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R-projectivity

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Abstract: A module over a ring R is R-projective if it is projective relative to R. This module-theoretic notion is dual to the notion of an R-injective module that plays a key role in the classic Baer's Criterion for Injectivity. This Thesis is concerned with the validity of dual version of Baer's Criterion. It also introduces a concept of projectivity in a general category-theoretic setting.

DBC is known to hold for all perfect rings. However, DBC either fails or it is undecidable in ZFC for non-perfect rings. In this Thesis we deal with the subclass of non-perfect rings, which are small, regular, semiartinian and have primitive factors artinian. Trlifaj showed that there is an extension of ZFC in which DBC holds for such rings. Especially, it is enough to consider extension of ZFC in which the weak version of Jensen's Diamond Principle holds. This combinatorial principle is known as the Weak Diamond Principle.

Apart from an overview of the properties of rings mentioned above and introduction of the necessary set-theoretic notions, the Thesis also contains a proof of this new result by Trlifaj published in the paper "Weak diamond, weak projectivity, and transfinite extensions of simple artinian rings" in the J. Algebra in 2022.

Keywords: semiartinian ring, projective module, set-theoretic homological algebra, Weak Diamond Principle

Contents

List of Abbreviations

Ab	category of abelian groups
BC	Baer's Criterion
$\mathscr{C}^{op}$	dual category of a category $\mathcal{C}$
$\mathscr{C}(A, B)$	all morphisms in $\mathcal{C}$ from an object A to an object B
$\mathrm{card}(A)$	cardinality of a set A
$\mathrm{cf}(A)$	cofinality of a set A
CH	Continuum Hypothesis
DBC	Dual Baer's Criterion
$\mathrm{E}(M)$	injective envelope of a module M
$\mathrm{Ext}^n_R(-,-)$	Ext functor
$f\restriction_A$	restriction of a map f to a set A
GCH	Generalized Continuum Hypothesis
$\mathrm{gen}(M)$	minimal cardinality of a set of generators of a module M
$\mathrm{Hom}_R(M, N)$	abelian group of all R-homomorphisms from M to N
$\mathrm{Hom}_R(-,-)$	Hom bifunctor
id_M	identity homomorphism of a module M
$\mathrm{Im}(f)$	image of a morphism f
$\mathrm{Ker}(f)$	kernel of a morphism f
$K[x_1, ..., x_n]$	polynomial ring over a field K in n variables
L	constructible universe
Mod-R	category of right R-modules
M^*	character left module of a right module M
$ob(\mathcal{C})$	all objects of a category $\mathcal{C}$
$\mathcal{P}r^{-1}(M)$	projectivity domain of a module M
$\mathbb{Q}$	rational numbers
$\mathbb{R}$	real numbers
R^I	product of I copies of R
$R^{(I)}$	direct sum of I copies of R
Set	category of sets
$\mathrm{Soc}(R)$	socle of a ring R
sup	supremum
SUP	Shelah's Uniformization Principle
V=L	Axiom of Constructibility
$\mathbb{Z}$	integers
ZFC	Zermelo–Fraenkel set theory with the Axiom of Choice
$\alpha + 1$	successor of an ordinal α
κ^+	successor of a cardinal κ
ω	natural numbers (as an ordinal)
$\aleph_0$	cardinality of ω
$\aleph_{n+1}$	successor of $\aleph_n$
$\diamondsuit$	Jensen's Diamond Principle (see Definition 3.11)
Φ	Weak Diamond Principle (see Definition 3.21)

$\lvert A \rvert$	cardinality of a set A
$\langle G \rangle$	module generated by a set G
$\{a \in A; \varphi(a)\}$	set of elements of A satisfying the condition φ
${}^{A}B$	all maps from a set A to a set B
$\cap$	intersection
$\cup$	union
$\times$	cartesian product
$\oplus$	module direct sum
$\emptyset$	empty set
$\rightarrow$	morphism
$\twoheadrightarrow$	epic, in particular epimorphism
$\hookrightarrow$	monic, in particular monomorphism
$\leq$	well ordering of ordinals by relation $\in$

Conventions and Notation

If not stated otherwise, all rings are associative with unit. A ring R is also viewed as a right R-module over itself. By the (co)limit we sometimes denote just the object of the (co)limit, especially if we are talking about the kernel or the image. All statements and their proofs are valid in ZFC, except for a few places in Chapter 3 where it is explicitly stated that the results are proved in an extension of ZFC.

Introduction

R-projectivity is a property of modules over a ring R. It is dual to R-injectivity, which is equivalent to injectivity by celebrated Baer's Criterion (BC). This Criterion, also known as Baer's test, allows us to test for injectivity of a module over an arbitrary ring only at ideals instead of at all modules. This enables classification of injective modules over commutative noetherian rings which in turn led to introduction of Bass invariants of modules. For more about the role of injective modules we refer the reader to monographs of Matsumura [1989] and Enochs and Jenda [2000].

Let us return to projectivity. If we state BC dually as an equivalence between projectivity and R-projectivity for all modules, we obtain the statement known as Dual Baer's Criterion (DBC). Although BC holds true in the Zermelo–Fraenkel set theory with the Axiom of Choice (ZFC) for each ring R, the validity of DBC in ZFC depends on properties of the ring R.

It was shown by Sandomierski [1964] (we also refer the reader to Ketkar and Vanaja [1981]) that DBC holds true if R is perfect. We recall that R is perfect provided that every module has a projective cover. It was first noted by Faith that it is unknown for which non-perfect rings does DBC hold true (Faith [1976]).

On the one hand, Alhilali et al. [2017] showed using Trlifaj [1996] that the failure of DBC is consistent with ZFC for every non perfect ring. That is, if R is non perfect, then the existence of a module for which DBC fails is consistent with ZFC (such a module needs to be infinitely generated, because for finitely generated modules DBC is easily provable in ZFC).

On the other hand, the consistency of DBC with ZFC has been also proved for some commutative non-perfect rings. Examples of such rings appear in Trlifaj [2019] and Trlifaj [2020]. This implies that there are just two classes of non-perfect rings, the first one for which DBC fails in ZFC and the second for which DBC is independent of ZFC. The exact boundary between these two classes of non-perfect rings is still not known.

It has turned out that many rings belong in the first class mentioned above, including commutative noetherian rings (see Hamsher [1966], Theorem 1), semilocal right noetherian rings (Alhilali et al. [2017], Proposition 2.11), and commutative domains (Trlifaj [2019], Lemma 1).

In this Thesis we are interested in the second class of rings. Our goal is to present the new result from Trlifaj [2022] that DBC is independent on ZFC for a subclass of non-perfect rings consisting of regular, semiartinian rings of non-trivial Loewy length, which have primitive factors artinian. Such a class of rings can be viewed as a class of transfinite extensions of simple artinian rings.

The Thesis has the following structure. We first, in Section 1.1, focus on the projectivity in a general sense as a property of objects in an exact category. Following that, throughout sections 1.2 and 1.3, we present basic facts corresponding to projective and injective modules. In Section 1.3 we also formulate Baer's Criterion which serves as an inspiration to the definition of R-projectivity and which in Section 1.4 leads us back to Faith's Problem about the validity of DBC.

In Chapter 2 we introduce regular semiartinian rings with primitive factors artinian and some of their properties. For the defining sequence of semiartinian rings, also known as Loewy rings, we use the term “socle sequence”, though it is also sometimes referred to as “Loewy series”.

The main part of Chapter 3, the proof of the Trlifaj’s result mentioned above, is based on the application of two mutually inconsistent combinatorial principles, Shelah’s Uniformization Principle and Weak Diamond Principle. These principles and the necessary set-theoretic notions are introduced in Section 3.1. Finally, in Section 3.2, we prove this result from Trlifaj [2022] under the assumption of the Weak Diamond Principle.

1. Projectivity and Injectivity

The topic of this Thesis is R-projectivity, a property of modules which is a relative version of the well-known projectivity. However, the notion of projectivity makes sense in more general category-theoretic settings. Thus, in Section 1.1, we will first introduce projectivity as a property of objects of a general category. Then we will briefly discuss exact and abelian categories, whose properties allow us to better understand projectivity.

There are theorems, which imply that these more general categories are not too far from categories of modules. The Gabriel-Quillen embedding theorem states that any small exact category can be fully faithfully embedded in an abelian one and the Freyd-Mitchell embedding theorem states that any small abelian category can be fully faithfully embedded in a category of right modules, Mod-R, for some ring R. The important property of these embeddings is that they preserves exactness. The sketches of proofs of these theorems appear at the end of Section 1.1.

For the rest of this Thesis, starting from Section 1.2, we will stay in the category of right modules over a ring R and work with projectivity only as the property of modules. The reader not interested in exact categories can safely skip the first section. In the following sections of this chapter we will define projectivity for modules, we will also introduce the dual property, the injectivity. There is a well-known criterion for testing for injectivity, Baer's Criterion, which leads us to the question of Faith's Problem, i.e., back to projectivity and to the possibility of dualizing Baer's Criterion.

1.1 Projectivity as a categorical notion

We assume knowledge of basics of category theory, e.g., at the level of Leinster [2014]. We use the notation from that book.

Definition 1.1. Let $\mathscr{C}$ be a category. An object $P \in ob(\mathscr{C})$ is *projective* if for any two objects $A, B \in ob(\mathscr{C})$ and any epic $\pi \in \mathscr{C}(A, B)$ the following holds: For every $f \in \mathscr{C}(P, B)$ there exists $g \in \mathscr{C}(P, A)$ such that the diagram

$$\begin{tikzcd} P \arrow[d, dotted, "g"'] \arrow[dr, "f"] & \\ A \arrow[r, two heads, "\pi"'] & B \end{tikzcd}$$

commutes, i.e., $\pi g = f$.

Notice that projective objects do not have to exist in a general category $\mathscr{C}$.

Proposition 1.2. *Let $\mathscr{C}$ be a category. An object $P \in ob(\mathscr{C})$ is projective iff the functor $\mathscr{C}(P, -) : \mathscr{C} \to \mathbf{Set}$ preserves epics.*

Proof. The functor $\mathscr{C}(P, -) : \mathscr{C} \to \mathbf{Set}$ preserves epics iff for any two objects $A, B \in ob(\mathscr{C})$ and any epic $\pi \in \mathscr{C}(A, B)$, $\mathscr{C}(P, \pi) \in \mathbf{Set}(\mathscr{C}(P, A), \mathscr{C}(P, B))$ is an epic in $\mathbf{Set}$. Because epics are surjections in the category of sets, $\mathscr{C}(P, \pi)$ is an epic if and only if for every $f \in \mathscr{C}(P, B)$ there exists $g \in \mathscr{C}(P, A)$ such that $\mathscr{C}(P, \pi)(g) = f$. Since $\mathscr{C}(P, \pi)(g) = \pi g$, it is equivalent to the fact that P is a projective object in $\mathscr{C}$.

$\square$

Definition 1.3. Let $\mathscr{C}$ be a category. An object $A \in ob(\mathscr{C})$ is *a zero object* if $|\mathscr{C}(A, B)| = |\mathscr{C}(B, A)| = 1$ for each $B \in ob(\mathscr{C})$.

Notice that a zero object is unique up to isomorphism. We denote this object by 0. Also note, that if a zero object exists, it determines a zero morphism: For any two objects B and B' there is exactly one morphism in $\mathscr{C}(B, 0)$ and exactly one in $\mathscr{C}(0, B')$. The composition of these two morphisms is called the zero morphism from B to B'.

Definition 1.4. A category $\mathscr{C}$ is called *abelian* if

1. a zero object exists in $\mathscr{C}$,
2. the product and the sum exist in $\mathscr{C}$ for each pair of objects,
3. the kernel and the cokernel exist in $\mathscr{C}$ for each morphism, and
4. each monic (resp. epic) is the kernel (resp. the cokernel) of some morphism.

As the terminology indicates, in an abelian category the set of morphisms between any two objects has the structure of an abelian group, see Freyd [1964] (chapter 2, section 2.3). The existence of the zero object, the kernels, and the cokernels, and therefore the images, allows us to introduce exactness of a sequence in the same way we are used to from homological algebra in categories of modules.

Definition 1.5. A sequence of objects of an abelian category and morphisms between them

$$\ldots \to A_{i+1} \overset{d_{i+1}}{\to} A_i \overset{d_i}{\to} A_{i-1} \overset{d_{i-1}}{\to} \ldots$$

is *an exact sequence* if $\mathrm{Ker}(d_i) = \mathrm{Im}(d_{i+1})$ for each $i < \omega$. If it is a finite exact sequence of the form $0 \to A \to B \to C \to 0$, it is called *a short exact sequence*.

The question of how to study exactness with less requirements on a category led to several definitions of an exact category. The first one, due to D. A. Buchsbaum, was presented in Appendix to Cartan and Eilenberg [1956]. In what follows, we will focus our attention on the definition from Quillen [1972].

Quillen starts with the idea of preservation of the class of exact sequences and determination them just as the kernel-cokernel pairs (Definition 1.6). Later, in Keller [1990], the definition of Quillen's exact category was reduced to the minimal set of axioms. Thus we will not discuss Quillen's axioms, rather we will introduce an exact category using Keller's axioms as in Bühler [2010].

Another example of an exact category was introduced in Barr [1973]. It is also known as a regular category but in the additive case (an additive category is a category which satisfies just conditions 1. and 2. from Definition 1.4) it coincides with the notion of an abelian one.

Definition 1.6. Let $\mathscr{C}$ be an additive category. A pair (i, p) is *a kernel-cokernel pair in $\mathscr{C}$* if i, p are composable morphisms in the category $\mathscr{C}$ such that i is the kernel of p and p is the cokernel of i.

An epic p (a monic i) is called *admissible* with respect to ϵ iff there exists i (resp. p) such that $(i, p) \in \epsilon$, where ϵ is a class of kernel-cokernel pairs. We will omit "with respect to ϵ" if the class ϵ is clear from the context.

Definition 1.7. Let $\mathscr{C}$ be an additive category. A class ϵ of kernel-cokernel pairs in $\mathscr{C}$ is *an exact structure on* $\mathscr{C}$ if

1. ϵ is closed under isomorphisms,
2. an identity morphism 1_A is an admissible isomorphism for each $A \in ob(\mathscr{C})$,
3. the composition of admissible epics (resp. monics) is also an admissible epic (resp. monic), and
4. the push-out (resp. pull-back) of an admissible monic (resp. epic) along an arbitrary morphism exists and yields an admissible monic (resp. epic).

Remark 1.8. Notice that all conditions in the definition of an exact structure are self-dual. This implies that ϵ is an exact structure on $\mathscr{C}$ iff ϵ^{op} is an exact structure on $\mathscr{C}^{op}$.

Let us recall the definition of a splitting property of a sequence, a monic, and an epic in an abelian category.

Definition 1.9. A short exact (with respect to the exact structure of all kernel-cokernel pairs) sequence in an abelian category $\mathscr{C}$

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

is *a splitting sequence* if any of the following equivalent conditions holds:

1. There exists $f' \in \mathscr{C}(B, A)$ such that $f'f = 1_A$, that is, *the monic f splits.*
2. There exists $g' \in \mathscr{C}(C, B)$ such that $g'g = 1_C$, that is, *the epic g splits.*

For a fixed additive category $\mathscr{C}$ the sequences, that are short exact splitting after an embedding $(\mathscr{C}, \epsilon)$ into an abelian category, are exact with respect to ϵ (see Bühler [2010], Remark 2.8). Moreover, they form an exact structure, so they are the smallest exact structure which is contained in every other one. In the opposite direction all exact structures are contained in a class of all kernel-cokernel pairs but such a class does not have to be an exact structure.

It turns out that exact structures on an additive category $\mathscr{C}$ form a bounded complete lattice, so the maximal exact structure always exists. For more about the existence and form of the maximal exact structure we refer to Crivei [2022].

Definition 1.10. The pair ($\mathscr{C}$, ϵ) is *an exact category (in the sense of Quillen)* if $\mathscr{C}$ is an additive category and ϵ is an exact structure on $\mathscr{C}$. The elements of ϵ are called *short exact sequences.*

Since in an exact category the important epics and monics are just the admissible ones, we will employ a stricter definition of the projective object than in a general category. This will be necessary for the characterization of projective objects in Lemma 1.12. Nevertheless, in an abelian category, there are all epics and monics admissible with respect to the exact structure of all kernel-cokernel pairs. Therefore, in such a category, Definition 1.1 is equivalent to the following one.

Definition 1.11. Let $(\mathscr{C}, \epsilon)$ be an exact category. An object $P \in ob(\mathscr{C})$ is called *projective in* $(\mathscr{C}, \epsilon)$ if for any two objects $A, B \in ob(\mathscr{C})$ and any **admissible** epic $\pi \in \mathscr{C}(A, B)$ the following holds: For every $f \in \mathscr{C}(P, B)$ there exists $g \in \mathscr{C}(P, A)$ such that the diagram

$$\begin{tikzcd} P \arrow[d, dotted, "g"'] \arrow[dr, "f"] & \\ A \arrow[r, two heads, "\pi"'] & B \end{tikzcd}$$

commutes, i.e., $\pi g = f$.

Lemma 1.12. *Let $(\mathscr{C}, \epsilon)$ be an exact category and P be its object. Then the following are equivalent:*

1. *P is projective.*
2. *The functor $\mathscr{C}(P, -)$ is exact, in the sense that $\mathscr{C}(P, -)$ maps each element of ϵ to a short exact sequence in* **Set**.
3. *Any admissible epic whose codomain is P splits.*

Proof. Let $(\mathscr{C}, \epsilon)$ be an exact category and P be its object.

