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Properties and interpretation of black hole spacetimes

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In Prague, April 5, 2022 Eliška Polášková

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Title: Properties and interpretation of black hole spacetimes

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Abstract: In this thesis, we study a limit of the Kerr–(A)dS spacetime in a general dimension where an arbitrary number of its rotational parameters is set equal. The resulting metric after the limit formally splits into two parts: the first part has the form of the Kerr–NUT–(A)dS metric analogous to the metric of the entire spacetime, but only for the directions not subjected to the limit, and the second part can be interpreted as the Kähler metrics. However, this separation is only valid for tangent spaces and it is not integrable, thus it does not lead to independent manifolds. We also reconstruct the original number of explicit and hidden symmetries associated with Killing vectors and Killing tensors. Therefore, the resulting spacetime represents a special case of the generalized Kerr–NUT–(A)dS metric studied before that also retains the full Killing tower of symmetries. In $D = 6$, we present evidence of an enhanced symmetry structure after the limit. Namely, we find additional Killing vectors and show that one of the Killing tensors becomes reducible as it can be decomposed into Killing vectors.

Keywords: higher-dimensional black holes, Kerr–NUT–(A)dS metric, equal-spin limit, explicit and hidden symmetries

Contents

Introduction

Four-dimensional black holes have been widely studied for more than a hundred years. Nowadays, they are particularly interesting from the astrophysical point of view as several breakthrough observations were made in recent years. These include the first detection of gravitational waves from a binary black hole merger [1], the first image of the supermassive black hole candidate in the center of the galaxy M87, made by the Event Horizon Telescope [2, 3], or the explanation of the star motion near the black hole in the centre of our galaxy [4, 5]. The standard model of the black hole used in such astrophysical situations is the Kerr solution of the Einstein equations in four-dimensional general relativity.

From the mathematical point of view, Kerr black holes [6] are included in a large family of solutions known as the Plebański–Demiański metric [7]. This metric represents spacetimes of algebraic type D that solve the vacuum Einstein equations with the cosmological constant, and it is characterized by seven arbitrary parameters, which can be interpreted as the cosmological constant, mass, NUT parameter, rotation, acceleration, electric and magnetic charge. It contains many well-known spacetimes as special cases: apart from the Kerr metric, which describes an axially symmetric rotating black hole, it also includes for example the Taub–NUT (Newman–Unti–Tamburino) solution [8, 9] with one NUT parameter as well as accelerating black holes represented by the C-metric [10].

Unlike the Kerr spacetime, the Taub–NUT metric does not have a clear physical interpretation — the presence of a NUT parameter in a four-dimensional spacetime leads to pathologies such as the existence of closed timelike curves [11]. However, some of these pathologies can be regarded as an unphysical feature of the idealized inner solution, which disappears when a realistic matter source for the outer solution is introduced.

In this work, we study generalization of these black holes to higher dimensions. The motivation for studying higher-dimensional metrics in general is their connection with string theory, the AdS/CFT correspondence and brane-world models. Moreover, the perspective of a general dimension may deepen the understanding of studied solutions. Last but not least, they are interesting from the mathematical point of view. An extensive review of higher-dimensional black hole solutions can be found in [12].

One of the interesting higher-dimensional solutions, which generalizes the black hole solutions known in four dimensions, is called the *Kerr–NUT–(A)dS metric* [13]. It is characterized by the cosmological constant, mass, rotational and NUT parameters, however, it does not include acceleration and electric/magnetic charge. Therefore, a generalization of the Plebański–Demiański metric to higher dimensions is yet to be discovered. The Kerr–NUT–(A)dS metric can describe various geometries of both the Euclidean and the Lorentzian signature, such as maximally symmetric spaces, so-called Euclidean instantons and black holes. It also includes well-known higher-dimensional solutions as special cases, for example the Myers–Perry black hole [14] (generalization of the Kerr black hole), the Kerr–(A)dS metric [15, 16] (generally rotating black hole in an asymptotically (anti)-de Sitter spacetime) and the higher-dimensional Taub–NUT–(A)dS metric [17, 18].

Higher-dimensional rotating black holes display many similar properties to their four-dimensional counterparts. This is caused by the fact that both types of spacetimes admit a special geometrical object, which we refer to as the *principal tensor* [19–22]. It is defined as a non-degenerate closed conformal Killing–Yano tensor. The very existence of the principal tensor significantly restricts the geometry — the most general geometry consistent with the existence of this tensor is the off-shell Kerr–NUT–(A)dS geometry. Here, the attribute "off-shell" refers to a general form of the metric that does not require the vacuum Einstein equations. The principal tensor generates a rich symmetry structure called the Killing tower [23, 24], which includes Killing vectors and Killing tensors associated with explicit and hidden symmetries of the spacetime. Moreover, it uniquely determines canonical coordinates in which the Hamilton–Jacobi [25] and the Klein–Gordon equations [26–28] as well as the Dirac [29–31] and the Maxwell equations [32–35] are fully separable, and therefore the geodesic motion is completely integrable [23, 36, 37]. Separability has been demonstrated also for higher-form fields [38]. As one can see, the principal tensor indeed plays a very important role in higher-dimensional black hole physics. For an extensive review of the role of the principal tensor and other properties of the Kerr–NUT–(A)dS geometry, see [39].

Apart from the Kerr–NUT–(A)dS spacetime and its properties, several limit cases of the general metric were also studied, such as the near-horizon limits [40–43]. Furthermore, the limit where some of the black hole's rotations are switched off was investigated [44]. Such a limit leads to warped spaces deformed and twisted by the NUT parameters, which thus do not maintain their unphysical properties when present in a space with the Euclidean signature. Another limit case where particular roots of the metric functions degenerate was studied [45], which results in geometries such as the Taub–NUT–(A)dS metric and the extreme near-horizon geometry. Therefore, these papers have demonstrated that not only can performing various limits of the general metric shed light on the role of NUT charges, but it can also lead to new interesting geometries. Moreover, the resulting spacetimes are expected to possess an enhanced symmetry structure after the limit, which is manifested in the presence of additional Killing vectors and also in the reducibility of Killing tensors that can be decomposed into Killing vectors. Reducibility properties of Killing tensors were also studied in four dimensions for near-horizon geometries [46, 47].

However, performing a limiting procedure is not in general a trivial task since certain regions of the spacetime can shrink or expand during the limit and become degenerate. Therefore, it is usually necessary to accompany the limiting procedure by a suitable rescaling of coordinates and parameters.

This thesis is focused on a particular limit case of the general Kerr–NUT–(A)dS metric that has not been thoroughly investigated yet. Namely, we study the *equal-spin limit*, that is a limit where an arbitrary number of rotational parameters of the spacetime coincides. As was mentioned above, the limiting procedure also includes an appropriate parametrization and rescaling of the parameters and the coordinates that cease to be well-defined after the limit.

The main results presented here were published in [48]. This work contains additional details regarding the derivation of the results and we also discuss the context of the limiting procedure more thoroughly.

The thesis is organized as follows. In Chapter 1, which is an overview of already known results, we introduce the Kerr–NUT–(A)dS spacetime and summarize its properties. The next two chapters are dedicated to a general equal-spin limit. Namely, in Chapter 2 we introduce the parametrization of the limit and apply it to the metric, while Chapter 3 discusses the limit form of the principal tensor, Killing vectors and Killing tensors. Chapter 4 presents explicit examples of the general results obtained in Chapters 2 and 3 — it focuses on black holes with all the rotational parameters set equal. Additional technical results and detailed calculations are provided in the appendices. Appendix A summarizes definitions and useful identities concerning auxiliary functions that appear in the metric before and after performing the limit. Appendix B gives proofs of selected results in the main text, which were obtained after employing the limiting procedure.

Notation

Let us summarize the conventional notation we will be using throughout this thesis. Notation specific to the Kerr–NUT–(A)dS spacetime and related topics is explained in the text when it is used for the first time.

Latin letters from the beginning of the alphabet represent general spacetime indices. In $D = 2N$ dimensions, they go over the ranges

$$a, b, \ldots = 1, \ldots, 2N\,.$$

We do employ the Einstein summation convention for them. Indices inside round brackets are symmetrized and square brackets denote antisymmetrization.

A dot $\cdot$ denotes a contraction of two tensors in adjacent indices. For example, if $\boldsymbol{X}$ is a vector and $\boldsymbol{h}$ is a 2-form, then $\boldsymbol{X} \cdot \boldsymbol{h}$ represents a 1-form with the components $X^b h_{ba}$.

The spacetime metric $\boldsymbol{g}$ is used to raise and lower indices. We do not usually indicate explicitly whether a tensor is covariant or contravariant, however, we do our best to make it clear from the context.

The covariant derivative is denoted by ∇. It is torsion-free and compatible with the spacetime metric $\boldsymbol{g}$, i.e. it satisfies

$$\nabla \boldsymbol{g} = 0\,.$$

1. Kerr–NUT–(A)dS spacetime

This chapter introduces the general Kerr–NUT–(A)dS spacetime in higher dimensions and describes some of its interesting properties. Namely, we introduce the metric in its general form and two types of coordinates it can be written in. We also discuss parameters of the metric and their interpretation. Subsequently, we describe geometries of both Euclidean and Lorentzian signatures that can be obtained from the general metric as special cases, and in each case we discuss suitable ranges of the coordinates and the parameters. Finally, the chapter is concluded with the analysis of explicit and hidden symmetries associated with Killing vectors and Killing tensors, and we show how these objects can be generated from the principal tensor.

The purpose of this chapter is to overview already known results regarding the Kerr–NUT–(A)dS spacetime, and it is based mainly on the review [39]. Our original findings are discussed in Chapters 2—4, see also [48].

For simplicity, we restrict ourselves to even dimensions $D = 2N$. However, the generalization to odd dimensions is straightforward — a corresponding term is added to the metric and other related quantities (see [39]). Otherwise, the analysis remains the same for both cases.

1.1 Metric

1.1.1 Canonical form of the metric

A metric describing the Kerr–NUT–(A)dS geometry can be written in the form [13]

$$\boldsymbol{g} = \sum_{\mu} \left[\frac{U_\mu}{X_\mu} \mathbf{d}x_\mu^2 + \frac{X_\mu}{U_\mu} \left(\sum_k A_\mu^{(k)} \mathbf{d}\psi_k \right)^2 \right] , \tag{1.1}$$

with Greek and Latin indices from the middle of the alphabet going over slightly different ranges

$$\begin{aligned} \mu, \nu, \ldots &= 1, \ldots, N \, , \\ k, l, \ldots &= 0, \ldots, N-1 \, . \end{aligned}$$

The Einstein summation convention is not used for these indices. Also, we do not indicate their ranges explicitly in sums or products, unless they differ from the default above. Greek indices label independent 2-planes, into which the metric (1.1) can be naturally split. Latin indices, on the other hand, indicate the powers of x_μ^2 that are present in metric functions.

The functions U_μ and $A_\mu^{(k)}$ that appear in the metric are defined as polynomials in the coordinates x_μ

$$U_\mu = \prod_{\substack{\nu \\ \nu \neq \mu}} \left(x_\nu^2 - x_\mu^2 \right) , \qquad A_\mu^{(k)} = \sum_{\substack{\nu_1, \ldots, \nu_k \\ \nu_1 < \cdots < \nu_k \\ \nu_i \neq \mu}} x_{\nu_1}^2 \ldots x_{\nu_k}^2 \, . \tag{1.2}$$

Each function $X_\mu = X_\mu(x_\mu)$ is dependent on a single coordinate x_μ. If these functions are left unspecified, the metric is then referred to as *off-shell*. On the other hand, if the functions X_μ are also defined as polynomials, namely

$$X_\mu = \lambda \mathcal{J}(x_\mu^2) - 2b_\mu x_\mu \,, \tag{1.3}$$

where $\mathcal{J}(x_\mu^2)$ reads

$$\mathcal{J}(x_\mu^2) = \prod_\nu \left(a_\nu^2 - x_\mu^2\right) \,, \tag{1.4}$$

then the metric (1.1) satisfies the Einstein equations in vacuum and we refer to it as *on-shell*. The complete list of metric functions as well as important relations between them can be found in Appendix A.1.

The canonical coordinates we used are divided into two sets. In the black hole case, which is discussed in detail in Section 1.3, the coordinates x_μ represent radius and latitudinal angles whereas the coordinates ψ_k represent time and longitudinal angles. Moreover, since the metric functions are independent of ψ_k, they are also the *Killing coordinates*.

Both types of canonical coordinates are uniquely determined by the principal tensor $\boldsymbol{h}$, see Section 1.4.1. Namely, x_μ are its eigenvalues and ψ_k are associated with Killing vectors generated by the principal tensor. Moreover, since the metric in these coordinates is rather simple, they are suitable for constructing the Killing tower and studying explicit and hidden symmetries as is shown later, in Section 1.4.

The inverse metric reads

$$\boldsymbol{g}^{-1} = \sum_\mu \left[\frac{X_\mu}{U_\mu} \left(\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x_\mu} \right)^2 + \frac{U_\mu}{X_\mu} \left(\sum_k \frac{(-x_\mu^2)^{N-1-k}}{U_\mu} \frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\psi_k} \right)^2 \right] \,,$$

which can be proved using the identities (A.7) and (A.8).

1.1.2 Alternative form of the metric

Instead of ψ_k, we can also use another set of angular coordinates ϕ_μ defined as

$$\phi_\mu = \lambda a_\mu \sum_k \mathcal{A}_\mu^{(k)} \psi_k \,, \qquad \psi_k = \sum_\mu \frac{(-a_\mu^2)^{N-1-k}}{\lambda a_\mu \mathcal{U}_\mu} \phi_\mu \,, \tag{1.5}$$

where the functions $\mathcal{U}_\mu$ and $\mathcal{A}_\mu^{(k)}$ are defined similarly to U_μ and $A_\mu^{(k)}$ in Eq. (1.2), only x_μ are replaced with a_μ (see also Eqs. (A.4) and (A.6))

$$\mathcal{U}_\mu = \prod_{\substack{\nu \\ \nu\neq\mu}} \left(a_\nu^2 - a_\mu^2\right) \,, \qquad \mathcal{A}_\mu^{(k)} = \sum_{\substack{\nu_1,\dots,\nu_k \\ \nu_1<\dots<\nu_k \\ \nu_i\neq\mu}} a_{\nu_1}^2 \dots a_{\nu_k}^2 \,.$$

The metric (1.1) then obtains the form

$$\boldsymbol{g} = \sum_\mu \left[\frac{U_\mu}{X_\mu} \mathbf{d}x_\mu^2 + \frac{X_\mu}{U_\mu} \left(\sum_\nu \frac{J_\mu(a_\nu^2)}{\lambda a_\nu \mathcal{U}_\nu} \mathbf{d}\phi_\nu \right)^2 \right] \,, \tag{1.6}$$

where $J_\mu(a_\nu^2)$ read (see also (A.3))

$$J_\mu(a_\nu^2) = \prod_{\substack{\kappa \\ \kappa\neq\mu}} \left(x_\kappa^2 - a_\nu^2\right) \,.$$

Since ϕ_μ are simply linear combinations of ψ_k with constant coefficients, they are the Killing coordinates as well. Moreover, their corresponding Killing vectors have fixed points and thus define the axes of rotational symmetry [49]. Therefore, the coordinates ϕ_μ are better suited for the physical interpretation of the metric.

1.1.3 Parameters

The on-shell metric is described by the parameters a_μ, b_μ and λ, but only the very last one can be interpreted straightforwardly — it is related to the cosmological constant Λ as

$$\Lambda = (2N-1)(N-1)\lambda\,.$$

Regarding the others, in general we can say that the parameters a_μ somehow describe rotations and the parameters b_μ encode mass and NUT charges. However, their interpretation strongly depends on specific other choices that can be made. This will be demonstrated in Sections 1.2 and 1.3 where various geometries that can be obtained from the general metric are discussed.

It is also worth mentioning that a_μ and b_μ are not independent as there exists a one-parametric gauge freedom in rescaling the coordinates, the parameters and the metric functions, which preserves the metric. The transformation reads

$$\begin{aligned} x_\mu &\to s x_\mu\,, & \psi_k &\to s^{-(2k+1)}\psi_k\,, & \phi_\mu &\to \phi_\mu\,,\\ a_\mu &\to s a_\mu\,, & b_\mu &\to s^{2N-1} b_\mu\,, & \lambda &\to \lambda\,,\\ X_\mu &\to s^{2N} X_\mu\,, & U_\mu &\to s^{2(N-1)} U_\mu\,, & A^{(k)}_\mu &\to s^{2k} A^{(k)}_\mu \end{aligned} \tag{1.7}$$

with s being the scaling parameter. This transformation enables us to set one of the parameters a_μ to a suitable value, which will be used to our advantage in the black hole case by imposing the condition (1.20).

Therefore, for a fixed cosmological constant the on-shell Kerr–NUT–(A)dS metric in $D = 2N$ dimensions contains $2N-1$ independent parameters. For a black hole, they represent mass, $N-1$ rotations and $N-1$ NUT charges.

1.1.4 Orthogonal frames

It is useful to introduce the following orthogonal frames of 1-forms

$$\begin{aligned} \boldsymbol{e}^\mu &= \left(\frac{U_\mu}{X_\mu}\right)^{\frac{1}{2}} \boldsymbol{\epsilon}^\mu = \left(\frac{U_\mu}{X_\mu}\right)^{\frac{1}{2}} \mathbf{d}x_\mu\,,\\ \hat{\boldsymbol{e}}^\mu &= \left(\frac{X_\mu}{U_\mu}\right)^{\frac{1}{2}} \hat{\boldsymbol{\epsilon}}^\mu = \left(\frac{X_\mu}{U_\mu}\right)^{\frac{1}{2}} \sum_k A^{(k)}_\mu \mathbf{d}\psi_k\\ &= \left(\frac{X_\mu}{U_\mu}\right)^{\frac{1}{2}} \sum_\nu \frac{J_\mu(a_\nu^2)}{\lambda a_\nu \mathcal{U}_\nu} \mathbf{d}\phi_\nu\,, \end{aligned} \tag{1.8}$$

where $\{\boldsymbol{e}^\mu, \hat{\boldsymbol{e}}^\mu\}$ is normalized and $\{\boldsymbol{\epsilon}^\mu, \hat{\boldsymbol{\epsilon}}^\mu\}$ is not normalized. In this chapter, we will be mostly using the orthonormal frame $\{\boldsymbol{e}^\mu, \hat{\boldsymbol{e}}^\mu\}$. However, when performing the equal-spin limit in the next two chapters, it will be more convenient to use the orthogonal one $\{\boldsymbol{\epsilon}^\mu,\ \hat{\boldsymbol{\epsilon}}^\mu\}$ since it has a simpler form. Similarly, dual orthogonal frames of vectors read

$$\begin{aligned}
\boldsymbol{e}_\mu &= \left(\frac{X_\mu}{U_\mu}\right)^{\frac{1}{2}} \boldsymbol{\epsilon}_\mu = \left(\frac{X_\mu}{U_\mu}\right)^{\frac{1}{2}} \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x_\mu}\,, \\
\hat{\boldsymbol{e}}_\mu &= \left(\frac{U_\mu}{X_\mu}\right)^{\frac{1}{2}} \hat{\boldsymbol{\epsilon}}_\mu = \left(\frac{U_\mu}{X_\mu}\right)^{\frac{1}{2}} \sum_k \frac{(-x_\mu^2)^{N-1-k}}{U_\mu} \frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\psi_k} \\
&= \left(\frac{U_\mu}{X_\mu}\right)^{\frac{1}{2}} \sum_\nu \frac{\lambda a_\nu \mathcal{J}_\nu(x_\mu^2)}{U_\mu} \frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\phi_\nu}\,,
\end{aligned} \tag{1.9}$$

where $\{\boldsymbol{e}_\mu,\ \hat{\boldsymbol{e}}_\mu\}$ is a normalized and $\{\boldsymbol{\epsilon}_\mu,\ \hat{\boldsymbol{\epsilon}}_\mu\}$ is an unnormalized frame. The duality stems from the properties of the metric functions described in Appendix A.1, namely the identities (A.7) and (A.10).

Using these frames, the metric and its inverse can be written simply as

$$\begin{aligned}
\boldsymbol{g} &= \sum_\mu \left(\boldsymbol{e}^\mu \boldsymbol{e}^\mu + \hat{\boldsymbol{e}}^\mu \hat{\boldsymbol{e}}^\mu\right) = \sum_\mu \left(\frac{U_\mu}{X_\mu} \boldsymbol{\epsilon}^\mu \boldsymbol{\epsilon}^\mu + \frac{X_\mu}{U_\mu} \hat{\boldsymbol{\epsilon}}^\mu \hat{\boldsymbol{\epsilon}}^\mu\right)\,, \\
\boldsymbol{g}^{-1} &= \sum_\mu \left(\boldsymbol{e}_\mu \boldsymbol{e}_\mu + \hat{\boldsymbol{e}}_\mu \hat{\boldsymbol{e}}_\mu\right) = \sum_\mu \left(\frac{X_\mu}{U_\mu} \boldsymbol{\epsilon}_\mu \boldsymbol{\epsilon}_\mu + \frac{U_\mu}{X_\mu} \hat{\boldsymbol{\epsilon}}_\mu \hat{\boldsymbol{\epsilon}}_\mu\right)\,.
\end{aligned} \tag{1.10}$$

Although these expressions suggest that the metric is positive definite, it is not necessarily the case since some of the frame 1-forms (or vectors) can be imaginary. A detailed discussion of suitable coordinate and parameter choices to obtain the Lorentzian signature is provided in Section 1.3.

1.2 Geometries with Euclidean signature

As was indicated above, the general metric can have both the Euclidean and the Lorentzian signature, depending on our choice of the coordinate ranges and values of the parameters. This section focuses on the Euclidean signature, namely we will show how to obtain the maximally symmetric geometry of a homogeneous sphere and also a Euclidean instanton geometry as special cases. The next section focuses on the Lorentzian signature.

1.2.1 Homogeneous sphere

In order to obtain a metric describing a spherical geometry, we assume that the mass and the NUT charges vanish, i.e. $b_\mu = 0$, while a_μ remain unrestricted and $\lambda > 0$. Then the on-shell metric functions (1.3) simplify to

$$X_\mu = \lambda \mathcal{J}(x_\mu^2)\,, \tag{1.11}$$

and using the orthogonality relations (A.12) to simplify the angular part of the metric (1.6), it becomes

$$\boldsymbol{g} = \sum_\mu \left[\frac{U_\mu}{\lambda \mathcal{J}(x_\mu^2)} \mathbf{d}x_\mu^2 - \frac{J(a_\mu^2)}{\lambda a_\mu^2 \mathcal{U}_\mu} \mathbf{d}\phi_\mu^2\right]\,. \tag{1.12}$$

The function $J(a_\mu^2)$ is defined similarly to $\mathcal{J}(x_\mu^2)$ in Eq. (1.4), only with x_μ and a_μ interchanged (see also (A.1))

$$J(a_\mu^2) = \prod_\nu \left(x_\nu^2 - a_\mu^2\right) .$$

We can introduce $N+1$ coordinates ρ_μ, $\mu = 0, 1, \ldots, N$, instead of N coordinates x_μ employing the Jacobi transformation

$$\lambda\rho_\mu^2 = \frac{J(a_\mu^2)}{-a_\mu^2 \mathcal{U}_\mu} , \qquad \lambda\rho_0^2 = \frac{J(0)}{\mathcal{J}(0)} = \frac{A^{(N)}}{\mathcal{A}^{(N)}} . \tag{1.13}$$

The functions $A^{(k)}$ and $\mathcal{A}^{(k)}$ are defined as in (A.2). It can be shown that the new coordinates are restricted by the constraint

$$\sum_{\mu=0}^{N} \rho_\mu^2 = \frac{1}{\lambda} \tag{1.14}$$

and the metric (1.12) simplifies to

$$\boldsymbol{g} = \mathbf{d}\rho_0^2 + \sum_\mu \left(\mathbf{d}\rho_\mu^2 + \rho_\mu^2 \mathbf{d}\phi_\mu^2\right) . \tag{1.15}$$

Therefore, we obtained a $2N$-dimensional sphere given by the constraint (1.14) embedded in a $(2N+1)$-dimensional flat space described by the metric (1.15), where the metric is written in the multi-cylindrical coordinates $\{\rho_0, \rho_\mu, \phi_\mu\}$.

It turns out that this metric (together with the constraint) describes the sphere of the same radius regardless of the choice of the parameters a_μ as it does not depend on them. This means that a_μ do not actually parametrize the sphere itself, but rather a choice of coordinates. Namely, they characterize the freedom in choice of the coordinates x_μ in the inverse Jacobi transformation.

1.2.2 Euclidean instanton

The general metric can also have the Euclidean signature and describe a non-trivial geometry of an instanton if we make the following assumptions for the parameters and the coordinates. Let $\lambda > 0$ and the parameters a_μ and b_μ as well as the coordinates x_μ and ψ_k be real. Moreover, we assume that a_μ are ordered as

$$0 < a_1 < \ldots < a_{N-1} < a_N . \tag{1.16}$$

The metric (1.1) has the Euclidean signature and U_μ are non-singular if and only if

$$\frac{U_\mu}{X_\mu} > 0 ,$$

which determines the suitable ranges of x_μ.

Let us first discuss the case of vanishing mass and NUT parameters, i.e. $b_\mu = 0$. In this case, the functions X_μ have the form (1.11) and their roots are precisely the parameters a_μ. Therefore, each coordinate is restricted by

$$a_{\mu-1} < x_\mu < a_\mu , \tag{1.17}$$

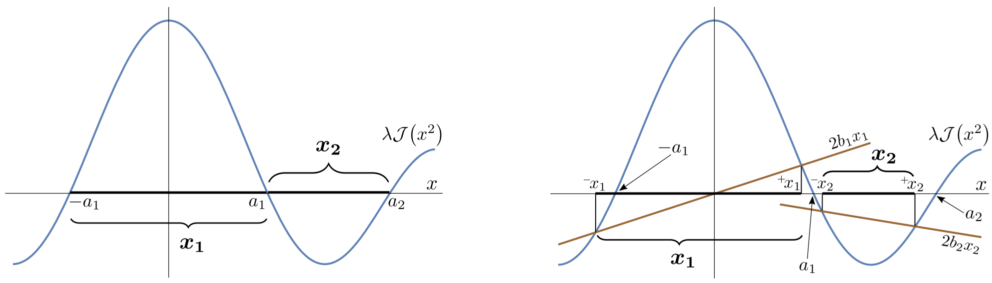
with the only exception of $x_1 \in (-a_1, a_1)$.

For non-vanishing mass and NUT charges, the coordinate ranges need to be modified. If b_μ are small, the ranges of x_μ change only slightly — they become restricted by the roots ${}^{\pm}x_\mu$ of the functions X_μ in the form (1.3). Therefore, the ranges in this case are

$$a_{\mu-1} < {}^{-}x_\mu < x_\mu < {}^{+}x_\mu < a_\mu \,,$$

with the only exception being ${}^{-}x_1$ since ${}^{-}x_1 < -a_1$, see Figure 1.1.

Unlike for a homogeneous sphere, in the case of a Euclidean instanton both the parameters a_μ and b_μ encode geometry deformations. Namely, b_μ encode how the geometry deviates from the spherical one and a_μ do not parametrize only a coordinate transformation, but they affect the geometry as well (for $b_\mu \neq 0$). Moreover, there is no curvature singularity in the case of non-zero b_μ as it occurs outside the coordinate ranges established above.



(a) (b)

Figure 1.1: Ranges of the coordinates x_μ for (a) zero and (b) non-zero mass and NUT parameters b_μ. In the case (a), the ranges are given by the roots of the metric function $\mathcal{J}(x_\mu^2)$, which are precisely the parameters a_μ. In the case (b), the ranges are determined by the roots ${}^{\pm}x_\mu$ of the metric functions X_μ in the form (1.3). In other words, ${}^{\pm}x_\mu$ are the values at which the polynomial $\lambda\mathcal{J}(x_\mu^2)$ intersects the lines $2b_\mu x_\mu$.

1.3 Black holes

Let us now focus on the form of the general metric that can describe spacetimes interesting from the physical point of view. First and foremost, it is necessary that the metric has the Lorentzian signature. In order to obtain this signature, some of the coordinates and the parameters need to be Wick-rotated, and the interpretation of the metric depends on their particular choice. Following [39], we choose x_N and ϕ_N to be imaginary and define

$$x_N = ir \,, \qquad \phi_N = \lambda a_N t \,, \tag{1.18}$$

where the radial coordinate r and the temporal coordinate t acquire real values. We also Wick-rotate the parameter b_N to a real-valued mass parameter M as

$$b_N = iM \,. \tag{1.19}$$

Finally, we make use of the metric invariance under the scaling transformation (1.7) and set

$$a_N^2 = -\frac{1}{\lambda}\,, \tag{1.20}$$

which guarantees that ϕ_N is Wick-rotated only for $\lambda > 0$.

1.3.1 Generalized Boyer–Lindquist coordinates

With the choices introduced above, the metric (1.6) can be written as

$$\begin{aligned}
\boldsymbol{g} = &-\frac{\Delta_r}{\Sigma}\left[\prod_{\bar\nu}\frac{1+\lambda x_{\bar\nu}^2}{1+\lambda a_{\bar\nu}^2}\mathbf{d}t - \sum_{\bar\nu}\frac{\bar J(a_{\bar\nu}^2)}{a_{\bar\nu}\left(1+\lambda a_{\bar\nu}^2\right)\bar{\mathcal{U}}_{\bar\nu}}\mathbf{d}\phi_{\bar\nu}\right]^2 \\
&+\frac{\Sigma}{\Delta_r}\mathbf{d}r^2 + \sum_{\bar\mu}\frac{\left(r^2+x_{\bar\mu}^2\right)\bar U_{\bar\mu}}{\Delta_{\bar\mu}}\mathbf{d}x_{\bar\mu}^2 \\
&+\sum_{\bar\mu}\frac{\Delta_{\bar\mu}}{\left(r^2+x_{\bar\mu}^2\right)\bar U_{\bar\mu}}\left[\frac{1-\lambda r^2}{1+\lambda x_{\bar\mu}^2}\prod_{\bar\nu}\frac{1+\lambda x_{\bar\nu}^2}{1+\lambda a_{\bar\nu}^2}\mathbf{d}t + \sum_{\bar\nu}\frac{\left(r^2+a_{\bar\nu}^2\right)\bar J_{\bar\mu}(a_{\bar\nu}^2)}{a_{\bar\nu}\left(1+\lambda a_{\bar\nu}^2\right)\bar{\mathcal{U}}_{\bar\nu}}\mathbf{d}\phi_{\bar\nu}\right]^2\,,
\end{aligned}$$

where barred indices go over the ranges

$$\begin{aligned}
\bar\mu, \bar\nu, \ldots &= 1, \ldots, \bar N\,, \\
\bar k, \bar l, \ldots &= 0, \ldots, \bar N - 1\,, \\
\bar N &= N - 1\,.
\end{aligned}$$

This notation has been introduced in order to separate the temporal and the radial coordinate from the angular coordinates. The metric functions are defined as

$$\begin{aligned}
\Delta_r &= -X_N = \left(1-\lambda r^2\right)\prod_{\bar\nu}\left(r^2+a_{\bar\nu}^2\right) - 2Mr\,, & \Sigma = U_N &= \prod_{\bar\nu}\left(r^2+x_{\bar\nu}^2\right)\,, \\
\Delta_{\bar\mu} &= -X_{\bar\mu} = \left(1+\lambda x_{\bar\mu}^2\right)\bar{\mathcal{J}}(x_{\bar\mu}^2) + 2b_{\bar\mu}x_{\bar\mu}\,, & \bar U_{\bar\mu} &= \prod_{\substack{\bar\nu \\ \bar\nu\neq\bar\mu}}\left(x_{\bar\nu}^2 - x_{\bar\mu}^2\right)\,.
\end{aligned} \tag{1.21}$$

Barred functions are defined in the same way as their unbarred counterparts in Appendix A.1, only with modified sets of coordinates and parameters (i.e. without x_N and a_N), for example

$$\bar J(a_{\bar\mu}^2) = \prod_{\bar\nu}\left(x_{\bar\nu}^2 - a_{\bar\mu}^2\right)\,.$$

We refer to $\{t,\, r,\, x_{\bar\mu},\, \phi_{\bar\mu}\}$ as the *generalized Boyer–Lindquist coordinates*.

As for the coordinate ranges, the discussion is very similar to the Euclidean case, so we will just briefly summarize the results. We assume that $a_{\bar\mu}$ are ordered as

$$0 < a_1 < \ldots < a_{\bar N-1} < a_{\bar N}\,.$$

For vanishing NUT parameters (and non-zero mass), i.e. $b_{\bar\mu} = 0$, the metric has the desired signature and U_μ are non-singular if the coordinates are restricted by

$$a_{\bar\mu-1} < x_{\bar\mu} < a_{\bar\mu}\,,$$

with the same exception of $x_1 \in (-a_1, a_1)$. For non-vanishing NUT charges, the coordinate ranges are

$$a_{\bar{\mu}-1} < {}^{-}x_{\bar{\mu}} < x_{\bar{\mu}} < {}^{+}x_{\bar{\mu}} < a_{\bar{\mu}} \tag{1.22}$$

again except for ${}^{-}x_1$ (see Figure 1.1).

The horizon structure is given by the roots of the metric function Δ_r defined in Eq. (1.21). For $\lambda \leq 0$, there exist at most two horizons: an inner and an outer one. The two horizons can also coincide, thus forming a single extremal horizon, or we can obtain a naked singularity in case there are no horizons. For $\lambda > 0$, there is usually one additional cosmological horizon, see Figure 1.2.

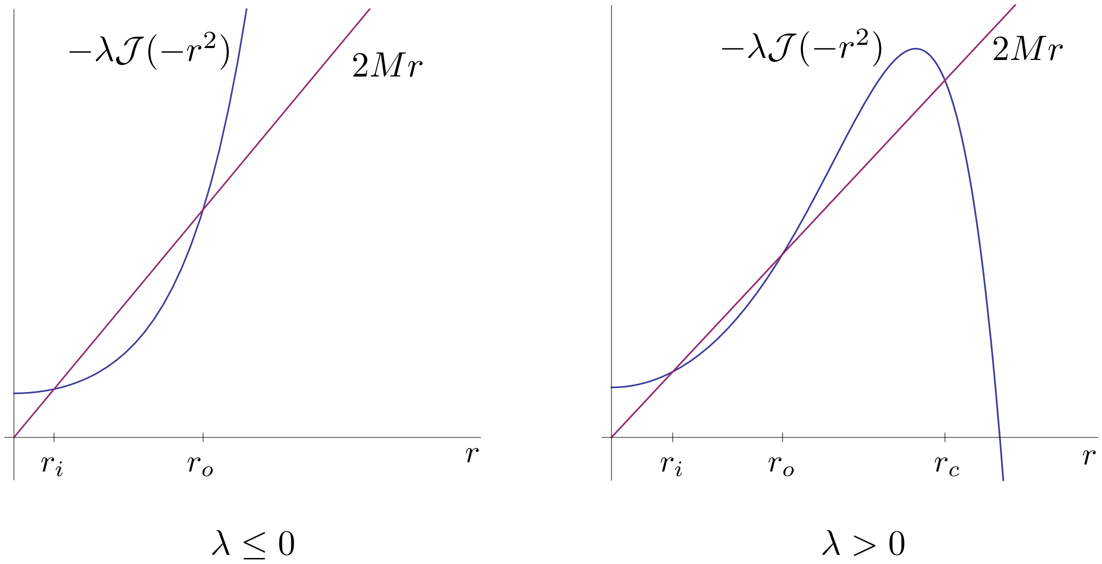


Figure 1.2: Black hole horizons. They are given by the roots of the metric function Δ_r defined in (1.21), i.e. by the condition $\Delta_r = -\lambda\mathcal{J}(-r^2) - 2Mr = 0$. This condition is satisfied when the even polynomial $-\lambda\mathcal{J}(-r^2)$ intersects the line $2Mr$. For $\lambda \leq 0$, there are at most two such intersections, which define an inner horizon r_i and an outer horizon r_o. If the line is tangential to the polynomial, i.e. there is only one intersection, we obtain an extremal horizon. In case the line does not intersect the polynomial curve at all, we obtain a naked singularity without horizons. For $\lambda > 0$, there is usually one additional intersection that defines a cosmological horizon r_c. However, for large M the outer horizon can extend beyond the cosmological horizon, thus leaving the black hole with only an inner horizon.

