hookrightarrow Y(0,1)$, *but the embedding is not almost compact.*

Proof. The theorem can be proved in a similar way to Theorem 2.3.14. We just note that (cf. [44, Lemma 1.5])

$$\|f\|_{\Lambda^\infty(v)} = \operatorname*{ess\,sup}_{t\in(0,1)} f^*(t)\tilde{v}(t) \quad \text{for every } f \in \mathfrak{M}^+(0,1),$$

and so, since $\frac{1}{\tilde{v}}$ is a positive, nonincreasing function on $(0,1)$,

$$\|g\|_{Z'(0,1)} = \sup_{\|f\|_{Z(0,1)}\leq 1} \int_0^1 f^*(t)g^*(t)\,\mathrm{d}t \approx \int_0^1 \frac{g^*(t)}{\tilde{v}(t)}\,\mathrm{d}t \quad \text{for every } g \in \mathfrak{M}^+(0,1)$$

thanks to (1.10). □

Theorem 2.3.14 and Theorem 2.3.15 combined with straightforward computations yield the following particular examples.

Corollary 2.3.16. *Let $p, q \in (1,\infty]$, $\alpha, \beta \in \mathbb{R}$, and $r \in [1,q)$. Set*

$$Z(0,1) = \begin{cases} L^{p,q;\alpha}(0,1), & p \in (1,\infty); \\ L^{\infty,q;\alpha}(0,1), & p = \infty,\ \alpha + \frac{1}{q} < 0; \\ L^{\infty,q;-\frac{1}{q},\beta}(0,1), & p = \infty,\ \alpha + \frac{1}{q} = 0,\ \beta + \frac{1}{q} < 0, \end{cases}$$

and

$$Y(0,1) = \begin{cases} Z(0,1) + L^{p,r;\alpha+\frac{1}{q}-\frac{1}{r}}(0,1), & p \in (1,\infty); \\ Z(0,1) + L^{\infty,r;\alpha+\frac{1}{q}-\frac{1}{r},\frac{1}{q}-\frac{1}{r}}(0,1), & p = \infty,\ \alpha + \frac{1}{q} < 0; \\ Z(0,1) + L^{\infty,r;-\frac{1}{r},\beta+\frac{1}{q}-\frac{1}{r},\frac{1}{q}-\frac{1}{r}}(0,1), & p = \infty,\ \alpha + \frac{1}{q} = 0,\ \beta + \frac{1}{q} < 0. \end{cases}$$

We have that $Z(0,1) \hookrightarrow Y(0,1)$, $\lim_{t\to 0^+} \frac{\varphi_{Y(0,1)}(t)}{\varphi_{Z(0,1)}(t)} = 0$, but the embedding is not almost compact.

3. Classical operators of harmonic analysis and optimality

The question of when classical operators of harmonic analysis on $\mathbb{R}^n$, such as the Hardy–Littlewood maximal operator, various fractional operators or singular integral operators, are bounded between various function spaces has been studied for decades (e.g. [87, 89, 70, 71, 34, 82, 16, 22, 43, 11], to name some classic references). In this chapter we shall focus on the sharpness of such results within the class of rearrangement-invariant function spaces.

We say that a rearrangement-invariant function space $Y(\mathbb{R}^n)$ is the *optimal target space* for an operator T and a rearrangement-invariant function space $X(\mathbb{R}^n)$ if $T\colon X(\mathbb{R}^n) \to Y(\mathbb{R}^n)$ is bounded (in particular, T is well defined on $X(\mathbb{R}^n)$) and $Y(\mathbb{R}^n) \hookrightarrow Z(\mathbb{R}^n)$ whenever $Z(\mathbb{R}^n)$ is a rearrangement-invariant function space such that $T\colon X(\mathbb{R}^n) \to Z(\mathbb{R}^n)$ is bounded. We say that a rearrangement-invariant function space $X(\mathbb{R}^n)$ is the *optimal domain space* for an operator T and a rearrangement-invariant function space $Y(\mathbb{R}^n)$ if $T\colon X(\mathbb{R}^n) \to Y(\mathbb{R}^n)$ is bounded (in particular, T is well defined on $X(\mathbb{R}^n)$) and $Z(\mathbb{R}^n) \hookrightarrow X(\mathbb{R}^n)$ whenever $Z(\mathbb{R}^n)$ is a rearrangement-invariant function space such that $T\colon Z(\mathbb{R}^n) \to Y(\mathbb{R}^n)$ is bounded.

3.1 The Hilbert and Riesz transforms

The Hilbert transform H on $\mathbb{R}$ is a textbook example of a singular integral operator with odd kernel (this statement, while subjective in nature, is “proved” by the fact that E.M. Stein used the Hilbert transform to illustrate the theory of singular integrals in [87, Chapter 2]) . The Hilbert transform is defined as

$$Hf(x) = \frac{1}{\pi} \lim_{\varepsilon \to 0^+} \int_{|x-t| \geq \varepsilon} \frac{f(t)}{x-t} \,\mathrm{d}t,\ x \in \mathbb{R},$$

for those locally integrable functions f on $\mathbb{R}$ for which the limit exists for a.e. $x \in \mathbb{R}$.

Analogues of the Hilbert transform in $\mathbb{R}^n$, $n \in \mathbb{N}$, $n \geq 2$, are the Riesz transforms R_j, $j \in \{1, 2, \dots, n\}$, that are defined as

$$R_j(x) = \frac{1}{\pi \omega_{n-1}} \lim_{\varepsilon \to 0^+} \int_{|x-y| \geq \varepsilon} \frac{(x_j - y_j) f(y)}{|x-y|^{n+1}} \,\mathrm{d}y,\ x \in \mathbb{R}^n,$$

for those locally integrable functions f on $\mathbb{R}^n$ for which the limit exists for a.e. $x \in \mathbb{R}^n$, where ω_{n-1} denotes the volume of the unit ball in $\mathbb{R}^{n-1}$. For both of these transformations an (in a sense) sharp pointwise estimate on their rearrangements is known.

More generally, let T be the singular integral operator defined as

$$Tf(x) = \lim_{\varepsilon \to 0^+} \int_{|x-y| \geq \varepsilon} \frac{\Omega(x-y) f(y)}{|x-y|^n} \,\mathrm{d}y,\ x \in \mathbb{R}^n,$$

where Ω is an odd function on $\mathbb{R}^n$, which is not identically zero, that is homogeneous of degree zero, that is, $\Omega(\alpha x) = \Omega(x)$ for every $\alpha \in (0, \infty)$ and every $x \in \mathbb{R}^n$, and that satisfies the following “Dini condition” (cf. [87, p. 39]):

$$\int_0^1 \frac{\omega(\delta)}{\delta}\,\mathrm{d}\delta < \infty,$$

where $\omega(\delta) = \sup_{\substack{|x-y|\leq\delta\\ |x|=|y|=1}} |\omega(x)-\omega(y)|$, $\delta\in(0,1)$. Furthermore, let $\mathcal{T}$ be the corresponding maximal operator, that is,

$$\mathcal{T}f(x) = \sup_{\varepsilon>0}\Big|\int_{|x-y|\geq\varepsilon} \frac{\Omega(x-y)f(y)}{|x-y|^n}\,\mathrm{d}y\Big|,\ x\in\mathbb{R}^n.$$

The Hilbert and Riesz transforms are particular instances of such singular integral operators ($\Omega(x)=\frac{x}{\pi|x|}$ and $\Omega(x)=\frac{x_j}{\pi\omega_{n-1}|x|}$, respectively). Throughout this section, T and $\mathcal{T}$ denote a singular integral operator and its corresponding maximal operator, respectively, as above.

It is known ([6, Theorem 16.12], cf. [6, Theorem 16.6], [74, Section 3]) that T is well defined, that is, the limit exists (and is finite) for a.e. $x\in\mathbb{R}^n$, for every locally integrable function f on $\mathbb{R}^n$ such that

$$\int_0^1 f^*(t)\,\mathrm{d}t + \int_1^\infty f^*(t)\,\frac{\mathrm{d}t}{t} < \infty, \tag{3.1}$$

and, for such functions, we have that

$$(Tf)^*(t) \leq (\mathcal{T}f)^*(t) \lesssim \frac{1}{t}\int_0^t f^*(s)\,\mathrm{d}s + \int_t^\infty f^*(s)\,\frac{\mathrm{d}s}{s} \tag{3.2}$$

for every $t\in(0,\infty)$, where the multiplicative constant depends only on Ω and n. Moreover, we have that ([21, p. 55], cf. [83, Lemma 2.1]), for every nonincreasing function $g\in\mathfrak{M}^+(0,\infty)$ satisfying (3.1) with f^* replaced by g, there is a locally integrable function f on $\mathbb{R}^n$ that is equimeasurable with g (and so f also satisfies (3.1)) and

$$\frac{1}{t}\int_0^t f^*(s)\,\mathrm{d}s + \int_t^\infty f^*(s)\,\frac{\mathrm{d}s}{s} \lesssim (Tf)^*(t) \quad \text{for every } t\in(0,\infty), \tag{3.3}$$

where the multiplicative constant depends only on Ω and n.

We shall utilize pointwise estimates (3.2) and (3.3) to characterize optimal rearrangement-invariant function spaces for T and $\mathcal{T}$. It should be noted that the optimal target space for the Hilbert transform H within the class of certain symmetric quasi-Banach function spaces was obtained in a somewhat implicit form in [92]. Furthermore, the optimal rearrangement-invariant function spaces for the Hilbert transform H and the Riesz potential I_γ, which we shall study in the next section, were also obtained in the author’s Master’s thesis [65] by ad hoc methods.

Having the pointwise estimates at our disposal, we can make the following useful observation.

Proposition 3.1.1. *Let $X(\mathbb{R}^n)$ and $Y(\mathbb{R}^n)$ be rearrangement-invariant function spaces. Let S_σ be the Calderón operator S_σ defined by* (2.19) *with $\sigma=(1,1,0,0)$ (and so $m=1$). The following four statements are equivalent:*

(i) $T\colon X(\mathbb{R}^n) \to Y(\mathbb{R}^n)$ *is bounded;*

(ii) $\mathcal{T}\colon X(\mathbb{R}^n) \to Y(\mathbb{R}^n)$ *is bounded;*

(iii) $S_\sigma\colon X(0,\infty) \to Y(0,\infty)$ *is bounded;*

(iv) $S_\sigma\colon Y'(0,\infty) \to X'(0,\infty)$ *is bounded.*

Proof. (*i*) *implies* (*iii*). Assume that $T\colon X(\mathbb{R}^n) \to Y(\mathbb{R}^n)$ is bounded. In particular, T is well defined on $X(\mathbb{R}^n)$. We would like to show that $S_\sigma\colon X(0,\infty) \to Y(0,\infty)$ is bounded. Owing to (2.20), it is sufficient to show that

$$\|S_\sigma(g^*)\|_{Y(0,\infty)} \lesssim \|g\|_{X(0,\infty)} \quad \text{for every } g \in \mathfrak{M}^+(0,\infty). \tag{3.4}$$

Let $g \in \mathfrak{M}^+(0,\infty)$. There is a function $f \in \mathfrak{M}^+(\mathbb{R}^n)$ such that $f^* = g^*$ ([8, Chapter 2, Corollary 7.8]). Let $\{f_k\}_{k=1}^\infty$ be a sequence of nonnegative simple functions on $\mathbb{R}^n$ such that $f_k \nearrow f$, $k \to \infty$. Since f_k, $k \in \mathbb{N}$, are simple functions, they satisfy (3.1); hence we have that

$$S_\sigma(f_k^*)(t) \lesssim (Tf_k)^*(t) \quad \text{for every } t \in (0,\infty) \text{ and every } k \in \mathbb{N}$$

thanks to (3.3). Consequently, we have that

$$\begin{aligned} \|S_\sigma(f_k^*)\|_{Y(0,\infty)} &\lesssim \|(Tf_k)^*\|_{Y(0,\infty)} = \|Tf_k\|_{Y(\mathbb{R}^n)} \\ &\lesssim \|f_k\|_{X(\mathbb{R}^n)} \leq \|f\|_{X(\mathbb{R}^n)} \\ &= \|g\|_{X(0,\infty)} \end{aligned} \tag{3.5}$$

for every $k \in \mathbb{N}$. Desired inequality (3.4) now follows from (3.5) combined with the monotone convergence theorem and the fact that $(f_k)^* \nearrow f^* = g^*$, $k \to \infty$.

(*iii*) *implies* (*i*). We first observe that (3.1) is satisfied for every $f \in X(\mathbb{R}^n)$, and so Tf is well defined on $X(\mathbb{R}^n)$. Indeed, thanks to the boundedness of S_σ, we have that

$$\|S_\sigma(f^*)\|_{Y(0,\infty)} \lesssim \|f^*\|_{X(0,\infty)} < \infty \quad \text{for every } f \in X(\mathbb{R}^n),$$

whence $S_\sigma(f^*)(t) < \infty$ for every $t \in (0,\infty)$ owing to (1.12) and the fact that $S_\sigma(f^*)$ is nonincreasing. In particular, $S_\sigma(f^*)(1) < \infty$ for every $f \in X(\mathbb{R}^n)$. The fact that $T\colon X(\mathbb{R}^n) \to Y(\mathbb{R}^n)$ is bounded now follows immediately from (*iii*) combined with (3.2).

(*ii*) *is equivalent to* (*iii*). This can be proved in the same way as the equivalence of (*i*) and (*iii*), and so we omit the proof.

(*iii*) *is equivalent to* (*iv*). Since $\sigma' = \sigma$, where σ' is defined by (2.24), this follows immediately from Proposition 2.2.20. □

The following theorem describes the optimal target space for T and $\mathcal{T}$.

Theorem 3.1.2. *Let* $X(\mathbb{R}^n)$ *be a rearrangement-invariant function space such that*

$$\eta \in X'(0,\infty), \tag{3.6}$$

where

$$\eta(t) = \log\Big(\frac{e}{t}\Big)\chi_{(0,1)}(t) + \frac{1}{t}\chi_{[1,\infty)}(t),\ t \in (0,\infty). \tag{3.7}$$

The rearrangement-invariant function space $Y(\mathbb{R}^n)$ whose associate function norm is defined as

$$\|f\|_{Y'(0,\infty)} = \|S_\sigma(f^*)\|_{X'(0,\infty)}, \ f \in \mathfrak{M}^+(0,\infty),$$

where $\sigma = (1,1,0,0)$, is the optimal target space for the operator T and the space $X(\mathbb{R}^n)$.

Moreover, if (3.6) is not satisfied, then there is no rearrangement-invariant function space $Z(\mathbb{R}^n)$ such that $T\colon X(\mathbb{R}^n) \to Z(\mathbb{R}^n)$ is bounded.

Finally, exactly the same holds when T is replaced by $\mathcal{T}$.

Proof. Proposition 3.1.1 tells us that $Y(\mathbb{R}^n)$ is the optimal target space for the operator T and the space $X(\mathbb{R}^n)$ if and only if $Y(0,\infty)$ is the optimal target space for the operator S_σ and the space $X(0,\infty)$. Consequently, the theorem follows from Proposition 2.2.21 (note that $\sigma = \sigma'$). □

When $X(\mathbb{R}^n)$ is a Lorentz–Zygmund space, the optimal target space for T (or $\mathcal{T}$) and $X(\mathbb{R}^n)$ reads as follows. We obtain the result by combining Proposition 3.1.1 and Proposition 2.2.52.

Theorem 3.1.3. *Assume that $\|\cdot\|_{L^{p,q;\mathbb{A}}(0,\infty)}$ is equivalent to a rearrangement-invariant function norm, that is, the parameters p, q and $\mathbb{A} = (\alpha_0, \alpha_\infty)$ satisfy one of conditions (1.28) (with $\beta_0 = \beta_\infty = 0$). We have that*

$$T\colon L^{p,q;\mathbb{A}}(\mathbb{R}^n) \to Y(\mathbb{R}^n)$$

is bounded, where

$$Y(\mathbb{R}^n) = \begin{cases} L^{1,1;\mathbb{A}-1}(\mathbb{R}^n), & p = q = 1,\ \alpha_0 \geq 1,\ \alpha_\infty < 0; \\ Y_1(\mathbb{R}^n), & p = q = 1,\ \alpha_0 \geq 1,\ \alpha_\infty = 0; \\ L^{p,q;\mathbb{A}}(\mathbb{R}^n), & p \in (1,\infty); \\ L^{\infty,\infty;\mathbb{A}-1}(\mathbb{R}^n), & p = q = \infty,\ \alpha_0 \leq 0,\ \alpha_\infty > 1; \\ Y_2(\mathbb{R}^n), & p = \infty,\ q = 1,\ \alpha_0 + 1 < 0,\ \alpha_\infty \geq 0 \text{ or} \\ & p = \infty,\ q \in (1,\infty),\ \alpha_0 + \frac{1}{q} < 0,\ \alpha_\infty > 1 - \frac{1}{q}, \end{cases}$$

where $Y_1(\mathbb{R}^n)$ and $Y_2(\mathbb{R}^n)$ are the rearrangement-invariant function spaces whose associate function norms satisfy

$$\|f\|_{Y_1'(0,\infty)} \approx \sup_{t \in (0,1)} \ell^{-\alpha_0+1}(t) f^*(t) + \int_1^\infty f^{**}(s) \frac{\mathrm{d}s}{s},$$

$$\|f\|_{Y_2'(0,\infty)} \approx \left\| t^{-\frac{1}{q'}} \ell^{-\mathbb{A}-1}(t) \int_0^t f^{**}(s) \, \mathrm{d}s \right\|_{L^{q'}(0,\infty)},$$

for every $f \in \mathfrak{M}^+(0,\infty)$. Moreover, $Y(\mathbb{R}^n)$ is the optimal target space for T and the space $L^{p,q;\mathbb{A}}(\mathbb{R}^n)$. Furthermore, if $p = q = 1$, $\alpha_0 \in [0,1)$, $\alpha_\infty \leq 0$, or $p = \infty$, $q = 1$, $\alpha_0 + 1 < 0$, $\alpha_\infty < 0$, or $p = \infty$, $q \in (1,\infty)$, $\alpha_0 + \frac{1}{q} < 0$, $\alpha_\infty \leq 1 - \frac{1}{q}$, or $p = q = \infty$, $\alpha_0 \leq 0$, $\alpha_\infty \leq 1$, then there is no rearrangement-invariant function space $Z(\mathbb{R}^n)$ such that $T\colon L^{p,q;\mathbb{A}}(\mathbb{R}^n) \to Z(\mathbb{R}^n)$ is bounded.

Finally, exactly the same holds when T is replaced by $\mathcal{T}$.

The following theorem describes the optimal domain space for T and $\mathcal{T}$.

Theorem 3.1.4. *Let $Y(\mathbb{R}^n)$ be a rearrangement-invariant function space such that*

$$\eta \in Y(0,\infty), \tag{3.8}$$

where η is defined by (3.7)*. The rearrangement-invariant function space $X(\mathbb{R}^n)$ whose function norm is defined as*

$$\|f\|_{X(0,\infty)} = \|S_\sigma(f^*)\|_{Y(0,\infty)},\ f \in \mathfrak{M}^+(0,\infty),$$

where $\sigma = (1,1,0,0)$, is the optimal domain space for the operator T and the space $Y(\mathbb{R}^n)$.

Moreover, if (3.8) *is not satisfied, then there is no rearrangement-invariant function space $Z(\mathbb{R}^n)$ such that $T\colon Z(\mathbb{R}^n) \to Y(\mathbb{R}^n)$ is bounded.*

Finally, exactly the same holds when T is replaced by $\mathcal{T}$.

Proof. Proposition 3.1.1 tells us that $X(\mathbb{R}^n)$ is the optimal domain space for the operator T and the space $Y(\mathbb{R}^n)$ if and only if $X(0,\infty)$ is the optimal domain space for the operator S_σ and the space $Y(0,\infty)$. Consequently, the theorem follows from Proposition 2.2.19. ☐

When $Y(\mathbb{R}^n)$ is a Lorentz–Zygmund space, the optimal domain space for T (or $\mathcal{T}$) and $Y(\mathbb{R}^n)$ reads as follows. We obtain the result by combining Proposition 3.1.1 and Proposition 2.2.48.

Theorem 3.1.5. *Assume that $\|\cdot\|_{L^{p,q;\mathbb{A}}(0,\infty)}$ is equivalent to a rearrangement-invariant function norm, that is, the parameters p, q and $\mathbb{A} = (\alpha_0, \alpha_\infty)$ satisfy one of conditions* (1.28) *(with $\beta_0 = \beta_\infty = 0$). We have that*

$$T\colon X(\mathbb{R}^n) \to L^{p,q;\mathbb{A}}(\mathbb{R}^n)$$

is bounded, where

$$X(\mathbb{R}^n) = \begin{cases} L^{1,1;\mathbb{A}+1}(\mathbb{R}^n), & p=1,\ q=1,\ \alpha_0 \geq 0,\ \alpha_\infty + 1 < 0; \\ L^{p,q;\mathbb{A}}(\mathbb{R}^n), & p \in (1,\infty); \\ L^{\infty,q;\mathbb{A}+1}(\mathbb{R}^n), & p=\infty,\ q\in[1,\infty),\ \alpha_0 + 1 + \frac{1}{q} < 0,\ \alpha_\infty + \frac{1}{q} > 0 \text{ or} \\ & p=q=\infty,\ \alpha_0 + 1 \leq 0,\ \alpha_\infty > 0; \\ X_1(\mathbb{R}^n), & p=\infty,\ q\in[1,\infty),\ \alpha_0 + 1 + \frac{1}{q} < 0,\ \alpha_\infty + \frac{1}{q} \leq 0 \text{ or} \\ & p=q=\infty,\ \alpha_0 + 1 \leq 0,\ \alpha_\infty \leq 0, \end{cases}$$

where $X_1(\mathbb{R}^n)$ is the rearrangement-invariant function space whose function norm satisfies

$$\|f\|_{X_1(0,\infty)} \approx \left\| t^{-\frac{1}{q}} \ell^{\mathbb{A}}(t) \int_t^\infty f^{**}(s) s^{-1}\, \mathrm{d}s \right\|_{L^q(0,\infty)} \quad \text{for every } f \in \mathfrak{M}^+(0,\infty).$$

Moreover, $X(\mathbb{R}^n)$ is the optimal domain space for T and the space $L^{p,q;\mathbb{A}}(\mathbb{R}^n)$. Furthermore, if $p = q = 1$, $\alpha_0 \geq 0$ and $\alpha_\infty \in [-1,0]$, or $p = \infty$, $q \in [1,\infty)$, $\alpha_0 \in [-1-\frac{1}{q}, -\frac{1}{q})$, or $p = q = \infty$, $\alpha_0 \in (-1,0]$, then there is no rearrangement-invariant function space $Z(\mathbb{R}^n)$ such that $T\colon Z(\mathbb{R}^n) \to L^{p,q;\mathbb{A}}(\mathbb{R}^n)$ is bounded.

Finally, exactly the same holds when T is replaced by $\mathcal{T}$.

We conclude this section with the following observation.

Proposition 3.1.6. *If $X(\mathbb{R}^n)$ is a rearrangement-invariant function space such that $T\colon X(\mathbb{R}^n) \to X(\mathbb{R}^n)$ is bounded, then $X(\mathbb{R}^n)$ is the optimal partner space for T, that is, $X(\mathbb{R}^n)$ is the optimal domain space for the operator T and the space $X(\mathbb{R}^n)$ and, simultaneously, $X(\mathbb{R}^n)$ is the optimal target space for the operator T and the space $X(\mathbb{R}^n)$.*

Moreover, exactly the same holds when T is replaced by $\mathcal{T}$.

Proof. Assume that $T\colon X(\mathbb{R}^n) \to X(\mathbb{R}^n)$ is bounded. Let $Y(\mathbb{R}^n)$ and $Z(\mathbb{R}^n)$ be rearrangement-invariant function spaces such that $T\colon X(\mathbb{R}^n) \to Y(\mathbb{R}^n)$ and $T\colon Z(\mathbb{R}^n) \to X(\mathbb{R}^n)$ are bounded. We need to show that $X(\mathbb{R}^n) \hookrightarrow Y(\mathbb{R}^n)$ and $Z(\mathbb{R}^n) \hookrightarrow X(\mathbb{R}^n)$. Thanks to Proposition 3.1.1, we know that $S_\sigma\colon X(0,\infty) \to Y(0,\infty)$ and $S_\sigma\colon X'(0,\infty) \to Z'(0,\infty)$, where $\sigma = (1,1,0,0)$, are bounded. Consequently, we have that

$$\|f\|_{Y(0,\infty)} \le \|f^{**}\|_{Y(0,\infty)} \le \|S_\sigma(f^*)\|_{Y(0,\infty)} \lesssim \|f\|_{X(0,\infty)}$$

and that

$$\|f\|_{Z'(0,\infty)} \le \|f^{**}\|_{Z'(0,\infty)} \le \|S_\sigma(f^*)\|_{Z'(0,\infty)} \lesssim \|f\|_{X'(0,\infty)}$$

for every $f \in \mathfrak{M}^+(0,\infty)$. Hence $X(\mathbb{R}^n) \hookrightarrow Y(\mathbb{R}^n)$ and $X'(\mathbb{R}^n) \hookrightarrow Z'(\mathbb{R}^n)$. Finally, recall that the latter is equivalent to $Z(\mathbb{R}^n) \hookrightarrow X(\mathbb{R}^n)$ thanks to (1.15). ☐

3.2 The Riesz potential

The Riesz potential I_γ of order $\gamma \in (0,n)$, $n \ge 2$, is defined as

$$I_\gamma f(x) = c(\gamma,n) \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\gamma}} \,\mathrm{d}y,\ x \in \mathbb{R}^n,$$

for all locally integrable functions f on $\mathbb{R}^n$ for which the integral converges for a.e. $x \in \mathbb{R}^n$, where $c(\gamma,n)$ is a certain positive constant depending on n and γ, whose exact value is completely immaterial for our purposes (cf. [87, p. 117]). If f is a locally integrable function on $\mathbb{R}^n$ such that

$$\int_0^1 f^*(s)\,\mathrm{d}s + \int_1^\infty f^*(s) s^{-1+\frac{\gamma}{n}}\,\mathrm{d}s < \infty, \tag{3.9}$$

then it follows from the O'Neil convolution inequality [73, Lemma 1.5] that $I_\gamma f$ is well defined and

$$(I_\gamma f)^*(t) \lesssim t^{\frac{\gamma}{n}-1} \int_0^t f^*(s)\,\mathrm{d}s + \int_t^\infty f^*(s) s^{-1+\frac{\gamma}{n}}\,\mathrm{d}s \quad \text{for every } t \in (0,\infty), \tag{3.10}$$

where the multiplicative constant depends only on n and γ. On the other hand, for every nonincreasing function $g \in \mathfrak{M}^+(0,\infty)$ satisfying (3.9) with f^* replaced by g, there is a function $f \in \mathfrak{M}^+(\mathbb{R}^n)$ equimeasurable with g (namely $f(x) = g(\omega_n|x|^n)$, $x \in \mathbb{R}^n$) such that

$$t^{\frac{\gamma}{n}-1} \int_0^t f^*(s)\,\mathrm{d}s + \int_t^\infty f^*(s) s^{-1+\frac{\gamma}{n}}\,\mathrm{d}s \lesssim (I_\gamma f)^*(t) \quad \text{for every } t \in (0,\infty), \tag{3.11}$$

where the multiplicative constant depends only on n and γ. This fact was basically observed already by R. O'Neil ([73, p. 142]), and its proof can be found, for example, in [38, Lemma 3.4].

Having the (in a sense) sharp pointwise estimate of the nonincreasing rearrangement of the Riesz pontential at our disposal, we are now ready to characterize optimal rearrangement-invariant function spaces for I_γ. The proofs of the following results concerning I_γ are carried out along the same lines as their counterparts in Section 3.1. We start off with a reduction principle for I_γ.

Proposition 3.2.1. *Let $n \in \mathbb{N}$, $n \geq 2$, and $\gamma \in (0,n)$. Let $X(\mathbb{R}^n)$ and $Y(\mathbb{R}^n)$ be rearrangement-invariant function spaces. Let S_σ be the Calderón operator S_σ defined by* (2.19) *with $\sigma = (1, \frac{n-\gamma}{n}, \frac{\gamma}{n}, 0)$ (and so $m = 1$). The following three statements are equivalent:*

(i) $I_\gamma\colon X(\mathbb{R}^n) \to Y(\mathbb{R}^n)$ *is bounded;*

(ii) $S_\sigma\colon X(0,\infty) \to Y(0,\infty)$ *is bounded;*

(iii) $S_\sigma\colon Y'(0,\infty) \to X'(0,\infty)$ *is bounded.*

Proof. (*i*) *implies* (*ii*). Assume that $I_\gamma\colon X(\mathbb{R}^n) \to Y(\mathbb{R}^n)$ is bounded. In particular, I_γ is well defined on $X(\mathbb{R}^n)$. We would like to show that $S_\sigma\colon X(0,\infty) \to Y(0,\infty)$ is bounded. Owing to (2.20), it is sufficient to show that

$$\|S_\sigma(g^*)\|_{Y(0,\infty)} \lesssim \|g\|_{X(0,\infty)} \quad \text{for every } g \in \mathfrak{M}^+(0,\infty). \tag{3.12}$$

Let $g \in \mathfrak{M}^+(0,\infty)$. There is a function $f \in \mathfrak{M}^+(\mathbb{R}^n)$ such that $f^* = g^*$ ([8, Chapter 2, Corollary 7.8]). Let $\{f_k\}_{k=1}^\infty$ be a sequence of nonnegative simple functions on $\mathbb{R}^n$ such that $f_k \nearrow f$, $k \to \infty$. Since f_k, $k \in \mathbb{N}$, are simple functions, they satisfy (3.9); hence we have that

$$S_\sigma(f_k^*)(t) \lesssim (I_\gamma f_k)^*(t) \quad \text{for every } t \in (0,\infty) \text{ and every } k \in \mathbb{N}$$

thanks to (3.11). Consequently, we have that

$$\begin{aligned} \|S_\sigma(f_k^*)\|_{Y(0,\infty)} &\lesssim \|(I_\gamma f_k)^*\|_{Y(0,\infty)} = \|I_\gamma f_k\|_{Y(\mathbb{R}^n)} \\ &\lesssim \|f_k\|_{X(\mathbb{R}^n)} \leq \|f\|_{X(\mathbb{R}^n)} \\ &= \|g\|_{X(0,\infty)} \end{aligned} \tag{3.13}$$

for every $k \in \mathbb{N}$. Desired inequality (3.12) now follows from (3.13) combined with the monotone convergence theorem and the fact that $(f_k)^* \nearrow f^* = g^*$, $k \to \infty$.

(*ii*) *implies* (*i*). We first observe that (3.9) is satisfied for every $f \in X(\mathbb{R}^n)$, and so $I_\gamma f$ is well defined on $X(\mathbb{R}^n)$. Indeed, thanks to the boundedness of S_σ, we have that

$$\|S_\sigma(f^*)\|_{Y(0,\infty)} \lesssim \|f^*\|_{X(0,\infty)} < \infty \quad \text{for every } f \in X(\mathbb{R}^n),$$

whence $S_\sigma(f^*)(t) < \infty$ for every $t \in (0,\infty)$ owing to (1.12) and the fact that $S_\sigma(f^*)$ is nonincreasing. In particular, $S_\sigma(f^*)(1) < \infty$ for every $f \in X(\mathbb{R}^n)$. The fact that $I_\gamma\colon X(\mathbb{R}^n) \to Y(\mathbb{R}^n)$ is bounded now follows immediately from (*ii*) combined with (3.10).

(*ii*) *is equivalent to* (*iii*). Since $\sigma' = \sigma$, where σ' is defined by (2.24), this follows immediately from Proposition 2.2.20. □

The following theorem describes the optimal target space for I_γ.

Theorem 3.2.2. *Let $n \in \mathbb{N}$, $n \geq 2$, and $\gamma \in (0, n)$. Let $X(\mathbb{R}^n)$ be a rearrangement-invariant function space such that*

$$\eta \in X'(0, \infty), \tag{3.14}$$

where

$$\eta(t) = (t+1)^{-1+\frac{\gamma}{n}}, \ t \in (0, \infty). \tag{3.15}$$

The rearrangement-invariant function space $Y(\mathbb{R}^n)$ whose associate function norm is defined as

$$\|f\|_{Y'(0,\infty)} = \|S_\sigma(f^*)\|_{X'(0,\infty)}, \ f \in \mathfrak{M}^+(0, \infty),$$

where $\sigma = (1, \frac{n-\gamma}{n}, \frac{\gamma}{n}, 0)$, is the optimal target space for the operator I_γ and the space $X(\mathbb{R}^n)$.

Moreover, if (3.14) *is not satisfied, then there is no rearrangement-invariant function space $Z(\mathbb{R}^n)$ such that $I_\gamma \colon X(\mathbb{R}^n) \to Z(\mathbb{R}^n)$ is bounded.*

Proof. Proposition 3.2.1 tells us that $Y(\mathbb{R}^n)$ is the optimal target space for the operator I_γ and the space $X(\mathbb{R}^n)$ if and only if $Y(0, \infty)$ is the optimal target space for the operator S_σ and the space $X(0, \infty)$. Consequently, the assertion follows from Proposition 2.2.21 (note that $\sigma = \sigma'$). ☐

While the theorem above, describing the optimal target space for I_γ and a rearrangement-invariant function space $X(\mathbb{R}^n)$, is quite implicit, the next theorem provides us with an explicit description of the optimal target space (save for one case) when $X(\mathbb{R}^n)$ is a Lorentz–Zygmund space. The result is a consequence of Proposition 2.2.50 with $\beta = \frac{n-\gamma}{n}$ together with Proposition 3.2.1.