(1. $\iff$ 2.): The functor $\mathscr{C}(A, -)$ preserves monics for all $A \in ob(\mathscr{C})$. Consider some monic $i \in \mathscr{C}(B, C)$, where $B, C \in ob(\mathscr{C})$. If $\mathscr{C}(A, i)(g) = \mathscr{C}(A, i)(g')$ for some $g, g' \in \mathscr{C}(A, B)$, then $ig = ig'$ and since i is a monic, $g = g'$. So $\mathscr{C}(A, i)$ is an injection in **Set**, which means that it is a monic.

Let $(i, p) \in \epsilon$ and $A, B, C \in ob(\mathscr{C})$ be such that $i \in \mathscr{C}(A, B)$ and $p \in \mathscr{C}(B, C)$. The equality between $\mathrm{Im}(\mathscr{C}(P, i))$ and $\mathrm{Ker}(\mathscr{C}(P, p))$ follows from the fact that i is the kernel of p since $\mathrm{Im}(\mathscr{C}(P, i)) = \{f \in \mathscr{C}(P, B);\ \exists g \in \mathscr{C}(P, A),\ f = ig\} = \{f \in \mathscr{C}(P, B); pf = pig = 0g = 0\} = \{f \in \mathscr{C}(P, B); \mathscr{C}(P, p)f = 0\} = \mathrm{Ker}(\mathscr{C}(P, p))$.

The proof of this part is finished by Proposition 1.2; $\mathscr{C}(P, -)$ preservers epics iff P is projective.

(2. $\implies$ 3.): Let P satisfy Condition 2 and π be an admissible epic with P as a codomain. Then $\mathscr{C}(P, \pi)$ is a surjection. In particular, there exists some g a pre-image for 1_P. Then g is a witness for the splitting of π because $\pi g = \mathscr{C}(P, \pi)g = 1_P$.

(3. $\implies$ 1.): Let P satisfy the Condition 3, A, B be objects of $\mathscr{C}$, and π be an admissible epic in $\mathscr{C}(A, B)$. Consider $f \in \mathscr{C}(P, B)$. Let (D, d_1, d_2) be a triple such that the diagram

$$\begin{tikzcd} D \arrow[r, two heads, "d_2"] \arrow[d, "d_1"] & P \arrow[d, "f"] \\ A \arrow[r, two heads, "\pi"] & B \end{tikzcd} \text{ is the pull-back of } \begin{tikzcd} & P \arrow[d, "f"] \\ A \arrow[r, two heads, "\pi"] & B \end{tikzcd}.$$

Then d_2 is an admissible epic whose codomain is P, hence there exists $g \in \mathscr{C}(P, D)$ such that $d_2 g = 1_P$. Since $\pi(d_1 g) = f$, P is projective.

$\square$

Definition 1.13. An exact category $(\mathscr{C}, \epsilon)$ *has enough projectives* if for each object $A \in ob(\mathscr{C})$ there exists a projective object $P \in ob(\mathscr{C})$ and an admissible epic $\pi \in \mathscr{C}(P, A)$.

Lemma 1.14. *Let $(\mathscr{C}, \epsilon)$ be an exact category which has enough projectives and $A \xrightarrow{i} B \xrightarrow{p} C$ be a pair of composable morphisms with their domains and codomains in $\mathscr{C}$. Then (i,p) belongs to ϵ iff the sequence*

$$0 \longrightarrow \mathscr{C}(P,A) \xrightarrow{\mathscr{C}(P,i)} \mathscr{C}(P,B) \xrightarrow{\mathscr{C}(P,p)} \mathscr{C}(P,C) \longrightarrow 0$$

is exact in **Set** *for every projective object P in $ob(\mathscr{C})$.*

Notice that the zero object in **Set** is the empty set.

Proof. The implication that P is projective then $\mathscr{C}(P,-)$ is exact is satisfied in any exact category by Lemma 1.12, (1. $\implies$ 2.).

For the reverse implication we assume a short exact sequence

$$0 \longrightarrow \mathscr{C}(P,A) \overset{\mathscr{C}(P,i)}{\longrightarrow} \mathscr{C}(P,B) \overset{\mathscr{C}(P,p)}{\longrightarrow} \mathscr{C}(P,C) \longrightarrow 0$$

for each projective object $P \in ob(\mathscr{C})$.

First, we want that i is a monic. Let X be an object in $\mathscr{C}$ and $f, g \in \mathscr{C}(X,A)$. We will show that if $if = ig$, then $f = g$. Since $(\mathscr{C}, \epsilon)$ has enough projectives, there exists a projective object $P \in ob(\mathscr{C})$ and an epic $j \in \mathscr{C}(P,X)$. If $if = ig$, then $ifj = igj$ which is $\mathscr{C}(P,i)(fj) = \mathscr{C}(P,i)(gj)$. Since $\mathscr{C}(P,i)$ is monic, we obtain $fj = gj$, and since j is epic, we conclude that $f = g$.

Second, observe that i is a kernel of p. The property of having enough projectives also gives us that $\mathscr{C}(P,A)$ is not the zero object, moreover, there exists an epic $f \in \mathscr{C}(P,A)$. From the exactness we infer that $\mathscr{C}(P,i)\mathscr{C}(P,p) = 0$, so $ipf = \mathscr{C}(P,i)\mathscr{C}(P,p)(f) = 0 = 0f$, and hence because f is epic, we conclude that $ip = 0$. Consider some $i' \in \mathscr{C}(X,B)$ such that $i'p = 0$. We will show that there is a unique $f \in \mathscr{C}(X,A)$ such that $if = i'$. The uniqueness follows from the proven fact that i is a monic. The existence follows from the fact the given exact sequence is exact in **Set**, from which we obtain $\mathrm{Ker}(\mathscr{C}(P,p)) = \mathrm{Im}(\mathscr{C}(P,i))$. So, if $i'p = 0$, then $i' \in \mathrm{Ker}(\mathscr{C}(P,p))$. Hence $i' \in \mathrm{Im}(\mathscr{C}(P,i))$, which implies that there is $f \in \mathscr{C}(P,A)$ such that $if = \mathscr{C}(P,i)(f) = i'$.

Then applying an obscure axiom (from Quillen's axioms which are equivalent to our definition of exact category, see Bühler [2010]), which says that if p has kernel, pi is epic then p is epic, finishes the proof.

□

For the sake of completeness, it should be mentioned that Grothendieck [1957] introduced another important class of categories, nowadays called Grothendieck categories. Within the framework of his AB-hierarchy, he defined them as AB5 categories with generators. Note that abelian categories are labelled by AB2. The category AB5 is an abelian one with its filtered colimits exact and in which the direct sum of any family $(A_i)_{i \in I}$ of its objects exists.

The main example of a Grothendieck category is formed by quasi-coherent sheaves on a scheme X, usually denoted by $QCoh(X)$. These are widely used in homological algebra and algebraic geometry.

It is interesting to note that there is an abelian category which has a generator despite containing no non-zero projective objects. While in any category $\mathscr{C}$, an object A is projective iff $\mathscr{C}(A,-)$ preserves epics, the generator is characterized by the fact that $\mathscr{C}(A, -)$ reflects epics. We will see that in the category of modules the non-zero free modules are both projective and generators.

Definition 1.15. Let $\mathscr{C}$ be an abelian category. An object G is *a generator* if $\mathscr{C}(G, -) : \mathscr{C} \to \mathbf{Set}$ is a faithful functor.

For more detailed information about generators see Freyd [1964] (section 3.3). The projective generators play an important role in embedding theorems, pointed out at the beginning of this chapter.

The embedding from the Gabriel-Quillen theorem is the well-known contravariant Yoneda embedding. A small exact category $(\mathscr{C}, \epsilon)$ is thus embedded in the abelian category of left exact functors from $\mathscr{C}^{op}$ to the category of abelian groups. Such an embedding is fully faithful and preserves exactness. The proof of the Gabriel-Quillen theorem was first published in Laumon [1983] (1.0.3) and is based on Grothendieck's theory of sheafification. Notice that any full subcategory of an abelian category which is closed under extensions is an exact category. The embedding theorem says that there are no other small exact categories. See Bühler [2010] (Theorem A.1 and Remark A.3).

The proof of the Freyd-Mitchell embedding theorem is done in two steps. First, Freyd showed that any exact category $(\mathscr{C}, \epsilon)$ is contravariantly embedded into a full subcategory of left-exact functors, which is a complete abelian category with an injective cogenerator. This embedding is fully faithful and preserves exactness. The proof can be seen in Freyd [1964] (Section 7.3). If we consider the dual of such a target category, we obtain that $(\mathscr{C}, \epsilon)$ is covariantly embedded into a (complete abelian) category with a projective generator. As the second step, Mitchell proved that an abelian category which has a projective generator is exactly faithfully embedded into a category of modules over the ring of endomorphisms of a projective generator. This embedding is full for a complete abelian category. See Mitchell [1964] (Theorem 3.1). The proof can be also seen in Freyd [1964], (Theorem 4.44 and Exercise D on the page 104).

1.2 Projectivity in the category of modules

We assume that the reader know terms as a module, submodule, direct sum of modules, etc. , for such and others simple notions from module theory we refer to Anderson and Fuller [1992]. At the end of this section and in the following one we will also use a few notions from homological algebra, which we will introduce very briefly. For more details of such notions we refer to Weibel [1994].

The category of modules over a ring R is a category whose objects are right R-modules and whose morphisms are R-homomorphisms. The composition of morphisms is naturally the composition of maps and the identity morphisms are the identity R-homomorphisms denoted by id_M for every right R-module M. We use Mod-R to denote such a category.

The left R-modules form a category too. This category is denoted by R-Mod but it is actually the category of right modules over the opposite ring of R, so we will concentrate just on the right modules. We will often write *a module* instead of a right R-module, if there is no ambiguity. Similarly, we will use *a homomorphism* instead of an R-homomorphism. For epics and monics we will use words *epimorphisms* and *monomorphisms*.

The category Mod-R is the fundamental example of an abelian category. Thus homomorphisms between any two modules have the structure of an abelian group. Therefore, we will view the bifunctor Mod-$R\,(-,-) : (\text{Mod-}R)^{op} \times \text{Mod-}R \to \mathbf{Set}$, usually denoted by $\mathrm{Hom}_R(-,-)$, as a functor whose codomain is $\mathbf{Ab}$. Where $\mathbf{Set}$ denotes the category of sets and $\mathbf{Ab}$ the category of abelian groups. Furthermore, there exists the zero object, namely the module 0. Kernels and cokernels exist for every homomorphism too, moreover, they can be defined by elements of modules: For $f \in \mathrm{Hom}_R(M,N)$ $\mathrm{Ker}(f) := \{m \in M; f(m) = 0\}$, $\mathrm{Im}(f) := \{n \in N; \exists m \in M, f(m) = n\}$.

Definition 1.16. Let R be a ring. A module P is *projective* if for any two modules M, N and any epimorphism $\pi \in \mathrm{Hom}_R(M,N)$ the following holds: For every $f \in \mathrm{Hom}_R(P,N)$ there exists $g \in \mathrm{Hom}_R(P,M)$ such that the diagram

$$\begin{tikzcd} P \arrow[d, dotted, "g"'] \arrow[dr, "f"] & \\ M \arrow[r, two heads, "\pi"'] & N \end{tikzcd}$$

commutes, i.e., $\pi g = f$.

Analogously to Lemma 1.12 for objects of an exact category, we have several equivalent conditions for projectivity of modules.

Lemma 1.17. *Let R be a ring and P a module. Then following are equivalent.*

1. *P is projective.*
2. *The functor* $\mathrm{Hom}_R(P,-)$ *preserves short exact sequences.*
3. *Any epimorphism with codomain P splits.*

Proof. (1. $\iff$ 2.): Since $\mathrm{Hom}_R(P,\pi)(g) = \pi g$, projectivity of the module P is equivalent to surjectivity of $\mathrm{Hom}_R(P,\pi)$ for every epimorphism π.

If we broaden the bottom arrow from the diagram in the definition of projective module to a short exact sequence

$$\begin{tikzcd} & & P \arrow[d, dotted, "g"'] \arrow[dr, "f"] & & \\ 0 \arrow[r] & A \arrow[r, hook] & M \arrow[r, two heads, "\pi"'] & N \arrow[r] & 0 \end{tikzcd},$$

which is always possible by setting $A = \mathrm{Ker}(\pi)$, we see (because $\mathrm{Hom}_R(A,-)$ preserves monomorphisms for every module A) that projectivity of P can be equivalently defined as the fact that $\mathrm{Hom}_R(P,-)$ preserves short exact sequences. That is, whenever $0 \to A \to M \xrightarrow{\pi} N \to 0$ is short exact sequence of modules then there is an exact sequence:

$$0 \to \mathrm{Hom}_R(P,A) \hookrightarrow \mathrm{Hom}_R(P,M) \overset{\mathrm{Hom}_R(P,\pi)}{\twoheadrightarrow} \mathrm{Hom}_R(P,N) \to 0.$$

The fact that $\mathrm{Ker}(\mathrm{Hom}_R(P,\pi))$ is the image of the arrow in front of it follows from the equality between $\mathrm{Ker}(\pi)$ and the image of the arrow in front of π (let us denote it by ι): Since

$$\begin{aligned}
&\mathrm{Ker}(\mathrm{Hom}_R(P,\pi)) = \{f \in \mathrm{Hom}_R(P,M); \pi f = 0\} = \\
&\{f \in \mathrm{Hom}_R(P,M); f(m) \in \mathrm{Ker}(\pi) \text{ for all } m \in M\} = \\
&\{f \in \mathrm{Hom}_R(P,M); f(m) \in \mathrm{Im}(\iota) \text{ for all } m \in M\} = \\
&\{f \in \mathrm{Hom}_R(P,M); f \in \mathrm{Im}(\mathrm{Hom}_R(P,\iota))\} = \mathrm{Im}(\mathrm{Hom}_R(P,\iota)).
\end{aligned}$$

(1. $\Longrightarrow$ 3.): Let $P \in ob(\mathscr{C})$ be projective and $\pi \in \mathrm{Hom}_R(M,P)$ be an epimorphism. Then from projectivity of P we obtain a morphism $g \in \mathrm{Hom}_R(P,M)$ for $f = id_P$ such that the diagram

$$\begin{tikzcd}
P \arrow[d, dotted, "g"'] \arrow[dr, "id_P"] & \\
M \arrow[r, two heads, "\pi"'] & P
\end{tikzcd}$$

commutes, hence $\pi g = id_P$. This proves that π splits.

(3. $\Longrightarrow$ 1.): Let P be such that any epimorphism with P as a codomain splits. Consider an epimorphism between two arbitrary modules $\pi \in \mathrm{Hom}_R(M,N)$. And consider $f \in \mathrm{Hom}_R(P,N)$. Let (D,f',π') be the pull-back of f and π. Since π splits, we obtain a commutative diagram

$$\begin{tikzcd}
D \arrow[r, two heads, shift left, "\pi'"] \arrow[dr, "f'"'] & P \arrow[l, shift left, "\varphi"] \arrow[dr, "f"] & \\
& M \arrow[r, two heads, "\pi"'] & N
\end{tikzcd}$$

in which $\pi(f'\varphi) = f$, so P is projective.

$\square$

An important subclass of projective modules is formed by free modules. Free structures are generally objects constructed from the set in a naive way in some sense. Precisely, they are images of the free functor, a left adjoint to the forgetful functor.

Definition 1.18. Let R be a ring. A module M is *free* if there exists a set $X \subseteq M$ such that for every module N and every function x from X to M there exists a unique $f \in \mathrm{Hom}_R(M,N)$ such that $f\restriction_X = x$ and the diagram

$$\begin{tikzcd}
M \arrow[r, dotted, "f"] & N \\
X \arrow[u, hook, "\subseteq"] \arrow[ur, "x"'] &
\end{tikzcd}$$

commutes. Any such set X is called *a free basis* of the module M.

Remark 1.19. It is not hard to see that M is free iff $M \cong R^{(X)}$ for some set X (where $R^{(X)}$ is the direct sum of X copies of R).

From the remark above several trivial propositions follow.

Proposition 1.20. *Let R be a ring. Every module M is the homomorphic image of some free module.*

Proof. Let M be a module and G its set of generators. Consider the free module $R^{(G)}$. Then the function $x : G \to M$ defined by $z(1_g) := g$ (where 1_g is the element of $R^{(G)}$ which has 1 at the g-th coordinate and 0 on the others) has exactly one extension to a homomorphism from $R^{(G)}$ to M, say f. Since $\mathrm{Im}(f) \subseteq \langle G \rangle = M$, f is an epimorphism.

□

Proposition 1.21. *Let R be a ring. A module M is projective iff it is isomorphic to a direct summand of some free module.*

Proof. Let R be a ring and M be a module. By the previous proposition there exists a free module F and an epimorphism $f \in \mathrm{Hom}_R(F, M)$ which fits to a short exact sequence $0 \to \mathrm{Ker}(f) \xrightarrow{\iota} F \xrightarrow{f} M \to 0$.

If M is projective, then f splits by Lemma 1.12, therefore, $C \oplus \mathrm{Ker}(\iota) \cong F$ for some $C \cong \mathrm{Im}(f)$. Since $M = \mathrm{Im}(f)$, M is isomorphic to a direct summand of a free module.