As for the interpretation of the metric parameters, for zero NUT charges $b_{\bar{\mu}} = 0$, $a_{\bar{\mu}}$ can be identified with the rotational parameters of the black hole. On the other hand, when the NUT parameters are non-trivial, then both $a_{\bar{\mu}}$ and $b_{\bar{\mu}}$ deform the geometry, however, their exact role remains elusive. This is where studying various limit cases might prove useful as they can help clarify this issue.

1.3.2 Myers–Perry coordinates

Assuming that the NUT charges vanish, i.e. $b_{\bar{\mu}} = 0$, the metric can be transformed into another set of coordinates. The metric functions $X_{\bar{\mu}}$ then simplify to the form (1.11) and $X_N = -\Delta_r$ has an additional term proportional to mass M as in Eq. (1.21). Using the orthogonality relations (A.12) to simplify the angular part of the metric (1.6) in the same way as in the case of a homogeneous sphere geometry in Section 1.2.1, we obtain

$$\begin{aligned}\boldsymbol{g} =& \sum_{\bar\mu} \frac{U_{\bar\mu}}{\lambda \mathcal{J}(x_{\bar\mu}^2)} \mathbf{d}x_{\bar\mu}^2 + \frac{\Sigma}{\Delta_r}\mathbf{d}r^2 - \sum_{\bar\mu} \frac{J(a_{\bar\mu}^2)}{\lambda a_{\bar\mu}^2 \mathcal{U}_{\bar\mu}} \mathbf{d}\phi_{\bar\mu}^2 - \lambda \frac{J(a_N^2)}{\mathcal{U}_N}\mathbf{d}t^2 \\ &+ \frac{2Mr}{\Sigma}\left[\sum_{\bar\mu} \frac{J_N(a_{\bar\mu}^2)}{\lambda a_{\bar\mu}\mathcal{U}_{\bar\mu}}\mathbf{d}\phi_{\bar\mu} + \frac{J_N(a_N^2)}{\mathcal{U}_N}\mathbf{d}t\right]^2 .\end{aligned} \tag{1.23}$$

We have also separated the temporal and the radial coordinate from the angular coordinates, employed the Wick rotation and the gauge fixing (1.18)—(1.20) and used the metric functions (1.21).

Similarly to the homogeneous sphere, let us introduce $\bar N + 1$ coordinates $\mu_{\bar\nu}$, $\bar\nu = 0, 1, \ldots, \bar N$ instead of $\bar N$ coordinates $x_{\bar\kappa}$ using the Jacobi transformation in the form

$$\mu_{\bar\nu}^2 = \frac{\bar J(a_{\bar\nu}^2)}{-a_{\bar\nu}^2 \bar{\mathcal{U}}_{\bar\nu}}, \qquad \mu_0^2 = \frac{\bar J(0)}{\bar{\mathcal{J}}(0)} = \frac{\bar A^{(\bar N)}}{\bar{\mathcal{A}}^{(\bar N)}}, \tag{1.24}$$

which satisfy the following constraint

$$\sum_{\bar\nu=0}^{\bar N} \mu_{\bar\nu}^2 = 1 . \tag{1.25}$$

It can be shown that the new coordinates are related to the coordinates ρ_μ defined in (1.13) as

$$\begin{aligned}\lambda \rho_{\bar\nu}^2 &= \frac{r^2 + a_{\bar\nu}^2}{a_{\bar\nu}^2 - a_N^2}\mu_{\bar\nu}^2 , \\ 1 - \lambda R^2 \equiv \lambda \rho_N^2 &= \left(1 - \lambda r^2\right)\left(\mu_0^2 + \sum_{\bar\nu} \frac{\mu_{\bar\nu}^2}{1 + \lambda a_{\bar\nu}^2}\right) .\end{aligned} \tag{1.26}$$

We refer to $\{t,\, r,\, \mu_0,\, \mu_{\bar\nu},\, \phi_{\bar\nu}\}$ as the *Myers–Perry coordinates.*

The metric (1.23) in these coordinates becomes

$$\begin{aligned}\boldsymbol{g} =& -\left(1 - \lambda R^2\right)\mathbf{d}t^2 + \frac{2Mr}{\Sigma}\left[\mathbf{d}t + \sum_{\bar\nu}\frac{a_{\bar\nu}\mu_{\bar\nu}^2}{1 + \lambda a_{\bar\nu}^2}\left(\mathbf{d}\phi_{\bar\nu} - \lambda a_{\bar\nu}\mathbf{d}t\right)\right]^2 \\ &+ \frac{\Sigma}{\Delta_r}\mathbf{d}r^2 + r^2 \mathbf{d}\mu_0^2 + \sum_{\bar\nu}\frac{r^2 + a_{\bar\nu}^2}{1 + \lambda a_{\bar\nu}^2}\left(\mathbf{d}\mu_{\bar\nu}^2 + \mu_{\bar\nu}^2 \mathbf{d}\phi_{\bar\nu}^2\right) \\ &+ \frac{\lambda}{1 - \lambda R^2}\left(r^2 \mu_0 \mathbf{d}\mu_0 + \sum_{\bar\nu}\frac{r^2 + a_{\bar\nu}^2}{1 + \lambda a_{\bar\nu}^2}\mu_{\bar\nu}\mathbf{d}\mu_{\bar\nu}\right)^2 ,\end{aligned} \tag{1.27}$$

where

$$\begin{aligned}\Delta_r &= \left(1 - \lambda r^2\right)\prod_{\bar\nu}\left(r^2 + a_{\bar\nu}^2\right) - 2Mr , \\ \Sigma &= \left(\mu_0^2 + \sum_{\bar\nu}\frac{r^2 \mu_{\bar\nu}^2}{r^2 + a_{\bar\nu}^2}\right)\prod_{\bar\mu}\left(r^2 + a_{\bar\mu}^2\right) .\end{aligned} \tag{1.28}$$

We have thus obtained the *Kerr–(A)dS metric* [15, 16].

Let us consider a vacuum subcase of the general Kerr–(A)dS solution, i.e. $\lambda = 0$. The metric can be then expressed as

$$\begin{aligned}\boldsymbol{g} = &-\mathbf{d}t^2 + \frac{2Mr}{\Sigma}\left(\mathbf{d}t + \sum_{\bar{\nu}} a_{\bar{\nu}}\mu_{\bar{\nu}}^2\mathbf{d}\phi_{\bar{\nu}}\right)^2 + \frac{\Sigma}{\Delta_r}\mathbf{d}r^2\\ &+ r^2\mathbf{d}\mu_0^2 + \sum_{\bar{\nu}}\left(r^2 + a_{\bar{\nu}}^2\right)\left(\mathbf{d}\mu_{\bar{\nu}}^2 + \mu_{\bar{\nu}}^2\mathbf{d}\phi_{\bar{\nu}}^2\right),\end{aligned}$$

where

$$\begin{aligned}\Delta_r &= \prod_{\bar{\nu}}\left(r^2 + a_{\bar{\nu}}^2\right) - 2Mr\,,\\ \Sigma &= \left(\mu_0^2 + \sum_{\bar{\nu}}\frac{r^2\mu_{\bar{\nu}}^2}{r^2 + a_{\bar{\nu}}^2}\right)\prod_{\bar{\mu}}\left(r^2 + a_{\bar{\mu}}^2\right).\end{aligned}$$

This solution is known as the *Myers–Perry spacetime* [14].

1.4 Explicit and hidden symmetries

The Kerr–NUT–(A)dS spacetime possesses symmetries of two kinds: explicit symmetries, which are represented by well-known Killing vector fields, and hidden symmetries, which in our case will be described by Killing tensors. Let us briefly mention basic definitions and properties of these objects.

If the spacetime metric $\boldsymbol{g}$ is Lie-constant along a vector field $\boldsymbol{\xi}$, i.e. it satisfies

$$\mathcal{L}_{\boldsymbol{\xi}}\boldsymbol{g} = 0\,,$$

then $\boldsymbol{\xi}$ is referred to as a Killing vector of the spacetime. This condition is equivalent to the Killing vector equation

$$\nabla_{(a}\xi_{b)} = 0\,,$$

where ∇ denotes the covariant derivative, and indices inside the round brackets are symmetrized[1]. When studying particle dynamics in a curved spacetime, it can be shown that the integrals of geodesic motion that are generated by Killing vectors are necessarily linear in particle's momentum [39].

Hidden symmetries, on the other hand, are associated with Killing tensors, which generate integrals of geodesic motion that are of higher order in particle's momenta. A completely symmetric tensor $\boldsymbol{k}$ of rank s is a Killing tensor if it satisfies the Killing tensor equation

$$\nabla^{(a_0}k^{a_1\ldots a_s)} = 0\,.$$

The metric itself constitutes a trivial example of a Killing tensor that is present in every spacetime, moreover, a Killing tensor of rank $s = 1$ reduces to a Killing

[1] Let us remind the reader that Latin letters from the beginning of the alphabet label general spacetime indices $a, b, \ldots = 1, \ldots, 2N$, and we do employ the Einstein summation convention for them.

vector. Unlike Killing vectors, the action of Killing tensors of rank $s \geq 2$ does not create a spacetime diffeomorphism, therefore in that sense they represent symmetries that are not encoded only in the spacetime manifold, hence "hidden". They can be "discovered", however, when studying the particle motion in the phase space.

There exists an object which generates a rich structure of both types of symmetries — the principal tensor. It is a crucial object that does not only uniquely determine the canonical form of the metric (1.1), but it also generates a set of Killing vectors and Killing tensors, which we refer to as the *Killing tower*.

1.4.1 Principal tensor

The principal tensor $\boldsymbol{h}$ is defined as a closed conformal Killing–Yano 2-form that is also non-degenerate. A closed conformal Killing–Yano form satisfies

$$\boldsymbol{\nabla}_{\boldsymbol{X}}\boldsymbol{h} = \boldsymbol{X} \wedge \boldsymbol{\xi} \qquad \Leftrightarrow \qquad \nabla_a h_{bc} = g_{ab}\xi_c - g_{ac}\xi_b \,,$$

where $\boldsymbol{\xi}$ is given by the equation

$$\boldsymbol{\xi} = \frac{1}{D-1}\boldsymbol{\nabla}\cdot\boldsymbol{h} \qquad \Leftrightarrow \qquad \xi^a = \frac{1}{D-1}\nabla_b h^{ba} \,, \tag{1.29}$$

and $\boldsymbol{X}$ is a general vector. The non-degeneracy condition means that $\boldsymbol{h}$ has the maximal possible matrix rank and also the maximum number of functionally independent eigenvalues. Namely, in $D = 2N$ dimensions it possesses N eigenvalues that are non-constant and N pairs of conjugate eigenvectors.

Using the orthonormal frame $\{\boldsymbol{e}^\mu,\ \hat{\boldsymbol{e}}^\mu\}$ defined in (1.8), the principal tensor can be written in the form

$$\boldsymbol{h} = \sum_\mu x_\mu \boldsymbol{e}^\mu \wedge \hat{\boldsymbol{e}}^\mu \,. \tag{1.30}$$

If we explicitly denote the principal tensor with the first index raised by $\sharp\boldsymbol{h}$, i.e. it is given by

$$(\sharp\boldsymbol{h})^a{}_b = g^{ac}h_{cb} \,,$$

then it satisfies the following eigenvalue equations

$$\sharp\boldsymbol{h}\cdot\boldsymbol{m}_\mu = -ix_\mu\boldsymbol{m}_\mu \,, \qquad \sharp\boldsymbol{h}\cdot\bar{\boldsymbol{m}}_\mu = ix_\mu\bar{\boldsymbol{m}}_\mu \,.$$

The eigenvalues of the principal tensor are thus $\pm ix_\mu$, and its eigenvectors are related to the orthonormal frame (1.9) as

$$\boldsymbol{m}_\mu = \frac{1}{\sqrt{2}}\left(\hat{\boldsymbol{e}}_\mu + i\boldsymbol{e}_\mu\right) \,, \qquad \bar{\boldsymbol{m}}_\mu = \frac{1}{\sqrt{2}}\left(\hat{\boldsymbol{e}}_\mu - i\boldsymbol{e}_\mu\right) \,.$$

The non-degeneracy condition means that x_μ are functionally independent. Therefore, they can be used as canonical coordinates in the metric (1.1). As for the eigenvectors, they are null and satisfy the null-orthonormality relations

$$\boldsymbol{m}_\mu\cdot\boldsymbol{m}_\nu = \bar{\boldsymbol{m}}_\mu\cdot\bar{\boldsymbol{m}}_\nu = 0 \,, \qquad \boldsymbol{m}_\mu\cdot\bar{\boldsymbol{m}}_\nu = \delta_{\mu\nu} \,.$$

We can also apply the principal tensor to the orthonormal frame of vectors, obtaining

$$\sharp\boldsymbol{h}\cdot\boldsymbol{e}_\mu = -x_\mu\hat{\boldsymbol{e}}_\mu\,, \qquad \sharp\boldsymbol{h}\cdot\hat{\boldsymbol{e}}_\mu = x_\mu\boldsymbol{e}_\mu\,.$$

Finally, it turns out that the vector $\boldsymbol{\xi}$ given by (1.29) is a Killing vector and can be written in the canonical coordinates as

$$\boldsymbol{\xi} = \frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\psi_0}\,.$$

Considering that it will be used to construct the whole tower of other Killing vectors, we refer to it as the *principal Killing vector.*

1.4.2 Killing tower

As was mentioned before, the importance of the principal tensor lies, besides other things, in its ability to generate the entire tower of Killing objects that represent both explicit and hidden symmetries of the Kerr–NUT–(A)dS spacetime. The Killing tower can be constructed either directly or using generating functions.

Direct method of construction

Let us first describe how to construct the Killing tower directly [23, 24, 50], using the following steps.

(i) We define *closed conformal Killing–Yano forms* as wedge powers of the principal tensor $\boldsymbol{h}$

$$\boldsymbol{h}^{(k)} = \frac{1}{k!}\boldsymbol{h}^{\wedge k}\,,$$

for $k = 0, \ldots N$. The resulting forms $\boldsymbol{h}^{(k)}$ are of increasing rank $2k$ and it can be shown that

$$\boldsymbol{h}^{(0)} = 1\,, \qquad \boldsymbol{h}^{(N)} = \sqrt{A^{(N)}}\boldsymbol{\varepsilon}\,,$$

with $\boldsymbol{\varepsilon}$ denoting the Levi-Civita tensor.

(ii) The next step is to calculate *Killing–Yano forms*, which are given as the Hodge duals of closed conformal Killing–Yano forms $\boldsymbol{h}^{(k)}$

$$\boldsymbol{f}^{(k)} = *\boldsymbol{h}^{(k)}\,.$$

The tensors $\boldsymbol{f}^{(k)}$ are of rank $D-2k$ and we have

$$\boldsymbol{f}^{(0)} = \boldsymbol{\varepsilon}\,, \qquad \boldsymbol{f}^{(N)} = \sqrt{A^{(N)}}\,.$$

(iii) Using Killing–Yano forms $\boldsymbol{f}^{(k)}$, we can introduce *Killing tensors* as partial contractions of their squares

$$k^{ab}_{(k)} = \frac{1}{(D-2k-1)!}f^{(k)a}{}_{c_1\ldots c_{D-2k-1}}f^{(k)bc_1\ldots c_{D-2k-1}}\,.$$

The resulting tensors $\boldsymbol{k}_{(k)}$ are of rank 2 and they satisfy

$$\boldsymbol{k}_{(0)} = \boldsymbol{g}\,, \qquad \boldsymbol{k}_{(N)} = \boldsymbol{0}\,.$$

(iv) We can also define rank-2 *conformal Killing tensors* as partial contractions of closed conformal Killing–Yano forms $\boldsymbol{h}^{(k)}$

$$Q^{ab}_{(k)} = \frac{1}{(2k-1)!} h^{(k)a}{}_{c_1 \dots c_{2k-1}} h^{(k)bc_1 \dots c_{2k-1}} \,. \tag{1.31}$$

We have

$$\boldsymbol{Q}_{(0)} = \boldsymbol{0} \,, \qquad \boldsymbol{Q}_{(N)} = A^{(N)} \boldsymbol{g} \,,$$

and it can be shown that

$$\boldsymbol{k}_{(k)} + \boldsymbol{Q}_{(k)} = A^{(k)} \boldsymbol{g} \,,$$

where the symmetric polynomials $A^{(k)}$ are defined as in (A.2). Therefore, conformal Killing tensors $\boldsymbol{Q}_{(k)}$ and Killing tensors $\boldsymbol{k}_{(k)}$ contain basically the same information.

(v) Finally, we calculate *Killing vectors* as contractions of Killing tensors $\boldsymbol{k}_{(k)}$ with the principal Killing vector $\boldsymbol{\xi}$

$$\boldsymbol{l}_{(k)} = \boldsymbol{k}_{(k)} \cdot \boldsymbol{\xi} \,. \tag{1.32}$$

They satisfy

$$\boldsymbol{l}_{(0)} = \boldsymbol{\xi} \,, \qquad \boldsymbol{l}_{(N)} = \boldsymbol{0} \,. \tag{1.33}$$

By this construction we have thus obtained N Killing vectors and the same number of Killing tensors. Together they generate the required number of Poisson-bracket-commuting conserved quantities so that the geodesic motion of a particle in the spacetime is fully integrable.

Method of generating functions

We shall now focus on the second method of constructing the Killing tower using generating functions [39]. Namely, we introduce auxiliary β-dependent Killing tensors and Killing vectors such that regular Killing tensors and Killing vectors form coefficients in the β-expansion of these generating functions.

Let us first define a β-dependent conformal Killing tensor as

$$\boldsymbol{q}(\beta) = \boldsymbol{g} + \beta^2 \boldsymbol{Q} \,,$$

where β is a real parameter and $Q^{ab} = Q^{ab}_{(1)} = h^a{}_c h^{bc}$ is the first conformal Killing tensor introduced in (1.31). We also define a scalar function

$$A(\beta) = \sqrt{\frac{\det \boldsymbol{q}(\beta)}{\det \boldsymbol{g}}} \,.$$

Using these definitions, we can introduce generating functions for Killing tensors and Killing vectors, respectively, in the form

$$\boldsymbol{k}(\beta) = A(\beta) \boldsymbol{q}^{-1}(\beta) \,, \qquad \boldsymbol{l}(\beta) = \boldsymbol{k}(\beta) \cdot \boldsymbol{\xi} \,.$$

The β-expansion of these functions can be then written as

$$\boldsymbol{k}(\beta) = \sum_{k=0}^{N} \beta^{2k} \boldsymbol{k}_{(k)} \,, \qquad \boldsymbol{l}(\beta) = \sum_{k=0}^{N} \beta^{2k} \boldsymbol{l}_{(k)} \,, \tag{1.34}$$

thus generating Killing tensors $\boldsymbol{k}_{(k)}$ and Killing vectors $\boldsymbol{l}_{(k)}$. Similarly, the function $A(\beta)$ generates the polynomials $A^{(k)}$

$$A(\beta) = \sum_{k=0}^{N} \beta^{2k} A^{(k)} \,.$$

For a fixed parameter β, $\boldsymbol{k}(\beta)$ is a linear combination of Killing tensors, therefore, $\boldsymbol{k}(\beta)$ itself is a Killing tensor. Likewise, $\boldsymbol{l}(\beta)$ is a Killing vector.

Generating functions for other objects from the Killing tower can be constructed in a similar manner, however, we will not need them in this thesis (they can be found in [39]).

Explicit form of the Killing tower

We shall conclude this chapter by explicitly writing Killing vectors and Killing tensors generated from the principal tensor, using the orthogonal frame of vectors $\{\boldsymbol{\epsilon}_\mu,\ \hat{\boldsymbol{\epsilon}}_\mu\}$ defined in Eq. (1.9). As was mentioned earlier, this frame is more convenient for performing the limit, especially for these objects.

Defining an alternative set of Killing vectors $\boldsymbol{s}_{(\mu)}$, which are, up to normalization, the coordinate vectors,

$$\boldsymbol{s}_{(\mu)} = \lambda a_\mu \frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\phi_\mu} = \sum_\nu \frac{J_\nu(a_\mu^2)}{\mathcal{U}_\mu}\, \hat{\boldsymbol{\epsilon}}_\nu \,, \tag{1.35}$$

Killing vectors $\boldsymbol{l}_{(k)}$ can be written as

$$\boldsymbol{l}_{(k)} = \sum_\mu \mathcal{A}_\mu^{(k)} \boldsymbol{s}_{(\mu)} = \sum_\mu A_\mu^{(k)} \hat{\boldsymbol{\epsilon}}_\mu = \frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\psi_k} \,. \tag{1.36}$$

The second equality in (1.36) can be derived from the relation (1.5) between the angular coordinates ψ_k and ϕ_μ, and the third equality stems from the definition (1.9) of the frame vectors $\hat{\boldsymbol{\epsilon}}_\mu$. As we can see, the canonical coordinates ψ_k are directly associated with the Killing vectors defined in (1.32), and as was mentioned before, they are also the Killing coordinates. Moreover, since $\mathcal{A}_\mu^{(k)}$ in the first equality are just constants (see the definition (A.4)), $\boldsymbol{s}_{(\mu)}$ are indeed Killing vectors — and they are more suitable for the limiting procedure as is shown in Section 3.2. The principal Killing vector acquires a simple form

$$\boldsymbol{\xi} = \sum_\mu \boldsymbol{s}_{(\mu)} = \sum_\mu \hat{\boldsymbol{\epsilon}}_\mu = \frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\psi_0} \,. \tag{1.37}$$

The corresponding generating function reads

$$\boldsymbol{l}(\beta) = \sum_\mu \mathcal{A}_\mu(\beta) \boldsymbol{s}_{(\mu)} = \sum_\mu A_\mu(\beta) \hat{\boldsymbol{\epsilon}}_\mu = \sum_k \beta^{2k} \frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\psi_k} \,, \tag{1.38}$$

where the functions $A_\mu(\beta)$ and $\mathcal{A}_\mu(\beta)$ generate the polynomials $A_\mu^{(k)}$ and $\mathcal{A}_\mu^{(k)}$, respectively, and they can be expressed in several ways

$$\begin{aligned} A_\mu(\beta) &= \sum_k \beta^{2k} A_\mu^{(k)} = \prod_{\substack{\nu \\ \nu\neq\mu}} \left(1 + \beta^2 x_\nu^2\right) = \frac{A(\beta)}{1 + \beta^2 x_\mu^2} \,, \\ \mathcal{A}_\mu(\beta) &= \sum_k \beta^{2k} \mathcal{A}_\mu^{(k)} = \prod_{\substack{\nu \\ \nu\neq\mu}} \left(1 + \beta^2 a_\nu^2\right) = \frac{\mathcal{A}(\beta)}{1 + \beta^2 a_\mu^2} \,. \end{aligned} \tag{1.39}$$

The functions $A(\beta)$ and $\mathcal{A}(\beta)$ have the form

$$A(\beta) = \prod_\nu \left(1+\beta^2 x_\nu^2\right) \,, \qquad \mathcal{A}(\beta) = \prod_\nu \left(1+\beta^2 a_\nu^2\right) \,. \tag{1.40}$$

Motivated by the structure of Killing vectors, we also introduce a new set of Killing tensors $\boldsymbol{r}_{(\mu)}$

$$\boldsymbol{r}_{(\mu)} = \sum_\nu \frac{J_\nu(a_\mu^2)}{\mathcal{U}_\mu} \boldsymbol{\pi}_\nu \,, \tag{1.41}$$

where $\boldsymbol{\pi}_\mu$ are defined as

$$\boldsymbol{\pi}_\mu = \frac{X_\mu}{U_\mu}\, \boldsymbol{\epsilon}_\mu \boldsymbol{\epsilon}_\mu + \frac{U_\mu}{X_\mu}\, \hat{\boldsymbol{\epsilon}}_\mu \hat{\boldsymbol{\epsilon}}_\mu \,, \tag{1.42}$$

and they denote “frame” 2-tensors appearing explicitly in the inverse metric (1.10)

$$\boldsymbol{g}^{-1} = \sum_\mu \boldsymbol{\pi}_\mu \,.$$

Eq. (1.41) can be proved using Eqs. (A.3) and (A.7).

Tensors $\boldsymbol{r}_{(\mu)}$ form a base of Killing tensors alternative to $\boldsymbol{k}_{(k)}$ introduced above. Their equivalence can be observed from the relations analogous to (1.36)

$$\boldsymbol{k}_{(k)} = \sum_\mu \mathcal{A}_\mu^{(k)} \boldsymbol{r}_{(\mu)} = \sum_\mu A_\mu^{(k)} \boldsymbol{\pi}_\mu \,. \tag{1.43}$$

Similarly to $\boldsymbol{s}_{(\mu)}$ being Killing vectors, $\boldsymbol{r}_{(\mu)}$ are indeed Killing tensors due to $\mathcal{A}_\mu^{(k)}$ being constants — and similarly, they are better suited for performing the limit.

Finally, the generating function for Killing tensors is

$$\boldsymbol{k}(\beta) = \sum_\mu \mathcal{A}_\mu(\beta) \boldsymbol{r}_{(\mu)} = \sum_\mu A_\mu(\beta) \boldsymbol{\pi}_\mu \,. \tag{1.44}$$

2. Equal-spin limit of the Kerr–(A)dS spacetime

This and the following chapter include the main results of our work. These results have also been published in [48]. Namely, we focus on a particular limit case of the spacetime with general spin where an arbitrary number of its rotational parameters coincides. In this chapter, we introduce an appropriate parametrization of the limit, which also includes modifying the index notation used throughout Chapter 1, and apply the limiting procedure to the general metric. Furthermore, we study the cases of a homogeneous sphere and a black hole using suitable coordinate systems.

In Chapter 1, we have introduced the Kerr–NUT–(A)dS spacetime, which is the most general metric in higher dimensions that possesses the principal tensor. However, we will hereafter restrict our analysis to the case of vanishing NUT parameters and non-zero mass, i.e. $b_{\bar{\mu}} = 0$, thus studying the equal-spin limit of the Kerr–(A)dS spacetime since it is the most relevant case from the physical point of view. Moreover, let us remind the reader that we consider only even dimensions $D = 2N$ as in the previous chapter.

Let us emphasize that any limit of spacetime always strongly depends on the choice of the limiting procedure and used parametrization; see the classical work of Geroch [51] and an illustration in, e.g., [52]. One has to always carefully choose a limit interesting from the physical point of view. Different choices of the limiting procedure might focus on different aspects and thus would lead to different spacetimes after the limit. For example, one can zoom in on the regions near the black hole horizon by including a suitable rescaling during the limit, which would result in a near-horizon limit [40–43, 45], or rescale asymptotic regions, which could reveal the asymptotic structure of spacetime. Our limiting procedure preserves (and possibly enhances) the symmetry structure of the spacetime. Moreover, all the outer regions of the black hole remain non-degenerate after the limit.

2.1 Preliminaries

Before performing the actual equal-spin limit, let us address several matters. Firstly, we need to adjust the indices used to label quantities to better suit our needs, and secondly, it is necessary to rescale the parameters and the coordinates, which become degenerate after the limit, using a suitable parametrization.

2.1.1 Double indexing and equal-spin blocks

In order to perform the limit, it will be convenient to first modify the indexing of parameters and coordinates to reflect the structure that will emerge after the limit. Namely, assuming that the rotational parameters a_μ are ordered as in (1.16), we group them into $\tilde{N}$ "equal-spin" blocks so that within each block all the rotations approach the same value. This means that instead of using a single Greek index μ (or ν, κ...), it will be more natural to use two Greek indices:

one from the beginning of the alphabet α (β, γ...)[1] to label the block of equal rotations and the second from the later parts of the alphabet ρ (σ, τ...) to distinguish between the rotations inside the block.

The first rotation in a block is labeled as $a_{\alpha,0}$. It will remain unchanged after the limit and all the other rotations in the block, denoted by $a_{\alpha,\rho}$, will approach $a_{\alpha,0}$ as shown in Figure 2.1. The indices go over the ranges

$$\begin{aligned} \alpha, \beta, \ldots &= 1, \ldots, \tilde{N} \,, \\ \rho, \sigma, \ldots &= 1, \ldots, {}^{\alpha}N \,, \end{aligned}$$

where $\tilde{N}$ is the number of blocks, and therefore the number of distinct rotational parameters remaining in the spacetime after performing the limit, while ${}^{\alpha}N$ is the number of parameters in the block α subjected to the limit, and thus the number of additional rotations approaching the value $a_{\alpha,0}$. These numbers satisfy

$$\tilde{N} + \sum_{\alpha} {}^{\alpha}N = N \,.$$

Moreover, in the Lorentzian case we assume

$${}^{\tilde{N}}N = 0 \,,$$

which means that the last block contains only a single parameter $a_{\tilde{N},0}$ not subjected to the limit. This will enable us to obtain the Lorentzian signature of the metric in a similar way to Section 1.3, see Eqs. (2.3) and (2.4).

The ranges of the coordinates $\{x_{\alpha,0},\, x_{\alpha,\rho}\}$ remain as in (1.17) — or written in our new notation

$$\begin{aligned} a_{\alpha-1,{}^{\alpha-1}N} < x_{\alpha,0} < a_{\alpha,0} \,, \\ a_{\alpha,\rho-1} < x_{\alpha,\rho} < a_{\alpha,\rho} \,, \end{aligned} \tag{2.1}$$

with the only exception being $x_{1,0} \in (-a_{1,0}, a_{1,0})$.

All the other quantities such as other coordinates and metric functions will be indexed in the same way.

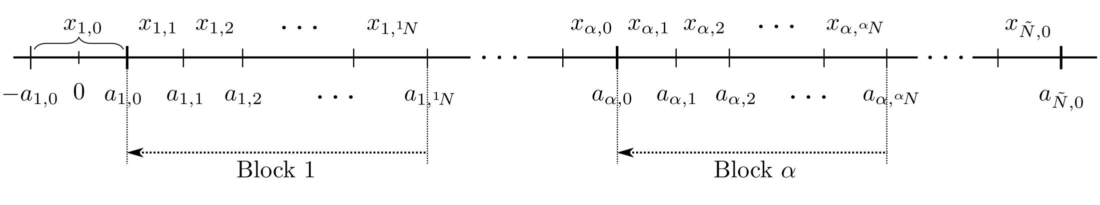


Figure 2.1: New indexing and grouping of the rotational parameters $\{a_{\alpha,0},\, a_{\alpha,\rho}\}$ into blocks that have equal spin once the limit $a_{\alpha,\rho} \to a_{\alpha,0}$ has been performed. The coordinates $\{x_{\alpha,0},\, x_{\alpha,\rho}\}$ remain restricted by the rotational parameters as in (2.1).

[1] Strictly speaking, we should be using tilded indices $\tilde{\alpha}$ ($\tilde{\beta}$, $\tilde{\gamma}$...) to label the blocks of equal rotations in order to clearly distinguish between indices and quantities before and after the limit. However, for the sake of simplicity and better readability, we will use tildes only over the names of the relevant quantities after the limit, e.g. $\tilde{x}_{\alpha}$, $\tilde{a}_{\alpha}$.

2.1.2 Parametrization of the limit

When performing the limit $a_{\alpha,\rho} \to a_{\alpha,0}$, the rotational parameters $a_{\alpha,\rho}$ and the ranges of the coordinates $x_{\alpha,\rho}$ degenerate. Therefore, we will rescale them using the following parametrization

$$\begin{aligned} a_{\alpha,0} &= \tilde{a}_\alpha\,, & x_{\alpha,0} &= \tilde{x}_\alpha\,, & \phi_{\alpha,0} &= \tilde{\phi}_\alpha\,, \\ a_{\alpha,\rho} &= \tilde{a}_\alpha + {}^\alpha a_\rho\,\varepsilon\,, & x_{\alpha,\rho} &= \tilde{a}_\alpha + {}^\alpha x_\rho\,\varepsilon\,, & \phi_{\alpha,\rho} &= {}^\alpha\phi_\rho\,, \end{aligned} \tag{2.2}$$

where $\varepsilon \ll 1$ is a small parameter. We have denoted the quantities that do not change after performing the limit using tildes and we will refer to the corresponding directions as *primary coordinate directions*. We have also introduced new rescaled parameters ${}^\alpha a_\rho$ and coordinates ${}^\alpha x_\rho$, which remain well-defined after the limit, and we will refer to the corresponding directions as *secondary coordinate directions*. Within one block, ${}^\alpha x_\rho$ are ordered as

$$0 < {}^\alpha x_1 < {}^\alpha a_1 < {}^\alpha x_2 < {}^\alpha a_2 < \ldots < {}^\alpha x_{{}^\alpha N} < {}^\alpha a_{{}^\alpha N}\,.$$

Let us note that the angular coordinates $\{\phi_{\alpha,0},\, \phi_{\alpha,\rho}\}$ are unambiguously related to the corresponding rotational parameter $a_{\alpha,0}$ or $a_{\alpha,\rho}$. In practice this means that they are labeled with the same type of indices — the Greek ones. Therefore, it is more convenient to use these coordinates in the limiting procedure instead of the coordinates ψ_k introduced in (1.1). The coordinates ψ_k non-trivially mix different equal-spin blocks and thus are not suitable for our limit.

As was mentioned earlier, the last block can be Wick-rotated to obtain the Lorentzian signature in a similar way to before the limit (see Eqs. (1.18)—(1.20))

$$\tilde{x}_{\tilde{N}} = ir\,, \qquad \tilde{\phi}_{\tilde{N}} = \lambda \tilde{a}_{\tilde{N}} t\,, \qquad \tilde{b}_{\tilde{N}} = iM\,, \tag{2.3}$$

and we also set

$$\tilde{a}^2_{\tilde{N}} = -\frac{1}{\lambda}\,. \tag{2.4}$$

2.2 Limiting procedure

2.2.1 Metric

Our goal is to perform the limit of the metric. For that we need to introduce two types of metric functions corresponding to the two sets of directions: tilded functions include only the coordinates and the parameters in the primary directions and functions with an upper left index are constructed using only variables in the secondary directions. They are defined in a similar manner to the metric functions before the limit (see Appendix A.1), only the sets of coordinates and parameters are modified, for example

$$\tilde{J}(\tilde{a}^2_\alpha) = \prod_\beta \left(\tilde{x}^2_\beta - \tilde{a}^2_\alpha\right)\,, \qquad {}^\alpha J({}^\alpha a_\rho) = {}^\alpha\prod_\sigma \left({}^\alpha x_\sigma - {}^\alpha a_\rho\right)\,,$$

where we have introduced a new notation for sums and products of quantities in the secondary directions using an upper left index

$${}^\alpha\sum_\rho \equiv \sum_{\rho=1}^{{}^\alpha N}\,.$$

An important difference is that the functions in the secondary directions, i.e. those including the rescaled variables ${}^{\alpha}x_{\rho}$ and ${}^{\alpha}a_{\rho}$ are not defined using squares, but only first powers. Moreover, since we set $b_{\bar{\mu}} = 0$, the on-shell metric functions after the limit are defined as

$$
\begin{aligned}
&\tilde{X}_{\bar{\alpha}} = \lambda\tilde{\mathcal{J}}(\tilde{x}_{\bar{\alpha}}^2)\,, \qquad \tilde{X}_{\tilde{N}} = \lambda\tilde{\mathcal{J}}(\tilde{x}_{\tilde{N}}^2) - 2\tilde{b}_{\tilde{N}}\tilde{x}_{\tilde{N}}\prod_{\bar{\alpha}}\left(\tilde{a}_{\bar{\alpha}}^2 - \tilde{x}_{\tilde{N}}^2\right)^{-{}^{\bar{\alpha}}N}\,,\\
&{}^{\bar{\alpha}}X_{\rho} = 2\,{}^{\bar{\alpha}}x_{\rho}\,{}^{\bar{\alpha}}\mathcal{J}({}^{\bar{\alpha}}x_{\rho})\,,
\end{aligned}
\tag{2.5}
$$

and ${}^{\tilde{N}}X_{\rho}$ does not exist in the Lorentzian case since $\rho \in \emptyset$ in the last block. Barred indices are used in the same way as before, namely, to skip the Lorentzian sector (see Section 1.3), i.e. they go over the ranges

$$
\begin{aligned}
\bar{\alpha}, \bar{\beta}, \ldots &= 1, \ldots, \bar{\tilde{N}}\,,\\
\bar{\tilde{N}} &= \tilde{N} - 1\,.
\end{aligned}
$$

Further details of the limiting procedure applied to individual functions that appear in the metric are provided in Appendix A.2.