Theorem 3.2.3. *Let $\gamma \in (0, n)$. Assume that $\| \cdot \|_{L^{p,q;\mathbb{A}}(0,\infty)}$ is equivalent to a rearrangement-invariant function norm, that is, the parameters p, q and $\mathbb{A} = (\alpha_0, \alpha_\infty)$ satisfy one of conditions* (1.28) *(with $\beta_0 = \beta_\infty = 0$). We have that*

$$I_\gamma \colon L^{p,q;\mathbb{A}}(\mathbb{R}^n) \to Y(\mathbb{R}^n)$$

is bounded, where

$$Y(\mathbb{R}^n) = \begin{cases} L^{\frac{n}{n-\gamma},\infty}(\mathbb{R}^n), & p = q = 1, \ \alpha_0 = \alpha_\infty = 0; \\ Y_1(\mathbb{R}^n), & p = q = 1, \ \alpha_0 \geq 0, \ \alpha_\infty \leq 0, \ |\alpha_0| + |\alpha_\infty| > 0; \\ L^{\frac{np}{n-\gamma p},q;\mathbb{A}}(\mathbb{R}^n), & p \in (1, \frac{n}{\gamma}); \\ L^{\infty,q;\mathbb{A}-1}(\mathbb{R}^n), & p = \frac{n}{\gamma}, \ \alpha_0 < 1 - \frac{1}{q}, \ \alpha_\infty > 1 - \frac{1}{q}; \\ Y_2(\mathbb{R}^n), & p = \frac{n}{\gamma}, \ q \in [1, \infty), \ \alpha_0 > 1 - \frac{1}{q}, \ \alpha_\infty > 1 - \frac{1}{q}; \\ L^{\infty,\infty;(0,\alpha_\infty-1)}(\mathbb{R}^n), & p = \frac{n}{\gamma}, \ q = \infty, \ \alpha_0 > 1, \ \alpha_\infty > 1; \\ L^{\infty,1;(-1,\alpha_\infty-1),(-1,0),(-1,0)}(\mathbb{R}^n), & p = \frac{n}{\gamma}, \ q = 1, \ \alpha_0 = 0, \ \alpha_\infty > 0; \\ L^{\infty,q;(-\frac{1}{q},\alpha_\infty-1),(-1,0)}(\mathbb{R}^n), & p = \frac{n}{\gamma}, \ q \in (1, \infty], \ \alpha_0 = 1 - \frac{1}{q}, \ \alpha_\infty > 1 - \frac{1}{q}; \\ Y_3(\mathbb{R}^n), & p = \frac{n}{\gamma}, \ q = 1, \ \alpha_0 < 0, \ \alpha_\infty = 0; \\ L^\infty(\mathbb{R}^n), & p = \frac{n}{\gamma}, \ q = 1, \ \alpha_0 \geq 0, \ \alpha_\infty = 0, \end{cases}$$

where $Y_1(\mathbb{R}^n)$ is the rearrangement-invariant function space whose associate function norm satisfies

$$\|f\|_{Y_1'(0,\infty)} \approx \sup_{t\in(0,\infty)} \ell^{-\mathbb{A}}(t) \int_t^\infty f^{**}(s) s^{\frac{\gamma}{n}-1}\,\mathrm{d}s \quad \text{for every } f \in \mathfrak{M}^+(0,\infty),$$

and where $Y_2(\mathbb{R}^n)$ and $Y_3(\mathbb{R}^n)$ are the rearrangement-invariant function spaces whose function norms satisfy

$$\begin{aligned} \|f\|_{Y_2(0,\infty)} &\approx \|t^{-\frac{1}{q}} \ell^{\alpha_\infty - 1}(t) f^*(t) \chi_{(1,\infty)}(t)\|_{L^q(0,\infty)} + \|f\|_{L^\infty(0,\infty)}, \\ \|f\|_{Y_3(0,\infty)} &\approx \|t^{-1} \ell^{\alpha_0 - 1}(t) f^*(t)\|_{L^1(0,1)}, \end{aligned}$$

for every $f \in \mathfrak{M}(0,\infty)$. Moreover, $Y(\mathbb{R}^n)$ is the optimal target space for I_γ and the space $L^{p,q;\mathbb{A}}(\mathbb{R}^n)$. Furthermore, if $p = \frac{n}{\gamma}$, $q = 1$, $\alpha_\infty < 0$, or $p = \frac{n}{\gamma}$, $q \in (1,\infty]$, $\alpha_\infty \le 1 - \frac{1}{q}$, or $p > \frac{n}{\gamma}$, then there is no rearrangement-invariant function space $Z(\mathbb{R}^n)$ such that $I_\gamma \colon L^{p,q;\mathbb{A}}(\mathbb{R}^n) \to Z(\mathbb{R}^n)$ is bounded.

The following theorem describes the optimal domain space for I_γ.

Theorem 3.2.4. *Let $n \in \mathbb{N}$, $n \ge 2$, and $\gamma \in (0,n)$. Let $Y(\mathbb{R}^n)$ be a rearrangement-invariant function space such that*

$$\eta \in Y(0,\infty), \tag{3.16}$$

where η is defined by (3.15). The rearrangement-invariant function space $X(\mathbb{R}^n)$ whose function norm is defined as

$$\|f\|_{X(0,\infty)} = \|S_\sigma(f^*)\|_{Y(0,\infty)},\ f \in \mathfrak{M}^+(0,\infty),$$

where $\sigma = (1, \frac{n-\gamma}{n}, \frac{\gamma}{n}, 0)$, is the optimal domain space for the operator I_γ and the space $Y(\mathbb{R}^n)$.

Moreover, if (3.16) is not satisfied, then there is no rearrangement-invariant function space $Z(\mathbb{R}^n)$ such that $I_\gamma \colon Z(\mathbb{R}^n) \to Y(\mathbb{R}^n)$ is bounded.

Proof. Proposition 3.2.1 tells us that $X(\mathbb{R}^n)$ is the optimal domain space for the operator I_γ and the space $Y(\mathbb{R}^n)$ if and only if $X(0,\infty)$ is the optimal domain space for the operator S_σ and the space $Y(0,\infty)$. Consequently, the theorem follows from Proposition 2.2.19. ☐

We obtain a description of the optimal domain space for I_γ and a Lorentz–Zygmund space $Y(\mathbb{R}^n)$ by combining Proposition 3.2.1 and Proposition 2.2.46 with $\beta = \frac{n-\gamma}{n}$.

Theorem 3.2.5. *Let $\gamma \in (0,n)$. Assume that $\|\cdot\|_{L^{p,q;\mathbb{A}}(0,\infty)}$ is equivalent to a rearrangement-invariant function norm, that is, the parameters p, q and $\mathbb{A} = (\alpha_0, \alpha_\infty)$ satisfy one of conditions (1.28) (with $\beta_0 = \beta_\infty = 0$). We have that*

$$I_\gamma \colon X(\mathbb{R}^n) \to L^{p,q;\mathbb{A}}(\mathbb{R}^n)$$

is bounded, where

$$X(\mathbb{R}^n) = \begin{cases} L^{1,1;(0,\alpha_\infty+1)}(\mathbb{R}^n), & p = \frac{n}{n-\gamma},\ q = 1,\ \alpha_0 + 1 < 0,\ \alpha_\infty + 1 < 0; \\ L^{1,1;\mathbb{A}+1}(\mathbb{R}^n), & p = \frac{n}{n-\gamma},\ q = 1,\ \alpha_0 + 1 > 0,\ \alpha_\infty + 1 < 0; \\ L^{1,1;(0,\alpha_\infty+1),(1,0)}(\mathbb{R}^n), & p = \frac{n}{n-\gamma},\ q = 1,\ \alpha_0 + 1 = 0,\ \alpha_\infty + 1 < 0; \\ L^{(1,q;\mathbb{A})}(\mathbb{R}^n), & p = \frac{n}{n-\gamma},\ q \in (1,\infty),\ \alpha_\infty + \frac{1}{q} < 0 \text{ or} \\ & p = \frac{n}{n-\gamma},\ q = \infty,\ \alpha_\infty \le 0; \\ L^{\frac{np}{n+\gamma p},q;\mathbb{A}}(\mathbb{R}^n), & p \in (\frac{n}{n-\gamma},\infty); \\ L^{\frac{n}{\gamma},1}(\mathbb{R}^n), & p = q = \infty,\ \alpha_0 = \alpha_\infty = 0; \\ X_1(\mathbb{R}^n), & p = \infty,\ q \in [1,\infty),\ \alpha_0 + \frac{1}{q} < 0 \text{ or} \\ & p = q = \infty,\ \alpha_0 \le 0,\ |\alpha_0| + |\alpha_\infty| > 0, \end{cases}$$

where $X_1(\mathbb{R}^n)$ is the rearrangement-invariant function space whose function norm satisfies

$$\|f\|_{X_1(0,\infty)} \approx \left\| t^{-\frac{1}{q}} \ell^{\mathbb{A}}(t) \int_t^\infty f^{**}(s) s^{\frac{\gamma}{n}-1}\,\mathrm{d}s \right\|_{L^q(0,\infty)} \quad \text{for every } f \in \mathfrak{M}^+(0,\infty).$$

Moreover, $X(\mathbb{R}^n)$ is the optimal domain space for I_γ and the space $L^{p,q;\mathbb{A}}(\mathbb{R}^n)$. Furthermore, if $p = \frac{n}{n-\gamma}$, $q = \infty$, $\alpha_\infty > 0$, or $p = \frac{n}{n-\gamma}$, $q \in [1,\infty)$, $\alpha_\infty + \frac{1}{q} \ge 0$, or $p < \frac{n}{n-\gamma}$, then there is no rearrangement-invariant function space $Z(\mathbb{R}^n)$ such that $I_\gamma \colon Z(\mathbb{R}^n) \to L^{p,q;\mathbb{A}}(\mathbb{R}^n)$ is bounded.

3.3 The Hardy–Littlewood maximal operator

The Hardy–Littlewood maximal operator M, which is defined for every $f \in \mathfrak{M}(\mathbb{R}^n)$ as

$$Mf(x) = \sup_{x \ni Q} \frac{1}{|Q|} \int_Q |f(y)|\,\mathrm{d}y,\ x \in \mathbb{R}^n,$$

where the supremum is taken over all cubes in $\mathbb{R}^n$ whose edges are parallel to the coordinate axes of $\mathbb{R}^n$, and its numerous variations have been indispensable in various areas of mathematical analysis ever since G.H. Hardy and J.E. Littlewood introduced it (in the one-dimensional setting) in [47]. The following sharp pointwise estimate of the nonincreasing rearrangement of M is well known. We have that

$$(Mf)^*(t) \approx f^{**}(t) \quad \text{for every } t \in (0,\infty), \tag{3.17}$$

where the multiplicative constants depend only on n. The upper bound on $(Mf)^*$ dates back to the 1930s, during which it was established by F. Riesz in [81] ($n = 1$) and N. Wiener in [96] ($n \in \mathbb{N}$). The lower bound was proved a few decades afterwards by C. Hertz ($n = 1$) in [49] and C. Bennett and R. Sharpley ($n \in \mathbb{N}$) in [7].

We shall combine (3.17) with results obtained in Section 2.2.1 to characterize optimal rearrangement-invariant function spaces for M. It should be noted that optimal function spaces for M within a certain (more general) class of Banach function spaces were also characterize in [35, 86]. It should also be noted that the results of this section could be obtained by substantially more direct methods without the “heavy machinery” of Section 2.2.1. However, since we already have the machinery at our disposal (and we do not pay for using it), we will take advantage of it. Furthermore, while we could get by without the machinery in this section, we would not get far without it in Section 3.4. For the interested reader, we took a considerably more elementary approach to characterizing optimal function spaces for M in [40].

Throughout this section, we set $R = R_{u,v,\nu}$ and $H = H_{u,v,\nu}$, where $R_{u,v,\nu}$ and $H_{u,v,\nu}$ are the operators defined by (2.1) and (2.2), respectively, with $u \equiv 1$, $v(t) = t^{-1}$, $t \in (0,\infty)$, $\nu = 1$ and $L = \infty$.

Proposition 3.3.1. *Let $X(\mathbb{R}^n)$ and $Y(\mathbb{R}^n)$ be rearrangement-invariant function spaces. The following three statements are equivalent:*

(i) $M\colon X(\mathbb{R}^n) \to Y(\mathbb{R}^n)$ *is bounded;*

(ii) $R\colon X(0,\infty) \to Y(0,\infty)$ *is bounded;*

(iii) $H\colon Y'(0,\infty) \to X'(0,\infty)$ *is bounded.*

Proof. (*i*) *implies* (*ii*). Note that $f^{**} = R(f^*)$ for every $f \in \mathfrak{M}(\mathbb{R}^n)$. Let $g \in \mathfrak{M}(0,\infty)$. Set $f(x) = g^*(\omega_n|x|^n)$, $x \in \mathbb{R}^n$. Note that f is equimeasurable with g. Furthermore, we have that

$$\begin{aligned}\|Rg\|_{Y(0,\infty)} \leq \|R(f^*)\|_{Y(0,\infty)} &\lesssim \|(Mf)^*\|_{Y(0,\infty)} = \|Mf\|_{Y(\mathbb{R}^n)}\\ &\lesssim \|f\|_{X(\mathbb{R}^n)} = \|g\|_{X(0,\infty)}\end{aligned}$$

thanks to the Hardy–Littlewood inequality (1.3), (3.17) and (*i*), whence (*ii*) follows.

(*ii*) *implies* (*i*). This is an immediate consequence of (3.17).

(*iii*) *is equivalent to* (*ii*). This is Proposition 2.2.2. ☐

The following theorem describes the optimal target space for M.

Theorem 3.3.2. *Let $X(\mathbb{R}^n)$ be a rearrangement-invariant function space such that*

$$\eta \in X'(0,\infty), \tag{3.18}$$

where

$$\eta(t) = \log\Big(\frac{1}{t}\Big)\chi_{(0,1)}(t),\ t \in (0,\infty).$$

The rearrangement-invariant function space $Y(\mathbb{R}^n)$ whose associate function norm is defined as

$$\|f\|_{Y'(0,\infty)} = \Big\| \int_t^\infty f^*(s)\,\frac{\mathrm{d}s}{s} \Big\|_{X'(0,\infty)},\ f \in \mathfrak{M}^+(0,\infty),$$

is the optimal target space for the operator M and the space $X(\mathbb{R}^n)$.

Moreover, if (3.18) *is not satisfied, then there is no rearrangement-invariant function space $Z(\mathbb{R}^n)$ such that $M\colon X(\mathbb{R}^n) \to Z(\mathbb{R}^n)$ is bounded.*

Proof. Owing to Proposition 3.3.1, $Y(\mathbb{R}^n)$ is the optimal target space for M and the space $X(\mathbb{R}^n)$ if and only if $Y(0,\infty)$ is the optimal target space for R and the space $X(0,\infty)$. Thanks to that, the claim follows immediately from Theorem 2.2.6. To this end, note that (2.37) is actually equality in our setting because $T_\varphi f = f^*$, where T_φ is defined by (2.17) with $\varphi \equiv 1$, and so its norm on $X(0,\infty)$ is equal to one. □

When $X(\mathbb{R}^n)$ is a Lorentz–Zygmund space, Theorem 3.3.2 reads as follows. The result follows from Proposition 3.3.1 and Proposition 2.2.30 with $\nu = 1$.

Theorem 3.3.3. *Assume that* $\|\cdot\|_{L^{p,q;\mathbb{A}}(0,\infty)}$ *is equivalent to a rearrangement-invariant function norm, that is, the parameters* p, q *and* $\mathbb{A} = (\alpha_0, \alpha_\infty)$ *satisfy one of conditions* (1.28) *(with* $\beta_0 = \beta_\infty = 0$*). We have that*

$$M\colon L^{p,q;\mathbb{A}}(\mathbb{R}^n) \to Y(\mathbb{R}^n)$$

is bounded, where

$$Y(\mathbb{R}^n) = \begin{cases} L^{1,1;\mathbb{A}-1}(\mathbb{R}^n), & p=q=1,\ \alpha_0 \geq 1,\ \alpha_\infty < 0; \\ Y_1(\mathbb{R}^n), & p=q=1,\ \alpha_0 \geq 1,\ \alpha_\infty = 0; \\ L^{p,q;\mathbb{A}}(\mathbb{R}^n), & p \in (1,\infty) \text{ or} \\ & p=\infty,\ q\in[1,\infty),\ \alpha_0 + \frac{1}{q} < 0 \text{ or} \\ & p=q=\infty,\ \alpha_0 \leq 0, \end{cases}$$

where $Y_1(\mathbb{R}^n)$ *is the rearrangement-invariant function space whose associate function norm satisfies*

$$\|f\|_{Y_1'(0,\infty)} \approx \sup_{t\in(0,1)} \ell^{-\alpha_0}(t) \int_t^\infty f^*(s)\,\frac{\mathrm{d}s}{s} + \int_1^\infty f^*(s)\,\frac{\mathrm{d}s}{s},\ f \in \mathfrak{M}^+(0,\infty).$$

Moreover, $Y(\mathbb{R}^n)$ *the optimal target spaces for* M *and the space* $L^{p,q;\mathbb{A}}(\mathbb{R}^n)$*. Furthermore, if* $p=q=1$, $\alpha_0 \in [0,1)$ *and* $\alpha_\infty \leq 0$*, then there is no rearrangement-invariant function space* $Z(\mathbb{R}^n)$ *such that* $M\colon L^{p,q;\mathbb{A}}(\mathbb{R}^n) \to Z(\mathbb{R}^n)$ *is bounded.*

The following theorem describes the optimal domain space for M.

Theorem 3.3.4. *Let* $Y(\mathbb{R}^n)$ *be a rearrangement-invariant function space such that*

$$\eta \in Y(0,\infty), \tag{3.19}$$

where

$$\eta(t) = \frac{1}{t+1},\ t\in(0,\infty).$$

The rearrangement-invariant function space $X(\mathbb{R}^n)$ *whose function norm is defined as*

$$\|f\|_{X(0,\infty)} = \|f^{**}\|_{Y(0,\infty)},\ f\in\mathfrak{M}^+(0,\infty),$$

is the optimal domain space for the operator M *and the space* $Y(\mathbb{R}^n)$*.*

Moreover, if (3.19) *is not satisfied, then there is no rearrangement-invariant function space* $Z(\mathbb{R}^n)$ *such that* $M\colon Z(\mathbb{R}^n) \to Y(\mathbb{R}^n)$ *is bounded.*

Proof. Owing to Proposition 3.3.1, $X(\mathbb{R}^n)$ is the optimal domain space for M and the space $Y(\mathbb{R}^n)$ if and only if $X(0,\infty)$ is the optimal domain space for R and the space $Y(0,\infty)$. Therefore, the claim follows from Proposition 2.2.1. □

By combining Proposition 3.3.1 and Proposition 2.2.23 with $\nu = 1$, we obtain the next theorem, which describes the optimal domain space for M and a Lorentz–Zygmund space $Y(\mathbb{R}^n)$.

Theorem 3.3.5. *Assume that $\|\cdot\|_{L^{p,q;\mathbb{A}}(0,\infty)}$ is equivalent to a rearrangement-invariant function norm, that is, the parameters p, q and $\mathbb{A} = (\alpha_0, \alpha_\infty)$ satisfy one of conditions (1.28) (with $\beta_0 = \beta_\infty = 0$). We have that*

$$M\colon X(\mathbb{R}^n) \to L^{p,q;\mathbb{A}}(\mathbb{R}^n)$$

is bounded, where

$$X(\mathbb{R}^n) = \begin{cases} L^{1,1;\mathbb{A}+1}(\mathbb{R}^n), & p = 1,\ q = 1,\ \alpha_0 \geq 0,\ \alpha_\infty + 1 < 0; \\ L^{p,q;\mathbb{A}}(\mathbb{R}^n), & p \in (1,\infty) \ \textit{or} \\ & p = \infty,\ \alpha_0 + \frac{1}{q} < 0 \ \textit{or} \\ & p = q = \infty,\ \alpha_0 \leq 0. \end{cases}$$

Moreover, $X(\mathbb{R}^n)$ is the optimal domain space for M and the space $L^{p,q;\mathbb{A}}(\mathbb{R}^n)$. Furthermore, if $p = q = 1$, $\alpha_0 \geq 0$ and $\alpha_\infty \in [-1, 0]$, then there is no rearrangement-invariant function space $Z(\mathbb{R}^n)$ such that $M\colon Z(\mathbb{R}^n) \to L^{p,q;\mathbb{A}}(\mathbb{R}^n)$ is bounded.

We conclude this section with the following observation.

Proposition 3.3.6. *If $X(\mathbb{R}^n)$ is a rearrangement-invariant function space such that $M\colon X(\mathbb{R}^n) \to X(\mathbb{R}^n)$ is bounded, then $X(\mathbb{R}^n)$ is the optimal partner space for M, that is, $X(\mathbb{R}^n)$ is the optimal domain space for the operator M and the space $X(\mathbb{R}^n)$ and, simultaneously, $X(\mathbb{R}^n)$ is the optimal target space for the operator M and the space $X(\mathbb{R}^n)$.*

Proof. Assume that $M\colon X(\mathbb{R}^n) \to X(\mathbb{R}^n)$ is bounded. Let $Y(\mathbb{R}^n)$ and $Z(\mathbb{R}^n)$ be rearrangement-invariant function spaces such that $M\colon X(\mathbb{R}^n) \to Y(\mathbb{R}^n)$ and $M\colon Z(\mathbb{R}^n) \to X(\mathbb{R}^n)$ are bounded. We need to show that $X(\mathbb{R}^n) \hookrightarrow Y(\mathbb{R}^n)$ and $Z(\mathbb{R}^n) \hookrightarrow X(\mathbb{R}^n)$. Thanks to Proposition 3.3.1, we know that $R\colon X(0,\infty) \to Y(0,\infty)$ and $H\colon X'(0,\infty) \to Z'(0,\infty)$ are bounded. Consequently, we have that

$$\|f\|_{Y(0,\infty)} \leq \|f^{**}\|_{Y(0,\infty)} = \|R(f^*)\|_{Y(0,\infty)} \lesssim \|f\|_{X(0,\infty)}$$

and that

$$\begin{aligned} \|f\|_{Z'(0,\infty)} &\approx \Big\| f^*(t) \int_{\frac{t}{2}}^{t} \frac{\mathrm{d}s}{s} \Big\|_{Z'(0,\infty)} \leq \Big\| \int_{\frac{t}{2}}^{\infty} f^*(s) \frac{\mathrm{d}s}{s} \Big\|_{Z'(0,\infty)} \\ &\approx \|H(f^*)\|_{Z'(0,\infty)} \lesssim \|f\|_{X'(0,\infty)} \end{aligned}$$

for every $f \in \mathfrak{M}^+(0,\infty)$, where we used (1.13) in the second equivalence. Hence $X(\mathbb{R}^n) \hookrightarrow Y(\mathbb{R}^n)$ and $X'(\mathbb{R}^n) \hookrightarrow Z'(\mathbb{R}^n)$. Finally, recall that the latter is equivalent to $Z(\mathbb{R}^n) \hookrightarrow X(\mathbb{R}^n)$ thanks to (1.15). □

3.4 The fractional maximal operator

Let $\gamma \in (0,n)$. The fractional maximal operator M_γ of order γ is defined, for every $f \in \mathfrak{M}(\mathbb{R}^n)$, as

$$M_\gamma f(x) = \sup_{x \ni Q} \frac{1}{|Q|^{1-\frac{\gamma}{n}}} \int_Q |f(y)| \,\mathrm{d}y, \ x \in \mathbb{R}^n, \tag{3.20}$$

where the supremum is taken over all cubes in $\mathbb{R}^n$ whose edges are parallel to the coordinate axes of $\mathbb{R}^n$. The fractional maximal operator M_γ in some sense controls the Riesz potential I_γ (e.g. [3, Section 3.6]), as with the Hardy–Littlewood maximal operator and the Hilbert transform (e.g. [89, Chapter 4]). Although (3.20) with $\gamma = 0$ gives the Hardy–Littlewood maximal operator M, the behavior of M_γ, $\gamma > 0$, is substantially different from the behavior of M, and we consider these operators separately.

It was not until 2000 that a sharp pointwise estimate on the nonincreasing rearrangement of M_γ was obtained. In [28, Theorem 1.1], A. Cianchi, R. Kerman, L. Pick and B. Opic proved the following sharp pointwise estimate. For every $f \in \mathfrak{M}(\mathbb{R}^n)$, we have that

$$(M_\gamma f)^*(t) \lesssim \sup_{s \in [t,\infty)} s^{\frac{\gamma}{n}} f^{**}(s) \quad \text{for every } t \in (0,\infty), \tag{3.21}$$

where the multiplicative constant depends only on n and γ. Furthermore, for every nonincreasing function $g \in \mathfrak{M}^+(\mathbb{R}^n)$, there is a function $f \in \mathfrak{M}^+(\mathbb{R}^n)$ equimeasurable with g (namely $f(x) = g(\omega_n |x|^n)$, $x \in \mathbb{R}^n$) such that

$$\sup_{s \in [t,\infty)} s^{\frac{\gamma}{n}} f^{**}(s) \lesssim (M_\gamma f)^*(t) \quad \text{for every } t \in (0,\infty), \tag{3.22}$$

where the multiplicative constant depends only on n and γ.

The inevitable presence of the supremum brings a number of problems, which we did not face when dealing with the Hardy–Littlewood maximal operator. Fortunately, the heavy lifting was already done in Section 2.2.1, and we can just reap the fruits of our labor. We start off with a reduction principle for M_γ. Throughout this section, we set $R = R_{u,v,\nu}$ and $H = H_{u,v,\nu}$, where $R_{u,v,\nu}$ and $H_{u,v,\nu}$ are the operators defined by (2.1) and (2.2), respectively, with $u \equiv 1$, $v(t) = t^{\frac{\gamma}{n}-1}$, $t \in (0,\infty)$, $\nu = 1$ and $L = \infty$.

Proposition 3.4.1. *Let $X(\mathbb{R}^n)$ and $Y(\mathbb{R}^n)$ be rearrangement-invariant function spaces. Let $\gamma \in (0,n)$. The following three statements are equivalent:*

(i) $M_\gamma \colon X(\mathbb{R}^n) \to Y(\mathbb{R}^n)$ *is bounded;*

(ii) $R \colon X(0,\infty) \to Y(0,\infty)$ *is bounded;*

(iii) $H \colon Y'(0,\infty) \to X'(0,\infty)$ *is bounded.*

Proof. (*i*) *implies* (*ii*). Let $g \in \mathfrak{M}(\mathbb{R}^n)$. Set $f(x) = g^*(\omega_n |x|^n)$, $x \in \mathbb{R}^n$. Owing to (3.22), we have that

$$\begin{aligned} \|Rg\|_{Y(0,\infty)} &\le \|R(f^*)\|_{Y(0,\infty)} = \|t^{\frac{\gamma}{n}} f^{**}(t)\|_{Y(0,\infty)} \le \|\sup_{s\in[t,\infty)} s^{\frac{\gamma}{n}} f^{**}(s)\|_{Y(0,\infty)} \\ &\lesssim \|(M_\gamma f)^*\|_{Y(0,\infty)} = \|M_\gamma f\|_{Y(\mathbb{R}^n)} \lesssim \|f\|_{X(\mathbb{R}^n)} = \|g\|_{X(0,\infty)}, \end{aligned}$$

where we used the Hardy–Littlewood inequality (1.3) in the first inequality. Hence $R\colon X(0,\infty)\to Y(0,\infty)$ is bounded.

(ii) implies (i). Note that the function $(0,\infty)\ni t\mapsto t^{\frac{\gamma}{n}}$ is quasiconcave, and so we have that

$$\|\sup_{s\in[t,\infty)} s^{\frac{\gamma}{n}} f^{**}(s)\|_{Y(0,\infty)} \lesssim \|t^{\frac{\gamma}{n}} f^{**}(t)\|_{Y(0,\infty)} \quad \text{for every } f\in\mathfrak{M}(0,\infty) \tag{3.23}$$

thanks to Proposition 2.2.8. By combining (3.21) and (3.23), we obtain that

$$\begin{aligned}\|Mf\|_{Y(\mathbb{R}^n)} &= \|(Mf)^*\|_{Y(0,\infty)} \lesssim \|\sup_{s\in[t,\infty)} s^{\frac{\gamma}{n}} f^{**}(s)\|_{Y(0,\infty)}\\ &\lesssim \|t^{\frac{\gamma}{n}} f^{**}(t)\|_{Y(0,\infty)} = \|R(f^*)\|_{Y(0,\infty)} \lesssim \|f^*\|_{X(0,\infty)}\\ &= \|f\|_{X(\mathbb{R}^n)}\end{aligned}$$

for every $f\in\mathfrak{M}(\mathbb{R}^n)$.

(iii) is equivalent to (ii). This is Proposition 2.2.2. □

The next theorem characterize the optimal target space for M_γ.

Theorem 3.4.2. *Let $\gamma\in(0,n)$. Let $X(\mathbb{R}^n)$ be a rearrangement-invariant function space such that*

$$\inf_{t\in[1,\infty)} \varphi_{X(0,\infty)}(t)t^{-\frac{\gamma}{n}} > 0. \tag{3.24}$$

The rearrangement-invariant function space $Y(\mathbb{R}^n)$ whose associate function norm is defined as

$$\|f\|_{Y'(0,\infty)} = \sup_{g\sim f} \left\| \int_t^\infty g(s)s^{\frac{\gamma}{n}-1}\,\mathrm{d}s \right\|_{X'(0,\infty)},\quad f\in\mathfrak{M}^+(0,\infty), \tag{3.25}$$

where the supremum is taken over all $g\in\mathfrak{M}^+(0,\infty)$ equimeasurable with f, is the optimal target space for the operator M_γ and the space $X(\mathbb{R}^n)$.

Moreover, if (3.24) is not satisfied, then there is no rearrangement-invariant function space $Z(\mathbb{R}^n)$ such that $M_\gamma\colon X(\mathbb{R}^n)\to Z(\mathbb{R}^n)$ is bounded.

Proof. Owing to Proposition 3.4.1, $Y(\mathbb{R}^n)$ is the optimal target space for M_γ and the space $X(\mathbb{R}^n)$ if and only if $Y(0,\infty)$ is the optimal target space for R and the space $X(0,\infty)$. Note that

$$\left\|\chi_{(0,1)}(t)\int_t^1 s^{\frac{\gamma}{n}-1}\,\mathrm{d}s\right\|_{X'(0,\infty)} \lesssim \|\chi_{(0,1)}\|_{X'(0,\infty)} < \infty$$

and that

$$\inf_{t\in[1,\infty)} \varphi_{X(0,\infty)}(t)t^{-\frac{\gamma}{n}} > 0$$

if and only if

$$\sup_{t\in[1,\infty)} \varphi_{X'(0,\infty)}(t)t^{\frac{\gamma}{n}-1} < \infty$$

thanks to (1.18). Hence the claim follows from Proposition 2.2.3. □

Although the general description of the optimal target space for M_γ is complicated, it can often be considerably simplified. Throughout this section, we define the operator T_α, $\alpha \in (0,1)$, as

$$T_\alpha f(t) = t^{-\alpha} \sup_{s\in[t,\infty)} s^\alpha f^*(s),\ t \in (0,\infty),\ f \in \mathfrak{M}(0,\infty). \tag{3.26}$$

Note that $T_\alpha = T_\varphi$, where T_φ is the operator defined by (2.17) with $\varphi(t) = t^\alpha$, $t \in (0,\infty)$.

Theorem 3.4.3. *Let $\gamma \in (0,n)$. Let $X(\mathbb{R}^n)$ be a rearrangement-invariant function space such that*

$$T_{\frac{\gamma}{n}} \colon X(0,\infty) \to X(0,\infty) \tag{3.27}$$

is bounded. The rearrangement-invariant function space $Y(\mathbb{R}^n)$ whose associate function norm is defined as

$$\|f\|_{Y'(0,\infty)} = \sup_{\|g\|_{X(0,\infty)}\leq 1} \int_0^\infty f^*(t)t^{\frac{\gamma}{n}-1} \int_0^t T_{\frac{\gamma}{n}} g(s)\,\mathrm{d}s\,\mathrm{d}t,\ f \in \mathfrak{M}^+(0,\infty),$$

is the optimal target space for the operator M_γ and the space $X(\mathbb{R}^n)$. Furthermore, $\|\cdot\|_{Y'(0,\infty)}$ is equivalent to the functional

$$\mathfrak{M}^+(0,\infty) \ni f \mapsto \Big\| \int_t^\infty f^*(s)s^{\frac{\gamma}{n}-1}\,\mathrm{d}s \Big\|_{X'(0,\infty)}. \tag{3.28}$$

Moreover, the equivalence constants depend only on n, γ and the operator norm of $T_{\frac{\gamma}{n}}$ on $X(0,\infty)$.

Proof. Since, owing to Proposition 3.4.1, $Y(\mathbb{R}^n)$ is the optimal target space for M_γ and the space $X(\mathbb{R}^n)$ if and only if $Y(0,\infty)$ is the optimal target space for R and the space $X(0,\infty)$, the claim follows immediately from Theorem 2.2.6. ☐

Remark 3.4.4. Assumption (3.27) is actually reasonable and natural because the endpoint estimates for the operator $T_{\frac{\gamma}{n}}$, which have the form ([46, Theorem 3.2], cf. [52, Lemma 3.5])

$$T_{\frac{\gamma}{n}} \colon L^1(0,\infty) \to L^1(0,\infty) \quad \text{and} \quad T_{\frac{\gamma}{n}} \colon L^{\frac{n}{\gamma},\infty}(0,\infty) \to L^{\frac{n}{\gamma},\infty}(0,\infty),$$

are consistent with those for M_γ, which have the form ([93, Chapter VI, (2.19)] and [28, (2.2)])

$$M_\gamma \colon L^1(\mathbb{R}^n) \to L^{\frac{n}{n-\gamma},\infty}(\mathbb{R}^n) \quad \text{and} \quad M_\gamma \colon L^{\frac{n}{\gamma},\infty}(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n).$$

However, assumption (3.27) is strictly stronger than assumption (3.24).

