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DIPLOMOVÁ PRÁCE

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Použití traktorového kalkulu pro parabolické geometrie

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Děkuji profesoru Součkovi za vedení mé práce, svým rodičům za jejich podporu, bez níž by mé studium nebylo možné, Svatopluku Krýslovi a Petrovi Franekovi za mnohé konzultace a pomoc a i dalším svým přátelům za povzbuzování a kritiku.

Prohlašuji, že jsem svou diplomovou práci napsal samostatně a výhradně s použitím citovaných pramenů. Souhlasím se zapůjčováním práce.

V Praze dne 8. srpna 2008 Vít Tuček

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Název práce: Použití traktorového kalkulu pro parabolické geometrie
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Abstrakt: Operátory symetrií zachovávají prostor harmonických funkcí a asociativní algebra těchto operátorů má zajímavé vlastnosti jak z matematického, tak fyzikálního pohledu. Definice a základní výsledky o této algebře jsou podány v článku Micheala Eastwooda [Eas05a]. Jím odvozené rekurentní formule pro konformně invariantní operátory na libovolné Riemannovské varietě M, které se pro $M = \mathbb{R}^n$ shodují s operátory symetrií, vycházejí z klasické ad hoc metody používané v konformní geometrii a jak sám autor udává, jedná se o "brute-force calculation". V této práci je rozveden výše zmíněný článek a explicitní tvar operátorů algebry vyšších symetrií pro Laplaceovu rovnici je spočten pomocí ambientní konstrukce a adaptovaného repéru.
Klíčová slova: operátor vyšší symetrie, Laplaceův operátor, ambientní konstrukce

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Abstract: The symmetry operators preserve the space of harmonic functions and the associative algebra of these operators has some interesting properties from the mathematical as well as physical point of view. Definition and basic results about this algebra has been given in the paper [Eas05a] by Micheal Eastwood. He derived recurent formulae for conformally invariant operators on any Riemannian manifold M which coincide with the symmetry operators in the case $M = \mathbb{R}^n$. He used the classical ad hoc method of conformal geometry and as he himself admits, it is a "brute-force calculation". This work elaborates the aforementioned paper and the explicit form of the symmetry operators for Laplace equation is calculated using ambient construction and suitable adapted frame.
Keywords: higher symmetry operator, Laplace operator, ambient construction

1 Introduction

One way of obtaining a (smooth or analytic) solution for a partial differential equation is to try to find the so called symmetry group of the equation. Then one can construct new solutions from the known ones simply by the action of the symmetry group. One may also look for the invariant solutions. It is not always possible to find such a group, nevertheless it is the case in many problems arising in physics. The reason behind this fact is that a lot of physical laws can be formulated as a symmetry or invariance condition. This is the way how Lie groups (and a great deal of differential geometry with them) enter the physics.

Another classical approach to (smooth) solutions of the partial differential equations is the method of separation of variables. The word smooth is rightfully in parentheses because this several hundreds years old method provides useful ingredients even for weak or numerical solutions since the separated solutions can often serve as an orthonormal basis in appropriate Hilbert space.

Surprisingly these two methods are tightly interconnected. For example the classical coordinate systems in $\mathbb{R}^3$ that separate the Laplace equation $\Delta f = 0$ are characterized by pairs of commuting second-order differential operators in the so called symmetry algebra of the PDE in question [Mil77]. This symmetry algebra arises from the Lie derivative of the action of PDE's symmetry group.

The aim of this work is to explicitly describe all symmetry operators of the Laplacian on $\mathbb{R}^n$ as introduced in [Eas05a]. Since the symmetry group of Laplacian is the conformal group, we find ourselves in the realm of conformal geometry. There are two important approaches to the conformal geometry - the ambient metric construction and tractor bundles. Because we are concerned here only with the flat (in the Riemannian sense) $\mathbb{R}^n$, the ambient metric construction gives us basically the classical Kleinian description of the conformal geometry as a homogeneous model G/P where the $G = \mathrm{SO}(n+1, 1)$ and P is a parabolic subgroup of G. The tractor approach, while more general and conceptual than the ambient construction, gives nothing new in this simplest case. Thus we will stick to the ambient model which leads to the result more directly and intuitively.

As none of the results depends on the signature of the flat metric chosen in $\mathbb{R}^n$ we will actually work with the Laplace–Beltrami operator[1] on $\mathbb{R}^{p,q}$. The second section is devoted to notation used in this work, while the fourth section serves the purpose of giving the definition of parabolic geometry. The

[1]Please note that this includes hyperbolic operators like the wave operator.

third sections introduces the algebra of symmetry operators of Δ. Description of the ambient model of the conformal geometry on $\mathbb{R}^{p,q}$, which is used for calculations, is presented in the fifth section. Sixth section is devoted to the main results of this work — the calculation of the explicit form of the symmetry operators of the Laplace–Beltrami operator. The work concludes in the section seven with presentation of two interesting theorems concerning the symmetry operators and the symmetry algebra of the Laplacian.

2 Notation

We need to fix the notation first. So $\mathbb{R}$ denotes the set of real numbers and $\mathbb{R}^{p,q}$ stands for (pseudo)euclidean space $\mathbb{R}^{p+q} = \mathbb{R}^n$ with a bilinear form of signature $(p, q, 0)$. All differential operators considered are linear with smooth coefficients unless stated otherwise.

Almost all calculations in this work are done via Penrose's abstract index formalism as introduced in [PR84]. Tensors are (very much by definition) multilinear mappings from a Cartesian product of vector spaces and its duals to the underlying field. The abstract indices serve the purpose of identifying the type of the tensor, they are not the components of the tensor with respect to some basis. Upper indices denote the inputs from the vector space itself, while the lower ones denote the inputs from the dual. The order of the indices indicates the order of the inputs. Hence $T_a{}^b$ is a bilinear mapping $V \times V^* \to \mathbb{R}$ whereas $T^b{}_a$ is a bilinear mapping $V^* \times V \to \mathbb{R}$. Different vector spaces will be distinguished by different scripts in the following sections. Big latin letters are reserved for $\mathbb{R}^{n+2}$, the small ones are for $\mathbb{R}^n$. For the concrete indices we reserve arabic numerals, the symbol ∞ and bold latin letters. If we choose two linearly independent vectors in $\mathbb{R}^{n+2}$ then we can decompose this space as $\mathbb{R} \oplus \mathbb{R}^n \oplus \mathbb{R}$. This will reflect in the notation via

$$v^A = \begin{pmatrix} v^0 \\ v^a \\ v^\infty \end{pmatrix}$$

where v^0 and v^∞ are coordinates of v^A with respect to the two chosen vectors and v^a is a vector from $\mathbb{R}^n$. As usual, even indices can have their own (numerical) indices in order to keep track of their numbers but these will be of course always concrete.

Abstract index notation brings a great advantage to the calculations because the symbols, with which we represent the tensors, form an associative commutative unital algebra where multiplication is defined via string concatenation. This concatenation corresponds naturally to the tensor product. It really is commutative because the tensors $t_{ab}{}^c = v_a u_b{}^c$ and $s_{ab}{}^c = u_b{}^c v_a$ equal the same bilinear mapping.

The round brackets around indices denote the symmetrisation of the tensor while the square brackets represent the antisymmetrisation. Specifically

$$v^{(a_1 \cdots a_n)} = \frac{1}{n!} \sum_{\pi \in \mathrm{S}_n} v^{a_{\pi(1)} \cdots a_{\pi(n)}}$$

$$v^{[a_1 \cdots a_n]} = \frac{1}{n!} \sum_{\pi \in \mathrm{S}_n} \mathrm{sgn}(\pi) v^{a_{\pi(1)} \cdots a_{\pi(n)}}$$

where S_n denotes the group of all permutations on the set $\{1, \dots, n\}$.

The identity mapping on a vector space will be denoted by $\delta_a{}^b$. In the coordinates

$$\delta_{\mathbf{a}}{}^{\mathbf{b}} = \begin{cases} 0 & \mathbf{a} \neq \mathbf{b} \\ 1 & \mathbf{a} = \mathbf{b} \end{cases}$$

Repetition of indices represents the evaluation of the natural pairing between the vector space and its dual. Hence $T_a{}^a$ denotes the trace of the tensor $T_a{}^b$. We can raise or lower the indices of tensor in the presence of metric g_{ab}:

$$T_a{}^b g_{bc} = T_{ac} \qquad T_a{}^b g^{ac} = T^{bc},$$

where g^{ab} is related to g_{ab} by $g^{ab} g_{bc} = \delta^a{}_c$. Finally the tensor $T^{a_1 \cdots a_k}{}_{b_1 \cdots b_l}$ will be called trace-free if any of its traces is zero.

The symbol ∇_A will denote the (Levi–Civita) affine connection on a Riemannian manifold M. It is an operator

$$\nabla_A : TM \otimes_{\mathbb{R}} E \to E$$

or equivalently

$$\nabla_A : E \to TM^* \otimes_{\mathbb{R}} E$$

where E is any tensor bundle made of copies of the tangent and the cotangent bundle of M. Action of this operator is uniquely determined by its action on the tangent bundle. If this operator is used more than once in a formula, the natural interpretation "from the right" is used. Hence $\nabla_A \nabla_B f$ means, that we differentiate function f to obtain a one-form and then we differentiate this one-form to obtain two-form. If a tensor field is written in a formula which contains also covariant derivative, it is understood as the multiplication operator and the Leibniz rule applies. For example

$$\begin{aligned} [T_A, \nabla_B] f &= T_A \nabla_B f - \nabla_B T_A f \\ &= T_A \nabla_B f - (\nabla_B T_A) f - T_A \nabla_B f \\ &= -(\nabla_B T_A) f \end{aligned}$$

Torsion $T_{AB}{}^C$ is defined by the following equality for all smooth functions on the manifold

$$\nabla_A \nabla_B f - \nabla_B \nabla_A f = T_{AB}{}^C \nabla_C f,$$

and *curvature* $R_{AB}{}^C$ is for a torsion-free affine connection ∇ defined by

$$\nabla_A \nabla_B X - \nabla_B \nabla_A X = R_{AB}{}^C \nabla_C X \tag{1}$$

which must hold for all smooth vector fields X. For more thorough discussion of abstract indices and for justification of these formulae see [PR84].