The first step in the limiting procedure is to rewrite the quantities using double indices introduced in Section 2.1.1. Let us begin with the metric in the form (1.10) using the unnormalized orthogonal frame. Applying the modified index labeling it reads

$$
\boldsymbol{g} = \sum_{\alpha}\left(\frac{U_{\alpha,0}}{X_{\alpha,0}}\boldsymbol{\epsilon}^{\alpha,0}\boldsymbol{\epsilon}^{\alpha,0} + {}^{\alpha}\!\sum_{\rho}\frac{U_{\alpha,\rho}}{X_{\alpha,\rho}}\boldsymbol{\epsilon}^{\alpha,\rho}\boldsymbol{\epsilon}^{\alpha,\rho} + \frac{X_{\alpha,0}}{U_{\alpha,0}}\hat{\boldsymbol{\epsilon}}^{\alpha,0}\hat{\boldsymbol{\epsilon}}^{\alpha,0} + {}^{\alpha}\!\sum_{\rho}\frac{X_{\alpha,\rho}}{U_{\alpha,\rho}}\hat{\boldsymbol{\epsilon}}^{\alpha,\rho}\hat{\boldsymbol{\epsilon}}^{\alpha,\rho}\right)\,,
$$

where we have separated the primary directions from the secondary directions, which are subjected to the limit. The unnormalized frame of 1-forms (1.8) can be written as

$$
\begin{aligned}
\boldsymbol{\epsilon}^{\alpha,0} &= \mathbf{d}x_{\alpha,0}\,, \qquad \hat{\boldsymbol{\epsilon}}^{\alpha,0} = \sum_{\beta}\left[\frac{J_{\alpha,0}(a_{\beta,0}^2)}{\lambda a_{\beta,0}\mathcal{U}_{\beta,0}}\mathbf{d}\phi_{\beta,0} + {}^{\beta}\!\sum_{\sigma}\frac{J_{\alpha,0}(a_{\beta,\sigma}^2)}{\lambda a_{\beta,\sigma}\mathcal{U}_{\beta,\sigma}}\mathbf{d}\phi_{\beta,\sigma}\right]\,,\\
\boldsymbol{\epsilon}^{\alpha,\rho} &= \mathbf{d}x_{\alpha,\rho}\,, \qquad \hat{\boldsymbol{\epsilon}}^{\alpha,\rho} = \sum_{\beta}\left[\frac{J_{\alpha,\rho}(a_{\beta,0}^2)}{\lambda a_{\beta,0}\mathcal{U}_{\beta,0}}\mathbf{d}\phi_{\beta,0} + {}^{\beta}\!\sum_{\sigma}\frac{J_{\alpha,\rho}(a_{\beta,\sigma}^2)}{\lambda a_{\beta,\sigma}\mathcal{U}_{\beta,\sigma}}\mathbf{d}\phi_{\beta,\sigma}\right]\,.
\end{aligned}
$$

Now we can perform the limit by expanding the orthogonal frame in the limiting parameter ε. Applying the parametrization (2.2), it becomes

$$
\begin{aligned}
\boldsymbol{\epsilon}^{\alpha,0} &\approx \tilde{\boldsymbol{\epsilon}}^{\alpha}\,, \qquad \hat{\boldsymbol{\epsilon}}^{\alpha,0} \approx \hat{\tilde{\boldsymbol{\epsilon}}}^{\alpha}\,,\\
\boldsymbol{\epsilon}^{\alpha,\rho} &\approx \varepsilon\,{}^{\alpha}\boldsymbol{\epsilon}^{\rho}\,, \qquad \hat{\boldsymbol{\epsilon}}^{\alpha,\rho} \approx \frac{1}{\varepsilon}\frac{\tilde{J}(\tilde{a}_{\alpha}^2)}{2\lambda\tilde{a}_{\alpha}^2\tilde{\mathcal{U}}_{\alpha}}\,{}^{\alpha}\hat{\boldsymbol{\epsilon}}^{\rho}\,,
\end{aligned}
$$

where $\approx$ denotes equality in the leading-order terms in ε. The 1-forms defined using the rescaled quantities on the right-hand sides of the equations read

$$
\begin{aligned}
&\tilde{\boldsymbol{\epsilon}}^{\alpha} = \mathbf{d}\tilde{x}_{\alpha}\,, \qquad \hat{\tilde{\boldsymbol{\epsilon}}}^{\alpha} = \sum_{\beta}\frac{\tilde{J}_{\alpha}(\tilde{a}_{\beta}^2)}{\lambda\tilde{a}_{\beta}\tilde{\mathcal{U}}_{\beta}}\tilde{\boldsymbol{\Phi}}^{\beta} = \sum_{r}\tilde{A}_{\alpha}^{(r)}\tilde{\boldsymbol{\Psi}}^{r}\,,\\
&{}^{\alpha}\boldsymbol{\epsilon}^{\rho} = \mathbf{d}\,{}^{\alpha}x_{\rho}\,, \qquad {}^{\alpha}\hat{\boldsymbol{\epsilon}}^{\rho} = \frac{{}^{\alpha}J_{\rho}(0)}{{}^{\alpha}\mathcal{J}(0)}\mathbf{d}\tilde{\phi}_{\alpha} - {}^{\alpha}\!\sum_{\sigma}\frac{{}^{\alpha}J_{\rho}({}^{\alpha}a_{\sigma})}{{}^{\alpha}a_{\sigma}\,{}^{\alpha}\mathcal{U}_{\sigma}}\mathbf{d}\,{}^{\alpha}\phi_{\sigma}\,.
\end{aligned}
\tag{2.6}
$$

We have written $\hat{\tilde{\boldsymbol{\epsilon}}}^\alpha$ so that they have a similar form to $\hat{\boldsymbol{\epsilon}}^\mu$ before the limit in Eq. (1.8), only instead of simple gradients $\mathbf{d}\phi_\mu$ we had to introduce 1-forms $\tilde{\boldsymbol{\Phi}}^\alpha$ given by

$$\tilde{\boldsymbol{\Phi}}^\alpha = \frac{{}^\alpha J(0)}{{}^\alpha\mathcal{J}(0)}\mathbf{d}\tilde{\phi}_\alpha - {}^\alpha\!\sum_\rho \frac{{}^\alpha J({}^\alpha a_\rho)}{{}^\alpha a_\rho {}^\alpha\mathcal{U}_\rho}\mathbf{d}{}^\alpha\phi_\rho\,, \tag{2.7}$$

and instead of $\mathbf{d}\psi_k$ we have used $\tilde{\boldsymbol{\Psi}}^r$, which are related to $\tilde{\boldsymbol{\Phi}}^\alpha$ as

$$\tilde{\boldsymbol{\Phi}}^\alpha = \lambda\tilde{a}_\alpha \sum_r \tilde{\mathcal{A}}^{(r)}_\alpha \tilde{\boldsymbol{\Psi}}^r\,, \qquad \tilde{\boldsymbol{\Psi}}^r = \sum_\alpha \frac{(-\tilde{a}_\alpha^2)^{\tilde{N}-1-r}}{\lambda\tilde{a}_\alpha\tilde{\mathcal{U}}_\alpha}\tilde{\boldsymbol{\Phi}}^\alpha\,.$$

The index r goes over the range

$$r = 0,\dots,\tilde{N}-1\,, \tag{2.8}$$

and the functions $\tilde{A}^{(r)}_\alpha$ and $\tilde{\mathcal{A}}^{(r)}_\alpha$ are given by (A.13). Notice that the relations between $\tilde{\boldsymbol{\Phi}}^\alpha$ and $\tilde{\boldsymbol{\Psi}}^r$ are analogous to the transformation formulae (1.5) between the two types of angular coordinates ϕ_μ and ψ_k before the limit. Moreover, in Section 3.1 we will show that $\tilde{\boldsymbol{\Phi}}^\alpha$ are Kähler potentials.

Using the same limiting procedure for the dual unnormalized orthogonal frame of vectors (1.9), it becomes

$$\begin{aligned}
\boldsymbol{\epsilon}_{\alpha,0} &\approx \tilde{\boldsymbol{\epsilon}}_\alpha\,, & \hat{\boldsymbol{\epsilon}}_{\alpha,0} &\approx \hat{\tilde{\boldsymbol{\epsilon}}}_\alpha\,,\\
\boldsymbol{\epsilon}_{\alpha,\rho} &\approx \frac{1}{\varepsilon}{}^\alpha\boldsymbol{\epsilon}_\rho\,, & \hat{\boldsymbol{\epsilon}}_{\alpha,\rho} &\approx \varepsilon\frac{2\lambda\tilde{a}_\alpha^2\tilde{\mathcal{U}}_\alpha}{\tilde{J}(\tilde{a}_\alpha^2)}{}^\alpha\hat{\boldsymbol{\epsilon}}_\rho\,,
\end{aligned}$$

where the frame vectors defined in terms of the rescaled quantities after the limit are

$$\begin{aligned}
\tilde{\boldsymbol{\epsilon}}_\alpha &= \frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\tilde{x}_\alpha}\,, & \hat{\tilde{\boldsymbol{\epsilon}}}_\alpha &= \sum_\beta \frac{\tilde{\mathcal{J}}_\beta(\tilde{x}_\alpha^2)}{\tilde{U}_\alpha}\tilde{\boldsymbol{\Phi}}_\beta\,,\\
{}^\alpha\boldsymbol{\epsilon}_\rho &= \frac{\boldsymbol{\partial}}{\boldsymbol{\partial}{}^\alpha x_\rho}\,, & {}^\alpha\hat{\boldsymbol{\epsilon}}_\rho &= \frac{{}^\alpha\mathcal{J}({}^\alpha x_\rho)}{{}^\alpha U_\rho}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\tilde{\phi}_\alpha} - {}^\alpha\!\sum_\sigma \frac{{}^\alpha x_\rho {}^\alpha\mathcal{J}_\sigma({}^\alpha x_\rho)}{{}^\alpha U_\rho}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}{}^\alpha\phi_\sigma}\,,
\end{aligned} \tag{2.9}$$

and the vectors $\tilde{\boldsymbol{\Phi}}_\alpha$ read

$$\tilde{\boldsymbol{\Phi}}_\alpha = \lambda\tilde{a}_\alpha\left(\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\tilde{\phi}_\alpha} + {}^\alpha\!\sum_\rho \frac{\boldsymbol{\partial}}{\boldsymbol{\partial}{}^\alpha\phi_\rho}\right)\,. \tag{2.10}$$

It turns out that they are Killing vectors, as will be shown in Section 3.2. Notice that $\hat{\tilde{\boldsymbol{\epsilon}}}_\alpha$ have a similar form to $\hat{\boldsymbol{\epsilon}}_\mu$ in (1.9).

The orthogonal frame after the limit separates into two sets: the *primary frame directions* $\{\tilde{\boldsymbol{\epsilon}}^\alpha, \hat{\tilde{\boldsymbol{\epsilon}}}^\alpha\}$ and the *secondary frame directions* $\{{}^\alpha\boldsymbol{\epsilon}^\rho, {}^\alpha\hat{\boldsymbol{\epsilon}}^\rho\}$[2]. This separation is valid only in the sense of tangent spaces since these directions do not correspond directly to the primary and the secondary coordinate directions

[2]The index notation in the secondary directions is as follows. The left index (indicating which block a direction belongs to) is always placed at the top, whereas the position of the right index (distinguishing between the directions inside the block) reveals in a standard manner whether the concerned object is a form or a vector.

— the hatted 1-forms $\hat{\tilde{\boldsymbol{\epsilon}}}^{\alpha}$ and ${}^{\alpha}\hat{\boldsymbol{\epsilon}}^{\rho}$ contain angular coordinates in both primary and secondary coordinate directions. Moreover, primary and secondary frame directions are not integrable distributions of subspaces in the tangent spaces.

The splitting is respected by the duality relations between the frame of vectors and the frame of 1-forms

$$\begin{aligned} \tilde{\boldsymbol{\epsilon}}_{\alpha} \cdot \tilde{\boldsymbol{\epsilon}}^{\beta} &= \delta_{\alpha\beta} \,, & \hat{\tilde{\boldsymbol{\epsilon}}}_{\alpha} \cdot \hat{\tilde{\boldsymbol{\epsilon}}}^{\beta} &= \delta_{\alpha\beta} \,, \\ {}^{\alpha}\boldsymbol{\epsilon}_{\rho} \cdot {}^{\beta}\boldsymbol{\epsilon}^{\sigma} &= \delta_{\alpha\beta}\delta_{\rho\sigma} \,, & {}^{\alpha}\hat{\boldsymbol{\epsilon}}_{\rho} \cdot {}^{\beta}\hat{\boldsymbol{\epsilon}}^{\sigma} &= \delta_{\alpha\beta}\delta_{\rho\sigma} \,, \end{aligned} \tag{2.11}$$

with all the other products being zero. The frames of the primary directions $\{\tilde{\boldsymbol{\epsilon}}^{\alpha}, \hat{\tilde{\boldsymbol{\epsilon}}}^{\alpha}\}$ and in the individual blocks of the secondary directions $\{{}^{\alpha}\boldsymbol{\epsilon}^{\rho}, {}^{\alpha}\hat{\boldsymbol{\epsilon}}^{\rho}\}$ become independent orthogonal frames.

The metric after the limit obtains the form

$$\boldsymbol{g} \;\approx\; \tilde{\boldsymbol{g}} \;-\; \sum_{\alpha} \frac{\tilde{J}(\tilde{a}_{\alpha}^{2})}{2\lambda\tilde{a}_{\alpha}^{2}\tilde{\mathcal{U}}_{\alpha}}\, {}^{\alpha}\boldsymbol{g} \,, \tag{2.12}$$

where

$$\tilde{\boldsymbol{g}} = \sum_{\alpha} \left(\frac{\tilde{U}_{\alpha}}{\tilde{X}_{\alpha}} \tilde{\boldsymbol{\epsilon}}^{\alpha}\tilde{\boldsymbol{\epsilon}}^{\alpha} + \frac{\tilde{X}_{\alpha}}{\tilde{U}_{\alpha}} \hat{\tilde{\boldsymbol{\epsilon}}}^{\alpha}\hat{\tilde{\boldsymbol{\epsilon}}}^{\alpha} \right) \,, \tag{2.13}$$

$${}^{\bar{\alpha}}\boldsymbol{g} = {}^{\bar{\alpha}}\!\sum_{\rho} \left(\frac{{}^{\bar{\alpha}}U_{\rho}}{{}^{\bar{\alpha}}X_{\rho}}\, {}^{\bar{\alpha}}\boldsymbol{\epsilon}^{\rho}\, {}^{\bar{\alpha}}\boldsymbol{\epsilon}^{\rho} + \frac{{}^{\bar{\alpha}}X_{\rho}}{{}^{\bar{\alpha}}U_{\rho}}\, {}^{\bar{\alpha}}\hat{\boldsymbol{\epsilon}}^{\rho}\, {}^{\bar{\alpha}}\hat{\boldsymbol{\epsilon}}^{\rho} \right) \,, \qquad {}^{\tilde{N}}\boldsymbol{g} = 0 \,. \tag{2.14}$$

The last equality holds because in the Lorentzian case we have reserved the block $\alpha = \tilde{N}$ to be potentially Wick-rotated to the temporal and radial coordinates, and thus did not subject it to the limit.

The inverse metric in its limit form can be written as

$$\boldsymbol{g}^{-1} \;\approx\; \tilde{\boldsymbol{g}}^{-1} \;-\; \sum_{\alpha} \frac{2\lambda\tilde{a}_{\alpha}^{2}\tilde{\mathcal{U}}_{\alpha}}{\tilde{J}(\tilde{a}_{\alpha}^{2})}\, {}^{\alpha}\boldsymbol{g}^{-1} \,, \tag{2.15}$$

with

$$\tilde{\boldsymbol{g}}^{-1} = \sum_{\alpha} \left(\frac{\tilde{X}_{\alpha}}{\tilde{U}_{\alpha}} \tilde{\boldsymbol{\epsilon}}_{\alpha}\tilde{\boldsymbol{\epsilon}}_{\alpha} + \frac{\tilde{U}_{\alpha}}{\tilde{X}_{\alpha}} \hat{\tilde{\boldsymbol{\epsilon}}}_{\alpha}\hat{\tilde{\boldsymbol{\epsilon}}}_{\alpha} \right) \,, \tag{2.16}$$

$${}^{\bar{\alpha}}\boldsymbol{g}^{-1} = {}^{\bar{\alpha}}\!\sum_{\rho} \left(\frac{{}^{\bar{\alpha}}X_{\rho}}{{}^{\bar{\alpha}}U_{\rho}}\, {}^{\bar{\alpha}}\boldsymbol{\epsilon}_{\rho}{}^{\bar{\alpha}}\boldsymbol{\epsilon}_{\rho} + \frac{{}^{\bar{\alpha}}U_{\rho}}{{}^{\bar{\alpha}}X_{\rho}}\, {}^{\bar{\alpha}}\hat{\boldsymbol{\epsilon}}_{\rho}{}^{\bar{\alpha}}\hat{\boldsymbol{\epsilon}}_{\rho} \right) \,, \qquad {}^{\tilde{N}}\boldsymbol{g}^{-1} = 0 \,. \tag{2.17}$$

Note, that the tensors (2.16) and (2.17) are individually inverse to the metrics (2.13) and (2.14) on the respective subspaces spanned on the primary and the secondary frame directions, as can be seen using the duality relations (2.11). Combined together, it gives that $\boldsymbol{g}^{-1}$ is indeed the inverse of $\boldsymbol{g}$.

2.2.2 Relation to generalized Kerr–NUT–(A)dS spacetimes

It turns out that the metric (2.12) is in the form that represents a special case of a more general metric described by Houri et al. in [53]. In this paper, the authors study the *generalized Kerr–NUT–(A)dS metric*, which possesses the principal tensor $\boldsymbol{h}$ with both non-constant and constant eigenvalues, thus the tensor is not necessarily non-degenerate.

Let us remember that in our case the principal tensor before the limit has non-constant and functionally independent eigenvalues x_μ [22] (see Section 1.4.1). However, after employing the limiting procedure, some of these eigenvalues become constant. Namely, all the eigenvalues $x_{\alpha,\rho}$ from the secondary blocks degenerate into the respective constant values $\tilde{a}_\alpha$ after the limit, while $\tilde{x}_\alpha$ in the primary directions remain non-constant. Therefore, the geometry we have obtained by applying the equal-spin limiting procedure is indeed a subcase of the results published in [53]. This also confirms the results of Oota and Yasui [54].

The authors of [53] write the metric in the form

$$\boldsymbol{g} = \sum_{\alpha=1}^{n} \frac{\mathbf{d}x_\alpha^2}{P_\alpha(x)} + \sum_{\alpha=1}^{n} P_\alpha(x) \left[\sum_{r=0}^{n-1} \sigma_r(\hat{x}_\alpha)\boldsymbol{\theta}_r\right]^2 + \sum_{i=1}^{n'} \prod_{\alpha=1}^{n} \left(x_\alpha^2 - \xi_i^2\right) \boldsymbol{g}^{(i)} + \sigma_n \boldsymbol{g}^{(0)} \,, \tag{2.18}$$

where the functions $P_\alpha(x)$ are defined as

$$P_\alpha(x) = \frac{X_\alpha}{x_\alpha^K \prod_{i=1}^{n'} \left(x_\alpha^2 - \xi_i^2\right)^{m_i} U_\alpha} \,, \qquad U_\alpha = \prod_{\substack{\beta=1 \\ \beta\neq\alpha}}^{n} \left(x_\alpha^2 - x_\beta^2\right) \,,$$

and the functions $X_\alpha = X_\alpha(x_\alpha)$ are arbitrary. The number of non-constant eigenvalues x_α of the principal tensor is n, the number of constant non-zero eigenvalues ξ_i is n' and the number of zero eigenvalues is K. Each ξ_i has the multiplicity m_i and $\boldsymbol{g}^{(i)}$ are the Kähler metrics on $2m_i$-dimensional Kähler manifolds, whereas $\boldsymbol{g}^{(0)}$ is in general any metric on a K-dimensional manifold.

In our case, the number of non-constant and distinct constant non-zero eigenvalues is the same and it corresponds to $\tilde{N}$ in our notation. Moreover, since we have made the assumption that the rotational parameters are non-zero, $K = 0$ in this case, which implies that $\boldsymbol{g}^{(0)} \equiv 0$. The equivalence between the symbols used in [53] and our notation can be thus summarized as follows

$$\begin{gathered} n = n' \leftrightarrow \tilde{N} \,, \qquad m_i \leftrightarrow {}^{\alpha}\!N \,, \qquad x_\alpha \leftrightarrow \tilde{x}_\alpha \,, \qquad \xi_i \leftrightarrow \tilde{a}_\alpha \,, \\ \sigma_r(\hat{x}_\alpha) \leftrightarrow \tilde{A}_\alpha^{(r)} \,, \qquad P_\alpha(x) \leftrightarrow \frac{\tilde{X}_\alpha}{\tilde{U}_\alpha} \,, \qquad U_\alpha \leftrightarrow (-1)^{\tilde{N}-1}\tilde{U}_\alpha \,, \\ \boldsymbol{\theta}_r \leftrightarrow \tilde{\boldsymbol{\Psi}}^r \,, \qquad \boldsymbol{g}^{(i)} \leftrightarrow -\frac{{}^{\alpha}\boldsymbol{g}}{2\lambda\tilde{a}_\alpha^2\tilde{\mathcal{U}}_\alpha} \,. \end{gathered} \tag{2.19}$$

Since the form of our metrics ${}^{\alpha}\boldsymbol{g}$ in the secondary blocks differs from the authors' $\boldsymbol{g}^{(i)}$ only by a constant factor, we will refer to ${}^{\alpha}\boldsymbol{g}$ as *Kähler metrics*. Furthermore, the metric $\tilde{\boldsymbol{g}}$ defined in Eq. (2.13) has the form of the Kerr–NUT–(A)dS metric analogous to the metric of the entire spacetime (1.10) but for the primary directions only. Thus we shall refer to it as the *Kerr–NUT–(A)dS part*.

However, we should emphasize that ${}^{\alpha}\boldsymbol{g}$ represent the Kähler metrics only formally as there seems to be no decomposition of the original Kerr–NUT–(A)dS manifold into a direct product of the Kerr–NUT–(A)dS part and the Kähler manifolds. As was already mentioned, the primary and the secondary directions form only subspaces of tangent spaces. Metrics ${}^{\alpha}\boldsymbol{g}$ act on these secondary subspaces of the tangent space, but these subspaces are not integrable to form independent Kähler manifolds.

2.2.3 Homogeneous sphere

We have already set all the NUT charges to zero at the beginning of this chapter. Let us simplify the situation even more by setting the mass parameter to zero as well, and perform the equal-spin limit for the maximally symmetric case of a homogeneous sphere geometry.

When applying the limiting procedure to the Jacobi transformation (1.13), it turns out that we can define analogous transformations for the primary and the secondary directions separately. Indeed, the limit form of (1.13) reads

$$\lambda\rho^2_{\alpha,0} \approx \lambda\tilde{\rho}^2_\alpha\ {}^\alpha\!\rho^2_0\,, \qquad \lambda\rho^2_{\alpha,\sigma} \approx \lambda\tilde{\rho}^2_\alpha\ {}^\alpha\!\rho^2_\sigma\,, \qquad \lambda\rho^2_0 \approx \lambda\tilde{\rho}^2_0\,, \tag{2.20}$$

where we have denoted

$$\begin{aligned} \lambda\tilde{\rho}^2_\alpha &= \frac{\tilde{J}(\tilde{a}^2_\alpha)}{-\tilde{a}^2_\alpha\tilde{\mathcal{U}}_\alpha}\,, \qquad & \lambda\tilde{\rho}^2_0 &= \frac{\tilde{J}(0)}{\tilde{\mathcal{J}}(0)} = \frac{\tilde{A}^{(\tilde{N})}}{\tilde{\mathcal{A}}^{(\tilde{N})}}\,, \\ {}^\alpha\!\rho^2_\sigma &= \frac{{}^\alpha\! J({}^\alpha\! a_\sigma)}{-{}^\alpha\! a_\sigma {}^\alpha\mathcal{U}_\sigma}\,, \qquad & {}^\alpha\!\rho^2_0 &= \frac{{}^\alpha\! J(0)}{{}^\alpha\mathcal{J}(0)} = \frac{{}^\alpha\! A^{({}^\alpha\! N)}}{{}^\alpha\mathcal{A}^{({}^\alpha\! N)}}\,. \end{aligned} \tag{2.21}$$

The advantage of defining the two sets of multi-cylindrical coordinates in this way is that a constraint similar to (1.14) holds for both of them independently, i.e. they satisfy

$$\sum_{\alpha=0}^{\tilde{N}} \tilde{\rho}^2_\alpha = \frac{1}{\lambda}\,, \qquad \sum_{\sigma=0}^{{}^\alpha\! N} {}^\alpha\!\rho^2_\sigma = 1\,. \tag{2.22}$$

The limit form of the metric in these coordinates can be achieved by employing one of two methods. Using the first method, we perform the coordinate transformation with all the quantities already in their limit form — following the same steps as in Section 1.2.1. First, we realize that setting $M = 0$ implies

$$\tilde{X}_\alpha = \lambda\tilde{\mathcal{J}}(\tilde{x}^2_\alpha)\,,$$

and apply orthogonality relations (tilded version of (A.12)) to simplify the angular sector of the Kerr–NUT–(A)dS part (2.13), obtaining

$$\tilde{\boldsymbol{g}} = \sum_\alpha \left[\frac{\tilde{U}_\alpha}{\lambda\tilde{\mathcal{J}}(\tilde{x}^2_\alpha)}\mathbf{d}\tilde{x}^2_\alpha - \frac{\tilde{J}(\tilde{a}^2_\alpha)}{\lambda\tilde{a}^2_\alpha\tilde{\mathcal{U}}_\alpha}\left(\tilde{\boldsymbol{\Phi}}^\alpha\right)^2\right]\,. \tag{2.23}$$

Subsequently, we apply the Jacobi transformation (2.21), which is defined after the limit, to the metric (2.12). This leads to

$$\boldsymbol{g} \approx \mathbf{d}\tilde{\rho}^2_0 + \sum_\alpha \left(\mathbf{d}\tilde{\rho}^2_\alpha + \tilde{\rho}^2_\alpha\ {}^\alpha\boldsymbol{g}_{\text{Eucl}}\right)\,. \tag{2.24}$$

where ${}^\alpha\boldsymbol{g}_{\text{Eucl}}$ denote $(2\,{}^\alpha\! N{+}2)$-dimensional Euclidean metrics in the multi-polar coordinates

$${}^\alpha\boldsymbol{g}_{\text{Eucl}} = \mathbf{d}{}^\alpha\!\rho^2_0 + {}^\alpha\!\rho^2_0\mathbf{d}\tilde{\phi}^2_\alpha + {}^\alpha\!\sum_\sigma \left(\mathbf{d}{}^\alpha\!\rho^2_\sigma + {}^\alpha\!\rho^2_\sigma\mathbf{d}{}^\alpha\!\phi^2_\sigma\right)\,. \tag{2.25}$$

Although the Kähler metrics ${}^\alpha\boldsymbol{g}$ defined in (2.14) probably cannot be simplified using orthogonality relations in the same way as the Kerr–NUT–(A)dS part, they can be transformed into the multi-cylindrical coordinates directly. The proof and further details of this coordinate transformation can be found in Appendix B.1.

Following the second method, we start with the metric (1.15) before the limit, but already transformed into the multi-cylindrical coordinates, and apply the limit of these coordinates (2.20) to it. This gives us an expression identical to (2.24) above.

As one can see, the secondary blocks become spherically symmetric after the limit. In particular, they can be viewed as $(2\,{}^{\alpha}\!N{+}1)$-dimensional spheres, given by the constraints (2.22), embedded in $(2\,{}^{\alpha}\!N{+}2)$-dimensional flat spaces, described by the metrics ${}^{\alpha}\boldsymbol{g}_{\text{Eucl}}$ (2.25). Moreover, in the full spacetime metric (2.24), each sphere is coupled solely to the coordinate $\tilde{\rho}_\alpha$ in the corresponding primary direction. Therefore, the full metric after the limit has a similar form to the metric (1.15) before the limit, only simple 2-forms $\mathbf{d}\phi_\mu^2$ have been replaced with the spheres ${}^{\alpha}\boldsymbol{g}_{\text{Eucl}}$.

Notice that the metric in the multi-cylindrical coordinates no longer clearly separates into the Kerr–NUT–(A)dS part $\tilde{\boldsymbol{g}}$ and the Kähler metrics ${}^{\alpha}\boldsymbol{g}$, but they are rather combined together, as can be seen from the proof in Appendix B.1.

2.2.4 Black hole

When focusing on the physical interpretation of the limited metric, it is convenient to use the Myers–Perry coordinates defined in Section 1.3.2. Further details of the following results are provided in Appendix B.2. We shall proceed in a similar manner to the homogeneous sphere case described in the previous section. First, let us apply the limit to the Jacobi transformation (1.24), which gives

$$\mu_{\bar{\alpha},0}^2 \approx \tilde{\mu}_{\bar{\alpha}}^2 \, {}^{\bar{\alpha}}\mu_0^2 \,, \qquad \mu_{\bar{\alpha},\rho}^2 \approx \tilde{\mu}_{\bar{\alpha}}^2 \, {}^{\bar{\alpha}}\mu_\rho^2 \,, \qquad \mu_0^2 \approx \tilde{\mu}_0^2 \,, \tag{2.26}$$

having denoted

$$\begin{aligned} \tilde{\mu}_{\bar{\alpha}}^2 &= \frac{\tilde{\bar{J}}(\tilde{a}_{\bar{\alpha}}^2)}{-\tilde{a}_{\bar{\alpha}}^2 \tilde{\bar{\mathcal{U}}}_{\bar{\alpha}}} \,, \qquad & \tilde{\mu}_0^2 &= \frac{\tilde{\bar{J}}(0)}{\tilde{\bar{\mathcal{J}}}(0)} = \frac{\tilde{\bar{A}}^{(\tilde{\bar{N}})}}{\tilde{\bar{\mathcal{A}}}^{(\tilde{\bar{N}})}} \,, \\ {}^{\bar{\alpha}}\mu_\rho^2 &= \frac{{}^{\bar{\alpha}}J({}^{\bar{\alpha}}a_\rho)}{-{}^{\bar{\alpha}}a_\rho {}^{\bar{\alpha}}\mathcal{U}_\rho} \,, \qquad & {}^{\bar{\alpha}}\mu_0^2 &= \frac{{}^{\bar{\alpha}}J(0)}{{}^{\bar{\alpha}}\mathcal{J}(0)} = \frac{{}^{\bar{\alpha}}A^{({}^{\bar{\alpha}}N)}}{{}^{\bar{\alpha}}\mathcal{A}^{({}^{\bar{\alpha}}N)}} \,. \end{aligned} \tag{2.27}$$

Functions decorated with both a bar and a tilde are defined analogously to barred functions in Section 1.3, i.e. without $\tilde{x}_{\tilde{N}}$ and $\tilde{a}_{\tilde{N}}$. As before, the two sets of Myers–Perry coordinates satisfy constraints similar to (1.25) independently

$$\sum_{\bar{\alpha}=0}^{\tilde{\bar{N}}} \tilde{\mu}_{\bar{\alpha}}^2 = 1 \,, \qquad \sum_{\rho=0}^{{}^{\bar{\alpha}}N} {}^{\bar{\alpha}}\mu_\rho^2 = 1 \,. \tag{2.28}$$

Moreover, they are related to the multi-cylindrical coordinates analogously to Eq. (1.26)

$$\begin{aligned} \lambda \tilde{\rho}_{\bar{\alpha}}^2 &= \frac{r^2 + \tilde{a}_{\bar{\alpha}}^2}{\tilde{a}_{\bar{\alpha}}^2 - \tilde{a}_{\tilde{N}}^2} \tilde{\mu}_{\bar{\alpha}}^2 \,, \\ 1 - \lambda \tilde{R}^2 \equiv \lambda \tilde{\rho}_{\tilde{N}}^2 &= \left(1 - \lambda r^2\right) \left(\tilde{\mu}_0^2 + \sum_{\bar{\alpha}} \frac{\tilde{\mu}_{\bar{\alpha}}^2}{1 + \lambda \tilde{a}_{\bar{\alpha}}^2} \right) . \end{aligned} \tag{2.29}$$

Let us first follow the same steps as in Section 1.3.2, only with all the quantities already in their limit form. To begin with, we split the sums into temporal/radial and angular parts, employ the Wick rotation (2.3) and the gauge fixing (2.4), and use orthogonality relations (tilded version of (A.12)) for the Kerr–NUT–(A)dS part (2.13), which becomes

$$\begin{aligned}\tilde{\boldsymbol{g}} =&\sum_{\bar{\alpha}} \frac{\tilde{U}_{\bar{\alpha}}}{\lambda \tilde{\mathcal{J}}(\tilde{x}_{\bar{\alpha}}^2)}\mathbf{d}\tilde{x}_{\bar{\alpha}}^2 + \frac{\tilde{\Sigma}}{\tilde{\Delta}_r}\mathbf{d}r^2 - \sum_{\bar{\alpha}} \frac{\tilde{J}(\tilde{a}_{\bar{\alpha}}^2)}{\lambda \tilde{a}_{\bar{\alpha}}^2 \tilde{\mathcal{U}}_{\bar{\alpha}}}\left(\tilde{\boldsymbol{\Phi}}^{\bar{\alpha}}\right)^2 - \lambda \frac{\tilde{J}(\tilde{a}_{\tilde{N}}^2)}{\tilde{\mathcal{U}}_{\tilde{N}}}\mathbf{d}t^2 \\ &+ \frac{2Mr}{\tilde{\Sigma}}\prod_{\bar{\beta}}\left(r^2+\tilde{a}_{\bar{\beta}}^2\right)^{-\bar{\beta}N}\left[\sum_{\bar{\alpha}}\frac{\tilde{J}_{\tilde{N}}(\tilde{a}_{\bar{\alpha}}^2)}{\lambda \tilde{a}_{\bar{\alpha}}\tilde{\mathcal{U}}_{\bar{\alpha}}}\tilde{\boldsymbol{\Phi}}^{\bar{\alpha}} + \frac{\tilde{J}_{\tilde{N}}(\tilde{a}_{\tilde{N}}^2)}{\tilde{\mathcal{U}}_{\tilde{N}}}\mathbf{d}t\right]^2 .\end{aligned} \tag{2.30}$$

Applying (2.27) to the metric (2.12) then leads to

$$\begin{aligned}\boldsymbol{g} \approx &-\left(1-\lambda\tilde{R}^2\right)\mathbf{d}t^2 + \frac{2Mr}{\tilde{\Sigma}}\prod_{\bar{\beta}}\left(r^2+\tilde{a}_{\bar{\beta}}^2\right)^{-\bar{\beta}N}\left[\mathbf{d}t + \sum_{\bar{\alpha}}\frac{\tilde{a}_{\bar{\alpha}}\tilde{\mu}_{\bar{\alpha}}^2}{1+\lambda\tilde{a}_{\bar{\alpha}}^2}\left(\tilde{\boldsymbol{\Phi}}^{\bar{\alpha}} - \lambda\tilde{a}_{\bar{\alpha}}\mathbf{d}t\right)\right]^2 \\ &+ \frac{\tilde{\Sigma}}{\tilde{\Delta}_r}\mathbf{d}r^2 + r^2\mathbf{d}\tilde{\mu}_0^2 + \sum_{\bar{\alpha}}\frac{r^2+\tilde{a}_{\bar{\alpha}}^2}{1+\lambda\tilde{a}_{\bar{\alpha}}^2}\left(\mathbf{d}\tilde{\mu}_{\bar{\alpha}}^2 + \tilde{\mu}_{\bar{\alpha}}^2\,{}^{\bar{\alpha}}\boldsymbol{g}_{\text{Eucl}}\right) \\ &+ \frac{\lambda}{1-\lambda\tilde{R}^2}\left(r^2\tilde{\mu}_0\mathbf{d}\tilde{\mu}_0 + \sum_{\bar{\alpha}}\frac{r^2+\tilde{a}_{\bar{\alpha}}^2}{1+\lambda\tilde{a}_{\bar{\alpha}}^2}\tilde{\mu}_{\bar{\alpha}}\mathbf{d}\tilde{\mu}_{\bar{\alpha}}\right)^2 ,\end{aligned} \tag{2.31}$$

where ${}^{\bar{\alpha}}\boldsymbol{g}_{\text{Eucl}}$ denote $(2\,{}^{\bar{\alpha}}N{+}2)$-dimensional Euclidean metrics, this time in the following multi-polar coordinates

$${}^{\bar{\alpha}}\boldsymbol{g}_{\text{Eucl}} = \mathbf{d}{}^{\bar{\alpha}}\mu_0^2 + {}^{\bar{\alpha}}\mu_0^2\mathbf{d}\tilde{\phi}_{\bar{\alpha}}^2 + {}^{\bar{\alpha}}\sum_{\rho}\left(\mathbf{d}{}^{\bar{\alpha}}\mu_\rho^2 + {}^{\bar{\alpha}}\mu_\rho^2\mathbf{d}{}^{\bar{\alpha}}\phi_\rho^2\right) , \tag{2.32}$$

and the metric functions are given by

$$\begin{aligned}\tilde{\Delta}_r &= -\tilde{X}_{\tilde{N}} = \left(1-\lambda r^2\right)\prod_{\bar{\alpha}}\left(r^2+\tilde{a}_{\bar{\alpha}}^2\right) - 2Mr\prod_{\bar{\alpha}}\left(r^2+\tilde{a}_{\bar{\alpha}}^2\right)^{-\bar{\alpha}N} , \\ \tilde{\Sigma} &= \tilde{U}_{\tilde{N}} = \left(\tilde{\mu}_0^2 + \sum_{\bar{\alpha}}\frac{r^2\tilde{\mu}_{\bar{\alpha}}^2}{r^2+\tilde{a}_{\bar{\alpha}}^2}\right)\prod_{\bar{\beta}}\left(r^2+\tilde{a}_{\bar{\beta}}^2\right) .\end{aligned} \tag{2.33}$$

Notice that these expressions are similar to Eq. (1.28) — except for the second term of $\tilde{\Delta}_r$, which is multiplied by an additional factor emerging from applying the limit on quantities in the secondary directions. The 1-forms $\tilde{\boldsymbol{\Phi}}^{\bar{\alpha}}$ in these coordinates read

$$\tilde{\boldsymbol{\Phi}}^{\bar{\alpha}} = {}^{\bar{\alpha}}\mu_0^2\mathbf{d}\tilde{\phi}_{\bar{\alpha}} + {}^{\bar{\alpha}}\sum_{\rho}{}^{\bar{\alpha}}\mu_\rho^2\mathbf{d}{}^{\bar{\alpha}}\phi_\rho .$$

Alternatively, starting with the metric (1.27) in the Myers–Perry coordinates before the limit and employing (2.26), we obtain an expression identical to (2.31) above.