First, the fact that (3.27) implies (3.24) follows from comparing the preceding two theorems and taking into account the "moreover part" of Theorem 3.4.2. Alternatively, it follows from the following. If (3.27) is satisfied, then, in particular,

$$\|T_{\frac{\gamma}{n}}\chi_{(0,a)}\|_{X(0,\infty)} \lesssim \varphi_{X(0,\infty)}(a) \quad \text{for every } a \in (0,\infty). \tag{3.29}$$

Note that

$$T_{\frac{\gamma}{n}}\chi_{(0,a)}(t)=\chi_{(0,a)}(t)a^{\frac{\gamma}{n}}t^{-\frac{\gamma}{n}}\quad\text{for every } a\in(0,\infty) \text{ and every } t\in(0,\infty). \tag{3.30}$$

By combining (3.29) and (3.30), we obtain that

$$\varphi_{X(0,\infty)}(a)a^{-\frac{\gamma}{n}}\gtrsim\|\chi_{(0,a)}(t)t^{-\frac{\gamma}{n}}\|_{X(0,\infty)}\geq\|\chi_{(0,1)}(t)t^{-\frac{\gamma}{n}}\|_{X(0,\infty)}>0$$

for every $a\in[1,\infty)$, whence (3.24) follows.

Second, if (for example) $X(0,\infty)=L^{\frac{n}{\gamma},q}(0,\infty)$ with $q\in[1,\infty)$, then (3.24) is satisfied, but $T_{\frac{\gamma}{n}}$ is not bounded on $X(0,\infty)$ (this can be obtained from [46, Theorem 3.2]).

When $X(\mathbb{R}^n)$ is a Lorentz–Zygmund space, the optimal target space for M_γ and $X(\mathbb{R}^n)$ reads as follows. The result follows from Proposition 3.4.1 and Proposition 2.2.28 with $\nu=1$ and $\widetilde{\gamma}=\frac{\gamma}{n}$.

Theorem 3.4.5. *Let $\gamma\in(0,n)$. Assume that $\|\cdot\|_{L^{p,q;\mathbb{A}}(0,\infty)}$ is equivalent to a rearrangement-invariant function norm, that is, the parameters p,q and $\mathbb{A}=(\alpha_0,\alpha_\infty)$ satisfy one of conditions* (1.28) *(with $\beta_0=\beta_\infty=0$). We have that*

$$M_\gamma\colon L^{p,q;\mathbb{A}}(\mathbb{R}^n)\to Y(\mathbb{R}^n)$$

is bounded, where

$$Y(\mathbb{R}^n)=\begin{cases}L^{\frac{n}{n-\gamma},\infty}(\mathbb{R}^n), & p=q=1,\ \alpha_0=\alpha_\infty=0;\\ Y_1(\mathbb{R}^n), & p=q=1,\ \alpha_0\geq0,\ \alpha_\infty\leq0,\ |\alpha_0|+|\alpha_\infty|>0;\\ L^{\frac{np}{n-\gamma p},q;\mathbb{A}}(\mathbb{R}^n), & p\in(1,\frac{n}{\gamma});\\ L^{\infty,\infty;\mathbb{A}}(\mathbb{R}^n), & p=\frac{n}{\gamma},\ q=\infty,\ \alpha_0\leq0,\ \alpha_\infty\geq0;\\ L^\infty(\mathbb{R}^n), & p=\frac{n}{\gamma},\ q\in[1,\infty),\ \alpha_0=\alpha_\infty=0;\\ Y_2(\mathbb{R}^n), & p=\frac{n}{\gamma},\ q\in[1,\infty),\ \alpha_\infty\geq0,\ |\alpha_0|+|\alpha_\infty|>0 \text{ or}\\ & p=\frac{n}{\gamma},\ q=\infty,\ \alpha_0>0,\ \alpha_\infty\geq0,\end{cases}$$

where $Y_1(\mathbb{R}^n)$ and $Y_2(\mathbb{R}^n)$ are the rearrangement-invariant function spaces whose associate function norms satisfy

$$\|f\|_{Y_1'(0,\infty)}\approx\sup_{t\in(0,\infty)}\ell^{-\mathbb{A}}(t)\int_t^\infty f^*(s)s^{\frac{\gamma}{n}-1}\,\mathrm{d}s,\ f\in\mathfrak{M}^+(0,\infty),$$

$$\|f\|_{Y_2'(0,\infty)}\approx\sup_{g\sim f}\left\|t^{\frac{1}{q}-\frac{\gamma}{n}}\ell^{-\mathbb{A}}(t)\int_t^\infty g(s)s^{\frac{\gamma}{n}-1}\,\mathrm{d}s\right\|_{L^{q'}(0,\infty)},\ f\in\mathfrak{M}^+(0,\infty), \tag{3.31}$$

where the supremum in (3.31) *is taken over all $g\in\mathfrak{M}^+(0,\infty)$ equimeasurable with f. Moreover, $Y(\mathbb{R}^n)$ is the optimal target space for M_γ and the space $L^{p,q;\mathbb{A}}(\mathbb{R}^n)$. Furthermore, if $p=\frac{n}{\gamma}$ and $\alpha_\infty<0$, or $p>\frac{n}{\gamma}$, then there is no rearrangement-invariant function space $Z(\mathbb{R}^n)$ such that $M_\gamma\colon L^{p,q;\mathbb{A}}(\mathbb{R}^n)\to Z(\mathbb{R}^n)$ is bounded.*

Remark 3.4.6. By virtue of Theorem 2.2.13 (see also Remark 2.2.14(ii)), the supremum in (3.31) is inevitable.

The following theorem describes the optimal domain space for M_γ.

Theorem 3.4.7. *Let $\gamma \in (0, n)$. Let $Y(\mathbb{R}^n)$ be a rearrangement-invariant function space such that*

$$\eta \in Y(0, \infty), \tag{3.32}$$

where

$$\eta(t) = (t + 1)^{\frac{\gamma}{n} - 1}, \ t \in (0, \infty).$$

The rearrangement-invariant function space $X(\mathbb{R}^n)$ whose function norm is defined as

$$\|f\|_{X(0,\infty)} = \|t^{\frac{\gamma}{n}} f^{**}(t)\|_{Y(0,\infty)}, \ f \in \mathfrak{M}^+(0, \infty),$$

is the optimal domain space for the operator M_γ and the space $Y(\mathbb{R}^n)$.

Moreover, if (3.32) *is not satisfied, then there is no rearrangement-invariant function space $Z(\mathbb{R}^n)$ such that $M_\gamma \colon Z(\mathbb{R}^n) \to Y(\mathbb{R}^n)$ is bounded.*

Proof. Owing to Proposition 3.4.1, $X(\mathbb{R}^n)$ is the optimal domain space for M_γ and the space $Y(\mathbb{R}^n)$ if and only if $X(0, \infty)$ is the optimal domain space for R and the space $Y(0, \infty)$. Therefore, the claim follows from Proposition 2.2.1. □

When $Y(\mathbb{R}^n)$ is a Lorentz–Zygmund space, the optimal domain space for M_γ and $Y(\mathbb{R}^n)$ reads as follows. The result follows from Proposition 3.4.1 and Proposition 2.2.22 with $\nu = 1$ and $\widetilde{\gamma} = \frac{\gamma}{n}$.

Theorem 3.4.8. *Let $\gamma \in (0, n)$. Assume that $\| \cdot \|_{L^{p,q;\mathbb{A}}(0,\infty)}$ is equivalent to a rearrangement-invariant function norm, that is, the parameters p, q and $\mathbb{A} = (\alpha_0, \alpha_\infty)$ satisfy one of conditions* (1.28) *(with $\beta_0 = \beta_\infty = 0$). We have that*

$$M_\gamma \colon X(\mathbb{R}^n) \to L^{p,q;\mathbb{A}}(\mathbb{R}^n)$$

is bounded, where

$$X(\mathbb{R}^n) = \begin{cases} L^{1,1;(0,\alpha_\infty+1)}(\mathbb{R}^n), & p = \frac{n}{n-\gamma},\ q = 1,\ \alpha_0 + 1 < 0,\ \alpha_\infty + 1 < 0; \\ L^{1,1;\mathbb{A}+1}(\mathbb{R}^n), & p = \frac{n}{n-\gamma},\ q = 1,\ \alpha_0 + 1 > 0,\ \alpha_\infty + 1 < 0; \\ L^{1,1;(0,\alpha_\infty+1),(1,0)}(\mathbb{R}^n), & p = \frac{n}{n-\gamma},\ q = 1,\ \alpha_0 + 1 = 0,\ \alpha_\infty + 1 < 0; \\ L^{(1,q;\mathbb{A})}(\mathbb{R}^n), & p = \frac{n}{n-\gamma},\ q \in (1, \infty),\ \alpha_\infty + \frac{1}{q} < 0 \text{ or} \\ & p = \frac{n}{n-\gamma},\ q = \infty,\ \alpha_\infty \le 0; \\ L^{\frac{np}{n+\gamma p},q;\mathbb{A}}(\mathbb{R}^n), & p \in (\frac{n}{n-\gamma}, \infty); \\ L^{\frac{n}{\gamma},\infty;\mathbb{A}}(\mathbb{R}^n), & p = q = \infty,\ \alpha_0 \le 0,\ \alpha_\infty \ge 0; \\ X_1(\mathbb{R}^n), & p = q = \infty,\ \alpha_0 \le 0,\ \alpha_\infty < 0 \text{ or} \\ & p = \infty,\ q \in [1, \infty),\ \alpha_0 + \frac{1}{q} < 0, \end{cases}$$

where $X_1(\mathbb{R}^n)$ is the rearrangement-invariant function space whose function norm satisfies

$$\|f\|_{X_1(0,\infty)} \approx \|t^{\frac{\gamma}{n}} f^{**}(t)\|_{L^{\infty,q;\mathbb{A}}(0,\infty)}, \ f \in \mathfrak{M}^+(0, \infty).$$

Moreover, $X(\mathbb{R}^n)$ is the optimal domain space for M_γ and the space $L^{p,q;\mathbb{A}}(\mathbb{R}^n)$. Furthermore, if $p = \frac{n}{n-\gamma}$, $q = \infty$, $\alpha_\infty > 0$, or $p = \frac{n}{n-\gamma}$, $q \in [1, \infty)$, $\alpha_\infty + \frac{1}{q} \ge 0$, or $p < \frac{n}{n-\gamma}$, then there is no rearrangement-invariant function space $Z(\mathbb{R}^n)$ such that $M \colon Z(\mathbb{R}^n) \to L^{p,q;\mathbb{A}}(\mathbb{R}^n)$ is bounded.

When $T_{\frac{\gamma}{n}}\colon X(0,\infty) \to (0,\infty)$ is bounded, Theorem 3.4.3 tells us that the associate function norm of the optimal target space for M_γ and $X(\mathbb{R}^n)$ is equivalent to the functional given by (3.28), which is considerably more manageable than (3.25). The following proposition tells us that, if $T_{\frac{\gamma}{n}}\colon X(0,\infty) \to X(0,\infty)$ is not bounded, then the supremum in (3.25) is indeed essential. The proposition follows from Proposition 3.4.1 and Theorem 2.2.13 (see also Remark 2.2.14(ii)).

Proposition 3.4.9. *Let* $\gamma \in (0,n)$. *Let* $X(\mathbb{R}^n)$ *be a rearrangement-invariant function space. The following three statements are equivalent:*

(i) $T_{\frac{\gamma}{n}}\colon X(0,\infty) \to X(0,\infty)$ *is bounded;*

(ii) the function norm given by (3.25) *is equivalent to the functional given by* (3.28)*;*

(iii) $X(\mathbb{R}^n)$ *is the optimal domain space for the operator* M_γ *and some rearrangement-invariant function space* $Z(\mathbb{R}^n)$*.*

Remark 3.4.10. By Proposition 3.4.1 and Remark 2.2.14(i), if $X(\mathbb{R}^n)$ is the optimal domain space for the operator M_γ and some rearrangement-invariant function space $Z(\mathbb{R}^n)$, then $X(\mathbb{R}^n)$ is actually the optimal domain space for the operator M_γ and its own optimal target space.

4. Optimal Sobolev-type embeddings on $\mathbb{R}^n$

In this chapter, we study Gagliardo–Nirenberg–Sobolev-type inequalities

$$\|u\|_{Y(\mathbb{R}^n)} \leq C\|\nabla^m u\|_{X(\mathbb{R}^n)} \tag{4.1}$$

in the framework of rearrangement-invariant function spaces, where C is a positive constant independent of u and u is a m-times weakly differentiable function whose m-th order gradient belongs to $X(\mathbb{R}^n)$. Obviously, (4.1) cannot hold for every $u \in V^m X(\mathbb{R}^n)$ (as for the notation, recall Section 1.2). Should there be any chance for (4.1) to hold, we need to limit ourselves only to those functions $u \in V^m X(\mathbb{R}^n)$ that have some decay at infinity or to subtract an appropriate polynomial from u. We shall investigate both of these possible approaches. Embeddings of Sobolev-type spaces on $\mathbb{R}^n$ within the class of rearrangement-invariant function spaces were also studied in [5, 95] but with the right-hand side of (4.1) involving the full gradient (that is, derivatives of all orders). It turns out that the optimal rearrangement-invariant function norm on the left-hand side of the inequality $\|u\|_{Y(\mathbb{R}^n)} \leq C\|u\|_{W^m X(\mathbb{R}^n)}$ behaves, loosely speaking, like the optimal target rearrangement-invariant function norm for Sobolev embeddings on bounded (regular) domains (see [52]) locally and like the norm on $X(\mathbb{R}^n)$ itself "near infinity". This time, however, there is no "localization".

Throughout this chapter, we assume that $n \in \mathbb{N}$, $n \geq 2$.

4.1 Functions with some decay at infinity

In this section, we study the inequality having the form

$$\|u\|_{Y(\mathbb{R}^n)} \leq C\|\nabla^m u\|_{X(\mathbb{R}^n)} \quad \text{for every } u \in V_0^m X(\mathbb{R}^n), \tag{4.2}$$

where $m \in \mathbb{N}$, $1 \leq m < n$, and C is a positive constant independent of u. Recall that the subscript 0 in $V_0^m X(\mathbb{R}^n)$ means that $|\{x \in \mathbb{R}^n \colon |\nabla^k u(x)| > \lambda\}| < \infty$ for every $\lambda > 0$ and $k = 0, 1, \ldots, m-1$, which is in a sense the most general condition on decay of u at infinity that ensures that (4.1) may hold for appropriate pairs of rearrangement-invariant function spaces.

We say that a rearrangement-invariant function space $Y(\mathbb{R}^n)$ is the *optimal target space* for a rearrangement-invariant function space $X(\mathbb{R}^n)$ in (4.2) if (4.2) is satisfied and, whenever (4.2) is satisfied with $Y(\mathbb{R}^n)$ replaced by a rearrangement-invariant function space $Z(\mathbb{R}^n)$, $Z(\mathbb{R}^n)$ is larger than $Y(\mathbb{R}^n)$, that is, $Y(\mathbb{R}^n) \hookrightarrow Z(\mathbb{R}^n)$. We say that a rearrangement-invariant function space $X(\mathbb{R}^n)$ is the *optimal domain space* for a rearrangement-invariant function space $Y(\mathbb{R}^n)$ in (4.2) if (4.2) is satisfied and, whenever (4.2) is satisfied with $X(\mathbb{R}^n)$ replaced by a rearrangement-invariant function space $Z(\mathbb{R}^n)$, $Z(\mathbb{R}^n)$ is smaller than $X(\mathbb{R}^n)$, that is, $Z(\mathbb{R}^n) \hookrightarrow X(\mathbb{R}^n)$.

The first theorem of this section characterizes when, for a given rearrangement-invariant function space $X(\mathbb{R}^n)$, there is a rearrangement-invariant function space $Y(\mathbb{R}^n)$ that renders (4.2) true by a condition on the associate space of $X(\mathbb{R}^n)$, and, if the condition is satisfied, it describes the optimal target space for $X(\mathbb{R}^n)$.

Theorem 4.1.1. *Let $m \in \mathbb{N}$, $m < n$. Let $X(\mathbb{R}^n)$ be a rearrangement-invariant function space such that*

$$(t+1)^{\frac{m}{n}-1} \in X'(0,\infty). \tag{4.3}$$

There is a positive constant C, which depends only on m and n, such that

$$\|u\|_{Y_{\mathrm{targ}(X,m)}(\mathbb{R}^n)} \leq C\|\nabla^m u\|_{X(\mathbb{R}^n)} \quad \text{for every } u \in V_0^m X(\mathbb{R}^n), \tag{4.4}$$

where $Y_{\mathrm{targ}(X,m)}(\mathbb{R}^n)$ is the rearrangement-invariant function space whose associate function norm is defined as

$$\|f\|_{Y'_{\mathrm{targ}(X,m)}(0,\infty)} = \|t^{\frac{m}{n}} f^{**}(t)\|_{X'(0,\infty)}, \ f \in \mathfrak{M}^+(0,\infty). \tag{4.5}$$

Furthermore, $Y_{\mathrm{targ}(X,m)}(\mathbb{R}^n)$ is the optimal target space for $X(\mathbb{R}^n)$ in (4.2).

Finally, if $X(\mathbb{R}^n)$ is such that (4.3) is not satisfied, there is no rearrangement-invariant function space $Y(\mathbb{R}^n)$ making (4.2) true.

Proof. The fact that (4.5) is indeed a rearrangement-invariant function norm follows from Proposition 2.1.1 with $u \equiv 1$, $v(t) = t^{\frac{m}{n}-1}$, $t \in (0,\infty)$, and $\nu = 1$.

Before we start proving (4.4), we make the following important observation. If $k \in \mathbb{N}$ is such that $k \leq m$, then (4.3) is also satisfied with m replaced by k (and so $\|\cdot\|_{Y'_{\mathrm{targ}(X,k)}(0,\infty)}$, too, is a rearrangement-invariant function norm). Furthermore, if $l \in \mathbb{N}$ is such that $k + l = m$, then

$$(t+1)^{\frac{l}{n}-1} \in Y'_{\mathrm{targ}(X,k)}(0,\infty) \tag{4.6}$$

and

$$\begin{aligned} \|f\|_{Y'_{\mathrm{targ}(X,m)}(0,\infty)} &= \|t^{\frac{m}{n}} f^{**}(t)\|_{X'(0,\infty)} \\ &\approx \|t^{\frac{l}{n}} f^{**}(t)\|_{Y'_{\mathrm{targ}(X,k)}(0,\infty)} \\ &= \|f\|_{Y'_{\mathrm{targ}(Y_{\mathrm{targ}(X,k)},l)}(0,\infty)} \end{aligned} \tag{4.7}$$

for every $f \in \mathfrak{M}^+(0,\infty)$. Note that (4.7) is equivalent to

$$Y_{\mathrm{targ}(X,m)}(\mathbb{R}^n) = Y_{\mathrm{targ}(Y_{\mathrm{targ}(X,k)},l)}(\mathbb{R}^n) \tag{4.8}$$

by (1.9). The first part of the observation above is trivial inasmuch as $(t+1)^{\frac{k}{n}-1} \leq (t+1)^{\frac{m}{n}-1}$ for every $t \in (0,\infty)$. Furthermore, we have that

$$\begin{aligned} \|(t+1)^{\frac{l}{n}-1}\|_{Y'_{\mathrm{targ}(X,k)}(0,\infty)} &= \Big\|t^{\frac{k}{n}-1}\int_0^t (s+1)^{\frac{l}{n}-1}\,\mathrm{d}s\Big\|_{X'(0,\infty)} \\ &\approx \big\|t^{\frac{k}{n}-1}[(t+1)^{\frac{l}{n}}-1]\big\|_{X'(0,\infty)} \\ &\leq \big\|t^{\frac{k}{n}-1}[(t+1)^{\frac{l}{n}}-1]\chi_{(0,1)}(t)\big\|_{X'(0,\infty)} \\ &\quad + \big\|t^{\frac{k+l}{n}-1}\chi_{(1,\infty)}(t)\big\|_{X'(0,\infty)} \\ &\leq \big\|t^{\frac{k}{n}-1}[(t+1)^{\frac{l}{n}}-1]\big\|_{L^\infty(0,1)}\|\chi_{(0,1)}\|_{X'(0,\infty)} \\ &\quad + \big\|t^{\frac{k+l}{n}-1}\chi_{(1,\infty)}(t)\big\|_{X'(0,\infty)} \\ &< \infty. \end{aligned}$$

Finally, the validity of (4.7) is much more involved. Nevertheless, we are well prepared for the task. In order to prove (4.7), we actually need to show that

$$\|t^{\frac{m}{n}} f^{**}(t)\|_{X'(0,\infty)} \approx \|t^{\frac{k}{n}}[s^{\frac{l}{n}} f^{**}(s)]^{**}(t)\|_{X'(0,\infty)} \quad \text{for every } f \in \mathfrak{M}^+(0,\infty),$$

which, however, follows from Theorem 2.2.16 (see Remark 2.2.17(i) with $\gamma_1 = \frac{k}{n}$, $\gamma_2 = \frac{l}{n}$, $\nu_1 = \nu_2 = 1$). Moreover, the multiplicative constants depend only on m and n.

We are now ready to prove (4.4) by induction on m. First, assume that $m = 1$. Since we have that

$$\Big\|t^{\frac{1}{n}-1}\int_0^t f(s)\,\mathrm{d}s\Big\|_{X'(0,\infty)} \leq \Big\|t^{\frac{1}{n}-1}\int_0^t f^*(s)\,\mathrm{d}s\Big\|_{X'(0,\infty)} = \|f\|_{Y'_{\mathrm{targ}(X,1)}(0,\infty)}$$

for every $f \in \mathfrak{M}^+(0,\infty)$, where we used the Hardy–Littlewood inequality (1.3), we have that

$$\Big\|\int_t^\infty f(s)s^{\frac{1}{n}-1}\,\mathrm{d}s\Big\|_{Y_{\mathrm{targ}(X,1)}(0,\infty)} \leq \|f\|_{X(0,\infty)} \quad \text{for every } f \in \mathfrak{M}^+(0,\infty) \tag{4.9}$$

owing to (2.31). Let $u \in V_0^1X(\mathbb{R}^n)$. Since $|\{x \in \mathbb{R}^n : |u(x)| > \lambda\}| < \infty$ for every $\lambda \in (0,\infty)$, we have that $\lim\limits_{t\to\infty} u^*(t) = 0$. Furthermore, the generalized Pólya-Szegő principle [23, Lemma 4.1] tells us that u^* is locally absolutely continuous and

$$\Big\|-\frac{\mathrm{d}u^*}{\mathrm{d}s}(s)s^{1-\frac{1}{n}}\Big\|_{X(0,\infty)} \lesssim \|\nabla u\|_{X(\mathbb{R}^n)}, \tag{4.10}$$

where the multiplicative constant depends only on n. Therefore, by combining (4.9) and (4.10), we have that

$$\begin{aligned}
\|u\|_{Y_{\mathrm{targ}(X,1)}(\mathbb{R}^n)} = \|u^*\|_{Y_{\mathrm{targ}(X,1)}(0,\infty)} &= \Big\|\int_t^\infty -\frac{\mathrm{d}u^*}{\mathrm{d}s}(s)\,\mathrm{d}s\Big\|_{Y_{\mathrm{targ}(X,1)}(0,\infty)} \\
&= \Big\|\int_t^\infty \Big(-\frac{\mathrm{d}u^*}{\mathrm{d}s}(s)s^{1-\frac{1}{n}}\Big)s^{\frac{1}{n}-1}\,\mathrm{d}s\Big\|_{Y_{\mathrm{targ}(X,1)}(0,\infty)} \\
&\leq \Big\|-\frac{\mathrm{d}u^*}{\mathrm{d}s}(s)s^{1-\frac{1}{n}}\Big\|_{X(0,\infty)} \\
&\lesssim \|\nabla u\|_{X(\mathbb{R}^n)}.
\end{aligned}$$

Next, assume that $1 < m < n$ and that we have already proved (4.4) for all smaller values of m. Let $u \in V_0^mX(\mathbb{R}^n)$. For each $i \in \{1,\dots,n\}$ we have that $\frac{\partial u}{\partial x_i} \in V_0^{m-1}X(\mathbb{R}^n)$ and, by the inductive hypothesis,

$$\Big\|\frac{\partial u}{\partial x_i}\Big\|_{Y_{\mathrm{targ}(X,m-1)}(\mathbb{R}^n)} \lesssim \Big\|\nabla^{m-1}\frac{\partial u}{\partial x_i}\Big\|_{X(\mathbb{R}^n)} \leq \|\nabla^m u\|_{X(\mathbb{R}^n)}.$$

Hence

$$\|\nabla u\|_{Y_{\mathrm{targ}(X,m-1)}(\mathbb{R}^n)} \lesssim \|\nabla^m u\|_{X(\mathbb{R}^n)}, \tag{4.11}$$

and so $u \in V_0^1Y_{\mathrm{targ}(X,m-1)}(\mathbb{R}^n)$. Owing to (4.6) with $l = 1$, $k = m-1$, and the inductive hypothesis, we have that

$$\|u\|_{Y_{\mathrm{targ}(Y_{\mathrm{targ}(X,m-1)},1)}(\mathbb{R}^n)} \lesssim \|\nabla u\|_{Y_{\mathrm{targ}(X,m-1)}(\mathbb{R}^n)}. \tag{4.12}$$

Finally, by combining (4.8), (4.12) and (4.11), we obtain that

$$\begin{aligned}\|u\|_{Y_{\mathrm{targ}(X,m)}(\mathbb{R}^n)} \approx \|u\|_{Y_{\mathrm{targ}(Y_{\mathrm{targ}(X,m-1)},1)}(\mathbb{R}^n)} &\lesssim \|\nabla u\|_{Y_{\mathrm{targ}(X,m-1)}(\mathbb{R}^n)}\\ &\lesssim \|\nabla^m u\|_{X(\mathbb{R}^n)}.\end{aligned}$$

This finishes the inductive step.

Next, we shall prove the optimality of $Y_{\mathrm{targ}(X,m)}$. Assume that $Y(\mathbb{R}^n)$ is a rearrangement-invariant function space such that

$$\|u\|_{Y(\mathbb{R}^n)} \lesssim \|\nabla^m u\|_{X(\mathbb{R}^n)} \quad \text{for every } u \in V_0^m X(\mathbb{R}^n). \tag{4.13}$$

We claim that (4.13) implies that

$$\Big\| \int_t^\infty f(s) s^{\frac{m}{n}-1}\,\mathrm{d}s \Big\|_{Y(0,\infty)} \lesssim \|f\|_{X(0,\infty)} \quad \text{for every } f \in \mathfrak{M}^+(0,\infty). \tag{4.14}$$

The proof of (4.14) is carried out along the lines of that of [5, Theorem 3.3]. Since every function $f \in \mathfrak{M}^+(0,\infty)$ can be pointwise approximated by a nondecreasing sequence of nonnegative functions with bounded supports, it is sufficient to establish (4.14) for every $f \in \mathfrak{M}^+(0,\infty)$ having bounded support. Furthermore, we may assume that $\|f\|_{X(0,\infty)} < \infty$ because otherwise there is nothing to prove. Let $f \in \mathfrak{M}^+(0,\infty)$ be such a function. We define the function g as

$$g(t) = \int_{\omega_n t^n}^\infty \int_{s_1}^\infty \cdots \int_{s_{m-1}}^\infty f(s_m) s_m^{\frac{m}{n}-m}\,\mathrm{d}s_m \cdots \mathrm{d}s_1,\ t \in (0,\infty),$$

where ω_n denotes the volume of the unit ball in $\mathbb{R}^n$. Routine, albeit slightly tedious, computations show that, for every $k \in \{1,\dots,m-1\}$,

$$|g^{(k)}(t)| \lesssim \sum_{l=1}^k t^{ln-k} \int_{\omega_n t^n}^\infty f(s) s^{\frac{m}{n}-l-1}\,\mathrm{d}s \quad \text{for every } t \in (0,\infty) \tag{4.15}$$

and that

$$|g^{(m)}(t)| \lesssim f(\omega_n t^n) + \sum_{l=1}^{m-1} t^{ln-m} \int_{\omega_n t^n}^\infty f(s) s^{\frac{m}{n}-l-1}\,\mathrm{d}s \quad \text{for a.e. } t \in (0,\infty). \tag{4.16}$$

Set

$$u(x) = g(|x|),\ x \in \mathbb{R}^n.$$

The function u is m-times weakly differentiable on $\mathbb{R}^n$ and, by straightforward induction on $j = 1,\dots,m$, one can show that $\frac{\partial^j u}{\partial^{\alpha_1} x_1 \cdots \partial^{\alpha_n} x_n}$, where $\alpha_1 + \cdots + \alpha_n = j$, is a linear combination of functions having the form

$$x_{i_1} \cdots x_{i_l} g^{(k)}(|x|)|x|^{-j-l+k} \quad \text{for a.e. } x \in \mathbb{R}^n,$$

where $l \in \{0,\dots,j\}$ and $k \in \{1,\dots,j\}$. Therefore,

$$\Big| \frac{\partial^m u}{\partial^{\alpha_1} x_1 \cdots \partial^{\alpha_n} x_n}(x) \Big| \lesssim \sum_{k=1}^m |g^{(k)}(|x|)| |x|^{k-m} \quad \text{for a.e. } x \in \mathbb{R}^n, \tag{4.17}$$

where $\alpha_1 + \cdots + \alpha_n = m$. Hence, by combining (4.15) and (4.16) with (4.17), we obtain that

$$|\nabla^m u(x)| \lesssim f(\omega_n|x|^n) + \sum_{l=1}^{m-1} |x|^{ln-m} \int_{\omega_n|x|^n}^{\infty} f(s)s^{\frac{m}{n}-l-1}\,\mathrm{d}s \quad \text{for a.e. } x \in \mathbb{R}^n. \tag{4.18}$$

Note that the sum on the right-hand side of (4.18) vanishes if $m = 1$. If $m > 2$, we define, for each $l \in \{1, \dots, m-1\}$, the linear operator T_l as

$$T_l h(t) = t^{l-\frac{m}{n}} \int_t^{\infty} h(s)s^{\frac{m}{n}-l-1}\,\mathrm{d}s,\ h \in (L^1 + L^\infty)(0,\infty),\ t \in (0,\infty).$$

It can be easily verified that T_l is bounded on $L^1(0,\infty)$ and $L^\infty(0,\infty)$. Moreover, the corresponding operator norms depend only on l and n. Therefore, T_l is bounded on every rearrangement-invariant function space over $(0,\infty)$ by [8, Chapter 3, Theorem 2.2]. In particular, it is bounded on $X(0,\infty)$. Moreover, the operator norm of T_l on $X(0,\infty)$ can be bounded from above by a constant depending only on m and n. Hence, by (4.18), we have that

$$\|\nabla^m u\|_{X(\mathbb{R}^n)} \lesssim \|f\|_{X(0,\infty)} + \sum_{l=1}^{m-1} \|T_l f\|_{X(0,\infty)} \lesssim \|f\|_{X(0,\infty)}, \tag{4.19}$$

where the multiplicative constants depend only on m and n. Hence $u \in V^m X(\mathbb{R}^n)$. Furthermore, since the support of f is bounded, it follows that $u \in V_0^m X(\mathbb{R}^n)$. By Fubini's theorem

$$u(x) = \frac{1}{(m-1)!} \int_{\omega_n|x|^n}^{\infty} f(s)s^{\frac{m}{n}-m}(s-\omega_n|x|^n)^{m-1}\,\mathrm{d}s \quad \text{for every } x \in \mathbb{R}^n,$$

whence

$$\begin{aligned} \|u\|_{Y(\mathbb{R}^n)} &\gtrsim \Big\| \int_{2t}^{\infty} f(s)s^{\frac{m}{n}-m}(s-t)^{m-1}\,\mathrm{d}s \Big\|_{Y(0,\infty)} \\ &\gtrsim \Big\| \int_{2t}^{\infty} f(s)s^{\frac{m}{n}-m}s^{m-1}\,\mathrm{d}s \Big\|_{Y(0,\infty)} = \Big\| \int_{2t}^{\infty} f(s)s^{\frac{m}{n}-1}\,\mathrm{d}s \Big\|_{Y(0,\infty)}, \end{aligned} \tag{4.20}$$

where the second inequality follows from the simple fact that $-t \geq -\frac{s}{2}$ for $s \geq 2t$. We are finally ready to establish (4.14). Indeed, by virtue of the boundedness of the dilation operator on rearrangement-invariant function spaces over $(0,\infty)$ (see (1.13)), (4.13), (4.19) and (4.20), we obtain that

$$\begin{aligned} \Big\| \int_t^{\infty} f(s)s^{\frac{m}{n}-1}\,\mathrm{d}s \Big\|_{Y(0,\infty)} &\lesssim \Big\| \int_{2t}^{\infty} f(s)s^{\frac{m}{n}-1}\,\mathrm{d}s \Big\|_{Y(0,\infty)} \lesssim \|u\|_{Y(\mathbb{R}^n)} \\ &\lesssim \|\nabla^m u\|_{X(\mathbb{R}^n)} \lesssim \|f\|_{X(0,\infty)}. \end{aligned}$$

Thanks to (2.31), (4.5) and (4.14), we have that

$$\|f\|_{Y'_{\mathrm{targ}(X,m)}(0,\infty)} = \|t^{\frac{m}{n}} f^{**}(t)\|_{X'(0,\infty)} \lesssim \|f\|_{Y'(0,\infty)} \quad \text{for every } f \in \mathfrak{M}^+(0,\infty),$$

whence $Y_{\mathrm{targ}(X,m)}(\mathbb{R}^n) \hookrightarrow Y(\mathbb{R}^n)$ owing to (1.15).

Last, we observe that, if there is any rearrangement-invariant function space $Y(\mathbb{R}^n)$ such that (4.2) is valid, then (4.3), too, is valid. Indeed, we already proved above that, if (4.2) is valid for a pair of rearrangement-invariant function spaces $X(\mathbb{R}^n)$ and $Y(\mathbb{R}^n)$, then

$$\|t^{\frac{m}{n}} f^{**}(t)\|_{X'(0,\infty)} \lesssim \|f\|_{Y'(0,\infty)} \quad \text{for every } f \in \mathfrak{M}^+(0,\infty),$$

whence the necessity of (4.3) follows (we just consider $f = \chi_{(0,1)}$). $\square$

For instance, assumption (4.3) is satisfied when $X(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ with $p \in [1, \frac{n}{m})$ or $X(\mathbb{R}^n) = L^{\frac{n}{m},1}(\mathbb{R}^n)$.