The symbols ∂_A and ∂_a are reserved for the Levi–Civita connection on $\mathbb{R}^{p+1,q+1}$ and $\mathbb{R}^{p,q}$ respectively. Since the metrics are constant it means that $\partial_a = \frac{\partial}{\partial x^a}$ – i.e. it is just a differentiation in coordinates.

Product or sum over empty set is defined to be 1 for convenience. For example $\sum_{i=1}^{0} A_i = \prod_{i,j\in\emptyset} B_{ij} = 1$. Hat over index or symbol in an expression means that this term is omitted. Sometimes the shorthand $v_{A_1 \cdots A_n}$ will be used for $v_{A_1} \cdots v_{A_n}$.

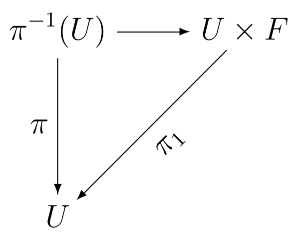
Lets finish this section with a few definitions:

2.1 Definition Let x_0 be a fixed vector in $\mathbb{R}^n$. Then the map

$$x \mapsto \frac{x - x_0}{\|x - x_0\|^2}$$

defined for all $x \in \mathbb{R}^n$ such that $\|x - x_0\| \neq 0$ is called *special conformal transformation.*

2.2 Definition Let E,B and F be smooth manifolds and let $\pi : E \to B$ be a smooth submersion. Then the *fiber bundle* is the four tuple (E, π, B, F) where E is called the *total space*, B is the *base space* and F is the fiber. The preimage $\pi^{-1}(x)$ is diffeomorphic to F for any $x \in B$ and moreover for any chart (U, ϕ) of B there is a commutative diagram



where π_1 is the projection onto the first component.

3 Symmetries of differential operators

Sophus Lie in his attempt to unify various methods of solutions of ordinary and partial differential equations introduced a structure which he called *the infinitesimal group*. Nowadays the standard term for this strucutre is the Lie algebra. For introduction to Lie algebras and Lie groups please consult [Hum72].

Throughout this chapter $\mathcal{L}$ denotes an arbitrary linear differential operator with smooth coefficients on $\mathbb{R}^n$.

3.1 Definition Let G be a Lie group and consider its (right) transitive and effective action on $\mathbb{R}^n$.

$$\rho : \mathbb{R}^n \times G \to \mathbb{R}^n \quad \rho(x, gh) = \rho(\rho(x, g), h)\ \forall x \in \mathbb{R}$$

We will say that (G, ρ) is *the symmetry group* of the differential operator $\mathcal{L}$ if the following is satisfied:

$$\mathcal{L}f(x) = 0 \implies \mathcal{L}(f(\rho(x, g))) = 0\ \forall g \in G \tag{2}$$

Let us use xg to denote the action of $\rho(\ , g)$ on $x \in \mathbb{R}$ from now on. For the next definition we will need the basic identification of the Lie algebra $\mathfrak{g}$ as the tangent space of a Lie group G in the neutral element.

3.2 Definition Let (G, ρ) be a symmetry group of $\mathcal{L}$. The *infinitesimal group symmetry* of $\mathcal{L}$ is the first order differential operator defined by

$$\mathcal{D}_{\mathrm{u}}\, f(x) = \frac{\mathrm{d}}{\mathrm{d}t} f(x \exp(tu))|_{t=0}, \tag{3}$$

i.e. $\mathcal{D}_u$ is just the action of $\mathfrak{g}$ induced by the representation ρ. It is straightforward to verify that this operator preserves the kernel of $\mathcal{L}$.

$$\begin{aligned}
\mathcal{L}\,\mathcal{D}_{\mathrm{u}}\, f(x) &= \mathcal{L}\,\frac{\mathrm{d}}{\mathrm{d}t} f(x \exp(tu))\Big|_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\,\mathcal{L}\, f(x \exp(tu))\Big|_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\, 0\,\Big|_{t=0} \qquad \text{by (2)} \\
&= 0
\end{aligned}$$

This induced representation of $\mathfrak{g}$ extends to the representation of its universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ [Dix96]. Thus we have unital associative filtered algebra of differential operators which preserve the kernel of $\mathcal{L}$. On the other hand one may bypass the group G and define infinitesimal symmetries as those differential operators (of any order) which preserve the kernel of $\mathcal{L}$.

3.3 Definition Differential operator $\mathcal{D}$ is called *infinitesimal symmetry* of $\mathcal{L}$ if the following holds

$$\mathcal{L}u = 0 \implies \mathcal{L}\,\mathcal{D}\,u = 0$$

These operators also form an associative unital algebra filtered by order and in view of the preceding comment, the universal algebra $\mathfrak{U}(\mathfrak{g})$ can be realised as its (sub)algebra in the case where the symmetry group G is present. Infinitesimal symmetries can be used for explicit construction of solutions of the PDE in question (i.e. $\mathcal{L}f = 0$). For details see [Olv93]. In order to find these operators one must however solve a system of partial differential equations.[2] Hence we constrain ourselves to more manageable operators.

3.4 Definition The *symmetry operator* of $\mathcal{L}$ is a differential operator $\mathcal{D}$ such that

$$\mathcal{L}\,\mathcal{D} = \delta\,\mathcal{L}$$

holds for some differential operator δ.

We can offhand provide a plenty of symmetry operators in the form $\mathcal{P}\mathcal{L}$ for $\mathcal{P}$ being any linear differential operator. Nevertheless these operators act trivially on the solution space of $\mathcal{L}$. Ergo it is useful to introduce certain relation of equivalence on the vector space of all symmetry operators of $\mathcal{L}$ to prune away such trivialities. One may also embrace the viewpoint that we care only for operators which have distinguished actions on the solution space of $\mathcal{L}u = 0$.

3.5 Definition Two symmetry operators $\mathcal{D}$ and $\widetilde{\mathcal{D}}$ of the operator $\mathcal{L}$ are called equivalent if there exists a differential operator $\mathcal{P}$ such that $\mathcal{D} - \widetilde{\mathcal{D}} = \mathcal{P}\mathcal{L}$.

3.6 Lemma *The vector space of equivalence classes of symmetry operators forms an associative unital algebra with multiplication being the composition of differential operators.*

[2] And it can very well happen that this system is harder to solve than the original PDE.

Proof We need to show that the operation of composition of differential operators preserves the vector space of symmetry operators and respects their equivalence. If $\mathcal{D}_1$ and $\mathcal{D}_2$ are two symmetry operators, we must find a differential operator δ such that

$$\mathcal{L}\,\mathcal{D}_1\,\mathcal{D}_2 = \delta\,\mathcal{L}\,. \tag{4}$$

From the definition of a symmetry operator it follows that there are differential operators δ_1 and δ_2 which satisfy the relations

$$\mathcal{L}\,\mathcal{D}_1 = \delta_1\,\mathcal{L} \quad \mathcal{L}\,\mathcal{D}_2 = \delta_2\,\mathcal{L}$$

If we substitute these relations into (4)

$$\mathcal{L}\,\mathcal{D}_1\,\mathcal{D}_2 = \delta_1\,\mathcal{L}\,\mathcal{D}_2 = \delta_1\delta_2\,\mathcal{L}$$

we immediately see that our desired δ equals to $\delta_1\delta_2$.

The second part states that the composition $\widetilde{\mathcal{D}}_1\widetilde{\mathcal{D}}_2$ of two symmetry operators $\widetilde{\mathcal{D}}_1, \widetilde{\mathcal{D}}_2$ (equivalent to $\mathcal{D}_1$ and $\mathcal{D}_2$ respectively) is equivalent to $\mathcal{D}_1\,\mathcal{D}_2$. In other words we are looking for some differential operator $\mathcal{P}$ such that $\mathcal{D}_1\,\mathcal{D}_2 - \widetilde{\mathcal{D}}_1\widetilde{\mathcal{D}}_2 = \mathcal{P}\mathcal{L}$. If we establish operators $\mathcal{P}_1, \mathcal{P}_2, \delta_2$ in an obvious manner we get

$$\begin{aligned}
\mathcal{D}_1\,\mathcal{D}_2 - \widetilde{\mathcal{D}}_1\widetilde{\mathcal{D}}_2 &= (\mathcal{P}_1\,\mathcal{L} + \widetilde{\mathcal{D}}_1)\,\mathcal{D}_2 - \widetilde{\mathcal{D}}_1(\mathcal{D}_2 - \mathcal{P}_2\,\mathcal{L}) = \\
&= \mathcal{P}_1\mathcal{L}\mathcal{D}_2 + \widetilde{\mathcal{D}}_1\mathcal{P}_2\,\mathcal{L} = \\
&= (\mathcal{P}_1\delta_2 + \widetilde{\mathcal{D}}_1\mathcal{P}_2)\,\mathcal{L}
\end{aligned}$$

and hence we can set $\mathcal{P} = \mathcal{P}_1\delta_2 + \widetilde{\mathcal{D}}_1\mathcal{P}_2$. □

3.7 Definition The *symmetry algebra* $\mathscr{A}_{\mathcal{L}}$ of a differential operator $\mathcal{L}$ consists of equivalence classes of symmetry operators.