For the opposite implication we assume M to be isomorphic via φ to a summand of some free module F. Let $\pi \in \mathrm{Hom}_R(A, M)$ be an epimorphism and let ι be a canonical embedding of $\varphi(M)$ to $\varphi(M) \oplus C = F$. Then $\iota\varphi\pi$ is an epimorphism in $\mathrm{Hom}_R(A, F)$. Since F is free, there is some free basis X of F. Next, define a function $z : X \to A$ by taking $z(x)$ as some element of $(\iota\varphi\pi)^{-1}(x)$. Then there exists a unique extension of z to a homomorphism in $\mathrm{Hom}_R(F, A)$, say g. This g satisfies $\iota\varphi\pi g(x) = id_F(x)$ for every $x \in X$, so $\iota\varphi\pi g = id_F$. Since ι is the canonical embedding, $\varphi\pi g = id_{\varphi(M)}$ and since φ is an isomorphism, we get $\pi g = id_M$. This implies that π splits, and hence M is projective.

□

Corollary 1.22. *Every free module is projective.*

Remark 1.23. The converse does not hold in general. Although every free module is isomorphic to a direct sum of copies of R, it is not true that every projective module M is a direct sum of some copies of R. As a counterexample it is enough to consider $M = \mathbb{Z}$ with the structure of $(\mathbb{Z}\oplus\mathbb{Z})$-module given by $(a \oplus b) \cdot x = ax$.

Nevertheless, for modules over a local ring, projective modules actually coincide with free ones, see Anderson and Fuller [1992] (Corollary 26.7).

The iteration of Proposition 1.20 together with the corollary above gives us an important representation for any module M:

Definition 1.24. A long exact sequence

$$... \to P_{n+1} \to P_n \to P_{n-1} \to ... \to P_0 \to M \to 0$$

is *a projective resolution of* M if P_i is a projective module for every $i < \omega$.

Note that in a category of abelian groups **Ab**, *a chain complex* is a long sequence of abelian groups and group homomorphisms between them

$$... \to A_{i+1} \xrightarrow{d_{i+1}} A_i \xrightarrow{d_i} A_{i-1} \xrightarrow{d_{i-1}} ...$$

where $d_i d_{i+1} = 0$ for each $i < \omega$.

The condition $d_i d_{i+1} = 0$ is equivalent to $\mathrm{Ker}(d_i) \supseteq \mathrm{Im}(d_{i+1})$, hence a chain complex can be viewed as a long sequence which is half exact.

Now, we can define the functor $\mathrm{Ext}^n_R(-, N) : (\text{Mod-}R)^{op} \to \mathbf{Ab}$ derived from $\mathrm{Hom}_R(-, N)$. We define $\mathrm{Ext}^n_R(M, N)$ as the n-th cohomology group of a chain complex of abelian groups:

$$0 \to \mathrm{Hom}_R(P_0, N) \to \mathrm{Hom}_R(P_1, N) \to ...$$
$$... \to \mathrm{Hom}_R(P_{n-1}, N) \to \mathrm{Hom}_R(P_n, N) \to \mathrm{Hom}_R(P_{n+1}, N) \to ...$$

For detailed introduction to derived functors we refer the reader to Weibel [1994].

For us, the important property of Ext^1_R is that for any short exact sequence $0 \to A \to B \to C \to 0$ there is a long exact sequence:

$$0 \to \mathrm{Hom}_R(M, A) \to \mathrm{Hom}_R(M, B) \to \mathrm{Hom}_R(M, C) \to \mathrm{Ext}^1_R(M, A)$$
$$\to \mathrm{Ext}^1_R(M, B) \to \mathrm{Ext}^1_R(M, C) \to \mathrm{Ext}^2_R(M, A) \to \mathrm{Ext}^2_R(M, B) \to ...$$

Proposition 1.25. *Let R be a ring and P be a module. Then P is projective iff $\mathrm{Ext}^1_R(P, A) = 0$ for every module A.*

Proof. It is obvious that if $\mathrm{Ext}^1_R(M, A) = 0$ for every module A, then M is projective, because then the functor $\mathrm{Hom}_R(M, -)$ preserves short exact sequences.

The reverse implication also holds since $\mathrm{Ext}^1_R(M, A)$ can be computed from

$$... \to 0 \to 0 \to M \to M \to 0,$$

which is a projective resolution of M if M is projective, as the 1-th cohomology of a chain complex of abelian groups:

$$0 \to \mathrm{Hom}_R(M, A) \to \mathrm{Hom}_R(0, A) \to \mathrm{Hom}_R(0, A) \to ...$$

Since $\mathrm{Hom}_R(0, A) = 0$, $\mathrm{Ext}^1_R(M, A) = 0$.

□

In agreement with the general Definition 1.15, we can define the notion of a generator in the module category as in Anderson and Fuller [1992] as it follows.

Definition 1.26. Let R be a ring. A module G is *a generator* if for every module M there exist a set X and an epimorphism in $\mathrm{Hom}_R(G^{(X)}, M)$. Or, equivalently, if the functor $\mathrm{Hom}_R(G, -)$ reflects epimorphisms.

Recall that preservation of epimorphisms by $\mathrm{Hom}_R(P, -)$ characterizes projective modules. And as we saw in the previous proposition

M is projective iff M is isomorphic to a direct summand of $R^{(I)}$

while from the definition of generator and the fact that R is projective (by Corollary 1.22)

M is generator iff R is isomorphic to a direct summand of $M^{(I)}$.

The set I is possibly infinite.

Corollary 1.27. *Every non-zero free module is a projective generator.*

In module theory the main general result about the structure of projective modules is the Kaplansky theorem. For a proof we refer the reader to Anderson and Fuller [1992] (Corollary 26.2).

Theorem 1.28 (Kaplansky)**.** *Let R be a ring. Every projective module is isomorphic to a direct sum of countably generated modules.*

The decomposition from the theorem above is not unique in general. For example, there exist Dedekind domains R with a non-principal ideal I such that $I \oplus I^{-1} \cong R^2$.

1.3 Injectivity, Baer's Criterion, and R-injectivity

Injectivity is the dual notion to projectivity in the sense that a module I is injective in Mod-R if and only if it is a projective object in the category dual to Mod-R. However, the category (Mod-$R)^{op}$ is not a category of modules, it is not even AB5 (AB5 categories were mentioned at the end of Section 1.1).

Definition 1.29. Let R be a ring. A module I is *injective* if for every pair of modules M, N and every monomorphism $\iota \in \mathrm{Hom}_R(M, N)$ the following holds: For every morphism $f \in \mathrm{Hom}_R(M, I)$ there exists a morphism $g \in \mathrm{Hom}_R(N, I)$ such that the diagram

$$\begin{tikzcd} M \arrow[r, hook, "\iota"] \arrow[dr, "f"'] & N \arrow[d, dotted, "g"] \\ & I \end{tikzcd}$$

commutes, i.e., $g\iota = f$.

Analogously to the projective case we can reformulate this definition in terms of properties of the Hom and Ext functors.

Lemma 1.30. *Let R be a ring and I be a module. Then the following conditions are equivalent.*

1. *I is injective.*

2. $\mathrm{Hom}_R(-, I)$ *preserves short exact sequences.*

3. *Any monomorphism whose domain is I splits.*

Proof. The proof is dual to the one of Lemma 1.17.

□

There is a useful criterion for injectivity known as Baer's Criterion:

Theorem 1.31. *(Baer's Criterion, BC) A module M over a ring R is injective iff every module homomorphism from a right ideal of R to M can be extended to a homomorphism from R to M.*

This allows us to test for injectivity of a module just at ideals of R instead of at all modules. This condition, which might seem weaker but from Baer's Criterion we know that it is equivalent to injectivity, defines R-injectivity.

Definition 1.32. A module I is *R-injective* if for every ideal $J \subseteq R$ and for the canonical embedding of J to R, ι, the following holds: For every morphism $f \in \mathrm{Hom}_R(J, I)$ there exists a morphism $g \in \mathrm{Hom}_R(R, I)$ such that $g\iota = f$, i.e., the diagram

$$\begin{tikzcd} J \arrow[r, hook, "\iota"] \arrow[dr, "f"'] & R \arrow[d, dotted, "g"] \\ & I \end{tikzcd}$$

commutes.

Therefrom, we can see that Baer's Criterion just says that a module M is injective iff it is R-injective.

One of many corollaries of BC is the characterization of injective modules over a Dedekind domain, in particular, of injective abelian groups.

Corollary 1.33. *Let R be a Dedekind domain. Then M is injective iff M is divisible.*

Remark 1.34. In particular, divisible $\mathbb{Z}$-modules, which are groups of the form $\mathbb{Q}^{(I)} \oplus \bigoplus_p \mathbb{Z}_{p^\infty}^{(I_p)}$ where p runs over all prime integers and I, I_p are some sets, coincide with the injective groups.

With the dualization of Proposition 1.20 and some properties of the group $\mathbb{Q}/\mathbb{Z}$ BC provides the following.

Proposition 1.35. *Let R be a ring. Every module M is embedded in some injective module.*

Proof. Let R be a ring and M be a module. The module M has the natural structure of a left $\mathbb{Z}$-module. Consider the functor $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z}) : \mathbb{Z}\text{-Mod} \to \mathbb{Z}\text{-Mod}$ and by M^* denote the module $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$. Then M^* has also the structure of a left R-module given by

$$r \cdot f := (m \mapsto (mr)f).$$

By Proposition 1.20, there is a free left R-module F and an epimorphism $\pi \in \mathrm{Hom}_R(F, M^*)$. Applying $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$ to $F \xrightarrow{\pi} M^*$, we obtain

$$M^{**} = \mathrm{Hom}_{\mathbb{Z}}(M^*, \mathbb{Q}/\mathbb{Z}) \stackrel{\pi^* = \mathrm{Hom}_Z(\pi, \mathbb{Q}/\mathbb{Z})}{\longrightarrow} \mathrm{Hom}_{\mathbb{Z}}(F, \mathbb{Q}/\mathbb{Z}) = F^* = (R^{(I)})^*.$$

The last equality is satisfied by Remark 1.19.

The $\mathbb{Z}$-homomorphism π^* is also an R-homomorphism since $(m)(\pi^*(r \cdot g) = (m)(\pi(r \cdot g)) = ((m)\pi)(r \cdot g) = ((m)\pi r)g = ((mr)\pi)g = (mr)(\pi g) = (m)(r \cdot (\pi g)) = (m)(r \cdot (\pi^*(g)))$ for every $g \in M^{**}, r \in R$ and $m \in M^*$.

Consider the evaluation R-homomorphism

$$\begin{aligned} \nu : M &\to M^{**} \\ m &\mapsto (f \mapsto (m)f). \end{aligned}$$

Using the fact that $\mathbb{Q}/\mathbb{Z}$ is a cogenerator in $\mathbb{Z}$-Mod we obtain $\pi^*\nu \in \mathrm{Hom}_R(M, F^*)$ is an epimorphism. This is the embedding which we are looking for. All that remains is to show that F^* is injective.

From the properties of the Hom functor

$$F^* = (R^{(I)})^* = \mathrm{Hom}_{\mathbb{Z}}(R^{(I)}, \mathbb{Q}/\mathbb{Z}) \cong \prod_I \mathrm{Hom}_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z}) = \prod_I R^* = (R^*)^{(I)}.$$

So it is enough to show that R^* is injective.

From BC we can test for injectivity of R^* just on right ideals. Let P be a right ideal of R. Consider $f \in \mathrm{Hom}_R(P, R^*)$. Define $h \in \mathrm{Hom}_{\mathbb{Z}}(P, \mathbb{Q}/\mathbb{Z})$ as $(P)h := (f(P))(1)$. Since $\mathbb{Q}/\mathbb{Z}$ is divisible abelian group and hence injective, there exists $h' \in \mathrm{Hom}_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z}) = R^*$ which extends h. We claim that the morphism which sends $r \in R$ to $r \cdot h'$ is an extension of f to R. This claim follows from the equation $p(r) \cdot h' = (pr)h' = (pr)h = (f(pr))(1) = (f(p)(r))(1) = (f(p))(r)$ for every $r \in R$ and $p \in P$. This proves that R^* (and thus F^*) is injective.

□

Remark 1.36. The module $M^* = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ from the proof of previous lemma is called *the character left module associated to the right module M*. In some sources it is also denoted by M^+.

Moreover, there always exists such an embedding of a module M, which is in some sense the best one and thus determines an injective envelope:

Definition 1.37. Let R be a ring, I a module and M a submodule of I. The module M is *essential* in I if for every non-zero submodule of I its intersection with M is non-zero. In such case, we also say that the embedding of M into I is *essential.* If I is moreover injective, then it is called *an injective envelope* of M.

The injective envelope of a module M is uniquely determined up to isomorphism and it is usually denoted by $E(M)$. In some texts it is called *the injective hull* of M.

Baer's Criterion has several other corollaries.

Corollary 1.38. *Let R be a right noetherian ring. Then every injective module is a unique direct sum of copies of indecomposable injective modules, which are of the form $E(R/I)$, where I are irreducible ideals of R.*

Corollary 1.39. *Let R be a commutative noetherian ring. Then every injective module is a unique direct sum of copies of $E(R/p)$, where p is a prime ideal of R.*

The uniqueness from Corollaries 1.38 and 1.39 is understood up to ordering of the indecomposable summands. Nevertheless, the form of such summands in Corollary 1.38 is not unique. A proof of the former corollary appears in Matlis [1958] (Theorem 2.4 and 2.5). The latter one comes also from Matlis [1958] (Proposition 3.1), which states that over commutative noetherian rings there is a bijection between modules of the form $E(R/p)$, where p are the prime ideals of R, and injective indecomposable modules.

The iteration of Proposition 1.35 together with the uniqueness of an injective envelope gives us the important representation of any module M:

Definition 1.40. A long exact sequence

$$0 \to M \xrightarrow{f_0} I_0 \xrightarrow{f_1} \ldots \xrightarrow{f_{n-1}} I_{n-1} \xrightarrow{f_n} I_n \xrightarrow{f_{n+1}} I_{n+1} \to \ldots,$$

is *the minimal injective resolution* if I_i is an injective envelope of $\mathrm{Im}(f_i)$ for every $i < \omega$.

It is well-known that for the functor $\mathrm{Ext}^n_R(N, -) : (\text{Mod-}R) \to \mathbf{Ab}$, derived from $\mathrm{Hom}_R(N, -)$ and defined as the n-th cohomology of the chain complex of abelian groups

$$\begin{gathered} 0 \to \mathrm{Hom}_R(N, M) \to \mathrm{Hom}_R(N, I_0) \to \ldots \\ \ldots \to \mathrm{Hom}_R(N, I_{n-1}) \to \mathrm{Hom}_R(N, I_n) \to \ldots, \end{gathered}$$

it holds that $\mathrm{Ext}^n_R(N, -)(M) = \mathrm{Ext}^n_R(N, M) = \mathrm{Ext}^n_R(-, M)(N)$ for every pair of modules M, N.

For any short exact sequence $0 \to A \to B \to C \to 0$ there is again a long exact sequence:

$$\begin{aligned} 0 &\to \mathrm{Hom}_R(C, M) \to \mathrm{Hom}_R(B, M) \to \mathrm{Hom}_R(A, M) \to \mathrm{Ext}^1_R(C, M) \\ &\to \mathrm{Ext}^1_R(B, M) \to \mathrm{Ext}^1_R(A, M) \to \mathrm{Ext}^2_R(C, M) \to \mathrm{Ext}^2_R(B, M) \to \ldots \end{aligned}$$

In analogy to Proposition 1.25 there is the following characterization of injective modules:

Proposition 1.41. *Let R be a ring. A module I is injective iff the* $\mathrm{Ext}^1_R(C, I)$ *vanishes for every module C.*

Let us also note that the analogue to the Kaplansky direct sum decomposition of projective modules into a dirrect sum of modules of bounded cardinality fails for injective modules over non-right noetherian rings since the Faith-Walker theorem states that such decomposition exists exactly for injective modules over a right noetherian ring. A proof appear, for example, in Anderson and Fuller [1992] (Theorem 25.6).

1.4 Faith's Problem (DBC and R-projectivity)

If we weaken projectivity in the same way as we did for injectivity in the previous section, there arises the question of whether the analogous claim to Baer's Criterion (known as Dual Baer's Criterion, DBC for short) holds. Let us have a look at this situation in detail.

Definition 1.42. Let R be a ring. A module M is *R-projective* if any homomorphism $f \in \mathrm{Hom}_R(M, R/J)$ factorizes through the canonical projection $\pi_J \in \mathrm{Hom}_R(R, R/J)$ for each right ideal $J \subseteq R$.

This means nothing else than that the functor $\mathrm{Hom}_R(M, -)$ is exact at each short exact sequence with R as the middle term. So naturally we define this property more generally as in Anderson and Fuller [1992] (§16).

Definition 1.43. Let R be a ring and N a module. A module M is *N-projective* if $\mathrm{Hom}_R(M, -)$ is exact at each short exact sequence with N as the middle term.

The set $\mathcal{P}r^{-1}(M) := \{N \in \text{Mod-}R; M \text{ is } N\text{-projective}\}$ known as *the projectivity domain of the module M* has a nice property:

Lemma 1.44. *Let M be a module. The projectivity domain of M is closed under submodules, homomorphic images, and finite direct sums.*

Proof. For a proof we refer the reader to Anderson and Fuller [1992] (Proposition 16.12).