Similarly to the homogeneous sphere case, the secondary blocks become spherically symmetric in the black hole case as well. In particular, they can be viewed as

$(2\,{}^{\bar{\alpha}}N{+}1)$-dimensional spheres given by the constraints (2.28), embedded in $(2\,{}^{\bar{\alpha}}N{+}2)$-dimensional flat spaces, described by the metrics ${}^{\bar{\alpha}}\boldsymbol{g}_{\text{Eucl}}$ (2.32). In the full spacetime metric (2.31), each sphere is also coupled solely to the coordinate $\tilde{\mu}_{\bar{\alpha}}$ in the corresponding primary direction. The full metric after the limit thus has a similar form to the metric (1.27) before the limit, but the 2-forms $\mathbf{d}\phi_{\nu}^{2}$ have been replaced with the spheres ${}^{\bar{\alpha}}\boldsymbol{g}_{\text{Eucl}}$. The only other occurrence of the secondary directions is in the 1-forms $\tilde{\boldsymbol{\Phi}}^{\bar{\alpha}}$, which play the role of Kähler potentials as is discussed in Section 3.1.

The black hole bears another similarity to the homogeneous sphere: the metric in the Myers–Perry coordinates cannot be decomposed into the Kerr–NUT–(A)dS part $\tilde{\boldsymbol{g}}$ and the Kähler metrics ${}^{\alpha}\boldsymbol{g}$, unlike the metric in the generalized Boyer–Lindquist coordinates $\{t,\, r,\, \tilde{x}_{\bar{\alpha}},\, \tilde{\phi}_{\bar{\alpha}},\, {}^{\bar{\alpha}}x_{\rho},\, {}^{\bar{\alpha}}\phi_{\rho}\}$.

3. Reconstructing original symmetries

This chapter focuses on explicit and hidden symmetries of the resulting spacetime after the limit $a_{\alpha,\rho} \to a_{\alpha,0}$ has been performed. We reconstruct the original number of Killing vectors and Killing tensors, thus showing that the symmetry group has not been reduced during the limiting procedure. We expect the symmetry group to be enhanced, however, this will be shown explicitly only in the case of six dimensions discussed in Section 4.2 and is yet to be proved in a general dimension.

3.1 Principal tensor

Let us first discuss the limit form of the principal tensor. Applying the limiting procedure as before, the principal tensor (1.30) becomes

$$\boldsymbol{h} \approx \sum_{\alpha} \left[\tilde{x}_\alpha \tilde{\boldsymbol{\epsilon}}^\alpha \wedge \hat{\tilde{\boldsymbol{\epsilon}}}^\alpha + \frac{\tilde{J}(\tilde{a}_\alpha^2)}{2\lambda \tilde{a}_\alpha \tilde{\mathcal{U}}_\alpha} \boldsymbol{\omega}^\alpha \right] , \tag{3.1}$$

where $\boldsymbol{\omega}^\alpha$ are defined as

$$\boldsymbol{\omega}^{\bar{\alpha}} = {}^{\bar{\alpha}}\!\sum_{\rho} {}^{\bar{\alpha}}\boldsymbol{\epsilon}^\rho \wedge {}^{\bar{\alpha}}\hat{\boldsymbol{\epsilon}}^\rho , \qquad \boldsymbol{\omega}^{\tilde{N}} = 0 .$$

Similarly to the metric, we have obtained the principal tensor in the form that represents a special case of the results published in [53]. The authors write the principal tensor in the form

$$\boldsymbol{h} = \sum_{\alpha=1}^{n} x_\alpha \mathbf{d}x_\alpha \wedge \left[\sum_{r=0}^{n-1} \sigma_r(\hat{x}_\alpha) \boldsymbol{\theta}_r \right] + \sum_{i=1}^{n'} \xi_i \prod_{\alpha=1}^{n} \left(x_\alpha^2 - \xi_i^2 \right) \boldsymbol{\omega}^{(i)} ,$$

where $\boldsymbol{\omega}^{(i)}$ are the Kähler forms corresponding to the Kähler metrics $\boldsymbol{g}^{(i)}$ in (2.18). The symbols used by the authors correspond to our notation as in (2.19) and

$$\boldsymbol{\omega}^{(i)} \leftrightarrow \frac{\boldsymbol{\omega}^\alpha}{2\lambda \tilde{a}_\alpha^2 \tilde{\mathcal{U}}_\alpha} .$$

Our tensors $\boldsymbol{\omega}^\alpha$ differ from $\boldsymbol{\omega}^{(i)}$ only by a constant factor. Therefore, $\boldsymbol{\omega}^\alpha$ formally represent the Kähler forms.

According to its general definition, a Kähler form is closed. It can be shown that $\mathbf{d}\boldsymbol{\omega}^\alpha = 0$, thus in our case this requirement is indeed satisfied. Moreover, there exists a Kähler potential, which is in our case represented by $\tilde{\boldsymbol{\Phi}}^\alpha$ defined earlier in (2.7). Namely, the following equality holds

$$\boldsymbol{\omega}^\alpha = \mathbf{d}\tilde{\boldsymbol{\Phi}}^\alpha .$$

3.2 Killing vectors

We shall now perform the limit $a_{\alpha,\rho} \to a_{\alpha,0}$ for Killing vectors. It is shown in Eq. (1.36) that we can define two types of Killing vectors $\boldsymbol{l}_{(k)}$ and $\boldsymbol{s}_{(\mu)}$, associated with two types of angular coordinates ψ_k and ϕ_μ, respectively. As was already mentioned, our limiting procedure is better suited for the latter choice since these coordinates are in one-to-one correspondence with the rotational parameters.

The limit of the Killing vectors $\boldsymbol{s}_{(\mu)}$ is, in the new indexing scheme,

$$\boldsymbol{s}_{(\alpha,0)} \approx \tilde{\boldsymbol{s}}_{(\alpha)} \,, \qquad \boldsymbol{s}_{(\alpha,\rho)} \approx {}^{\alpha}\boldsymbol{s}_{(\rho)} \,,$$

with $\tilde{\boldsymbol{s}}_{(\alpha)}$ and ${}^{\alpha}\boldsymbol{s}_{(\rho)}$ defined similarly to (1.35)

$$\tilde{\boldsymbol{s}}_{(\alpha)} = \lambda \tilde{a}_\alpha \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} \tilde{\phi}_\alpha} \,, \qquad {}^{\alpha}\boldsymbol{s}_{(\rho)} = \lambda \tilde{a}_\alpha \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} {}^{\alpha}\phi_\rho} \,.$$

The Killing vectors $\tilde{\boldsymbol{\Phi}}_\alpha$ introduced in Eq. (2.10) can now be rewritten as

$$\tilde{\boldsymbol{\Phi}}_\alpha = \tilde{\boldsymbol{s}}_{(\alpha)} + {}^{\alpha}\sum_{\rho} {}^{\alpha}\boldsymbol{s}_{(\rho)} \,. \tag{3.2}$$

We have thus obtained $\tilde{N}$ Killing vectors $\tilde{\boldsymbol{s}}_{(\alpha)}$ in the primary directions and $\sum_\alpha {}^{\alpha}N$ Killing vectors ${}^{\alpha}\boldsymbol{s}_{(\rho)}$ in the secondary directions, which gives in total N explicit symmetries — the same number as before the limit.

Although the Killing vectors $\boldsymbol{l}_{(k)}$ are not suitable for the limiting procedure itself, it proves useful to define their equivalents after the limit. Inspired by the first equality in (1.36), we introduce Killing vectors $\tilde{\boldsymbol{l}}_{(r)}$ and ${}^{\alpha}\boldsymbol{l}_{(p)}$ as linear combinations of the coordinate Killing vectors $\tilde{\boldsymbol{s}}_{(\alpha)}$ and ${}^{\alpha}\boldsymbol{s}_{(\rho)}$ in the Kerr–NUT–(A)dS part $\tilde{\boldsymbol{g}}$ and the Kähler parts ${}^{\alpha}\boldsymbol{g}$ of the metric, respectively,

$$\begin{aligned} \tilde{\boldsymbol{l}}_{(r)} &= \sum_{\alpha} \tilde{\mathcal{A}}^{(r)}_{\alpha} \tilde{\boldsymbol{s}}_{(\alpha)} \,, \\ {}^{\alpha}\boldsymbol{l}_{(p)} &= {}^{\alpha}\sum_{\rho} {}^{\alpha}\mathcal{A}^{(p)}_{\rho} \, {}^{\alpha}\boldsymbol{s}_{(\rho)} = {}^{\alpha}\sum_{\rho} {}^{\alpha}A^{(p)}_{\rho} \, {}^{\alpha}\hat{\boldsymbol{\epsilon}}_{\rho} \,. \end{aligned}$$

Here, the index r acquires the values as in (2.8), the index p goes over the ranges

$$p = 0, \ldots, {}^{\alpha}N - 1 \,,$$

and the functions $\tilde{\mathcal{A}}^{(r)}_{\alpha}$, ${}^{\alpha}\mathcal{A}^{(p)}_{\rho}$ and ${}^{\alpha}A^{(p)}_{\rho}$ are as in (A.13). Similarly to (1.33) and (1.37), we also introduce the principal Killing vectors $\tilde{\boldsymbol{\xi}}$ and ${}^{\alpha}\boldsymbol{\xi}$

$$\begin{aligned} \tilde{\boldsymbol{\xi}} &= \tilde{\boldsymbol{l}}_{(0)} = \sum_{\alpha} \tilde{\boldsymbol{s}}_{(\alpha)} \,, \\ {}^{\alpha}\boldsymbol{\xi} &= {}^{\alpha}\boldsymbol{l}_{(0)} = {}^{\alpha}\sum_{\rho} {}^{\alpha}\boldsymbol{s}_{(\rho)} = {}^{\alpha}\sum_{\rho} {}^{\alpha}\hat{\boldsymbol{\epsilon}}_{\rho} \,. \end{aligned} \tag{3.3}$$

Notice, that we have not listed relations between $\tilde{\boldsymbol{l}}_{(r)}$ and $\hat{\tilde{\boldsymbol{\epsilon}}}_\alpha$ similar to the second equality in (1.36). Indeed, such relations do not exist. It turns out that the Killing vectors $\tilde{\boldsymbol{l}}_{(r)}$ and $\tilde{\boldsymbol{s}}_{(\alpha)}$ need to be “improved” in order to fulfill such relations.

To show this, let us approach the limit from a different direction. We perform the limit of the generating function for Killing vectors (1.38), which reads

$$\boldsymbol{l}(\beta) \approx \prod_{\gamma} \left(1 + \beta^2 \tilde{a}_{\gamma}^2\right)^{\gamma N} \tilde{\boldsymbol{L}}(\beta) \,, \tag{3.4}$$

with

$$\tilde{\boldsymbol{L}}(\beta) = \sum_{\alpha} \tilde{\mathcal{A}}_{\alpha}(\beta) \tilde{\boldsymbol{\Phi}}_{\alpha} \,. \tag{3.5}$$

Realizing that the multiplicative prefactor $\prod_{\gamma}(1+\beta^2\tilde{a}_{\gamma}^2)^{\gamma N}$ in (3.4) is constant on the manifold, we can understand $\tilde{\boldsymbol{L}}(\beta)$ also as a generating function for Killing vectors. Namely, it is the generating function for new Killing vectors $\tilde{\boldsymbol{L}}_{(r)}$ (see (1.34))

$$\tilde{\boldsymbol{L}}(\beta) = \sum_{r} \beta^{2r} \tilde{\boldsymbol{L}}_{(r)} \,. \tag{3.6}$$

These new Killing vectors are actually dual to the 1-forms $\tilde{\boldsymbol{\Psi}}^r$ introduced in Eq. (2.6) as they satisfy

$$\tilde{\boldsymbol{L}}_{(r)} \cdot \tilde{\boldsymbol{\Psi}}^s = \delta_{rs} \,.$$

But most importantly, we find $\tilde{\boldsymbol{L}}_{(r)}$ to be fully analogous to (1.36) since they are related to the Killing vectors $\tilde{\boldsymbol{\Phi}}_{\alpha}$ and to the frame vectors $\hat{\tilde{\boldsymbol{\epsilon}}}_{\alpha}$ as

$$\tilde{\boldsymbol{L}}_{(r)} = \sum_{\alpha} \tilde{\mathcal{A}}_{\alpha}^{(r)} \tilde{\boldsymbol{\Phi}}_{\alpha} = \sum_{\alpha} \tilde{A}_{\alpha}^{(r)} \hat{\tilde{\boldsymbol{\epsilon}}}_{\alpha} \,, \tag{3.7}$$

with the functions $\tilde{A}_{\alpha}^{(r)}$ given by (A.13). The principal Killing vector $\tilde{\boldsymbol{\Xi}}$ associated with $\tilde{\boldsymbol{L}}_{(r)}$ reads

$$\tilde{\boldsymbol{\Xi}} = \tilde{\boldsymbol{L}}_{(0)} = \sum_{\alpha} \tilde{\boldsymbol{\Phi}}_{\alpha} = \sum_{\alpha} \hat{\tilde{\boldsymbol{\epsilon}}}_{\alpha} \,. \tag{3.8}$$

Detailed calculations can be found in Appendix B.3.

The generating function $\tilde{\boldsymbol{L}}(\beta)$ thus provides us with the Killing vectors $\tilde{\boldsymbol{L}}_{(r)}$ or $\tilde{\boldsymbol{\Phi}}_{\alpha}$, which "improve" the Killing vectors $\tilde{\boldsymbol{l}}_{(r)}$ and $\tilde{\boldsymbol{s}}_{(\alpha)}$ in such a way that (3.7) holds. Of course, any of these sets of Killing vectors can be used since they carry the same information. Indeed, the "improved" Killing vectors are related to the previously defined Killing vectors as

$$\tilde{\boldsymbol{L}}_{(r)} = \tilde{\boldsymbol{l}}_{(r)} + \sum_{\alpha} \tilde{\mathcal{A}}_{\alpha}^{(r)}\, {}^{\alpha}\boldsymbol{\xi} \,, \tag{3.9}$$

$$\tilde{\boldsymbol{\Phi}}_{\alpha} = \tilde{\boldsymbol{s}}_{(\alpha)} + {}^{\alpha}\boldsymbol{\xi} \,, \tag{3.10}$$

and

$$\tilde{\boldsymbol{\Xi}} = \tilde{\boldsymbol{\xi}} + \sum_{\alpha} {}^{\alpha}\boldsymbol{\xi} \,. \tag{3.11}$$

3.3 Killing tensors

To calculate the limit form of Killing tensors, we shall proceed in a similar manner to Killing vectors and apply the limiting procedure to the Killing tensors $\boldsymbol{r}_{(\mu)}$ (1.41) instead of $\boldsymbol{k}_{(k)}$.

As a preliminary, we consider the limit of $\boldsymbol{\pi}_\mu$. Applying the limit $a_{\alpha,\rho} \to a_{\alpha,0}$ to their definition (1.42), one obtains

$$\boldsymbol{\pi}_{\alpha,0} \approx \tilde{\boldsymbol{\pi}}_\alpha\,, \qquad \boldsymbol{\pi}_{\alpha,\rho} \approx -\frac{2\lambda\tilde{a}_\alpha^2\tilde{\mathcal{U}}_\alpha}{\tilde{J}(\tilde{a}_\alpha^2)}\,{}^\alpha\boldsymbol{\pi}_\rho\,, \tag{3.12}$$

with the tensors $\tilde{\boldsymbol{\pi}}_\alpha$ and ${}^\alpha\boldsymbol{\pi}_\rho$ defined as

$$\begin{aligned}\tilde{\boldsymbol{\pi}}_\alpha &= \frac{\tilde{X}_\alpha}{\tilde{U}_\alpha}\tilde{\boldsymbol{\epsilon}}_\alpha\tilde{\boldsymbol{\epsilon}}_\alpha + \frac{\tilde{U}_\alpha}{\tilde{X}_\alpha}\hat{\tilde{\boldsymbol{\epsilon}}}_\alpha\hat{\tilde{\boldsymbol{\epsilon}}}_\alpha\,,\\ {}^\alpha\boldsymbol{\pi}_\rho &= \frac{{}^\alpha X_\rho}{{}^\alpha U_\rho}\,{}^\alpha\boldsymbol{\epsilon}_\rho{}^\alpha\boldsymbol{\epsilon}_\rho + \frac{{}^\alpha U_\rho}{{}^\alpha X_\rho}\,{}^\alpha\hat{\boldsymbol{\epsilon}}_\rho{}^\alpha\hat{\boldsymbol{\epsilon}}_\rho\,.\end{aligned}$$

It is apparent that the Kerr–NUT–(A)dS part (2.16) and the Kähler part (2.17) of the inverse metric after the limit can be expressed in terms of these tensors as

$$\tilde{\boldsymbol{g}}^{-1} = \sum_\alpha \tilde{\boldsymbol{\pi}}_\alpha\,, \qquad {}^\alpha\boldsymbol{g}^{-1} = {}^\alpha\!\sum_\rho {}^\alpha\boldsymbol{\pi}_\rho\,. \tag{3.13}$$

We also define auxiliary tensors $\tilde{\boldsymbol{k}}_{(r)}$ and $\tilde{\boldsymbol{r}}_{(\alpha)}$ using relations analogous to (1.43),

$$\tilde{\boldsymbol{k}}_{(r)} = \sum_\alpha \tilde{\mathcal{A}}_\alpha^{(r)}\tilde{\boldsymbol{r}}_{(\alpha)} = \sum_\alpha \tilde{A}_\alpha^{(r)}\tilde{\boldsymbol{\pi}}_\alpha\,, \tag{3.14}$$

with the corollary

$$\tilde{\boldsymbol{k}}_{(0)} = \tilde{\boldsymbol{g}}^{-1}\,.$$

The tensors $\tilde{\boldsymbol{r}}_{(\alpha)}$ can be expressed as

$$\tilde{\boldsymbol{r}}_{(\alpha)} = \sum_\beta \frac{\tilde{J}_\beta(\tilde{a}_\alpha^2)}{\tilde{\mathcal{U}}_\alpha}\tilde{\boldsymbol{\pi}}_\beta\,,$$

which is analogous to (1.41) before the limit. It should be emphasized that the tensors $\tilde{\boldsymbol{k}}_{(r)}$ and $\tilde{\boldsymbol{r}}_{(\alpha)}$ *are not*, in general, Killing tensors.

Inserting the results (3.12) along with the limit form of the metric functions (see Appendix A.2) into Eq. (1.41), we acquire the leading term in the expansion of the Killing tensors $\boldsymbol{r}_{(\mu)}$

$$\varepsilon\boldsymbol{r}_{(\alpha,0)} \approx -{}^\alpha\!\sum_\rho \frac{\lambda\tilde{a}_\alpha}{{}^\alpha a_\rho}\,{}^\alpha\boldsymbol{r}_{(\rho)}\,, \qquad \varepsilon\boldsymbol{r}_{(\alpha,\rho)} \approx \frac{\lambda\tilde{a}_\alpha}{{}^\alpha a_\rho}\,{}^\alpha\boldsymbol{r}_{(\rho)}\,, \tag{3.15}$$

with ${}^\alpha\boldsymbol{r}_{(\rho)}$ in the form

$${}^\alpha\boldsymbol{r}_{(\rho)} = {}^\alpha\!\sum_\sigma \frac{{}^\alpha J_\sigma({}^\alpha a_\rho)}{{}^\alpha\mathcal{U}_\rho}\,{}^\alpha\boldsymbol{\pi}_\sigma\,,$$

see Appendix B.3 for the proof. They are independent Killing tensors after the limit associated with the secondary directions. Again, in analogy with (1.43) we also introduce Killing tensors ${}^\alpha\boldsymbol{k}_{(p)}$ as

$${}^\alpha\boldsymbol{k}_{(p)} = {}^\alpha\!\sum_\rho {}^\alpha\mathcal{A}_\rho^{(p)}\,{}^\alpha\boldsymbol{r}_{(\rho)} = {}^\alpha\!\sum_\rho {}^\alpha A_\rho^{(p)}\,{}^\alpha\boldsymbol{\pi}_\rho\,. \tag{3.16}$$

In this case, both ${}^\alpha\boldsymbol{k}_{(p)}$ and ${}^\alpha\boldsymbol{r}_{(\rho)}$ *are* Killing tensors: ${}^\alpha\boldsymbol{r}_{(\rho)}$ have been obtained as a limit of Killing tensors and ${}^\alpha\boldsymbol{k}_{(p)}$ are just linear combinations of ${}^\alpha\boldsymbol{r}_{(\rho)}$ with constant coefficients. Moreover, they are both directly related to the Kähler parts of the metric ${}^{\alpha}\boldsymbol{g}$. For $p = 0$ we even have

$$ {}^{\alpha}\boldsymbol{k}_{(0)} = {}^{\alpha}\boldsymbol{g}^{-1}\,, \tag{3.17} $$

thus the Kähler metrics are Killing tensors as well.

Inspecting the first expansion in (3.15), we see that the limiting procedure for $\boldsymbol{r}_{(\mu)}$ extracts only the Killing tensors ${}^{\alpha}\boldsymbol{r}_{(\rho)}$ in the secondary directions. It does not provide any additional information about the primary directions. Therefore, we need a different approach to obtain Killing tensors related to the primary directions.

Similarly to Killing vectors, let us apply the limiting procedure to the generating function $\boldsymbol{k}(\beta)$ for Killing tensors (1.44). The proof of the following results can be found in Appendix B.3. We obtain

$$ \boldsymbol{k}(\beta) \approx \prod_{\gamma} \left(1 + \beta^2 \tilde{a}_{\gamma}^2\right)^{{}^{\gamma}N} \tilde{\boldsymbol{K}}(\beta)\,. \tag{3.18} $$

Since $\boldsymbol{k}(\beta)$ and $\tilde{\boldsymbol{K}}(\beta)$ differ only by a constant factor, $\tilde{\boldsymbol{K}}(\beta)$ also generates Killing tensors. We use this generating function to generate Killing tensors associated with the primary directions after the limit. $\tilde{\boldsymbol{K}}(\beta)$ can be written in the form

$$ \tilde{\boldsymbol{K}}(\beta) = \sum_{r} \beta^{2r} \tilde{\boldsymbol{K}}_{(r)} - \beta^{2\tilde{N}} \sum_{\alpha} \frac{2\lambda \tilde{a}_{\alpha}^2 \tilde{\mathcal{U}}_{\alpha}}{1 + \beta^2 \tilde{a}_{\alpha}^2}\, {}^{\alpha}\boldsymbol{g}^{-1}\,, $$

where we also separated the first $\tilde{N}$ powers of β^2 from the remaining ones. Because of the higher powers of β^2 in the second sum, $\tilde{\boldsymbol{K}}(\beta)$ is no longer a direct analogue of its before-the-limit counterpart $\boldsymbol{k}(\beta)$ in (1.34). Nevertheless, extra Killing tensors corresponding to these higher powers of β^2 are rather trivial — they are just linear combinations of the Killing tensors ${}^{\alpha}\boldsymbol{g}^{-1}$ with constant coefficients. The more interesting part of $\tilde{\boldsymbol{K}}(\beta)$ is given by the first $\tilde{N}$ powers of β^2 contained in the first sum. The coefficients define new Killing tensors $\tilde{\boldsymbol{K}}_{(r)}$, which are given by

$$ \tilde{\boldsymbol{K}}_{(r)} = \tilde{\boldsymbol{k}}_{(r)} - \sum_{\alpha} \tilde{B}^{(r)}(\tilde{a}_{\alpha}^2) \frac{2\lambda \tilde{a}_{\alpha}^2 \tilde{\mathcal{U}}_{\alpha}}{\tilde{J}(\tilde{a}_{\alpha}^2)}\, {}^{\alpha}\boldsymbol{g}^{-1}\,, \tag{3.19} $$

with the functions $\tilde{B}^{(r)}(\tilde{a}_{\alpha}^2)$ defined as partial sums (see Eq. (B.11))

$$ \tilde{B}^{(r)}(\tilde{a}_{\alpha}^2) = \sum_{n=0}^{r} \tilde{A}^{(n)} (-\tilde{a}_{\alpha}^2)^{r-n}\,. $$

Since these are non-trivial functions on spacetime, the linear combination of the Killing tensors ${}^{\alpha}\boldsymbol{g}^{-1}$ in (3.19) does not have constant coefficients and, therefore, $\tilde{\boldsymbol{k}}_{(r)}$ cannot be expected to be Killing tensors.

For $r = 0$, we have

$$ \tilde{\boldsymbol{K}}_{(0)} \approx \boldsymbol{g}^{-1}\,, $$

thus the Killing tensors $\tilde{\boldsymbol{K}}_{(r)}$ are related to the full spacetime metric $\boldsymbol{g}$.

Finally, we can introduce another set of Killing tensors $\tilde{\boldsymbol{R}}_{(\alpha)}$, again using a formula analogous to (1.43)

$$ \tilde{\boldsymbol{K}}_{(r)} = \sum_{\alpha} \tilde{\mathcal{A}}_{\alpha}^{(r)} \tilde{\boldsymbol{R}}_{(\alpha)}\,. \tag{3.20} $$

It can be shown that $\tilde{\boldsymbol{R}}_{(\alpha)}$ read

$$\tilde{\boldsymbol{R}}_{(\alpha)} = \tilde{\boldsymbol{r}}_{(\alpha)} + \sum_{\substack{\beta \\ \beta\neq\alpha}} \frac{2\lambda \tilde{a}_\beta^2}{\tilde{a}_\alpha^2 - \tilde{a}_\beta^2} \frac{\tilde{\mathcal{U}}_\beta}{\tilde{\mathcal{U}}_\alpha} \frac{\tilde{J}(\tilde{a}_\alpha^2) - \tilde{J}(\tilde{a}_\beta^2)}{\tilde{J}(\tilde{a}_\beta^2)} \, {}^{\beta}\boldsymbol{g}^{-1} - 2\lambda \tilde{a}_\alpha^2 \sum_\beta \left(\tilde{x}_\beta^2 - \tilde{a}_\alpha^2\right)^{-1} {}^{\alpha}\boldsymbol{g}^{-1} \,, \tag{3.21}$$

see Appendix B.3 for the proof.

To summarize, we have obtained $\tilde{N}$ Killing tensors $\tilde{\boldsymbol{K}}_{(r)}$ in the primary directions and $\sum_\alpha {}^{\alpha}N$ Killing tensors ${}^{\alpha}\boldsymbol{k}_{(p)}$ in the secondary directions, thus reconstructing the original number of N hidden symmetries. Alternatively, we can use equivalent sets of Killing tensors $\tilde{\boldsymbol{R}}_{(\alpha)}$ in the primary directions and ${}^{\alpha}\boldsymbol{r}_{(\rho)}$ in the secondary directions.

Let us conclude this chapter by showing that the relations between Killing vectors, Killing tensors and the principal Killing vector are preserved in the primary and the secondary directions separately. Indeed, it can be shown that similar equations to (1.32) hold for the objects related to the full spacetime metric $\boldsymbol{g}$ and the Kähler parts ${}^{\alpha}\boldsymbol{g}$, respectively

$$\begin{aligned} \tilde{\boldsymbol{L}}_{(r)} &= \tilde{\boldsymbol{K}}_{(r)} \cdot \tilde{\boldsymbol{\Xi}} \,, \\ {}^{\alpha}\boldsymbol{l}_{(p)} &= {}^{\alpha}\boldsymbol{k}_{(p)} \cdot {}^{\alpha}\boldsymbol{\xi} \,. \end{aligned} \tag{3.22}$$

Moreover, the other set of Killing vectors and Killing tensors, labeled with Greek indices, satisfies analogous identities

$$\begin{aligned} \tilde{\boldsymbol{\Phi}}_{\alpha} &= \tilde{\boldsymbol{R}}_{(\alpha)} \cdot \tilde{\boldsymbol{\Xi}} \,, \\ {}^{\alpha}\boldsymbol{s}_{(\rho)} &= {}^{\alpha}\boldsymbol{r}_{(\rho)} \cdot {}^{\alpha}\boldsymbol{\xi} \,. \end{aligned} \tag{3.23}$$

Such relations are not satisfied, however, when the objects related to the Kerr–NUT–(A)dS part $\tilde{\boldsymbol{g}}$ of the metric are considered — an additional term is present in this case

$$\begin{aligned} \tilde{\boldsymbol{l}}_{(r)} &= \tilde{\boldsymbol{k}}_{(r)} \cdot \tilde{\boldsymbol{\xi}} - \sum_\alpha \tilde{\mathcal{A}}_\alpha^{(r)} \, {}^{\alpha}\boldsymbol{\xi} \,, \\ \tilde{\boldsymbol{s}}_{(\alpha)} &= \tilde{\boldsymbol{r}}_{(\alpha)} \cdot \tilde{\boldsymbol{\xi}} - {}^{\alpha}\boldsymbol{\xi} \,, \end{aligned} \tag{3.24}$$

see Appendix B.3 for the proof. This should not come as a surprise since $\tilde{\boldsymbol{k}}_{(r)}$ and $\tilde{\boldsymbol{r}}_{(\alpha)}$ are not actually Killing tensors of the full spacetime metric.

4. Examples: Equally-spinning black holes

This chapter includes two explicit examples of the general results obtained in Chapters 2 and 3 that are interesting from the physical point of view. Namely, we consider metrics with the Lorentzian signature describing higher-dimensional black holes, and we set all their rotational parameters equal. Section 4.1 focuses on a black hole in $D = 2N$ and Section 4.2 describes a six-dimensional case.