The definition of the symmetry algebra of $\mathcal{L}$ can be imitated for the infinitesimal symmetries as follows.

3.8 Definition Two infinitesimal symmetries $\mathcal{D}$ and $\widetilde{\mathcal{D}}$ of $\mathcal{L}$ are called equivalent if

$$(\mathcal{D} - \widetilde{\mathcal{D}})u = 0$$

for all u such that $\mathcal{L}u = 0$.

The *algebra of infinitesimal symmetries* $\mathscr{I}_{\mathcal{L}}$ of a differential operator $\mathcal{L}$ consists of equivalence classes of infinitesimal symmetries.

3.9 Remark In general $\mathscr{A}_{\mathcal{L}} \subseteq \mathscr{I}_{\mathcal{L}}$ because if $\mathcal{L}u = 0$ then we have

$$\mathcal{L}\,\mathcal{D}\,u = \delta\mathcal{L}u = \delta\,0$$

for any $\mathcal{D} \in \mathscr{A}_{\mathcal{L}}$. Moreover it is trivial to see that the translations belong to the symmetry group of any $\mathcal{L}$ with constant coefficients.

If we look at the first order symmetry operators, we see that they form a Lie algebra[3] which is clearly a subalgebra of the Lie algebra of first order infinitesimal symmetries.

The infinitesimal symmetries (as well as the symmetry operators) of order one or less form a Lie algebra with the Lie bracket being the commutator of differential operators. The associative algebras generated by these operators can be identified with the universal enveloping algebras of the appropriate Lie algebras [Dix96]. This point of view allows the use of the representation theory of Lie algebras for classification of invariant solutions [Mil77].

The relationship of these algebras is unclear in the general case. In addition, there are differential operators whose symmetry algebra is not generated by the first order symmetry operators [Mil77].

3.10 Example The time-dependent Schrödinger equation

$$(i\partial_t + \partial_{xx} + \partial_{yy} - \frac{\alpha}{x^2} - \frac{\beta}{y^2})\psi = 0$$

admits second-order symmetries but no nontrivial first-order symmetries. Boyer showed that this equation R-separates[4] in 25 coordinate systems for $\alpha = 0,\ \beta \neq 0$ and in 15 coordinate systems for $\alpha \neq 0,\ \beta \neq 0$. Moreover, he found that each separable coordinate systems corresponds to a pair of commuting second-order symmetry operators.

Nevertheless in the case of the Laplace equation $\sum_i \partial_{ii} f = 0$ the symmetry algebra is generated by the first order symmetry operators and the 13 separable coordinate systems on $\mathbb{R}^3$ are also characterised by commuting pairs of second-order symmetry operators. For the wave equation on $\mathbb{R}^3$ there is one commuting pair which doesn't correspond to any separable system [Mil77].

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[3] Since the commutator of two first order operators is again a first order operator.

[4] For definition see the final section.

4 Parabolic geometries

4.1 Cartan geometries

Let us start this section by a citation of Felix Klein who in his work [Kle93] redefined geometry as the study of coset spaces of Lie groups.

> The following comprehensive problem then arises as a generalisation of geometry: "Given a manifoldness and a group of transformations of the same; to investigate the configurations belonging to the manifoldness with regard to such properties as are not altered by the transformation of the group."
>
> Felix Klein, 1872

In other words we are led to investigation of G-invariant objects defined on a manifold M where G is a Lie group that has a (left) action on M. If we assume that this action is transitive then we can for any fixed $m \in M$ construct the map $a_m : g \mapsto gm$ which is onto M. The inverse image $H_m := a_m^{-1}(m) = \{g \in G \,|\, gm = m\}$ is a closed subgroup of G called the stabiliser of m. Moreover if we take $g_1, g_2 \in a_m^{-1}(m')$ then $g_2^{-1}g_1 \in H_m$ and thus $g_1 \in g_2 H_m$. It follows that M is isomorphic to G/H_m. We can summarise this as follows: "Description of a geometry with base point as a pair (M, m) together with the Lie group G is equivalent to description of pairs (G, H) where H is a closed subgroup of G." If we choose another basepoint $m' \in M$ and take the map $a_{m'}$ instead of a_m then the subgroups H_m and H'_m are conjugate by an inner automorphism.

The action of G on M is a group homomorphism $G \to \mathrm{Diff}(M)$. Kernel of this homomorphism is of course a normal subgroup of G and consists of the elements $\{g \in G \,|\, gm = m \;\forall m \in M\}$. Thus it is also a subgroup of H_m. Moreover it is the largest subgroup of H_m which is normal in G.

4.1 Definition A *Klein geometry* is a pair (G, H), where G is a Lie group and $H \subset G$ closed subgroup[5].

The *kernel* of a Klein geometry (G, H) is the largest subgroup K of H that is normal in G. A Klein geometry is called *effective*[6] if $K = 1$ and *locally effective* if K is discrete.

The space $M := G/H$ is called the *space of the Klein geometry.*

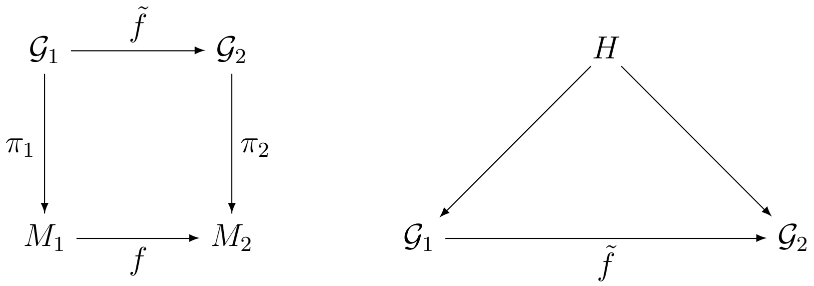
[5] Usually the space G/H is assumed to be connected. If the connectedness assumption is dropped then we speak of *homogeneous spaces*.

[6] The noneffective geometries are plays the role when describing spin phenomena.

It is not a priori clear whether G/H is a smooth manifold or not. In fact, even more is true — the group G is a principal H bundle over G/H (4.6) which means that not only is the quotient G/H smooth manifold but the projection mapping $G \to G/H$ is a smooth submersion.

4.2 Definition A *principal* right H *bundle* is a smooth fibre bundle (2.2) $(\mathcal{G}, \pi, M, F)$ together with a right action $\mathcal{G} \times H \to \mathcal{G}$ that is fibre preserving and acts simply transitively on each fibre[7].

The *morphism* of two principal H bundles $(\mathcal{G}_i, \pi_i, M_i, F)$, $i \in \{1, 2\}$ is a pair of smooth maps $(f, \tilde{f})$ such that the following diagrams commute



Now we will give characterisation of principal H bundles in terms of the properties of the right H action on $\mathcal{G}$.

4.3 Definition Let $\mathcal{G}$ be a smooth manifold, H a Lie group and $\mathcal{G} \times H \to \mathcal{G}$ a smooth action.

1. The action is called *free* if $gh = g$ for some $g \in \mathcal{G}$ implies $h = 1$.

2. The action is called *proper* if $\{h \in H \mid Ah \cap B \neq \emptyset\}$ is compact for any A and B compact.

4.4 Theorem *Let $\mathcal{G}$ be a smooth manifold, H a Lie group, and $\mu : \mathcal{G} \times H \to \mathcal{G}$ a smooth, free, proper right action. Then*

1. *$\mathcal{G}/H$ with the quotient topology is a topological manifold ($\dim \mathcal{G}/H = \dim \mathcal{G} - \dim H$).*

2. *$\mathcal{G}/H$ has a unique smooth structure for which the canonical projection $\pi : \mathcal{G} \to \mathcal{G}/H$ is a submersion.*

3. *$(\mathcal{G}, \pi, \mathcal{G}/H, H)$ is a smooth principal right H bundle.*

Proof See [Sha97] Appendix E. □

[7] Simple action has (by definition) only trivial stabilisers and hence F is in bijective correspondence with H.

The next two propositions, whose proof can also be found in [Sha97], show that we have really have a characterisation of principal bundles and that the Klein geometries are indeed principal bundles.

4.5 Proposition *Let $(\mathcal{G}, \pi, M, H)$ be a right principal H bundle. Then the action $\mathcal{G} \times H \to \mathcal{G}$ is free and proper.*

4.6 Lemma *Let G be a Lie group and H its closed subgroup. Then the action $G \times H \to G$ given by multiplication is free and proper.*

Of course one manifold M can carry multiple different Klein geometries as the next two simple examples show.

