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Betr.: Doctoral Thesis, Petr Lukáš, review

Numerical Solution of Convection-dominated Problems

submitted by

Petr Lukáš

The thesis is concerned with the reliable approximation of solutions to convection-reaction-diffusion equations of the type

$$-\varepsilon\Delta u + \boldsymbol{b}\cdot\nabla u + cu = f \quad \text{in } \Omega \subset \mathbb{R}^2,$$

in the convection dominated regime, i.e. when $0 < \varepsilon \ll \|\boldsymbol{b}\|$. To this end the candidate employs stabilised finite-element methods (FEM).

The general aim of the techniques presented in the thesis is to supress non-physical oscillations from which most numerical techniques suffer. The approach persued is that of adding artificial diffusion (higher order viscosity or other penalisation terms) in certain regions of the domain. This is done in an adaptive, iterative, non-linear manner depending on properties of computed numerical approximations.

The novelty of the approach pursued by the candidate (together with his supervisor and further collaborators) is in the use of optimisation techniques (reviewed in §5) to tune the free parameters in the stabilisation terms. They are optimised such that certain error indicators are minimised. The resulting numerical methods are rather complex, but dealt with by the candidate in a creative and systematic manner. He employs the full tool box of optimisation methods for high-dimensional problems.

The thesis is primarily based on numerical experiments (presented in §6). Most of the approaches are heuristical in nature. There are no theoretical results concerning convergence or even stability of the methods. For example §2.4 introduces a few versions of SOLD-methods, but there is no mentioning of well-posedness or even convergence. This might be a non-trivial problem because the method is non-linear, but I vaguely remember results by Codina and Lube that study such methods.

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In §6 the candidate presents and discusses results of many numerical tests. There are a lot of pictures, but no systematic studies of convergence properties of the various methods. I consider this a severe shortcoming.

Also I am missing a comparison of the efficiency of the methods using optimisation with methods that do not. The optimisation is rather expensive. Thus, a standard stabilised method on a much finer mesh might be similarly expensive, but give better results.

The conclusions of the candidate are way to optimistic. He should have judged the results more carefully and critically. For example, he writes:

> ,,From the investigated examples we found out that all error indicators (4.1), (4.3), and (4.10) allow us to obtain a high-quality solution of the scalar convection-diffusion equation with dominant convection even on a relatively coarse mesh. The reason that it works is that the spurious oscillations in the crosswind direction remaining from the SUPG method (2.5) are cured by employing an adaptive method based on an error indicator."

But looking at the results for Example 5 in Figures 6.18 and 6.20 one sees a significant smearing of the sharp layers and oscillations too! In the case of I_h^{cross} this is predictable and a design fault. That indicator penalises large crosswind derivatives. However, at chracteristic layers the crosswind derivative is large. Therefore, the method must fail in situations where there are characteristic layers. The candidate points this out when discussing Figure 6.8, p.51, but does not give the correct explanation.

Despite the shortcomings just mentioned, I consider the thesis a valuable contribution to the numerical treatment of convection-dominated problems. The candidate produced a number of interesting results that provide a lot of material for further analysis. The complex combination of FE-discretisations with optimisation methods poses challenges when developing the software.

I do not consider the thesis a strong one. Nonetheless, **I recommend the candidate to be awarded a PhD** based on the research work presented in the thesis.

Prof. Dr. Torsten Lin"s

Further comments and remarks

- §2.1: $h := \max_{T \in \mathcal{T}_h} h_T$ should be added.
- The proof of Theorem 1 and the material on p. 12 is reproduced from the literature. Proper reference must ge given.
- (2.5), (2.11), Theorem 1: There are τ, τ_T, τ_h, non of which has beee properly introduced.
- (2.19) – quote: ,,h_T is the characteristic length of T". What's this? Why ,,characteristic"? Before h_T denoted the diameter of T.
- §3.1.2 is hard to follow. What is the nature of the objects U, α, J, I, etc. From where to where do they map? What is the functional to minimise? What is the admissible set? What are the equality restrictions? The formulation might be acceptable in engineering publications, but not in a mathematical paper.
  (3.12): There is a j on the right-hand side of the equation, but not on the left-hand side. How should this be understood?
- §6.3: Where are the anisotropic meshes?