4.1 $D = 2N$

Setting all the rotational parameters equal in the Lorentzian case means that there are only two equal-spin blocks $\alpha \in \{1, \tilde{N}\}$, with the latter being reserved for the Wick rotation. Therefore, only the secondary directions within the first block are subjected to the limiting procedure, and their number is ${}^1\!N = N - 2$. The limit in this case is characterized by $a_{1,\rho} \to a_{1,0}$, where the index labeling the secondary directions acquires the values

$$\rho = 1, \ldots, N-2\,,$$

unless indicated otherwise. The parametrization (2.2) thus adopts the form

$$\begin{aligned} a_{1,0} &= \tilde{a}_1\,, & x_{1,0} &= \tilde{x}_1\,, & \phi_{1,0} &= \tilde{\phi}_1\,, \\ a_{1,\rho} &= \tilde{a}_1 + {}^1a_\rho\,\varepsilon\,, & x_{1,\rho} &= \tilde{a}_1 + {}^1x_\rho\,\varepsilon\,, & \phi_{1,\rho} &= {}^1\phi_\rho\,, \\ a_{\tilde{N},0} &= \tilde{a}_{\tilde{N}}\,, & x_{\tilde{N},0} &= \tilde{x}_{\tilde{N}}\,, & \phi_{\tilde{N},0} &= \tilde{\phi}_{\tilde{N}}\,. \end{aligned} \tag{4.1}$$

We shall proceed from the generally spinning metric after the limit and use its form (2.31) in the Myers–Perry coordinates $\{t,\ r,\ \tilde{\mu}_0,\ \tilde{\mu}_1,\ \tilde{\phi}_1,\ {}^1\mu_0,\ {}^1\mu_\rho,\ {}^1\phi_\rho\}$. Since the equally-spinning black hole retains a single rotational parameter after the limit (in the leading order terms in ε), let us write

$$\tilde{a}_1 \equiv \tilde{a}\,. \tag{4.2}$$

Moreover, the second limiting block is related just to the Lorentzian sector and it is renamed in the Wick rotation (2.3) and by imposing the gauge condition (2.4). Thus only a single primary direction and the corresponding block of secondary directions remain. Therefore, we shall further simplify the notation of the angular coordinates $\{\tilde{\phi}_1,\ {}^1\phi_\rho\}$ and the coordinates $\{{}^1\mu_0,\ {}^1\mu_\rho\}$ in the secondary directions by dropping the index "1", labeling the only limiting block

$$\begin{aligned} \tilde{\phi}_1 &\equiv \tilde{\phi}\,, & {}^1\phi_\rho &\equiv \phi_\rho\,, \\ {}^1\mu_0 &\equiv \mu_0\,, & {}^1\mu_\rho &\equiv \mu_\rho\,. \end{aligned} \tag{4.3}$$

As in the general case, the coordinates $\{\tilde{\mu}_0,\ \tilde{\mu}_1\}$ in the primary directions and $\{\mu_0,\ \mu_\rho\}$ in the secondary directions are not independent — they are constrained by the conditions

$$\tilde{\mu}_0^2 + \tilde{\mu}_1^2 = 1\,, \qquad \sum_{\rho=0}^{N-2} \mu_\rho^2 = 1\,. \tag{4.4}$$

The metric of an equally-spinning black hole then becomes

$$
\begin{aligned}
\boldsymbol{g} \approx & -\left(1-\lambda \tilde{R}^2\right) \mathbf{d}t^2 + \frac{2Mr}{\tilde{\Sigma}}\left(r^2+\tilde{a}^2\right)^{2-N}\left[\mathbf{d}t + \frac{\tilde{a}\tilde{\mu}_1^2}{1+\lambda\tilde{a}^2}\left(\tilde{\boldsymbol{\Phi}} - \lambda\tilde{a}\mathbf{d}t\right)\right]^2 \\
& + \frac{\tilde{\Sigma}}{\tilde{\Delta}_r}\mathbf{d}r^2 + r^2\mathbf{d}\tilde{\mu}_0^2 + \frac{r^2+\tilde{a}^2}{1+\lambda\tilde{a}^2}\left(\mathbf{d}\tilde{\mu}_1^2 + \tilde{\mu}_1^2\boldsymbol{g}_{\text{Eucl}}\right) \\
& + \frac{\lambda}{1-\lambda\tilde{R}^2}\left(r^2\tilde{\mu}_0\mathbf{d}\tilde{\mu}_0 + \frac{r^2+\tilde{a}^2}{1+\lambda\tilde{a}^2}\tilde{\mu}_1\mathbf{d}\tilde{\mu}_1\right)^2 ,
\end{aligned}
\tag{4.5}
$$

where the 1-form $\tilde{\boldsymbol{\Phi}}$ is given by

$$
\tilde{\boldsymbol{\Phi}} \equiv \tilde{\boldsymbol{\Phi}}^1 = \mu_0^2\mathbf{d}\tilde{\phi} + \sum_\rho \mu_\rho^2 \mathbf{d}\phi_\rho ,
$$

and $\boldsymbol{g}_{\text{Eucl}}$ denotes a $(2N-2)$-dimensional Euclidean metric in the multi-polar coordinates, which can be expressed as

$$
\boldsymbol{g}_{\text{Eucl}} = \mathbf{d}\mu_0^2 + \mu_0^2\mathbf{d}\tilde{\phi}^2 + \sum_\rho\left(\mathbf{d}\mu_\rho^2 + \mu_\rho^2\mathbf{d}\phi_\rho^2\right) . \tag{4.6}
$$

The metric functions adopt the form

$$
\begin{aligned}
1-\lambda\tilde{R}^2 &= \left(1-\lambda r^2\right)\left(\tilde{\mu}_0^2 + \frac{\tilde{\mu}_1^2}{1+\lambda\tilde{a}^2}\right) , \\
\tilde{\Delta}_r &= \left(1-\lambda r^2\right)\left(r^2+\tilde{a}^2\right) - 2Mr\left(r^2+\tilde{a}^2\right)^{2-N} , \\
\tilde{\Sigma} &= r^2 + \tilde{a}^2\tilde{\mu}_0^2 .
\end{aligned}
$$

We see that the secondary directions enter the full metric only through the terms $\boldsymbol{g}_{\text{Eucl}}$ and $\tilde{\boldsymbol{\Phi}}$.

Remembering that the coordinates are restricted by the constraint (4.4), we conclude that the secondary block essentially obtains the geometry of a $(2N-3)$-dimensional sphere. Indeed, it can be viewed as the sphere given by the constraint (4.4) embedded in a $(2N-2)$-dimensional flat space described by the metric $\boldsymbol{g}_{\text{Eucl}}$ (4.6). Similarly to the general case, this sphere is coupled only to the primary coordinate $\tilde{\mu}_1$.

Besides this metric piece, the secondary directions enter the full spacetime metric also in time-related terms through the Kähler potential $\tilde{\boldsymbol{\Phi}}$. It is related to the common rotation of the secondary directions.

4.2 Myers–Perry black hole in $D = 6$

In this section, we simplify the situation even more and study the equal-spin limit of a rotating black hole in vacuum — also called a Myers–Perry black hole — in six dimensions. This was also discussed by Ortaggio [55]. Such a simplification proves useful as we are able to find additional Killing vectors that emerge after the limit, thus providing evidence of an enhanced symmetry structure of the resulting spacetime. We expect the symmetry group to become enlarged also in the general limit case discussed in Chapters 2 and 3.

We focus on a vacuum case, i.e. we set $\lambda = 0$. Moreover, there are only two equal-spin blocks $\alpha \in \{1, \tilde{N}\}$, and since in six dimensions a black hole described by the Lorentzian metric has only two rotational parameters, the first block has ${}^1N = 1$ parameter subjected to the limit. Therefore, the limiting procedure in this case is characterized by $a_{1,1} \to a_{1,0}$. Since this is a special case of the limit discussed in Section 4.1 with $\rho = 1$, we shall use the parametrization (4.1) in the form

$$\begin{aligned}
&a_{1,0} = \tilde{a}_1 \equiv \tilde{a}\,, && x_{1,0} = \tilde{x}_1 \equiv \tilde{x}\,, && \phi_{1,0} = \tilde{\phi}_1 \equiv \tilde{\phi}\,,\\
&a_{1,1} = \tilde{a}_1 + {}^1a_1\,\varepsilon \equiv \tilde{a} + a\varepsilon\,, && x_{1,1} = \tilde{a}_1 + {}^1x_1\,\varepsilon \equiv \tilde{a} + x\varepsilon\,, && \phi_{1,1} = {}^1\phi_1 \equiv \phi\,,\\
&a_{\tilde{N},0} = \tilde{a}_{\tilde{N}}\,, && x_{\tilde{N},0} = \tilde{x}_{\tilde{N}}\,, && \phi_{\tilde{N},0} = \tilde{\phi}_{\tilde{N}}\,,
\end{aligned} \tag{4.7}$$

where we have renamed the coordinates and the parameters similarly to (4.2) and (4.3), and also dropped the index $\rho = 1$. Moreover, $\tilde{x}_{\tilde{N}}$ and $\tilde{\phi}_{\tilde{N}}$ are Wick-rotated as in (2.3), where we must perform the limit $\lambda \to 0$ assuming the gauge condition (2.4) for $\tilde{a}_{\tilde{N}}$.

4.2.1 Metric

Employing the parametrization (4.7), the metric (2.12) becomes

$$\begin{aligned}
\boldsymbol{g} \approx &-\mathbf{d}t^2 + \frac{2Mr}{\Sigma}\left[\mathbf{d}t - \frac{\tilde{x}^2 - \tilde{a}^2}{\tilde{a}}\left(\frac{x}{a}\mathbf{d}\tilde{\phi} - \frac{x-a}{a}\mathbf{d}\phi\right)\right]^2 + \frac{\Sigma}{\Delta_r}\mathbf{d}r^2\\
&- \frac{r^2 + \tilde{x}^2}{\tilde{x}^2 - \tilde{a}^2}\mathbf{d}\tilde{x}^2 + \frac{(r^2 + \tilde{a}^2)(\tilde{x}^2 - \tilde{a}^2)}{\tilde{a}^2}\left[\frac{\mathbf{d}x^2}{4x(x-a)} - \frac{x}{a}\mathbf{d}\tilde{\phi}^2 + \frac{x-a}{a}\mathbf{d}\phi^2\right],
\end{aligned} \tag{4.8}$$

with the metric functions in their limit form

$$\begin{aligned}
\Delta_r &\approx \left(r^2 + \tilde{a}^2\right)\tilde{\Delta}_r = \left(r^2 + \tilde{a}^2\right)^2 - 2Mr\,,\\
\Sigma &\approx \left(r^2 + \tilde{a}^2\right)\tilde{\Sigma} = \left(r^2 + \tilde{a}^2\right)\left(r^2 + \tilde{x}^2\right),
\end{aligned}$$

and $\tilde{\Delta}_r$ with $\tilde{\Sigma}$ defined as in Eq. (2.33), which confirms the results of [55]. This is the metric expressed in the generalized Boyer–Lindquist coordinates $\{t,\, r,\, \tilde{x},\, \tilde{\phi},\, x,\, \phi\}$, which are suitable for performing the limiting procedure. However, considering the physical interpretation of the resulting spacetime, another set of coordinates proves more useful.

Spherical-like coordinates

Instead of $\{\tilde{x},\, x\}$ we shall introduce new angular coordinates $\{\vartheta,\, \chi\}$, which are better suited for analysing physical properties of the black hole spacetime obtained after the limit. Namely, let us define

$$\begin{aligned}
\tilde{x} &= \tilde{a}\cos\vartheta\,,\\
x &= a\cos^2\chi\,.
\end{aligned} \tag{4.9}$$

Moreover, let us rename the angular coordinates $\{\tilde{\phi}, \phi\}$ in the following manner

$$\tilde{\phi} \equiv \varphi_1 \,,$$
$$\phi \equiv \varphi_2 \,.$$

Employing the coordinate transformation (4.9), the metric (4.8) then reads

$$\begin{aligned} \boldsymbol{g} \approx & - \mathbf{d}t^2 + \frac{2Mr}{\Sigma} \left[\mathbf{d}t + \tilde{a} \sin^2 \vartheta \left(\cos^2 \chi \mathbf{d}\varphi_1 + \sin^2 \chi \mathbf{d}\varphi_2\right)\right]^2 \\ & + \frac{\Sigma}{\Delta_r} \mathbf{d}r^2 + \left(r^2 + \tilde{a}^2 \cos^2 \vartheta\right) \mathbf{d}\vartheta^2 + \left(r^2 + \tilde{a}^2\right) \sin^2 \vartheta \mathbf{dS}_3 \,, \end{aligned} \tag{4.10}$$

where $\mathbf{dS}_3$ denotes the metric of a 3-sphere, which can be expressed as[1]

$$\mathbf{dS}_3 = \mathbf{d}\chi^2 + \cos^2 \chi \mathbf{d}\varphi_1^2 + \sin^2 \chi \mathbf{d}\varphi_2^2 \,. \tag{4.11}$$

The metric functions are

$$\Delta_r \approx \left(r^2 + \tilde{a}^2\right)^2 - 2Mr \,,$$
$$\Sigma \approx \left(r^2 + \tilde{a}^2\right) \left(r^2 + \tilde{a}^2 \cos^2 \vartheta\right) \,.$$

Notice that the metric (4.10) in these coordinates no longer contains the parameter a from the secondary block. This parameter controls how fast $a_{1,1}$ approaches $a_{1,0}$. However, in case the limit is applied to a single rotational parameter, such a scale is irrelevant.

The inverse metric can be written as

$$\begin{aligned} \boldsymbol{g}^{-1} \approx & - \left(\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} t}\right)^2 - \frac{2Mr}{\Sigma} \frac{(r^2 + \tilde{a}^2)^2}{\Delta_r} \left[\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} t} - \frac{\tilde{a}}{r^2 + \tilde{a}^2} \left(\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\varphi_1} + \frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\varphi_2}\right)\right]^2 \\ & + \frac{\Delta_r}{\Sigma} \left(\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} r}\right)^2 + \frac{1}{r^2 + \tilde{a}^2 \cos^2 \vartheta} \left(\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\vartheta}\right)^2 + \frac{1}{(r^2 + \tilde{a}^2) \sin^2 \vartheta} \mathbf{dS}_3^{-1} \,, \end{aligned}$$

with the inverse metric of a 3-sphere given simply by

$$\mathbf{dS}_3^{-1} = \left(\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\chi}\right)^2 + \frac{1}{\cos^2 \chi} \left(\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\varphi_1}\right)^2 + \frac{1}{\sin^2 \chi} \left(\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\varphi_2}\right)^2 \,.$$

4.2.2 Killing vectors

Before the limit, the Myers–Perry black hole in six dimensions has three explicit symmetries associated with the following Killing vectors

$$\begin{aligned} \boldsymbol{\xi} &= \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} t} \,, \\ \boldsymbol{s}_+ &= \frac{1}{2} \left(\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\varphi_1} + \frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\varphi_2}\right) = \frac{1}{2\lambda} \left(\frac{\boldsymbol{s}_{(1)}}{a_1} + \frac{\boldsymbol{s}_{(2)}}{a_2}\right) \,, \\ \boldsymbol{s}_- &= \frac{1}{2} \left(\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\varphi_1} - \frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\varphi_2}\right) = \frac{1}{2\lambda} \left(\frac{\boldsymbol{s}_{(1)}}{a_1} - \frac{\boldsymbol{s}_{(2)}}{a_2}\right) \,, \end{aligned} \tag{4.12}$$

[1] The coordinates in which the metric of a 3-sphere has this particular form are called the *Hopf coordinates*.

where $\boldsymbol{\xi}$ denotes the principal Killing vector (1.37). Note that $\boldsymbol{s}_+$ and $\boldsymbol{s}_-$ are simply linear combinations of the vectors $\boldsymbol{s}_{(1)}$ and $\boldsymbol{s}_{(2)}$ defined in (1.35)[2]. However, they form well-known commutation relations with new Killing vectors, which we will identify next.

As was mentioned earlier, the most important result of this section is that we have been able to obtain additional Killing vectors emerging after the limit. Namely, we found two new Killing vectors, which can be written in the spherical-like coordinates as

$$\boldsymbol{u} = \frac{1}{2}\left[\cos(\varphi_2-\varphi_1)\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\chi} - \sin(\varphi_2-\varphi_1)\left(\tan\chi\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\varphi_1} + \cot\chi\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\varphi_2}\right)\right],$$
$$\boldsymbol{v} = \frac{1}{2}\left[\sin(\varphi_2-\varphi_1)\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\chi} + \cos(\varphi_2-\varphi_1)\left(\tan\chi\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\varphi_1} + \cot\chi\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\varphi_2}\right)\right].$$

These vectors are independent of the original Killing vectors (4.12) and they Lie-preserve the full spacetime metric (4.10). Furthermore, it can be shown that they are Killing vectors of a 3-sphere represented by the metric (4.11).

Let us now discuss how the symmetry group changes after applying the limiting procedure. Before the limit, the spacetime symmetries form a $\mathbb{R}\times U(1)\times U(1)$ group as there are three commuting Killing vectors $\boldsymbol{\xi}$, $\boldsymbol{s}_+$ and $\boldsymbol{s}_-$. However, the algebraic structure emerging after the limit indicates that the symmetry of the resulting spacetime is indeed further enhanced. In fact, the vectors $\boldsymbol{s}_-$, $\boldsymbol{u}$ and $\boldsymbol{v}$ generate the algebra of an $SO(3)$ group and the vectors $\boldsymbol{\xi}$ and $\boldsymbol{s}_+$ commute with all the other vectors as can be seen from their Lie brackets[3]

$$\begin{aligned} &[\boldsymbol{s}_-,\boldsymbol{u}] = \boldsymbol{v}\,, \qquad [\boldsymbol{u},\boldsymbol{v}] = \boldsymbol{s}_-\,, \qquad [\boldsymbol{v},\boldsymbol{s}_-] = \boldsymbol{u}\,, \\ &[\boldsymbol{s}_+,\boldsymbol{s}_-] = 0\,, \qquad [\boldsymbol{s}_+,\boldsymbol{u}] = 0\,, \qquad [\boldsymbol{s}_+,\boldsymbol{v}] = 0\,. \end{aligned} \tag{4.13}$$

Therefore, the symmetry group of the spacetime decouples and is enhanced from the original $\mathbb{R}\times U(1)\times U(1)$ to $\mathbb{R}\times U(1)\times SO(3)$ after the limit.

4.2.3 Killing tensors

Another proof of an enhanced symmetry structure after performing the equal-spin limit can be found when studying the limit form of Killing tensors. In six dimensions, the Myers–Perry black hole after the limit has three Killing tensors defined in Eqs. (3.19) and (3.16). Let us simplify the notation used in the general case and rename the tensors as follows

$$\tilde{\boldsymbol{K}}_{(0)} \equiv \boldsymbol{k}_{(0)}\,,$$
$$\tilde{\boldsymbol{K}}_{(1)} \equiv \boldsymbol{k}_{(1)}\,,$$
$${}^{1}\boldsymbol{k}_{(0)} \equiv \boldsymbol{k}_{(2)}\,.$$

[2] Let us note that the factor $1/\lambda$ in the expressions for $\boldsymbol{s}_\pm$ in terms of $\boldsymbol{s}_{(1,2)}$ does not present a problem in the limit $\lambda\to 0$ since it is compensated by λ in the definition (1.35).

[3] Commutation relations with $\boldsymbol{\xi}$ are trivial since all the other Killing vectors are time-independent.

In the spherical-like coordinates, they obtain the form[4]

$$\boldsymbol{k}_{(0)} \approx \boldsymbol{g}^{-1} ,$$

$$\begin{aligned}
\boldsymbol{k}_{(1)} = &- \left(\tilde{a}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} t}\right)^2 + \tilde{a}\,\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} t} \vee \left(\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\varphi_1} + \frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\varphi_2}\right) \\
&- \frac{2Mr}{\Sigma}\frac{\left[(r^2+\tilde{a}^2)\,\tilde{a}\cos\vartheta\right]^2}{\Delta_r}\left[\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} t} - \frac{\tilde{a}}{r^2+\tilde{a}^2}\left(\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\varphi_1} + \frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\varphi_2}\right)\right]^2 \\
&+ \frac{\Delta_r}{\Sigma}\left(\tilde{a}\cos\vartheta\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} r}\right)^2 - \frac{1}{r^2+\tilde{a}^2\cos^2\vartheta}\left(r\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\vartheta}\right)^2 - \frac{r^2+\tilde{a}^2\sin^2\vartheta}{(r^2+\tilde{a}^2)\sin^2\vartheta}\mathbf{dS}_3^{-1} ,
\end{aligned}$$

$$\boldsymbol{k}_{(2)} = {}^{1}\boldsymbol{g}^{-1} = \frac{1}{2}\left[\left(\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\chi}\right)^2 + \left(\tan\chi\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\varphi_1} - \cot\chi\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\varphi_2}\right)^2\right] ,$$

where ${}^{1}\boldsymbol{g}^{-1}$ is the Kähler part of the full spacetime metric $\boldsymbol{g}$. It turns out that $\boldsymbol{k}_{(2)}$ decouples into a sum of direct products of Killing vectors in the following manner

$$\boldsymbol{k}_{(2)} = 2\left(\boldsymbol{u}\boldsymbol{u} + \boldsymbol{v}\boldsymbol{v} + \boldsymbol{s}_-\boldsymbol{s}_- - \boldsymbol{s}_+\boldsymbol{s}_+\right) ,$$

thus the Killing tensor $\boldsymbol{k}_{(2)}$ becomes reducible. Therefore, the corresponding hidden symmetry splits into a combination of explicit symmetries characterized by the Killing vectors $\boldsymbol{s}_+$, $\boldsymbol{s}_-$, $\boldsymbol{u}$ and $\boldsymbol{v}$.

[4] Here, $\boldsymbol{\alpha} \vee \boldsymbol{\beta} = \boldsymbol{\alpha}\boldsymbol{\beta} + \boldsymbol{\beta}\boldsymbol{\alpha}$ is a normalized symmetric tensor product analogous to the antisymmetric wedge operation.

Conclusion

This thesis is devoted to the Kerr–NUT–(A)dS spacetime and to a particular limit case of the general metric, where an arbitrary number of its rotational parameters coincides. The results presented here have been published in [48].

The importance of the Kerr–NUT–(A)dS spacetime lies in the fact that it is the most general solution to the vacuum Einstein equations in higher dimensions with the cosmological constant that also possesses the principal tensor. The principal tensor is an important geometrical object that generates a rich symmetry structure demonstrated by the existence of the Killing tower of Killing vectors and Killing tensors.

The limiting spacetime inherits this symmetry structure. The symmetry is even enhanced since some of the hidden symmetries represented by Killing tensors factorize, which leads to a higher number of explicit symmetries represented by Killing vectors.

While Chapter 1 is primarily introductory as it presents the Kerr–NUT–(A)dS spacetime and its already known properties, Chapter 2 contains the main results of our work concerning the equal-spin limit of the Kerr–(A)dS metric (we considered vanishing NUT charges since it is the most relevant case from the physical point of view). Although the limiting procedure is not trivial since some of the parameters and the coordinate ranges become degenerate, we managed to find a suitable parametrization of the limit, in which the metric remains regular. We defined primary and secondary *coordinate* directions, which refer to the coordinates (and the parameters) that do not change after the limit and to those subjected to the limit, respectively. We then applied the limiting procedure to the orthogonal frame $\{\boldsymbol{\epsilon}^\mu, \hat{\boldsymbol{\epsilon}}^\mu\}$ and found out that after the limit it separates into two independent orthogonal frames $\{\tilde{\boldsymbol{\epsilon}}^\alpha, \hat{\tilde{\boldsymbol{\epsilon}}}^\alpha\}$ and $\{{}^\alpha\boldsymbol{\epsilon}^\rho, {}^\alpha\hat{\boldsymbol{\epsilon}}^\rho\}$, which we referred to as primary and secondary *frame* directions, respectively. This structure corresponds to the expected form of the generalized Kerr–NUT–(A)dS spacetimes [53], where spacetimes with the principal tensor that has constant eigenvalues have been discussed. In [53], the most general metric allowing such a principal tensor has been identified. It was shown that certain parts of the metric have formal properties of the Kähler metrics, which we confirmed in our limit. We thus kept the terminology of [53]. However, the orthogonal separation of the metric is only valid on the level of tangent spaces and is not integrable. Thus the resulting geometry cannot be understood as a product of independent manifolds and one cannot talk about true Kähler submanifolds. The resulting geometry is rather a certain kind of multi-warped product.

Applying the limit to the metric $\boldsymbol{g}$, we discovered that it splits into two parts — the primary Kerr–NUT–(A)dS part $\tilde{\boldsymbol{g}}$ and the secondary Kähler metrics ${}^\alpha\boldsymbol{g}$. The Kerr–NUT–(A)dS part has the form analogous to the Kerr–NUT–(A)dS metric of the entire spacetime but only in the primary directions, while the Kähler metrics describe special parts of the geometry that emerged in the limiting procedure. However, both pieces are interlaced together because of the fine inner structure of the primary and secondary orthogonal frames. Thus, we repeat once more, that the secondary metrics ${}^\alpha\boldsymbol{g}$ can be called Kähler metrics only formally since they do not live on independent submanifolds.

As a particular case we studied the limit of a homogeneous sphere geometry with vanishing mass parameter and the limit of a black hole geometry with the explicit Lorentzian signature. In these cases, the interlacing of the primary Kerr–NUT–(A)dS block and the secondary Kähler blocks can be demonstrated in an explicit form. It turns out that after the limit the secondary blocks simplify to spatial spheres, and each sphere is coupled to the coordinate in the corresponding primary direction. Additionally, for the Lorentzian metric, the secondary blocks are coupled to the primary metric in the rotational term linear in $\mathbf{d}t$ through the Kähler potentials.

In Chapter 3, we discussed the equal-spin limit of the Killing tower. We were able to obtain the same number of Killing vectors and Killing tensors as before the limit, thus having reconstructed the original explicit and hidden symmetries associated with these objects. Moreover, we showed that Killing vectors can be obtained as contractions of Killing tensors with the principal Killing vector in the primary and the secondary directions separately.

It turned out that applying the equal-spin limit to the Kerr–(A)dS metric leads to a special subclass of the generalized Kerr–NUT–(A)dS metrics studied in [53]. In our case, the principal tensor after the limit becomes degenerate since some of its eigenvalues reduce to constants. However, our metric represents a particularly interesting subcase as it displays a unique property that these metrics do not have in general. Namely, we showed that it retains the full tower of explicit and hidden symmetries, thus inheriting the complete integrability of the geodesic particle motion from the original geometry — unlike the generalized Kerr–NUT–(A)dS metrics, which do not necessarily admit the full Killing tower, and they thus possess a weaker symmetry. This makes our case suited to studying the symmetries represented by Killing vectors and Killing tensors.

In general, we expect that the limiting metric has even enhanced symmetry structure, i.e. that it possesses more explicit symmetries, making some of the hidden symmetries reducible.

Two examples of the general results were presented in Chapter 4, where we studied the metric of a black hole with all its rotational parameters set equal in a general (even) dimension and in six dimensions. In the six-dimensional case, we found an enhanced symmetry structure after the limit. Namely, we discovered two additional Killing vectors that emerge after performing the limiting procedure. These new vectors are independent of the original Killing vectors and they Lie-preserve the full spacetime metric. Moreover, combined with the original Killing vectors they generate the algebra of an $SO(3)$ group. Therefore, the symmetry group of the spacetime is enhanced from the original $\mathbb{R} \times U(1) \times U(1)$ to $\mathbb{R} \times SO(3) \times U(1)$ after the limit. Furthermore, it turned out that one of the Killing tensors becomes reducible as it decouples into a sum of direct products of Killing vectors, therefore, the associated hidden symmetry splits into a combination of explicit symmetries.

Future work

Let us conclude with several open problems concerning the equal-spin limit of the Kerr–NUT–(A)dS spacetime.

We expect the symmetry group to become enlarged in the general limit case.

Namely, we expect that after the limit the symmetries of spherically symmetric spatial parts emerge on the level of the full spacetime metric. This should be possible to prove using a higher-dimensional generalization of the spherical-like coordinates (4.9). While the task is rather simple when the secondary block contains a single coordinate, with more coordinates the complexity of the expressions grows considerably, and identifying spheres in these coordinates thus becomes non-trivial.

In this work, we mostly assumed vanishing NUT charges for simplicity. Some of the results could be generalized to the case of non-vanishing NUT charges, but it would require much more careful fine-tuning of the metric parameters and the coordinate ranges and one would need to investigate the relation of the resulting symmetry structure with the NUT-related singular structure of the axes.

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A. Metric functions

A.1 Definitions and properties

In Chapter 1 we have introduced several auxiliary functions that are then often used to describe the spacetime metric and other quantities before the limit and with some modifications also after the limit (see Appendix A.2). These functions are polynomials either in the coordinates x_μ (denoted by J, A, U) or in the parameters a_μ (denoted by $\mathcal{J}$, $\mathcal{A}, \mathcal{U}$). They are defined as [1]

$$\begin{aligned} J(a^2) &= \prod_{\nu} \left(x_\nu^2 - a^2\right) = \sum_{k=0}^{N} A^{(k)}(-a^2)^{N-k} \,, \\ \mathcal{J}(x^2) &= \prod_{\nu} \left(a_\nu^2 - x^2\right) = \sum_{k=0}^{N} \mathcal{A}^{(k)}(-x^2)^{N-k} \,. \end{aligned} \tag{A.1}$$

Considering these definitions, it follows that

$$\begin{aligned} A^{(k)} &= \sum_{\substack{\nu_1,\dots,\nu_k \\ \nu_1<\dots<\nu_k}} x_{\nu_1}^2 \dots x_{\nu_k}^2 \,, \\ \mathcal{A}^{(k)} &= \sum_{\substack{\nu_1,\dots,\nu_k \\ \nu_1<\dots<\nu_k}} a_{\nu_1}^2 \dots a_{\nu_k}^2 \,. \end{aligned} \tag{A.2}$$

Similarly, we define functions with the μ-th variable omitted as

$$\begin{aligned} J_\mu(a^2) &= \prod_{\substack{\nu \\ \nu\neq\mu}} \left(x_\nu^2 - a^2\right) = \sum_{k} A_\mu^{(k)}(-a^2)^{N-1-k} \,, \\ \mathcal{J}_\mu(x^2) &= \prod_{\substack{\nu \\ \nu\neq\mu}} \left(a_\nu^2 - x^2\right) = \sum_{k} \mathcal{A}_\mu^{(k)}(-x^2)^{N-1-k} \,, \end{aligned} \tag{A.3}$$

where

$$\begin{aligned} A_\mu^{(k)} &= \sum_{\substack{\nu_1,\dots,\nu_k \\ \nu_1<\dots<\nu_k \\ \nu_i\neq\mu}} x_{\nu_1}^2 \dots x_{\nu_k}^2 \,, \\ \mathcal{A}_\mu^{(k)} &= \sum_{\substack{\nu_1,\dots,\nu_k \\ \nu_1<\dots<\nu_k \\ \nu_i\neq\mu}} a_{\nu_1}^2 \dots a_{\nu_k}^2 \,. \end{aligned} \tag{A.4}$$

We set

$$A^{(0)} = A_\mu^{(0)} = 1 \,, \qquad \mathcal{A}^{(0)} = \mathcal{A}_\mu^{(0)} = 1 \tag{A.5}$$

[1] Let us remind the reader that indices in sums and products go over the "default" ranges unless indicated otherwise explicitly, i.e.

$$\sum_{\mu} = \sum_{\mu=1}^{N} \,, \qquad \sum_{k} = \sum_{k=0}^{N-1} \,.$$

and we also assume that the functions $A_\mu^{(k)}$ vanish if the index k "overflows", e.g. $A_\mu^{(N)} = 0$; the same applies to the functions $\mathcal{A}_\mu^{(k)}$. Finally, we define

$$\begin{aligned} U_\mu &= J_\mu(x_\mu^2) = \prod_{\substack{\nu \\ \nu\neq\mu}} \left(x_\nu^2 - x_\mu^2\right) , \\ \mathcal{U}_\mu &= \mathcal{J}_\mu(a_\mu^2) = \prod_{\substack{\nu \\ \nu\neq\mu}} \left(a_\nu^2 - a_\mu^2\right) \end{aligned} \tag{A.6}$$

as a special case of (A.3).

The functions defined above satisfy

$$\begin{aligned} J(x_\mu^2) &= 0 , & \mathcal{J}(a_\mu^2) &= 0 , \\ J_\mu(x_\nu^2) &= 0 , & \mathcal{J}_\mu(a_\nu^2) &= 0 , \qquad \text{if } \mu \neq \nu . \end{aligned}$$

If $A_\mu^{(k)}$ is understood as an $N\times N$ matrix, it is possible to write down its inverse

$$\sum_k A_\mu^{(k)} \frac{(-x_\nu^2)^{N-1-k}}{U_\nu} = \delta_{\mu\nu} , \tag{A.7}$$

$$\sum_\mu A_\mu^{(k)} \frac{(-x_\mu^2)^{N-1-l}}{U_\mu} = \delta_{kl} . \tag{A.8}$$

The following identity holds for the polynomials $A^{(k)}$ and $A_\mu^{(k)}$

$$\sum_\mu \frac{A_\mu^{(k)}}{x_\mu^2 U_\mu} = \frac{A^{(k)}}{A^{(N)}} . \tag{A.9}$$

The functions $\mathcal{A}^{(k)}$ and $\mathcal{A}_\mu^{(k)}$ satisfy analogous identities with x_μ and U_μ replaced by a_μ and $\mathcal{U}_\mu$, respectively.