4.7 Example (Euclidean plane) In this case $M = \mathbb{R}^n$. The Lie group of *symmetries* or *congruences* or *rigid motions* is

$$G = Euc_n(\mathbb{R}) = \left\{ \begin{pmatrix} 1 & 0 \\ v & A \end{pmatrix} \middle| \ AA^T = I, v \in \mathbb{R}^n \right\}$$

which acts on M by the formula

$$\begin{pmatrix} 1 & 0 \\ v & A \end{pmatrix} \cdot x = Ax + v, \quad \text{where} \quad x \in \mathbb{R}^n.$$

An easy calculation shows that the stabiliser of the origin of $\mathbb{R}^n$ is the subgroup of orthogonal transformations

$$\mathrm{O}_n(\mathbb{R}) \approx H = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \middle| \ A \in \mathrm{O}_n(\mathbb{R}) \right\}.$$

Then we have the basic identification $\mathrm{Euc}_n(\mathbb{R})/\mathrm{O}_n(\mathbb{R}) \approx \mathbb{R}^n$. In fact, $\mathrm{Euc}_n(\mathbb{R})$ is isomorphic to $\mathrm{O}_n(\mathbb{R}) \ltimes \mathbb{R}^n$ (the semidirect product with respect to the standard representation of $\mathrm{O}_n(\mathbb{R})$ on $\mathbb{R}^n$).

4.8 Example (Affine space) Here the base space is also $\mathbb{R}^n$ and the group G is the n-dimensional affine group:

$$G = \mathrm{Aff}_n^+(\mathbb{R}) = \left\{ \begin{pmatrix} 1 & 0 \\ v & A \end{pmatrix} \middle| \ A \in \mathrm{GL}_n(\mathbb{R}), \det A > 0, v \in \mathbb{R}^n \right\}.$$

The action on M is given by the formula

$$\begin{pmatrix} 1 & 0 \\ v & A \end{pmatrix} \cdot x = Ax + v,$$

where x and v are column vectors. Again, it is easily verified that the stabiliser of the origin is

$$H = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \in \mathrm{GL}_{n+1}(\mathbb{R}) \,\middle|\, A \in \mathrm{GL}_n(\mathbb{R}), \det A > 0 \right\} \approx \mathrm{GL}_n(\mathbb{R})$$

and $G \approx H \ltimes \mathbb{R}^2$(semidirect product).

Note that in the first example are the groups G and H disconnected, while in the second one they're both connected. We will give one more example. This time of the non-Euclidean geometry in the sense that it violates the fifth axiom.

4.9 Example (Hyperbolic plane) In this case $M = \{z = x + iy \in \mathbb{C}\ y > 0\}$. The group is the group of Möbius transformations $G = \mathrm{SL}_2(\mathbb{R})$, which acts on M by the formula

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az + b}{cz + d}.$$

It is easy to check that this is indeed an action on M and also that the stabiliser of the point $i \in M$ is the circle group $\mathrm{SO}_2(\mathbb{R})$.

The last fact needed to lead our way to the (global) definition of a Cartan geometry is the existence and properties of the Maurer-Cartan form on any Lie group G.

4.10 Definition Let G be a Lie group and $\mathfrak{g}$ its associated Lie algebra[8]. Let $L_g : G \to G$ denote the left translation map $f \mapsto gf$. The Maurer-Cartan form ω_G is the left-invariant $\mathfrak{g}$ valued one form on G defined by

$$\omega_G(v) = L_{g^{-1}*}(v)$$

for $v \in T_g(G)$.

Now we can proceed to the generalisation of Klein geometry as a principal bundle equipped with one form with values in Lie algebra.

4.11 Definition (Cartan geometry) Let $R_h : H \to H$ denote the right translation map $f \mapsto fh$. Let G be a Lie group with Lie subgroup H. Let $\mathfrak{g}$ and $\mathfrak{h}$ denote the appropriate Lie algebras of G and H respectively. A *Cartan geometry* $(\mathcal{G}, \omega)$ on M modelled on (G, H) consists of the following data:

[8] Ie $\mathfrak{g}$ is the tangent space of G at the identity element.

1. a smooth manifold M;
2. a principal right H bundle $\mathcal{G}$ over M with the right smooth action $\mu : \mathcal{G} \times H \to \mathcal{G}$;
3. a $\mathfrak{g}$-valued 1-form ω on $\mathcal{G}$ satisfying the following conditions:
    (a) $\omega_g : T_g(\mathcal{G}) \to \mathfrak{g}$ is linear isomorphism for each $g \in \mathcal{G}$;
    (b) $(R_h)^*\omega = \mathrm{Ad}(h^{-1})\omega$ for all $h \in H$;
    (c) $\omega(X^\dagger) = X$ for all $X \in \mathfrak{h}$, where $(X^\dagger)_g = \mu_{*(p,1)}(0, X)$.

The *morphism* of two Cartan geometries $(\mathcal{G}_1, \omega_1), (\mathcal{G}_2, \omega_2)$ modelled on M_1 and M_2 respectively is a morphism $(f, \tilde{f})$ of principal H bundles such that f is local diffeomorphism and the pullback of ω_2 is ω_1.

The *flat model* of a Cartan geometry $(\mathcal{G}, \omega)$ modelled on (G, H) is the Klein geometry (G, H). We say that the Cartan geometry is *effective* if the model geometry is effective.

4.12 Remark The vector field $X^\dagger$ is called the *fundamental vector field* and in the case when $\mathcal{G}$ is in fact a Lie group it is the left invariant vector field associated to X which is defined by $(X^\dagger)_g = L_{g*}X$.

We present only definition of the global form of Cartan geometry. The local form is based on the notion of a Cartan gauge and as the word "gauge" suggest this form is preferred among physicists. These two definitions agree only when the model Klein geometry is effective. For details consult [Sha97].

It is evident, that the knowledge of the whole group G is not necessary to give definition of the Cartan geometry since it relates only to the flat model of the geometry in question. Thus one speaks about geometry modelled on $(\mathfrak{g}, H)$ usually. On the other hand, in practice one starts with some geometrical data (lets say a positive definite section of $\bigodot^2 T^*M$ or family of connections) and tries to find the appropriate Cartan geometry. Sometimes in these cases it is clear what should be the group G and the definition of H follows by demanding the invariance of the geometrical data. Also see remark 4.23.

It is shown in [Sha97] that a Maurer-Cartan form satisfies the three conditions given in the preceding definition. Moreover it also satisfies the so called Maurer-Cartan equation:

$$\mathrm{d}\omega + \frac{1}{2}[\omega, \omega] = 0.$$

4.13 Definition Let $(\mathcal{G}, \omega)$ be a Cartan geometry on M modelled on $(\mathfrak{g}, H)$. Then the *curvature* of this geometry is a two form defined by

$$\Omega = \mathrm{d}\omega + \frac{1}{2}[\omega, \omega].$$

Cartan geometries with vanishing curvature are called *flat* and it can be shown ([ČS08]) that they are locally isomorphic to the flat model.

If $\rho : \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ is the canonical projection, then $\rho(\Omega)$ is called the *torsion*. Hence if the form Ω takes values only in the subalgebra $\mathfrak{h}$ then the geometry is *torsion-free*.

4.2 Parabolic subgroups

Now we will compile the basic definitions and facts about representations of semisimple Lie algebras.

Let $\mathfrak{g}$ be a (real or complex) *semisimple Lie algebra*. Let's fix a maximal commutative subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ called *Cartan subalgebra* The elements of $\mathfrak{h}^*$ are called *weights*.

A *representation* of $\mathfrak{g}$ is a vector space $\mathbb{V}$ that is a $\mathfrak{g}$-module, i.e. there is a homomorphisms of Lie algebras $\varphi : \mathfrak{g} \to \mathfrak{gl}(\mathbb{V})$. For a representation $\mathbb{V}$ and $\mu \in \mathfrak{h}^*$, we define the *weight space* $\mathbb{V}^\mu$ of weight μ by

$$\mathbb{V}^\mu := \{v \in \mathbb{V},\ \forall h \in \mathfrak{h}\ \ \varphi(h)(v) = \mu(h)v\ \}.$$

A representation $\mathbb{V}$ is called a *highest weight module*, if it is generated by a weight vector v_μ that is annihilated by all the positive root spaces in $\mathfrak{g}$.

The *roots* of $\mathfrak{g}$ are the weights of the adjoint representation of $\mathfrak{g}$ for which the weight space is nontrivial. Let Φ denote the set of all roots and define the set Φ^+ of *positive roots* for $(\mathfrak{g}, \mathfrak{h})$ as any subset of Φ which satisfies two conditions:

1. for each root $\alpha \in \Phi$ exactly one of the roots $\alpha, -\alpha$ is contained in Φ^+
2. for any $\alpha, \beta \in \Phi^+$ such that $\alpha + \beta$ is a root is $\alpha + \beta \in \Phi^+$

The *root space* $\mathfrak{g}_\phi$ corresponding to the root ϕ is one-dimensional and consists of elements e_ϕ such that $[h, e_\phi] = \phi(h)e_\phi$ for each $h \in \mathfrak{h}$.

It is well known that, for $\mathfrak{g}$ semisimple, $\mathfrak{g} = \mathfrak{h} \oplus_{\phi \in \Phi} \mathfrak{g}_\phi$. For a fixed $(\mathfrak{g}, \mathfrak{h}, \Phi^+)$, we define $\Delta = \{\alpha_1, \dots, \alpha_n\}$ to be the set of *simple roots* (basis of $\mathfrak{h}^*$ so that each positive root is an integral combination of α_i's with nonnegative coefficients).

The *Killing form* $(a, b) \mapsto \mathrm{Tr}(\mathrm{ad}(a)\mathrm{ad}(b))$ defines a duality between $\mathfrak{h}$ and $\mathfrak{h}^*$. For each root ϕ we define the *coroot* $H_\phi := \frac{2}{(\phi,\phi)}\phi \in \mathfrak{h}$ where we identified ϕ with an element of $\mathfrak{h}$ via the Killing form.