□

A module M is projective iff every module belongs to $\mathcal{P}r^{-1}(M)$. Whereas R-projectivity of M means simply that R belongs to $\mathcal{P}r^{-1}(M)$.

Remark 1.45. We can ask whether R-projectivity is actually weaker property than projectivity. It is known that it is not over any right perfect ring.

Therefore, consider R a non-right perfect ring. Since every non-zero projective module contains a maximal submodule (Anderson and Fuller [1992], Proposition 17.14), it is enough to consider a ring in which there exists an R-projective non-zero module without any maximal submodule.

Every commutative noetherian ring in which all submodules have a maximal submodule is artinian (Hamsher [1966], Lemma 1,2 and Theorem). Thus every commutative noetherian ring which is not artinian contains a non-zero module without maximal submodules. This module is clearly not projective.

Moreover, in Trlifaj [2020] (Lemma 2.4 (1)), it is proved that such a module, say M, is R-projective because submodules of cyclic modules over a noetherian ring are finitely generated and a module with no maximal submodules has no non-zero finitely generated homomorphic images and hence $\mathrm{Hom}_R(M, R/I) = 0$ for each ideal I in R. Example of such a ring is $\mathbb{Z}$, where such a module is $\mathbb{Q}$.

However, projectivity and R-projectivity coincide for finitely generated modules over an arbitrary ring R as it follows from Lemma 1.44.

Corollary 1.46. *Let R be a ring and M be a finitely generated R-projective module. Then M is projective.*

Proof. Let M be a finitely generated R-projective module. There exist $n < \omega$ and the monomorphism $\pi \in \mathrm{Hom}_R(R^n, M)$ by the proof of Proposition 1.20. Applying Lemma 1.44 we obtain that M is R^n-projective. That is, the identity morphism on M factorizes trough π and hence M is a summand in a free module R^n. By Proposition 1.21, M is projective.

□

The following example shows that the class of all R-projective modules is not closed under submodules.

Example 1.47. Let R be the polynomial ring over a field K in n commuting variables, $R = K[x_1, ...x_n]$, where $1 < n < \omega$. A ring R as a module is trivially R-projective. Since the right global dimension of $K[x_1, ...x_n]$ is n (Hilbert's syzygy theorem), the ring $K[x_1, ...x_n]$ is not right hereditary (see Assem et al. [2006], Theorem VII.1.4). So there exists a non-projective finitely generated ideal $I \subsetneq R$.

The existence of a non-projective ideal in R can be proven directly. Consider the ideal $I = x_1R + x_2R$. We want to show that the free resolution of I,

$$0 \to R \xrightarrow{f} R^2 \xrightarrow{g} I \to 0,$$

where f and g are defined by $f(1) = (x_2, x_1), g((1,0)) = x_1$ and $g((0,1)) = -x_2$, is not a splitting sequence, which implies that the projective dimension of I is 1 and hence I is not projective.

Let us assume for contradiction that f splits, i.e., there is $h \in \mathrm{Hom}_R(R^2, R)$ such that $hf = id_R$. Then $1 = hf(1) = h((x_2, x_1)) = x_2h((1,0)) + x_1h((0,1)) \in I$ which is clearly a contradiction since $I \neq R$.

The Hilbert's basis theorem implies that $K[x_1, ...x_n]$ is right noetherian. Hence I is finitely generated and Corollary 1.46 applies.

Both Hilbert's theorems mentioned above were first proved by Hilbert [1890]. For a proof of the Hilbert's basis theorem we refer the reader to Atiyah and MacDonald [1969] (Theorem 7.5). For a modern proof of Hilbert's syzygy theorem by Gröbner bases we refer to Ene and Herzog [2012] (Theorem 4.18).

Definition 1.48. Let R be a ring. Dual Baer's Criterion (DBC) for the ring R is the following claim:

A module M is projective iff it is R-projective.

The question: "For what rings R does Dual Baer's Criterion hold?" or equivalently "When is projectivity equivalent to R-projectivity?", originally formulated by Faith [1976], is known as Faith's Problem.

Let us recall what we have mentioned in introduction. If R is an arbitrary ring, then DBC holds for every finitely generated module (see Corollary 1.46). However, whether DBC is satisfied depends on ring properties. It is also known that DBC holds when R is a (right) perfect ring (Sandomierski [1964]).

If R is a non-(right) perfect ring, then DBC either does not hold or its validity is independent of ZFC. The exact boundary is unknown, thus Faith's Problem restricted to non-(right) perfect rings remains open.

However, DBC fails for rings which are commutative noetherian (see Hamsher [1966], Theorem 1) or semilocal right noetherian (Alhilali et al. [2017], Proposition 2.11), and for commutative domains (Trlifaj [2019], Lemma 1).

About the rings for which DBC is independent the best result achieved so far is from Trlifaj [2022]. It says that for the class of small regular, semiartinian rings with primitive factors artinian DBC is independent in ZFC+GCH. The proof of independence employs two combinatorial principles, SUP and $\diamondsuit$. For this reason we will explore properties of the rings mentioned above in the following chapter. In the last chapter we will present the aforementioned combinatorial principles together with the proof ot the independence of DBC for this class of rings.

2. Non-perfect Rings

Throughout this Thesis we will consider regularity of a ring in the sense of von Neumann as defined below.

Definition 2.1. A ring R is said to be *regular* if for each $r \in R$ there exists $s \in R$ such that $rsr = r$.

The trivial example of a regular ring is a field because the inverse element exists there for each non-zero element. The ring of square matrices over a field is also regular due to the existence of pseudo-inverse matrices. The more interesting example is the endomorphism ring of a vector space (possibly infinite dimensional) over a skew field. There also exist rings which are not regular, for example a polynomial ring over a field.

The important equivalent characterization of regular rings is that every principal ideal is generated by an idempotent element. Another notable equivalent condition is that every R-module is flat. For more information we refer to the classic monograph on regular rings, Goodearl [1979]. We will use some more notions from module theory without an explanation as a completely reducible module, socle and Jacobson radical of a module, and others. For definition and more about properties of such terms we refer the reader to Anderson and Fuller [1992].

Let us mention the statement from Goodearl's monograph which we will use.

Theorem 2.2. *Let R be a regular ring and P a projective module. Then for each finitely generated submodule $M \subseteq P$ there exists a module C such that $M \oplus C = P$.*

Proof. For a proof see Goodearl [1979] (Theorem 1.11).

□

2.1 The Loewy Length

Definition 2.3. Let R be a ring and M be a module. A sequence of (right) submodules $(S_\alpha)_{\alpha \le \tau}$ is called *a (right) socle sequence of M* if it is the largest sequence such that $S_0 = 0$, $S_{\alpha+1} \supsetneq S_\alpha$, $S_{\alpha+1}/S_\alpha = \operatorname{Soc}(R/S_\alpha)$ for each $\alpha < \tau$, and $S_\alpha = \bigcup_{\beta<\alpha} S_\beta$ for a limit ordinal $\alpha \le \tau$.

Notice that the socle sequence always exist and is unique by definition.

Definition 2.4. Let R be a ring. A module M is *(right) semiartinian* if there exists its (right) socle sequence $(S_\alpha)_{\alpha \le \tau}$ such that $S_\tau = M$. A ring R is said to be *(right) semiartinian* if R is semiartinian as a (right) module over itself.

Definition 2.5. Let R be a ring, M be a semiartinian module and $(S_\alpha)_{\alpha \le \tau}$ be its (right) socle sequence. The ordinal τ is called *a (right) Loewy length of M.*

If a module M is finitely generated (in particular if $M = R$), then the (right) Loewy length is a non-limit ordinal, for this reason we will write $\sigma + 1$ instead of τ.

In a regular ring every simple right ideal is irreducible and direct summand in R, hence generated by a primitive idempotent. Such an idempotent generates also a left ideal, which is again an irreducible direct summand in R, hence simple. Since the socle of a module is a two-sided ideal, we obtain that right and left socle sequences of R coincide. Hence for a regular ring the right Loewy length is equal to the left Loewy length and we need not to distinguish the sides.

In general a right semiartinian ring does not have to be left semiartinian but a ring R with the finite right Loewy length n is necessarily left semiartinian and the left Loewy length of R can be at most $2^n - 1$ (see Camillo and Fuller [1974], Proposition 1.2 and Theorem 1.3).

Semiartinian rings are also called Loewy, because a ring R is semiartinian iff every module has the Loewy length. It is a direct consequence of the following proposition.

Proposition 2.6. *A ring R is (right) semiartinian iff $\mathrm{Soc}(M) \neq 0$ for each (right) module $M \neq 0$.*

Proof. Let R be a (right) semiartinian ring and M be a module. First observe that any free module is semiartinian with the same Loewy length as R. Then, by Proposition 1.20, there exist a free module F and a morphism $f \in \mathrm{Hom}_R(F, M)$ such that $M \cong F/\mathrm{Ker}(f)$. Let $(S_\alpha)_{\alpha \leq \tau}$ be the socle sequence of F.

Let us define the sequence $(T_\alpha)_{\alpha \leq \tau}$ by $T_\alpha := (S_\alpha + G)/G$ for each $\alpha \leq \tau$. Then it forms a continuous (not necessary strictly) increasing sequence such that $T_0 = 0$ and $T_\tau = M$.

Applying twice the third isomorphism theorem, we obtain

$$T_{\alpha+1}/T_\alpha = ((S_{\alpha+1} + G)/G)/((S_\alpha + G)/G) \cong$$
$$S_{\alpha+1}/(S_{\alpha+1} \cap (S_\alpha + G)) \cong (S_{\alpha+1}/S_\alpha)/((S_{\alpha+1} \cap (S_\alpha + G))/S_\alpha).$$

Hence $T_{\alpha+1}/T_\alpha$ is a completely reducible module since it is a factor of $S_{\alpha+1}/S_\alpha$. Then refining this sequence we can obtain a transfinite composition series of M.

There need to be at least one simple submodule of M, hence $\mathrm{Soc}(M) \neq 0$. The reverse implication is trivial.

□

If we denote by $(U_\alpha)_{\alpha \leq \rho}$ the socle sequence of M from the proof of the proposition above, it can be shown by induction that $T_\alpha \subseteq U_\alpha$ for each $\alpha \leq \rho$. Then $\rho \leq \tau$, thus it follows that if the right Loewy length of a ring R is τ, then the right Loewy length of an arbitrary module is at most τ.

Consider a regular ring R. If the ring has Loewy length $\sigma + 1 = 1$, it is completely reducible because $R = R/0 = S_1/S_0 = \mathrm{Soc}(R/S_0) = \mathrm{Soc}(R)$. We say that R has the trivial Loewy length. In other cases (i.e., $\sigma > 0$), the ring is not completely reducible and if such a ring R is regular, its Jacobson radical is zero hence trivially nilpotent. Then the characterization of perfect rings (Anderson and Fuller [1992], Theorem[Bass] 28.4) gives us that R is non-perfect.

From now on, we will consider only regular semiartinian rings with non-trivial Loewy length.

2.2 Regular Semiartinian Rings with Primitive Factors Artinian

Definition 2.7. Let R be a ring. An ideal of R is *(right) primitive* if it is the annihilater of some (right) simple module. A ring R is said to be *with primitive factors artinian* if R/P is (right) artinian for each (right) primitive ideal P of R.

We will use this for regular rings for which the definition is left-right symmetric and equivalent to a claim that all prime factors of R are artinian (see Goodearl [1979], Theorem 6.2). Also we will use the following well-known results.

Lemma 2.8. *Let R be a regular ring with primitive factors artinian. Then R/I is also regular with primitive factors artinian for each ideal I.*

Proof. Let R be a regular ring with primitive factors artinian and I be an ideal. Then the factor of regular ring R/I is trivially regular. We want to prove an equivalent condition to R/I having primitive factors artinian, that is, for every countable set of orthogonal idempotents in R/I, $\{e_n \in R/I; n < \omega\}$, and for every set of elements $\{x_n \in R/I; n < \omega\}$ there exists $0 < n_0 < \omega$ such that $e_1x_1 \cdot \ldots \cdot e_{n_0}x_{n_0} = 0$ where 0 is the zero element of R/I (see Goodearl [1979], Theorem 6.2 d)).

By Goodearl [1979] (Proposition 2.18), there exists a countable set of orthogonal idempotents in R, $\{f_n; n < \omega\}$, such that $f_n + I = e_n$ for each $n < \omega$. Then there exists $y_n \in R$ for each $n < \omega$ such that $x_n = y_n + I$. By Goodearl [1979] (Theorem 6.2 d)) the assumed properties of R yields $n_0 < \omega$ such that $f_1y_1 \cdot \ldots \cdot f_{n_0}y_{n_0} = 0$ where 0 is the zero element in R. Therefore $e_1x_1 \cdot \ldots \cdot e_{n_0}x_{n_0} = (f_1 + I)(y_1 + I) \cdot \ldots \cdot (f_{n_0} + I)(y_{n_0} + I) = 0 + I =$ the zero element in R/I, which finishes the proof.

□

Proposition 2.9. *Let R be a regular ring. Then R has primitive factors artinian iff each homogenous semisimple module is injective.*

Proof. See Goodearl [1979] (Proposition 6.18).

□

The characterization of the hereditary case is given below. The proof of Lemma 2.10 follows Trlifaj [2020] (Lemma 3.10) and the proof of reverse implication stated in Remark 2.11 comes from Trlifaj [2022] (Lemma 2.3).

Lemma 2.10. *Let R be a regular, semiartinian ring with primitive factors artinian and $(S_\alpha)_{\alpha\leq\sigma+1}$ be its socle sequence. If σ is countable and $S_{\alpha+1}/S_\alpha$ is countably generated for each $0 < \alpha < \sigma$, then R is hereditary.*

Proof. Let R be a regular, semiartinian ring with primitive factors artinian. Assume that its socle sequence $(S_\alpha)_{\alpha\leq\sigma+1}$ is such that σ is countable and $S_{\alpha+1}/S_\alpha$ is countably generated for each $\alpha < \sigma$.

Let I be a right ideal of R. Since $(S_\alpha)_{\alpha\leq\sigma+1}$ is the socle sequence of R, $I = \bigcup_{\alpha\leq\sigma+1}(I \cap S_\alpha)$. Further $(I \cap S_{\alpha+1})/(I \cap S_\alpha)$ is isomorphic to a submodule of $S_{\alpha+1}/S_\alpha$ for each $0 < \alpha \leq \sigma$, which is completely reducible module and which we assume to be countably generated. Therefore $I/(I \cap S_1)$ is countably generated.

From the third isomorphism theorem, $(I+S_1)/S_1 \cong I/(I \cap S_1)$. So $(I+S_1)/S_1$ is countably generated. Then there exists a countable set $\{e_n \in I; n < \omega\}$ such that $\{e_n + S_1; n < \omega\}$ generates $(I + S_1)/S_1$. We can w.l.o.g. assume that e_n is an idempotent for each $n < \omega$ thanks to Goodearl [1979] (Proposition 2.14) where as A we use R and as B we use $(I + S_1)/S_1$. Thus we have a countable set of idempotents such that $((\bigoplus_{n<\omega} e_n R) + S_1) = I + S_1$.

Let $P = S_1/((\bigoplus_{n<\omega} e_n R) \cap S_1)$. Then P is a direct summand of $S_1 = \mathrm{Soc}(R)$, hence P is a completely reducible projective module. Again by the third isomorphism theorem $((\bigoplus_{n<\omega} e_n R) + S_1)/(\bigoplus_{n<\omega} e_n R) \cong P$. Therefore, $I + S_1 \cong (\bigoplus_{n<\omega} e_n R) \oplus P$. This, together with the fact that each idempotent e_n belongs to I, implies that $I \cong (\bigoplus_{n<\omega} e_n R) \oplus (I \cap Q)$ where $Q \cong P$. Since P is completely reducible, $I \cap Q$ is isomorphic to a direct summand of P. Hence $I \cap Q$ is projective and $(\bigoplus_{n<\omega} e_n R)$ as a countable direct sum of projective modules is also projective. This proves the projectivity of I. For a left ideal, the proof is analogous.

□

Remark 2.11. If S_1 is countably generated, then the reverse implication to Lemma 2.10 also holds.

Proof. Assume for contradiction that a regular semiartinian ring R with primitive factors artinian is right hereditary but σ is not countable or $S_{\alpha+1}/S_\alpha$ is not countably generated for some $0 < \alpha < \sigma$. In both cases there exists $0 < \delta \leq \sigma$ such that S_δ is not countably generated: If σ is not countable, then $\sigma = \delta + n$ for some uncountable ordinal $0 < \delta \leq \sigma$ and $n < \omega$. Then S_δ is uncountably generated. If $S_{\alpha+1}/S_\alpha$ is not countably generated, then S_δ is not countably generated for $\delta = \alpha + 1$.

$S_1 = \mathrm{Soc}(R)$ is countably generated because each homogenous component of $\mathrm{Soc}(R)$ is finitely generated by Proposition 2.9.

Since R is hereditary, S_δ is projective. And the regularity of R gives us an uncountable set of non-zero idempotents $\{e_\gamma \in R; \gamma < \kappa\}$ such that $S_\delta \cong \bigoplus_{\gamma<\kappa} e_\gamma R$.