Finally, the following orthogonality relations are satisfied

$$\sum_\kappa \frac{J_\mu(a_\kappa^2)}{\mathcal{U}_\kappa} \frac{\mathcal{J}_\kappa(x_\nu^2)}{U_\nu} = \delta_{\mu\nu} , \tag{A.10}$$

$$\sum_\kappa \frac{J_\mu(a_\kappa^2) J_\nu(a_\kappa^2)}{J(a_\kappa^2)\mathcal{U}_\kappa} = -\frac{U_\mu}{\mathcal{J}(x_\mu^2)} \delta_{\mu\nu} , \tag{A.11}$$

$$\sum_\kappa J_\kappa(a_\mu^2) J_\kappa(a_\nu^2) \frac{\mathcal{J}(x_\kappa^2)}{U_\kappa} = -J(a_\mu^2)\mathcal{U}_\mu \delta_{\mu\nu} . \tag{A.12}$$

A.2 Equal-spin limit

When employing the limiting procedure introduced in Chapter 2, first we rewrite the metric functions using the double indexing described in Section 2.1.1, and then apply the parametrization (2.2). The functions J and U along with their counterparts $\mathcal{J}$ and $\mathcal{U}$ can be written as [2] [3]

$$\mathcal{J}(x_{\alpha,0}^2) = \prod_{\gamma}\left(a_{\gamma,0}^2 - x_{\alpha,0}^2\right) \cdot \prod_{\gamma}{}^{\gamma}\prod_{\tau}\left(a_{\gamma,\tau}^2 - x_{\alpha,0}^2\right) ,$$

$$\mathcal{J}(x_{\alpha,\rho}^2) = \prod_{\gamma}\left(a_{\gamma,0}^2 - x_{\alpha,\rho}^2\right) \cdot \prod_{\gamma}{}^{\gamma}\prod_{\tau}\left(a_{\gamma,\tau}^2 - x_{\alpha,\rho}^2\right) ,$$

$$J_{\alpha,0}(a_{\beta,0}^2) = \prod_{\substack{\gamma\\ \gamma\neq\alpha}}\left(x_{\gamma,0}^2 - a_{\beta,0}^2\right) \cdot \prod_{\gamma}{}^{\gamma}\prod_{\tau}\left(x_{\gamma,\tau}^2 - a_{\beta,0}^2\right) ,$$

$$J_{\alpha,0}(a_{\beta,\sigma}^2) = \prod_{\substack{\gamma\\ \gamma\neq\alpha}}\left(x_{\gamma,0}^2 - a_{\beta,\sigma}^2\right) \cdot \prod_{\gamma}{}^{\gamma}\prod_{\tau}\left(x_{\gamma,\tau}^2 - a_{\beta,\sigma}^2\right) ,$$

$$J_{\alpha,\rho}(a_{\beta,0}^2) = \prod_{\gamma}\left(x_{\gamma,0}^2 - a_{\beta,0}^2\right) \cdot \prod_{\substack{\gamma\\ \gamma\neq\alpha}}{}^{\gamma}\prod_{\tau}\left(x_{\gamma,\tau}^2 - a_{\beta,0}^2\right) \cdot {}^{\alpha}\prod_{\substack{\tau\\ \tau\neq\rho}}\left(x_{\alpha,\tau}^2 - a_{\beta,0}^2\right) ,$$

$$J_{\alpha,\rho}(a_{\beta,\sigma}^2) = \prod_{\gamma}\left(x_{\gamma,0}^2 - a_{\beta,\sigma}^2\right) \cdot \prod_{\substack{\gamma\\ \gamma\neq\alpha}}{}^{\gamma}\prod_{\tau}\left(x_{\gamma,\tau}^2 - a_{\beta,\sigma}^2\right) \cdot {}^{\alpha}\prod_{\substack{\tau\\ \tau\neq\rho}}\left(x_{\alpha,\tau}^2 - a_{\beta,\sigma}^2\right) ,$$

$$\mathcal{J}_{\beta,0}(x_{\alpha,0}^2) = \prod_{\substack{\gamma\\ \gamma\neq\beta}}\left(a_{\gamma,0}^2 - x_{\alpha,0}^2\right) \cdot \prod_{\gamma}{}^{\gamma}\prod_{\tau}\left(a_{\gamma,\tau}^2 - x_{\alpha,0}^2\right) ,$$

$$\mathcal{J}_{\beta,\sigma}(x_{\alpha,0}^2) = \prod_{\gamma}\left(a_{\gamma,0}^2 - x_{\alpha,0}^2\right) \cdot \prod_{\substack{\gamma\\ \gamma\neq\beta}}{}^{\gamma}\prod_{\tau}\left(a_{\gamma,\tau}^2 - x_{\alpha,0}^2\right) \cdot {}^{\beta}\prod_{\substack{\tau\\ \tau\neq\sigma}}\left(a_{\beta,\tau}^2 - x_{\alpha,0}^2\right) ,$$

$$\mathcal{J}_{\beta,0}(x_{\alpha,\rho}^2) = \prod_{\substack{\gamma\\ \gamma\neq\beta}}\left(a_{\gamma,0}^2 - x_{\alpha,\rho}^2\right) \cdot \prod_{\gamma}{}^{\gamma}\prod_{\tau}\left(a_{\gamma,\tau}^2 - x_{\alpha,\rho}^2\right) ,$$

$$\mathcal{J}_{\beta,\sigma}(x_{\alpha,\rho}^2) = \prod_{\gamma}\left(a_{\gamma,0}^2 - x_{\alpha,\rho}^2\right) \cdot \prod_{\substack{\gamma\\ \gamma\neq\beta}}{}^{\gamma}\prod_{\tau}\left(a_{\gamma,\tau}^2 - x_{\alpha,\rho}^2\right) \cdot {}^{\beta}\prod_{\substack{\tau\\ \tau\neq\sigma}}\left(a_{\beta,\tau}^2 - x_{\alpha,\rho}^2\right) ,$$

$$U_{\alpha,0} = \prod_{\substack{\gamma\\ \gamma\neq\alpha}}\left(x_{\gamma,0}^2 - x_{\alpha,0}^2\right) \cdot \prod_{\gamma}{}^{\gamma}\prod_{\tau}\left(x_{\gamma,\tau}^2 - x_{\alpha,0}^2\right) ,$$

$$U_{\alpha,\rho} = \prod_{\gamma}\left(x_{\gamma,0}^2 - x_{\alpha,\rho}^2\right) \cdot \prod_{\substack{\gamma\\ \gamma\neq\alpha}}{}^{\gamma}\prod_{\tau}\left(x_{\gamma,\tau}^2 - x_{\alpha,\rho}^2\right) \cdot {}^{\alpha}\prod_{\substack{\tau\\ \tau\neq\rho}}\left(x_{\alpha,\tau}^2 - x_{\alpha,\rho}^2\right) ,$$

$$\mathcal{U}_{\alpha,0} = \prod_{\substack{\gamma\\ \gamma\neq\alpha}}\left(a_{\gamma,0}^2 - a_{\alpha,0}^2\right) \cdot \prod_{\gamma}{}^{\gamma}\prod_{\tau}\left(a_{\gamma,\tau}^2 - a_{\alpha,0}^2\right) ,$$

$$\mathcal{U}_{\alpha,\rho} = \prod_{\gamma}\left(a_{\gamma,0}^2 - a_{\alpha,\rho}^2\right) \cdot \prod_{\substack{\gamma\\ \gamma\neq\alpha}}{}^{\gamma}\prod_{\tau}\left(a_{\gamma,\tau}^2 - a_{\alpha,\rho}^2\right) \cdot {}^{\alpha}\prod_{\substack{\tau\\ \tau\neq\rho}}\left(a_{\alpha,\tau}^2 - a_{\alpha,\rho}^2\right) .$$

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[2] Let us remind the reader that indices introduced in Section 2.1.1 go over the "default" ranges unless indicated otherwise explicitly in the sum or product, i.e.

$$\prod_{\alpha} = \prod_{\alpha=1}^{\tilde{N}} , \qquad {}^{\alpha}\prod_{\rho} = \prod_{\rho=1}^{{}^{\alpha}N} .$$

[3] Note that the above list is not exhaustive — we calculate the limit only of those functions that appear in the metric or in related quantities.

After the limit, these functions become

$$\mathcal{J}(x^2_{\alpha,0}) \approx \tilde{\mathcal{J}}(\tilde{x}^2_\alpha) \prod_{\gamma} \left(\tilde{a}^2_\gamma - \tilde{x}^2_\alpha\right)^{\gamma N} ,$$

$$\mathcal{J}(x^2_{\alpha,\rho}) \approx -\tilde{\mathcal{U}}_\alpha {}^\alpha x_\rho {}^\alpha \mathcal{J}({}^\alpha x_\rho)\, (2\tilde{a}_\alpha \varepsilon)^{\alpha N+1} \prod_{\substack{\gamma \\ \gamma\neq\alpha}} \left(\tilde{a}^2_\gamma - \tilde{a}^2_\alpha\right)^{\gamma N} ,$$

$$J_{\alpha,0}(a^2_{\beta,0}) \approx \tilde{J}_\alpha(\tilde{a}^2_\beta) {}^\beta J(0)\, (2\tilde{a}_\beta \varepsilon)^{\beta N} \prod_{\substack{\gamma \\ \gamma\neq\beta}} \left(\tilde{a}^2_\gamma - \tilde{a}^2_\beta\right)^{\gamma N} ,$$

$$J_{\alpha,0}(a^2_{\beta,\sigma}) \approx \tilde{J}_\alpha(\tilde{a}^2_\beta) {}^\beta J({}^\beta a_\sigma)\, (2\tilde{a}_\beta \varepsilon)^{\beta N} \prod_{\substack{\gamma \\ \gamma\neq\beta}} \left(\tilde{a}^2_\gamma - \tilde{a}^2_\beta\right)^{\gamma N} ,$$

$$J_{\alpha,\rho}(a^2_{\beta,0}) \approx \frac{\tilde{J}(\tilde{a}^2_\beta)}{\tilde{a}^2_\alpha - \tilde{a}^2_\beta} {}^\beta J(0)\, (2\tilde{a}_\beta \varepsilon)^{\beta N} \prod_{\substack{\gamma \\ \gamma\neq\beta}} \left(\tilde{a}^2_\gamma - \tilde{a}^2_\beta\right)^{\gamma N} , \qquad \alpha \neq \beta ,$$

$$J_{\alpha,\rho}(a^2_{\beta,\sigma}) \approx \frac{\tilde{J}(\tilde{a}^2_\beta)}{\tilde{a}^2_\alpha - \tilde{a}^2_\beta} {}^\beta J({}^\beta a_\sigma)\, (2\tilde{a}_\beta \varepsilon)^{\beta N} \prod_{\substack{\gamma \\ \gamma\neq\beta}} \left(\tilde{a}^2_\gamma - \tilde{a}^2_\beta\right)^{\gamma N} , \qquad \alpha \neq \beta ,$$

$$J_{\alpha,\rho}(a^2_{\alpha,0}) \approx \tilde{J}(\tilde{a}^2_\alpha) {}^\alpha J_\rho(0)\, (2\tilde{a}_\alpha \varepsilon)^{\alpha N-1} \prod_{\substack{\gamma \\ \gamma\neq\alpha}} \left(\tilde{a}^2_\gamma - \tilde{a}^2_\alpha\right)^{\gamma N} ,$$

$$J_{\alpha,\rho}(a^2_{\alpha,\sigma}) \approx \tilde{J}(\tilde{a}^2_\alpha) {}^\alpha J_\rho({}^\alpha a_\sigma)\, (2\tilde{a}_\alpha \varepsilon)^{\alpha N-1} \prod_{\substack{\gamma \\ \gamma\neq\alpha}} \left(\tilde{a}^2_\gamma - \tilde{a}^2_\alpha\right)^{\gamma N} ,$$

$$\mathcal{J}_{\beta,0}(x^2_{\alpha,0}) \approx \tilde{\mathcal{J}}_\beta(\tilde{x}^2_\alpha) \prod_{\gamma} \left(\tilde{a}^2_\gamma - \tilde{x}^2_\alpha\right)^{\gamma N} ,$$

$$\mathcal{J}_{\beta,\sigma}(x^2_{\alpha,0}) \approx \tilde{\mathcal{J}}_\beta(\tilde{x}^2_\alpha) \prod_{\gamma} \left(\tilde{a}^2_\gamma - \tilde{x}^2_\alpha\right)^{\gamma N} ,$$

$$\mathcal{J}_{\beta,0}(x^2_{\alpha,\rho}) \approx \frac{\tilde{\mathcal{U}}_\alpha}{\tilde{a}^2_\alpha - \tilde{a}^2_\beta} {}^\alpha x_\rho {}^\alpha \mathcal{J}({}^\alpha x_\rho)\, (2\tilde{a}_\alpha \varepsilon)^{\alpha N+1} \prod_{\substack{\gamma \\ \gamma\neq\alpha}} \left(\tilde{a}^2_\gamma - \tilde{a}^2_\alpha\right)^{\gamma N} , \qquad \alpha \neq \beta ,$$

$$\mathcal{J}_{\beta,\sigma}(x^2_{\alpha,\rho}) \approx \frac{\tilde{\mathcal{U}}_\alpha}{\tilde{a}^2_\alpha - \tilde{a}^2_\beta} {}^\alpha x_\rho {}^\alpha \mathcal{J}({}^\alpha x_\rho)\, (2\tilde{a}_\alpha \varepsilon)^{\alpha N+1} \prod_{\substack{\gamma \\ \gamma\neq\alpha}} \left(\tilde{a}^2_\gamma - \tilde{a}^2_\alpha\right)^{\gamma N} , \qquad \alpha \neq \beta ,$$

$$\mathcal{J}_{\alpha,0}(x^2_{\alpha,\rho}) \approx \tilde{\mathcal{U}}_\alpha {}^\alpha \mathcal{J}({}^\alpha x_\rho)\, (2\tilde{a}_\alpha \varepsilon)^{\alpha N} \prod_{\substack{\gamma \\ \gamma\neq\alpha}} \left(\tilde{a}^2_\gamma - \tilde{a}^2_\alpha\right)^{\gamma N} ,$$

$$\mathcal{J}_{\alpha,\sigma}(x^2_{\alpha,\rho}) \approx -\tilde{\mathcal{U}}_\alpha {}^\alpha x_\rho {}^\alpha \mathcal{J}_\sigma({}^\alpha x_\rho)\, (2\tilde{a}_\alpha \varepsilon)^{\alpha N} \prod_{\substack{\gamma \\ \gamma\neq\alpha}} \left(\tilde{a}^2_\gamma - \tilde{a}^2_\alpha\right)^{\gamma N} ,$$

$$U_{\alpha,0} \approx \tilde{U}_\alpha \prod_\gamma \left(\tilde{a}_\gamma^2 - \tilde{x}_\alpha^2\right)^{{}^\gamma\! N} ,$$

$$U_{\alpha,\rho} \approx \tilde{J}(\tilde{a}_\alpha^2) {}^\alpha U_\rho \left(2\tilde{a}_\alpha\varepsilon\right)^{{}^\alpha\! N-1} \prod_{\substack{\gamma \\ \gamma\neq\alpha}} \left(\tilde{a}_\gamma^2 - \tilde{a}_\alpha^2\right)^{{}^\gamma\! N} ,$$

$$\mathcal{U}_{\alpha,0} \approx \tilde{\mathcal{U}}_\alpha {}^\alpha\mathcal{J}(0) \left(2\tilde{a}_\alpha\varepsilon\right)^{{}^\alpha\! N} \prod_{\substack{\gamma \\ \gamma\neq\alpha}} \left(\tilde{a}_\gamma^2 - \tilde{a}_\alpha^2\right)^{{}^\gamma\! N} ,$$

$$\mathcal{U}_{\alpha,\rho} \approx -\tilde{\mathcal{U}}_\alpha {}^\alpha a_\rho {}^\alpha\mathcal{U}_\rho \left(2\tilde{a}_\alpha\varepsilon\right)^{{}^\alpha\! N} \prod_{\substack{\gamma \\ \gamma\neq\alpha}} \left(\tilde{a}_\gamma^2 - \tilde{a}_\alpha^2\right)^{{}^\gamma\! N} .$$

The functions after the limit, such as $\tilde{J}$, $\tilde{A}$, $\tilde{U}$ and ${}^\alpha J$, ${}^\alpha A$, ${}^\alpha U$ are defined similarly to the functions before the limit in Appendix A.1, only the sets of coordinates and parameters they include are restricted to $\tilde{x}_\alpha$, $\tilde{a}_\alpha$ and ${}^\alpha x_\rho$, ${}^\alpha a_\rho$, respectively. Moreover, the latter are defined using first powers of variables instead of their squares. For example, the definitions (A.3) and (A.4) are modified as

$$\tilde{J}_\alpha(\tilde{a}^2) = \prod_{\substack{\beta \\ \beta\neq\alpha}} \left(\tilde{x}_\beta^2 - \tilde{a}^2\right) = \sum_r \tilde{A}_\alpha^{(r)} (-\tilde{a}^2)^{\tilde{N}-1-r} ,$$

$$\tilde{\mathcal{J}}_\alpha(\tilde{x}^2) = \prod_{\substack{\beta \\ \beta\neq\alpha}} \left(\tilde{a}_\beta^2 - \tilde{x}^2\right) = \sum_r \tilde{\mathcal{A}}_\alpha^{(r)} (-\tilde{x}^2)^{\tilde{N}-1-r} ,$$

$${}^\alpha J_\rho({}^\alpha a) = {}^\alpha\prod_{\substack{\sigma \\ \sigma\neq\rho}} ({}^\alpha x_\sigma - {}^\alpha a) = \sum_p {}^\alpha A_\rho^{(p)} (-{}^\alpha a)^{{}^\alpha N-1-p} ,$$

$${}^\alpha\mathcal{J}_\rho({}^\alpha x) = {}^\alpha\prod_{\substack{\sigma \\ \sigma\neq\rho}} ({}^\alpha a_\sigma - {}^\alpha x) = \sum_p {}^\alpha\mathcal{A}_\rho^{(p)} (-{}^\alpha x)^{{}^\alpha N-1-p} ,$$

where

$$\begin{aligned} \tilde{A}_\alpha^{(r)} &= \sum_{\substack{\beta_1,\dots,\beta_r \\ \beta_1<\dots<\beta_r \\ \beta_i\neq\alpha}} \tilde{x}_{\beta_1}^2 \dots \tilde{x}_{\beta_r}^2 , & {}^\alpha A_\rho^{(p)} &= \sum_{\substack{\sigma_1,\dots,\sigma_p \\ \sigma_1<\dots<\sigma_p \\ \sigma_i\neq\rho}} {}^\alpha x_{\sigma_1} \dots {}^\alpha x_{\sigma_p} , \\ \tilde{\mathcal{A}}_\alpha^{(r)} &= \sum_{\substack{\beta_1,\dots,\beta_r \\ \beta_1<\dots<\beta_r \\ \beta_i\neq\alpha}} \tilde{a}_{\beta_1}^2 \dots \tilde{a}_{\beta_r}^2 , & {}^\alpha\mathcal{A}_\rho^{(p)} &= \sum_{\substack{\sigma_1,\dots,\sigma_p \\ \sigma_1<\dots<\sigma_p \\ \sigma_i\neq\rho}} {}^\alpha a_{\sigma_1} \dots {}^\alpha a_{\sigma_p} , \end{aligned} \tag{A.13}$$

and the Latin indices go over the ranges

$$\begin{aligned} r &= 0, \dots, \tilde{N}-1 , \\ p &= 0, \dots, {}^\alpha N - 1 . \end{aligned}$$

The other definitions are modified accordingly. These functions also satisfy analogous identities and orthogonality relations to (A.7)—(A.12). Furthermore, we define ${}^\alpha J({}^\alpha a) = 1$ and ${}^\alpha\mathcal{J}({}^\alpha x) = 1$ if ${}^\alpha N = 0$. In particular, in the Lorentzian case we have

$${}^{\tilde{N}}J({}^\alpha a) = 1 , \qquad {}^{\tilde{N}}\mathcal{J}({}^\alpha x) = 1 .$$

B. Proofs of selected results

The results in the main text of Chapters 2 and 3 are summarized without explicit evidence of their validity. We provide the proofs of selected formulae in this appendix.

B.1 Metric in multi-cylindrical coordinates

Section 2.2.3 describes the transformation of the limit form of the metric, which represents a maximally symmetric space with the geometry of a homogeneous sphere, into the multi-cylindrical coordinates $\{\tilde{\rho}_0,\, \tilde{\rho}_\alpha,\, \tilde{\phi}_\alpha,\, {}^\alpha\!\rho_0,\, {}^\alpha\!\rho_\sigma,\, {}^\alpha\!\phi_\sigma\}$. In this appendix, we shall provide further details of this coordinate transformation.

It is useful to formally extend the sets of rotational parameters in the primary and the secondary directions by adding

$$\tilde{a}_0 = 0\,, \qquad {}^\alpha a_0 = 0\,. \tag{B.1}$$

This does not change the actual expressions, however, it enables us to write them in a compact form and also use (modified) identities for the metric functions. Let us also introduce the following notation for sums (and products) including the zeroth variable

$$\overset{\circ}{\sum_{\alpha}} \equiv \sum_{\alpha=0}^{\tilde{N}}\,, \qquad {}^\alpha\overset{\circ}{\sum_{\rho}} \equiv \sum_{\rho=0}^{{}^\alpha N}\,. \tag{B.2}$$

Similarly, the metric functions decorated with a circle include the zeroth variable, for example

$$\overset{\circ}{\tilde{\mathcal{U}}}_\alpha = \overset{\circ}{\prod_{\substack{\beta\\ \beta\neq\alpha}}} \left(\tilde{a}_\beta^2 - \tilde{a}_\alpha^2\right)\,, \qquad {}^\alpha\overset{\circ}{\mathcal{U}}_\rho = {}^\alpha\overset{\circ}{\prod_{\substack{\sigma\\ \sigma\neq\rho}}} \left({}^\alpha a_\sigma - {}^\alpha a_\rho\right)\,. \tag{B.3}$$

Using this notation, the Jacobi transformation (2.21) can be written compactly as

$$\lambda\tilde{\rho}_\alpha^2 = \frac{\tilde{J}(\tilde{a}_\alpha^2)}{\overset{\circ}{\tilde{\mathcal{U}}}_\alpha}\,, \qquad {}^\alpha\!\rho_\sigma^2 = \frac{{}^\alpha J({}^\alpha a_\sigma)}{{}^\alpha\overset{\circ}{\mathcal{U}}_\sigma}\,,$$

where $\alpha = 0, 1, \ldots, \tilde{N}$ and $\sigma = 0, 1, \ldots, {}^\alpha N$. The constraints (2.22) can be then verified using the following steps

$$\sum_{\alpha}^{\circ} \lambda\tilde{\rho}_{\alpha}^2 = \sum_{\alpha}^{\circ} \frac{\mathring{\tilde{J}}_0(\tilde{a}_{\alpha}^2)}{\mathring{\tilde{\mathcal{U}}}_{\alpha}} \overset{\text{(A.3)}}{=} \sum_{\alpha}^{\circ} \frac{1}{\mathring{\tilde{\mathcal{U}}}_{\alpha}} \sum_{r}^{\circ} \mathring{\tilde{A}}_0^{(r)}(-\tilde{a}_{\alpha}^2)^{\mathring{\tilde{N}}-1-r}$$

$$\overset{\text{(A.5)}}{=} \sum_{r}^{\circ} \mathring{\tilde{A}}_0^{(r)} \sum_{\alpha}^{\circ} \mathring{\tilde{\mathcal{A}}}_{\alpha}^{(0)} \frac{(-\tilde{a}_{\alpha}^2)^{\mathring{\tilde{N}}-1-r}}{\mathring{\tilde{\mathcal{U}}}_{\alpha}} \overset{\text{(A.8)}}{=} \sum_{r}^{\circ} \mathring{\tilde{A}}_0^{(r)} \delta_{r0} = \mathring{\tilde{A}}_0^{(0)} = 1\,,$$

$${}^{\alpha}\sum_{\sigma}^{\circ} {}^{\alpha}\rho_{\sigma}^2 = {}^{\alpha}\sum_{\sigma}^{\circ} \frac{{}^{\alpha}\mathring{J}_0({}^{\alpha}a_{\sigma})}{{}^{\alpha}\mathring{\mathcal{U}}_{\sigma}} \overset{\text{(A.3)}}{=} {}^{\alpha}\sum_{\sigma}^{\circ} \frac{1}{{}^{\alpha}\mathring{\mathcal{U}}_{\sigma}} {}^{\alpha}\sum_{p}^{\circ} {}^{\alpha}\mathring{A}_0^{(p)}(-{}^{\alpha}a_{\sigma})^{{}^{\alpha}\mathring{N}-1-p}$$

$$\overset{\text{(A.5)}}{=} {}^{\alpha}\sum_{p}^{\circ} {}^{\alpha}\mathring{A}_0^{(p)} {}^{\alpha}\sum_{\sigma}^{\circ} {}^{\alpha}\mathring{\mathcal{A}}_{\sigma}^{(0)} \frac{(-{}^{\alpha}a_{\sigma})^{{}^{\alpha}\mathring{N}-1-p}}{{}^{\alpha}\mathring{\mathcal{U}}_{\sigma}} \overset{\text{(A.8)}}{=} {}^{\alpha}\sum_{p}^{\circ} {}^{\alpha}\mathring{A}_0^{(p)} \delta_{p0} = {}^{\alpha}\mathring{A}_0^{(0)} = 1\,,$$

where we have denoted

$$\sum_{r}^{\circ} \equiv \sum_{r=0}^{\mathring{\tilde{N}}-1}\,, \qquad {}^{\alpha}\sum_{p}^{\circ} \equiv \sum_{p=0}^{{}^{\alpha}\mathring{N}-1}\,,$$

with $\mathring{\tilde{N}} = \tilde{N}+1$ and ${}^{\alpha}\mathring{N} = {}^{\alpha}N+1$ as they include the additional zeroth variables. Such a notation may seem unnecessarily complicated, however, it enables us to employ analogous definitions and identities to those listed in Appendix A.1, adapted to our set of coordinates and parameters. For example, in the preceding calculation we have used the identities analogous to those indicated above the equal signs.

In order to obtain the metric (2.24) in the multi-cylindrical coordinates, let us first calculate the following differentials

$$\begin{aligned}
\sum_{\alpha}^{\circ} \mathbf{d}\tilde{\rho}_{\alpha}^2 &= \sum_{\alpha}^{\circ} (\tilde{\rho}_{\alpha}\mathbf{d}\tilde{\rho}_{\alpha})^2 \frac{1}{\tilde{\rho}_{\alpha}^2} = \sum_{\alpha}^{\circ} \left[\sum_{\beta} \frac{\tilde{J}_{\beta}(\tilde{a}_{\alpha}^2)\tilde{x}_{\beta}\mathbf{d}\tilde{x}_{\beta}}{\lambda\mathring{\tilde{\mathcal{U}}}_{\alpha}}\right]^2 \frac{\lambda\mathring{\tilde{\mathcal{U}}}_{\alpha}}{\tilde{J}(\tilde{a}_{\alpha}^2)} \\
&= \frac{1}{\lambda}\sum_{\beta}\sum_{\gamma}\sum_{\alpha}^{\circ} \frac{\tilde{J}_{\beta}(\tilde{a}_{\alpha}^2)\tilde{J}_{\gamma}(\tilde{a}_{\alpha}^2)}{\tilde{J}(\tilde{a}_{\alpha}^2)\mathring{\tilde{\mathcal{U}}}_{\alpha}} \tilde{x}_{\beta}\tilde{x}_{\gamma}\mathbf{d}\tilde{x}_{\beta}\mathbf{d}\tilde{x}_{\gamma} \\
&\overset{\text{(A.11)}}{=} -\frac{1}{\lambda}\sum_{\beta}\sum_{\gamma} \frac{\tilde{U}_{\beta}}{\mathring{\tilde{\mathcal{J}}}(\tilde{x}_{\beta}^2)} \delta_{\beta\gamma}\tilde{x}_{\beta}\tilde{x}_{\gamma}\mathbf{d}\tilde{x}_{\beta}\mathbf{d}\tilde{x}_{\gamma} = \sum_{\beta} \frac{\tilde{U}_{\beta}}{\lambda\tilde{\mathcal{J}}(\tilde{x}_{\beta}^2)}\mathbf{d}\tilde{x}_{\beta}^2\,,
\end{aligned} \tag{B.4}$$

$$\begin{aligned}
{}^{\alpha}\sum_{\sigma}^{\circ} \mathbf{d}{}^{\alpha}\rho_{\sigma}^2 &= {}^{\alpha}\sum_{\sigma}^{\circ} ({}^{\alpha}\rho_{\sigma}\mathbf{d}{}^{\alpha}\rho_{\sigma})^2 \frac{1}{{}^{\alpha}\rho_{\sigma}^2} = {}^{\alpha}\sum_{\sigma}^{\circ} \left[{}^{\alpha}\sum_{\tau} \frac{{}^{\alpha}J_{\tau}({}^{\alpha}a_{\sigma})\mathbf{d}{}^{\alpha}x_{\tau}}{2\,{}^{\alpha}\mathring{\mathcal{U}}_{\sigma}}\right]^2 \frac{{}^{\alpha}\mathring{\mathcal{U}}_{\sigma}}{{}^{\alpha}J({}^{\alpha}a_{\sigma})} \\
&= \frac{1}{4}{}^{\alpha}\sum_{\tau}{}^{\alpha}\sum_{\omega}{}^{\alpha}\sum_{\sigma}^{\circ} \frac{{}^{\alpha}J_{\tau}({}^{\alpha}a_{\sigma})\,{}^{\alpha}J_{\omega}({}^{\alpha}a_{\sigma})}{{}^{\alpha}J({}^{\alpha}a_{\sigma})\,{}^{\alpha}\mathring{\mathcal{U}}_{\sigma}}\mathbf{d}{}^{\alpha}x_{\tau}\mathbf{d}{}^{\alpha}x_{\omega} \\
&\overset{\text{(A.11)}}{=} -\frac{1}{4}{}^{\alpha}\sum_{\tau}{}^{\alpha}\sum_{\omega} \frac{{}^{\alpha}U_{\tau}}{{}^{\alpha}\mathring{\mathcal{J}}({}^{\alpha}x_{\tau})}\delta_{\tau\omega}\mathbf{d}{}^{\alpha}x_{\tau}\mathbf{d}{}^{\alpha}x_{\omega} = {}^{\alpha}\sum_{\tau} \frac{{}^{\alpha}U_{\tau}}{4{}^{\alpha}x_{\tau}{}^{\alpha}\mathcal{J}({}^{\alpha}x_{\tau})}\mathbf{d}{}^{\alpha}x_{\tau}^2\,.
\end{aligned}$$

Moreover, let us transform the following expression into the multi-cylindrical coordinates

$$
\begin{aligned}
&\tilde{\mathbf{\Phi}}^{\alpha}\tilde{\mathbf{\Phi}}^{\alpha} + {}^{\alpha}\!\sum_{\sigma} \frac{{}^{\alpha}\!x_{\sigma}\,{}^{\alpha}\mathcal{J}({}^{\alpha}\!x_{\sigma})}{{}^{\alpha}U_{\sigma}}\,{}^{\alpha}\hat{\boldsymbol{\epsilon}}^{\sigma}\,{}^{\alpha}\hat{\boldsymbol{\epsilon}}^{\sigma} \\
&= \left({}^{\alpha}\rho_0^2\mathbf{d}\tilde{\phi}_{\alpha} + {}^{\alpha}\!\sum_{\tau}{}^{\alpha}\rho_{\tau}^2\mathbf{d}{}^{\alpha}\phi_{\tau}\right)\left({}^{\alpha}\rho_0^2\mathbf{d}\tilde{\phi}_{\alpha} + {}^{\alpha}\!\sum_{\omega}{}^{\alpha}\rho_{\omega}^2\mathbf{d}{}^{\alpha}\phi_{\omega}\right) + {}^{\alpha}\!\sum_{\sigma}\frac{{}^{\alpha}\!x_{\sigma}\,{}^{\alpha}\mathcal{J}({}^{\alpha}\!x_{\sigma})}{{}^{\alpha}U_{\sigma}}\times \\
&\qquad\times\left(\frac{{}^{\alpha}\rho_0^2}{{}^{\alpha}\!x_{\sigma}}\mathbf{d}\tilde{\phi}_{\alpha} + {}^{\alpha}\!\sum_{\tau}\frac{{}^{\alpha}\rho_{\tau}^2}{{}^{\alpha}\!x_{\sigma} - {}^{\alpha}a_{\tau}}\mathbf{d}{}^{\alpha}\phi_{\tau}\right)\left(\frac{{}^{\alpha}\rho_0^2}{{}^{\alpha}\!x_{\sigma}}\mathbf{d}\tilde{\phi}_{\alpha} + {}^{\alpha}\!\sum_{\omega}\frac{{}^{\alpha}\rho_{\omega}^2}{{}^{\alpha}\!x_{\sigma} - {}^{\alpha}a_{\omega}}\mathbf{d}{}^{\alpha}\phi_{\omega}\right) \\
&= \left[1 + {}^{\alpha}\!\sum_{\sigma}\frac{{}^{\alpha}\mathcal{J}({}^{\alpha}\!x_{\sigma})}{{}^{\alpha}\!x_{\sigma}{}^{\alpha}U_{\sigma}}\right]{}^{\alpha}\rho_0^4\mathbf{d}\tilde{\phi}_{\alpha}^2 \\
&\quad + {}^{\alpha}\!\sum_{\tau}\left[1 + {}^{\alpha}\!\sum_{\sigma}\frac{{}^{\alpha}\mathcal{J}({}^{\alpha}\!x_{\sigma})}{({}^{\alpha}\!x_{\sigma} - {}^{\alpha}a_{\tau})\,{}^{\alpha}U_{\sigma}}\right]2\,{}^{\alpha}\rho_0^2\,{}^{\alpha}\rho_{\tau}^2\mathbf{d}\tilde{\phi}_{\alpha}\mathbf{d}{}^{\alpha}\phi_{\tau} \\
&\quad + {}^{\alpha}\!\sum_{\tau}{}^{\alpha}\!\sum_{\omega}\left[1 + {}^{\alpha}\!\sum_{\sigma}\frac{{}^{\alpha}\mathcal{J}({}^{\alpha}\!x_{\sigma})}{{}^{\alpha}U_{\sigma}}\frac{{}^{\alpha}\!x_{\sigma}}{({}^{\alpha}\!x_{\sigma} - {}^{\alpha}a_{\tau})({}^{\alpha}\!x_{\sigma} - {}^{\alpha}a_{\omega})}\right]{}^{\alpha}\rho_{\tau}^2\,{}^{\alpha}\rho_{\omega}^2\mathbf{d}{}^{\alpha}\phi_{\tau}\mathbf{d}{}^{\alpha}\phi_{\omega} \\
&= {}^{\alpha}\rho_0^2\mathbf{d}\tilde{\phi}_{\alpha}^2 + {}^{\alpha}\!\sum_{\tau}{}^{\alpha}\rho_{\tau}^2\mathbf{d}{}^{\alpha}\phi_{\tau}^2\,.
\end{aligned}
\tag{B.5}
$$

We started with $\tilde{\mathbf{\Phi}}^{\alpha}$ and ${}^{\alpha}\hat{\boldsymbol{\epsilon}}^{\sigma}$ in the form (2.7) and (2.6), respectively, and substituted the expressions in the square brackets with their simplifed forms

$$
\begin{aligned}
1 + {}^{\alpha}\!\sum_{\sigma}\frac{{}^{\alpha}\mathcal{J}({}^{\alpha}\!x_{\sigma})}{{}^{\alpha}\!x_{\sigma}{}^{\alpha}U_{\sigma}} &\overset{(A.1)}{=} 1 + {}^{\alpha}\!\sum_{\sigma}\frac{1}{{}^{\alpha}\!x_{\sigma}{}^{\alpha}U_{\sigma}}\,{}^{\alpha}\!\sum_{p}^{\circ}{}^{\alpha}\mathcal{A}^{(p)}(-{}^{\alpha}\!x_{\sigma})^{{}^{\alpha}\!N-p} \\
&\overset{(A.5)}{=} 1 - {}^{\alpha}\!\sum_{p}{}^{\alpha}\mathcal{A}^{(p)}\,{}^{\alpha}\!\sum_{\sigma}{}^{\alpha}\!A_{\sigma}^{(0)}\frac{(-{}^{\alpha}\!x_{\sigma})^{{}^{\alpha}\!N-1-p}}{{}^{\alpha}U_{\sigma}} + {}^{\alpha}\mathcal{A}^{({}^{\alpha}\!N)}\,{}^{\alpha}\!\sum_{\sigma}\frac{1}{{}^{\alpha}\!x_{\sigma}{}^{\alpha}U_{\sigma}} \\
&\overset{(A.8)(A.9)}{=} 1 - {}^{\alpha}\!\sum_{p}{}^{\alpha}\mathcal{A}^{(p)}\delta_{p0} + \frac{{}^{\alpha}\mathcal{A}^{({}^{\alpha}\!N)}}{{}^{\alpha}\!A^{({}^{\alpha}\!N)}} = \frac{{}^{\alpha}\mathcal{A}^{({}^{\alpha}\!N)}}{{}^{\alpha}\!A^{({}^{\alpha}\!N)}} = \frac{1}{{}^{\alpha}\rho_0^2}\,,
\end{aligned}
$$

$$
\begin{aligned}
1 + {}^{\alpha}\!\sum_{\sigma}\frac{{}^{\alpha}\mathcal{J}({}^{\alpha}\!x_{\sigma})}{({}^{\alpha}\!x_{\sigma} - {}^{\alpha}a_{\tau})\,{}^{\alpha}U_{\sigma}} &= 1 - {}^{\alpha}\!\sum_{\sigma}\frac{{}^{\alpha}\mathcal{J}_{\tau}({}^{\alpha}\!x_{\sigma})}{{}^{\alpha}U_{\sigma}} \\
&\overset{(A.3)(A.5)}{=} 1 - {}^{\alpha}\!\sum_{p}{}^{\alpha}\mathcal{A}_{\tau}^{(p)}\,{}^{\alpha}\!\sum_{\sigma}{}^{\alpha}\!A_{\sigma}^{(0)}\frac{(-{}^{\alpha}\!x_{\sigma})^{{}^{\alpha}\!N-1-p}}{{}^{\alpha}U_{\sigma}} \\
&\overset{(A.8)}{=} 1 - {}^{\alpha}\!\sum_{p}{}^{\alpha}\mathcal{A}_{\tau}^{(p)}\delta_{p0} = 1 - {}^{\alpha}\mathcal{A}_{\tau}^{(0)} = 0\,,
\end{aligned}
\tag{B.6}
$$

$$1+{}^{\alpha}\!\sum_{\sigma}\frac{{}^{\alpha}\!\mathcal{J}({}^{\alpha}\!x_\sigma)}{{}^{\alpha}\!U_\sigma}\frac{{}^{\alpha}\!x_\sigma}{({}^{\alpha}\!x_\sigma-{}^{\alpha}\!a_\tau)({}^{\alpha}\!x_\sigma-{}^{\alpha}\!a_\omega)}$$
$$=1+{}^{\alpha}\!\sum_{\sigma}\frac{{}^{\alpha}\!\mathcal{J}({}^{\alpha}\!x_\sigma)}{{}^{\alpha}\!U_\sigma}\frac{{}^{\alpha}\!x_\sigma-{}^{\alpha}\!a_\omega+{}^{\alpha}\!a_\omega}{({}^{\alpha}\!x_\sigma-{}^{\alpha}\!a_\tau)({}^{\alpha}\!x_\sigma-{}^{\alpha}\!a_\omega)}$$
$$=1+{}^{\alpha}\!\sum_{\sigma}\frac{{}^{\alpha}\!\mathcal{J}({}^{\alpha}\!x_\sigma)}{({}^{\alpha}\!x_\sigma-{}^{\alpha}\!a_\tau)\,{}^{\alpha}\!U_\sigma}+{}^{\alpha}\!\sum_{\sigma}\frac{{}^{\alpha}\!\mathcal{J}({}^{\alpha}\!x_\sigma)}{{}^{\alpha}\!U_\sigma}\frac{{}^{\alpha}\!a_\omega}{({}^{\alpha}\!x_\sigma-{}^{\alpha}\!a_\tau)({}^{\alpha}\!x_\sigma-{}^{\alpha}\!a_\omega)}$$
$$\overset{(B.6)}{=}\frac{{}^{\alpha}\!a_\omega}{{}^{\alpha}\!J({}^{\alpha}\!a_\tau)\,{}^{\alpha}\!J({}^{\alpha}\!a_\omega)}{}^{\alpha}\!\sum_{\sigma}{}^{\alpha}\!J_\sigma({}^{\alpha}\!a_\tau)\,{}^{\alpha}\!J_\sigma({}^{\alpha}\!a_\omega)\frac{{}^{\alpha}\!\mathcal{J}({}^{\alpha}\!x_\sigma)}{{}^{\alpha}\!U_\sigma}$$
$$\overset{(A.12)}{=}-\frac{{}^{\alpha}\!a_\omega}{{}^{\alpha}\!J({}^{\alpha}\!a_\tau)\,{}^{\alpha}\!J({}^{\alpha}\!a_\omega)}{}^{\alpha}\!J({}^{\alpha}\!a_\tau){}^{\alpha}\!\mathcal{U}_\tau\delta_{\tau\omega}=\frac{1}{{}^{\alpha}\!\rho_\tau^2}\delta_{\tau\omega}\,.$$

Starting with the metric $\boldsymbol{g}$ in the form (2.12), where its parts $\tilde{\boldsymbol{g}}$ and ${}^{\alpha}\!\boldsymbol{g}$ are expressed as in (2.23) and (2.14), respectively, and using the results derived above, it becomes

$$\boldsymbol{g}\approx\sum_{\alpha}\frac{\tilde{U}_\alpha}{\lambda\tilde{\mathcal{J}}(\tilde{x}_\alpha^2)}\mathbf{d}\tilde{x}_\alpha^2+\sum_{\alpha}\frac{\tilde{J}(\tilde{a}_\alpha^2)}{-\lambda\tilde{a}_\alpha^2\tilde{\mathcal{U}}_\alpha}{}^{\alpha}\!\sum_{\sigma}\frac{{}^{\alpha}\!U_\sigma}{4\,{}^{\alpha}\!x_\sigma{}^{\alpha}\!\mathcal{J}({}^{\alpha}\!x_\sigma)}\mathbf{d}{}^{\alpha}\!x_\sigma^2$$
$$+\sum_{\alpha}\frac{\tilde{J}(\tilde{a}_\alpha^2)}{-\lambda\tilde{a}_\alpha^2\tilde{\mathcal{U}}_\alpha}\left[\tilde{\boldsymbol{\Phi}}^\alpha\tilde{\boldsymbol{\Phi}}^\alpha+{}^{\alpha}\!\sum_{\sigma}\frac{{}^{\alpha}\!x_\sigma{}^{\alpha}\!\mathcal{J}({}^{\alpha}\!x_\sigma)}{{}^{\alpha}\!U_\sigma}{}^{\alpha}\!\hat{\boldsymbol{\epsilon}}^\sigma\,{}^{\alpha}\!\hat{\boldsymbol{\epsilon}}^\sigma\right]$$
$$=\sum_{\alpha}^{\circ}\mathbf{d}\tilde{\rho}_\alpha^2+\sum_{\alpha}\tilde{\rho}_\alpha^2\,{}^{\alpha}\!\sum_{\sigma}^{\circ}\mathbf{d}{}^{\alpha}\!\rho_\sigma^2+\sum_{\alpha}\tilde{\rho}_\alpha^2\left({}^{\alpha}\!\rho_0^2\mathbf{d}\tilde{\phi}_\alpha^2+{}^{\alpha}\!\sum_{\sigma}{}^{\alpha}\!\rho_\sigma^2\mathbf{d}{}^{\alpha}\!\phi_\sigma^2\right)$$
$$=\mathbf{d}\tilde{\rho}_0^2+\sum_{\alpha}\left\{\mathbf{d}\tilde{\rho}_\alpha^2+\tilde{\rho}_\alpha^2\left[\mathbf{d}{}^{\alpha}\!\rho_0^2+{}^{\alpha}\!\rho_0^2\mathbf{d}\tilde{\phi}_\alpha^2+{}^{\alpha}\!\sum_{\sigma}\left(\mathbf{d}{}^{\alpha}\!\rho_\sigma^2+{}^{\alpha}\!\rho_\sigma^2\mathbf{d}{}^{\alpha}\!\phi_\sigma^2\right)\right]\right\}.$$

As one can see, the explicit separation of the Kerr–NUT–(A)dS part $\tilde{\boldsymbol{g}}$ and the Kähler metrics ${}^{\alpha}\!\boldsymbol{g}$ is not preserved after the Jacobi transformation since their angular parts have been combined together in order to obtain the terms (B.5).