The *fundamental weights* $\varpi_1, \dots, \varpi_n$ are elements of $\mathfrak{h}^*$ dual to the simple coroots $H_{\alpha_1}, \dots, H_{\alpha_n}$. Fundamental weights form a basis of $\mathfrak{h}^*$. We define an ordering on the weights by $\mu \leq \lambda$ if and only if $\lambda - \mu$ is a sum of positive roots with nonnegative integral coefficients. When we write weight μ as $(a_1, a_2, \dots, a_n)$ we are reffering to the coefficients of μ with respect to the fundamental weights, ie $\mu = \sum_{i=1}^n a_i \varpi_i$.

A weight $\mu \in \mathfrak{h}$ is said to be *dominant*, if $H_\alpha(\mu) \geq 0$ for all $\alpha \in \Delta$ and *strictly dominant*, if $H_\alpha(\mu) > 0$ for all $\alpha \in \Delta$. The set of dominant weights is sometimes called *fundamental Weyl chamber*. A weight $\mu \in \mathfrak{h}$ is called to be *integral*, if $H_\alpha(\mu) \in \mathbb{Z}$ for all $\alpha \in \Delta$. We denote by P the set of integral weights (it is called *weight lattice*) and by P^{++} the set of dominant integral weights.

We see that a weight is integral, if it is an integral combination of fundamental weights and dominant, if it is a nonnegative combination of fundamental weights.

The following statements can be found in, e.g. [GW98, FH91, Hum72]:

4.14 Theorem *For $\mathfrak{g}$ semisimple, each finite-dimensional representation $\mathbb{V}$ of $\mathfrak{g}$ is a direct sum of irreducible representations. $\mathbb{V}$ is a direct sum of its weight spaces. There is a one to one correspondence between isomorphism classes of finite dimensional irreducible representations and P^{++}. This correspondence assigns to each $\mu \in P^{++}$ an irreducible finite dimensional highest weight module $\mathbb{V}_\mu$ with highest weight μ (and all such modules are isomorphic).*

4.15 Definition (Parabolic subalgebra) For the triple $(\mathfrak{g}, \mathfrak{h}, \Phi^+)$ where $\mathfrak{g}$ is semisimple, $\mathfrak{h}$ a Cartan subalgebra and Φ^+ the set of positive roots, we define the Lie algebra $\mathfrak{n} := \oplus_{\phi \in \Phi^+} \mathfrak{g}_\phi$. Further, we define the *Borel subalgebra*[9] $\mathfrak{b} := \mathfrak{h} \bigoplus \mathfrak{n}$.

We call any subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ a *parabolic subalgebra*, if it contains $\mathfrak{b}$ (associated to some choice of $(\mathfrak{h}, \Phi^+)$. In that case, $\mathfrak{h}$ is also a Cartan subalgebra of $\mathfrak{p}$ and Φ^+ is a set of roots for $(\mathfrak{p}, \mathfrak{h})$ as well (however, $\mathfrak{p}$ usually does not contain all negative root spaces of $\mathfrak{g}$).

4.16 Remark There is a $1 \leftrightarrow 1$ correspondence between parabolic subalgebras of $\mathfrak{g}$ (by fixed $\mathfrak{h}, \Phi^+$) and subsets of Δ. To any $\Sigma \subset \Delta$ we assign the

[9]The Borel subalgebra can also be defined as the maximal solvable subgroup of $\mathfrak{g}$. Analogously the *Borel subgroup* is the maximal solvable subgroup of G.

parabolic Lie algebra $\mathfrak{p}_\Sigma := \sum_{\phi \in A} \mathfrak{g}_{-\phi} \oplus \mathfrak{b}$ where $A \subset \Phi^+$ consists of those positive roots that can be expressed as a sum of simple roots that are not in Σ. Each parabolic subalgebra $\mathfrak{p}$ is of this type.

The following lemma (whose proof can be found in [ČS08]) shows that the parabolic subalgebras are equivalent to gradings of the lie algebra $\mathfrak{g}$.

4.17 Lemma *There is a bijective correspondence between parabolic subalgebras of $\mathfrak{g}$ and gradings $\mathfrak{g} = \oplus_{i=-k}^{k} \mathfrak{g}_i$ of $\mathfrak{g}$.*

Given $\Sigma \subset \Delta$, the set $\mathfrak{g}_i$ $(i \neq 0)$ is defined to be $\oplus_{\phi \in A_i} \mathfrak{g}_\phi$, where A_i contains elements $\phi = \sum_{\alpha_j \in \Delta} c_j \alpha_j$ such that $\sum_{\{j:\alpha_j \in \Sigma\}} c_j = i$, and $\mathfrak{g}_0 = \mathfrak{h} \oplus_{\phi \in A_0} \mathfrak{g}_\phi$.

Given a grading $\oplus_j \mathfrak{g}_j$, the parabolic subalgebra is then $\mathfrak{p} = \oplus_{j \geq 0} \mathfrak{g}_j$.

The proof of the following lemma is taken from [ČS08].

4.18 Lemma *Let $\mathfrak{g} = \oplus_{i=-k}^{k} \mathfrak{g}_i$ of $\mathfrak{g}$ be a $|k|$-graded semisimple Lie algebra over $\mathbb{R}$ or $\mathbb{C}$. There is a unique element $E \in \mathfrak{g}$ such that $[E, X] = jX, \quad \forall X \in \mathfrak{g}_j$. The element E lies in the center of the subalgebra $\mathfrak{g}_0 \leq \mathfrak{g}$.*

Proof Define $D : \mathfrak{g} \to \mathfrak{g}$ given by $D(X) := jX$ for $X \in \mathfrak{g}_j$. Since $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$ we have

$$D([X,Y]) = (i+j)[X,Y] = i[X,Y] + j[X,Y] = [D(X),Y] + [X, D(Y)]$$

for $X \in \mathfrak{g}_i$ and $Y \in \mathfrak{g}_j$. It follows that D is a derivation of $\mathfrak{g}$ - that is $D([X,Y]) = [D(X),Y] + [X,D(Y)], \quad \forall X, Y \in \mathfrak{g}$. Since any derivation in a semisimple Lie algebra is inner [ČS08], there exists unique $E \in \mathfrak{g}$ such that $D(X) = [E, X], \quad \forall X \in \mathfrak{g}$.

Decomposing $E = E_{-k} + \cdots + E_k$ with $E_i \in \mathfrak{g}_i$, we get

$$0 = [E, E] = \sum_{j=-k}^{k} [E, E_j] = \sum_{j=-k}^{k} jE_j.$$

This implies that all $E_i = 0$ except E_0 and thus $E = E_0$. Since by definition $[E, X] = 0X$ for all $X \in \mathfrak{g}_0$ we see, that E is in the center of $\mathfrak{g}_0$. □

It can be shown that $\mathfrak{g}_0 = \mathfrak{g}_0^{ss} \oplus \mathfrak{z}$ where $\mathfrak{g}_0^{ss}$ is semisimple and $\mathfrak{z}$ is the center of $\mathfrak{g}_0$. Dimension of $\mathfrak{z}$ is equal to the cardinality of Σ, so in case $\Sigma = \{\alpha_k\}$, $\mathfrak{z}$ is generated by the grading element. Clearly, $\mathfrak{h}$ is a Cartan subalgebra for $\mathfrak{g}_0$, $\Delta - \Sigma$ is a set of simple roots and $\{\varpi_j;\ \alpha_j \notin \Sigma\}$ is a set of fundamental weights for it. Any irreducible representation $\mathbb{V}$ of $\mathfrak{g}_0^{ss}$ can be extended to a representation of $\mathfrak{g}_0$ by letting $\mathfrak{z}$ act by $z \cdot v = \nu(z)v$ where $\nu \in \mathfrak{z}^*$ is arbitrary.

Let μ be any weight of $\mathbb{V}$. The number $\mu(E)$ is called *generalized conformal weight* (because $\mathbb{V}$ is generated, as a $\mathfrak{g}_0$-module, by a highest weight vector and E is in the center of $\mathfrak{g}_0$, this is independent of the choice of μ). Since the conformal geometry is 1-graded, this notation is justified by the fact that $\mu(E)$ equals minus the 'usual' conformal weight.

Further, any irreducible $\mathfrak{g}_0$-module $\mathbb{V}$ can be extended to an irreducible representation of the whole $\mathfrak{p}$, letting $\mathfrak{p}^+ = \sum_{j>0} \mathfrak{g}_j$ act trivially. On the other hand, if $\mathbb{V}$ is an irreducible representation of $\mathfrak{p}$, the action of $\mathfrak{p}^+$ must be trivial on it (it follows from Engel's theorem on nilpotent Lie algebras), so finite-dimensional irreducible representations of $\mathfrak{p}$ are completely described by the highest weight of $\mathbb{V}$ as a $\mathfrak{g}_0$-module.

Let us denote by $P_\mathfrak{p}$ the set of weights μ such that $H_\alpha(\mu) \in \mathbb{Z}$ for all $\alpha \in \Delta - \Sigma$ and call it $\mathfrak{p}$-integral weights (or $\mathfrak{g}_0^{ss}$-integral). We say that a weight μ is $\mathfrak{p}$-dominant (or $\mathfrak{g}_0^{ss}$-dominant) if $H_\alpha(\mu) \geq 0$ for all $\alpha \in \Delta - \Sigma$. Similarly, we define a strictly $\mathfrak{p}$-dominant weight. We denote by $P_\mathfrak{p}^{++}$ the set of $\mathfrak{p}$-integral and $\mathfrak{p}$-dominant weights. Note that $P_\mathfrak{p}$ is not a lattice, but it consists of $\dim(\mathfrak{z})$-dimensional planes in $\mathfrak{h}^*$. A weight is in $P_\mathfrak{p}^{++}$ exactly if it is expressed as $\sum_j c_j \varpi_j$ so that c_j is a nonnegative integer for each j such that $\alpha_j \notin \Sigma$. We see that there is a $1 \leftrightarrow 1$ correspondence between $P_\mathfrak{p}^{++}$ and the set of isomorphism classes of irreducible finite dimensional $\mathfrak{p}$-modules. Clearly $P^{++} \subset P_\mathfrak{p}^{++}$.