A somewhat surprising property of optimal target spaces in (4.2) is that they are stable under iteration (cf. [26, Theorem 1.5], [30, Theorem 5.7]). This was already proved at the beginning of the proof of Theorem 4.1.1, but it is worth stating the *sharp iteration principle* as a separate theorem.

Theorem 4.1.2. *Let $k, l \in \mathbb{N}$ be such that $k+l < n$. If $X(\mathbb{R}^n)$ is a rearrangement-invariant function space such that* (4.3) *with $m = k + l$ is satisfied, then*

$$(t+1)^{\frac{k}{n}-1} \in X'(0,\infty) \quad \textit{and} \quad t^{\frac{l}{n}-1}\chi_{(1,\infty)}(t) \in (Y_{\mathrm{targ}(X,k)})'(0,\infty),$$

where $Y_{\mathrm{targ}(X,k)}(0,\infty)$ is the rearrangement-invariant function space whose associate function norm is defined by (4.5) *with $m = k$, and*

$$Y_{\mathrm{targ}(Y_{\mathrm{targ}(X,k)},l)}(\mathbb{R}^n) = Y_{\mathrm{targ}(X,k+l)}(\mathbb{R}^n),$$

where the constants of the norm equivalence depend only on k, l and n.

The following theorem tells us that not only can the validity of (4.2) be reduced to the validity of certain one-dimensional Hardy-type inequalities, but the validity of (4.2) is also equivalent to the boundedness of the fractional maximal operator between associate spaces.

Theorem 4.1.3. *Let $m \in \mathbb{N}$, $m < n$. Let $X(\mathbb{R}^n)$ and $Y(\mathbb{R}^n)$ be rearrangement-invariant function spaces. The following five inequalities are equivalent:*

$$\|u\|_{Y(\mathbb{R}^n)} \lesssim \|\nabla^m u\|_{X(\mathbb{R}^n)} \quad \textit{for every } u \in V_0^m X(\mathbb{R}^n); \tag{4.21}$$

$$\left\| \int_t^\infty f(s) s^{\frac{m}{n}-1}\, \mathrm{d}s \right\|_{Y(0,\infty)} \lesssim \|f\|_{X(0,\infty)} \quad \textit{for every } f \in \mathfrak{M}^+(0,\infty); \tag{4.22}$$

$$\left\| \int_t^\infty f^*(s) s^{\frac{m}{n}-1}\, \mathrm{d}s \right\|_{Y(0,\infty)} \lesssim \|f\|_{X(0,\infty)} \quad \textit{for every } f \in \mathfrak{M}^+(0,\infty); \tag{4.23}$$

$$\|t^{\frac{m}{n}} g^{**}(t)\|_{X'(0,\infty)} \lesssim \|g\|_{Y'(0,\infty)} \quad \textit{for every } g \in \mathfrak{M}^+(0,\infty); \tag{4.24}$$

$$\|M_m g\|_{X'(\mathbb{R}^n)} \lesssim \|g\|_{Y'(\mathbb{R}^n)} \quad \textit{for every } g \in \mathfrak{M}(\mathbb{R}^n), \tag{4.25}$$

where M_m is the fractional maximal operator defined by (3.20). *Moreover, the multiplicative constants depend only on each other, m and n.*

Proof. The fact that (4.21) implies (4.22) was established in the proof of Theorem 4.1.1. The equivalence of (4.22), (4.23) and (4.24) follows from Corollary 2.2.12. Furthermore, the equivalence of (4.22) and (4.25) follows from

Proposition 3.4.1. Therefore, it is sufficient to show that (4.24) implies (4.21). Assume that (4.24) is valid. In particular, (4.3) is satisfied inasmuch as

$$\begin{aligned}\|(t+1)^{\frac{m}{n}-1}\|_{X'(0,\infty)} &= \|t^{\frac{m}{n}-1}\chi_{(1,\infty)}(t)\|_{X'(0,\infty)} \leq \|t^{\frac{m}{n}}\chi_{(0,1)}^{**}(t)\|_{X'(0,\infty)} \\ &\lesssim \|\chi_{(0,1)}\|_{Y'(0,\infty)} < \infty.\end{aligned}$$

Furthermore, we have that

$$\|g\|_{Y'_{\mathrm{targ}(X,m)}(0,\infty)} = \|t^{\frac{m}{n}} g^{**}(t)\|_{X'(0,\infty)} \lesssim \|g\|_{Y'(0,\infty)} \quad \text{for every } g \in \mathfrak{M}^+(0,\infty),$$

whence $Y_{\mathrm{targ}(X,m)}(\mathbb{R}^n) \hookrightarrow Y(\mathbb{R}^n)$ owing to (1.15). Hence we have that

$$\|u\|_{Y(\mathbb{R}^n)} \lesssim \|u\|_{Y_{\mathrm{targ}(X,m)}(\mathbb{R}^n)} \lesssim \|\nabla^m u\|_{X(\mathbb{R}^n)} \quad \text{for every } u \in V_0^m X(\mathbb{R}^n)$$

by virtue of Theorem 4.1.1. □

The following proposition provides a necessary condition (sometimes called "a condition of Muckenhoupt type" in the literature, cf. [9, Theorem 1] or [37, Lemma 1]) on a pair of rearrangement-invariant function spaces $X(\mathbb{R}^n)$ and $Y(\mathbb{R}^n)$ for the validity of (4.2). This condition can be useful for singling out pairs of rearrangement-invariant function spaces for which (4.2) cannot hold.

Proposition 4.1.4. *Let $m \in \mathbb{N}$, $m < n$. If $X(\mathbb{R}^n)$ and $Y(\mathbb{R}^n)$ are rearrangement-invariant function spaces such that* (4.2) *is valid for them, then*

$$\sup_{0<a<\infty} \varphi_{Y(0,\infty)}(a)\|t^{\frac{m}{n}-1}\chi_{(a,\infty)}(t)\|_{X'(0,\infty)} < \infty.$$

Proof. For every $a > 0$ we have that

$$\begin{aligned}\|\chi_{(0,a)}\|_{Y(0,\infty)}\|t^{\frac{m}{n}-1}\chi_{(a,\infty)}(t)\|_{X'(0,\infty)} &= \|\chi_{(0,a)}\|_{Y(0,\infty)} \sup_{\|f\|_{X(0,\infty)}\leq 1} \int_a^\infty |f(s)|s^{\frac{m}{n}-1}\,\mathrm{d}s \\ &\leq \sup_{\|f\|_{X(0,\infty)}\leq 1} \left\|\chi_{(0,a)}(t)\int_t^\infty |f(s)|s^{\frac{m}{n}-1}\,\mathrm{d}s\right\|_{Y(0,\infty)} \\ &\leq \sup_{\|f\|_{X(0,\infty)}\leq 1} \left\|\int_t^\infty |f(s)|s^{\frac{m}{n}-1}\,\mathrm{d}s\right\|_{Y(0,\infty)} \\ &< \infty,\end{aligned}$$

where we used (1.7) and Theorem 4.1.3. □

Complementing Theorem 4.1.1, the following theorem characterizes when, for a given rearrangement-invariant function space $Y(\mathbb{R}^n)$, there is a rearrangement-invariant function space $X(\mathbb{R}^n)$ rendering (4.2) true by a condition on the fundamental function of the space $Y(\mathbb{R}^n)$, and, if the condition is satisfied, it describes the optimal domain space.

Theorem 4.1.5. *Let $m \in \mathbb{N}$, $m < n$. Let $Y(\mathbb{R}^n)$ be a rearrangement-invariant function space such that*

$$\inf_{1\leq t<\infty} \frac{t^{1-\frac{m}{n}}}{\varphi_{Y(0,\infty)}(t)} > 0. \tag{4.26}$$

There is a positive constant C, which depends only on m and n, such that

$$\|u\|_{Y(\mathbb{R}^n)} \le C\|\nabla^m u\|_{X_{\mathrm{dom}(Y,m)}(\mathbb{R}^n)} \quad \text{for every } u \in V_0^m X_{\mathrm{dom}(Y,m)}(\mathbb{R}^n),$$

where $X_{\mathrm{dom}(Y,m)}(\mathbb{R}^n)$ is the rearrangement-invariant function space whose function norm is defined as

$$\|f\|_{X_{\mathrm{dom}(Y,m)}(0,\infty)} = \sup_{\substack{h\sim f\\ h\ge 0}} \left\| \int_t^\infty h(s)s^{\frac{m}{n}-1}\,\mathrm{d}s \right\|_{Y(0,\infty)}, \ f \in \mathfrak{M}^+(0,\infty), \tag{4.27}$$

where the supremum is taken over all $h \in \mathfrak{M}^+(0,\infty)$ equimeasurable with f. Furthermore, $X_{\mathrm{dom}(Y,m)}(\mathbb{R}^n)$ is the optimal domain space for $Y(\mathbb{R}^n)$ in (4.2).

Finally, if $Y(\mathbb{R}^n)$ is such that (4.26) *is not satisfied, then there is no rearrangement-invariant function space $X(\mathbb{R}^n)$ making* (4.2) *true.*

Proof. Owing to Theorem 4.1.3, $X(\mathbb{R}^n)$ is the optimal domain space for a rearrangement-invariant function space $Y(\mathbb{R}^n)$ in (4.2) if and only if $X'(\mathbb{R}^n)$ is the optimal target space for the fractional maximal operator M_m and the space $Y'(\mathbb{R}^n)$. Hence the theorem follows from Theorem 3.4.2 combined with (1.9). ☐

The general description of the optimal domain norm given by (4.27) is quite complicated. Fortunately, it can be significantly simplified in many customary situations.

Theorem 4.1.6. *Let $m \in \mathbb{N}$, $m < n$. Let $Y(\mathbb{R}^n)$ be a rearrangement-invariant function space such that*

$$T_{\frac{m}{n}} : Y'(0,\infty) \to Y'(0,\infty)$$

is bounded, where the operator $T_{\frac{m}{n}}$ is defined by (3.26). *The rearrangement-invariant function norm $\|\cdot\|_{X_{\mathrm{dom}(Y,m)}(0,\infty)}$ defined by* (4.27) *is equivalent to the functional*

$$\mathfrak{M}^+(0,\infty) \ni f \mapsto \left\| \int_t^\infty f^*(s)s^{\frac{m}{n}-1}\,\mathrm{d}s \right\|_{Y(0,\infty)}. \tag{4.28}$$

Moreover, the equivalence constants depend only on the operator norm of $T_{\frac{m}{n}}$ on $Y'(0,\infty)$, m and n.

Proof. Similarly to Theorem 4.1.5, this theorem follows from the corresponding theorem for the fractional maximal operator M_m, namely Theorem 3.4.3. ☐

Theorem 4.1.6 can be used, for example, when $Y(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ with $p \in (\frac{n}{n-m}, \infty]$ or $Y(\mathbb{R}^n) = L^{\frac{n}{n-m},1}(\mathbb{R}^n)$ (see the proof of Proposition 2.2.28, also [66, Proposition 5.4]).

The next theorem is an optimal domain counterpart to Theorem 4.1.2.

Theorem 4.1.7. *Let $k, l \in \mathbb{N}$ be such that $k+l < n$. If $Y(\mathbb{R}^n)$ is a rearrangement-invariant function space such that* (4.26) *with $m = k + l$ is satisfied and*

$$T_{\frac{k}{n}} : Y'(0,\infty) \to Y'(0,\infty)$$

is bounded, then

$$\inf_{1\le t<\infty} \frac{t^{1-\frac{l}{n}}}{\varphi_{X_{\mathrm{dom}(Y,k)}(0,\infty)}(t)} > 0,$$

where $X_{\mathrm{dom}(Y,k)}(0,\infty)$ *is the rearrangement-invariant function space whose associate function norm is defined by (4.27) with* $m = k$*, and*

$$X_{\mathrm{dom}(X_{\mathrm{dom}(Y,k)},l)}(\mathbb{R}^n) = X_{\mathrm{dom}(Y,k+l)}(\mathbb{R}^n), \tag{4.29}$$

where the constants of the norm equivalence depend only on the operator norm of $T_{\frac{k}{n}}$ *on* $Y'(0,\infty)$*,* k*,* l *and* n*.*

Proof. Since (4.26) with $m = k + l$ is satisfied, Theorem 4.1.5 tells us that $X_{\mathrm{dom}(Y,k+l)}(\mathbb{R}^n)$ exists and its function norm is defined as

$$\|f\|_{X_{\mathrm{dom}(Y,k+l)}(0,\infty)} = \sup_{\substack{h\sim f\\ h\ge 0}} \left\| \int_t^\infty h(s)s^{\frac{k+l}{n}-1}\,\mathrm{d}s \right\|_{Y(0,\infty)}, \quad f\in\mathfrak{M}^+(0,\infty). \tag{4.30}$$

Furthermore, since $T_{\frac{k}{n}}: Y'(0,\infty)\to Y'(0,\infty)$ is bounded, Theorem 4.1.6 guarantees that $X_{\mathrm{dom}(Y,k)}(\mathbb{R}^n)$ exists and its function norm satisfies

$$\|f\|_{X_{\mathrm{dom}(Y,k)}(0,\infty)} \approx \left\| \int_t^\infty f^*(s)s^{\frac{k}{n}-1}\,\mathrm{d}s \right\|_{Y(0,\infty)} \quad \text{for every } f\in\mathfrak{M}^+(0,\infty). \tag{4.31}$$

Observe that

$$\begin{aligned}
\inf_{1\le a<\infty} \frac{a^{1-\frac{l}{n}}}{\varphi_{X_{\mathrm{dom}(Y,k)}(0,\infty)}(a)} &\approx \inf_{1\le a<\infty} \frac{a^{1-\frac{l}{n}}}{\|\int_t^\infty \chi_{(0,a)}(s)s^{\frac{k}{n}-1}\,\mathrm{d}s\|_{Y(0,\infty)}}\\
&\ge \inf_{1\le a<\infty} \frac{a^{1-\frac{l}{n}}}{\|\chi_{(0,a)}(t)\int_0^a s^{\frac{k}{n}-1}\,\mathrm{d}s\|_{Y(0,\infty)}}\\
&\approx \inf_{1\le a<\infty} \frac{a^{1-\frac{l}{n}}}{a^{\frac{k}{n}}\varphi_{Y(0,\infty)}(a)} > 0
\end{aligned}$$

owing to the fact that (4.26) with $m = k + l$ is satisfied. Therefore, thanks to Theorem 4.1.5, $X_{\mathrm{dom}(X_{\mathrm{dom}(Y,k)},l)}(\mathbb{R}^n)$ exists and its function norm is defined as

$$\|f\|_{X_{\mathrm{dom}(X_{\mathrm{dom}(Y,k)},l)}(0,\infty)} = \sup_{\substack{h\sim f\\ h\ge 0}} \left\| \int_t^\infty h(s)s^{\frac{l}{n}-1}\,\mathrm{d}s \right\|_{X_{\mathrm{dom}(Y,k)}(0,\infty)}, \quad f\in\mathfrak{M}^+(0,\infty). \tag{4.32}$$

By combining (4.31) and (4.32), we obtain that

$$\|f\|_{X_{\mathrm{dom}(X_{\mathrm{dom}(Y,k)},l)}(0,\infty)} \approx \sup_{\substack{h\sim f\\ h\ge 0}} \|H_{v_1,1}(H_{v_2,1}h)\|_{Y(0,\infty)} \tag{4.33}$$

for every $f\in\mathfrak{M}^+(0,\infty)$, where $v_1(t) = t^{-1+\frac{k}{n}}$, $v_2(t) = t^{-1+\frac{l}{n}}$, $t\in(0,\infty)$. Furthermore, we have that

$$\sup_{\substack{h\sim f\\ h\ge 0}} \|H_{v_1,1}(H_{v_2,1}h)\|_{Y(0,\infty)} \approx \sup_{\substack{h\sim f\\ h\ge 0}} \left\| \int_t^\infty h(s)s^{-1+\frac{k+l}{n}}\,\mathrm{d}s \right\|_{Y(0,\infty)} \tag{4.34}$$

for every $f\in\mathfrak{M}^+(0,\infty)$ by virtue of Proposition 2.2.18. Hence (4.29) is true thanks to (4.30), (4.33) and (4.34). □

Remarks 4.1.8.

(i) Assume that $Y(0,\infty) = L^{p,q,\mathbb{A}}(0,\infty)$ is a Lorentz–Zygmund space. Note that (4.26) with $m = k + l$ is satisfied, which is necessary should $X_{\mathrm{dom}(Y,k+l)}(\mathbb{R}^n)$ exist, if and only if either $p > \frac{n}{n-(k+l)}$ or if $p = \frac{n}{n-(k+l)}$ and $\alpha_\infty \leq 0$ (see (2.99)). Furthermore, the validity of (4.26) with $m = k + l$ implies that $T_{\frac{k}{n}} \colon Y'(0,\infty) \to Y'(0,\infty)$ is bounded, and so we may use Proposition 4.1.7 whenever it makes sense. Indeed, it follows from [46, Theorem 3.2] that $T_{\frac{k}{n}} \colon Y'(0,\infty) \to Y'(0,\infty)$ is bounded if and only if $p > \frac{n}{n-k}$ or if $p = \frac{n}{n-k}$, $q = 1$, $\alpha_0 \geq 0$ and $\alpha_\infty \leq 0$ (cf. the proof of Proposition 2.2.28, also [66, Proposition 5.4]); hence $T_{\frac{k}{n}} \colon Y'(0,\infty) \to Y'(0,\infty)$ is bounded inasmuch as $p \geq \frac{n}{n-(k+l)} > \frac{n}{n-k}$ if (4.26) with $m = k + l$ is satisfied.

(ii) It may happen (see (iii) below) that (4.26) with $m = k + l$ is satisfied (and so $X_{\mathrm{dom}(Y,k+l)}(\mathbb{R}^n)$ and $X_{\mathrm{dom}(Y,k)}(\mathbb{R}^n)$ exist) and

$$\inf_{1 \leq a < \infty} \frac{a^{1-\frac{l}{n}}}{\varphi_{X_{\mathrm{dom}(Y,k)}(0,\infty)}(a)} > 0$$

(and so $X_{\mathrm{dom}(X_{\mathrm{dom}(Y,k)},l)}(\mathbb{R}^n)$ exists), but $T_{\frac{k}{n}} \colon Y'(0,\infty) \to Y'(0,\infty)$ is not bounded. If this is the case, then, instead of (4.33), we only have that

$$\|f\|_{X_{\mathrm{dom}(X_{\mathrm{dom}(Y,k)},l)}(0,\infty)} \geq \sup_{\substack{h\sim f\\ h\geq 0}} \|H_{v_1,1}(H_{v_2,1}h)\|_{Y(0,\infty)}$$

for every $f \in \mathfrak{M}^+(0,\infty)$, whence it follows (by arguing as in the proof of Theorem 4.1.7) that

$$X_{\mathrm{dom}(X_{\mathrm{dom}(Y,k)},l)}(\mathbb{R}^n) \hookrightarrow X_{\mathrm{dom}(Y,k+l)}(\mathbb{R}^n).$$

However, it remains an open question whether the opposite embedding, that is, $X_{\mathrm{dom}(Y,k+l)}(\mathbb{R}^n) \hookrightarrow X_{\mathrm{dom}(X_{\mathrm{dom}(Y,k)},l)}(\mathbb{R}^n)$, is/(can be) also true. That would amount to show (or disprove) that there is a positive constant C such that

$$\sup_{\substack{h\sim f\\ h\geq 0}} \sup_{\substack{g\sim H_{v_2,1}h\\ g\geq 0}} \|H_{v_1,1}(g)\|_{Y(0,\infty)} \leq C \sup_{\substack{h\sim f\\ h\geq 0}} \|H_{v_1,1}(H_{v_2,1}h)\|_{Y(0,\infty)}$$

for every $f \in \mathfrak{M}^+(0,\infty)$.

(iii) The situation described in (ii) happens, for example, when $Y(0,\infty) = L^1(0,\infty) + L^\infty(0,\infty)$. We have that

$$\inf_{1 \leq a < \infty} \frac{a^{1-\frac{k+l}{n}}}{\varphi_{Y(0,\infty)}(a)} = \inf_{1 \leq a < \infty} \frac{a^{1-\frac{k+l}{n}}}{\min\{1, a\}} = 1 > 0.$$

Furthermore, since we have that

$$\begin{aligned} X_{\mathrm{dom}(Y,k)}(0,\infty)(a) &= \sup_{\substack{h\sim \chi_{(0,a)}\\ h\geq 0}} \int_0^1 \int_t^\infty h(s) s^{-1+\frac{k}{n}} \,\mathrm{d}s\,\mathrm{d}t \\ &\leq \sup_{\substack{h\sim \chi_{(0,a)}\\ h\geq 0}} \int_0^\infty h(s) s^{-1+\frac{k}{n}} \,\mathrm{d}s \\ &\leq \int_0^\infty \chi_{(0,a)}(s) s^{-1+\frac{k}{n}} \,\mathrm{d}s = \frac{n}{k} a^{\frac{k}{n}} \end{aligned}$$

for every $a > 0$ thanks to the Hardy–Littlewood inequality (1.3), we also have that

$$\inf_{1 \le a < \infty} \frac{a^{1-\frac{l}{n}}}{\varphi_{X_{\mathrm{dom}(Y,k)}(0,\infty)}(a)} > 0.$$

However, $T_{\frac{k}{n}}$ is not bounded on $Y'(0,\infty) = L^1(0,\infty) \cap L^\infty(0,\infty)$ ([8, Chapter 2, Theorem 6.4]), because $T_{\frac{k}{n}}\chi_{(0,a)} \notin L^\infty(0,\infty)$ for any $a > 0$.

The next proposition, which follows from Theorem 4.1.3 and Proposition 3.4.9, tells us that, if $T_{\frac{m}{n}}$ is not bounded on $Y'(0,\infty)$, then (4.27) cannot be simplified to (4.28).

Proposition 4.1.9. *Let $m \in \mathbb{N}$, $m < n$. Let $Y(\mathbb{R}^n)$ be a rearrangement-invariant function space. The following three statements are equivalent:*

- *(i) $T_{\frac{m}{n}} \colon Y'(0,\infty) \to Y'(0,\infty)$ is bounded;*
- *(ii) the function norm given by (4.27) is equivalent to the functional given by (4.28);*
- *(iii) $Y(\mathbb{R}^n)$ is the optimal target space in (4.2) for some rearrangement-invariant function space $X(\mathbb{R}^n)$.*

We conclude this section with particular examples of optimal rearrangement-invariant function spaces in (4.2). Since, thanks to Theorem 4.1.3, (4.2) is valid for a pair of rearrangement-invariant function spaces $X(\mathbb{R}^n)$ and $Y(\mathbb{R}^n)$ if and only if $H_{v,1} \colon X(0,\infty) \to Y(0,\infty)$ with $v(t) = t^{\frac{m}{n}-1}$, $t \in (0,\infty)$, is bounded, we obtain the following two theorems from Proposition 2.2.40 and Proposition 2.2.36 with $\gamma = \frac{m}{n}$ and $\nu = 1$.

Theorem 4.1.10. *Let $m \in \mathbb{N}$, $m < n$. Assume that $\|\cdot\|_{L^{p,q;\mathbb{A}}(0,\infty)}$ is equivalent to a rearrangement-invariant function norm, that is, the parameters p, q and $\mathbb{A} = (\alpha_0, \alpha_\infty)$ satisfy one of conditions (1.28) (with $\beta_0 = \beta_\infty = 0$). Set*

$$Y(\mathbb{R}^n) = \begin{cases} L^{\frac{np}{n-mp},q;\mathbb{A}}(\mathbb{R}^n), & p = q = 1,\ \alpha_0 \ge 0,\ \alpha_\infty \le 0 \text{ or} \\ & p \in (1, \frac{n}{m}); \\ L^{\infty,q;\mathbb{A}-1}(\mathbb{R}^n), & p = \frac{n}{m},\ \alpha_0 < 1 - \frac{1}{q},\ \alpha_\infty > 1 - \frac{1}{q}; \\ Y_1(\mathbb{R}^n), & p = \frac{n}{m},\ q \in [1,\infty),\ \alpha_0 > 1 - \frac{1}{q},\ \alpha_\infty > 1 - \frac{1}{q}; \\ L^{\infty,\infty;(0,\alpha_\infty-1)}(\mathbb{R}^n), & p = \frac{n}{m},\ q = \infty,\ \alpha_0 > 1,\ \alpha_\infty > 1; \\ L^{\infty,1;(-1,\alpha_\infty-1),(-1,0),(-1,0)}(\mathbb{R}^n), & p = \frac{n}{m},\ q = 1,\ \alpha_0 = 0,\ \alpha_\infty > 0; \\ L^{\infty,q;(-\frac{1}{q},\alpha_\infty-1),(-1,0)}(\mathbb{R}^n), & p = \frac{n}{m},\ q \in (1,\infty],\ \alpha_0 = 1 - \frac{1}{q},\ \alpha_\infty > 1 - \frac{1}{q}; \\ Y_2(\mathbb{R}^n), & p = \frac{n}{m},\ q = 1,\ \alpha_0 < 0,\ \alpha_\infty = 0; \\ L^\infty(\mathbb{R}^n), & p = \frac{n}{m},\ q = 1,\ \alpha_0 \ge 0,\ \alpha_\infty = 0, \end{cases} \tag{4.35}$$

where $Y_1(\mathbb{R}^n)$ and $Y_2(\mathbb{R}^n)$ are the rearrangement-invariant function spaces whose function norms satisfy

$$\|f\|_{Y_1(0,\infty)} \approx \|t^{-\frac{1}{q}}\ell^{\alpha_\infty-1}(t) f^*(t)\chi_{(1,\infty)}(t)\|_{L^q(0,\infty)} + \|f\|_{L^\infty(0,\infty)},$$
$$\|f\|_{Y_2(0,\infty)} \approx \|t^{-1}\ell^{\alpha_0-1}(t) f^*(t)\|_{L^1(0,1)},$$

for every $f \in \mathfrak{M}^+(0,\infty)$. *The space* $Y(\mathbb{R}^n)$ *is the optimal target space in* (4.2) *for* $X(\mathbb{R}^n) = L^{p,q;\mathbb{A}}(\mathbb{R}^n)$. *Furthermore, if* $p = \frac{n}{m}$, $q = 1$, $\alpha_\infty < 0$, *or if* $p = \frac{n}{m}$, $q \in (1,\infty]$, $\alpha_\infty \le 1 - \frac{1}{q}$, *or if* $p \in (\frac{n}{m}, \infty]$, *then there is no rearrangement-invariant function space* $Y(\mathbb{R}^n)$ *for which* (4.2) *with* $X(\mathbb{R}^n) = L^{p,q;\mathbb{A}}(\mathbb{R}^n)$ *is true.*

The final theorem of this subsection describes the optimal domain spaces in (4.2) for a Lorentz–Zygmund space.

Theorem 4.1.11. *Let* $m \in \mathbb{N}$, $m < n$. *Assume that* $\|\cdot\|_{L^{p,q;\mathbb{A}}(0,\infty)}$ *is equivalent to a rearrangement-invariant function norm, that is, the parameters* p, q *and* $\mathbb{A} = (\alpha_0, \alpha_\infty)$ *satisfy one of conditions* (1.28) *(with* $\beta_0 = \beta_\infty = 0$*). Set*

$$X(\mathbb{R}^n) = \begin{cases} L^{1,1;\mathbb{A}}(\mathbb{R}^n), & p = \frac{n}{n-m},\ q = 1,\ \alpha_0 \ge 0,\ \alpha_\infty \le 0; \\ L^1(\mathbb{R}^n), & p = \frac{n}{n-m},\ \alpha_0 = \alpha_\infty = 0; \\ X_1(\mathbb{R}^n), & p = \frac{n}{n-m},\ q = 1,\ \alpha_0 < 0,\ \alpha_\infty \le 0 \text{ or} \\ & p = \frac{n}{n-m},\ q \in (1,\infty],\ |\alpha_0| + |\alpha_\infty| > 0,\ \alpha_\infty \le 0; \\ L^{\frac{np}{n+mp},q;\mathbb{A}}(\mathbb{R}^n), & p \in (\frac{n}{n-m}, \infty); \\ L^{\frac{n}{m},1}(\mathbb{R}^n), & p = q = \infty,\ \alpha_0 = \alpha_\infty = 0; \\ X_2(\mathbb{R}^n), & p = q = \infty,\ \alpha_0 \le 0,\ |\alpha_0| + |\alpha_\infty| > 0 \text{ or} \\ & p = \infty,\ q \in [1,\infty),\ \alpha_0 + \frac{1}{q} < 0, \end{cases}$$

where $X_1(\mathbb{R}^n)$ *and* $X_2(\mathbb{R}^n)$ *are the rearrangement-invariant function spaces whose function norms satisfy*

$$\|f\|_{X_1(0,\infty)} \approx \sup_{g \sim f} \left\| t^{1-\frac{m}{n}-\frac{1}{q}} \ell^{\mathbb{A}}(t) \int_t^\infty g(s) s^{-1+\frac{m}{n}} \, \mathrm{d}s \right\|_{L^q(0,\infty)}, \tag{4.36}$$

$$\|f\|_{X_2(0,\infty)} \approx \left\| t^{-\frac{1}{q}} \ell^{\mathbb{A}}(t) \int_t^\infty f^*(s) s^{-1+\frac{m}{n}} \, \mathrm{d}s \right\|_{L^q(0,\infty)},$$

for every $f \in \mathfrak{M}^+(0,\infty)$. *The space* $X(\mathbb{R}^n)$ *is the optimal domain space in* (4.2) *for* $Y(\mathbb{R}^n) = L^{p,q;\mathbb{A}}(\mathbb{R}^n)$. *Moreover, the supremum in* (4.36) *is inevitable. Furthermore, if either* $p = \frac{n}{n-m}$ *and* $\alpha_\infty > 0$ *or* $p \in [1, \frac{n}{n-m})$, *then there is no rearrangement-invariant function space* $X(\mathbb{R}^n)$ *for which* (4.2) *with* $Y(\mathbb{R}^n) = L^{p,q;\mathbb{A}}(\mathbb{R}^n)$ *is true.*

4.2 Functions with no restriction on their decay at infinity

The standard Poincaré-Sobolev inequality on balls (e.g. [60, Corollary 1.64]) can be stated as follows: if $p \in [1,n)$, $n \ge 2$, and $q \in [1, \frac{np}{n-p}]$, then there is a constant C, depending only on p,q and the dimension n, such that

$$\|u - \bar{u}_B\|_{L^q(B)} \le C(n,p,q) r^{1+\frac{n}{q}-\frac{n}{p}} \|\nabla u\|_{L^p(B)} \tag{4.37}$$

for every ball $B \subseteq \mathbb{R}^n$ with the radius r and every $u \in V^1L^p(B)$, where $\bar{u}_B$ is the integral mean of u over B. With q equal to the Sobolev exponent $\frac{np}{n-p}$, (4.37) reads as

$$\|u - \bar{u}_B\|_{L^{\frac{np}{n-p}}(B)} \le C(n,p) \|\nabla u\|_{L^p(B)}. \tag{4.38}$$

Note that the inequality no longer depends on the radius r. If $p = n$, then (4.37) holds for each $q \in [1,\infty)$, but there is no optimal Lebesgue exponent q that would render that inequality independent of r as was possible with $p \in [1,n)$ and $q = \frac{np}{n-p}$. Nevertheless, the situation can be salvaged by substituting the critical Lebesgue space $L^n(B)$ with the smaller Lorentz space $L^{n,1}(B)$ because we have that (see [73, 76])

$$\|u - \bar{u}_B\|_{L^\infty(B)} \leq C(n)\|\nabla u\|_{L^{n,1}(B)}. \tag{4.39}$$

Inequalities (4.38) and (4.39) suggest that Poincaré-Sobolev-type inequalities might be possibly extended to the case where balls are replaced with the entire $\mathbb{R}^n$ provided that we find an appropriate replacement for $\bar{u}_B$. It is known (e.g. [60, Theorem 1.78], cf. [51]) that, if $p \in [1,n)$, then for every $u \in V^1L^p(\mathbb{R}^n)$, there is a (unique) $\lambda \in \mathbb{R}$, depending on u, such that

$$\|u - \lambda\|_{L^{\frac{np}{n-p}}(\mathbb{R}^n)} \leq C\|\nabla u\|_{L^p(\mathbb{R}^n)},$$

where the constant C is independent of u. However, the method of their proof cannot be used for extending the inequality (4.39) to the entire space. Nevertheless, it was proved in [80, Theorem 3.7] by different techniques that there is a positive constant C such that for every $u \in V^1L^{n,1}(\mathbb{R}^n)$ there is $\lambda \in \mathbb{R}$, depending on u, such that

$$\|u - \lambda\|_{L^\infty(\mathbb{R}^n)} \leq C\|\nabla u\|_{L^{n,1}(\mathbb{R}^n)}.$$

In this section, we will provide a (in a sense) sharp, general version of such inequalities within the class of rearrangement-invariant function spaces.

We start off by establishing a Poincaré-Sobolev-type inequality on balls in rearrangement-invariant function spaces. Although a certain version of a Poincaré-Sobolev-type inequality on balls in rearrangement-invariant function spaces was established in [18, Lemma 4.2] (cf. [24, Theorem 3.1]), their version is not sufficient for our purposes, because we need a better control over the multiplicative constant appearing there. In the proof of the following theorem, we will make use of the *signed* nonincreasing rearrangement of a function. If $u \in \mathfrak{M}(\mathbb{R}^n)$, we define its signed nonincreasing rearrangement $u^\circ\colon (0,\infty) \to [-\infty,\infty]$ as

$$u^\circ(t) = \inf\{\lambda \in \mathbb{R}\colon |\{x \in \mathbb{R}^n\colon u(x) > \lambda\}| \leq t\},\ t \in (0,\infty).$$

Note that $(u^\circ)^* = u^*$. In particular, u and u° are equimeasurable.