Thus $\mathrm{Soc}(S_\delta) \cong \bigoplus_{\gamma<\kappa} \mathrm{Soc}(e_\gamma R)$ and since R is semiartinian, $\mathrm{Soc}(e_\gamma R) \neq 0$ for all $\gamma < \kappa$. Hence the uncountably generated $\mathrm{Soc}(S_\delta)$ is isomorphic to a direct summand in countably generated $S_1 = \mathrm{Soc}(R)$ which is a contradiction. □

For regular semiartinian rings with primitive factors artinian we know the structure of $S_{\alpha+1}/S_\alpha$ for the modules S_α from the socle sequence of R.

Lemma 2.12. *Let R be a semiartinian, regular ring with primitive factors artinian. For its socle sequence $(S_\alpha)_{\alpha\leq\sigma+1}$ we have the following:*

For each $\alpha \leq \sigma$ there are a cardinal λ_α, positive integers $n_{\alpha,\beta}$ ($\beta < \lambda_\alpha$), and skew-fields $K_{\alpha,\beta}$ ($\beta < \lambda_\alpha$) such that $S_{\alpha+1}/S_\alpha \overset{\varphi_\alpha}{\cong} \bigoplus_{\beta<\lambda_\alpha} M_{n_{\alpha,\beta}}(K_\alpha, \beta)$ as rings without unit. Moreover, λ_α is infinite iff $\alpha < \sigma$. The pre-image of $M_{n_{\alpha,\beta}}(K_{\alpha,\beta})$ in the isomorphism φ_α coincides with the β-th homogenous component of $\mathrm{Soc}(R/S_\alpha)$ and it is finitely generated as a right R/S_α-module for all $\beta < \lambda_\alpha$.

Proof. Let R be a semiartinian, regular ring with primitive factors artinian and $(S_\alpha)_{\alpha\leq\sigma+1}$ be its socle sequence. Then R/S_α is also regular with primitive factors artinian (see Lemma 2.8). Therefore Proposition 2.9 implies that the homogenous components of $\mathrm{Soc}(R/S_\alpha) = S_{\alpha+1}/S_\alpha$ are injective.

Begin with

$$S_{\alpha+1}/S_\alpha = \mathrm{Soc}(R/S_\alpha) \cong \bigoplus_{J \in Simp-R/S_\alpha} J^{(n_J)} = \bigoplus_{\beta<\lambda_\alpha} J_{\alpha,\beta}^{(n_{\alpha,\beta})}$$

where λ_α is a cardinality of set of those $J \in Simp-R/S_\alpha$ for which n_J is positive. Put $H_{\alpha,\beta} = J_{\alpha,\beta}^{(n_{\alpha,\beta})}$ the β-th homogenous component of $\mathrm{Soc}(R/S_\alpha)$. Then we can write

$$S_{\alpha+1}/S_\alpha \cong \bigoplus_{\beta<\lambda_\alpha} H_{\alpha,\beta}$$

as an isomorphism of modules.

For $\alpha = \sigma$, $S_{\alpha+1}/S_\alpha = R/S_\sigma$ which is completely reducible and hence the Wedderburn-Artin theorem implies $S_{\sigma+1}/S_\sigma = \prod_{\beta=1}^{\lambda_\sigma} M_{n_{\sigma,\beta}}(K_{\sigma,\beta})$ where λ_σ, $n_{\sigma,\beta}$ are finite and $K_{\sigma,\beta}$ are a skew-fields such that each of them is a skew-field of endomorphisms of β-th homogenous component of $S_{\sigma+1}/S_\sigma$.

Consider $\alpha < \sigma$. Since R is semiartinian and so $\mathrm{Soc}(R/S_\alpha) \neq 0$, $\mathrm{Soc}(R/S_\alpha)$ is not a direct summand of R/S_α (otherwise there exists some non-zero module $C \neq R/S_\alpha$ such that $\mathrm{Soc}(R/S_\alpha) \oplus C = R/S_\alpha$, which contradicts Proposition 2.6.

We conclude that λ_α is infinite because if not then $S_{\alpha+1}/S_\alpha$ is injective as a finite sum of injective modules and hence it is a direct summand in R/S_α. Since the homogenous components $H_{\alpha,\beta}$ are direct summands in R/S_α and hence the ring direct summands, we can use the Wedderburn Theorem to the simple homogenous rings $H_{\alpha,\beta}$ and obtain an integers $n_{\alpha,\beta}$ and a skew-fields $K_{\alpha,\beta}$ such that $H_{\alpha,\beta} = M_{n_{\alpha,\beta}}(K_{\alpha,\beta})$ for each $\beta < \lambda_\alpha$, hence $S_{\alpha+1}/S_\alpha \cong \bigoplus_{\beta<\lambda_\alpha} M_{n_{\alpha,\beta}}(K_{\alpha,\beta})$ as an isomorphism of rings without units. There $K_{\alpha,\beta}$ is again a skew-field of endomorphisms of $H_{\alpha,\beta}$ for each $\beta < \lambda_\alpha$ and $\alpha < \sigma$. □

For semiartinian rings the structure in the lemma above characterizes the regular rings with primitive factors artinian. We can find this result in this generality in Růžička et al. [1998]. Note that therefore we can think about semiartinian rings with primitive factors artinian as transfinite extensions of simple artinian rings, which full matrix rings over skew field are.

Remark 2.13. Let R be a semiartinian, regular ring with primitive factors artinian and $(S_\alpha)_{\alpha\leq\sigma+1}$ be its socle sequence. For each $\alpha \leq \sigma$ denote by $\mathcal{F}_\alpha$ the set of all finite subsets of λ_α. For each $F \in \mathcal{F}_\alpha$, let $N_{\alpha,F}$ be the submodule of $S_{\alpha+1}$ containing S_α such that

$$N_{\alpha,F}/S_\alpha \cong \bigoplus_{\beta\in F} M_{n_{\alpha,\beta}}(K_{\alpha,\beta}).$$

Let us denote by B and N the following modules

$$B = \prod_{\substack{0<\alpha\leq\sigma \\ F\in\mathcal{F}_\alpha}} N_{\alpha,F}$$

$$N = \prod_{\substack{0<\alpha\leq\sigma \\ F\in\mathcal{F}_\alpha}} N_{\alpha,F}/S_\alpha$$

and let π be the canonical projection between them. Note that the module $N_{\alpha,F}/S_\alpha$ is a finite direct sum of injective modules and hence it is injective for each $0 < \alpha \leq \sigma$ and $F \in \mathcal{F}_\alpha$. Therefore, N as their product is an injective module too. Also note that the cardinality of B is less than $\mathrm{card}(R)^{\sigma \times \mathrm{card}(\mathcal{F}_\alpha)}$. So if $\sigma+1$ is countable, it is bounded by $\mathrm{card}(R)^{\max\{\omega, \mathrm{card}(\mathcal{F}_\alpha)\}}$. If in addition $S_{\alpha+1}/S_\alpha$ is countably generated, then $\mathcal{F}_\alpha$ is countable. Then $\mathrm{card}(B) \leq \mathrm{card}(R)^\omega$.

Lemma 2.14. *There exists a regular semiartinian hereditary ring with primitive factors artinian which has Loewy length α for arbitrary $\alpha < \omega_1$.*

Proof. The proof can be found in Trlifaj [2022] (Example 2.4). The idea of this proof is to construct such a ring inductively for all $\alpha < \omega_1$. Moreover, such that R_α has countably generated layers (to obtain hereditarity by Lemma 2.10) and some other properties to obtain a semiartinian regular ring as in Eklof et al. [1997] (2.4).

For the base step there is considered the K-algebra of all eventually constant sequences of elements of a fixed field K, denoted by $\mathrm{ECS}(K)$. It is a semiartinian ring, due to $\mathrm{Soc}(ECS(K)) = K^{(\omega)}$ and

$$\mathrm{Soc}(ECS(K)/K^\omega) = \mathrm{Soc}(K) = K = ECS(K)/K^{(\omega)}.$$

So the socle sequence is $(0, K^{(\omega)}, ECS(K))$. By Lemma 2.12 it is a regular ring with primitive factors artinian. And by Lemma 2.10 it is hereditary.

In the induction step $R_{\alpha+1}$ is set to be an embedding of $\bigoplus_{n<\omega} R_\alpha$ to $\bigoplus_{n<\omega} K^\omega$ $\subseteq (K^\omega)^\omega = K^{\omega\times\omega}$ as K-algebra without unit to which is directly sum $1 \cdot K$ and which is isomorphously mapped into K^ω, since $\omega \times \omega$ is countable.

In the limit step, since for $\alpha < \omega_1$ there are countable set of ordinals β_n such that $\alpha = \sup_{n<\omega} \beta_n$, R_α is an image of again extended embedding of $\varinjlim_{n<\omega} R_{\beta_n}$ to K^ω as K-algebra without unit to a unital K-algebra embedding.

□

Remark 2.15. If we refrain from calling to be hereditary and have primitive factors artinian, there exists a ring of Loewy length $\alpha+1$ for an arbitrary ordinal α. It is known from the construction in Fuchs [1969] that there exists such a semiartinian ring, which is commutative and has zero Jacobson radical. It follows from Nastasescu and Popescu [1968] that each such a ring is regular.

2.3 A Weakening of R-projectivity

Sometimes it is useful to weaken a definition of some object to obtain further properties for this object. The same happens with R-projective modules. It will be convenient to have a class of modules which is closed under submodules. R-projective modules do not form such a class (Example 1.47), but there is a way how to weaken R-projectivity to achieve this property.

To do this, let us introduce the equivalent condition for R-projectivity, which we have in the case of the finite Loewy length as in Trlifaj [2020] (Theorem 3.4).

Theorem 2.16. *Let R be a regular semiartinian ring with the finite Loewy length and let $(S_\alpha)_{\alpha\leq\sigma+1}$ be its socle sequence. Then a module M is R-projective if and only if every homomorphism $f \in \mathrm{Hom}_R(M, S_{\alpha+1}/S_\alpha)$ factorizes through the canonical projection $\pi_\alpha : S_{\alpha+1} \to S_{\alpha+1}/S_\alpha$ for each $\alpha \leq \sigma$.*

Proof. The "only if" part holds in general (i.e., without the assumption that σ is finite) because $\mathcal{P}r^{-1}(R)$ is closed under submodules (see Lemma 1.44), so any R-projective module is also S_α-projective, which implies that M satisfies the required condition.

For the "if" part define $n(J) := max\{\alpha \leq \sigma; S_\alpha \subseteq J\}$ for each ideal $J \subsetneq R$. We will prove that $f \in \mathrm{Hom}_R(M, R/J)$ factorizes through the canonical projection $\pi_J : R \to R/J$ for every ideal J such that $n(J) = \alpha$ by downward induction on $\alpha \leq \sigma$.

If $\alpha = \sigma$ then $S_{\alpha+1} = R$ and $S_{\alpha+1}/S_\alpha = R/S_\sigma$ is completely reducible, hence the epimorphism $\rho : R/S_\sigma \to R/J$ splits, i.e., there exists a homomorphism $\eta : R/J \to R/S_\sigma$ such that $\rho\eta = id_{R/S_\sigma}$. The situation is represented by the diagram below.

$$\begin{tikzcd}
 & M \arrow[dl, dotted, "g"'] \arrow[dr, "f"] & \\
R \arrow[r, "\pi_\sigma"'] \arrow[rr, bend right, "\pi_J"'] & R/S_\sigma \arrow[r, shift right, "\rho"'] & R/J \arrow[l, shift right, "\eta"']
\end{tikzcd}$$

The homomorphism ηf factorizes through π_σ by assumption, so there exists $g \in \mathrm{Hom}_R(M, R)$ such that $\pi_\sigma g = \eta f$. And so f factorizes through π_J because $\pi_J g = \rho\pi_\sigma g = \rho\eta f = f$.

If $\alpha < \sigma$ and for each $\alpha < \beta \leq \sigma$ the assertion holds true, consider the canonical projection $\rho : R/J \to R/(S_{\alpha+1} + J)$. Because $n(S_{\alpha+1} + J) \geq \alpha + 1$, there exists by the inductive premise $\tilde{g} \in \mathrm{Hom}_R(M, R)$ such that $\pi_{(S_{\alpha+1}+J)}\tilde{g} = \rho f$. So we have the following diagram, in which the right and the external triangle commute but the left triangle does not have to commute.

$$\begin{tikzcd}
 & M \arrow[dl, dotted, "\tilde{g}"'] \arrow[d, "f"] \arrow[dr, "\rho f"] & \\
R \arrow[r, "\pi_J"'] \arrow[rr, bend right, "\pi_{S_{\alpha+1}+J}"'] & R/J \arrow[r, "\rho"'] & R/(S_{\alpha+1} + J)
\end{tikzcd}$$

The question whether the left triangle in the diagram above commutes can be translated to the question whether the difference $f - \pi_J\tilde{g}$ does factorize through π_J. Furthermore, $\rho(f - \pi_J\tilde{g}) = \rho f - \rho\pi_J\tilde{g} = \pi_{(S_{\alpha+1}+J)}\tilde{g} - \pi_{(S_{\alpha+1}+J)}\tilde{g} = 0$ implies that $\mathrm{Im}(f - \pi_J\tilde{g}) \subseteq \mathrm{Ker}(\rho) = (S_{\alpha+1} + J)/J$, so it is enough to show that $f - \pi_J\tilde{g}$ factorizes through $\pi_J\restriction_{S_{\alpha+1}+J}$.

$$\begin{tikzcd}
 & M \arrow[dl, dotted, "?"'] \arrow[d, "f-\pi_J\tilde{g}"] \\
S_{\alpha+1} \arrow[r, "\pi_J\restriction_{S_{\alpha+1}}"'] & (S_{\alpha+1} + J)/J
\end{tikzcd}$$

Consider the canonical projection $\theta : S_\alpha \to S_\alpha/(S_\alpha \cap J)$ and the isomorphism

$$\begin{aligned} i_\alpha : S_\alpha/(S_\alpha \cap K) &\to (S_\alpha + K)/K \\ s + S_\alpha \cap K &\mapsto s + K. \end{aligned}$$

Since $\theta = i_{\alpha+1}^{-1}\pi_J\restriction_{S_{\alpha+1}+J}$, it remains to show that $i_{\alpha+1}^{-1}(f - \pi_J\tilde{g})$ factorizes through θ.

$$\begin{tikzcd}
 & & M \arrow[dll, dotted, "?"'] \arrow[d, "f-\pi_J\tilde{g}"] \\
S_{\alpha+1} \arrow[drr, "\theta"] & & (S_{\alpha+1}+J)/J \arrow[d, "i_{\alpha+1}^{-1}"] \\
 & & S_{\alpha+1}/(S_{\alpha+1}\cap J)
\end{tikzcd}$$

Note that $S_{\alpha+1}/S_\alpha$ is completely reducible and hence the canonical projection $\varphi : S_{\alpha+1}/S_\alpha \to S_{\alpha+1}/(S_{\alpha+1}\cap J)$ splits. So there exists a homomorphism $\psi \in \mathrm{Hom}_R(S_{\alpha+1}/(S_{\alpha+1}\cap J), S_{\alpha+1}/S_\alpha)$ such that $\varphi\psi = id_{S_{\alpha+1}/(S_{\alpha+1}\cap J)}$.

$$\begin{tikzcd}
 & M \arrow[dl, dotted, "?"'] \arrow[d, "f-\pi_J\tilde{g}"] \\
S_{\alpha+1} \arrow[d, "\pi_\alpha"] \arrow[dr, "\theta"] & (S_{\alpha+1}+J)/J \arrow[d, "i_{\alpha+1}^{-1}"] \\
S_{\alpha+1}/S_\alpha \arrow[r, shift right, "\varphi"'] & S_{\alpha+1}/(S_{\alpha+1}\cap J) \arrow[l, shift right, "\psi"']
\end{tikzcd}$$

But now the question about factorizability is clear because $\psi i_{\alpha+1}^{-1}(f-\pi_J\tilde{g}) \in \mathrm{Hom}_R(M, S_{\alpha+1}/S_\alpha)$ and the assumed property from wording of this lemma tells us that this homomorphism factorizes through π_α, i.e., there exists a homomorphism $g \in \mathrm{Hom}_R(M, S_{\alpha+1})$ such that $\pi_\alpha g = \psi i_{\alpha+1}^{-1}(f-\pi_J\tilde{g})$. By applying φ to this equation we obtain $\varphi\pi_\alpha g = i_{\alpha+1}^{-1}(f-\pi_J\tilde{g})$, where the left side is nothing but θg. This proves that $i_{\alpha+1}^{-1}(f-\pi_J\tilde{g})$ factorizes through θ and hence f factorizes through π_J, which means that M is R-projective.

□

This gives us an idea how to weaken R-projectivity. We can use the condition from Theorem 2.16 and restrict it to morphisms with finitely generated images.

Definition 2.17. Let R be a semiartinian regular ring with the socle sequence $(S_\alpha)_{\alpha\leq\sigma+1}$. An R-module M is said to be *weakly R-projective* if for every $\alpha \leq \sigma$ it holds that every $f \in \mathrm{Hom}_R(M, S_{\alpha+1}/S_\alpha)$ with a finitely generated image factorizes through the canonical projection $\pi_\alpha \in \mathrm{Hom}_R(S_{\alpha+1}, S_{\alpha+1}/S_\alpha)$.

Proposition 2.18. *Let R be a ring and M be an R-projective module. Then M is weakly R-projective.*

Proof. Since $\mathcal{P}r^{-1}(R)$ is closed under submodules (see Lemma 1.44), every R-projective module M is also S_α-projective for each $\alpha \leq \sigma$. In particular the functor $\mathrm{Hom}_R(M, -)$ is exact on an exact sequence $0 \to S_\alpha \to S_{\alpha+1} \to S_{\alpha+1}/S_\alpha$, hence M is weakly R-projective.