4.3 Tractor bundles

4.19 Definition Let G be a semisimple Lie group and let $\mathfrak{p}$ be a parabolic subgroup of $\mathfrak{g}$. Any subgroup of G which has $\mathfrak{p}$ as its Lie algebra is called *parabolic subgroup* of G. The *parabolic geometry* on a manifold M is the Cartan geometry modelled on $(\mathfrak{g}, P)$ for P parabolic subgroup of G.

4.20 Definition Let $(\mathcal{G}, \pi, M, F)$ be a principal right H bundle and let there be a left action of H on a manifold N. Then the *associated bundle* $\mathcal{G} \times_H N$ is defined as the quotient of $\mathcal{G} \times N$ by the relation $(g, n) \simeq (gh, h^{-1}n)$. The elements of $\mathcal{G} \times N$ will be denoted by square brackets - $[g, n] = [gh, h^{-1}n]$. Since H acts fiberwise on $\mathcal{G}$ the projection π is well defined on $\mathcal{G} \times N$.

Let G be a Lie group and let N be a linear representation $\mathbb{V}$ of the Lie group $H \leq G$. Denoting the representation of H on $\mathbb{V}$ by ρ we write the associated bundle as $G \times_\rho \mathbb{V}$ and we call it the *homogeneous vector* bundle.

4.21 Lemma *Any smooth section of $\mathcal{G} \times_H N$ can be identified with the smooth function $f : \mathcal{G} \to \mathbb{V}$ which is H invariant*

$$f(gh) = h^{-1} f(g) \quad \forall g \in \mathcal{G}, \forall h \in H.$$

Proof Let γ be a smooth section of $\mathcal{G} \times_H N$. Then for any $m \in M$ we can write $\gamma(m) = [g, f(g)]$, where $g \in \mathcal{G}$ such that $\pi(g) = m$ and $f(g)$ is in N. Choosing another representative $[gh, h^{-1}f(g)]$ shows that the section γ defines a function $g \mapsto f(g)$ which has the appropriate invariance property.

On the other hand, with such an invariant function f we can define the section γ by setting $\gamma(m) := [g, f(g)]$. □

4.22 Definition Let $(\mathcal{G}, \omega)$ be a Cartan geometry on M modelled on (G, P). The *adjoint tractor bundle* of this geometry is the bundle associated to the adjoint representation of P on $\mathfrak{g}$

$$\mathcal{A} = \mathcal{G} \times_{\mathrm{Ad}} \mathfrak{g}.$$

Let $\mathbb{V}$ be a representation of P such that the induced representation of $\mathfrak{p}$ extends to a effective representation of $\mathfrak{g}$[10]. Then we define the $\mathbb{V}$-*tractor bundle* $\mathcal{T}_{\mathbb{V}}$ to be the associated bundle $\mathcal{G} \times_P \mathbb{V}$.

The *standard tractor bundle* of parabolic geometry modelled on $(\mathfrak{g}, P)$ is the bundle associated to the standard representation of G.

4.23 Remark In [ČG02] the authors start with G and a $|k|$-grading of its Lie algebra $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_k$ to which they associate decreasing filtration by $\mathfrak{g}^i = \mathfrak{g}_i \oplus \cdots \oplus g_k$. The adjoint tractor bundle for a manifold of dimension $\dim \mathfrak{g}/\mathfrak{g}^0$ as a smooth bundle $\mathcal{A}$ endowed with decreasing filtration $\mathcal{A}^{-k} \supset \mathcal{A}^{-k+1} \supset \cdots \supset \mathcal{A}^k$ by smooth subbundles and an algebraic Lie bracket $\mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$, such that this bundle is locally trivial bundle of filtered Lie algebras modelled on $\mathfrak{g}$.

If one defines $P = \{g \in G \,|\, \mathrm{Ad}(g)(\mathfrak{g}^i) \subset \mathfrak{g}^i \quad \forall i = -k, \ldots, k\}$ they are able to prove that for any adjoint tractor bundle $\mathcal{A}$ there is a canonical adapted frame bundle $\mathcal{G}$ with structure group P such that $\mathcal{A} = \mathcal{G} \times_{\mathrm{Ad}} \mathfrak{g}$. Moreover they recover the Cartan form $\omega : \mathcal{G} \to \mathfrak{g}$ from a suitably defined linear connection ∇ on $\mathcal{A}$. This implies that the pair $(\mathcal{A}, \nabla)$ is equivalent description of the Cartan geometry $(\mathcal{G}, \omega)$.

[10] Surely any infinitesimally effective representation of $\mathfrak{g}$ is of this type and in the case where $\mathfrak{g}$ is simple, effectivity is equivalent to non triviality.

5 The Ambient construction

It is a well known fact that the space of harmonic functions is invariant with respect to the action of the conformal group. I.e. for $f(x)$ harmonic we have also $f(xg)$ harmonic whenever g is translation, rotation, dilatation, special conformal transformation or composition of any of these actions. Since there is a close interrelationship between the symmetry group and the symmetry algebra of any differential operator it is expectable that we will need simple and efficient description of the action of conformal group on $\mathbb{R}^n$ for calculation of the symmetry algebra of the Laplacian. Such a representation comes from the Klein homogeneous model of conformal geometry.

Since none of what will follow depends on the signature of the flat metric chosen on $\mathbb{R}^n$ we will consider conformal geometry modelled on $\mathbb{R}^{p,q}$. Hence the 'Laplacian' can be (ultra-)hyperbolic. Nevertheless we need to restrict the dimension of the space, because the conformal geometry has rather special behaviour for $n \leq 2$. For example the space of local conformal transformations is infinite-dimensional in the case of $\mathbb{R}^2$ whereas for $n \geq 3$ one has only finite dimensional space spanned by the classical conformal transformations. By a local conformal transformation is understood a diffeomorphism whose Jacobian matrix is everywhere proportional to some orthogonal matrix. The Liouville theorem that states that, in the case of $n \geq 3$, every local transformation on $\mathbb{R}^n$ is either translation, rotation, dilatation or special conformal transformation [Slo92].

5.1 Definition Let g_{ab} be the flat metric of $\mathbb{R}^{p,q}$ with signature (p, q). The *Laplace operator* (or the Laplacian for short) is defined by

$$\Delta f = g^{ab}\partial_a\partial_b f = \partial^a\partial_a f \tag{5}$$

This is just a special case of the Laplace–Beltrami operator that is defined by an analogous formula $\nabla_a\nabla^a f$ on any Riemannian manifold.

5.2 Definition Let $\mathbb{R}^{p,q}$ be a (pseudo)euclidean space with non-degenerate symmetric bilinear form g_{ab}. The *ambient space* of $\mathbb{R}^{p,q}$ is the direct sum $\mathbb{R}^{p+1,q+1} = \mathbb{R} \oplus \mathbb{R}^{p,q} \oplus \mathbb{R}$ with non-degenerate symmetric bilinear form g_{AB} defined by

$$g_{AB}x^A y^B = x^0 y^\infty + x^\infty y^0 + g_{ab}x^a y^b \tag{6}$$

for

$$x^A = \begin{pmatrix} x^0 \\ x^a \\ x^\infty \end{pmatrix} \quad y^A = \begin{pmatrix} y^0 \\ y^a \\ y^\infty \end{pmatrix}.$$

The term ambient will be used when referring to the objects defined on some open subset of $\mathbb{R}^{n+2}$. The ambient Laplace operator will be distinguished by tilde $\tilde{\Delta} f = g^{AB}\partial_A\partial_B$.

If we want to lower the index of x^A we have

$$x_A = g_{AB}x^B = \big(x_\infty, x_a, x_0\big)$$

since

$$x_B y^B = g_{AB}x^A y^B = x^0 y^\infty + x^\infty y^0 + x_b y^b = (x_\infty, x_b, x_0)\begin{pmatrix} y^0 \\ y^b \\ y^\infty \end{pmatrix}$$

The ambient metric takes the form

$$g_{AB} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & g_{ab} & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and it is a matter of elementary calculation to show that the signature of this matrix is indeed $(p+1, q+1)$.

5.3 Definition Let

$$r = g_{AB}x^A x^B \tag{7}$$

be the quadratic form associated to the ambient metric g_{AB}. The *null cone* $\mathcal{N}$ is the zero set of r.

$$\mathcal{N} = \Big\{x \in \mathbb{R}^{p+1,q+1} \,|\, r(x) = 0\Big\}$$

Now consider the mapping $\phi : \mathbb{R}^{p,q} \longrightarrow \mathbb{R}^{p+1,q+1}$ given by

$$x^a \mapsto \begin{pmatrix} 1 \\ x^a \\ -x^a x_a/2 \end{pmatrix} =: \phi^A$$

It is easily seen that $\phi(\mathbb{R}^{p,q})$ lies on the null cone.