Theorem 4.2.1. *Let $X(\mathbb{R}^n)$ and $Y(\mathbb{R}^n)$ be rearrangement-invariant function spaces. If there is a positive constant C_1 such that*

$$\left\|\int_t^\infty f(s)s^{\frac{1}{n}-1}\,\mathrm{d}s\right\|_{Y(0,\infty)} \leq C_1\|f\|_{X(0,\infty)} \quad \text{for every } f \in \mathfrak{M}^+(0,\infty), \tag{4.40}$$

then there is a positive constant C_2, which depends only on C_1 and the dimension n, such that

$$\|(u - \bar{u}_B)\chi_B\|_{Y(\mathbb{R}^n)} \leq C_2\|(\nabla u)\chi_B\|_{X(\mathbb{R}^n)}$$

for every open ball $B \subseteq \mathbb{R}^n$ and every $u \in V^1X(B)$, where $\bar{u}_B$ denotes the integral mean of u over B, that is, $\bar{u}_B = \frac{1}{|B|}\int_B u(x)\,\mathrm{d}x$.

Proof. Fix an open ball $B \subseteq \mathbb{R}^n$ and a function $u \in V^1X(B)$. Note that $\bar{u}_B$ is a well-defined finite number, for u is integrable over B (see (1.34)). Since for any real number γ and a.e. $x \in B$ we have that

$$\begin{aligned}|u(x) - \bar{u}_B| &\le |u(x) - \gamma| + \frac{1}{|B|}\int_B |u(y) - \gamma| \,\mathrm{d}y \\ &\le |u(x) - \gamma| + \frac{1}{|B|}\|(u-\gamma)\chi_B\|_{Y(\mathbb{R}^n)}\|\chi_B\|_{Y'(\mathbb{R}^n)}\end{aligned}$$

by the Hölder inequality (1.14), it follows from this estimate and (1.18) that

$$\|(u - \bar{u}_B)\chi_B\|_{Y(\mathbb{R}^n)} \le 2\|(u-\gamma)\chi_B\|_{Y(\mathbb{R}^n)} \quad \text{for every } \gamma \in \mathbb{R}. \tag{4.41}$$

Since u° is locally absolutely continuous on $(0, |B|)$ ([27, Lemma 6.6]), we have that

$$\begin{aligned}\Big\|\Big(u - u^\circ\Big(\frac{|B|}{2}\Big)\Big)\chi_B\Big\|_{Y(\mathbb{R}^n)} &= \Big\|\Big(\Big(u - u^\circ\Big(\frac{|B|}{2}\Big)\Big)\chi_B\Big)^\circ\Big\|_{Y(0,\infty)} \\ &= \Big\|\Big(u^\circ - u^\circ\Big(\frac{|B|}{2}\Big)\Big)\chi_{(0,|B|)}\Big\|_{Y(0,\infty)} \\ &= \Big\|\chi_{(0,|B|)}(t)\int_t^{\frac{|B|}{2}} -\frac{\mathrm{d}u^\circ}{\mathrm{d}s}(s)\,\mathrm{d}s\Big\|_{Y(0,\infty)}.\end{aligned} \tag{4.42}$$

Hence, by virtue of (4.41) and (4.42), it is sufficient to prove that

$$\Big\|\chi_{(0,|B|}(t)\int_t^{\frac{|B|}{2}} -\frac{\mathrm{d}u^\circ}{\mathrm{d}s}(s)\,\mathrm{d}s\Big\|_{Y(0,\infty)} \lesssim \|(\nabla u)\chi_B\|_{X(\mathbb{R}^n)} \tag{4.43}$$

with a multiplicative constant depending only on C_1 and n. Since (4.40) is in force, we have that

$$\begin{aligned}\Big\|\chi_{(0,\frac{|B|}{2})}(t)\int_t^{\frac{|B|}{2}} -\frac{\mathrm{d}u^\circ}{\mathrm{d}s}(s)\,\mathrm{d}s\Big\|_{Y(0,|B|)} &= \Big\|\chi_{(0,\frac{|B|}{2})}(t)\int_t^{\infty} -\frac{\mathrm{d}u^\circ}{\mathrm{d}s}(s)\chi_{(0,\frac{|B|}{2})}(s)\,\mathrm{d}s\Big\|_{Y(0,|B|)} \\ &\le \Big\|\int_t^{\infty} -\frac{\mathrm{d}u^\circ}{\mathrm{d}s}(s)\chi_{(0,\frac{|B|}{2})}(s)\,\mathrm{d}s\Big\|_{Y(0,\infty)} \\ &\lesssim \Big\| -\frac{\mathrm{d}u^\circ}{\mathrm{d}t}(t)t^{1-\frac{1}{n}}\chi_{(0,\frac{|B|}{2})}(t)\Big\|_{X(0,\infty)},\end{aligned} \tag{4.44}$$

where the multiplicative constant depends only on C_1. It is well known (e.g. [97, Theorem 5.4.3]) that the isoperimetric function h_B of a ball $B \subseteq \mathbb{R}^n$ satisfies

$$h_B(s) \gtrsim \min\{s, |B| - s\}^{1-\frac{1}{n}} \quad \text{for every } s \in (0, |B|)$$

with a constant that depends only on the dimension n. Hence

$$\begin{aligned}\Big\| -\frac{\mathrm{d}u^\circ}{\mathrm{d}t}(t)t^{1-\frac{1}{n}}\chi_{(0,\frac{|B|}{2})}(t)\Big\|_{X(0,\infty)} &\lesssim \Big\| -\frac{\mathrm{d}u^\circ}{\mathrm{d}t}(t)h_B(t)\chi_{(0,\frac{|B|}{2})}(t)\Big\|_{X(0,\infty)} \\ &\le \Big\| -\frac{\mathrm{d}u^\circ}{\mathrm{d}t}(t)h_B(t)\chi_{(0,|B|)}(t)\Big\|_{X(0,\infty)} \\ &\le \|(\nabla u)\chi_B\|_{X(\mathbb{R}^n)},\end{aligned} \tag{4.45}$$

where the last inequality is valid thanks to [23, Lemma 4.1]. Exploiting the rearrangement invariance and the fact that the transformation $(0,|B|) \ni s \mapsto (|B|-s)$ is measure preserving on $(0,|B|)$ several times (cf. [8, Chapter 2, Proposition 7.2]) together with (4.40), we obtain in a similar way to (4.44) and (4.45) that

$$\begin{aligned}
\left\|\chi_{(\frac{|B|}{2},|B|)}(t)\int_t^{\frac{|B|}{2}} -\frac{\mathrm{d}u^\circ}{\mathrm{d}s}(s)\,\mathrm{d}s\right\|_{Y(0,\infty)} &= \left\|\chi_{(\frac{|B|}{2},|B|)}(t)\int_{|B|-t}^{\frac{|B|}{2}} -\frac{\mathrm{d}u^\circ}{\mathrm{d}s}(|B|-s)\,\mathrm{d}s\right\|_{Y(0,\infty)}\\
&= \left\|\chi_{(\frac{|B|}{2},|B|)}(|B|-t)\int_t^{\frac{|B|}{2}} -\frac{\mathrm{d}u^\circ}{\mathrm{d}s}(|B|-s)\,\mathrm{d}s\right\|_{Y(0,\infty)}\\
&= \left\|\chi_{(0,\frac{|B|}{2})}(t)\int_t^{\frac{|B|}{2}} -\frac{\mathrm{d}u^\circ}{\mathrm{d}s}(|B|-s)\,\mathrm{d}s\right\|_{Y(0,\infty)}\\
&\lesssim \left\| -\frac{\mathrm{d}u^\circ}{\mathrm{d}t}(|B|-t)t^{1-\frac{1}{n}}\chi_{(0,\frac{|B|}{2})}(t)\right\|_{X(0,\infty)}\\
&= \left\| -\frac{\mathrm{d}u^\circ}{\mathrm{d}t}(t)(|B|-t)^{1-\frac{1}{n}}\chi_{(0,\frac{|B|}{2})}(|B|-t)\right\|_{X(0,\infty)}\\
&= \left\| -\frac{\mathrm{d}u^\circ}{\mathrm{d}t}(t)(|B|-t)^{1-\frac{1}{n}}\chi_{(\frac{|B|}{2},|B|)}(t)\right\|_{X(0,\infty)}\\
&\lesssim \left\| -\frac{\mathrm{d}u^\circ}{\mathrm{d}t}(t)h_B(t)\chi_{(0,|B|)}(t)\right\|_{X(0,\infty)}\\
&\le \|(\nabla u)\chi_B\|_{X(\mathbb{R}^n)}.
\end{aligned}$$

Hence

$$\left\|\chi_{(\frac{|B|}{2},|B|)}(t)\int_t^{\frac{|B|}{2}} -\frac{\mathrm{d}u^\circ}{\mathrm{d}s}(s)\,\mathrm{d}s\right\|_{Y(0,\infty)} \lesssim \|(\nabla u)\chi_B\|_{X(\mathbb{R}^n)}. \tag{4.46}$$

Finally, by combining (4.44), (4.45) and (4.46), we obtain (4.43). □

The following auxiliary result will soon help us to establish the main result of this section.

Proposition 4.2.2. *Let $X(\mathbb{R}^n)$ be a rearrangement-invariant function space satisfying (4.3) with $m=1$. Let $u \in V^1X(\mathbb{R}^n)$ and set*

$$\lambda_k = \frac{1}{|B_k|}\int_{B_k} u(x)\,\mathrm{d}x,\ \ k \in \mathbb{N},$$

where B_k are balls in $\mathbb{R}^n$ such that $B_k \subseteq B_{k+1}$ for every $k \in \mathbb{N}$. The sequence $\{\lambda_k\}_{k=1}^\infty$ is bounded. In particular, it has a convergent subsequence.

Proof. Note that Theorem 3.2.2 guarantees that the Riesz potential I_1 is bounded from $X(\mathbb{R}^n)$ to some rearrangement-invariant function space over $\mathbb{R}^n$. In particular, it follows that

$$I_1(|\nabla u|)(x) < \infty \quad \text{and} \quad |u(x)| < \infty \quad \text{for a.e. } x \in \mathbb{R}^n \tag{4.47}$$

by (1.12) and $V^1X(\mathbb{R}^n) \subseteq L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Furthermore, since $u \in W^1L^1(B_k)$ for every $k \in \mathbb{N}$, we have that (e.g. [60, Lemma 1.50])

$$|u(x)-\lambda_k| \lesssim I_1\left(|\nabla u|\chi_{B_k}\right)(x) \quad \text{for every } k \in \mathbb{N} \text{ and a.e. } x \in B_k, \tag{4.48}$$

where the multiplicative constant depends only on the dimension n. By combining (4.47) and (4.48), we obtain that there is a point $x_0 \in B_1 \subseteq B_k$ such that

$$|u(x_0)| < \infty \quad \text{and} \quad |u(x_0) - \lambda_k| \lesssim I_1(|\nabla u|)(x_0) < \infty \quad \text{for every } k \in \mathbb{N},$$

whence the boundedness of $\{\lambda_k\}_{k=1}^{\infty}$ follows. $\square$

We are now ready to establish the main result of this subsection.

Theorem 4.2.3. *Let $m \in \mathbb{N}$, $m < n$. Let $X(\mathbb{R}^n)$ be a rearrangement-invariant function space satisfying* (4.3)*. There is a positive constant C, depending only on m and n, such that for each $u \in V^m X(\mathbb{R}^n)$ there is a polynomial P of order at most $m-1$, depending on u, that renders the inequality*

$$\|u - P\|_{Y_{\mathrm{targ}(X,m)}(\mathbb{R}^n)} \leq C \|\nabla^m u\|_{X(\mathbb{R}^n)} \tag{4.49}$$

true, where $Y_{\mathrm{targ}(X,m)}(\mathbb{R}^n)$ is the optimal target space in (4.2) *for $X(\mathbb{R}^n)$ given by Theorem 4.1.1. Furthermore, if P and $\widetilde{P}$ are such polynomials, then $P - \widetilde{P}$ is a constant polynomial. Moreover, if $\lim_{t\to\infty} \varphi_{Y_{\mathrm{targ}(X,m)}(0,\infty)}(t) = \infty$, then these polynomials are unique.*

Proof. We start by proving the uniqueness part. Assume that for $u \in V^m X(\mathbb{R}^n)$ there are polynomials P and $\widetilde{P}$ (of order at most $m-1$) such that

$$\|u - P\|_{Y_{\mathrm{targ}(X,m)}(\mathbb{R}^n)} \leq C \|\nabla^m u\|_{X(\mathbb{R}^n)}$$

and

$$\|u - \widetilde{P}\|_{Y_{\mathrm{targ}(X,m)}(\mathbb{R}^n)} \leq C \|\nabla^m u\|_{X(\mathbb{R}^n)}.$$

Hence $P - \widetilde{P} \in Y_{\mathrm{targ}(X,m)}(\mathbb{R}^n)$. Since $Y_{\mathrm{targ}(X,m)}(\mathbb{R}^n) \subseteq (L^1 + L^\infty)(\mathbb{R}^n)$ by (1.16), it follows that $P - \widetilde{P}$ is a constant polynomial. Moreover, we have that

$$\begin{aligned} |P - \widetilde{P}| \varphi_{Y_{\mathrm{targ}(X,m)}(0,\infty)}(k) &= \|(P - \widetilde{P})\chi_{E_k}\|_{Y_{\mathrm{targ}(X,m)}(\mathbb{R}^n)} \\ &\leq \|u - P\|_{Y_{\mathrm{targ}(X,m)}(\mathbb{R}^n)} + \|u - \widetilde{P}\|_{Y_{\mathrm{targ}(X,m)}(\mathbb{R}^n)} \\ &\lesssim \|\nabla^m u\|_{X(\mathbb{R}^n)} \\ &< \infty \end{aligned}$$

for every $k \in \mathbb{N}$, where $E_k \subseteq \mathbb{R}^n$ are such that $|E_k| = k$. Hence $|P - \widetilde{P}| = 0$ if $\lim_{t\to\infty} \varphi_{Y_{\mathrm{targ}(X,m)}(0,\infty)}(t) = \infty$.

We now prove the existence of such polynomials by induction on m. Assume that $m = 1$. Let $u \in V^1 X(\mathbb{R}^n)$. Set

$$\lambda_k = \frac{1}{|B_k|} \int_{B_k} u(x) \, \mathrm{d}x, \ k \in \mathbb{N},$$

where B_k is the ball in $\mathbb{R}^n$ centered at the origin having its radius equal to k. Owing to Proposition 4.2.2, we may assume, without loss of generality, that there is $\lambda \in \mathbb{R}$ such that

$$\lim_{k\to\infty} \lambda_k = \lambda. \tag{4.50}$$

By combining Proposition 2.2.1 and Proposition 2.2.2, we obtain that

$$\Big\| \int_t^\infty f(s) s^{\frac{1}{n}-1} \,\mathrm{d}s \Big\|_{Y_{\mathrm{targ}(X,1)}(0,\infty)} \leq \|f\|_{X(0,\infty)} \quad \text{for every } f \in \mathfrak{M}^+(0,\infty).$$

Hence, by virtue of Theorem 4.2.1, we have that

$$\|(u - \lambda_k)\chi_{B_k}\|_{Y_{\mathrm{targ}(X,1)}(\mathbb{R}^n)} \lesssim \|(\nabla u)\chi_{B_k}\|_{X(\mathbb{R}^n)} \quad \text{for every } k \in \mathbb{N},$$

where the multiplicative constant depends only on n. Consequently,

$$\|(u - \lambda_k)\chi_{B_k}\|_{Y_{\mathrm{targ}(X,1)}(\mathbb{R}^n)} \lesssim \|(\nabla u)\chi_{B_k}\|_{X(\mathbb{R}^n)} \leq \|\nabla u\|_{X(\mathbb{R}^n)} \tag{4.51}$$

for every $k \in \mathbb{N}$. Furthermore, we have that

$$\lim_{k\to\infty} (u(x) - \lambda_k)\chi_{B_k}(x) = u(x) - \lambda \quad \text{for a.e. } x \in \mathbb{R}^n$$

by (4.50). Hence

$$\|u - \lambda\|_{Y_{\mathrm{targ}(X,1)}(\mathbb{R}^n)} \leq \liminf_{k\to\infty} \|(u - \lambda_k)\chi_{B_k}\|_{Y_{\mathrm{targ}(X,1)}(\mathbb{R}^n)} \lesssim \|\nabla u\|_{X(\mathbb{R}^n)}$$

owing to Fatou's lemma [8, Chapter 1, Theorem 1.7] and (4.51).

We now carry out the inductive step. Assume that $1 < m < n$. Since $(t+1)^{\frac{1}{n}-1} \leq (t+1)^{\frac{m}{n}-1} \in X'(0,\infty)$, we are entitled to use the inductive hypothesis for $\widetilde{m} = 1$. Let $u \in V^m X(\mathbb{R}^n)$. We clearly have that $\frac{\partial^{|\alpha|} u}{\partial^\alpha x} \in V^1 X(\mathbb{R}^n)$ for every multi-index $\alpha \in \mathbb{N}_0^n$ such that $|\alpha| = \alpha_1 + \dots + \alpha_n = m-1$. By the inductive hypothesis for $\widetilde{m} = 1$, for every $\alpha \in \mathbb{N}_0^n$, $|\alpha| = m-1$, there is $\lambda_\alpha \in \mathbb{R}$ such that

$$\Big\| \frac{\partial^{|\alpha|} u}{\partial^\alpha x} - \lambda_\alpha \Big\|_{Y_{\mathrm{targ}(X,1)}(\mathbb{R}^n)} \lesssim \Big\| \nabla\Big(\frac{\partial^{|\alpha|} u}{\partial^\alpha x}\Big) \Big\|_{X(\mathbb{R}^n)} \leq \|\nabla^m u\|_{X(\mathbb{R}^n)}. \tag{4.52}$$

Set

$$v(x) = u(x) - \sum_{\substack{\alpha\in\mathbb{N}_0^n \\ |\alpha| = m-1}} \frac{\lambda_\alpha}{\alpha!} x^\alpha, \text{ a.e. } x \in \mathbb{R}^n,$$

where $\alpha! = \alpha_1! \cdot \alpha_2! \cdots \alpha_n!$. Since

$$\frac{\partial^{|\alpha|} v}{\partial^\alpha x} = \frac{\partial^{|\alpha|} u}{\partial^\alpha x} - \lambda_\alpha \quad \text{for every } \alpha \in \mathbb{N}_0^n,\ |\alpha| = m-1, \tag{4.53}$$

we have that

$$v \in V^{m-1} Y_{\mathrm{targ}(X,1)}(\mathbb{R}^n)$$

thanks to (4.52). Furthermore, since $(t+1)^{\frac{m-1}{n}-1} \in Y'_{\mathrm{targ}(X,1)}(0,\infty)$ (see Theorem 4.1.2), we are also entitled to use the inductive hypothesis for $\widetilde{m} = m-1$ and $\widetilde{X}(\mathbb{R}^n) = Y_{\mathrm{targ}(X,1)}(\mathbb{R}^n)$. Hence there is a polynomial Q of order at most $m-2$ such that

$$\|v - Q\|_{Y_{\mathrm{targ}(Y_{\mathrm{targ}(X,1)}, m-1)}(\mathbb{R}^n)} \lesssim \|\nabla^{m-1} v\|_{Y_{\mathrm{targ}(X,1)}(\mathbb{R}^n)},$$

where the multiplicative constant depends only on m and n. Consequently, we have that

$$\begin{aligned}
\Big\| u - \Big(Q + \sum_{|\alpha|=m-1} \frac{\lambda_\alpha}{\alpha!} x^\alpha \Big) \Big\|_{Y_{\mathrm{targ}(X,m)}(\mathbb{R}^n)} &= \|v - Q\|_{Y_{\mathrm{targ}(X,m)}(\mathbb{R}^n)} \\
&\approx \|v - Q\|_{Y_{\mathrm{targ}(Y_{\mathrm{targ}(X,1)},m-1)}(\mathbb{R}^n)} \\
&\lesssim \|\nabla^{m-1} v\|_{Y_{\mathrm{targ}(X,1)}(\mathbb{R}^n)} \\
&\approx \max_{|\alpha|=m-1} \Big\| \frac{\partial^{|\alpha|} v}{\partial^\alpha x} \Big\|_{Y_{\mathrm{targ}(X,1)}(\mathbb{R}^n)} \\
&= \max_{|\alpha|=m-1} \Big\| \frac{\partial^{|\alpha|} u}{\partial^\alpha x} - \lambda_\alpha \Big\|_{Y_{\mathrm{targ}(X,1)}(\mathbb{R}^n)} \\
&\lesssim \|\nabla^m u\|_{X(\mathbb{R}^n)},
\end{aligned}$$

where the first equivalence follows from Theorem 4.1.2, the second equality is true owing to (4.53) and the last inequality is true thanks to (4.52). We complete the proof by observing that $Q + \sum_{|\alpha|=m-1} \frac{\lambda_\alpha}{\alpha!} x^\alpha$ is of order at most $m-1$. □

We obtain the following particular examples by combining Theorem 4.2.3 and Theorem 4.1.10.

Theorem 4.2.4. *Let $m \in \mathbb{N}$, $m < n$. Let $p, q \in [1, \infty]$ and $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ be such that $p = q = 1$, $\alpha_0 \geq 0$, $\alpha_\infty \leq 0$, or $p \in (1, \frac{n}{m})$, or $p = \frac{n}{m}$, $q = 1$, $\alpha_\infty \geq 0$, or $p = \frac{n}{m}$, $q \in (1, \infty]$, $\alpha_\infty > 1 - \frac{1}{q}$. Let $Y(\mathbb{R}^n)$ be the rearrangement-invariant function space given by (4.35). There is a positive constant C such that for every $u \in V^m L^{p,q;\mathbb{A}}(\mathbb{R}^n)$ there is a polynomial P of order at most $m-1$ such that*

$$\|u - P\|_{Y(\mathbb{R}^n)} \leq C \|\nabla^m u\|_{L^{p,q;\mathbb{A}}(\mathbb{R}^n)}.$$

Moreover, these polynomials P are unique except for the case where $p = \frac{n}{m}$, $q = 1$ and $\alpha_\infty = 0$.

Remark 4.2.5. An immediate consequence of Theorem 4.2.4 is that, if $|\nabla u| \in L^{n,1}(\mathbb{R}^n)$, then u is bounded on $\mathbb{R}^n$ (more precisely, its continuous representative is). If $m \geq 2$ and $|\nabla^m u| \in L^{\frac{n}{m},1}(\mathbb{R}^n)$, then u obviously need not be bounded on $\mathbb{R}^n$ (consider, for example, $u(x) = x$), but u differs from a polynomial of order at most $m-1$ by a bounded function on $\mathbb{R}^n$. Therefore, if $|\nabla^m u| \in L^{\frac{n}{m},1}(\mathbb{R}^n)$ ($m \geq 2$) and u has "some decay at infinity", that is, $|\{x \in \mathbb{R}^n \colon |u(x)| > \lambda\}| < \infty$ for each $\lambda > 0$, it follows again that u is bounded on the entire space.

5. Compactness of Sobolev-type embeddings with measures

Throughout this chapter, we assume that $n \in \mathbb{N}$, $n \geq 2$. Let $\Omega \subseteq \mathbb{R}^n$ be an open set such that $|\Omega| < \infty$. We say that a finite (nontrivial) Borel measure μ on Ω is a *d-upper Ahlfors measure* on Ω, where $d \in (0, n]$, if

$$\sup_{x\in\mathbb{R}^n, r>0} \frac{\mu\left(B_r(x)\cap\Omega\right)}{r^d} < \infty, \tag{5.1}$$

where $B_r(x)$ denotes the open ball in $\mathbb{R}^n$ of radius r centered at x. A standard example of a d-upper Ahlfors measure is the restriction of the d-dimensional Hausdorff measure $\mathcal{H}^d$ to a d-dimensional subset of Ω. This includes, in particular, the restriction of the d-dimensional Hausdorff measure to the intersection of Ω and a d-dimensional affine subspace of $\mathbb{R}^n$. Another notable example of a d-upper Ahlfors measure is the absolutely continuous measure $\mathrm{d}\mu(x) = |x - x_0|^{d-n}\,\mathrm{d}x$, where x_0 is a fixed point in Ω. The class of upper Ahlfors measures provides a common framework for studying various Sobolev-type embeddings in a unified way instead of studying them separately as is often the case. We shall consider Sobolev-type embeddings having the form

$$W^m_0 X(\Omega) \hookrightarrow Y(\Omega, \mu). \tag{5.2}$$

Since a function from $W^m_0 X(\Omega)$ need not be a well-defined μ-measurable function on Ω, such embeddings need to be understood as *trace embeddings*, that is, $W^m_0 X(\Omega) \hookrightarrow Y(\Omega, \mu)$ actually means that there is a bounded linear operator $\mathrm{T}_\mu \colon W^m_0 X(\Omega) \to Y(\Omega, \mu)$ such that $\mathrm{T}_\mu u = \widetilde{u}$ whenever $u \in W^m_0 X(\Omega) \cap C(\Omega)$, where $\widetilde{u}$ is the continuous representative of u. While embeddings (5.2) were already thoroughly studied by D.R. Adams ([1, 2]) and V.G. Maz'ya ([62, 63]) within the class of Lebesgue spaces, it was not until recently that such embeddings were satisfactorily studied within the class of general rearrangement-invariant function spaces. In [31], A. Cianchi, L. Pick and L. Slavíková comprehensively investigated the validity of (5.2) in the framework of rearrangement-invariant function spaces. Since $W^n_0 L^1(\Omega) \hookrightarrow C_b(\Omega)$ (e.g. [4, Theorem 4.12]), it comes as no surprise that the situation where $m \geq n$ is not particularly interesting within the scope of rearrangement-invariant function spaces. Therefore, we limit ourselves to the case where $m < n$ for the most of this chapter and the rest of this introduction. Rather unsurprisingly, allowing general rearrangement-invariant function spaces in (5.2) leads to some difficulties. It turns out that the situation heavily depends on whether $d \in [n - m, n]$ or $d \in (0, n - m)$.

When μ satisfies (5.1) with $d \in [n - m, n]$, we say that μ is *fast decaying*. For fast decaying d-upper Ahlfors measures, their speed of decay on shrinking balls is actually all that matters. When μ is a fast decaying d-upper Ahlfors measure on Ω that decays exactly like r^d around at least one point, that is,

$$\inf_{r\in(0,R]} \frac{\mu\left(B_r(x_1)\cap\Omega\right)}{r^d} > 0 \quad \text{for some } x_1 \in \Omega \text{ and } R > 0, \tag{5.3}$$

the authors of [31] were able to completely characterize the validity of (5.2) (see [31, Theorem 4.1]) in terms of boundedness of a Hardy-type operator between the corresponding representation spaces. Furthermore, traces of functions from $W_0^m X(\Omega)$ on (Ω, μ) are always well defined owing to the sharp endpoint estimate ([31, Theorem 3.1])

$$\mathrm{T}_\mu \colon W_0^m L^1(\Omega) \to L^{\frac{d}{n-m},1}(\Omega, \mu)$$

and (1.17). However, the situation changes dramatically when $d \in (0, n-m)$.

A d-upper Ahlfors measure satisfying (5.1) with $d \in (0, n-m)$ (and not satisfying (5.1) with any $d \geq n-m$) is said to be *slowly decaying.* Unlike when $d \geq n-m$, traces of functions from $W_0^m X(\Omega)$ on (Ω, μ) need not be well defined unless $\|\cdot\|_{X(0,1)}$ is at least as strong as $\|\cdot\|_{L^{\frac{n-d}{n},1}(0,1)}$, that is, $X(0,1) \hookrightarrow L^{\frac{n-d}{n},1}(0,1)$ (see [25]). Furthermore, for slowly decaying d-upper Ahlfors measures, to know their speed of decay on shrinking balls is not enough for characterizing the validity of (5.2). This can be illustrated as follows. On the one hand, if $\mu = \mathcal{H}^d|_{\Omega_d}$, where $d \in (0, n-m) \cap \mathbb{N}$ and Ω_d is the (nonempty) intersection of Ω and a d-dimensional affine subspace of $\mathbb{R}^n$, then

$$\mathrm{T}_\mu \colon W_0^m L^{\frac{n-d}{m},1}(\Omega) \to L^{\frac{n-d}{m}}(\Omega, \mu)$$

and $\|\cdot\|_{L^{\frac{n-d}{m}}(0,1)}$ is the strongest rearrangement-invariant function norm $\|\cdot\|_{Y(0,1)}$ that renders $\mathrm{T}_\mu \colon W_0^m L^{\frac{n-d}{m},1}(\Omega) \to Y(\Omega, \mu)$ true ([31, Proposition 3.4]). On the other hand, if $\mathrm{d}\mu(x) = |x - x_0|^{d-n}\,\mathrm{d}x$, where $x_0 \in \Omega$ is a fixed point and $d \in (0, n-m)$, then ([31, Proposition 3.5])

$$\mathrm{T}_\mu \colon W_0^m L^{\frac{n-d}{m},1}(\Omega) \to L^{\frac{n-d}{m},1}(\Omega, \mu).$$

Note that $\|\cdot\|_{L^{\frac{n-d}{m},1}(0,1)}$ is a strictly stronger rearrangement-invariant function norm than $\|\cdot\|_{L^{\frac{n-d}{m}}(0,1)}$ inasmuch as $\frac{n-d}{m} > 1$. Therefore, instead of characterizing the validity of (5.2) for general slowly decaying d-upper Ahlfors measures, the authors of [31] provided sufficient conditions for the validity of (5.2) (see [31, Theorems 5.1 and 5.2]). However, if we know not only the speed of decay of μ on shrinking balls but also that μ has a radially nonincreasing density, that is, μ has the form

$$\mathrm{d}\mu(x) = g(|x - x_0|)\,\mathrm{d}x, \tag{5.4}$$

where $x_0 \in \Omega$ is a fixed point and $g \colon [0, \infty) \to [0, \infty)$, $g \not\equiv 0$, is a nonincreasing function, then the validity of (5.2) can be characterized again in terms of boundedness of a Hardy-type operator between the corresponding representation spaces similarly to the case where $d \in [n-m, n]$ (see [31, Theorem 5.2]).

In this chapter, we shall address the question of when the embedding (5.2) is compact, that is, when the trace operator

$$\mathrm{T}_\mu \colon W_0^m X(\Omega) \to Y(\Omega, \mu), \tag{5.5}$$

whose existence and boundedness were studied in [31], is compact. When μ is fast decaying or has the form (5.4), we shall characterize the compactness of (5.5). When μ is a general slowly decaying d-upper Ahlfors measure, we shall provide a sufficient condition for the compactness of (5.5).

It should be noted that the results concerning Sobolev embeddings (5.2) obtained in [31] have their "$W^m X(\Omega)$ counterparts" provided that certain regularity assumptions are imposed on Ω. Thanks to that, our results also remain valid in the following two situations. The proofs just need a few routine changes.

(i) When Ω is a bounded domain having the cone property, it is possible to replace (5.5) by

$$\mathrm{T}_\mu \colon W^m X(\Omega) \to Y(\Omega, \mu)$$

provided that $\mathrm{T}_\mu u = \widetilde{u}$ whenever $u \in W_0^m X(\Omega) \cap C(\Omega)$ is replaced by $\mathrm{T}_\mu u = \widetilde{u}$ for all $u \in W^m X(\Omega) \cap C(\Omega)$.

(ii) When Ω is a bounded Lipschitz domain and μ is a d-Ahlfors measure on $\overline{\Omega}$, $d \in (0, n]$, that is, μ is a finite Borel measure on $\overline{\Omega}$ satisfying

$$\sup_{x \in \mathbb{R}^n, r>0} \frac{\mu(B_r(x) \cap \overline{\Omega})}{r^d} < \infty,$$

it is possible to replace (5.5) by

$$\mathrm{T}_\mu \colon W^m X(\Omega) \to Y(\overline{\Omega}, \mu)$$

provided that (5.3) is replaced by

$$\inf_{r \in (0,R]} \frac{\mu(B_r(x_1) \cap \overline{\Omega})}{r^d} > 0 \quad \text{for some } x_1 \in \overline{\Omega} \text{ and } R > 0$$

and $\mathrm{T}_\mu u = \widetilde{u}$ whenever $u \in W_0^m X(\Omega) \cap C(\Omega)$ is replaced by $\mathrm{T}_\mu u = \widetilde{u}$ for all $u \in W^m X(\Omega) \cap C(\overline{\Omega})$.

In particular, if Ω is a bounded Lipschitz domain and $\mu = \mathcal{H}^{n-1}_{|\partial\Omega}$, this chapter also provides compactness results for boundary trace embeddings

$$W^m X(\Omega) \hookrightarrow Y(\partial\Omega).$$

5.1 General compactness results

We start with an auxiliary compactness result.

Proposition 5.1.1. *Let Ω be an open set in $\mathbb{R}^n$ such that $|\Omega| < \infty$. Let $\|\cdot\|_{X(0,1)}$ be a rearrangement-invariant function norm. Let $m \in \mathbb{N}$, $m < n$. Let μ be a d-upper Ahlfors measure on Ω.*

(i) If $d \in (0, n-m)$ and

$$\lim_{r \to 0^+} \|t^{-1+\frac{m}{n-d}} \chi_{(0,r)}(t)\|_{X'(0,1)} = 0, \tag{5.6}$$

then the trace operator

$$\mathrm{T}_\mu \colon W^m X(B) \to L^{\frac{n-d}{m}}(B, \mu)$$

is compact for every open ball $B \subseteq \Omega$. In particular, every bounded sequence in $W_0^m X(\Omega)$ contains a subsequence converging pointwise μ-a.e. in Ω to a function from $L^{\frac{n-d}{m}}(\Omega, \mu)$.