B.2 Metric in Myers–Perry coordinates

This appendix provides details of the coordinate transformation discussed in Section 2.2.4, that is the transformation of a black hole's metric with the Lorentzian signature into the Myers–Perry coordinates $\{t,\, r,\, \tilde{\mu}_0,\, \tilde{\mu}_{\bar{\alpha}},\, \tilde{\phi}_{\bar{\alpha}},\, {}^{\bar{\alpha}}\!\mu_0,\, {}^{\bar{\alpha}}\!\mu_\rho,\, {}^{\bar{\alpha}}\!\phi_\rho\}$.

We shall be using the notation (B.1)—(B.3) in a slightly modified form, adapted to the set of coordinates after the separation of the temporal/radial coordinate and the angular coordinates. We thus define

$$\tilde{a}_0=0\,,\qquad {}^{\bar{\alpha}}\!a_0=0\,.$$

The sums (and products) are denoted as[1]

$$\sum_{\bar{\alpha}}^{\circ}\equiv\sum_{\bar{\alpha}=0}^{\tilde{\tilde{N}}},\qquad {}^{\bar{\alpha}}\!\sum_{\rho}^{\circ}\equiv\sum_{\rho=0}^{{}^{\bar{\alpha}}\!N},$$

[1]Let us remind the reader that $\tilde{\tilde{N}}=\tilde{N}-1$.

and the metric functions decorated with a circle are in this case

$$\mathring{\tilde{\tilde{\mathcal{U}}}}_{\bar{\alpha}} = \mathring{\prod_{\substack{\bar{\beta} \\ \bar{\beta}\neq\bar{\alpha}}}} \left(\tilde{a}^2_{\bar{\beta}} - \tilde{a}^2_{\bar{\alpha}}\right) , \qquad {}^{\bar{\alpha}}\mathring{\mathcal{U}}_{\rho} = {}^{\bar{\alpha}}\mathring{\prod_{\substack{\sigma \\ \sigma\neq\rho}}} \left({}^{\bar{\alpha}}a_{\sigma} - {}^{\bar{\alpha}}a_{\rho}\right) .$$

Using this notation, the Jacobi transformation (2.27) can be written compactly as

$$\tilde{\mu}^2_{\bar{\alpha}} = \frac{\tilde{\tilde{J}}(\tilde{a}^2_{\bar{\alpha}})}{\mathring{\tilde{\tilde{\mathcal{U}}}}_{\bar{\alpha}}} , \qquad {}^{\bar{\alpha}}\mu^2_{\rho} = \frac{{}^{\bar{\alpha}}J({}^{\bar{\alpha}}a_{\rho})}{{}^{\bar{\alpha}}\mathring{\mathcal{U}}_{\rho}} ,$$

where $\bar{\alpha} = 0, 1, \ldots, \tilde{\tilde{N}}$ and $\rho = 0, 1, \ldots, {}^{\bar{\alpha}}N$. The constraints (2.28) can be then verified using similar steps as in Section B.1

$$\begin{aligned}
\mathring{\sum_{\bar{\alpha}}} \tilde{\mu}^2_{\bar{\alpha}} &= \mathring{\sum_{\bar{\alpha}}} \frac{\mathring{\tilde{\tilde{J}}}_0(\tilde{a}^2_{\bar{\alpha}})}{\mathring{\tilde{\tilde{\mathcal{U}}}}_{\bar{\alpha}}} \overset{\text{(A.3)}}{=} \mathring{\sum_{\bar{\alpha}}} \frac{1}{\mathring{\tilde{\tilde{\mathcal{U}}}}_{\bar{\alpha}}} \mathring{\sum_{\bar{r}}} \mathring{\tilde{\tilde{A}}}{}^{(\bar{r})}_0 (-\tilde{a}^2_{\bar{\alpha}})^{\mathring{\tilde{\tilde{N}}}-1-\bar{r}} \\
&\overset{\text{(A.5)}}{=} \mathring{\sum_{\bar{r}}} \mathring{\tilde{\tilde{A}}}{}^{(\bar{r})}_0 \mathring{\sum_{\bar{\alpha}}} \mathring{\tilde{\tilde{\mathcal{A}}}}{}^{(0)}_{\bar{\alpha}} \frac{(-\tilde{a}^2_{\bar{\alpha}})^{\mathring{\tilde{\tilde{N}}}-1-\bar{r}}}{\mathring{\tilde{\tilde{\mathcal{U}}}}_{\bar{\alpha}}} \overset{\text{(A.8)}}{=} \mathring{\sum_{\bar{r}}} \mathring{\tilde{\tilde{A}}}{}^{(\bar{r})}_0 \delta_{\bar{r}0} = \mathring{\tilde{\tilde{A}}}{}^{(0)}_0 = 1 ,
\end{aligned}$$

$$\begin{aligned}
{}^{\bar{\alpha}}\mathring{\sum_{\rho}} {}^{\bar{\alpha}}\mu^2_{\rho} &= {}^{\bar{\alpha}}\mathring{\sum_{\rho}} \frac{{}^{\bar{\alpha}}\mathring{J}_0({}^{\bar{\alpha}}a_{\rho})}{{}^{\bar{\alpha}}\mathring{\mathcal{U}}_{\rho}} \overset{\text{(A.3)}}{=} {}^{\bar{\alpha}}\mathring{\sum_{\rho}} \frac{1}{{}^{\bar{\alpha}}\mathring{\mathcal{U}}_{\rho}} {}^{\bar{\alpha}}\mathring{\sum_{p}} {}^{\bar{\alpha}}\mathring{A}{}^{(p)}_0 (-{}^{\bar{\alpha}}a_{\rho})^{{}^{\bar{\alpha}}\mathring{N}-1-p} \\
&\overset{\text{(A.5)}}{=} {}^{\bar{\alpha}}\mathring{\sum_{p}} {}^{\bar{\alpha}}\mathring{A}{}^{(p)}_0 {}^{\bar{\alpha}}\mathring{\sum_{\rho}} {}^{\bar{\alpha}}\mathring{\mathcal{A}}{}^{(0)}_{\rho} \frac{(-{}^{\bar{\alpha}}a_{\rho})^{{}^{\bar{\alpha}}\mathring{N}-1-p}}{{}^{\bar{\alpha}}\mathring{\mathcal{U}}_{\rho}} \overset{\text{(A.8)}}{=} {}^{\bar{\alpha}}\mathring{\sum_{p}} {}^{\bar{\alpha}}\mathring{A}{}^{(p)}_0 \delta_{p0} = {}^{\bar{\alpha}}\mathring{A}{}^{(0)}_0 = 1 ,
\end{aligned}$$

where we have denoted

$$\mathring{\sum_{\bar{r}}} \equiv \sum_{\bar{r}=0}^{\mathring{\tilde{\tilde{N}}}-1} , \qquad {}^{\bar{\alpha}}\mathring{\sum_{p}} \equiv \sum_{p=0}^{{}^{\bar{\alpha}}\mathring{N}-1} ,$$

with $\mathring{\tilde{\tilde{N}}} = \tilde{\tilde{N}} + 1$ and ${}^{\bar{\alpha}}\mathring{N} = {}^{\bar{\alpha}}N + 1$. As in the previous section, the identities used in the calculations are indicated above the respective equal signs.

In order to obtain the metric (2.31) in the Myers–Perry coordinates, let us begin with the following differentials

$$
\begin{aligned}
\overset{\circ}{\sum_{\bar\alpha}} \mathbf{d}\tilde\mu_{\bar\alpha}^2 &= \overset{\circ}{\sum_{\bar\alpha}} (\tilde\mu_{\bar\alpha}\mathbf{d}\tilde\mu_{\bar\alpha})^2 \frac{1}{\tilde\mu_{\bar\alpha}^2} = \overset{\circ}{\sum_{\bar\alpha}} \left[\sum_{\bar\beta} \frac{\bar{\tilde{\bar{J}}}_{\bar\beta}(\tilde a_{\bar\alpha}^2)\tilde x_{\bar\beta}\mathbf{d}\tilde x_{\bar\beta}}{\mathring{\bar{\tilde{\bar{\mathcal{U}}}}}_{\bar\alpha}}\right]^2 \frac{\mathring{\bar{\tilde{\bar{\mathcal{U}}}}}_{\bar\alpha}}{\bar{\tilde{\bar{J}}}(\tilde a_{\bar\alpha}^2)} \\
&= \sum_{\bar\beta}\sum_{\bar\gamma}\overset{\circ}{\sum_{\bar\alpha}} \frac{\bar{\tilde{\bar{J}}}_{\bar\beta}(\tilde a_{\bar\alpha}^2)\bar{\tilde{\bar{J}}}_{\bar\gamma}(\tilde a_{\bar\alpha}^2)}{\bar{\tilde{\bar{J}}}(\tilde a_{\bar\alpha}^2)\mathring{\bar{\tilde{\bar{\mathcal{U}}}}}_{\bar\alpha}} \tilde x_{\bar\beta}\tilde x_{\bar\gamma}\mathbf{d}\tilde x_{\bar\beta}\mathbf{d}\tilde x_{\bar\gamma} \\
&\overset{\text{(A.11)}}{=} -\sum_{\bar\beta}\sum_{\bar\gamma} \frac{\bar{\tilde{\bar{U}}}_{\bar\beta}}{\mathring{\bar{\tilde{\bar{\mathcal{J}}}}}(\tilde x_{\bar\beta}^2)} \delta_{\bar\beta\bar\gamma}\tilde x_{\bar\beta}\tilde x_{\bar\gamma}\mathbf{d}\tilde x_{\bar\beta}\mathbf{d}\tilde x_{\bar\gamma} = \sum_{\bar\beta} \frac{\bar{\tilde{\bar{U}}}_{\bar\beta}}{\bar{\tilde{\bar{\mathcal{J}}}}(\tilde x_{\bar\beta}^2)} \mathbf{d}\tilde x_{\bar\beta}^2 \,,
\end{aligned}
$$

$$
\begin{aligned}
{}^{\bar\alpha}\overset{\circ}{\sum_{\rho}} \mathbf{d}{}^{\bar\alpha}\mu_\rho^2 &= {}^{\bar\alpha}\overset{\circ}{\sum_{\rho}} \left({}^{\bar\alpha}\mu_\rho \mathbf{d}{}^{\bar\alpha}\mu_\rho\right)^2 \frac{1}{{}^{\bar\alpha}\mu_\rho^2} = {}^{\bar\alpha}\overset{\circ}{\sum_{\rho}} \left[{}^{\bar\alpha}\sum_{\sigma} \frac{{}^{\bar\alpha}J_\sigma({}^{\bar\alpha}a_\rho)\mathbf{d}{}^{\bar\alpha}x_\sigma}{2\,{}^{\bar\alpha}\mathring{\mathcal{U}}_\rho}\right]^2 \frac{{}^{\bar\alpha}\mathring{\mathcal{U}}_\rho}{{}^{\bar\alpha}J({}^{\bar\alpha}a_\rho)} \\
&= \frac{1}{4}\,{}^{\bar\alpha}\sum_{\sigma}{}^{\bar\alpha}\sum_{\tau}{}^{\bar\alpha}\overset{\circ}{\sum_{\rho}} \frac{{}^{\bar\alpha}J_\sigma({}^{\bar\alpha}a_\rho)\,{}^{\bar\alpha}J_\tau({}^{\bar\alpha}a_\rho)}{{}^{\bar\alpha}J({}^{\bar\alpha}a_\rho)\,{}^{\bar\alpha}\mathring{\mathcal{U}}_\rho} \mathbf{d}{}^{\bar\alpha}x_\sigma \mathbf{d}{}^{\bar\alpha}x_\tau \\
&\overset{\text{(A.11)}}{=} -\frac{1}{4}\,{}^{\bar\alpha}\sum_{\sigma}{}^{\bar\alpha}\sum_{\tau} \frac{{}^{\bar\alpha}U_\sigma}{{}^{\bar\alpha}\mathring{\mathcal{J}}({}^{\bar\alpha}x_\sigma)} \delta_{\sigma\tau}\mathbf{d}{}^{\bar\alpha}x_\sigma\mathbf{d}{}^{\bar\alpha}x_\tau = {}^{\bar\alpha}\sum_{\sigma} \frac{{}^{\bar\alpha}U_\sigma}{4\,{}^{\bar\alpha}x_\sigma\,{}^{\bar\alpha}\mathcal{J}({}^{\bar\alpha}x_\sigma)} \mathbf{d}{}^{\bar\alpha}x_\sigma^2 \,.
\end{aligned}
$$

To continue, let us show how the following expressions transform from the Myers–Perry coordinates into the generalized Boyer–Lindquist coordinates (to prove the transformation in this direction is a simpler task than in the opposite direction)

$$
\begin{aligned}
1-\lambda\tilde R^2 &= \left(1-\lambda r^2\right)\left(\tilde\mu_0^2 + \sum_{\bar\alpha}\frac{\tilde\mu_{\bar\alpha}^2}{1+\lambda\tilde a_{\bar\alpha}^2}\right) = \left(1-\lambda r^2\right)\overset{\circ}{\sum_{\bar\alpha}}\frac{\tilde\mu_{\bar\alpha}^2}{1+\lambda\tilde a_{\bar\alpha}^2} \\
&= \left(1-\lambda r^2\right)\overset{\circ}{\sum_{\bar\alpha}} \frac{\bar{\tilde{\bar{J}}}(\tilde a_{\bar\alpha}^2)}{\mathring{\bar{\tilde{\bar{\mathcal{U}}}}}_{\bar\alpha}} \frac{\mathring{\bar{\tilde{\bar{\mathcal{J}}}}}_{\bar\alpha}(\tilde a_{\tilde N}^2)}{\bar{\tilde{\bar{\mathcal{J}}}}(\tilde a_{\tilde N}^2)} = \frac{1-\lambda r^2}{\bar{\tilde{\bar{\mathcal{J}}}}(\tilde a_{\tilde N}^2)} \overset{\circ}{\sum_{\bar\alpha}} \frac{\mathring{\bar{\tilde{\bar{J}}}}_0(\tilde a_{\bar\alpha}^2)\mathring{\bar{\tilde{\bar{\mathcal{J}}}}}_{\bar\alpha}(\tilde a_{\tilde N}^2)}{\mathring{\bar{\tilde{\bar{\mathcal{U}}}}}_{\bar\alpha}} \\
&\overset{\text{(A.3)}}{=} \frac{1-\lambda r^2}{\bar{\tilde{\bar{\mathcal{J}}}}(\tilde a_{\tilde N}^2)} \overset{\circ}{\sum_{\bar r}}\overset{\circ}{\sum_{\bar s}} \mathring{\bar{\tilde{\bar{A}}}}{}_0^{(\bar r)}(-\tilde a_{\tilde N}^2)^{\mathring{\bar{\tilde{\bar{N}}}}-1-\bar s} \overset{\circ}{\sum_{\bar\alpha}} \mathring{\bar{\tilde{\bar{\mathcal{A}}}}}{}_{\bar\alpha}^{(\bar s)} \frac{(-\tilde a_{\bar\alpha}^2)^{\mathring{\bar{\tilde{\bar{N}}}}-1-\bar r}}{\mathring{\bar{\tilde{\bar{\mathcal{U}}}}}_{\bar\alpha}} \\
&\overset{\text{(A.8)}}{=} \frac{1-\lambda r^2}{\bar{\tilde{\bar{\mathcal{J}}}}(\tilde a_{\tilde N}^2)} \overset{\circ}{\sum_{\bar r}} \mathring{\bar{\tilde{\bar{A}}}}{}_0^{(\bar r)}(-\tilde a_{\tilde N}^2)^{\mathring{\bar{\tilde{\bar{N}}}}-1-\bar r} = \left(1-\lambda r^2\right)\frac{\bar{\tilde{\bar{J}}}(\tilde a_{\tilde N}^2)}{\bar{\tilde{\bar{\mathcal{J}}}}(\tilde a_{\tilde N}^2)} \\
&= \left(1-\lambda r^2\right)\frac{\tilde J_{\tilde N}(\tilde a_{\tilde N}^2)}{\tilde{\mathcal{U}}_{\tilde N}} = \lambda\frac{\tilde J(\tilde a_{\tilde N}^2)}{\tilde{\mathcal{U}}_{\tilde N}} \,,
\end{aligned}
\tag{B.7}
$$

$$
1 - \sum_{\bar\alpha}\frac{\lambda\tilde a_{\bar\alpha}^2\tilde\mu_{\bar\alpha}^2}{1+\lambda\tilde a_{\bar\alpha}^2} = \tilde\mu_0^2 + \sum_{\bar\alpha}\frac{\tilde\mu_{\bar\alpha}^2}{1+\lambda\tilde a_{\bar\alpha}^2} \overset{\text{(B.7)}}{=} \frac{\tilde J_{\tilde N}(\tilde a_{\tilde N}^2)}{\tilde{\mathcal{U}}_{\tilde N}} \,.
$$

The transformation (B.8) can be proved using the multi-cylindrical coordinates. Namely, the sum of differentials in the primary directions can be expressed in two different forms

$$\begin{aligned}\overset{\circ}{\sum_{\alpha}} \mathbf{d}\tilde{\rho}_{\alpha}^{2} &= \frac{r^2}{1-\lambda r^2} \overset{\circ}{\sum_{\bar{\alpha}}} \frac{\tilde{\mu}_{\bar{\alpha}}^2}{r^2+\tilde{a}_{\bar{\alpha}}^2} \mathbf{d}r^2 + \overset{\circ}{\sum_{\bar{\alpha}}} \frac{r^2+\tilde{a}_{\bar{\alpha}}^2}{1+\lambda \tilde{a}_{\bar{\alpha}}^2} \mathbf{d}\tilde{\mu}_{\bar{\alpha}}^2 \\ &\quad + \frac{\lambda}{1-\lambda \tilde{R}^2} \left(\overset{\circ}{\sum_{\bar{\alpha}}} \frac{r^2+\tilde{a}_{\bar{\alpha}}^2}{1+\lambda \tilde{a}_{\bar{\alpha}}^2} \tilde{\mu}_{\bar{\alpha}} \mathbf{d}\tilde{\mu}_{\bar{\alpha}} \right)^2 ,\end{aligned}$$

$$\overset{\circ}{\sum_{\alpha}} \mathbf{d}\tilde{\rho}_{\alpha}^{2} = \frac{r^2}{1-\lambda r^2} \overset{\circ}{\sum_{\bar{\alpha}}} \frac{\tilde{\mu}_{\bar{\alpha}}^2}{r^2+\tilde{a}_{\bar{\alpha}}^2} \mathbf{d}r^2 + \sum_{\bar{\alpha}} \frac{\tilde{U}_{\bar{\alpha}}}{\lambda \tilde{\mathcal{J}}(\tilde{x}_{\bar{\alpha}}^2)} \mathbf{d}\tilde{x}_{\bar{\alpha}}^2 ,$$

where we have differentiated (2.29) to obtain the first form and applied several non-trivial identities to (B.4) to obtain the second form. Comparing these formulae, the transformation can be then written as

$$\begin{aligned}\sum_{\bar{\alpha}} \frac{\tilde{U}_{\bar{\alpha}}}{\lambda \tilde{\mathcal{J}}(\tilde{x}_{\bar{\alpha}}^2)} \mathbf{d}\tilde{x}_{\bar{\alpha}}^2 &= \overset{\circ}{\sum_{\bar{\alpha}}} \frac{r^2+\tilde{a}_{\bar{\alpha}}^2}{1+\lambda \tilde{a}_{\bar{\alpha}}^2} \mathbf{d}\tilde{\mu}_{\bar{\alpha}}^2 + \frac{\lambda}{1-\lambda \tilde{R}^2} \left(\overset{\circ}{\sum_{\bar{\alpha}}} \frac{r^2+\tilde{a}_{\bar{\alpha}}^2}{1+\lambda \tilde{a}_{\bar{\alpha}}^2} \tilde{\mu}_{\bar{\alpha}} \mathbf{d}\tilde{\mu}_{\bar{\alpha}} \right)^2 \\ &= r^2 \mathbf{d}\tilde{\mu}_0^2 + \sum_{\bar{\alpha}} \frac{r^2+\tilde{a}_{\bar{\alpha}}^2}{1+\lambda \tilde{a}_{\bar{\alpha}}^2} \mathbf{d}\tilde{\mu}_{\bar{\alpha}}^2 \\ &\quad + \frac{\lambda}{1-\lambda \tilde{R}^2} \left(r^2 \tilde{\mu}_0 \mathbf{d}\tilde{\mu}_0 + \sum_{\bar{\alpha}} \frac{r^2+\tilde{a}_{\bar{\alpha}}^2}{1+\lambda \tilde{a}_{\bar{\alpha}}^2} \tilde{\mu}_{\bar{\alpha}} \mathbf{d}\tilde{\mu}_{\bar{\alpha}} \right)^2 .\end{aligned} \tag{B.8}$$

Combined angular sectors of the Kerr–NUT–(A)dS part and the Kähler metrics can be calculated using similar steps to (B.5)

$$\tilde{\boldsymbol{\Phi}}^{\bar{\alpha}} \tilde{\boldsymbol{\Phi}}^{\bar{\alpha}} + {}^{\bar{\alpha}}\!\sum_{\rho} \frac{{}^{\bar{\alpha}}x_{\rho}\, {}^{\bar{\alpha}}\mathcal{J}({}^{\bar{\alpha}}x_{\rho})}{{}^{\bar{\alpha}}U_{\rho}}\, {}^{\bar{\alpha}}\hat{\boldsymbol{\epsilon}}^{\rho}\, {}^{\bar{\alpha}}\hat{\boldsymbol{\epsilon}}^{\rho} = {}^{\bar{\alpha}}\mu_0^2 \mathbf{d}\tilde{\phi}_{\bar{\alpha}}^2 + {}^{\bar{\alpha}}\!\sum_{\rho} {}^{\bar{\alpha}}\mu_{\rho}^2 \mathbf{d}\, {}^{\bar{\alpha}}\phi_{\rho}^2 . \tag{B.9}$$

Starting with the metric $\boldsymbol{g}$ in the form (2.12), where its parts $\tilde{\boldsymbol{g}}$ and ${}^{\alpha}\boldsymbol{g}$ are expressed as in (2.30) and (2.14), respectively, and using the results derived above, it becomes

$$\begin{aligned}
\boldsymbol{g} \approx & -\lambda\frac{\tilde{J}(\tilde{a}_{\tilde{N}}^2)}{\tilde{\mathcal{U}}_{\tilde{N}}}\mathbf{d}t^2 + \frac{2Mr}{\tilde{\Sigma}}\prod_{\bar{\beta}}\left(r^2+\tilde{a}_{\bar{\beta}}^2\right)^{-\bar{\beta}N}\left[\sum_{\bar{\alpha}}\frac{\tilde{J}_{\tilde{N}}(\tilde{a}_{\bar{\alpha}}^2)}{\lambda\tilde{a}_{\bar{\alpha}}\tilde{\mathcal{U}}_{\bar{\alpha}}}\tilde{\boldsymbol{\Phi}}^{\bar{\alpha}} + \frac{\tilde{J}_{\tilde{N}}(\tilde{a}_{\tilde{N}}^2)}{\tilde{\mathcal{U}}_{\tilde{N}}}\mathbf{d}t\right]^2 \\
& + \frac{\tilde{\Sigma}}{\tilde{\Delta}_r}\mathbf{d}r^2 + \sum_{\bar{\alpha}}\frac{\tilde{U}_{\bar{\alpha}}}{\lambda\tilde{\mathcal{J}}(\tilde{x}_{\bar{\alpha}}^2)}\mathbf{d}\tilde{x}_{\bar{\alpha}}^2 + \sum_{\bar{\alpha}}\frac{\tilde{J}(\tilde{a}_{\bar{\alpha}}^2)}{-\lambda\tilde{a}_{\bar{\alpha}}^2\tilde{\mathcal{U}}_{\bar{\alpha}}}\,{}^{\bar{\alpha}}\!\sum_{\rho}\frac{{}^{\bar{\alpha}}U_\rho}{4\,{}^{\bar{\alpha}}x_\rho\,{}^{\bar{\alpha}}\mathcal{J}({}^{\bar{\alpha}}x_\rho)}\mathbf{d}\,{}^{\bar{\alpha}}x_\rho^2 \\
& + \sum_{\bar{\alpha}}\frac{\tilde{J}(\tilde{a}_{\bar{\alpha}}^2)}{-\lambda\tilde{a}_{\bar{\alpha}}^2\tilde{\mathcal{U}}_{\bar{\alpha}}}\left[\tilde{\boldsymbol{\Phi}}^{\bar{\alpha}}\tilde{\boldsymbol{\Phi}}^{\bar{\alpha}} + {}^{\bar{\alpha}}\!\sum_{\rho}\frac{{}^{\bar{\alpha}}x_\rho\,{}^{\bar{\alpha}}\mathcal{J}({}^{\bar{\alpha}}x_\rho)}{{}^{\bar{\alpha}}U_\rho}\,{}^{\bar{\alpha}}\boldsymbol{\epsilon}^\rho\,{}^{\bar{\alpha}}\boldsymbol{\epsilon}^\rho\right] \\
= & -\left(1-\lambda\tilde{R}^2\right)\mathbf{d}t^2 + \frac{2Mr}{\tilde{\Sigma}}\prod_{\bar{\beta}}\left(r^2+\tilde{a}_{\bar{\beta}}^2\right)^{-\bar{\beta}N}\left[\mathbf{d}t + \sum_{\bar{\alpha}}\frac{\tilde{a}_{\bar{\alpha}}\tilde{\mu}_{\bar{\alpha}}^2}{1+\lambda\tilde{a}_{\bar{\alpha}}^2}\left(\tilde{\boldsymbol{\Phi}}^{\bar{\alpha}} - \lambda\tilde{a}_{\bar{\alpha}}\mathbf{d}t\right)\right]^2 \\
& + \frac{\tilde{\Sigma}}{\tilde{\Delta}_r}\mathbf{d}r^2 + r^2\mathbf{d}\tilde{\mu}_0^2 + \sum_{\bar{\alpha}}\frac{r^2+\tilde{a}_{\bar{\alpha}}^2}{1+\lambda\tilde{a}_{\bar{\alpha}}^2}\mathbf{d}\tilde{\mu}_{\bar{\alpha}}^2 + \sum_{\bar{\alpha}}\frac{r^2+\tilde{a}_{\bar{\alpha}}^2}{1+\lambda\tilde{a}_{\bar{\alpha}}^2}\tilde{\mu}_{\bar{\alpha}}^2\,{}^{\bar{\alpha}}\!\sum_{\rho}^{\circ}\mathbf{d}\,{}^{\bar{\alpha}}\mu_\rho^2 \\
& + \sum_{\bar{\alpha}}\frac{r^2+\tilde{a}_{\bar{\alpha}}^2}{1+\lambda\tilde{a}_{\bar{\alpha}}^2}\tilde{\mu}_{\bar{\alpha}}^2\left({}^{\bar{\alpha}}\mu_0^2\mathbf{d}\tilde{\phi}_{\bar{\alpha}}^2 + {}^{\bar{\alpha}}\!\sum_{\rho}{}^{\bar{\alpha}}\mu_\rho^2\mathbf{d}\,{}^{\bar{\alpha}}\phi_\rho^2\right) \\
& + \frac{\lambda}{1-\lambda\tilde{R}^2}\left(r^2\tilde{\mu}_0\mathbf{d}\tilde{\mu}_0 + \sum_{\bar{\alpha}}\frac{r^2+\tilde{a}_{\bar{\alpha}}^2}{1+\lambda\tilde{a}_{\bar{\alpha}}^2}\tilde{\mu}_{\bar{\alpha}}\mathbf{d}\tilde{\mu}_{\bar{\alpha}}\right)^2 .
\end{aligned}$$

Similarly to the metric in the multi-cylindrical coordinates, angular sectors of the Kerr–NUT–(A)dS part $\tilde{\boldsymbol{g}}$ and the Kähler metrics ${}^{\alpha}\boldsymbol{g}$ together form the terms (B.9), therefore, the explicit separation of $\tilde{\boldsymbol{g}}$ and ${}^{\alpha}\boldsymbol{g}$ is not preserved after the Jacobi transformation.

Finally, let us write the limit form of the metric functions Δ_r and Σ defined in (1.28). After employing the limiting procedure, they become

$$\Delta_r \approx \prod_{\bar{\alpha}}\left(r^2+\tilde{a}_{\bar{\alpha}}^2\right)^{\bar{\alpha}N}\tilde{\Delta}_r\,,$$

$$\Sigma \approx \prod_{\bar{\alpha}}\left(r^2+\tilde{a}_{\bar{\alpha}}^2\right)^{\bar{\alpha}N}\tilde{\Sigma}\,,$$

where $\tilde{\Delta}_r$ and $\tilde{\Sigma}$ are given by (2.33). While $\tilde{\Delta}_r$ has the same form in both types of coordinates, $\tilde{\Sigma}$ can be transformed using the following steps

$$\begin{aligned}
\tilde{\Sigma} &= \left(\tilde{\mu}_0^2 + \sum_{\bar{\alpha}}\frac{r^2\tilde{\mu}_{\bar{\alpha}}^2}{r^2+\tilde{a}_{\bar{\alpha}}^2}\right)\prod_{\bar{\beta}}\left(r^2+\tilde{a}_{\bar{\beta}}^2\right) = \sum_{\bar{\alpha}}^{\circ}\frac{\tilde{\mu}_{\bar{\alpha}}^2}{r^2+\tilde{a}_{\bar{\alpha}}^2}\prod_{\bar{\beta}}^{\circ}\left(r^2+\tilde{a}_{\bar{\beta}}^2\right) \\
&= \sum_{\bar{\alpha}}^{\circ}\frac{\mathring{\tilde{\tilde{J}}}(\tilde{a}_{\bar{\alpha}}^2)}{\mathring{\tilde{\tilde{\mathcal{U}}}}_{\bar{\alpha}}}\mathring{\tilde{\tilde{\mathcal{J}}}}_{\bar{\alpha}}(\tilde{x}_{\tilde{N}}^2) = \sum_{\bar{\alpha}}^{\circ}\frac{\mathring{\tilde{\tilde{J}}}_0(\tilde{a}_{\bar{\alpha}}^2)}{\mathring{\tilde{\tilde{\mathcal{U}}}}_{\bar{\alpha}}}\mathring{\tilde{\tilde{\mathcal{J}}}}_{\bar{\alpha}}(\tilde{x}_{\tilde{N}}^2) \\
&\overset{(A.3)}{=} \sum_{\bar{r}}^{\circ}\sum_{\bar{s}}^{\circ}\mathring{\tilde{\tilde{A}}}_0^{(\bar{r})}(-\tilde{x}_{\tilde{N}}^2)^{\mathring{\tilde{\tilde{N}}}-1-\bar{s}}\sum_{\bar{\alpha}}^{\circ}\mathring{\tilde{\tilde{\mathcal{A}}}}_{\bar{\alpha}}^{(\bar{s})}\frac{(-\tilde{a}_{\bar{\alpha}}^2)^{\mathring{\tilde{\tilde{N}}}-1-\bar{r}}}{\mathring{\tilde{\tilde{\mathcal{U}}}}_{\bar{\alpha}}} \\
&\overset{(A.8)}{=} \sum_{\bar{r}}^{\circ}\mathring{\tilde{\tilde{A}}}_0^{(\bar{r})}(-\tilde{x}_{\tilde{N}}^2)^{\mathring{\tilde{\tilde{N}}}-1-\bar{r}} = \mathring{\tilde{\tilde{J}}}_0(\tilde{x}_{\tilde{N}}^2) = \tilde{J}_{\tilde{N}}(\tilde{x}_{\tilde{N}}^2) \overset{(A.6)}{=} \tilde{U}_{\tilde{N}}\,.
\end{aligned}$$

B.3 Killing vectors and Killing tensors

Let us provide proofs of selected results from Sections 3.2 and 3.3, which focus on Killing vectors and Killing tensors after the limit.