The conformal geometry[11] is an instance of parabolic geometry for $G = \mathrm{SO}(p+1, q+1)$ and P the stabiliser subgroup of $(1, 0, \ldots, 0)^{\mathrm{T}} \in \mathcal{N}$. Detailed treatment from this point of view can be found in [Sha97]. There it is shown,

[11] In fact we are describing only the oriented conformal geometry. The non oriented case corresponds to the group $G = \mathrm{O}(p+1, q+1)$ and one can define analogously also the conformal spin geometries.

that (under suitable normalisation condition on the Cartan curvature) there is a unique parabolic geometry of the type (G, P) for every conformal structure in the sense of the next definition. The Cartan connection there plays a role similar to the Levi-Civita connection in the Riemannian setting.

Here we will follow more elementary treatment, where not much more than multilinear algebra and calculus is needed.

5.4 Definition A *(pseudo)conformal structure* on a smooth manifold M is an equivalence class of (pseudo)riemannian metrics where two metrics $g, h \in \bigodot^2 T^*M$ are equivalent iff $g = \Omega^2 h$ for some $\Omega \in C^\infty(M)$ that is nonzero on M. Any conformal structure is thus given by a subbundle of the second symmetric power of the cotangent bundle. Moreover we can define a $\mathbb{R}_+$–action on this subbundle by $(t, g_{ab}) \mapsto t^2 g_{ab}$ fiberwise.

The positive rays of the null-cone (i.e. those which have $x^\infty > 0$) which intersect the embedded $\mathbb{R}^n$ also constitute a $\mathbb{R}_+$–bundle over $\mathbb{R}^n$. Namely

$$\mathcal{N}_+ = \left\{ \begin{pmatrix} t \\ tx^a \\ t\frac{x^a x_a}{2} \end{pmatrix} : t \in \mathbb{R}_+ \right\} = \mathbb{R}_+ \times \phi(\mathbb{R}^n)$$

and the projection map $\pi : \mathcal{N}_+ \to \mathbb{R}^{p,q}$ is defined as

$$\pi(\begin{pmatrix} t \\ tx^a \\ t\frac{x^a x_a}{2} \end{pmatrix}) = x^a$$

Since both of these bundles are trivial, they're easily seen to be isomorphic in the category of $\mathbb{R}_+$-bundles via the mapping

$$\begin{pmatrix} t \\ tx^a \\ t\frac{x^a x_a}{2} \end{pmatrix} \mapsto t^2 g_{ab}(x^a).$$

5.5 Lemma *The bundle $\mathcal{N}_+ \subset \mathbb{R}^{p+1,q+1}$ together with the ambient metric determines the conformal class on $\mathbb{R}^n$.*[12]

Proof

An arbitrary global section of $\mathcal{N}_+$ defines a smooth nowhere zero function Ω on $\mathbb{R}^n$ such that this section can be written as $\Omega(x^a)\phi(x^a)$. Since $\mathcal{N}_+ \subset \mathbb{R}^{n+2}$, this section can be viewed as an embedding of $\mathbb{R}^n$ into the ambient space. The claim is that the ambient metric induces the metric $\Omega^2 g_{ab}$ on $\mathbb{R}^n$.

[12] Global sections of $\mathcal{N}_+$ can be viewed as embeddings $\mathbb{R}^n \to \mathbb{R}^{n+2}$ and the ambient metric induces all the metrics in the conformal class of g_{ab} via these embeddings.

The tangent map of this embedding is

$$\partial_a \phi^B = \begin{pmatrix} \partial_a \Omega(x^b) \\ (\partial_a \Omega(x^b))x^b + \Omega(x^b){\delta_a}^b \\ -(\partial_a \Omega(x^b))\frac{x^b x_b}{2} - \Omega(x^b)x_a \end{pmatrix}$$

and the pullback at the point $x \in \mathbb{R}^n$ of the ambient metric via this embedding:

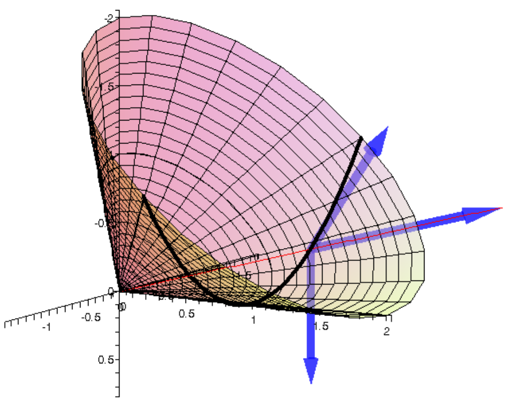
$$\begin{aligned} g_{AB}\partial_c \phi^A \partial_d \phi^B &= \partial_c \phi^{(0}\partial_d \phi^{\infty)} + g_{ab}\partial_c \phi^a \partial_d \phi^b \\ &= -\Omega_c(\Omega_d \tfrac{x^b x_b}{2} + \Omega x_d) - \Omega_d(\Omega_c \tfrac{x^b x_b}{2} + \Omega x_c) + \\ &\quad + g_{ab}(\Omega_c x^a + \Omega {\delta_c}^a)(\Omega_d x^b + \Omega {\delta_d}^b) \\ &= -\Omega_c \Omega_d x^b x_b - 2\Omega\Omega_{(c} x_{d)} + \Omega^2 g_{cd} + \Omega_c \Omega_d g_{ab} x^a x^b + \\ &\quad + g_{ab}(\Omega_c x^a \Omega {\delta_d}^b + \Omega {\delta_c}^a \Omega_d x^b) \\ &= \Omega^2 g_{cd} - 2\Omega\Omega_{(c} x_{d)} + \Omega_c \Omega g_{ad} x^a + \Omega \Omega_d g_{cb} x^b \\ &= \Omega^2 g_{cd} \end{aligned}$$

We can conclude that $\Omega\phi$ is isometrical embedding of $(\mathbb{R}^n, \Omega^2 g_{ab})$ into the ambient space and that the ambient metric induces g_{ab} in the case of $\Omega = 1$. This also identifies $(\mathcal{N}_+, g_{AB})$ with the conformal class of g_{ab}. $\square$

Let us introduce an adapted frame on $\mathbb{R}^{p+1,q+1}$ in order to be able to perform efficient calculations.

5.6 Definition Let $t, \rho \in \mathbb{R}$ and $x^a \in \mathbb{R}^{p,q}$

$$X^A = \begin{pmatrix} t \\ tx^a \\ t(\rho - \frac{x^a x_a}{2}) \end{pmatrix} \quad Y^A_b = \partial_b X^A = \begin{pmatrix} 0_b \\ t\delta^a_b \\ -tx_b \end{pmatrix} \quad Z^A = -\frac{1}{n}\partial^b Y^A_b = \begin{pmatrix} 0 \\ 0^a \\ t \end{pmatrix}$$



If we lower the indices via the ambient metric we have:

$$X_A = (t(\rho - \frac{x^a x_a}{2}), tx_a, 1) \quad Y_{bA} = (-tx_b, tg_{ab}, 0_b) \quad Z_A = (t, 0_a, 0)$$

One immediately sees that X^A is the embedding $\phi(x^a)$ when $t = 1$ and $\rho = 0$.

5.7 Lemma

$$t^2({\delta_A}^B + 2\rho {U_A}^B) = X_A Z^B + Z_A X^B + Y_A^c Y_c^B \tag{8}$$

where the tensor U^{AB} has only one nonzero component $U^{\infty\infty} = 1$.

Proof Let ${T_A}^B$ denote the right hand side of (8). For simplicity reasons we will prove an equivalent statement $T^{AB} = t^2(g^{AB} + 2\rho U^{AB})$ since then the coefficients with respect to the canonical basis of the first two terms of $X^{\mathbf{i}} Z^{\mathbf{j}} + Z^{\mathbf{i}} X^{\mathbf{j}} + Y^{\mathbf{i}c} Y_c^{\mathbf{j}}$ can be expressed in the form of Kronecker product of vectors

$$X^{\mathbf{i}} Z^{\mathbf{j}} = \begin{pmatrix} X^0 Z^0 & \cdots & X^0 Z^n & X^0 Z^\infty \\ X^1 Z^0 & \cdots & X^1 Z^n & X^1 Z^\infty \\ \vdots & \ddots & \vdots & \vdots \\ X^n Z^0 & \cdots & X^n Z^n & X^n Z^\infty \\ X^\infty Z^0 & \cdots & X^\infty Z^n & X^\infty Z^\infty \end{pmatrix} = t^2 \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & x^1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & x^n \\ 0 & \cdots & 0 & \rho - \frac{x^a x_a}{2} \end{pmatrix}$$

$$Z^{\mathbf{i}} X^{\mathbf{j}} = \begin{pmatrix} Z^0 X^0 & Z^0 X^1 & \cdots & Z^0 X^n & Z^0 X^\infty \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ Z^n X^0 & Z^n X^1 & \cdots & Z^n X^n & Z^n X^\infty \\ Z^\infty X^0 & Z^\infty X^1 & \cdots & Z^\infty X^n & Z^\infty X^\infty \end{pmatrix} = t^2 \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 1 & x^1 & \cdots & x^n & \rho - \frac{x^a x_a}{2} \end{pmatrix}$$

while the last term is in fact a product of $(n+2)$ by n matrix with its transpose

$$Y^{\mathbf{ik}} = \delta_A^{\mathbf{i}} Y^{Ac} \delta_c^{\mathbf{k}} = \begin{pmatrix} 0 & \cdots & 0 \\ tg^{1,1} & \cdots & tg^{1,n} \\ \vdots & \ddots & \vdots \\ tg^{n,1} & \cdots & tg^{n,n} \\ -tx^1 & \cdots & -tx^n \end{pmatrix} \quad Y_{\mathbf{k}}^{\mathbf{j}} = \begin{pmatrix} 0 & t & \cdots & 0 & -tx_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & t & -tx_n \end{pmatrix}$$

$$Y^{\mathbf{ik}}Y_{\mathbf{k}}^{\mathbf{j}} = \begin{pmatrix} 0 & \cdots & 0 \\ tg^{1,1} & \cdots & tg^{1,n} \\ \vdots & \ddots & \vdots \\ tg^{n,1} & \cdots & tg^{n,n} \\ -tx^1 & \cdots & -tx^n \end{pmatrix} \begin{pmatrix} 0 & t & \cdots & 0 & -tx_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & t & -tx_n \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & t^2g^{1,1} & \cdots & t^2g^{1,n} & -t^2x^1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & t^2g^{n,1} & \cdots & t^2g^{n,n} & -t^2x^n \\ 0 & -t^2x^1 & \cdots & -t^2x^n & t^2x^ax_a \end{pmatrix}$$