(ii) If $d \in [n-m, n]$ *and*

$$X(0,1) \neq L^1(0,1), \tag{5.7}$$

then the trace operator

$$\mathrm{T}_\mu \colon W^m X(B) \to L^{\frac{d}{n-m},1}(B,\mu)$$

is compact for every open ball $B \subseteq \Omega$*. In particular, every bounded sequence in* $W_0^m X(\Omega)$ *contains a subsequence converging pointwise* μ*-a.e. in* Ω *to a function from* $L^{\frac{d}{n-m},1}(\Omega,\mu)$*.*

Proof. First, assume that $d \in (0, n-m)$. Since

$$\|t^{-1+\frac{m}{n-d}}\|_{X'(0,1)} \leq \|t^{-1+\frac{m}{n-d}}\chi_{(0,r)}(t)\|_{X'(0,1)} + r^{-1+\frac{m}{n-d}}\|\chi_{(r,1)}\|_{X'(0,1)}$$

for every $r \in (0,1)$, (5.6) implies, in particular, that $\|t^{-1+\frac{m}{n-d}}\|_{X'(0,1)} < \infty$. Consequently, owing to the Hölder inequality (1.14), we have that

$$X(\Omega) \hookrightarrow L^{\frac{n-d}{m},1}(\Omega). \tag{5.8}$$

Note that (5.8) combined with (1.17) implies that $W^m X(B)$ is an intermediate space between $W^m L^{\frac{n-d}{m},1}(B)$ and $W^m L^\infty(B)$ ([8, Chapter 3, Definition 1.4]). By [31, Theorem 3.3], the trace operator

$$\mathrm{T}_\mu \colon W^m L^{\frac{n-d}{m},1}(B) \to L^{\frac{n-d}{m}}(B,\mu)$$

is bounded. Furthermore, the embedding $W^m L^\infty(B) \hookrightarrow C(\overline{B})$ is compact (e.g., [4, Theorem 6.3]). Hence, in particular,

$$\mathrm{T}_\mu \colon W^m L^\infty(B) \to L^{\frac{n-d}{m}}(B,\mu)$$

is compact. We plainly have that

$$K(t,u; W^m L^{\frac{n-d}{m},1}(B), W^m L^\infty(B)) \leq \psi(t)\|u\|_{W^m X(B)}$$

for every $u \in W^m X(B)$ and every $t > 0$, where

$$\psi(t) = \sup\left\{K(t,u; W^m L^{\frac{n-d}{m},1}(B), W^m L^\infty(B)) \colon \|u\|_{W^m X(B)} = 1\right\},\ t \in (0,\infty).$$

Furthermore, ψ is quasiconcave (cf. [10, Lemma 3.1.1]). It follows from [61, Theorem 1.1(i)] (cf. [33, Theorem 3.1(b)]) that $\mathrm{T}_\mu \colon W^m X(B) \to L^{\frac{n-d}{m}}(B,\mu)$ is compact provided that

$$\lim_{t\to 0^+} \psi(t) = 0. \tag{5.9}$$

Therefore, in order to prove the compactness of $\mathrm{T}_\mu \colon W^m X(B) \to L^{\frac{n-d}{m}}(B,\mu)$, it is sufficient to show that (5.9) is true. Thanks to (1.36), (1.33) and the Hölder inequality (1.14), we have that

$$\begin{aligned}
&\sup\left\{K(t,u; W^m L^{\frac{n-d}{m},1}(B), W^m L^\infty(B)) \colon \|u\|_{W^m X(B)} = 1\right\} \\
&\approx \sup\left\{\int_0^{t^{\frac{n-d}{m}}} s^{-1+\frac{m}{n-d}}\,|D^m u|^*(s)\,\mathrm{d}s \colon \|u\|_{W^m X(B)} = 1\right\} \\
&\lesssim \big\|s^{-1+\frac{m}{n-d}}\chi_{(0,t^{\frac{n-d}{m}})}(s)\big\|_{X'(0,1)}
\end{aligned}$$

for every $t \in (0,\infty)$; therefore, (5.9) follows from (5.6). Furthermore, if $\{u_j\}_{j=1}^\infty$ is a bounded sequence in $W_0^m X(\Omega)$, we obtain that $\{\mathrm{T}_\mu\, u_j\}_{j=1}^\infty$ contains a subsequence converging μ-a.e. to a function $g \in \mathfrak{M}(\Omega,\mu)$ by covering Ω with countably many open balls, using the compactness result that we have just established together with (1.12) and finally by employing the standard diagonalization argument. Moreover, thanks to Fatou's lemma ([8, Chapter 1, Theorem 1.7, (iii)]), we have that

$$\|g\|_{L^{\frac{n-d}{m}}(\Omega,\mu)} < \infty.$$

Second, assume that $d \in [n-m,n]$. The proof is similar to that for $d \in (0,n-m)$. Owing to (1.17), $W^m X(B)$ is an intermediate space between $W^m L^1(B)$ and $W^m L^\infty(B)$. The trace operator $\mathrm{T}_\mu\colon W^m L^1(B) \to L^{\frac{d}{n-m},1}(B,\mu)$ is bounded thanks to [32, Theorem 3.1]; furthermore, $\mathrm{T}_\mu\colon W^m L^\infty(B) \to L^{\frac{d}{n-m},1}(B,\mu)$ is compact for any open ball $B \subseteq \Omega$ ([4, Theorem 6.3]). Therefore, in order to establish the compactness of $\mathrm{T}_\mu\colon W^m X(B) \to L^{\frac{d}{n-m},1}(B,\mu)$, it is sufficient to prove that (5.9) holds with $W^m L^{\frac{n-d}{m},1}(B)$ replaced by $W^m L^1(B)$. To this end, note that, by combining (1.22) and (1.26) with (5.7), we have that

$$\lim_{a\to 0^+} \sup_{\|f\|_{X(0,1)}\le 1} \|f^*\,\chi_{(0,a)}\|_{L^1(0,1)} = 0.$$

Hence, by (1.36), (1.33) and the Hölder inequality (1.14), we have that

$$\begin{aligned}
&\lim_{t\to 0^+} \sup\big\{K(t,u;W^m L^1(B),W^m L^\infty(B))\colon \|u\|_{W^m X(B)} = 1\big\}\\
&\approx \lim_{t\to 0^+} \sup\left\{\int_0^t |D^m u|^*(s)\,\mathrm{d}s\colon \|D^m u\|_{X(B)} = 1\right\} = 0.
\end{aligned}$$

□

Remarks 5.1.2.

(i) If $d \in (0,n-m)$, then $\frac{m}{n-d} \in (0,1)$ and we have that

$$\begin{aligned}
\|t^{-1+\frac{m}{n-d}}\chi_{(0,r)}(t)\|_{X'(0,1)} &= \sup_{\|f\|_{X(0,1)}\le 1} \int_0^1 f^*(t)t^{-1+\frac{m}{n-d}}\chi_{(0,r)}(t)\,\mathrm{d}t\\
&= \sup_{\|f\|_{X(0,1)}\le 1} \|f^*\chi_{(0,r)}\|_{L^{\frac{n-d}{m},1}(0,1)}
\end{aligned}$$

for every $r \in (0,1)$, where we used (1.10) together with (1.9). Hence assumption (5.6) is equivalent to $X(0,1) \overset{*}{\hookrightarrow} L^{\frac{n-d}{m},1}(0,1)$.

(ii) By (1.21), assumption (5.7) is equivalent to

$$\lim_{t\to 0^+} \frac{t}{\varphi_{X(0,1)}(t)} = 0.$$

The next proposition provides us with a necessary condition for the compactness of (5.5) in terms of the Hardy-type operator $H_{v,\nu}$ defined by (2.2).

Proposition 5.1.3. *Let Ω be an open set in $\mathbb{R}^n$ such that $|\Omega| < \infty$. Let $\|\cdot\|_{X(0,1)}$ and $\|\cdot\|_{Y(0,1)}$ be rearrangement-invariant function norms. Let $m \in \mathbb{N}$, $m < n$. Let μ be a d-upper Ahlfors measure on Ω satisfying* (5.3)*. If the trace operator $\mathrm{T}_\mu \colon W^m_0 X(\Omega) \to Y(\Omega, \nu)$ is compact, then the operator $H_{v,\frac{d}{n}} \colon X(0,1) \to Y(0,1)$, where $v(t) = t^{-1+\frac{m}{n}}$, $t \in (0,1)$, is compact.*

Proof. We may, without loss of generality, assume that the R from (5.3) is such that $\overline{B}(x_1, R) \subseteq \Omega$. For each $f \in X(0,1)$, we define the function $u_f \colon \Omega \to \mathbb{R}$ as

$$u_f(x) = \begin{cases} \eta(x) \int_{\omega_n|x-x_1|^n}^{\omega_n R^n} \int_{r_1}^{\omega_n R^n} \cdots \int_{r_{m-1}}^{\omega_n R^n} f\big(\frac{r_m}{\omega_n R^n}\big) r_m^{-m+\frac{m}{n}} \,\mathrm{d}r_m \cdots \mathrm{d}r_1 & \text{if } m \geq 2, \\ \eta(x) \int_{\omega_n|x-x_1|^n}^{\omega_n R^n} f\big(\frac{r}{\omega_n R^n}\big) r^{-1+\frac{1}{n}} \,\mathrm{d}r & \text{if } m = 1, \end{cases}$$

where $\eta(x) = \chi_{(0,\omega_n R^n)}(\omega_n |x - x_1|^n)$, $x \in \Omega$. By [26, inequality (4.20)], we have that $u_f \in W^m_0 X(\Omega)$ and

$$\|u_f\|_{W^m X(\Omega)} \lesssim \|f\|_{X(0,1)} \quad \text{for every } f \in X(0,1). \tag{5.10}$$

Furthermore, owing to [31, inequalities (4.23) and (4.25)], we have that

$$\|H_{v,\frac{d}{n}} f\|_{Y(0,1)} \lesssim \|\mathrm{T}_\mu u_f\|_{Y(\Omega,\mu)} \quad \text{for every } f \in X(0,1). \tag{5.11}$$

Let $\{f_j\}_{j=0}^\infty$ be a bounded sequence in $X(0,1)$. Owing to (5.10), the sequences $\{u_{f_j^+}\}_{j=1}^\infty$ and $\{u_{f_j^-}\}_{j=1}^\infty$ are bounded in $W^m_0 X(\Omega)$. The compactness of $\mathrm{T}_\mu \colon W^m_0 X(\Omega) \to Y(\Omega, \nu)$ guarantees the existence of a subsequence $\{f_{j_k}\}_{k=1}^\infty$ of $\{f_j\}_{j=1}^\infty$ such that both $\{\mathrm{T}_\mu u_{f_{j_k}^+}\}_{k=1}^\infty$ and $\{\mathrm{T}_\mu u_{f_{j_k}^-}\}_{k=1}^\infty$ are Cauchy sequences in $Y(\Omega, \nu)$. Finally, since

$$\begin{aligned} \|H_{v,\frac{d}{n}} f - H_{v,\frac{d}{n}} g\|_{Y(0,1)} &\leq \|H_{v,\frac{d}{n}} f^+ - H_{v,\frac{d}{n}} g^+\|_{Y(0,1)} + \|H_{v,\frac{d}{n}} f^- - H_{v,\frac{d}{n}} g^-\|_{Y(0,1)} \\ &\leq \|H_{v,\frac{d}{n}} (f^+ - g^+)\|_{Y(0,1)} + \|H_{v,\frac{d}{n}} (f^- - g^-)\|_{Y(0,1)} \end{aligned}$$

and $u_{f-g} = u_f - u_g$ for every $f, g \in X(0,1)$, (5.11) implies that $\{H_{v,\frac{d}{n}}(f_{j_k})\}_{k=1}^\infty$ is a Cauchy sequence in $Y(0,1)$; hence $\{H_{v,\frac{d}{n}}(f_{j_k})\}_{k=1}^\infty$ converges in $Y(0,1)$. □

When μ is fast decaying or is of the special form (5.4), we manage to obtain a sharp compactness result. When the target function norm is equivalent to $\|\cdot\|_{L^\infty}$, the situation is different (and simpler) and will be dealt with later.

Theorem 5.1.4. *Let Ω be an open set in $\mathbb{R}^n$ such that $|\Omega| < \infty$. Let $\|\cdot\|_{X(0,1)}$ and $\|\cdot\|_{Y(0,1)}$ be rearrangement-invariant function norms. Assume that $\|\cdot\|_{Y(0,1)}$ is not equivalent to $\|\cdot\|_{L^\infty}$. Let $m \in \mathbb{N}$, $m < n$. Let μ be a d-upper Ahlfors measure on Ω. Assume that $d \in [n-m, n]$ or μ has the form* (5.4) *(and $d \in (0,n]$). Set $v(t) = t^{-1+\frac{m}{n}}$, $t \in (0,1)$, and consider the following six statements:*

(i) *the operator $H_{v,\frac{d}{n}} \colon X(0,1) \to Y(0,1)$ is compact;*

(ii) $\displaystyle\lim_{a\to 0^+} \sup_{\|f\|_{X(0,1)}\leq 1} \Big\| \int_{t^{\frac{n}{d}}}^1 |f(s)| s^{-1+\frac{m}{n}} \chi_{(0,a)}(s) \,\mathrm{d}s \Big\|_{Y(0,1)} = 0$*;*

(iii) $\displaystyle\lim_{a\to 0^+} \sup_{\|f\|_{X(0,1)}\leq 1} \Big\| \int_{t^{\frac{n}{d}}}^1 f^*(s) s^{-1+\frac{m}{n}} \chi_{(0,a)}(s) \,\mathrm{d}s \Big\|_{Y(0,1)} = 0$*;*

(iv) $\displaystyle\lim_{a\to 0^+}\sup_{\|f\|_{X(0,1)}\leq 1}\Big\|\chi_{(0,a)}(t)\int_{t^{\frac{n}{d}}}^{1}|f(s)|s^{-1+\frac{m}{n}}\,\mathrm{d}s\Big\|_{Y(0,1)}=0;$

(v) $\displaystyle\lim_{a\to 0^+}\sup_{\|f\|_{X(0,1)}\leq 1}\Big\|\chi_{(0,a)}(t)\int_{t^{\frac{n}{d}}}^{1}f^*(s)s^{-1+\frac{m}{n}}\,\mathrm{d}s\Big\|_{Y(0,1)}=0;$

(vi) we have that $Y_X(0,1)\overset{*}{\hookrightarrow}Y(0,1)$*, where* $Y_X(0,1)$ *is the optimal target space for the operator* $H_{v,\frac{d}{n}}$ *and the space* $X(0,1)$*, whose associate function norm satisfies*

$$\|f\|_{Y_X'(0,1)}=\Big\|t^{-1+\frac{m}{n}}\int_0^{t^{\frac{d}{n}}}f^*(s)\,\mathrm{d}s\Big\|_{X'(0,1)}\quad \textit{for every } f\in\mathfrak{M}^+(0,1).$$

These statements are equivalent, and each of them implies that $\mathrm{T}_\mu\colon W_0^mX(\Omega)\to Y(\Omega,\mu)$ *is compact. Furthermore, if, in addition,* μ *satisfies* (5.3)*, then each of* (i) - (vi) *is also necessary for* $\mathrm{T}_\mu\colon W_0^mX(\Omega)\to Y(\Omega,\mu)$ *to be compact.*

Proof. Note that the six statements are indeed equivalent owing to Proposition 2.3.2 (see also Remark 2.3.3). Furthermore, since $d\in[n-m,n]$ or μ has the form (5.4), we have that

$$\mathrm{T}_\mu\colon W_0^mX(\Omega)\to Y_X(\Omega,\mu) \tag{5.12}$$

is bounded by virtue of [31, Theorem 4.4] or [31, Theorem 5.3], respectively. Let $\{u_j\}_{j=1}^\infty$ be a bounded sequence in $W_0^mX(\Omega)$. We would like to show that $\{\mathrm{T}_\mu\,u_j\}_{j=1}^\infty$ contains a subsequence converging in $Y(\Omega,\mu)$. Thanks to [84, Theorem 3.1], in order to show that, it is sufficient to prove that $\{\mathrm{T}_\mu\,u_j\}_{j=1}^\infty$ contains a subsequence converging pointwise μ-a.e. in Ω inasmuch as we have that (5.12) and (by (vi))

$$Y_X(\Omega,\mu)\overset{*}{\hookrightarrow}Y(\Omega,\mu).$$

First, assume that $d\in(n-m,n]$. We claim that $\mathrm{T}_\mu\colon W^mX(B)\to L^1(B,\mu)$ is compact for every $B\subseteq\Omega$. To this end, note that, by [64, Theorem 1.4.6/1, the sufficiency part], every subset of $C^\infty(\overline{B})$ bounded in $W^mL^1(B)$ is precompact in $L^1(B,\mu)$. Since functions from $C^\infty(\overline{B})$ are dense in $W^mL^1(B)$, a routine approximation argument shows that $\mathrm{T}_\mu\colon W^mL^1(B)\to L^1(\Omega,\mu)$ is compact; hence $\mathrm{T}_\mu\colon W^mX(B)\to L^1(B,\mu)$ is also compact owing to (1.17). By covering Ω with countably many open balls and employing the standard diagonalization argument, we obtain that $\{\mathrm{T}_\mu\,u_j\}_{j=1}^\infty$ contains a subsequence converging pointwise μ-a.e. in Ω.

Next, assume that $d=n-m$. The fact that $\{\mathrm{T}_\mu\,u_j\}_{j=1}^\infty$ contains a subsequence converging pointwise μ-a.e. in Ω will follow from Proposition 5.1.1(ii) once we show that $X(0,1)\neq L^1(0,1)$. Owing to Proposition 2.2.43 with $\gamma=\frac{m}{n}$ and $\nu=\frac{d}{n}$, we have that

$$Y_{L^1}(0,1)=L^1(0,1). \tag{5.13}$$

Assumption (vi) together with (1.17) ensures that $Y_X(0,1)\overset{*}{\hookrightarrow}L^1(0,1)$, whence

$$Y_X(0,1)\neq L^1(0,1) \tag{5.14}$$

thanks to (1.26). By combining (5.13) and (5.14), we obtain that $X(0,1) \neq L^1(0,1)$.

Last, assume that $d \in (0, n-m)$ and μ has the form (5.4). Since μ is absolutely continuous with respect to the Lebesgue measure, we actually have that $\mathrm{T}_\mu u = u$ μ-a.e. in Ω for every $u \in W_0^m X(\Omega)$ (cf. [31, Theorem 5.3]). Furthermore, for every open ball $B \subseteq \Omega$, $W^m X(B) \hookrightarrow L^1(B)$ is compact by the Rellich–Kondrachov theorem combined with (1.17). Therefore, by covering Ω with countably many open balls and employing the standard diagonalization argument, we obtain that $\{u_j\}_{j=1}^\infty$ contains a subsequence converging pointwise a.e. in Ω. Hence $\{\mathrm{T}_\mu u_j\}_{j=1}^\infty$ contains a subsequence converging pointwise μ-a.e. in Ω.

Finally, if μ satisfies (5.3), then the necessity of (*i*) follows from Proposition 5.1.3. $\square$

For a general slowly decaying measure μ, we provide a sufficient condition for the compactness of (5.5), which can actually be used to produce sharp compactness results when, roughly speaking, the domain function norm is "stronger enough than" $\|\cdot\|_{L^{\frac{n-d}{m},1}(0,1)}$ (see Theorem 5.2.5).

Proposition 5.1.5. *Let Ω be an open set in $\mathbb{R}^n$ such that $|\Omega| < \infty$. Let $m \in \mathbb{N}$, $m < n$. Let μ be a d-upper Ahlfors measure on Ω. Assume that $d \in (0, n-m)$. Let $\|\cdot\|_{X(0,1)}$, $\|\cdot\|_{Y(0,1)}$ and $\|\cdot\|_{Z(0,1)}$ be rearrangement-invariant function norms such that $X(0,1) \overset{*}{\hookrightarrow} L^{\frac{n-d}{m},1}(0,1)$ and $Y(0,1) \overset{*}{\hookrightarrow} Z(0,1)$. If $\mathrm{T}_\mu \colon W_0^m X(\Omega) \to Y(\Omega,\mu)$ is bounded, then $\mathrm{T}_\mu \colon W_0^m X(\Omega) \to Z(\Omega,\mu)$ is compact.*

Proof. Owing to Remark 5.1.2(i), assumption $X(0,1) \overset{*}{\hookrightarrow} L^{\frac{n-d}{m},1}(0,1)$ is equivalent to (5.6). By combining that with Proposition 5.1.1, we obtain that every bounded sequence in $W_0^m X(\Omega)$ contains a subsequence converging pointwise μ-a.e. in Ω. Since we assume that $\mathrm{T}_\mu \colon W_0^m X(\Omega) \to Y(\Omega,\mu)$ is bounded and $Y(0,1) \overset{*}{\hookrightarrow} Z(0,1)$, the compactness of $\mathrm{T}_\mu \colon W_0^m X(\Omega) \to Z(\Omega,\mu)$ follows from [84, Theorem 3.1] (cf. [84, Theorem 3.2]). $\square$

Remarks 5.1.6.

(i) If $d \in (0, n-m)$, traces of functions from $W_0^m X(\Omega)$ on (Ω,μ) need not be well defined unless we assume that the rearrangement-invariant function norm $\|\cdot\|_{X(0,1)}$ is strong enough. When $\|\cdot\|_{X(0,1)}$ is at least as strong as the Lorentz norm $\|\cdot\|_{L^{\frac{n-d}{m},1}(0,1)}$, that is, $X(0,1) \hookrightarrow L^{\frac{n-d}{m},1}(0,1)$, the existence of $\mathrm{T}_\mu \colon W_0^m X(\Omega) \to L^1(\Omega,\mu)$ (we can actually replace $L^1(\Omega,\mu)$ by $L^{\frac{n-d}{m}}(\Omega,\mu)$) is guaranteed by [31, Theorem 3.3] (cf. [54]). In fact, this assumption on $\|\cdot\|_{X(0,1)}$ is also necessary if $d \in (0, n-m) \cap \mathbb{N}$, see [25]. In Proposition 5.1.5 the stronger relation $X(0,1) \overset{*}{\hookrightarrow} L^{\frac{n-d}{m},1}(0,1)$ is assumed. The stronger relation ensures that the space $X(0,1)$ is not "too close" to the endpoint space $L^{\frac{n-d}{m},1}(0,1)$, and so we can extract a subsequence converging pointwise μ-a.e. It is not, however, too restrictive. For example, if $X(0,1)$ is a Lorentz–Zygmund space $L^{p,q;\alpha,\beta}(0,1)$, then $L^{p,q;\alpha,\beta}(0,1) \overset{*}{\hookrightarrow} L^{\frac{n-d}{m},1}(0,1)$ if and only if $L^{p,q;\alpha,\beta}(0,1) \subsetneq L^{\frac{n-d}{m},1}(0,1)$.

(ii) In order to use Proposition 5.1.5 successfully, one needs to know when $\mathrm{T}_\mu \colon W_0^m X(\Omega) \to Y(\Omega,\mu)$ is bounded. Sufficient conditions for that are provided by [31, Theorems 5.1 and 5.2].

When the target function norm is equivalent to $\|\cdot\|_{L^\infty(0,1)}$, the situation changes significantly. It turns out that the speed of decay of μ on shrinking balls is immaterial.

Theorem 5.1.7. *Let Ω be an open set in $\mathbb{R}^n$ such that $|\Omega| < \infty$. Let $\|\cdot\|_{X(0,1)}$ be a rearrangement-invariant function norm. Let $m \in \mathbb{N}$, $m < n$. Let μ be a d-upper Ahlfors measure on Ω. Set $v(t) = t^{-1+\frac{m}{n}}$, $t \in (0,1)$, and consider the following five statements:*

(i) the operator $H_{v,\frac{d}{n}} \colon X(0,1) \to L^\infty(0,1)$ is compact;

(ii) $\lim\limits_{a\to 0^+} \sup\limits_{\|f\|_{X(0,1)}\le 1} \int_0^a |f(s)| s^{-1+\frac{m}{n}}\,\mathrm{d}s = 0$;

(iii) $\lim\limits_{a\to 0^+} \sup\limits_{\|f\|_{X(0,1)}\le 1} \int_0^a f^*(s) s^{-1+\frac{m}{n}}\,\mathrm{d}s = 0$;

(iv) $\lim\limits_{a\to 0^+} \|t^{-1+\frac{m}{n}}\chi_{(0,a)}(t)\|_{X'(0,1)} = 0$;

(v) $X(0,1) \overset{*}{\hookrightarrow} L^{\frac{n}{m},1}(0,1)$.

These statements are equivalent, and each of them implies that $\mathrm{T}_\mu \colon W_0^m X(\Omega) \to L^\infty(\Omega,\mu)$ is compact. Furthermore, if, in addition, μ satisfies (5.3), then each of (i) - (v) is also necessary for $\mathrm{T}_\mu \colon W_0^m X(\Omega) \to L^\infty(\Omega,\mu)$ to be compact.

Proof. The equivalence of the five statements follows from Proposition 2.3.4. Assume that (v) holds. By iterating the sharp Sobolev embedding due to Peetre ([76, Theorem 8.1]), we have that

$$W_0^1 L^{\frac{n}{m-j},1}(\Omega) \hookrightarrow L^{\frac{n}{m-j-1},1}(\Omega) \quad \text{for every } j \in \{0,1,\dots,m-2\}$$

if $m > 1$, whence we get that

$$W_0^m X(\Omega) \hookrightarrow W_0^m L^{\frac{n}{m},1}(\Omega) \hookrightarrow W_0^{m-1} L^{\frac{n}{m-1},1}(\Omega) \hookrightarrow \cdots \hookrightarrow W_0^1 L^{n,1}(\Omega).$$

Since $W_0^m X(\Omega) \hookrightarrow W_0^1 L^{n,1}(\Omega)$, we have that, owing to [88] combined with [23, Theorem 3.5],

$$W_0^m X(\Omega) \hookrightarrow C_b(\Omega); \tag{5.15}$$

consequently,

$$\mathrm{T}_\mu u = \tilde{u} \quad \text{for every } u \in W_0^m X(\Omega), \tag{5.16}$$

where $\tilde{u}$ is the continuous representative of u. Furthermore, $\mathrm{T}_\mu \colon W_0^m X(\Omega) \to L^\infty(\Omega,\mu)$ is bounded. Indeed, by (5.15) and (5.16), we have that

$$\|\mathrm{T}_\mu u\|_{L^\infty(\Omega,\mu)} = \|\tilde{u}\|_{L^\infty(\Omega,\mu)} \le \sup_{x\in\Omega} |\tilde{u}(x)| \lesssim \|u\|_{W_0^m X(\Omega)}.$$

As

$$\|\mathrm{T}_\mu u\|_{L^\infty(\Omega,\mu)} \le \|u\|_{L^\infty(\Omega)} \quad \text{for every } u \in W_0^m X(\Omega),$$

in order to prove that $W_0^m X(\Omega) \hookrightarrow L^\infty(\Omega,\mu)$ is compact, it is sufficient to prove that the embedding $W_0^m X(\Omega) \hookrightarrow L^\infty(\Omega)$ is compact. By [53, Theorem 1.2], this is equivalent to (*iii*) (and so also to (*v*)). Although embeddings on bounded Lipschitz domains with no restrictions on boundary values were considered there, one can readily check that their proof works for any open set Ω with $|\Omega| < \infty$ provided that we restrict ourselves to $W^m_0 X(\Omega)$ (see also [20, Proof of Theorem 1.2]).

Finally, if μ satisfies (5.3), then the necessity of (*i*) follows from Proposition 5.1.3. ☐

Remark 5.1.8. When $d = n$ and μ is the Lebesgue measure on Ω, Theorem 5.1.4 and Theorem 5.1.7 recover the sharp compactness results in the Euclidean setting obtained in [53] (cf. [85]). When $d \in [n - m, n] \cap \mathbb{N}$ and $\mu = \mathcal{H}^d_{|\Omega_d}$, where Ω_d is the (nonempty) intersection of Ω and a d-dimensional affine subspace of $\mathbb{R}^n$, these theorems recover the sharp compactness results for trace embeddings on d-dimensional affine subspaces obtained in [20].

For the sake of completeness, we conclude this section by covering the case where $m \geq n$. Not surprisingly, the speed of decay of μ on shrinking balls is entirely immaterial.

Proposition 5.1.9. *Let Ω be an open set in $\mathbb{R}^n$ such that $|\Omega| < \infty$. Let $\|\cdot\|_{X(0,1)}$ and $\|\cdot\|_{Y(0,1)}$ be rearrangement-invariant function norms. Let $m \in \mathbb{N}$. Let μ be a d-upper Ahlfors measure on Ω.*

- *(i) If $m = n$ and $\|\cdot\|_{X(0,1)}$ not equivalent to $\|\cdot\|_{L^1(0,1)}$, then the trace operator $\mathrm{T}_\mu \colon W^n_0 X(\Omega) \to Y(\Omega, \mu)$ is compact.*
- *(ii) If $m = n$ and $\|\cdot\|_{Y(0,1)}$ not equivalent to $\|\cdot\|_{L^\infty(0,1)}$, then the trace operator $\mathrm{T}_\mu \colon W^n_0 X(\Omega) \to Y(\Omega, \mu)$ is compact.*
- *(iii) If $m > n$, then the trace operator $\mathrm{T}_\mu \colon W^m_0 X(\Omega) \to Y(\Omega, \mu)$ is always compact.*

Proof. Recall that, when $m \geq n$,

$$W^m_0 L^1(\Omega) \hookrightarrow C_b(\Omega), \quad \text{and the embedding is compact if } m > n, \tag{5.17}$$

(e.g. [4, Theorems 4.12 and 6.28]). Hence, by (1.17), for every bounded sequence $\{u_j\}_{j=0}^\infty$ in $W^m_0 X(\Omega)$, the sequence $(\tilde{u}_j)_{j=1}^\infty$ is bounded in $L^\infty(\Omega, \mu)$, where $\tilde{u}_j$ is the continuous representative of u_j. We would like to prove that there is a subsequence $\{\tilde{u}_{j_k}\}_{k=1}^\infty$ that converges in $Y(\Omega, \mu)$.

If $m > n$, the assertion immediately follows from (1.17) and (5.17).

If $m = n$ and $\|\cdot\|_{Y(0,1)}$ is not equivalent to $\|\cdot\|_{L^\infty(0,1)}$, by (1.17) and the standard Sobolev embedding $W^n_0 L^1(\Omega) \hookrightarrow W^{n-1}_0 L^{\frac{n}{n-1}}(\Omega)$ (e.g. [4, Theorem 4.12]), Proposition 5.1.1 ensures that $\{\tilde{u}_j\}_{j=1}^\infty$ contains a subsequence $\{\tilde{u}_{j_k}\}_{k=1}^\infty$ that converges pointwise μ-a.e. in Ω. Note that $\frac{n}{n-1} > \max\{1, \frac{n-d}{n-1}\}$. Therefore, [84, Theorems 3.1 and 5.2] guarantee that $\{\tilde{u}_{j_k}\}_{k=1}^\infty$ converges in $Y(\Omega, \mu)$.

If $m = n$ and $\|\cdot\|_{X(0,1)}$ is not equivalent to $\|\cdot\|_{L^1(0,1)}$, we have that the embedding $W^n_0 X(\Omega) \hookrightarrow L^\infty(\Omega)$ is compact (e.g. [20, Theorem 1.2]). Hence, thanks to (5.17) and (1.17), the conclusion follows from a line of argument similar to that in the proof of Theorem 5.1.7. ☐

5.2 Particular examples

In this section, we shall provide compactness results for the trace operator

$$\mathrm{T}_\mu \colon W_0^m L^{p_1,q_1;\alpha_1}(\Omega) \to L^{p_2,q_2;\alpha_2,\beta}(\Omega,\mu).$$

We consider two tiers of logarithms in the target space so that we can capture even delicate limiting cases.

First, we consider fast decaying measures, that is, $d \in [n-m, n]$.