□

The reversed implication does not hold, it follows from the fact proved below that weak R-projectivity is closed under submodules while R- projectivity is not as we can saw in Example 1.47.

Proposition 2.19. *Let R be a regular semiartinian ring with primitive factors artinian. Let M be a weak R-projective module and $N \subseteq M$ be a submodule. Then N is weakly R-projective.*

Proof. Let $(S_\alpha)_{\alpha\leq\sigma+1}$ be the socle sequence of R and let π_α denote the canonical projection from $S_{\alpha+1}$ to $S_{\alpha+1}/S_\alpha$. Consider $f \in \mathrm{Hom}_R(N, S_{\alpha+1}/S_\alpha)$ with the finitely generated image. Since every finitely generated submodule of $S_{\alpha+1}/S_\alpha$ is injective (Proposition 2.9), we can use injectivity of $\mathrm{Im}(f)$ to obtain $\tilde{f} \in \mathrm{Hom}_R(M, \mathrm{Im}(f))$ such that $\tilde{f} = f\iota$, where ι is the canonical embedding of N to M. The weak R-projectivity of M yields $g \in \mathrm{Hom}_R(M, S_\alpha)$ such that $\pi_\alpha g = \tilde{f}$. So we have $\pi_\alpha g = f\iota$ and restriction to N gives us $\pi_\alpha g\restriction_N = f$, which proves that N is weakly R-projective.

□

The following lemma allows us to test for weak R-projectivity of a module just on the one exact sequence, $0 \to \mathrm{Ker}(\pi) \to B \xrightarrow{\pi} N \to 0$, where modules B and N are from Remark 2.13

Lemma 2.20. *Let B, N, and π be as in Remark 2.13 then a module M is weakly R-projective iff $\mathrm{Hom}_R(M, \pi)$ is surjective.*

Proof. Let M be weakly R-projective. Consider $f \in \mathrm{Hom}_R(M, N)$, then f is determined by $p_{\alpha,F}f$ $(\alpha \leq \sigma, F \in \mathcal{F}_\alpha)$, where $p_{\alpha,F}$ is the canonical projection form N to $N_{\alpha,F}/S_\alpha$. Since F is finite, the $\mathrm{Im}(p_{\alpha,F}f)$ is finitely generated submodule of $S_{\alpha+1}/S_\alpha$. Then, from weak R-projectivity of M, there is $g_{\alpha,F} \in \mathrm{Hom}_R(M, S_{\alpha+1})$ such that $\pi_\alpha g_{\alpha,F} = p_{\alpha,F}f$. Put $g := \Pi_{\alpha\leq\sigma,F\in\mathcal{F}_\alpha} g_{\alpha,F}$ then $g \in \mathrm{Hom}_R(M, B)$ because $\mathrm{Im}(g_{\alpha,F}) \subseteq S_{\alpha+1} \subseteq N_{\alpha+1,F}$, and $\pi g = f$ because

$$\pi g = \Pi_{\alpha\leq\sigma,F\in\mathcal{F}_\alpha} \pi\restriction_{N_{\alpha,F}} \Pi_{\alpha\leq\sigma,F\in\mathcal{F}_\alpha} g_{\alpha,F} = \Pi_{\alpha\leq\sigma,F\in\mathcal{F}_\alpha} (\pi\restriction_{N_{\alpha+1,F}} g_{\alpha,F}) =$$
$$\Pi_{\alpha\leq\sigma,F\in\mathcal{F}_\alpha} (\pi_\alpha g_{\alpha,F}) = \Pi_{\alpha\leq\sigma,F\in\mathcal{F}_\alpha} (p_{\alpha,F}f) = f.$$

This proves that $\mathrm{Hom}_R(M, \pi)$ is surjective.

To prove the reverse implication, assume $\mathrm{Hom}_R(M, \pi)$ is surjective. Consider $f \in \mathrm{Hom}_R(M, S_{\alpha+1}/S_\alpha)$ with the finitely generated image, then $\iota_{\alpha+1}f \in \mathrm{Hom}_R(M, N)$, where $\iota_{\alpha+1}$ is the canonical embedding of $N_{\alpha+1,F}/S_\alpha$ into N. From surjectivity of $\mathrm{Hom}_R(M, \pi)$, there exists $\tilde{g} \in \mathrm{Hom}_R(M, B)$ such that $\pi\tilde{g} = \iota_{\alpha+1}f$. Since $\mathrm{Im}(\iota_{\alpha+1}f)$ is a subset of the module, which is isomorphic to $S_{\alpha+1}/S_\alpha$, we conclude that $\mathrm{Im}(\varphi g) \subseteq S_{\alpha+1}$, where φ is the canonical embedding of B into $N_{\alpha+1,F}$. And hence for $g := \varphi\tilde{g}$ we obtain $\pi_\alpha g = f$. This proves that M is weakly R-projective.

□

Remark that we can go far and weaken the weak R-projectivity to a layer projectivity. Let us briefly introduce this weak condition, however, we will not use it any more in this Thesis. If R is regular semiartinian ring and M is a module, the factor modules $M_{\alpha+1}/M_\alpha$ are the layers of the module M, where $(M_\alpha)_{\alpha\leq\tau}$ is the socle sequence of the module M and τ is a Loewy length of M.

Definition 2.21. Let R be a regular semiartinian ring and M be a module with Loewy length τ. Then M is *layer projective* if the α-th layer of M is a projective R/S_α module, for each $\alpha < \tau$.

The weak R-projectivity implies the layer projectivity. Also layer projectivity is inherited to submodules. For proofs of these properties we refer the reader to Trlifaj [2020] (Lemma 3.7).

3. The Independence of DBC

In this chapter we will prove the assertion that it is independent on ZFC whether DBC holds for regular, semiartinian, small rings with primitive factors artinian. The proof will be based on employing two different extensions of ZFC. The first one will satisfy the Weak Diamond Principle, which implies that DBC holds true for this particular class of rings, and the second will be one in which Shelah's Uniformization Principle will hold and therefore DBC fails.

3.1 Set-theoretic notions

The goal of this section is to introduce several concepts from set theory which are needed to understand the relevant combinatorial principles; Shelah's Uniformization Principle and some variants of Jensen's Diamond Principle.

We assume knowledge of ordinals, cardinals, cofinality and other basic notions from set theory, for which we refer to Eklof and Mekler [2002].

Definition 3.1. Let δ be a limit ordinal. A subset $A \subseteq \delta$ is said to be *closed in* δ if $\alpha = \sup(A \cap \alpha)$ implies $\alpha \in A$ for every limit ordinal $\alpha < \delta$.

Equivalently, this means that for every sequence $(a_i)_{i\in I}$ of elements of A such that $\sup_{i\in I} a_i < \delta$ it holds that $\sup_{i\in I} a_i$ belongs also to A. So the closeness is a topological property and the considered topology there is the one induced by $\in$ (an ordering of ordinal numbers).

Definition 3.2. Let δ be a limit ordinal. A subset $A \subseteq \delta$ is said to be *unbounded in* δ if $\sup A = \delta$.

It means that for every $\alpha \in \delta$ there exists some $\beta \in A$ such that $\alpha \leq \beta$. We can equivalently say that A is unbounded in δ if A is cofinal with δ.

Definition 3.3. Let δ be a limit ordinal. A subset $C \subseteq \delta$ is said to be *a club in* δ if C is both closed and unbounded.

Example 3.4. Neither of these properties can be taken for granted. Moreover, none of them implies the other. The set $N = \{0, 1, \dots, \omega\}$ is closed in ω_1 but not unbounded. On the other hand, the set $M = \{\alpha + 1; \alpha \in \omega_1\}$ is unbounded in ω_1 but not closed.

Images of normal functions (i.e., strictly increasing and continuous) are typical examples of clubs.

Remark 3.5. For a regular cardinal δ it is true that images of normal functions $f : \delta \to \delta$ coincide with clubs in δ.

Proposition 3.6. *Let δ be a limit ordinal. An intersection of less than* $\mathrm{cf}(\delta)$ *clubs in δ is also a club in δ.*

Proof. If $\mathrm{cf}(\delta) \leq \omega$ it is obvious. For δ such that $\mathrm{cf}(\delta) > \omega$ see Eklof and Mekler [2002] (Chapter II, Lemma 4.3).

□

Stationary sets are not-omitted in a space of clubs in the following sense.

Definition 3.7. Let δ be a limit ordinal. A subset $S \subseteq \delta$ is said to be *stationary in* δ if for every club C in δ it holds that $S \cap C \neq \emptyset$.

It is clear (by Proposition 3.6) that every club is stationary. The following example shows us that the reverse implication does not hold in general.

Example 3.8. The set $\{\alpha < \aleph_2;\ \mathrm{cf}(\alpha) = \omega\}$ is stationary in $\aleph_2$ but not a club, because it is not closed in $\aleph_2$.

Remark 3.9. Also the set $E_\rho = \{\alpha < \kappa;\ \mathrm{cf}(\alpha) = \rho\}$ (where $\rho < \kappa$ is an infinite regular cardinal) is stationary in κ and for $\kappa > \aleph_1$ it is not a club as it is shown in Eklof and Mekler [2002] (Chapter II, Example 4.7).

The observation below, about a partition of a stationary set, is a direct corollary of Proposition 3.6.

Corollary 3.10. *Let δ be a limit ordinal such that* $\mathrm{cf}(\delta) > \omega$. *Let $S = \bigcup_{n<\omega} S_n$ be a stationary set in δ. Then at least one of the sets S_n is stationary in δ.*

Proof. Let δ be a limit ordinal such that $\mathrm{cf}(\delta) > \omega$. Let $S = \bigcup_{n<\omega} S_n$ be a stationary set in δ. Assume for contradiction that none of the sets S_n is stationary. Then there exist clubs C_n witnessing non-stationarity of S_n, that is, $S_n \cap C_n = \emptyset$ for each $n < \omega$. By Proposition 3.6 the set $C = \bigcap_{n<\omega} C_n$ is a club in δ. Since $S_n \cap C = \emptyset$ for every $n \in \omega$, also $S \cap C = \emptyset$, which contradicts the stationarity of the set S.

□

Now we can introduce combinatorial principles, known as diamonds. They are also known as prediction principles, because they guarantee the existence of sequences which stationarily often predict all initial segments of an arbitrary set.

Definition 3.11. Let κ be a regular uncountable cardinal and $S \subseteq \kappa$ be a stationary subset of κ. The principle $\diamondsuit_\kappa(S)$ is the following claim:

There exists a sequence $(W_\alpha)_{\alpha\in S}$, where $W_\alpha \subseteq \alpha$, such that
the set $\{\alpha \in S;\ W_\alpha = X \cap \alpha\}$ is stationary in κ,
for every set $X \subseteq \kappa$.

Jensen's Diamond Principle, which we denote by $\diamondsuit$, is the claim: For every regular uncountable cardinal κ and every stationary subset $S \subseteq \kappa$ the principle $\diamondsuit_\kappa(S)$ holds.

Notation. The sequence of sets $(W_\alpha)_{\alpha\in S}$ from the definition above is usually called a $\diamondsuit_\kappa(S)$-sequence or just a diamond sequence.

Lemma 3.12. $\diamondsuit_{\lambda^+}(\lambda^+)$ *implies* $2^\lambda = \lambda^+$ *for any cardinal* λ.

Proof. Assume $\diamondsuit_{\lambda^+}(\lambda^+)$. Then there exists a sequence $(W_\alpha \subseteq \alpha)_{\alpha\in\lambda^+}$ such that the set $\{\alpha \in \lambda^+;\ W_\alpha = X \cap \alpha\}$ is stationary in λ^+ for every set $X \subseteq \lambda^+$. This allows us to define a strictly increasing function from the power set of λ to λ^+ by sending $X \subseteq \lambda < \lambda^+$ to some $\alpha < \lambda^+$ such that $W_\alpha = X \cap \alpha$ and which is big enough. So $2^\lambda \leq \lambda^+$ and hence $2^\lambda = \lambda^+$.

□

In particular, $\diamondsuit_{\omega_1}(\omega_1)$ implies the Continuum Hypothesis (CH). About the reverse implication, the best result is the theorem below, proved by Shelah [2010].

Theorem 3.13 (Shelah)**.** *For any infinite cardinal λ such that $2^\lambda = \lambda^+$ and any stationary set S such that $S \subseteq \{\delta < \lambda^+; \mathrm{cf}(\delta) \neq \mathrm{cf}(\lambda)\}$ the principle $\diamondsuit_{\lambda^+}(S)$ holds.*

Remark 3.14. Jensen showed that CH is consistent with a negation of Jensen's Diamond Principle for ω_1, hence for a stationary set S for which the condition from Theorem 3.13 is not satisfied, i.e., $S \subseteq \{\delta < \lambda^+; \mathrm{cf}(\delta) = \mathrm{cf}(\lambda)\}$, the Generalized Continuum Hypothesis (GCH) does not imply $\diamondsuit_{\lambda^+}(S)$.

Jensen [1972] established the principle $\diamondsuit$ and showed that it holds in the constructible universe (L). Therefore, this combinatorial principle is consistent with ZFC + GCH. The relative consistence of the Axiom of Constructibility, $V = L$, was proved by Gödel [1938].

For our needs, it is better to reformulate $\diamondsuit$ from a claim about the existence of a diamond sequence to a claim about the existence of the Jensen-functions. For this purpose some more definitions will be useful.

Definition 3.15. Let κ be a regular uncountable cardinal and A be a set. A sequence of subsets of A, $(A_\alpha)_{\alpha\leq\kappa}$, is said to be *a κ-filtration of the set A* if it has the following properties

1. (increasing chain) $A_\alpha \subseteq A_{\alpha+1}$, for every $\alpha < \kappa$,
2. (continuity) $A_\alpha = \bigcup_{\beta<\alpha} A_\beta$ for every limit ordinal $\alpha < \kappa$,
3. $A_\kappa = A$, and
4. $\mathrm{card}(A_\alpha) < \kappa$ for every $\alpha < \kappa$.

Definition 3.16. Let κ be a regular uncountable cardinal and M be an module. A sequence of submodules of M, $(M_\alpha)_{\alpha\leq\kappa}$, is said to be *a κ-filtration of the module M* if it has the following properties

1. (increasing chain) $M_\alpha \subseteq M_{\alpha+1}$, for every $\alpha < \kappa$,
2. (continuity) $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$ for every limit ordinal $\alpha < \kappa$,
3. $M_0 = 0$, $M_\kappa = M$, and
4. the minimal cardinality of a set of generators of a module M, $\mathrm{gen}(M_\alpha)$, is less than κ for every $\alpha < \kappa$.

Remark 3.17. A κ-filtration of a module M exists for every M which is not more than κ generated. It is because we obtain a κ-filtration of M from the set of generators $\{g_\beta; \beta \leq \kappa\}$ as $(\langle\{g_\beta; \beta \leq \gamma\}\rangle)_{\gamma\leq\kappa}$. Any filtration of a module M is also a filtration of the supporting set of the module M whenever $\mathrm{card}(M_\alpha) < \kappa$ for every $\alpha < \kappa$.

Definition 3.18. For a κ-filtration $\mathcal{M} = (\mathcal{M}_\alpha)_{\alpha\leq\kappa}$ of a module M and for a normal function $f : \kappa \to \kappa$ we define *a sub-filtration of $\mathcal{M}$ induced by f* as a chain $(M_{f(\alpha)})_{\alpha\leq\kappa}$.

Note that the sub-filtration of M induced by f is again a κ-filtration of a module M. Of course, we extend f on κ by $f(\kappa) := \kappa$. The following proposition shows that every two κ-filtrations of the same module have a common sub-filtration.

Proposition 3.19. *Let κ be a regular uncountable cardinal. Assume that a module M has two κ-filtrations $(A_\alpha)_{\alpha<\kappa}$ and $(B_\alpha)_{\alpha<\kappa}$. Then the set*

$$C = \{\alpha < \kappa; A_\alpha = B_\alpha\}$$

is a club in κ.

Proof. Firstly, we will show that C is closed. Let $(\alpha_i)_{i\in I}$ be a sequence of elements of C such that $\sup_{i\in I}(\alpha_i) < \kappa$. Denote by γ this $\sup_{i\in I}(\alpha_i)$. From the continuity of a filtration $A_\gamma = \bigcup_{i\in I} A_{\alpha_i} = \bigcup_{i\in I} B_{\alpha_i} = B_\gamma$, so γ belongs to the set C.

Secondly, we will see that C is also unbounded. Let $\alpha < \kappa$, we want to find some $\beta \in C$ such that $\beta > \alpha$. Consider some δ such that $\alpha < \delta < \kappa$. Then either $A_\delta = B_\delta$ and we finish with $\beta := \delta$ or $A_\delta \neq B_\delta$, w.l.o.g. $A_\delta \subsetneq B_\delta$, and we will find $\delta < \delta^1 < \kappa$ such that $B_\delta \subseteq A_{\delta^1}$.

We can do this iteratively and if we do not stop in any finite step we will obtain $A_{\sup_{i\in\omega}(\delta^i)} = \bigcup_{i\in\omega} A_{\delta^i} = \bigcup_{i\in\omega} B_{\delta^i} = B_{\sup_{i\in\omega}(\delta^i)}$, so we win with $\beta := \sup_{i\in\omega}(\delta^i)$ which belongs to C.

