doc. RNDr. Mirko Rokyta
CSc. Dean

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Dear doc. RNDr. Mirko Rokyta

Report on the thesis: Numerical Solution of Convection Dominated Problems.

Please find below my report on the above mentioned thesis, authored by Mr. Petr Lukàš. As the title indicates, the ambition of this thesis is to study convection dominated problems. This refers to second order elliptic partial differential equations (pde) of advection–diffusion type, where the diffusion coefficient ε is small compared to the transport velocity, $\mathbf{b}$. In many cases of practical interest this leads to situations where the smallest scale h that is resolved computationally is much larger than the smallest scales of the flow, $\varepsilon/|\mathbf{b}|$. In this situation the commonly used standard finite element method, that performs well for elliptic problems in the case where $h < \varepsilon/|\mathbf{b}|$, has poor stability properties. A well known remedy to this is to add some linear stabilizing terms. These ideas were introduced during the 80s by the research group of Dr. T. Hughes and was analysed in the work of Dr. C. Johnson and co-workers. The conclusion was that for smooth solutions the addition of such stabilizing terms improve convergence and for solutions with layers the stabilization prevents spurious oscillations close to layers from penetrating into the zone where the exact solution is smooth. To further enhance underresolved solutions the addition of a nonlinear stabilizing term has been suggested, often termed "shock capturing" term, or in the work of John and Knobloch, "SOLD" term. The objective of this term is to further reduce spurious oscillations close to layers, ideally leading to approxmations with similar local stability properties as the continuous problem.

A vast literature exists on the design and analysis of the above mentioned methods. However it has been difficult to pin down certain parameters of the methods except in terms of their scaling with physical parameters and the mesh parameter. In most situations a non-dimensional constant remains that has to be fixed by the end user.

The topic of the present thesis is to develop approaches to set such parameters in an adaptive fashion. Adaptivity is a well-known concept in the finite element method when the local mesh-size is considered. Here the idea is to use similar ideas to set the stabilization parameter for a fixed h by minimizing some quantity that serves as a proxy for the computational error in the quantity of interest. The approach chosen is to cast the approximation problem on the form of a optimization problem constrained by the finite element formulation. The stabilization parameter is chosen so as to minimize the target quantity. This leads to a nonlinear minimization problem the cost of which exceeds the cost of solving the original linear finite element problem by orders of magnitude. However it is argued that the approximation method using nonlinear stabilization already is nonlinear and that furthermore in the case of nonlinear pde, the underlying physical problem is nonlinear as well, making the additional computational effort less important.

The thesis consists of six chapters, of which the four first revisit known results on stabilized methods, constrained optimization and error estimators. The last two chapters contain the novel elements of the thesis. In chapter 5 several approaches to the numerical solution of the optimization problem are discussed and in chap-

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ter 6 the different methods are applied to a number of model problems based on the linear advection–diffusion equation. This numerical study is very informative and convincingly shows that the accuracy of the approximation can be improved substantially by a accurate choice of the stabilization parameter. The superiority of quasi-Newton methods to steepest descent methods or nonlinear conjugate gradient methods is also evident from the numerical experiments.

This is a computational study of very high quality of a difficult problem in the approximation of pde. On the other hand the theoretical foundation that it is built upon appears weak, due to incomplete theory of the underlying stabilized methods. These theoretical issues are admittedly beyond the scope of the present thesis, but still makes the methods heuristic in nature. The residual quantities that are minimized are not uniform error bounds for the goal quantities and the residual based SOLD methods used are nonlinear exactly with the objective of choosing the parameter automatically as a function of the approximate solution. If the parameter is set using some other (reliable) method one could in principle question the whole design of the SOLD method. Nevertheless, within the bounds of the heuristic assumptions made, the thesis makes some interesting points and certainly gives evidence of the authors ability for creative work. The observations made through the computational experiments can provoke interesting discussions in the community working on stabilized finite element methods.

I recommend that the thesis is accepted unconditionally.

Sincerely yours,

Erik Burman