Theorem 5.2.1. *Let Ω be an open set in $\mathbb{R}^n$ such that $|\Omega| < \infty$. Let $m \in \mathbb{N}$, $m < n$. Let μ be a d-upper Ahlfors measure on Ω. Assume that $d \in [n-m, n]$. Assume that the parameters $p_1, q_1, p_2, q_2 \in [1,\infty]$, $\alpha_1, \alpha_2, \beta \in \mathbb{R}$ are such that $\|\cdot\|_{L^{p_1,q_1;\alpha_1}(0,1)}$ and $\|\cdot\|_{L^{p_2,q_2;\alpha_2,\beta}(0,1)}$ are equivalent to rearrangement-invariant function norms, that is, the parameters satisfy one of conditions (1.29). If one of the following conditions is true:*

- $p_1 = q_1 = 1$, $\alpha_1 \geq 0$, $d > n-m$ *and*
    - ⋆ $p_2 < \frac{d}{n-m}$ *or*
    - ⋆ $p_2 = \frac{d}{n-m}$, $\alpha_2 < \alpha_1$ *or*
    - ⋆ $p_2 = \frac{d}{n-m}$, $\alpha_2 = \alpha_1$, $\beta < 0$;
- $p_1 = q_1 = 1$, $\alpha_1 > 0$, $d = n-m$ *and*
    - ⋆ $p_2 = q_2 = 1$, $0 < \alpha_2 < \alpha_1$ *or*
    - ⋆ $p_2 = q_2 = 1$, $0 = \alpha_2 < \alpha_1$, $\beta \geq 0$ *or*
    - ⋆ $p_2 = q_2 = 1$, $0 < \alpha_2 = \alpha_1$, $\beta < 0$;
- $p_1 \in (1, \frac{n}{m})$ *and*
    - ⋆ $p_2 < \frac{dp_1}{n-mp_1}$ *or*
    - ⋆ $p_2 = \frac{dp_1}{n-mp_1}$, $\alpha_2 < \alpha_1 + \min\{\frac{1}{q_1} - \frac{1}{q_2}, 0\}$ *or*
    - ⋆ $p_2 = \frac{dp_1}{n-mp_1}$, $\alpha_2 = \alpha_1 + \min\{\frac{1}{q_1} - \frac{1}{q_2}, 0\}$, $\beta < \min\{\frac{1}{q_1} - \frac{1}{q_2}, 0\}$;
- $p_1 = \frac{n}{m}$, $\alpha_1 < 1 - \frac{1}{q_1}$ *and*
    - ⋆ $p_2 < \infty$ *or*
    - ⋆ $p_2 = \infty$, $\alpha_2 + \frac{1}{q_2} < \alpha_1 + \frac{1}{q_1} - 1$ *or*
    - ⋆ $p_2 = \infty$, $\alpha_2 + \frac{1}{q_2} = \alpha_1 + \frac{1}{q_1} - 1$, $\beta < \min\{\frac{1}{q_1} - \frac{1}{q_2}, 0\}$;
- $p_1 = \frac{n}{m}$, $\alpha_1 = 1 - \frac{1}{q_1}$ *and*
    - ⋆ $p_2 < \infty$ *or*
    - ⋆ $p_2 = \infty$, $\alpha_2 + \frac{1}{q_2} < 0$ *or*
    - ⋆ $p_2 = \infty$, $\alpha_2 + \frac{1}{q_2} = 0$, $\beta + \frac{1}{q_2} < -1 + \frac{1}{q_1}$;
- *either* $p_1 = \frac{n}{m}$, $\alpha_1 > 1 - \frac{1}{q_1}$ *or* $p_1 > \frac{n}{m}$,

then the trace operator $\mathrm{T}_\mu\colon W_0^m L^{p_1,q_1;\alpha_1}(\Omega) \to L^{p_2,q_2;\alpha_2,\beta}(\Omega,\mu)$ *is compact. Furthermore, if* μ *satisfies* (5.3)*, then these restrictions on the parameters are also necessary for* $\mathrm{T}_\mu\colon W_0^m L^{p_1,q_1;\alpha_1}(\Omega) \to L^{p_2,q_2;\alpha_2,\beta}(\Omega,\mu)$ *to be compact.*

Proof. The assertion is an immediate consequence of Theorem 5.1.4 and Theorem 5.1.7 combined with Proposition 2.3.6 with $\gamma = \frac{m}{n}$ and $\nu = \frac{d}{n}$. $\square$

Corollary 5.2.2. *Let* Ω *be an open set in* $\mathbb{R}^n$ *such that* $|\Omega| < \infty$*. Let* $m \in \mathbb{N}$*,* $m < n$*. Let* μ *be a* d*-upper Ahlfors measure on* Ω*. Assume that* $d \in [n-m, n]$*. Let* $p_1, q_1, p_2, q_2 \in [1,\infty]$*.*

- *If* $p_2 < \frac{d}{n-m}$ *and* $d > n-m$*, then* $\mathrm{T}_\mu\colon W_0^m L^1(\Omega) \to L^{p_2,1}(\Omega,\mu)$ *is compact. If* μ *satisfies* (5.3) *and* $d > n-m$*, then* $\mathrm{T}_\mu\colon W_0^m L^1(\Omega) \to L^{\frac{d}{n-m},\infty}(\Omega,\mu)$ *is not compact. If* μ *satisfies* (5.3) *and* $d = n-m$*, then* $\mathrm{T}_\mu\colon W_0^m L^1(\Omega) \to L^1(\Omega,\mu)$ *is not compact.*
- *If* $p_1 \in (1, \frac{n}{m})$ *and* $p_2 < \frac{dp_1}{n-mp_1}$*, then* $\mathrm{T}_\mu\colon W_0^m L^{p_1,\infty}(\Omega) \to L^{p_2,1}(\Omega,\mu)$ *is compact. If* $p_1 \in (1, \frac{n}{m})$ *and* μ *satisfies* (5.3)*, then* $\mathrm{T}_\mu\colon W_0^m L^{p_1,1}(\Omega) \to L^{\frac{dp_1}{n-mp_1},\infty}(\Omega,\mu)$ *is not compact.*
- *If* $p_2 < \infty$*, then* $\mathrm{T}_\mu\colon W_0^m L^{\frac{n}{m},\infty}(\Omega) \to L^{p_2,1}(\Omega,\mu)$ *is compact. If* μ *satisfies* (5.3)*, then* $\mathrm{T}_\mu\colon W_0^m L^{\frac{n}{m},1}(\Omega) \to L^\infty(\Omega,\mu)$ *is not compact.*
- *If* $p_1 > \frac{n}{m}$*, then* $\mathrm{T}_\mu\colon W_0^m L^{p_1,\infty}(\Omega) \to L^\infty(\Omega,\mu)$ *is compact.*

Second, we deal with the particular case where μ has the form (5.4). Since the case where $d \in [n-m, n]$ was already dealt with in the preceding theorem, we restrict ourselves to the case where $d < n-m$. The proof is the same as that of the preceding theorem.

Theorem 5.2.3. *Let* Ω *be an open set in* $\mathbb{R}^n$ *such that* $|\Omega| < \infty$*. Let* $m \in \mathbb{N}$*,* $m < n$*. Let* μ *be a* d*-upper Ahlfors measure on* Ω *having the form* (5.4)*. Assume that* $d \in (0, n-m)$*. Assume that the parameters* $p_1, q_1, p_2, q_2 \in [1,\infty]$*,* $\alpha_1, \alpha_2, \beta \in \mathbb{R}$ *are such that* $\|\cdot\|_{L^{p_1,q_1;\alpha_1}(0,1)}$ *and* $\|\cdot\|_{L^{p_2,q_2;\alpha_2,\beta}(0,1)}$ *are equivalent to rearrangement-invariant function norms, that is, the parameters satisfy one of conditions* (1.29)*. If one of the following conditions is true:*

- $p_1 = \frac{n}{m+d}$*,* $q_1 = 1$*,* $\alpha_1 > 0$ *and*
    - ⋆ $p_2 = q_2 = 1$*,* $0 < \alpha_2 < \alpha_1$ *or*
    - ⋆ $p_2 = q_2 = 1$*,* $0 = \alpha_2 < \alpha_1$*,* $\beta \geq 0$ *or*
    - ⋆ $p_2 = q_2 = 1$*,* $0 < \alpha_2 = \alpha_1$*,* $\beta < 0$*;*
- $p_1 = \frac{n}{m+d}$*,* $q \in (1,\infty]$*,* $\alpha_1 > 1 - \frac{1}{q_1}$ *and*
    - ⋆ $p_2 = q_2 = 1$*,* $0 < \alpha_2 < \alpha_1 - 1 + \frac{1}{q_1}$ *or*
    - ⋆ $p_2 = q_2 = 1$*,* $\alpha_2 = 0$ *and* $\beta \geq 0$ *or*
    - ⋆ $p_2 = q_2 = 1$*,* $\alpha_2 = \alpha_1 - 1 + \frac{1}{q_1}$ *and* $\beta < -1 + \frac{1}{q_1}$*;*
- $p_1 \in (\frac{n}{m+d}, \frac{n}{m})$ *and*

  - $\star$ $p_2 < \frac{dp_1}{n-mp_1}$ *or*
  - $\star$ $p_2 = \frac{dp_1}{n-mp_1}$, $\alpha_2 < \alpha_1 + \min\{\frac{1}{q_1} - \frac{1}{q_2}, 0\}$ *or*
  - $\star$ $p_2 = \frac{dp_1}{n-mp_1}$, $\alpha_2 = \alpha_1 + \min\{\frac{1}{q_1} - \frac{1}{q_2}, 0\}$, $\beta < \min\{\frac{1}{q_1} - \frac{1}{q_2}, 0\}$*;*

- $p_1 = \frac{n}{m}$, $\alpha_1 < 1 - \frac{1}{q_1}$ *and*
  - $\star$ $p_2 < \infty$ *or*
  - $\star$ $p_2 = \infty$, $\alpha_2 + \frac{1}{q_2} < \alpha_1 + \frac{1}{q_1} - 1$ *or*
  - $\star$ $p_2 = \infty$, $\alpha_2 + \frac{1}{q_2} = \alpha_1 + \frac{1}{q_1} - 1$, $\beta < \min\{\frac{1}{q_1} - \frac{1}{q_2}, 0\}$*;*

- $p_1 = \frac{n}{m}$, $\alpha_1 = 1 - \frac{1}{q_1}$ *and*
  - $\star$ $p_2 < \infty$ *or*
  - $\star$ $p_2 = \infty$, $\alpha_2 + \frac{1}{q_2} < 0$ *or*
  - $\star$ $p_2 = \infty$, $\alpha_2 + \frac{1}{q_2} = 0$, $\beta + \frac{1}{q_2} < -1 + \frac{1}{q_1}$*;*

- *either* $p_1 = \frac{n}{m}$, $\alpha_1 > 1 - \frac{1}{q_1}$ *or* $p_1 > \frac{n}{m}$*,*

then the trace operator $\mathrm{T}_\mu\colon W_0^m L^{p_1,q_1;\alpha_1}(\Omega) \to L^{p_2,q_2;\alpha_2,\beta}(\Omega,\mu)$ *is compact. Furthermore, if* μ *satisfies* (5.3)*, then these restrictions on the parameters are also necessary for* $\mathrm{T}_\mu\colon W_0^m L^{p_1,q_1;\alpha_1}(\Omega) \to L^{p_2,q_2;\alpha_2,\beta}(\Omega,\mu)$ *to be compact.*

Corollary 5.2.4. *Let* Ω *be an open set in* $\mathbb{R}^n$ *such that* $|\Omega| < \infty$*. Let* $m \in \mathbb{N}$*,* $m < n$*. Let* μ *be a* d*-upper Ahlfors measure on* Ω *having the form* (5.4)*. Assume that* $d \in (0, n-m)$*. Let* $p_1, q_1, p_2, q_2 \in [1,\infty]$*.*

- *If* μ *satisfies* (5.3)*, then* $\mathrm{T}_\mu\colon W_0^m L^{\frac{n}{m+d},1}(\Omega) \to L^1(\Omega,\mu)$ *is not compact.*
- *If* $p_1 \in (\frac{n}{m+d}, \frac{n}{m})$ *and* $p_2 < \frac{dp_1}{n-mp_1}$*, then* $\mathrm{T}_\mu\colon W_0^m L^{p_1,\infty}(\Omega) \to L^{p_2,1}(\Omega,\mu)$ *is compact. If* $p_1 \in (\frac{n}{m+d}, \frac{n}{m})$ *and* μ *satisfies* (5.3)*, then* $\mathrm{T}_\mu\colon W_0^m L^{p_1,1}(\Omega) \to L^{\frac{dp_1}{n-mp_1},\infty}(\Omega,\mu)$ *is not compact.*
- *If* $p_2 < \infty$*, then* $\mathrm{T}_\mu\colon W_0^m L^{\frac{n}{m},\infty}(\Omega) \to L^{p_2,1}(\Omega,\mu)$ *is compact. If* μ *satisfies* (5.3)*, then* $\mathrm{T}_\mu\colon W_0^m L^{\frac{n}{m},1}(\Omega) \to L^\infty(\Omega,\mu)$ *is not compact.*
- *If* $p_1 > \frac{n}{m}$*, then* $\mathrm{T}_\mu\colon W_0^m L^{p_1,\infty}(\Omega) \to L^\infty(\Omega,\mu)$ *is compact.*

Next, we consider general slowly decaying measures μ, that is, without the extra information that μ has the form (5.4).

Theorem 5.2.5. *Let* Ω *be an open set in* $\mathbb{R}^n$ *such that* $|\Omega| < \infty$*. Let* $m \in \mathbb{N}$*,* $m < n$*. Let* μ *be a* d*-upper Ahlfors measure on* Ω*. Assume that* $d \in (0, n-m)$*. Assume that the parameters* $p_1, q_1, p_2, q_2 \in [1,\infty]$*,* $\alpha_1, \alpha_2, \beta \in \mathbb{R}$ *are such that* $\|\cdot\|_{L^{p_1,q_1;\alpha_1}(0,1)}$ *and* $\|\cdot\|_{L^{p_2,q_2;\alpha_2,\beta}(0,1)}$ *are equivalent to rearrangement-invariant function norms, that is, the parameters satisfy one of conditions* (1.29)*. If one of the following conditions is true:*

- $p_1 = \frac{n-d}{m}$, $\alpha_1 > 1 - \frac{1}{q}$ *and*
  - $\star$ $p_2 < \frac{n-d}{m}$ *or*

- $\star$ $p_2 = \frac{n-d}{m}$, $\alpha_2 < \alpha_1 - 1 + \frac{1}{q_1} - \frac{1}{q_2}$ *or*
- $\star$ $p_2 = \frac{n-d}{m}$, $\alpha_2 = \alpha_1 - 1 + \frac{1}{q_1} - \frac{1}{q_2}$, $\beta < \min\{\frac{1}{q_1} - \frac{1}{q_2}, 0\}$*;*

- $p_1 \in (\frac{n-d}{m}, \frac{n}{m})$ *and*
  - $\star$ $p_2 < \frac{dp_1}{n-mp_1}$ *or*
  - $\star$ $p_2 = \frac{dp_1}{n-mp_1}$, $\alpha_2 < \alpha_1 + \min\{\frac{1}{q_1} - \frac{1}{q_2}, 0\}$ *or*
  - $\star$ $p_2 = \frac{dp_1}{n-mp_1}$, $\alpha_2 = \alpha_1 + \min\{\frac{1}{q_1} - \frac{1}{q_2}, 0\}$, $\beta < \min\{\frac{1}{q_1} - \frac{1}{q_2}, 0\}$*;*
- $p_1 = \frac{n}{m}$, $\alpha_1 < 1 - \frac{1}{q_1}$ *and*
  - $\star$ $p_2 < \infty$ *or*
  - $\star$ $p_2 = \infty$, $\alpha_2 + \frac{1}{q_2} < \alpha_1 + \frac{1}{q_1} - 1$ *or*
  - $\star$ $p_2 = \infty$, $\alpha_2 + \frac{1}{q_2} = \alpha_1 + \frac{1}{q_1} - 1$, $\beta < \min\{\frac{1}{q_1} - \frac{1}{q_2}, 0\}$*;*
- $p_1 = \frac{n}{m}$, $\alpha_1 = 1 - \frac{1}{q_1}$ *and*
  - $\star$ $p_2 < \infty$ *or*
  - $\star$ $p_2 = \infty$, $\alpha_2 + \frac{1}{q_2} < 0$ *or*
  - $\star$ $p_2 = \infty$, $\alpha_2 + \frac{1}{q_2} = 0$, $\beta + \frac{1}{q_2} < -1 + \frac{1}{q_1}$*;*
- *either* $p_1 = \frac{n}{m}$, $\alpha_1 > 1 - \frac{1}{q_1}$ *or* $p_1 > \frac{n}{m}$,

then the trace operator $\mathrm{T}_\mu \colon W_0^m L^{p_1,q_1;\alpha_1}(\Omega) \to L^{p_2,q_2;\alpha_2,\beta}(\Omega,\mu)$ *is compact. Furthermore, if* μ *satisfies* (5.3) *and* $p_1 > \frac{n-d}{m}$*, then these restrictions on the parameters are also necessary for* $\mathrm{T}_\mu \colon W_0^m L^{p_1,q_1;\alpha_1}(\Omega) \to L^{p_2,q_2;\alpha_2,\beta}(\Omega,\mu)$ *to be compact.*

Proof. Set $H = H_{v,\nu}$, where $v(t) = t^{-1+\frac{m}{n}}$, $t \in (0,1)$, $\nu = \frac{d}{n}$. We define the sublinear operator S as

$$Sf(t) = t^{-\frac{m}{n-d}} \int_0^{t^{\frac{n}{d}}} f^*(s) s^{-1+\frac{m}{n-d}}\,\mathrm{d}s,\ t \in (0,1),\ f \in \mathfrak{M}(0,1).$$

We denote by $Y_{L^{p_1,q_1;\alpha_1}}(0,1)$ the optimal target space for the operator H and the space $L^{p_1,q_1;\alpha_1}(0,1)$. Owing to Proposition 2.2.43, we have that (note that $\frac{n}{m+d} < \frac{n-d}{m}$ inasmuch as $d < n-m$)

$$Y_{L^{p_1,q_1;\alpha_1}}(0,1) = \begin{cases} L^{\frac{dp_1}{n-mp_1},q_1;\alpha_1}(0,1), & p_1 \in [\frac{n-d}{m}, \frac{n}{m});\\ L^{\infty,q_1;\alpha_1-1}(0,1), & p_1 = \frac{n}{m},\ \alpha_1 < 1 - \frac{1}{q_1};\\ L^{\infty,q_1;-\frac{1}{q_1},-1}(0,1), & p_1 = \frac{n}{m},\ q_1 \in (1,\infty],\ \alpha_1 = 1 - \frac{1}{q_1};\\ L^\infty(0,1), & p_1 = \frac{n}{m},\ \alpha_1 > 1 - \frac{1}{q_1} \text{ or}\\ & p_1 = \frac{n}{m},\ q_1 = 1,\ \alpha_1 \geq 0 \text{ or}\\ & p_1 > \frac{n}{m}. \end{cases}$$

First, if either $p_1 = \frac{n}{m}$, $\alpha_1 > 1 - \frac{1}{q_1}$ or $p_1 > \frac{n}{m}$, then $L^{p_1,q_1;\alpha_1}(0,1) \overset{*}{\hookrightarrow} L^{\frac{n}{m},1}(0,1)$ (see (2.133)). By Theorem 5.1.7, $\mathrm{T}_\mu \colon W_0^m L^{p_1,q_1;\alpha_1}(\Omega) \to L^\infty(\Omega,\mu)$ is compact; hence $\mathrm{T}_\mu \colon W_0^m L^{p_1,q_1;\alpha_1}(\Omega) \to L^{p_2,q_2;\alpha_2,\beta}(\Omega,\mu)$ is compact.

Second, assume that either $p_1 = \frac{n}{m}$, $\alpha_1 \leq 1 - \frac{1}{q_1}$ or $p_1 \in (\frac{n-d}{m}, \frac{n}{m})$. We claim that $S\colon L^{p_1,q_1;\alpha_1}(0,1) \to Y_{L^{p_1,q_1;\alpha_1}}(0,1)$ is bounded. To this end, note that we have the following endpoint estimates for S ([32, Proposition 3.5]):

$$S\colon L^{\frac{n-d}{m},1}(0,1) \to L^{\frac{n-d}{m},\infty}(0,1) \quad \text{and} \quad S\colon L^{\frac{n}{m},\infty}(0,1) \to L^{\infty}(0,1). \tag{5.18}$$

If $p_1 \in (\frac{n-d}{m}, \frac{n}{m})$, then the fact that $S\colon L^{p_1,q_1;\alpha_1}(0,1) \to L^{\frac{dp_1}{n-mp_1},q_1;\alpha_1}(0,1)$ is bounded follows from the endpoint estimates by non-limiting interpolation thanks to [41, Theorem 4.1]. If $p_1 = \frac{n}{m}$ and $\alpha_1 < 1 - \frac{1}{q_1}$, then we obtain the boundedness of $S\colon L^{\frac{n}{m},q_1;\alpha_1}(0,1) \to L^{\infty,q_1;\alpha_1}(0,1)$ from the endpoint estimates by single-limiting interpolation thanks to [41, Theorem 4.2]. Finally, if $p_1 = \frac{n}{m}$, $q \in (1,\infty]$ and $\alpha_1 = 1 - \frac{1}{q_1}$, we have that

$$\|Sf\|_{L^{\infty,q_1;-\frac{1}{q_1},-1}(0,1)} \lesssim \|Sf\|_{L^{\infty}(0,1)} \lesssim \|f\|_{L^{\frac{n}{m},\infty}(0,1)} \lesssim \|f\|_{L^{\frac{n}{m},q_1;1-\frac{1}{q_1}}(0,1)}$$

for every $f \in \mathfrak{M}(0,1)$ owing to the fact that $L^{\infty}(0,1) \hookrightarrow L^{\infty,q_1;-\frac{1}{q_1},-1}(0,1)$ and $L^{\frac{n}{m},q_1;1-\frac{1}{q_1}}(0,1) \hookrightarrow L^{\frac{n}{m},\infty}(0,1)$ ([75, Theorem 4.5]). Now that we know that $S\colon L^{p_1,q_1;\alpha_1}(0,1) \to Y_{L^{p_1,q_1;\alpha_1}}(0,1)$ is bounded ($H\colon L^{p_1,q_1;\alpha_1}(0,1) \to Y_{L^{p_1,q_1;\alpha_1}}(0,1)$ is, of course, also bounded), [31, Theorem 5.1] implies that

$$\mathrm{T}_{\mu}\colon W^m_0 L^{p_1,q_1;\alpha_1}(\Omega) \to Y_{L^{p_1,q_1;\alpha_1}}(\Omega,\mu)$$

is bounded. Therefore, thanks to Proposition 5.1.5,

$$\mathrm{T}_{\mu}\colon W^m_0 L^{p_1,q_1;\alpha_1}(\Omega) \to L^{p_2,q_2;\alpha_2,\beta}(\Omega,\mu)$$

is compact whenever $Y_{L^{p_1,q_1;\alpha_1}}(0,1) \overset{*}{\hookrightarrow} L^{p_2,q_2;\alpha_2,\beta}(0,1)$ (note that, since $p_1 > \frac{n-d}{m}$, $L^{p_1,q_1;\alpha_1}(0,1) \overset{*}{\hookrightarrow} L^{\frac{n-d}{m},1}(0,1)$).

Last, assume that $p_1 = \frac{n-d}{m}$ and $\alpha_1 > 1 - \frac{1}{q_1}$. Note that $L^{\frac{n-d}{m},q_1;\alpha_1}(0,1) \overset{*}{\hookrightarrow} L^{\frac{n-d}{m},1}(0,1)$ thanks to (2.133). Fix any $\gamma \in (-\infty, \min\{\frac{1}{q_1} - \frac{1}{q_2}, 0\})$. By combining (5.18) and [41, Theorem 4.2], we obtain that

$$S\colon L^{\frac{n-d}{m},q_1;\alpha_1}(0,1) \to L^{\frac{n-d}{m},q_2;\alpha_1-1+\frac{1}{q_1}-\frac{1}{q_2},\gamma}(0,1) \tag{5.19}$$

is bounded inasmuch as $\gamma < \min\{\frac{1}{q_1} - \frac{1}{q_2}, 0\}$. Furthermore, note that $Y_{L^{\frac{n-d}{m},q_1;\alpha_1}}(0,1) = L^{\frac{n-d}{m},q_1;\alpha_1}(0,1)$ and $L^{\frac{n-d}{m},q_1;\alpha_1}(0,1) \hookrightarrow L^{\frac{n-d}{m},q_2;\alpha_1-1+\frac{1}{q_1}-\frac{1}{q_2},\gamma}(0,1)$ ([75, Theorem 4.5]); hence we also have that

$$H\colon L^{\frac{n-d}{m},q_1;\alpha_1}(0,1) \to L^{\frac{n-d}{m},q_2;\alpha_1-1+\frac{1}{q_1}-\frac{1}{q_2},\gamma}(0,1). \tag{5.20}$$

Owing to (5.19) and (5.20), [31, Theorem 5.1] implies that

$$\mathrm{T}_{\mu}\colon W^m_0 L^{\frac{n-d}{m},q_1;\alpha_1}(\Omega) \to L^{\frac{n-d}{m},q_2;\alpha_1-1+\frac{1}{q_1}-\frac{1}{q_2},\gamma}(\Omega,\mu)$$

is bounded. Now, if $\alpha_2 < \alpha_1 - 1 + \frac{1}{q_1} - \frac{1}{q_2}$, then $L^{\frac{n-d}{m},q_2;\alpha_1-1+\frac{1}{q_1}-\frac{1}{q_2},\gamma}(0,1) \overset{*}{\hookrightarrow} L^{\frac{n-d}{m},q_2;\alpha_2,\beta}(0,1)$, no matter what γ is, thanks to (2.133). If $\alpha_2 = \alpha_1 - 1 + \frac{1}{q_1} - \frac{1}{q_2}$, then $L^{\frac{n-d}{m},q_2;\alpha_1-1+\frac{1}{q_1}-\frac{1}{q_2},\gamma}(0,1) \overset{*}{\hookrightarrow} L^{\frac{n-d}{m},q_2;\alpha_2,\beta}(0,1)$ thanks to (2.133) provided that $\beta < \gamma$. Hence

$$\mathrm{T}_\mu \colon W_0^m L^{\frac{n-d}{m},q_1;\alpha_1}(\Omega) \to L^{\frac{n-d}{m},q_2;\alpha_2,\beta}(\Omega,\mu)$$

is compact owing to Proposition 5.1.5 (if $\alpha_2 = \alpha_1 - 1 + \frac{1}{q_1} - \frac{1}{q_2}$, we take any $\gamma \in (\beta, \min\{\frac{1}{q_1} - \frac{1}{q_2}, 0\})$.

Finally, if μ satisfies (5.3) and $p_1 > \frac{n-d}{m}$, the necessity of the restrictions on the parameters follows from Proposition 5.1.3 combined with Proposition 2.3.6. $\square$

Corollary 5.2.6. *Let Ω be an open set in $\mathbb{R}^n$ such that $|\Omega| < \infty$. Let $m \in \mathbb{N}$, $m < n$. Let μ be a d-upper Ahlfors measure on Ω. Assume that $d \in (0, n-m)$. Let $p_1, q_1, p_2, q_2 \in [1,\infty]$.*

- *If $p_1 \in (\frac{n-d}{m}, \frac{n}{m})$ and $p_2 < \frac{dp_1}{n-mp_1}$, then $\mathrm{T}_\mu \colon W_0^m L^{p_1,\infty}(\Omega) \to L^{p_2,1}(\Omega,\mu)$ is compact. If $p_1 \in (\frac{n-d}{m}, \frac{n}{m})$ and μ satisfies (5.3), then $\mathrm{T}_\mu \colon W_0^m L^{p_1,1}(\Omega) \to L^{\frac{dp_1}{n-mp_1},\infty}(\Omega,\mu)$ is not compact.*
- *If $p_2 < \infty$, then $\mathrm{T}_\mu \colon W_0^m L^{\frac{n}{m},\infty}(\Omega) \to L^{p_2,1}(\Omega,\mu)$ is compact. If μ satisfies (5.3), then $\mathrm{T}_\mu \colon W_0^m L^{\frac{n}{m},1}(\Omega) \to L^\infty(\Omega,\mu)$ is not compact.*
- *If $p_1 > \frac{n}{m}$, then $\mathrm{T}_\mu \colon W_0^m L^{p_1,\infty}(\Omega) \to L^\infty(\Omega,\mu)$ is compact.*

5.3 A few "counterexamples"

Let Ω be an open set in $\mathbb{R}^n$ such that $|\Omega| < \infty$. Let $m \in \mathbb{N}$, $m < n$. Let μ be a d-upper Ahlfors measure on Ω, where $d \in [n-m, n]$, that satisfies (5.3). Let $p, q, r, s \in [1,\infty]$ be such that $L^{p,q}(0,1)$ and $L^{r,s}(0,1)$ are Lorentz spaces. Assume that $p < \frac{n}{m}$. It follows from Corollary 5.2.2 that the trace embedding

$$\mathrm{T}_\mu \colon W_0^m L^{p,q}(\Omega) \to L^{r,s}(\Omega,\mu) \tag{5.21}$$

is compact if and only if $r \in [1, \frac{dp}{n-mp})$. In other words, (5.21) is compact if and only if $L^{\frac{dp}{n-mp},\infty}(0,1) \subsetneq L^{r,s}(0,1)$. Note that, owing to (1.19), $L^{\frac{dp}{n-mp},\infty}(0,1)$ is the biggest rearrangement-invariant function space on the same "fundamental level" as $L^{\frac{dp}{n-mp},q}(0,1)$, the optimal target space in (5.21) for $X(0,1) = L^{p,q}(0,1)$ ([32, Theorem 3.1]). In yet other words, (5.21) is compact if and only if

$$\lim_{t\to 0^+} \frac{\varphi_{L^{r,s}(0,1)}(t)}{\varphi_{L^{\frac{dp}{n-mp},q}(0,1)}(t)} = 0.$$

We can see that the question of whether the trace embedding (5.21) is compact is completely determined by the fundamental functions of the corresponding Lorentz spaces. However, as was foreshadowed in Section 2.3.2, the situation is much more complicated in the general setting of rearrangement-invariant function spaces. The aim of this section is to provide a few "counterexamples" illustrating that some principles concerning compactness of Sobolev-type embeddings, which are valid and natural when we limit ourselves to Lebesgue or Lorentz spaces only, may fail terribly when general rearrangement-invariant function spaces are allowed.

Assume that Ω, m, μ are as above. Thanks to [31, Theorem 4.1, Remark 4.2], the trace operator (5.5) is bounded if and only if $H_{v,\frac{d}{n}}\colon X(0,1) \to Y(0,1)$ is bounded, where $v(t) = t^{-1+\frac{m}{n}}$, $t \in (0,1)$. Therefore, optimal spaces for T_μ are the same as those for $H_{v,\frac{d}{n}}$ (cf. [31, Theorem 4.4]). We shall use this fact throughout this subsection without further explicit reference.

First, we show that the fact that a target space in (5.5) is strictly bigger than the Marcinkiewicz space corresponding to the optimal target space in (5.5) does not guarantee the compactness of (5.5).

Theorem 5.3.1. *Let Ω be an open set in $\mathbb{R}^n$ such that $|\Omega| < \infty$. Let $m \in \mathbb{N}$, $m < n$. Let μ be a d-upper Ahlfors measure on Ω satisfying* (5.3)*, where $d \in [n-m, n]$. Let $\|\cdot\|_{X(0,1)}$ be a rearrangement-invariant function norm such that $t^{\frac{m}{n}-1} \notin X'(0,1)$. Furthermore, if $d = n-m$, we assume that $\|\cdot\|_{X(0,1)}$ is not equivalent to $\|\cdot\|_{L^1(0,1)}$. There is a rearrangement-invariant function norm $\|\cdot\|_{Y(0,1)}$ such that $M_{\varphi_{Y_X(0,1)}}(0,1) \subsetneq Y(0,1)$ and $\mathrm{T}_\mu\colon W_0^m X(\Omega) \to Y(\Omega,\mu)$ is bounded but not compact, where $\|\cdot\|_{Y_X(0,1)}$ is the optimal target norm in* (5.5)*.*

Proof. First, note that the assumption $t^{\frac{m}{n}-1} \notin X'(0,1)$ is equivalent to the fact that $L^{\frac{n}{n-m},\infty}(0,1) \not\subseteq X'(0,1)$, which is equivalent to $X(0,1) \not\subseteq L^{\frac{n}{m},1}(0,1)$ (see (1.27) and (1.15)); hence, since $L^{\frac{n}{m},1}(0,1)$ is the optimal domain space for $H_{v,\frac{d}{n}}$ and $L^\infty(0,1)$ (Proposition 2.2.38), we have that $Y_X(0,1) \neq L^\infty(0,1)$. Therefore $\varphi_{Y_X(0,1)}$ satisfies (2.137) owing to (1.20). Next, we claim that $\varphi_{Y_X(0,1)}$ also satisfies (2.136). Owing to (1.21), that amounts to proving that $Y_X(0,1) \neq L^1(0,1)$. If $d \in (n-m, n]$, then this follows from $Y_X(0,1) \subseteq Y_{L^1}(0,1) = L^{\frac{d}{n-m},1}(0,1) \subsetneq L^1(0,1)$ (Proposition 2.2.43, also [32, Theorem 3.1]). If $d = n-m$, then it is assumed that $X(0,1) \neq L^1(0,1)$, whence it follows that $X(0,1) \overset{*}{\hookrightarrow} L^1(0,1)$ thanks to (1.26). Therefore $Y_X(0,1) \neq L^1(0,1)$ (see (5.13) and (5.14)).

By virtue of Corollary 2.3.8, there is a Marcinkiewicz space $M_\psi(0,1)$ such that $M_{\varphi_{Y_X(0,1)}}(0,1) \subsetneq M_\psi(0,1)$, but the embedding is not almost compact. Since $Y_X(0,1) \hookrightarrow M_{\varphi_{Y_X(0,1)}}(0,1) \hookrightarrow M_\psi(0,1)$, we know that $\mathrm{T}_\mu\colon W_0^m X(\Omega) \to M_\psi(\Omega,\mu)$ is bounded; however the trace embedding is not compact (Theorem 5.1.4). ☐

Remark 5.3.2. Since the assumption $t^{\frac{m}{n}-1} \notin X'(0,1)$ is equivalent to the fact that $Y_X(0,1) \neq L^\infty(0,1)$, that assumption is completely natural in light of (1.25). Furthermore, if $d = n-m$ and $\|\cdot\|_{X(0,1)}$ is equivalent to $\|\cdot\|_{L^1(0,1)}$, then

$$Y_X(0,1) = L^{\frac{d}{n-m},1}(0,1) = L^1(0,1);$$

hence the assumption that $\|\cdot\|_{X(0,1)}$ is not equivalent to $\|\cdot\|_{L^1(0,1)}$ when $d = n-m$ is also inevitable owing to (1.26).

The following examples show that it may actually happen that

$$\lim_{t\to 0^+} \frac{\varphi_{Y(0,1)}(t)}{\varphi_{Y_X(0,1)}(t)} = 0$$

and $\mathrm{T}_\mu\colon W_0^m X(\Omega) \to Y(\Omega,\mu)$ is bounded, but the embedding is not compact. Moreover, it may even happen that $\lim_{t\to 0^+} \frac{\varphi_{Y(0,1)}(t)}{\varphi_{Y_X(0,1)}(t)} = 0$ and $M_{\varphi_{Y_X(0,1)}}(0,1) \subsetneq Y(0,1)$, but the trace embedding $\mathrm{T}_\mu\colon W_0^m X(\Omega) \to Y(\Omega,\mu)$, while bounded, is still not compact.