Starting with its form (1.38), the generating function for Killing vectors (3.4) after the limit can be calculated as follows ($\approx$ denotes equality in the leading-order terms in ε)

$$\begin{aligned}
\boldsymbol{l}(\beta) &= \mathcal{A}(\beta)\sum_\mu \frac{1}{1+\beta^2 a_\mu^2}\boldsymbol{s}_{(\mu)} = \prod_\nu \Big(1+\beta^2 a_\nu^2\Big)\sum_\mu \frac{1}{1+\beta^2 a_\mu^2}\boldsymbol{s}_{(\mu)} \\
&= \prod_\gamma \Big(1+\beta^2 a_{\gamma,0}^2\Big)\cdot \prod_\gamma {}^\gamma\prod_\sigma \Big(1+\beta^2 a_{\gamma,\sigma}^2\Big)\times \\
&\qquad \times\left(\sum_\alpha \frac{1}{1+\beta^2 a_{\alpha,0}^2}\boldsymbol{s}_{(\alpha,0)} + \sum_\alpha {}^\alpha\sum_\rho \frac{1}{1+\beta^2 a_{\alpha,\rho}^2}\boldsymbol{s}_{(\alpha,\rho)}\right) \\
&\approx \prod_\gamma \Big(1+\beta^2 \tilde{a}_{\gamma}^2\Big)\cdot \prod_\gamma {}^\gamma\prod_\sigma \Big(1+\beta^2 \tilde{a}_{\gamma}^2\Big)\times \\
&\qquad \times\left(\sum_\alpha \frac{1}{1+\beta^2 \tilde{a}_{\alpha}^2}\tilde{\boldsymbol{s}}_{(\alpha)} + \sum_\alpha {}^\alpha\sum_\rho \frac{1}{1+\beta^2 \tilde{a}_{\alpha}^2}{}^\alpha\boldsymbol{s}_{(\rho)}\right) \\
&= \prod_\gamma \Big(1+\beta^2 \tilde{a}_{\gamma}^2\Big)^{\gamma N}\tilde{\mathcal{A}}(\beta)\sum_\alpha \frac{1}{1+\beta^2 \tilde{a}_{\alpha}^2}\left(\tilde{\boldsymbol{s}}_{(\alpha)} + {}^\alpha\sum_\rho {}^\alpha\boldsymbol{s}_{(\rho)}\right) \\
&= \prod_\gamma \Big(1+\beta^2 \tilde{a}_{\gamma}^2\Big)^{\gamma N}\sum_\alpha \tilde{\mathcal{A}}_\alpha(\beta)\left(\tilde{\boldsymbol{s}}_{(\alpha)} + {}^\alpha\sum_\rho {}^\alpha\boldsymbol{s}_{(\rho)}\right) \\
&= \prod_\gamma \Big(1+\beta^2 \tilde{a}_{\gamma}^2\Big)^{\gamma N}\sum_\alpha \tilde{\mathcal{A}}_\alpha(\beta)\tilde{\boldsymbol{\Phi}}_\alpha = \prod_\gamma \Big(1+\beta^2 \tilde{a}_{\gamma}^2\Big)^{\gamma N}\tilde{\boldsymbol{L}}(\beta)\,,
\end{aligned}$$

where

$$\tilde{\boldsymbol{L}}(\beta) = \sum_\alpha \tilde{\mathcal{A}}_\alpha(\beta)\tilde{\boldsymbol{\Phi}}_\alpha = \sum_r \beta^{2r}\sum_\alpha \tilde{\mathcal{A}}_\alpha^{(r)}\tilde{\boldsymbol{\Phi}}_\alpha = \sum_r \beta^{2r}\tilde{\boldsymbol{L}}_{(r)}\,,$$

and

$$\tilde{\boldsymbol{L}}_{(r)} = \sum_\alpha \tilde{\mathcal{A}}_\alpha^{(r)}\tilde{\boldsymbol{\Phi}}_\alpha\,.$$

In these calculations, we have used properties (1.39) and (1.40) of the generating functions for polynomials and also the expression (3.2).

We shall now focus on Killing tensors. The limit form of the Killing tensors

(3.15) can be derived from (1.41) using the following steps

$$\boldsymbol{r}_{(\alpha,\rho)} = \sum_{\substack{\beta \\ \beta\neq\alpha}} \left(\frac{J_{\beta,0}(a^2_{\alpha,\rho})}{\mathcal{U}_{\alpha,\rho}} \boldsymbol{\pi}_{\beta,0} + {}^{\beta}\!\sum_{\sigma} \frac{J_{\beta,\sigma}(a^2_{\alpha,\rho})}{\mathcal{U}_{\alpha,\rho}} \boldsymbol{\pi}_{\beta,\sigma} \right)$$

$$+ \frac{J_{\alpha,0}(a^2_{\alpha,\rho})}{\mathcal{U}_{\alpha,\rho}} \boldsymbol{\pi}_{\alpha,0} + {}^{\alpha}\!\sum_{\sigma} \frac{J_{\alpha,\sigma}(a^2_{\alpha,\rho})}{\mathcal{U}_{\alpha,\rho}} \boldsymbol{\pi}_{\alpha,\sigma}$$

$$\approx {}^{\alpha}\!\sum_{\sigma} \frac{J_{\alpha,\sigma}(a^2_{\alpha,\rho})}{\mathcal{U}_{\alpha,\rho}} \boldsymbol{\pi}_{\alpha,\sigma} \approx -\frac{\lambda \tilde{a}_\alpha}{\varepsilon}\, {}^{\alpha}\!\sum_{\sigma} \frac{{}^{\alpha}\!J_\sigma({}^{\alpha}a_\rho)}{{}^{\alpha}\mathring{\mathcal{U}}_\rho}\, {}^{\alpha}\boldsymbol{\pi}_\sigma\,,$$

where $\rho = 0, 1, \ldots, {}^{\alpha}N$ and ${}^{\alpha}\mathring{\mathcal{U}}_\rho$ is given by (B.3). We have also employed (3.12) and the limit form of the metric functions from Appendix A.2. Separating the zeroth variable from the rest, the tensors can be written as

$$\varepsilon \boldsymbol{r}_{(\alpha,0)} \approx -\lambda \tilde{a}_\alpha\, {}^{\alpha}\!\sum_{\sigma} \frac{{}^{\alpha}\!J_\sigma(0)}{{}^{\alpha}\mathcal{J}(0)}\, {}^{\alpha}\boldsymbol{\pi}_\sigma = -\,{}^{\alpha}\!\sum_{\rho} \frac{\lambda \tilde{a}_\alpha}{{}^{\alpha}a_\rho}\, {}^{\alpha}\boldsymbol{r}_{(\rho)}\,,$$

$$\varepsilon \boldsymbol{r}_{(\alpha,\rho)} \approx \frac{\lambda \tilde{a}_\alpha}{{}^{\alpha}a_\rho}\, {}^{\alpha}\!\sum_{\sigma} \frac{{}^{\alpha}\!J_\sigma({}^{\alpha}a_\rho)}{{}^{\alpha}\mathcal{U}_\rho}\, {}^{\alpha}\boldsymbol{\pi}_\sigma = \frac{\lambda \tilde{a}_\alpha}{{}^{\alpha}a_\rho}\, {}^{\alpha}\boldsymbol{r}_{(\rho)}\,.$$

The second equation defines the tensors ${}^{\alpha}\boldsymbol{r}_{(\rho)}$. Moreover, to obtain the final form of the first expression we have used the following identity

$${}^{\alpha}\!\sum_{\rho} \frac{\lambda \tilde{a}_\alpha}{{}^{\alpha}a_\rho}\, {}^{\alpha}\boldsymbol{r}_{(\rho)} = \lambda \tilde{a}_\alpha\, {}^{\alpha}\!\sum_{\rho} \frac{{}^{\alpha}\!\mathcal{A}^{({}^{\alpha}N-1)}_\rho}{{}^{\alpha}\mathcal{J}(0)}\, {}^{\alpha}\!\sum_{\sigma} \frac{{}^{\alpha}\!J_\sigma({}^{\alpha}a_\rho)}{{}^{\alpha}\mathcal{U}_\rho}\, {}^{\alpha}\boldsymbol{\pi}_\sigma$$

$$\stackrel{\text{(A.3)}}{=} \lambda \tilde{a}_\alpha\, {}^{\alpha}\!\sum_{\sigma} {}^{\alpha}\!\sum_{p} \frac{{}^{\alpha}\!A^{(p)}_\sigma}{{}^{\alpha}\mathcal{J}(0)}\, {}^{\alpha}\!\sum_{\rho} {}^{\alpha}\!\mathcal{A}^{({}^{\alpha}N-1)}_\rho \frac{(-{}^{\alpha}a_\rho)^{{}^{\alpha}N-1-p}}{{}^{\alpha}\mathcal{U}_\rho}\, {}^{\alpha}\boldsymbol{\pi}_\sigma$$

$$\stackrel{\text{(A.8)}}{=} \lambda \tilde{a}_\alpha\, {}^{\alpha}\!\sum_{\sigma} \frac{{}^{\alpha}\!A^{({}^{\alpha}N-1)}_\sigma}{{}^{\alpha}\mathcal{J}(0)}\, {}^{\alpha}\boldsymbol{\pi}_\sigma = \lambda \tilde{a}_\alpha\, {}^{\alpha}\!\sum_{\sigma} \frac{{}^{\alpha}\!J_\sigma(0)}{{}^{\alpha}\mathcal{J}(0)}\, {}^{\alpha}\boldsymbol{\pi}_\sigma\,.$$

Using (1.39), (1.40) and (3.13), the generating function of Killing tensors

(3.18) can be obtained from (1.44) as

$$
\begin{aligned}
\boldsymbol{k}(\beta) &= A(\beta)\sum_\mu \frac{1}{1+\beta^2 x_\mu^2}\boldsymbol{\pi}_\mu = \prod_\nu \left(1+\beta^2 x_\nu^2\right)\sum_\mu \frac{1}{1+\beta^2 x_\mu^2}\boldsymbol{\pi}_\mu \\
&= \prod_\gamma \left(1+\beta^2 x_{\gamma,0}^2\right)\cdot \prod_\gamma {}^\gamma\prod_\sigma \left(1+\beta^2 x_{\gamma,\sigma}^2\right)\times \\
&\qquad \times\left(\sum_\alpha \frac{1}{1+\beta^2 x_{\alpha,0}^2}\boldsymbol{\pi}_{\alpha,0} + \sum_\alpha {}^\alpha\sum_\rho \frac{1}{1+\beta^2 x_{\alpha,\rho}^2}\boldsymbol{\pi}_{\alpha,\rho}\right) \\
&\approx \prod_\gamma \left(1+\beta^2 \tilde{x}_\gamma^2\right)\cdot \prod_\gamma {}^\gamma\prod_\sigma \left(1+\beta^2 \tilde{a}_\gamma^2\right)\times \\
&\qquad \times\left(\sum_\alpha \frac{1}{1+\beta^2 \tilde{x}_\alpha^2}\tilde{\boldsymbol{\pi}}_\alpha - \sum_\alpha \frac{1}{1+\beta^2 \tilde{a}_\alpha^2}\frac{2\lambda\tilde{a}_\alpha^2\tilde{\mathcal{U}}_\alpha}{\tilde{J}(\tilde{a}_\alpha^2)}\,{}^\alpha\sum_\rho {}^\alpha\boldsymbol{\pi}_\rho\right) \\
&= \prod_\gamma \left(1+\beta^2 \tilde{a}_\gamma^2\right)^{\gamma N}\tilde{A}(\beta)\sum_\alpha \left(\frac{1}{1+\beta^2 \tilde{x}_\alpha^2}\tilde{\boldsymbol{\pi}}_\alpha - \frac{1}{1+\beta^2 \tilde{a}_\alpha^2}\frac{2\lambda\tilde{a}_\alpha^2\tilde{\mathcal{U}}_\alpha}{\tilde{J}(\tilde{a}_\alpha^2)}\,{}^\alpha\boldsymbol{g}^{-1}\right) \\
&= \prod_\gamma \left(1+\beta^2 \tilde{a}_\gamma^2\right)^{\gamma N}\sum_\alpha \left(\tilde{A}_\alpha(\beta)\tilde{\boldsymbol{\pi}}_\alpha - \frac{\tilde{A}(\beta)}{1+\beta^2 \tilde{a}_\alpha^2}\frac{2\lambda\tilde{a}_\alpha^2\tilde{\mathcal{U}}_\alpha}{\tilde{J}(\tilde{a}_\alpha^2)}\,{}^\alpha\boldsymbol{g}^{-1}\right) \\
&= \prod_\gamma \left(1+\beta^2 \tilde{a}_\gamma^2\right)^{\gamma N}\tilde{\boldsymbol{K}}(\beta)\,,
\end{aligned}
$$

where

$$
\begin{aligned}
\tilde{\boldsymbol{K}}(\beta) &= \sum_\alpha \left(\tilde{A}_\alpha(\beta)\tilde{\boldsymbol{\pi}}_\alpha - \frac{\tilde{A}(\beta)}{1+\beta^2 \tilde{a}_\alpha^2}\frac{2\lambda\tilde{a}_\alpha^2\tilde{\mathcal{U}}_\alpha}{\tilde{J}(\tilde{a}_\alpha^2)}\,{}^\alpha\boldsymbol{g}^{-1}\right) \\
&= \sum_r \beta^{2r}\sum_\alpha \left[\tilde{A}_\alpha^{(r)}\tilde{\boldsymbol{\pi}}_\alpha - \tilde{B}^{(r)}(\tilde{a}_\alpha^2)\frac{2\lambda\tilde{a}_\alpha^2\tilde{\mathcal{U}}_\alpha}{\tilde{J}(\tilde{a}_\alpha^2)}\,{}^\alpha\boldsymbol{g}^{-1}\right] - \beta^{2\tilde{N}}\sum_\alpha \frac{2\lambda\tilde{a}_\alpha^2\tilde{\mathcal{U}}_\alpha}{1+\beta^2\tilde{a}_\alpha^2}\,{}^\alpha\boldsymbol{g}^{-1} \\
&= \sum_r \beta^{2r}\tilde{\boldsymbol{K}}_{(r)} - \beta^{2\tilde{N}}\sum_\alpha \frac{2\lambda\tilde{a}_\alpha^2\tilde{\mathcal{U}}_\alpha}{1+\beta^2\tilde{a}_\alpha^2}\,{}^\alpha\boldsymbol{g}^{-1}\,,
\end{aligned}
\tag{B.10}
$$

and

$$
\tilde{\boldsymbol{K}}_{(r)} = \sum_\alpha \left[\tilde{A}_\alpha^{(r)}\tilde{\boldsymbol{\pi}}_\alpha - \tilde{B}^{(r)}(\tilde{a}_\alpha^2)\frac{2\lambda\tilde{a}_\alpha^2\tilde{\mathcal{U}}_\alpha}{\tilde{J}(\tilde{a}_\alpha^2)}\,{}^\alpha\boldsymbol{g}^{-1}\right] = \tilde{\boldsymbol{k}}_{(r)} - \sum_\alpha \tilde{B}^{(r)}(\tilde{a}_\alpha^2)\frac{2\lambda\tilde{a}_\alpha^2\tilde{\mathcal{U}}_\alpha}{\tilde{J}(\tilde{a}_\alpha^2)}\,{}^\alpha\boldsymbol{g}^{-1}\,,
$$

which defines the tensors $\tilde{\boldsymbol{k}}_{(r)}$. In (B.10) we have used the following identity

$$
\begin{aligned}
\frac{\tilde{A}(\beta)}{1+\beta^2\tilde{a}_\alpha^2} &= \frac{1}{1+\beta^2\tilde{a}_\alpha^2}\sum_{r=0}^{\tilde{N}}\beta^{2r}\tilde{A}^{(r)} \\
&= \frac{1}{1+\beta^2\tilde{a}_\alpha^2}\left(\sum_{r=0}^{\tilde{N}}\beta^{2r}\sum_{n=0}^{r}\tilde{A}^{(n)}(-\tilde{a}_\alpha^2)^{r-n} - \sum_{r=1}^{\tilde{N}}\beta^{2r}\sum_{n=0}^{r-1}\tilde{A}^{(n)}(-\tilde{a}_\alpha^2)^{r-n}\right) \\
&= \frac{1}{1+\beta^2\tilde{a}_\alpha^2}\left(\sum_{r=0}^{\tilde{N}}\beta^{2r}\sum_{n=0}^{r}\tilde{A}^{(n)}(-\tilde{a}_\alpha^2)^{r-n} + \beta^2\tilde{a}_\alpha^2\sum_{r=0}^{\tilde{N}-1}\beta^{2r}\sum_{n=0}^{r}\tilde{A}^{(n)}(-\tilde{a}_\alpha^2)^{r-n}\right) \\
&= \frac{1}{1+\beta^2\tilde{a}_\alpha^2}\left(\sum_{r=0}^{\tilde{N}-1}\beta^{2r}\sum_{n=0}^{r}\tilde{A}^{(n)}(-\tilde{a}_\alpha^2)^{r-n} + \beta^2\tilde{a}_\alpha^2\sum_{r=0}^{\tilde{N}-1}\beta^{2r}\sum_{n=0}^{r}\tilde{A}^{(n)}(-\tilde{a}_\alpha^2)^{r-n}\right. \\
&\qquad\qquad \left. +\beta^{2\tilde{N}}\sum_{n=0}^{\tilde{N}}\tilde{A}^{(n)}(-\tilde{a}_\alpha^2)^{\tilde{N}-n}\right) \\
&\overset{(A.1)}{=} \sum_{r=0}^{\tilde{N}-1}\beta^{2r}\sum_{n=0}^{r}\tilde{A}^{(n)}(-\tilde{a}_\alpha^2)^{r-n} + \frac{\beta^{2\tilde{N}}}{1+\beta^2\tilde{a}_\alpha^2}\tilde{J}(\tilde{a}_\alpha^2) \\
&= \sum_r \beta^{2r}\tilde{B}^{(r)}(\tilde{a}_\alpha^2) + \frac{\beta^{2\tilde{N}}}{1+\beta^2\tilde{a}_\alpha^2}\tilde{J}(\tilde{a}_\alpha^2)\,,
\end{aligned}
\tag{B.11}
$$

where the last equality defines the functions $\tilde{B}^{(r)}(\tilde{a}_\alpha^2)$.

The alternative set of Killing tensors (3.21) can be acquired from (3.20) in the following manner

$$
\begin{aligned}
\tilde{\boldsymbol{R}}_{(\alpha)} &\overset{(A.7)}{=} \sum_r \frac{(-\tilde{a}_\alpha^2)^{\tilde{N}-1-r}}{\tilde{\mathcal{U}}_\alpha}\tilde{\boldsymbol{K}}_{(r)} \\
&= \sum_r \frac{(-\tilde{a}_\alpha^2)^{\tilde{N}-1-r}}{\tilde{\mathcal{U}}_\alpha}\tilde{\boldsymbol{k}}_{(r)} - \sum_r \frac{(-\tilde{a}_\alpha^2)^{\tilde{N}-1-r}}{\tilde{\mathcal{U}}_\alpha}\sum_\beta \tilde{B}^{(r)}(\tilde{a}_\beta^2)\frac{2\lambda\tilde{a}_\beta^2\tilde{\mathcal{U}}_\beta}{\tilde{J}(\tilde{a}_\beta^2)}\,{}^{\beta}\boldsymbol{g}^{-1} \\
&= \tilde{\boldsymbol{r}}_{(\alpha)} - \sum_{\substack{\beta\\ \beta\neq\alpha}}\sum_r \frac{(-\tilde{a}_\alpha^2)^{\tilde{N}-1-r}}{\tilde{\mathcal{U}}_\alpha}\tilde{B}^{(r)}(\tilde{a}_\beta^2)\frac{2\lambda\tilde{a}_\beta^2\tilde{\mathcal{U}}_\beta}{\tilde{J}(\tilde{a}_\beta^2)}\,{}^{\beta}\boldsymbol{g}^{-1} \\
&\qquad - \sum_r (-\tilde{a}_\alpha^2)^{\tilde{N}-1-r}\tilde{B}^{(r)}(\tilde{a}_\alpha^2)\frac{2\lambda\tilde{a}_\alpha^2}{\tilde{J}(\tilde{a}_\alpha^2)}\,{}^{\alpha}\boldsymbol{g}^{-1} \\
&= \tilde{\boldsymbol{r}}_{(\alpha)} + \sum_{\substack{\beta\\ \beta\neq\alpha}}\frac{2\lambda\tilde{a}_\beta^2}{\tilde{a}_\alpha^2-\tilde{a}_\beta^2}\frac{\tilde{\mathcal{U}}_\beta}{\tilde{\mathcal{U}}_\alpha}\frac{\tilde{J}(\tilde{a}_\alpha^2)-\tilde{J}(\tilde{a}_\beta^2)}{\tilde{J}(\tilde{a}_\beta^2)}\,{}^{\beta}\boldsymbol{g}^{-1} - 2\lambda\tilde{a}_\alpha^2\sum_\beta\left(\tilde{x}_\beta^2-\tilde{a}_\alpha^2\right)^{-1}\,{}^{\alpha}\boldsymbol{g}^{-1}\,,
\end{aligned}
$$

where we have defined

$$\tilde{\boldsymbol{r}}_{(\alpha)} = \sum_r \frac{(-\tilde{a}_\alpha^2)^{\tilde{N}-1-r}}{\tilde{\mathcal{U}}_\alpha} \tilde{\boldsymbol{k}}_{(r)} = \sum_\beta \frac{1}{\tilde{\mathcal{U}}_\alpha} \sum_r \tilde{A}_\beta^{(r)} (-\tilde{a}_\alpha^2)^{\tilde{N}-1-r} \tilde{\boldsymbol{\pi}}_\beta \overset{\text{(A.3)}}{=} \sum_\beta \frac{\tilde{J}_\beta(\tilde{a}_\alpha^2)}{\tilde{\mathcal{U}}_\alpha} \tilde{\boldsymbol{\pi}}_\beta \,.$$

Moreover, we have employed the following identity

$$\begin{aligned}
\sum_r (-\tilde{a}_\alpha^2)^{\tilde{N}-1-r} \tilde{B}^{(r)}(\tilde{a}_\beta^2) &= \sum_{r=0}^{\tilde{N}-1} \sum_{n=0}^{r} \tilde{A}^{(n)} (-\tilde{a}_\alpha^2)^{\tilde{N}-1-r} (-\tilde{a}_\beta^2)^{r-n} \\
&= \sum_{s=0}^{\tilde{N}-1} \tilde{A}^{(s)} (-\tilde{a}_\alpha^2)^{\tilde{N}-1-s} \sum_{m=s}^{\tilde{N}-1} \left(\frac{-\tilde{a}_\beta^2}{-\tilde{a}_\alpha^2}\right)^{m-s} \\
&= \sum_{s=0}^{\tilde{N}-1} \tilde{A}^{(s)} (-\tilde{a}_\alpha^2)^{\tilde{N}-1-s} \sum_{m=0}^{\tilde{N}-1-s} \left(\frac{-\tilde{a}_\beta^2}{-\tilde{a}_\alpha^2}\right)^{m} \,,
\end{aligned}$$

which can be further simplified in case $\tilde{a}_\beta \neq \tilde{a}_\alpha$ to

$$\begin{aligned}
\sum_r (-\tilde{a}_\alpha^2)^{\tilde{N}-1-r} \tilde{B}^{(r)}(\tilde{a}_\beta^2) &= \sum_{s=0}^{\tilde{N}-1} \tilde{A}^{(s)} (-\tilde{a}_\alpha^2)^{\tilde{N}-1-s} \sum_{m=0}^{\tilde{N}-1-s} \left(\frac{-\tilde{a}_\beta^2}{-\tilde{a}_\alpha^2}\right)^{m} \\
&= \sum_{s=0}^{\tilde{N}-1} \tilde{A}^{(s)} (-\tilde{a}_\alpha^2)^{\tilde{N}-1-s} \frac{1-\left(\frac{-\tilde{a}_\beta^2}{-\tilde{a}_\alpha^2}\right)^{\tilde{N}-s}}{1-\left(\frac{-\tilde{a}_\beta^2}{-\tilde{a}_\alpha^2}\right)} \\
&= -\frac{1}{\tilde{a}_\alpha^2 - \tilde{a}_\beta^2} \left(\sum_{s=0}^{\tilde{N}-1} \tilde{A}^{(s)} (-\tilde{a}_\alpha^2)^{\tilde{N}-s} - \sum_{s=0}^{\tilde{N}-1} \tilde{A}^{(s)} (-\tilde{a}_\beta^2)^{\tilde{N}-s} \right) \\
&= -\frac{1}{\tilde{a}_\alpha^2 - \tilde{a}_\beta^2} \left(\sum_{s=0}^{\tilde{N}} \tilde{A}^{(s)} (-\tilde{a}_\alpha^2)^{\tilde{N}-s} - \sum_{s=0}^{\tilde{N}} \tilde{A}^{(s)} (-\tilde{a}_\beta^2)^{\tilde{N}-s} \right) \\
&\overset{\text{(A.1)}}{=} -\frac{\tilde{J}(\tilde{a}_\alpha^2) - \tilde{J}(\tilde{a}_\beta^2)}{\tilde{a}_\alpha^2 - \tilde{a}_\beta^2} \,,
\end{aligned}$$

and in case $\tilde{a}_\beta = \tilde{a}_\alpha$ it can be written as

$$\begin{aligned}
\sum_r (-\tilde{a}_\alpha^2)^{\tilde{N}-1-r} \tilde{B}^{(r)}(\tilde{a}_\alpha^2) &= \sum_{s=0}^{\tilde{N}-1} \tilde{A}^{(s)} (-\tilde{a}_\alpha^2)^{\tilde{N}-1-s} (\tilde{N}-s) \\
&= \frac{\mathrm{d}}{\mathrm{d}(-\tilde{a}_\alpha^2)} \left[\sum_{s=0}^{\tilde{N}} \tilde{A}^{(s)} (-\tilde{a}_\alpha^2)^{\tilde{N}-s} \right] \\
&\overset{\text{(A.1)}}{=} \frac{\mathrm{d}}{\mathrm{d}(-\tilde{a}_\alpha^2)} \tilde{J}(\tilde{a}_\alpha^2) = \sum_\beta \tilde{J}_\beta(\tilde{a}_\alpha^2) \,.
\end{aligned}$$

Finally, the equations (3.22) can be proved using (3.3), (3.8) and the orthogonality relations (2.11) as follows

$$\tilde{\boldsymbol{K}}_{(r)} \cdot \tilde{\boldsymbol{\Xi}} = \sum_{\alpha} \left[\tilde{A}^{(r)}_{\alpha} \left(\tilde{\boldsymbol{\epsilon}}_{\alpha} \tilde{\boldsymbol{\epsilon}}^{\alpha} + \hat{\tilde{\boldsymbol{\epsilon}}}_{\alpha} \hat{\tilde{\boldsymbol{\epsilon}}}^{\alpha} \right) + \tilde{B}^{(r)}(\tilde{a}^2_{\alpha})\, {}^{\alpha}\!\sum_{\rho} \left({}^{\alpha}\boldsymbol{\epsilon}_{\rho}\, {}^{\alpha}\boldsymbol{\epsilon}^{\rho} + {}^{\alpha}\hat{\boldsymbol{\epsilon}}_{\rho}\, {}^{\alpha}\hat{\boldsymbol{\epsilon}}^{\rho} \right) \right] \cdot \sum_{\beta} \hat{\tilde{\boldsymbol{\epsilon}}}_{\beta}$$

$$= \sum_{\alpha} \tilde{A}^{(r)}_{\alpha} \hat{\tilde{\boldsymbol{\epsilon}}}_{\alpha} = \tilde{\boldsymbol{L}}_{(r)}\,,$$

$${}^{\alpha}\boldsymbol{k}_{(p)} \cdot {}^{\alpha}\boldsymbol{\xi} = {}^{\alpha}\!\sum_{\rho} {}^{\alpha}A^{(p)}_{\rho} \left({}^{\alpha}\boldsymbol{\epsilon}_{\rho}\, {}^{\alpha}\boldsymbol{\epsilon}^{\rho} + {}^{\alpha}\hat{\boldsymbol{\epsilon}}_{\rho}\, {}^{\alpha}\hat{\boldsymbol{\epsilon}}^{\rho} \right) \cdot {}^{\alpha}\!\sum_{\sigma} {}^{\alpha}\hat{\boldsymbol{\epsilon}}_{\sigma} = {}^{\alpha}\!\sum_{\rho} {}^{\alpha}A^{(p)}_{\rho}\, {}^{\alpha}\hat{\boldsymbol{\epsilon}}_{\rho} = {}^{\alpha}\boldsymbol{l}_{(p)}\,.$$

Analogous steps can be followed for the equations (3.23)

$$\tilde{\boldsymbol{R}}_{(\alpha)} \cdot \tilde{\boldsymbol{\Xi}} = \left\{ \sum_{\beta} \frac{\tilde{J}_{\beta}(\tilde{a}^2_{\alpha})}{\tilde{\mathcal{U}}_{\alpha}} \left[\tilde{\boldsymbol{\epsilon}}_{\beta} \tilde{\boldsymbol{\epsilon}}^{\beta} + \hat{\tilde{\boldsymbol{\epsilon}}}_{\beta} \hat{\tilde{\boldsymbol{\epsilon}}}^{\beta} + {}^{\alpha}\!\sum_{\rho} \left({}^{\alpha}\boldsymbol{\epsilon}_{\rho}\, {}^{\alpha}\boldsymbol{\epsilon}^{\rho} + {}^{\alpha}\hat{\boldsymbol{\epsilon}}_{\rho}\, {}^{\alpha}\hat{\boldsymbol{\epsilon}}^{\rho} \right) \right] \right.$$

$$\left. - \sum_{\substack{\beta \\ \beta \neq \alpha}} \frac{\tilde{J}(\tilde{a}^2_{\alpha}) - \tilde{J}(\tilde{a}^2_{\beta})}{\left(\tilde{a}^2_{\alpha} - \tilde{a}^2_{\beta} \right) \tilde{\mathcal{U}}_{\alpha}}\, {}^{\beta}\!\sum_{\rho} \left({}^{\beta}\boldsymbol{\epsilon}_{\rho}\, {}^{\beta}\boldsymbol{\epsilon}^{\rho} + {}^{\beta}\hat{\boldsymbol{\epsilon}}_{\rho}\, {}^{\beta}\hat{\boldsymbol{\epsilon}}^{\rho} \right) \right\} \cdot \sum_{\gamma} \hat{\tilde{\boldsymbol{\epsilon}}}_{\gamma}$$

$$= \sum_{\beta} \frac{\tilde{J}_{\beta}(\tilde{a}^2_{\alpha})}{\tilde{\mathcal{U}}_{\alpha}} \hat{\tilde{\boldsymbol{\epsilon}}}_{\beta} = \tilde{\boldsymbol{\Phi}}_{\alpha}\,,$$

$${}^{\alpha}\boldsymbol{r}_{(\rho)} \cdot {}^{\alpha}\boldsymbol{\xi} = {}^{\alpha}\!\sum_{\sigma} \frac{{}^{\alpha}J_{\sigma}({}^{\alpha}a_{\rho})}{{}^{\alpha}\mathcal{U}_{\rho}} \left({}^{\alpha}\boldsymbol{\epsilon}_{\sigma}\, {}^{\alpha}\boldsymbol{\epsilon}^{\sigma} + {}^{\alpha}\hat{\boldsymbol{\epsilon}}_{\sigma}\, {}^{\alpha}\hat{\boldsymbol{\epsilon}}^{\sigma} \right) \cdot {}^{\alpha}\!\sum_{\tau} {}^{\alpha}\hat{\boldsymbol{\epsilon}}_{\tau} = {}^{\alpha}\!\sum_{\sigma} \frac{{}^{\alpha}J_{\sigma}({}^{\alpha}a_{\rho})}{{}^{\alpha}\mathcal{U}_{\rho}}\, {}^{\alpha}\hat{\boldsymbol{\epsilon}}_{\sigma} = {}^{\alpha}\boldsymbol{s}_{(\rho)}\,.$$

And similarly for (3.24)

$$\tilde{\boldsymbol{k}}_{(r)} \cdot \tilde{\boldsymbol{\xi}} = \sum_{\alpha} \tilde{A}^{(r)}_{\alpha} \left(\tilde{\boldsymbol{\epsilon}}_{\alpha} \tilde{\boldsymbol{\epsilon}}^{\alpha} + \hat{\tilde{\boldsymbol{\epsilon}}}_{\alpha} \hat{\tilde{\boldsymbol{\epsilon}}}^{\alpha} \right) \cdot \sum_{\beta} \left(\hat{\tilde{\boldsymbol{\epsilon}}}_{\beta} - {}^{\beta}\!\sum_{\rho} {}^{\beta}\hat{\boldsymbol{\epsilon}}_{\rho} \right) = \sum_{\alpha} \tilde{A}^{(r)}_{\alpha} \hat{\tilde{\boldsymbol{\epsilon}}}_{\alpha}$$

$$= \tilde{\boldsymbol{L}}_{(r)} = \tilde{\boldsymbol{l}}_{(r)} + \sum_{\alpha} \tilde{\mathcal{A}}^{(r)}_{\alpha}\, {}^{\alpha}\boldsymbol{\xi}\,,$$

$$\tilde{\boldsymbol{r}}_{(\alpha)} \cdot \tilde{\boldsymbol{\xi}} = \sum_{\beta} \frac{\tilde{J}_{\beta}(\tilde{a}^2_{\alpha})}{\tilde{\mathcal{U}}_{\alpha}} \left(\tilde{\boldsymbol{\epsilon}}_{\beta} \tilde{\boldsymbol{\epsilon}}^{\beta} + \hat{\tilde{\boldsymbol{\epsilon}}}_{\beta} \hat{\tilde{\boldsymbol{\epsilon}}}^{\beta} \right) \cdot \sum_{\gamma} \left(\hat{\tilde{\boldsymbol{\epsilon}}}_{\gamma} - {}^{\gamma}\!\sum_{\rho} {}^{\gamma}\hat{\boldsymbol{\epsilon}}_{\rho} \right) = \sum_{\beta} \frac{\tilde{J}_{\beta}(\tilde{a}^2_{\alpha})}{\tilde{\mathcal{U}}_{\alpha}} \hat{\tilde{\boldsymbol{\epsilon}}}_{\beta}$$

$$= \tilde{\boldsymbol{\Phi}}_{\alpha} = \tilde{\boldsymbol{s}}_{(\alpha)} + {}^{\alpha}\boldsymbol{\xi}\,,$$

where we have used also (3.11) and (3.9).