$\square$

Theorem 3.20. *Let κ be a regular uncountable cardinal and assume a stationary subset $S \subseteq \kappa$. The principle $\diamondsuit_\kappa(S)$ is equivalent to the following claim:*

For every two κ-filtrations $A = \bigcup_{\alpha<\kappa} A_\alpha$, $B = \bigcup_{\alpha<\kappa} B_\alpha$,
there exist functions $f_\alpha \in {}^{A_\alpha}B_\alpha$, for all $\alpha \in S$, such that
the set $\{\alpha \in S; f_\alpha = f\restriction_{A_\alpha}\}$ is stationary in κ,
for every function $f \in {}^A B$.

Notation. These functions $(f_\alpha)_{\alpha\in S}$, are called *Jensen-functions*.

Proof. Let κ be a regular uncountable cardinal and $S \subseteq \kappa$. Assume $\diamondsuit_\kappa(S)$, then there exists a sequence $(W_\alpha)_{\alpha\in S}$ such that the set $\{\alpha \in S; W_\alpha = X \cap \alpha\}$ is stationary in κ, for every set $X \subseteq \kappa$. Let us have two sets A and B and their κ-filtrations $(A_\alpha)_{\alpha<\kappa}$ and $(B_\alpha)_{\alpha<\kappa}$. Because $\mathrm{card}(A \times B) = \mathrm{card}(\kappa \times \kappa) = \kappa$ we can identify $A \times B$ with κ, and then any function $f \subseteq A \times B$, considered as a graph, can be used as a set $X \subseteq \kappa$.

Clearly $A \times B = \bigcup_{\alpha<\kappa} A_\alpha \times \bigcup_{\alpha<\kappa} B_\alpha$, so $A_\alpha \times B_\alpha$ forms a κ-filtration of $A \times B$. Since we can w.l.o.g. assume that $A \times B = \kappa$, the sequence $(\alpha)_{\alpha<\kappa}$ is also a κ-filtration of the set $A \times B$. So by Proposition 3.19, we obtain a set C, which is a club in κ, such that $A_\alpha \times B_\alpha = \alpha$, for every $\alpha \in C$.

Let us define f_α by

$$f_\alpha = \begin{cases} W_\alpha & \text{if } \alpha \in S \cap C \text{ and } W_\alpha \text{ is a function,} \\ \emptyset & \text{otherwise.} \end{cases}$$

Now for an arbitrary function $f \in {}^A B$ we have

$$\{\alpha \in S; f_\alpha = f\restriction_{A_\alpha}\} \supseteq \{\alpha \in S \cap D; f_\alpha = f\restriction_{A_\alpha}\} =$$
$$\{\alpha \in S \cap D; W_\alpha = f \cap (A_\alpha \times B_\alpha)\} \supseteq \{\alpha \in S \cap D \cap C; W_\alpha = f \cap (A_\alpha \times B_\alpha)\} =$$
$$\{\alpha \in S \cap D \cap C; W_\alpha = f \cap \alpha\} = \{\alpha \in S; W_\alpha = f \cap \alpha\} \cap D \cap C,$$

where D is a club set of ordinals α such that $f\restriction_{A_\alpha}$ has the range in B_α. So the set $\{\alpha \in S; f_\alpha = f\restriction_{A_\alpha}\}$ is stationary because it contains an intersection of stationary set and clubs, which is stationary.

$\square$

Definition 3.21. Let κ be a regular uncountable cardinal and $S \subseteq \kappa$ be a stationary subset in κ. The principle $\Phi_\kappa(S)$ is stated by the following claim:

For every two λ-filtrations $A = \bigcup_{\alpha<\lambda} A_\alpha$, $B = \bigcup_{\alpha<\lambda} B_\alpha$,
for any family of colourings $(c_\alpha : {}^{A_\alpha}B_\alpha \to 2)_{\alpha<\kappa}$
there exist a function $c : \kappa \to 2$ such that
the set $\{\alpha \in S; f\restriction_{A_\alpha} \in {}^{A_\alpha}B_\alpha,\ c(\alpha) = c_\alpha(f\restriction_{A_\alpha})\}$
is stationary in κ, for every function $f \in {}^{A}B$.

By Φ we denote the following claim: For every regular uncountable cardinal κ and every stationary subset $S \subseteq \kappa$ the principle $\Phi_\kappa(S)$ holds.

Notation. The stationary set from the principle above will be denoted by $S(f)$.

Proposition 3.22. $\diamondsuit$ *implies* Φ.

Proof. Assume $\diamondsuit$. Let κ be a regular uncountable cardinal. Let $S \subseteq \kappa$. From $\diamondsuit_\kappa(S)$ (due to Theorem 3.20) for every two κ-filtrations $A = \bigcup_{\alpha<\kappa} A_\alpha$, $B = \bigcup_{\alpha<\kappa} B_\alpha$, there is a collection of functions $(f_\alpha \in {}^{A_\alpha}B_\alpha)_{\alpha\in S}$ such that the set $\{\alpha \in S; f_\alpha = f\restriction_{A_\alpha}\}$ is stationary in κ for every function $f \in {}^{A}B$.

Let $c_\alpha : {}^{A_\alpha}B_\alpha \to 2$ be an arbitrary colouring, for each $\alpha < \kappa$. Let us define $c : \kappa \to 2$ by putting $c(\alpha) = c_\alpha(f_\alpha)$. Then for any $f \in {}^{A}B$, the set $\{\alpha \in S; f\restriction_{A_\alpha} \in {}^{A_\alpha}B_\alpha$ and $c(\alpha) = c_\alpha(f\restriction_{A_\alpha})\}$ is stationary in κ because it contains the stationary set $\{\alpha \in S; f_\alpha = f\restriction_{A_\alpha}\}$ as a subset.

$\square$

The fact that Φ is weaker than $\diamondsuit$ follows from the theorem that Φ does not imply CH. Surprisingly, $\Phi_{\omega_1}(\omega_1)$ corresponds to a claim about cardinal arithmetic.

Theorem 3.23. $\Phi_{\omega_1}(\omega_1)$ *is equivalent with* $2^{\omega_0} < 2^{\omega_1}$.

For the proof we refer to Devlin and Shelah [1978].

Definition 3.24. Let E be a set of limit ordinals and $\alpha \in E$. Then *a ladder on* α is a function $l_\alpha : \omega \to \alpha$ which is strictly increasing and its domain is cofinal in α. By *a ladder system on E* we mean a sequence $L = (l_\alpha)_{\alpha\in E}$, where l_α is a ladder on α.

Definition 3.25. Let $\delta \geq 2$ and E be as above. Assume $L = (l_\alpha)_{\alpha\in E}$, a ladder system on E. Define *a δ-colouring of a ladder system L* as a sequence of functions $(c_\alpha : \omega \to \delta)_{\alpha\in E}$.

We are thinking about these colourings as a system which uses δ colours and gives us a colour $c_\alpha(i)$ for the i-th rung of the α-th ladder, i.e., for $l_\alpha(i)$.

Definition 3.26. Shelah's Uniformization Principle (SUP) is the following statement:
For every uncountable ordinal λ such that $\mathrm{cf}(\lambda) = \omega$, there exists a set $S \subseteq \{\alpha < \lambda^+; \mathrm{cf}(\alpha) = \omega\}$ stationary in λ^+ on which there exists a ladder system $L = (l_\alpha)_{\alpha \in S}$ such that for every cardinal $\delta < \lambda$ and for every δ-colouring $(f_\alpha)_{\alpha \in S}$ of this system L there exists a global colouring of λ^+ by δ colours, say $f : \lambda^+ \to \delta$, such that for each $\alpha \in S$ it holds that $f(l_\alpha(i)) = f_\alpha(i)$ for all but finitely many $i \in \omega$.

It is known due to Eklof and Shelah [1991] that if ZFC is consistent, then ZFC+GCH+SUP is also consistent.

The principle SUP can be generalized to the claim of the existence $S \subseteq \{\alpha < \lambda^+; \mathrm{cf}(\alpha) = \mathrm{cf}(\lambda)\}$ for a fixed λ not necessary of cofinality ω. In this case, we consider generalized ladders and δ-colourings in the sense that domain of these functions are $\mathrm{cf}(\alpha)$ instead of ω. The SUP generalized in this way is also consistent with ZFC + GCH as it is stated in Eklof and Mekler [2002] (chapter XIII, Theorem 3.11).

Lemma 3.27. *Assume SUP. Then Φ fails.*

Proof. Assume SUP. Then for a singular cardinal λ such that $\mathrm{cf}(\lambda) = \omega$ there exists a set $S \subseteq \{\alpha < \lambda^+; \mathrm{cf}(\alpha) = \omega\}$ which is stationary in λ^+ and there exists a ladder system $L = (l_\alpha)_{\alpha \in S}$ on this set S. We will prove the negation of $\Phi_{\lambda^+}(S)$. Let $(A_\gamma = \gamma)_{\gamma \leq \lambda^+}$ be a λ^+-filtration of $A = \lambda^+$ and $(B_\gamma = 2)_{\gamma \leq \lambda^+}$ be a λ^+-filtration of $B = 2$. For each $\alpha \in S$, if $x \in {}^{A_\alpha}B_\alpha$, set

$$c_\alpha(x) = \begin{cases} 1 & \text{if } \{i \in \omega; x(l_\alpha(i)) = 0\} \text{ is infinite,} \\ 0 & \text{otherwise.} \end{cases}$$

Fix an arbitrary function $c : S \to 2$. Define a δ-colouring of the ladder system L, $(h_\alpha)_{\alpha \in S}$, by setting

$$h_\alpha(l_\alpha(i)) := c(\alpha)$$

for every $\alpha \in S$, for every $i \in \omega$. For this δ-colouring we obtain (by SUP) $h : \lambda^+ \to \lambda$ such that $\{i < \omega; h_\alpha(l_\alpha(i)) \neq h(l_\alpha(i))\}$ is finite. This h is a function from A to B, it remains to show that a set $\tilde{S} = \{\alpha \in S; c_\alpha(h\restriction_{A_\alpha}) = c(\alpha)\}$ is not stationary in λ^+ which proves that $\Phi_{\lambda^+}(S)$ fails and hence Φ does not hold.

Consider $\alpha \in \tilde{S}$, then if $h(l_\alpha(i)) = 0$ for all but finitely many $i \in \omega$, the set $\{i \in \omega; h\restriction_{A_\alpha}(l_\alpha(i)) = 0\}$ is infinite, so $c_\alpha(h\restriction_{A_\alpha}) = 1$. And because α belongs to $\tilde{S}$, we obtain $c(\alpha) = 1$ which from the definition of h_α means that $h_\alpha(l_\alpha(i)) = 1$ for each $i \in \omega$. Since $\{i < \omega; h_\alpha(l_\alpha(i)) \neq h(l_\alpha(i))\}$ is finite, $h(l_\alpha(i)) = 1$ for all but finitely many $i \in \omega$, which is a contradiction. The second case, $h(l_\alpha(i)) = 1$, contradicts the choice of h_α analogically.

□

3.2 The Weak diamond and a positive answer to Faith's Problem

The fact that DBC holds true for some big class of non-perfect rings under the assumption of the Weak Diamond Principle, is the main result of this Thesis. The proof presented there is a reformulation of Trlifaj [2022](Theorem 3.2) without assuming CH. This is possible as observed by Jan Šaroch and as mentioned in Trlifaj [2022](Remark 3.3). However, we require the stronger meaning of being small, which demand to have $\mathrm{card}(R) \leq \aleph_1$ instead of $\mathrm{card}(R) \leq 2^\omega$ as was originally stated in the mentioned article. Theorem 3.2 generalized the previous result (Trlifaj [2020], Theorem 4.4) by using just the Weak Diamond Principle instead of Jensen's Diamond Principle and allowing an arbitrary countable Loewy length of the ring.

Definition 3.28. A regular semiartinian ring R with primitive factors artinian is said to be *small* if $\mathrm{card}(R) \leq \aleph_1$, σ is a countable ordinal and the $S_{\alpha+1}/S_\alpha$ is countably generated, for every $0 < \alpha < \sigma$, where $(S_\alpha)_{\alpha\leq\sigma+1}$ is the socle sequence for R.

Remark 3.29. Any regular semiartinian ring which is small is hereditary. This follows from Lemma 2.10. The requirement that a ring is of bounded cardinality will allow us to use Φ for an induction step in Theorem 3.31.

Lemma 3.30. *Let R be a regular semiartinian ring with primitive factors artinian and $(S_\alpha)_{\alpha\leq\sigma+1}$ be its socle sequence. Then any countably generated module M which is R-projective is also projective.*

Proof. Let R be a semiartinian ring with primitive factors artinian and $(S_\alpha)_{\alpha\leq\sigma+1}$ its socle sequence. Let M be an R-projective module which is countably generated, so $M = \bigcup_{n<\omega} F_n$, where $(F_n)_{n<\omega}$ is a chain of finitely generated submodules of M.

By Proposition 2.18 M is weakly R-projective. Since weak R-projectivity is inherited to submodules (see Proposition 2.19), the set F_n is also weakly R-projective for each $n < \omega$.

Let $n < \omega$ be arbitrary. By induction on $\alpha \leq \sigma + 1$, we will show that F_n is S_α-projective. In particular for $\alpha = \sigma + 1$, we will obtain that F_n is R-projective.

For the base case, $\alpha = 0$, the S_α-projectivity of F_n is obvious: Because $S_0 = 0$, for every submodule K of the module $S_0 = 0$ (which is just 0 module) $\mathrm{Hom}_R(F_n, S_0/K) = \{0\}$ and the zero-homomorphism trivially factorizes trough $\pi_0 = 0$.

For the non-limit induction step consider a submodule $K \subseteq S_{\alpha+1}$. And let us have these canonical projections:

$$\begin{aligned}
&\pi_\alpha : S_{\alpha+1} \to S_{\alpha+1}/S_\alpha,\\
&\pi_{\alpha,K} : S_{\alpha+1} \to S_{\alpha+1}/K,\\
&\rho : S_{\alpha+1} \to S_{\alpha+1}/(S_\alpha + K),\\
&\eta : S_{\alpha+1}/K \to S_{\alpha+1}/(S_\alpha + K),\\
&\theta_\alpha : S_\alpha \to S_\alpha/(S_\alpha \cap K), \text{ and}
\end{aligned}$$

the isomorphism

$$\begin{gathered}
i_\alpha : S_\alpha/(S_\alpha \cap K) \to (S_\alpha + K)/K\\
s + S_\alpha \cap K \mapsto s + K.
\end{gathered}$$

Observe that $\mathrm{Soc}(S_{\alpha+1}/S_\alpha) = \mathrm{Soc}(\mathrm{Soc}(R/S_\alpha)) = \mathrm{Soc}(R/S_\alpha) = S_{\alpha+1}/S_\alpha$, so $S_{\alpha+1}/S_\alpha$ is a completely reducible module and therefrom the canonical projection $p : S_{\alpha+1}/S_\alpha \to S_{\alpha+1}/(S_\alpha + K)$ splits. The splitting property yields $j \in \mathrm{Hom}_R(S_{\alpha+1}/(S_\alpha+K), S_{\alpha+1}/S_\alpha)$ such that pj is an identity on $S_{\alpha+1}/(S_\alpha+K)$.

Let $f \in \mathrm{Hom}_R(F_n, S_{\alpha+1}/K)$, then from weak R-projectivity of F_n we obtain for a homomorphism $j\sigma f \in \mathrm{Hom}_R(F_n, S_{\alpha+1}/S_\alpha)$ the existence of a homomorphism $h \in \mathrm{Hom}_R(F_n, S_{\alpha+1})$ such that $\pi_\alpha h = j\sigma f$.

Applying p to both sides of the equation we conclude that $p\pi_\alpha h = \sigma f$. Since $p\pi_\alpha = \rho = \sigma\pi_{\alpha,K}$ (all mentioned morphisms are the canonical projections), we obtain $\sigma\pi_{\alpha,K}h = \sigma f$ which means that $\sigma(f - \pi_{\alpha,K}h) = 0$ and hence

$$\mathrm{Im}(f - \pi_k h) \subseteq \mathrm{Ker}(\sigma) = (S_\alpha + K)/K$$

so $i_\alpha^{-1}(f - \pi_{\alpha,K}h) \in \mathrm{Hom}_R(F_n, S_\alpha/(S_\alpha \cap K))$.

Since F_n is S_α-projective by the inductive premise, there exists some $\epsilon \in \mathrm{Hom}_R(F_n, S_\alpha)$ such that $\theta_\alpha\epsilon = i_\alpha^{-1}(f - \pi_{\alpha,K}h)$. Applying i_α we obtain $i_\alpha\theta_\alpha\epsilon = f - \pi_{\alpha,K}h$. And because $i_\alpha\theta_\alpha = \pi_{\alpha,K}\restriction_{S_\alpha+K}$, we conclude that $\pi_{\alpha,K}\epsilon = f - \pi_{\alpha,K}h$ and hence $f = \pi_{\alpha,K}(\epsilon + h)$, which shows that f factorizes through $\pi_{\alpha,K}$ and so F_n is $S_{\alpha+1}$-projective.

For the limit induction step consider a submodule $K \subseteq S_\alpha = \bigcup_{\beta<\alpha} S_\beta$. Let f be a homomorphism from $\mathrm{Hom}_R(F_n, S_\alpha/K)$. Because F_n is finitely generated, there exists some $\beta < \alpha$ such that $f \in \mathrm{Hom}_R(F_n, (S_\beta + K)/K)$.