Proposition 5.3.3. *Let Ω be an open set in $\mathbb{R}^n$ such that $|\Omega| < \infty$. Let $m \in \mathbb{N}$, $m < n$. Let μ be a d-upper Ahlfors measure on Ω satisfying* (5.3)*, where $d \in [n-m, n]$. Let $p \in (1, \frac{n}{m}]$, $q \in (1, \infty]$ and $\alpha \in \mathbb{R}$. If $p = \frac{n}{m}$, we assume that $\alpha \le 1 - \frac{1}{q}$. Let $r \in [1, q)$. Set*

$$Y(0,1) = \begin{cases} L^{\frac{dp}{n-mp},q;\alpha}(0,1) + L^{\frac{dp}{n-mp},r;\alpha+\frac{1}{q}-\frac{1}{r}}(0,1), & p \in (1, \frac{n}{m}); \\ L^{\infty,q;\alpha-1}(0,1) + L^{\infty,r;\alpha-1+\frac{1}{q}-\frac{1}{r},\frac{1}{q}-\frac{1}{r}}(0,1), & p = \infty,\ \alpha < 1 - \frac{1}{q}; \\ L^{\infty,q;-\frac{1}{q},-1}(0,1) + L^{\infty,r;-\frac{1}{r},-1+\frac{1}{q}-\frac{1}{r},\frac{1}{q}-\frac{1}{r}}(0,1), & p = \infty,\ \alpha = 1 - \frac{1}{q}. \end{cases}$$

We have that $\lim_{t \to 0^+} \frac{\varphi_{Y(0,1)}(t)}{\varphi_{Y_{L^{p,q;\alpha}}(0,1)}(t)} = 0$ and $W_0^m L^{p,q;\alpha}(\Omega) \to Y(\Omega, \mu)$ is bounded but not compact.

Proof. Inasmuch as we have that (Proposition 2.2.43, cf. [32, Theorem 3.1])

$$Y_{L^{p,q;\alpha}}(0,1) = \begin{cases} L^{\frac{dp}{n-mp},q;\alpha}(0,1), & p \in (1, \frac{n}{m}); \\ L^{\infty,q;\alpha-1}(0,1), & p = \infty,\ \alpha < 1 - \frac{1}{q}; \\ L^{\infty,q;-\frac{1}{q},-1}(0,1), & p = \infty,\ \alpha = 1 - \frac{1}{q}, \end{cases}$$

the claim follows from Corollary 2.3.16. ☐

Remark 5.3.4. By taking $q = \infty$ in Proposition 5.3.3, we obtain examples of pairs of rearrangement-invariant function spaces $X(0,1)$ and $Y(0,1)$ such that $\lim_{t \to 0^+} \frac{\varphi_{Y(0,1)}(t)}{\varphi_{Y_X(0,1)}(t)} = 0$, $M_{\varphi_{Y_X(0,1)}}(0,1) \subsetneq Y(0,1)$ and the trace embedding $\mathrm{T}_\mu \colon W_0^m X(\Omega) \to Y(\Omega, \mu)$ is bounded but not compact.

We conclude this section with a collection of examples illustrating the unrelatedness of the compactness of (5.5) to the fundamental scale of the optimal target space in (5.5).

Proposition 5.3.5. *Let Ω be an open set in $\mathbb{R}^n$ such that $|\Omega| < \infty$. Let $m \in \mathbb{N}$, $m < n$. Let μ be a d-upper Ahlfors measure on Ω satisfying* (5.3)*, where $d \in [n-m, n]$. Let $p \in (\frac{d}{n-m}, \infty]$, $q \in (1, \infty]$ and $\alpha \in \mathbb{R}$. If $p = \infty$, we assume that $\alpha + 1 < 0$. Set*

$$X(0,1) = \begin{cases} L^{\frac{np}{d+mp},1;\alpha}(0,1) \cap L^{\frac{np}{d+mp},q;\alpha+1-\frac{1}{q}}(0,1), & p \in (\frac{d}{n-m}, \infty); \\ L^{\frac{n}{m},1;\alpha+1}(0,1) \cap X_q(0,1), & p = \infty \text{ and } \alpha + 1 < 0, \end{cases}$$

where $X_q(0,1)$ is the rearrangement-invariant function space whose function norm satisfies

$$\|f\|_{X_q(0,1)} \approx \left\| t^{-\frac{1}{q}} \ell(t)^{\alpha+1-\frac{1}{q}} \ell\ell(t)^{1-\frac{1}{q}} \int_{t^{\frac{n}{d}}}^{1} f^*(s) s^{-1+\frac{m}{n}} \, \mathrm{d}s \right\|_{L^q(0,1)}$$

for every $f \in \mathfrak{M}^+(0,1)$. We have that

$$Y_X(0,1) = \begin{cases} L^{p,1;\alpha}(0,1) \cap L^{p,q;\alpha+1-\frac{1}{q}}(0,1), & p \in (\frac{d}{n-m}, \infty); \\ L^{\infty,1;\alpha}(0,1) \cap L^{\infty,q;\alpha+1-\frac{1}{q},1-\frac{1}{q}}(0,1), & p = \infty \text{ and } \alpha + 1 < 0. \end{cases}$$

Furthermore, we have that:

- *the spaces* $X(0,1)$ *and* $Y_X(0,1)$ *are mutually optimal in* $\mathrm{T}_\mu\colon W_0^m X(\Omega) \to Y_X(\Omega,\mu)$*;*
- $\mathrm{T}_\mu\colon W_0^m X(\Omega) \to \Lambda_\psi(\Omega,\mu)$ *is bounded but not compact, where* ψ *is a quasiconcave function on* $[0,1]$ *such that*

$$\psi(t) \approx \begin{cases} t^{\frac{1}{p}}\ell(t)^\alpha & \text{if } p \in (\frac{d}{n-m}, \infty), \\ \ell(t)^{\alpha+1} & \text{if } p = \infty \text{ and } \alpha + 1 < 0; \end{cases}$$

- $\lim\limits_{t\to 0^+} \frac{\psi(t)}{\varphi_{Y_X(0,1)}(t)} = 0.$

Proof. Note that $\Lambda_\psi(0,1) = L^{p,1;\alpha}(0,1)$. Set

$$Z(0,1) = \begin{cases} L^{p,q;\alpha+1-\frac{1}{q}}(0,1) & \text{if } p \in (\frac{d}{n-m}, \infty), \\ L^{\infty,q;\alpha+1-\frac{1}{q},1-\frac{1}{q}}(0,1) & \text{if } p = \infty \text{ and } \alpha + 1 < 0. \end{cases}$$

Let $X_{\Lambda_\psi\cap Z}(0,1)$, $X_{\Lambda_\psi}(0,1)$ and $X_Z(0,1)$ be the optimal domain spaces for $H_{v,\frac{d}{n}}$ and $\Lambda_\psi(0,1)\cap Z(0,1)$, $\Lambda_\psi(0,1)$ and $Z(0,1)$, respectively. We have that (Proposition 2.2.38)

$$X_{\Lambda_\psi(0,1)}(0,1) = \begin{cases} L^{\frac{np}{d+mp},1;\alpha}(0,1) & \text{if } p \in (\frac{d}{n-m}, \infty), \\ L^{\frac{n}{m},1;\alpha+1}(0,1) & \text{if } p = \infty \text{ and } \alpha + 1 < 0, \end{cases}$$

and

$$X_{Z(0,1)}(0,1) = \begin{cases} L^{\frac{np}{d+mp},q;\alpha+1-\frac{1}{q}}(0,1) & \text{if } p \in (\frac{d}{n-m}, \infty), \\ X_q(0,1) & \text{if } p = \infty \text{ and } \alpha + 1 < 0. \end{cases}$$

Furthermore, it can be readily seen from (2.36) that $X_{\Lambda_\psi\cap Z}(0,1) = X_{\Lambda_\psi}(0,1) \cap X_Z(0,1)$.

Since $\Lambda_\psi(0,1)$ has absolutely continuous norm, we have that $(\Lambda_\psi(0,1) \cap Z(0,1))' = M_{\frac{t}{\psi(t)}}(0,1) + Z'(0,1)$ ([8, Chapter 3, Exercise 5]). As $p' < \frac{d}{d-n+m}$, the supremum operator T_φ defined by (2.17) with $\varphi(t) = t^{\frac{d-n+m}{d}}$, $t \in (0,1)$, is bounded on both $M_{\frac{t}{\psi(t)}}(0,1)$ and $Z'(0,1)$ owing to [46, Theorem 3.2]. It follows that T_φ is also bounded on $(\Lambda_\psi(0,1)\cap Z(0,1))'$. Hence the spaces in $H_{v,\nu}\colon X_{\Lambda_\psi\cap Z}(0,1) \to \Lambda_\psi \cap Z(0,1)$, $H_{v,\nu}\colon X_{\Lambda_\psi}(0,1) \to \Lambda_\psi(0,1)$ and $H_{v,\nu}\colon X_Z(0,1) \to Z(0,1)$ are mutually optimal in each of these embeddings by virtue of Theorem 2.2.13 and Remark 2.2.14(i). Therefore, the spaces in $\mathrm{T}_\mu\colon W_0^m X(\Omega) \to Y_X(\Omega,\mu)$ are indeed mutually optimal.

We plainly have that $Y_X(0,1) \hookrightarrow \Lambda_\psi(0,1)$, and so the trace embedding $\mathrm{T}_\mu\colon W_0^m X(\Omega) \to \Lambda_\psi(\Omega,\mu)$ is bounded. Furthermore, we have that (see [75, Lemma 3.14])

$$\lim_{t\to 0^+} \frac{\psi(t)}{\varphi_{Y_X(0,1)}(t)} \leq \lim_{t\to 0^+} \frac{\psi(t)}{\varphi_{Z(0,1)}(t)} = 0.$$

However, the embedding $Y_X(0,1) \hookrightarrow \Lambda_\psi(0,1)$ is not almost compact. Indeed, that follows from [42, Lemma 3.7] because $Z(0,1) \not\subseteq \Lambda_\psi(0,1)$ (see [75, Theorem 4.5]). Hence $\mathrm{T}_\mu\colon W_0^m X(\Omega) \to \Lambda_\psi(\Omega,\mu)$ is not compact owing to Theorem 5.1.4. $\square$

Bibliography

[1] D.R. Adams. Traces of potentials arising from translation invariant operators. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3)*, 25:203–217, 1971. ISSN 0391-173X.

[2] D.R. Adams. A trace inequality for generalized potentials. *Studia Math.*, 48: 99–105, 1973. ISSN 0039-3223. doi: 10.4064/sm-48-1-99-105.

[3] D.R. Adams and L.I. Hedberg. *Function spaces and potential theory*, volume 314 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences].* Springer-Verlag, Berlin, 1996. ISBN 3-540-57060-8. doi: 10.1007/978-3-662-03282-4.

[4] R.A. Adams and J.J.F. Fournier. *Sobolev spaces*, volume 140 of *Pure and Applied Mathematics (Amsterdam).* Elsevier/Academic Press, Amsterdam, second edition, 2003. ISBN 0-12-044143-8.

[5] A. Alberico, A. Cianchi, L. Pick, and L. Slavíková. Sharp Sobolev type embeddings on the entire Euclidean space. *Commun. Pure Appl. Anal.*, 17 (5):2011–2037, 2018. ISSN 1534-0392. doi: 10.3934/cpaa.2018096.

[6] C. Bennett and K. Rudnick. On Lorentz-Zygmund spaces. *Dissertationes Math. (Rozprawy Mat.)*, 175:67, 1980. ISSN 0012-3862.

[7] C. Bennett and R. Sharpley. Weak-type inequalities for H^p and BMO. In *Harmonic analysis in Euclidean spaces (Proc. Sympos. Pure Math., Williams Coll., Williamstown, Mass., 1978), Part 1*, Proc. Sympos. Pure Math., XXXV, Part, pages 201–229. Amer. Math. Soc., Providence, R.I., 1979.

[8] C. Bennett and R. Sharpley. *Interpolation of operators*, volume 129 of *Pure and Applied Mathematics.* Academic Press, Inc., Boston, MA, 1988. ISBN 0-12-088730-4.

[9] E.I. Berezhnoĭ. Sharp estimates for operators on cones in ideal spaces. *Trudy Mat. Inst. Steklov.*, 204(Issled. po Teor. Differ. Funktsiĭ Mnogikh Peremen. i ee Prilozh. 16):3–34, 1993. ISSN 0371-9685.

[10] J. Bergh and J. Löfström. *Interpolation spaces. An introduction.* Springer-Verlag, Berlin-New York, 1976. Grundlehren der Mathematischen Wissenschaften, No. 223.

[11] D.W. Boyd. The Hilbert transform on rearrangement-invariant spaces. *Canadian J. Math.*, 19:599–616, 1967. ISSN 0008-414X. doi: 10.4153/CJM-1967-053-7.

[12] J.S. Bradley. Hardy inequalities with mixed norms. *Canad. Math. Bull.*, 21 (4):405–408, 1978. ISSN 0008-4395. doi: 10.4153/CMB-1978-071-7.

[13] H. Brézis and S. Wainger. A note on limiting cases of Sobolev embeddings and convolution inequalities. *Comm. Partial Differential Equations*, 5(7): 773–789, 1980. ISSN 0360-5302. doi: 10.1080/03605308008820154.

[14] E. Buriánková, D.E. Edmunds, and L. Pick. Optimal function spaces for the Laplace transform. *Rev. Mat. Complut.*, 30(3):451–465, 2017. ISSN 1139-1138. doi: 10.1007/s13163-017-0234-5.

[15] A.-P. Calderón. Spaces between L^1 and L^∞ and the theorem of Marcinkiewicz. *Studia Math.*, 26:273–299, 1966. ISSN 0039-3223. doi: 10.4064/sm-26-3-301-304.

[16] M.J. Carro and J. Soria. The Hardy-Littlewood maximal function and weighted Lorentz spaces. *J. London Math. Soc. (2)*, 55(1):146–158, 1997. ISSN 0024-6107. doi: 10.1112/S0024610796004462.

[17] M.J. Carro, A. García del Amo, and J. Soria. Weak-type weights and normable Lorentz spaces. *Proc. Amer. Math. Soc.*, 124(3):849–857, 1996. ISSN 0002-9939. doi: 10.1090/S0002-9939-96-03214-5.

[18] P. Cavaliere and A. Cianchi. Classical and approximate Taylor expansions of weakly differentiable functions. *Ann. Acad. Sci. Fenn. Math.*, 39(2):527–544, 2014. ISSN 1239-629X. doi: 10.5186/aasfm.2014.3933.

[19] P. Cavaliere and Z. Mihula. Compactness of Sobolev-type embeddings with measures. Accepted for publication in *Commun. Contemp. Math.*

[20] P. Cavaliere and Z. Mihula. Compactness for Sobolev-type trace operators. *Nonlinear Anal.*, 183:42–69, 2019. ISSN 0362-546X. doi: 10.1016/j.na.2019.01.013.

[21] A. Cianchi. A note on two-weight inequalities for maximal functions and singular integrals. *Bull. London Math. Soc.*, 29(1):53–59, 1997. ISSN 0024-6093. doi: 10.1112/S0024609396001798.

[22] A. Cianchi. Strong and weak type inequalities for some classical operators in Orlicz spaces. *J. London Math. Soc. (2)*, 60(1):187–202, 1999. ISSN 0024-6107. doi: 10.1112/S0024610799007711.

[23] A. Cianchi and L. Pick. Sobolev embeddings into BMO, VMO, and L_∞. *Ark. Mat.*, 36(2):317–340, 1998. ISSN 0004-2080. doi: 10.1007/BF02384772.

[24] A. Cianchi and L. Pick. Optimal Gaussian Sobolev embeddings. *J. Funct. Anal.*, 256(11):3588–3642, 2009. ISSN 0022-1236. doi: 10.1016/j.jfa.2009.03.001.

[25] A. Cianchi and L. Pick. An optimal endpoint trace embedding. *Ann. Inst. Fourier (Grenoble)*, 60(3):939–951, 2010. ISSN 0373-0956. doi: 10.5802/aif.2543.

[26] A. Cianchi and L. Pick. Optimal Sobolev trace embeddings. *Trans. Amer. Math. Soc.*, 368(12):8349–8382, 2016. ISSN 0002-9947. doi: 10.1090/tran/6606.

[27] A. Cianchi, D.E. Edmunds, and P. Gurka. On weighted Poincaré inequalities. *Math. Nachr.*, 180:15–41, 1996. ISSN 0025-584X. doi: 10.1002/mana.3211800103.

[28] A. Cianchi, R. Kerman, B. Opic, and L. Pick. A sharp rearrangement inequality for the fractional maximal operator. *Studia Math.*, 138(3):277–284, 2000. ISSN 0039-3223. doi: 10.4064/sm-138-3-277-284.

[29] A. Cianchi, R. Kerman, and L. Pick. Boundary trace inequalities and rearrangements. *J. Anal. Math.*, 105:241–265, 2008. ISSN 0021-7670. doi: 10.1007/s11854-008-0036-2.

[30] A. Cianchi, L. Pick, and L. Slavíková. Higher-order Sobolev embeddings and isoperimetric inequalities. *Adv. Math.*, 273:568–650, 2015. ISSN 0001-8708. doi: 10.1016/j.aim.2014.12.027.

[31] A. Cianchi, L. Pick, and L. Slavíková. Sobolev embeddings, rearrangement-invariant spaces and Frostman measures. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 37(1):105–144, 2020. ISSN 0294-1449. doi: 10.1016/j.anihpc.2019.06.004.

[32] A. Cianchi, L. Pick, and L. Slavíková. Sobolev embeddings in Orlicz and Lorentz spaces with measures. *J. Math. Anal. Appl.*, 485(2):123827, 31, 2020. ISSN 0022-247X. doi: 10.1016/j.jmaa.2019.123827.

[33] F. Cobos, M. Cwikel, and P Matos. Best possible compactness results of Lions-Peetre type. *Proc. Edinb. Math. Soc. (2)*, 44(1):153–172, 2001. ISSN 0013-0915. doi: 10.1017/S0013091598001163.

[34] R.R. Coifman and C. Fefferman. Weighted norm inequalities for maximal functions and singular integrals. *Studia Math.*, 51:241–250, 1974. ISSN 0039-3223. doi: 10.4064/sm-51-3-241-250.

[35] O. Delgado and J. Soria. Optimal domain for the Hardy operator. *J. Funct. Anal.*, 244(1):119–133, 2007. ISSN 0022-1236. doi: 10.1016/j.jfa.2006.12.011.

[36] R. DeVore and K. Scherer. Interpolation of linear operators on Sobolev spaces. *Ann. of Math. (2)*, 109(3):583–599, 1979. ISSN 0003-486X. doi: 10.2307/1971227.

[37] D.E. Edmunds, P. Gurka, and L. Pick. Compactness of Hardy-type integral operators in weighted Banach function spaces. *Studia Math.*, 109(1):73–90, 1994. ISSN 0039-3223.

[38] D.E. Edmunds, P. Gurka, and B. Opic. Double exponential integrability of convolution operators in generalized Lorentz-Zygmund spaces. *Indiana Univ. Math. J.*, 44(1):19–43, 1995. ISSN 0022-2518. doi: 10.1512/iumj.1995.44.1977.

[39] D.E. Edmunds, R. Kerman, and L. Pick. Optimal Sobolev imbeddings involving rearrangement-invariant quasinorms. *J. Funct. Anal.*, 170(2):307–355, 2000. ISSN 0022-1236. doi: 10.1006/jfan.1999.3508.

[40] D.E. Edmunds, Z. Mihula, V. Musil, and L. Pick. Boundedness of classical operators on rearrangement-invariant spaces. *J. Funct. Anal.*, 278(4):108341, 56, 2020. ISSN 0022-1236. doi: 10.1016/j.jfa.2019.108341.

[41] W.D. Evans, B. Opic, and L. Pick. Interpolation of operators on scales of generalized Lorentz-Zygmund spaces. *Math. Nachr.*, 182:127–181, 1996. ISSN 0025-584X. doi: 10.1002/mana.19961820108.

[42] P. Fernández-Martínez, A. Manzano, and E. Pustylnik. Absolutely continuous embeddings of rearrangement-invariant spaces. *Mediterr. J. Math.*, 7(4):539–552, 2010. ISSN 1660-5446. doi: 10.1007/s00009-010-0039-y.

[43] D. Gallardo. Orlicz spaces for which the Hardy-Littlewood maximal operator is bounded. *Publ. Mat.*, 32(2):261–266, 1988. ISSN 0214-1493. doi: 10.5565/PUBLMAT_32288_09.

[44] A. Gogatishvili and L. Pick. Embeddings and duality theorems for weak classical Lorentz spaces. *Canad. Math. Bull.*, 49(1):82–95, 2006. ISSN 0008-4395. doi: 10.4153/CMB-2006-008-3.

[45] A. Gogatishvili and F. Soudský. Normability of Lorentz spaces—an alternative approach. *Czechoslovak Math. J.*, 64(139)(3):581–597, 2014. ISSN 0011-4642. doi: 10.1007/s10587-014-0120-y.

[46] A. Gogatishvili, B. Opic, and L. Pick. Weighted inequalities for Hardy-type operators involving suprema. *Collect. Math.*, 57(3):227–255, 2006. ISSN 0010-0757.

[47] G.H. Hardy and J.E. Littlewood. A maximal theorem with function-theoretic applications. *Acta Math.*, 54(1):81–116, 1930. ISSN 0001-5962. doi: 10.1007/BF02547518.

[48] G.H. Hardy, J.E. Littlewood, and G. Pólya. *Inequalities.* Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1988. ISBN 0-521-35880-9. Reprint of the 1952 edition.

[49] C. Hertz. The Hardy–Littlewood maximal theorem. *Symposium on Harmonic Analysis, University of Warwick*, 1968.

[50] T Holmstedt. Interpolation of quasi-normed spaces. *Math. Scand.*, 26:177–199, 1970. ISSN 0025-5521. doi: 10.7146/math.scand.a-10976.

[51] R. Hurri-Syrjänen. Unbounded Poincaré domains. *Ann. Acad. Sci. Fenn. Ser. A I Math.*, 17(2):409–423, 1992. ISSN 0066-1953. doi: 10.5186/aasfm.1992.1725.

[52] R. Kerman and L. Pick. Optimal Sobolev imbeddings. *Forum Math.*, 18(4):535–570, 2006. ISSN 0933-7741. doi: 10.1515/FORUM.2006.028.

[53] R. Kerman and L. Pick. Compactness of Sobolev imbeddings involving rearrangement-invariant norms. *Studia Math.*, 186(2):127–160, 2008. ISSN 0039-3223. doi: 10.4064/sm186-2-2.

[54] M.V. Korobkov and J. Kristensen. The trace theorem, the Luzin N- and Morse-Sard properties for the sharp case of Sobolev-Lorentz mappings. *J. Geom. Anal.*, 28(3):2834–2856, 2018. ISSN 1050-6926. doi: 10.1007/s12220-017-9936-7.

[55] M.A. Krasnosel′skiĭ and Ja.B. Rutickiĭ. *Convex functions and Orlicz spaces.* Translated from the first Russian edition by Leo F. Boron. P. Noordhoff Ltd., Groningen, 1961.

[56] S.G. Kreĭn, Yu. Ī. Petunīn, and E.M. Semënov. *Interpolation of linear operators*, volume 54 of *Translations of Mathematical Monographs.* American Mathematical Society, Providence, R.I., 1982. ISBN 0-8218-4505-7. Translated from the Russian by J. Szűcs.

[57] A. Kufner and L.-E. Persson. *Weighted inequalities of Hardy type.* World Scientific Publishing Co., Inc., River Edge, NJ, 2003. ISBN 981-238-195-3. doi: 10.1142/5129.

[58] M. Křepela. Almost compact embeddings of function spaces. Master's thesis, Charles University, Faculty of Mathematics and Physics, 2011. URL `https://is.cuni.cz/webapps/zzp/detail/71875/`.

[59] J. Lang, Z. Mihula, and L. Pick. Compactness of Sobolev embeddings and decay of norms. Submitted preprint. arXiv:2011.03450 [math.FA], 2020.

[60] J. Malý and W.P. Ziemer. *Fine regularity of solutions of elliptic partial differential equations*, volume 51 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI, 1997. ISBN 0-8218-0335-2. doi: 10.1090/surv/051.

[61] M. Mastyło. Interpolation of compact operators. *Funct. Approx. Comment. Math.*, 26:293–311, 1998. ISSN 0208-6573.

[62] V.G. Maz'ya. Certain integral inequalities for functions of several variables. In *Problems of mathematical analysis, No. 3: Integral and differential operators, Differential equations (Russian)*, pages 33–68. 1972. English translation: *J. Math. Sci.*,1:205–234, 1973. doi: 10.1007/BF01083775.

[63] V.G. Maz'ya. Strong capacity-estimates for "fractional" norms. *Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 70:161–168, 292, 1977. English translation: *J. Math. Sci.*, 23:1997–2003, 1983. doi: 10.1007/BF01093280.

[64] V.G. Maz'ya. *Sobolev spaces with applications to elliptic partial differential equations*, volume 342 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences].* Springer, Heidelberg, augmented edition, 2011. ISBN 978-3-642-15563-5. doi: 10.1007/978-3-642-15564-2.

[65] M. Mihula. Optimality of function spaces for classical integral operators. Master's thesis, Charles University, Faculty of Mathematics and Physics, 2017. URL `https://is.cuni.cz/webapps/zzp/detail/180597/`.

[66] Z. Mihula. Embeddings of homogeneous Sobolev spaces on the entire space. *Proc. Roy. Soc. Edinburgh Sect. A*, 151(1):296–328, 2021. ISSN 0308-2105. doi: 10.1017/prm.2020.14.

[67] Z. Mihula. Poincaré-Sobolev inequalities with rearrangement-invariant norms on the entire space. *Math. Z.*, Online First:1–18, 2021. doi: 10.1007/s00209-020-02652-z.

[68] J. Moser. A sharp form of an inequality by N. Trudinger. *Indiana Univ. Math. J.*, 20:1077–1092, 1971. ISSN 0022-2518. doi: 10.1512/iumj.1971.20.20101.

[69] B. Muckenhoupt. Hardy's inequality with weights. *Studia Math.*, 44:31–38, 1972. ISSN 0039-3223. doi: 10.4064/sm-44-1-31-38.

[70] B. Muckenhoupt. Weighted norm inequalities for the Hardy maximal function. *Trans. Amer. Math. Soc.*, 165:207–226, 1972. ISSN 0002-9947. doi: 10.2307/1995882.

[71] B. Muckenhoupt and R.L. Wheeden. Weighted norm inequalities for singular and fractional integrals. *Trans. Amer. Math. Soc.*, 161:249–258, 1971. ISSN 0002-9947. doi: 10.2307/1995940.

[72] V. Musil and R. Oľhava. Interpolation theorem for Marcinkiewicz spaces with applications to Lorentz gamma spaces. *Math. Nachr.*, 292(5):1106–1121, 2019. ISSN 0025-584X. doi: 10.1002/mana.201700452.

[73] R. O'Neil. Convolution operators and $L(p, q)$ spaces. *Duke Math. J.*, 30: 129–142, 1963. ISSN 0012-7094. doi: 10.1215/S0012-7094-63-03015-1.

[74] R. O'Neil and G. Weiss. The Hilbert transform and rearrangement of functions. *Studia Math.*, 23:189–198, 1963. ISSN 0039-3223. doi: 10.4064/sm-23-2-189-198.

[75] B. Opic and L. Pick. On generalized Lorentz-Zygmund spaces. *Math. Inequal. Appl.*, 2(3):391–467, 1999. ISSN 1331-4343. doi: 10.7153/mia-02-35.

[76] J. Peetre. Espaces d'interpolation et théorème de Soboleff. *Ann. Inst. Fourier (Grenoble)*, 16(1):279–317, 1966. ISSN 0373-0956. doi: 10.5802/aif.232.

[77] D. Peša. Lorentz–Karamata spaces. Preprint. arXiv:2006.14455v2 [math.FA], 2020.

[78] D. Peša. Reduction principle for a certain class of kernel-type operators. *Math. Nachr.*, 293(4):761–773, 2020. ISSN 0025-584X. doi: 10.1002/mana.201800510.

[79] L. Pick, A. Kufner, O. John, and S. Fučík. *Function spaces. Volume 1*, volume 1 of *De Gruyter Series in Nonlinear Analysis and Applications.* Berlin: de Gruyter, 2nd edition, 2013. doi: 10.1515/9783110250428.

[80] P.J. Rabier. Uniformly local spaces and refinements of the classical Sobolev embedding theorems. *Ark. Mat.*, 56(2):409–440, 2018. ISSN 0004-2080. doi: 10.4310/ARKIV.2018.v56.n2.a13.

[81] F. Riesz. Sur un Theoreme de Maximum de Mm. Hardy et Littlewood. *J. London Math. Soc.*, 7(1):10–13, 1932. ISSN 0024-6107. doi: 10.1112/jlms/s1-7.1.10.

[82] E. Sawyer. Boundedness of classical operators on classical Lorentz spaces. *Studia Math.*, 96(2):145–158, 1990. ISSN 0039-3223. doi: 10.4064/sm-96-2-145-158.

[83] R. Sharpley. Counterexamples for classical operators on Lorentz-Zygmund spaces. *Studia Math.*, 68(2):141–158, 1980. ISSN 0039-3223. doi: 10.4064/sm-68-2-141-158.

[84] L. Slavíková. Almost-compact embeddings. *Math. Nachr.*, 285(11-12):1500–1516, 2012. ISSN 0025-584X. doi: 10.1002/mana.201100286.

[85] L. Slavíková. Compactness of higher-order Sobolev embeddings. *Publ. Mat.*, 59(2):373–448, 2015. ISSN 0214-1493. doi: 10.5565/PUBLMAT_59215_06.

[86] J. Soria and P. Tradacete. Optimal rearrangement invariant range for Hardy-type operators. *Proc. Roy. Soc. Edinburgh Sect. A*, 146(4):865–893, 2016. ISSN 0308-2105. doi: 10.1017/S0308210515000669.

[87] E.M. Stein. *Singular integrals and differentiability properties of functions.* Princeton Mathematical Series, No. 30. Princeton University Press, Princeton, N.J., 1970.

[88] E.M. Stein. Editor's note: the differentiability of functions in $\mathbf{R}^n$. *Ann. of Math. (2)*, 113(2):383–385, 1981. ISSN 0003-486X.

[89] E.M. Stein and G. Weiss. *Introduction to Fourier analysis on Euclidean spaces.* Princeton University Press, Princeton, N.J., 1971. Princeton Mathematical Series, No. 32.

[90] V.D. Stepanov. The weighted Hardy's inequality for nonincreasing functions. *Trans. Amer. Math. Soc.*, 338(1):173–186, 1993. ISSN 0002-9947. doi: 10.2307/2154450.

[91] R.S. Strichartz. A note on Trudinger's extension of Sobolev's inequalities. *Indiana Univ. Math. J.*, 21:841–842, 1971/72. ISSN 0022-2518. doi: 10.1512/iumj.1972.21.21066.

[92] F. Sukochev, K. Tulenov, and D. Zanin. The optimal range of the Calderòn operator and its applications. *J. Funct. Anal.*, 277(10):3513–3559, 2019. ISSN 0022-1236. doi: 10.1016/j.jfa.2019.05.012.

[93] A. Torchinsky. *Real-variable methods in harmonic analysis*, volume 123 of *Pure and Applied Mathematics.* Academic Press, Inc., Orlando, FL, 1986. ISBN 0-12-695460-7; 0-12-695461-5.

[94] N.S. Trudinger. On imbeddings into Orlicz spaces and some applications. *J. Math. Mech.*, 17:473–483, 1967. doi: 10.1512/iumj.1968.17.17028.

[95] J. Vybíral. Optimal Sobolev embeddings on $\mathbb{R}^n$. *Publ. Mat.*, 51(1):17–44, 2007. ISSN 0214-1493. doi: 10.5565/PUBLMAT_51107_02.

[96] N. Wiener. The ergodic theorem. *Duke Math. J.*, 5(1):1–18, 1939. ISSN 0012-7094. doi: 10.1215/S0012-7094-39-00501-6.

[97] W.P. Ziemer. *Weakly differentiable functions*, volume 120 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1989. ISBN 0-387-97017-7. doi: 10.1007/978-1-4612-1015-3. Sobolev spaces and functions of bounded variation.

List of publications

Publications used in the thesis

- P. Cavaliere and Z. Mihula. Compactness of Sobolev-type embeddings with measures. Accepted for publication in *Commun. Contemp. Math.*
- P. Cavaliere and Z. Mihula. Compactness for Sobolev-type trace operators. *Nonlinear Anal.*, 183:42–69, 2019. ISSN 0362-546X. doi: 10.1016/j.na.2019.01.013.
- D.E. Edmunds, Z. Mihula, V. Musil, and L. Pick. Boundedness of classical operators on rearrangement-invariant spaces. *J. Funct. Anal.*, 278(4):108341, 56, 2020. ISSN 0022-1236. doi: 10.1016/j.jfa.2019.108341.
- J. Lang, Z. Mihula, and L. Pick. Compactness of Sobolev embeddings and decay of norms. Submitted preprint. arXiv:2011.03450 [math.FA], 2020.
- Z. Mihula. Embeddings of homogeneous Sobolev spaces on the entire space. *Proc. Roy. Soc. Edinburgh Sect. A*, 151(1):296–328, 2021. ISSN 0308-2105. doi: 10.1017/prm.2020.14.
- Z. Mihula. Poincaré-Sobolev inequalities with rearrangement-invariant norms on the entire space. *Math. Z.*, Online First:1–18, 2021. doi: 10.1007/s00209-020-02652-z.

Publications not used in the thesis

- D.E. Edmunds, J. Lang, and Z. Mihula. Measure of noncompactness of Sobolev embeddings on strip-like domains. Submitted preprint. arXiv:2006.07948 [math.FA], 2020.
- M. Křepela, Z. Mihula, and H. Turčinová. Discretization and antidiscretization of Lorentz norms with no restrictions on weights. Submitted preprint. arXiv:2011.00104 [math.FA], 2020.