By the inductive premise F_n is S_β-projective, so there exists some homomorphism $\epsilon \in \mathrm{Hom}_R(F_n, S_\beta)$ such that $\theta_\beta\epsilon = i_\beta^{-1}f$. Applying i_β we obtain $i_\beta\theta_\beta\epsilon = f$ and since $i_\beta\theta_\beta = \pi_{\beta,K}\restriction_{S_\beta}$, we conclude that $\pi_{\beta,K}\epsilon = f$, so F_n is S_α-projective.

As we said, for $\alpha = \sigma + 1$ this gives us R-projectivity of F_n and since this was proved for arbitrary $n < \omega$, we conclude that F_n is R-projective for each $n < \omega$. By Corollary 1.46 each F_n is projective because F_n is finitely generated for each $n < \omega$. Then regularity of the ring R yields sets C_n ($n < \omega$) such that $F_n \oplus C_n = F_{n+1}$ (Theorem 2.2). Finally, C_n is projective for each $n < \omega$, because projectivity is inherited to homomorphic images, and since a direct sum of projective modules is projective, $M = F_0 \bigoplus_{n<\omega} C_n$ is projective.

□

Theorem 3.31. *Assume Φ. Let R be a regular semiartinian ring with primitive factors artinian such that R is small. Then DBC holds true for R.*

Proof. Let $M \in Mod\text{-}R$ be R-projective. We will show that M is projective by induction on $\kappa = \mathrm{gen}(M)$. The base case for $\kappa \le \aleph_0$ is satisfied by Lemma 3.30. The induction step for a singular cardinal κ follows on from Shelah's Singular Compactness Theorem (see Eklof and Mekler [2002], Chapter IV, Theorem 3.7 see also chapter XII, Lemma 1.14) and from the Proposition 2.19. So it remains to prove the induction step for κ regular.

Let κ be a regular cardinal. The module M is R-projective, hence by Proposition 2.18 it is also weakly R-projective.

Since $\mathrm{gen}(M) = \kappa$, there exists a κ-filtration of the module M, say $(M_\gamma)_{\gamma\le\kappa}$, as it is described in Remark 3.17.

Assume that whenever M_β/M_γ is not weakly R-projective for some β such that $\gamma < \beta \le \kappa$, then $M_{\gamma+1}/M_\gamma$ is not weakly R-projective. If this property is not satisfied, then we choose some sub-filtration which satisfies it and replace the original one by this.

Let N be an injective module, $B = \Pi_{0<\alpha\leq\sigma, F\in\mathcal{F}_\alpha} N_{\alpha,F}$ a module such that $\operatorname{card}(N_{\alpha,F}) \leq \operatorname{card}(R)$, and $\pi_{\alpha,F} \in \operatorname{Hom}_R(N_{\alpha,F}, N)$ be an epimorphism for each $0 < \alpha \leq \sigma$ and $F \in \mathcal{F}_\alpha$, see Remark 2.13. Let us denote by $K_{\alpha,F}$ the kernel of $\pi_{\alpha,F}$ and by K the kernel of $\pi = \Pi_{0<\alpha\leq\sigma, F\in\mathcal{F}_\alpha} \pi_{\alpha,F}$.

Put $S := \{\gamma < \kappa; M_{\gamma+1}/M_\gamma$ is not weakly R-projective$\}$. We will show that this set S is not stationary.

Assume by contradiction that S is stationary. For each $\gamma \in S$ there exists $h_\gamma \in \operatorname{Hom}_R(M_{\gamma+1}/M_\gamma, N)$ such that it does not factorize through π (this exists because of Lemma 2.20). In particular, there are some $0 < \alpha \leq \sigma$ and $F \in \mathcal{F}_\alpha$ such that h_γ does not factorize through $\pi_{\alpha,F}$. Thus $S = \bigcup_{0<\alpha\leq\sigma, F\in\mathcal{F}_\alpha} S_{\alpha,F}$, where $S_{\alpha,F} = \{\gamma \in S; h_\gamma$ does not factorize through $\pi_{\alpha,F}\}$.

Notice that smallness of the ring R guarantees that σ is a countable ordinal and $S_{\alpha+1}/S_\alpha$ is countably generated for each $0 < \alpha \leq \sigma$ hence the Corollary 3.10 implies that at least one of these sets $S_{\alpha,F}$ is stationary in κ. Let us fix such α and F.

We will use $\Phi_\kappa(S_{\alpha,F})$ for sets A and $N_{\alpha,F}$ and some specific colourings c_γ, $\gamma < \kappa$. We already have a κ-filtration $(A_\gamma)_{\gamma\leq\kappa}$ for the set A. For the set $N_{\alpha,F}$ a κ-filtration exists because R is small. Let us fix $(B_\gamma)_{\gamma\leq\kappa}$ as an arbitrary κ-filtration of the set $N_{\alpha,F}$.

To define the colourings c_γ we will first fix two types of extensions of homomorphisms. From the injectivity of the module N, there exists some extension in $\operatorname{Hom}_R(M_{\gamma+1}, N)$ for every $g \in \operatorname{Hom}_R(M_\gamma, N)$, we fix some such extension and denote it by g^e. There exists also some extension in $\operatorname{Hom}_R(M_{\gamma+1}, B)$ for $f \in \operatorname{Hom}_R(M_\gamma, B)$, but we need to use the inductive premise for its existence, which says that $M_{\gamma+1}$ is projective for every $\gamma < \kappa$. To obtain the required extension we will use the projectivity of $M_{\gamma+1}$ with respect to a homomorphism $f\pi$ as it is illustrated in a right-hand diagram below.

$$\begin{tikzcd} M_\gamma \arrow[d, "f"'] \arrow[rr, "f\pi"] \arrow[drr, phantom, ""] & & M_{\gamma+1} \arrow[d, dotted, "(f\pi)^e"] \\ B \arrow[rr, "\pi"] & & N \end{tikzcd}
\qquad
\begin{tikzcd} M_\gamma \arrow[d, "f"'] & & M_{\gamma+1} \arrow[dll, dotted, "f^+"'] \arrow[d, "(f\pi)^e"] \\ B \arrow[rr, "\pi"] & & N \end{tikzcd}$$

Now we can define $c_\gamma : {}^{A_\gamma} B_\gamma \to 2$. Let $x \in {}^{A_\gamma} B_\gamma$, then there exists a unique extension $\overline{x} \in \operatorname{Hom}_R(M_\gamma, B)$. For this $\overline{x}$ we have $\overline{x}^+$. However, the extension ^{e} was fixed, so $(\pi\overline{x})^e \restriction_{M_\gamma} = \pi\overline{x}$. Therefore,

$$\pi(\overline{x}^+ \restriction_{M_\gamma} - \overline{x}) = \pi\overline{x}^+ \restriction_{M_\gamma} - \pi\overline{x} = (\pi\overline{x}^e) \restriction_{M_\gamma} - \pi\overline{x} = \pi\overline{x} - \pi\overline{x} = 0.$$

Than,

$$\delta_{\overline{x}} := \overline{x}^+ \restriction_{M_\gamma} - \overline{x} \in \operatorname{Hom}_R(M_\gamma, K).$$

Since $\operatorname{Im}(\overline{x}) \subseteq N_{\alpha,F}$, then $\delta_{\overline{x}}$ also belongs to $\operatorname{Hom}_R(M_\gamma, K_{\alpha,F})$. Put $c_\gamma(x) = 1$ iff $\delta_{\overline{x}}$ can be extended to a homomorphism in $\operatorname{Hom}_R(M_{\gamma+1}, K_{\alpha,F})$.

For this choice of the colourings c_γ the principle $\Phi_\kappa(S_{\alpha,F})$ gives us a function $c \in {}^{S_{\alpha,F}}2$ such that $S_{\alpha,F}(y) = \{\gamma \in S_{\alpha,F}; y\restriction_{A_\gamma},\ c(\gamma) = c_\gamma(y\restriction_{A_\gamma})\}$ is stationary in κ for every $y \in {}^A N_{\alpha,F}$.

This allows us to construct $g \in \mathrm{Hom}_R(M,N)$ such that $g \neq \pi_{\alpha,F} f$ for every $f \in \mathrm{Hom}_R(M, N_{\alpha,F})$ which contradicts our assumption that M is weakly R-projective. Therefore, every $g \in \mathrm{Hom}_R(M,N)$ factorizes through the projection $\pi_{\alpha,F}$.

We will construct such g inductively by defining $g\restriction_{M_\gamma}$ as $g_\gamma \in \mathrm{Hom}_R(M_\gamma, N)$ for all $\gamma < \kappa$. For the base case put $g_0 := 0$. In a non-limit $(\gamma+1)$th step, when g_γ is defined, we distinguish two cases:

1. if $\gamma \notin S_{\alpha,F}$ or $c(\gamma) = 0$, then $g_{\gamma+1} := (g_\gamma)^e$
2. if $\gamma \in S_{\alpha,F}$ and $c(\gamma) = 1$, then $g_{\gamma+1} := (g_\gamma)^e + h_\gamma \rho_\gamma$,

   where ρ_γ is the canonical projection from $M_{\gamma+1}$ to $M_{\gamma+1}/M_\gamma$.

We see that $g_{\gamma+1}\restriction_{M_\gamma} = g_\gamma$, so it is possible to put $g_\gamma := \bigcup_{\beta<\gamma} g_\beta$ in the limit γth step when g_β is defined for each $\beta < \gamma$ and finally put $g := \bigcup_{\gamma<\kappa} g_\gamma$.

We claim that $g \neq \pi_{\alpha,F} f$ for each $f \in \mathrm{Hom}_R(M, N_{\alpha,F})$. Assume there exists such f and use f_γ as an abbreviation for $f\restriction_{M_\gamma}$. For $\gamma \in S_{\alpha,F}(f \restriction A)$ assume $c(\gamma) = 0$. Then we are in Case 1. of construction and $g_{\gamma+1} = (g_\gamma)^e$. Thus, we obtain

$$\pi_{\alpha,F} f_{\gamma+1} = g_{\gamma+1} = (g_\gamma)^e = (\pi f_\gamma)^e = (\pi_{\alpha,F} f_\gamma)^e = \pi_{\alpha,F}(f_\gamma)^+,$$

where the first and the third equality follows from $g = \pi_{\alpha,F} f$ and the last one holds by the definition of $^+$. Hence $\pi_{\alpha,F}((f_\gamma)^+ - f_{\gamma+1}) = 0$, so $(f_\gamma)^+ - f_{\gamma+1}$ belongs to $\mathrm{Hom}_R(M_{\gamma+1}, K_{\alpha,F})$ and clearly extends $((f_\gamma)^+)\restriction M_\gamma - f_\gamma$.

In the case of $c_\gamma(f \restriction A_\gamma) = 0$ which means by definition of c_γ that

$$\delta_{\overline{f\restriction A_\gamma}} = \delta_{f_\gamma} = ((f_\gamma)^+)\restriction M_\gamma - f_\gamma$$

can not be extended to $M_{\gamma+1}$, it is not possible that $c(\gamma) = 0$. Reminding that $\gamma \in S_{\alpha,F}(f \restriction A)$, we conclude that $c(\gamma) = 1$.

However, if $c(\gamma) = 1$, then we are the Case 2. of construction and $g_{\gamma+1} = (g_\gamma)^e + h_\gamma\rho_\gamma$. In analogy to the previous case, $\pi_{\alpha,F} f_{\gamma+1} = \pi_{\alpha,F}(f_\gamma)^+ + h_\gamma\rho_\gamma$. Also from the definition of the colourings, $\delta_{\overline{f\restriction A_\gamma}} = \delta_{f_\gamma} = ((f_\gamma)^+)\restriction M_\gamma - f_\gamma$ can be extended to $M_{\gamma+1}$. Let us denote some such extension by $\Delta_{f_\gamma} \in \mathrm{Hom}_R(M_{\gamma+1}, K_{\alpha,F})$.

Note that $\pi_{\alpha,F}\Delta_{f_\gamma} = 0$, so we can see that

$$h_\gamma\rho_\gamma = \pi_{\alpha,F} f_{\gamma+1} - \pi_{\alpha,F}(f_\gamma)^+ = \pi_{\alpha,F}(f_{\gamma+1} - (f_\gamma)^+ + \Delta_{f_\gamma}).$$

To simplify our notation let $\square := f_{\gamma+1} - (f_\gamma)^+ + \Delta_{f_\gamma}$.

Now it is easy to see that

$$\begin{aligned}&\square\restriction_{M_\gamma} = f_{\gamma+1}\restriction_{M_\gamma} - ((f_\gamma)^+)\restriction_{M_\gamma} + \Delta_{f_\gamma}\restriction_{M_\gamma} = f_\gamma - ((f_\gamma)^+)\restriction_{M_\gamma} + \delta_{f_\gamma} = \\ &f_\gamma - ((f_\gamma)^+)\restriction_{M_\gamma} + ((f_\gamma)^+)\restriction M_\gamma - f_\gamma = 0.\end{aligned}$$

Hence there exists x_γ such that the first diagram below commutes.

$$\begin{tikzcd} & M_{\gamma+1} \arrow[r, "\rho_\gamma"] \arrow[dl, "\square"'] & M_{\gamma+1}/M_\gamma \arrow[dll, "x_\gamma"] \\ N_{\alpha,F} & & \end{tikzcd} \qquad \begin{tikzcd} & M_{\gamma+1} \arrow[r, "\rho_\gamma"] \arrow[dl, "\square"'] & M_{\gamma+1}/M_\gamma \arrow[dl, "h_\gamma"] \\ N_{\alpha,F} \arrow[r, "\pi_{\alpha,F}"] & N & \end{tikzcd}$$

Since we know that the second diagram commutes, we obtain that

$$\pi_{\alpha,F} x_\gamma \rho_\gamma = h_\gamma \rho_\gamma$$

and because ρ_γ is a surjection, we conclude that $\pi_{\alpha,F} x_\gamma = h_\gamma$. This will contradicts the non-factorizability of the homomorphism h_γ.

Then $S_{\alpha,F}$ and so S is not a stationary set. It means that there exists some club C in κ such that $C \cap S = \emptyset$. This C is the image of some normal function on κ which chooses from our filtration $(M_\gamma)_{\gamma \leq \kappa}$ a sub-filtration $(N_\gamma)_{\gamma \leq \kappa}$ whose factors $N_{\gamma+1}/N_\gamma$ are weakly R-projective, for every $\gamma < \kappa$, and hence by the inductive premise they are projective. We obtain that $N_{\gamma+1} = N_\gamma \oplus P_\gamma$ for some projective module P_γ and hence $M = N_0 \oplus \bigoplus_{\gamma<\kappa} P_\gamma$ is projective. This proves that DBC holds for each regular semiartinian small ring R with primitive factors artinian.

□

As it was mentioned in the previous section, there exists another combinatorial principle, namely SUP, which is inconsistent with Φ. It is not surprising that this principle gives an opposite result than Φ in Theorem 3.31.

Lemma 3.32. *Assume SUP. Let R be a non-right perfect ring. Then DBC fails for R.*

Proof. The proof in Alhilali et al. [2017](Lemma 2.4) gives the existence of a module M, which satisfies $\mathrm{Ext}^1_R(M, I) = 0$ for every right ideal I. Therefore M is R-projective but M has the projective dimension equal to 1, so it is not projective.

□

This lemma says that it is consistent with ZFC that DBC fails for every non-right perfect rings, in particular that it fails for a class of regular, semiartinian, small rings with primitive factors artinian. Together with Theorem 3.31 it gives the following result.

Corollary 3.33. *Let R be regular, semiartinian, small ring with primitive factors artinian whose Loewy length is non-trivial. Then DBC is independent of ZFC.*

Conclusion

In this Thesis we presented the class of rings for which Trlifaj [2020] proved that DBC is independent on ZFC+GCH. We also weakened the set-theoretic assumptions of this result using the Weak Diamond Principle instead of Jensen's functions as it is written in Trlifaj [2022]. We presented this proof without an assumption of CH as suggested by Jan Šaroch.

At the beginning, we pointed out that since projectivity is a local property, it is reasonable to view it in more general settings than simply as property of modules. Especially, projectivity in an exact category with enough projectives is worth more detailed study.

In the second chapter the theorem about structure of regular semiartinian rings with primitive factors artinian was presented. This theorem gives us an invariant called *the dimension sequence of a ring*,

$$\mathrm{D} = \{(\lambda_\alpha, \{(n_{\alpha,\beta}, K_{\alpha,\beta}); \beta < \lambda_\alpha\}); \alpha \leq \sigma\},$$

where $\sigma + 1$ is the Loewy length of such a ring, λ_α is a cardinal for each $\alpha \leq \sigma$, $n_{\alpha,\beta}$ is a positive integer for each $\alpha \leq \sigma$ and each $\beta < \lambda_\alpha$, and $K_{\alpha,\beta}$ is a skew field for each $\alpha \leq \sigma$ and each $\beta < \lambda_\alpha$, see Lemma 2.12. Although many constructions of semiartinian rings are known, the exact range of valuse of the dimension sequences of regular semiartinian rings with primitive factors artinian is still an open question.

The main part of this thesis, the independence of DBC for small regular semiartinian rings (of non-trivial Loewy length) with primitive factors artinian and its proof detailed in the last chapter is far from a complete answer to Faith's Problem. It remains open for non-perfect rings in the form of the following question: "Where is the exact border line within the class of all non-perfect rings, between those non-perfect rings for which DBC fails in ZFC, and those, for which it is independent of ZFC?"

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