Sum of these matrices clearly gives us the desired result $T^{\mathbf{ij}} = t^2(g^{\mathbf{ij}} + \rho U^{\mathbf{ij}})$ □

Of course we could have arrived at the same result by considering scalar products of the appropriate vectors

$$\begin{aligned} g^{AB}X_AX_B &= 2\cdot t^2(\rho - x^ax_a/2) + t^2g^{ab}x_ax_b = -t^2r + t^2r + 2t^2\rho = 2t^2\rho \\ g^{AB}Z_AZ_B &= 2\cdot 0 + g^{ab}0_a0_b = 0 \\ g^{AB}Y_A^cY_B^d &= 0^d \otimes (-tx^c) + 0^c \otimes (-tx^d) + g^{ab}t\delta_a^c t\delta_b^d = t^2g^{cd} \\ g^{AB}X_AZ_B &= t\cdot t + 0\cdot(-tx^ax_a/2) + tg^{ab}x_a0_b = t^2 \\ g^{AB}X_AY_B^c &= (-tx^ax_a/2)\cdot 0^c + (-tx^c)\cdot t + g^{ab}tx_at\delta_b^c = -t^2x^c + t^2x^c = 0^c \\ g^{AB}Z_AY_B^c &= t\cdot 0^c + 0\cdot(-tx^c) + g^{ab}t0_at\delta_b^c = 0^c \end{aligned} \tag{9}$$

since the equation (8) is an analogue of the standard decomposition of the identity on $\mathbb{R}^n$ into the projectors to some orthonormal basis.

5.8 Remark If we differentiate vector field X^A with respect to the real parameter t we get $\partial_t X^A = \phi(x^a)$. Because $Y_c^BY_B^d = t^2g^{cd}$ we only need to compute $\partial_\rho X^A = Z^A$ to see that the formula for X^A defines in fact a change of coordinates on the open half-space $\{t > 0\}$ of $\mathbb{R}^{p+1,q+1}$. Consequently X^A and x^A represent the same object[13] on $\mathbb{R}^{p+1,q+1}$ only in different coordinates. Let's explicitly define the mapping of the coordinate change:

$$\Phi(t, x^a, \rho) = \begin{pmatrix} t \\ tx^a \\ t(\rho - \frac{x^ax_a}{2}) \end{pmatrix} = \begin{pmatrix} y^0 \\ y^a \\ y^\infty \end{pmatrix} \tag{10}$$

[13]That is the identity vector field.

We see that $\phi(x^a) = \Phi(1, x^a, 0)$ and the identity (8) simplifies on the image of ϕ to

$$\delta_A{}^B = X_A Z^B + Z_A X^B + Y_A^c Y_c^B$$

This will be of a great use later on as well as the next lemma.

5.9 Lemma *The Euler operator in the new coordinates is equal to*

$$\mathbb{E} = t\partial_t \tag{11}$$

Proof For $f(y^A) \in \mathcal{C}^\infty(\mathbb{R}^n)$ we have

$$\begin{aligned} t\partial_t f(y^A(t, x^a, \rho)) &= t(\frac{\partial f}{\partial y^0} + x^a \frac{\partial f}{\partial y^a} + (\rho - \frac{x^b x_b}{2})\frac{\partial f}{\partial y^\infty}) \\ &= y^0(\frac{\partial f}{\partial y^0} + \frac{y^a}{y^0}\frac{\partial f}{\partial y^a} + \frac{y^\infty}{y^0}\frac{\partial f}{\partial y^\infty}) \\ &= (y^A \partial_A f) \circ \Phi(t, x^a, \rho) \end{aligned}$$

□

There is one more identity we will need later

$$\begin{aligned} Y_c^D \partial_D Y_b^A &= \begin{pmatrix} 0_c \\ t\delta_c^d \\ -tx_c \end{pmatrix} \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x^d}, \frac{\partial}{\partial \rho}\right) \begin{pmatrix} 0_b \\ t\delta_b^a \\ -tx_b \end{pmatrix} \\ &= \left(0_c \frac{\partial}{\partial t} + t\delta_c^d \frac{\partial}{\partial x^d} - tx_c \frac{\partial}{\partial \rho}\right) \begin{pmatrix} 0_b \\ t\delta_b^a \\ -tx_b \end{pmatrix} \\ &= \begin{pmatrix} 0_{cb} \\ 0_{cb} \\ t\frac{\partial}{\partial x^c}(-tx_b) \end{pmatrix} \\ &= \begin{pmatrix} 0_{cb} \\ 0_{cb} \\ -t^2 g_{cb} \end{pmatrix} \\ &= -tg_{cb} Z^A \end{aligned} \tag{12}$$

The fact that is of the most importance to us is that every local conformal transformation of $\mathbb{R}^{p,q}$ is given by the action of $SO(p+1, q+1)$ on $\mathcal{N}_+$. We should be working at this stage with the projectivisation of $\mathcal{N}$ that provides the conformal compactification of $\mathbb{R}^{p,q}$ via the embedding ϕ. Nevertheless we are aiming at differential operators and so we can restrict ourselves on the neighbourhood of $(1, 0, 0)^T$ and omit the technicalities arising from the projective structure. For details see [Slo92].

We need to incorporate smooth functions on $\mathbb{R}^{p,q}$ into the picture.

5.10 Definition Suppose that F is a smooth function defined on the neighbourhood of origin in $\mathbb{R}^{p,q}$. Then for any $w \in \mathbb{C}$

$$f(\Phi(t, x^a, 0)) = t^w F(x^a) \tag{13}$$

defines a smooth function on a conical neighbourhood of $(1, 0, 0)$ in the null-cone $\mathcal{N}$. Moreover it is a homogeneous function of degree w because $f(\lambda y^A) = \lambda^w f(y^A)$ for $\lambda > 0$. Conversely F may be recovered from f by setting $t = 1$. Hence, for fixed w, the functions F and f are equivalent[14].

In order to be able to apply ambient differential operators to f we need to extend it from the null cone to the whole space or at least to some open (in $\mathbb{R}^{p+1,q+1}$) neighbourhood of $(1, 0, 0)$. There are infinitely many choices for such an extension even if we stick to the homogeneous ones. Nevertheless any two such extensions will differ by a very convenient factor.

5.11 Lemma *Let f and $\hat{f}$ be two smooth w-homogeneous extensions of F on some open neighbourhood of $(1, 0, 0)$. Then there exist a smooth $(w-2)$-homogeneous function h such that $(f - \hat{f})(y^A) = r(y^A)h(y^A)$ where r is defined by (7).*

Proof If we perform coordinate transformation

$$(y^0, y^a, y^\infty) \mapsto (y^0, y^a, r) = (y^0, y^a, 2y^0y^\infty + y^a y_a)$$

we will be dealing with two functions that are equal on the hyperplane defined by $r = 0$. For any smooth function k on $\mathbb{R}^{p+1,q+1}$ holds

$$\begin{aligned} k(y^0, \ldots, y^n, r) &= k(y^0, \ldots, y^n, 0) + \int_0^1 \frac{d}{dt} k(y^0, \ldots, y^n, tr)dt = \\ &= k(y^0, \ldots, y^n, 0) + r \int_0^1 \frac{\partial k}{\partial r}(y^0, \ldots, y^n, tr)dt. \end{aligned}$$

So if we take k as the difference of two w-homogeneous extensions of F, we will have $k(y^0, \ldots, y^n, 0) = 0$ and thus it follows that $f - \hat{f} = rh$. This h is homogeneous of degree $w - 2$ because r has homogeneity 2 and $f - \hat{f}$ is w-homogeneous. □

5.12 Remark The classical chain rule formula which with regard to (13) gives

$$\partial_a F = \partial_a (f \circ \phi) = (\partial_a \phi^B \partial_B f) \circ \phi = (Y_a^B \partial_B f) \circ \phi \tag{14}$$

because $\partial_a \phi^B$ equals Y_a^B on the image of ϕ

[14] The homogeneity w is usually called the conformal weight of F when viewed on $\mathcal{N}$.

5.13 Lemma *For homogeneous function h on $\mathbb{R}^{p+1,q+1}$ of degree $w-2$ holds*

$$\tilde{\Delta}(rh) = r\,\tilde{\Delta}_h + 2(n+2w-2)h$$

Proof

$$\begin{aligned}\tilde{\Delta}(rh) &= g^{AB}\partial_A\partial_B(rf) = g^{AB}\partial_A(2x_Bh + r\partial_Bh) = \\ &= g^{AB}(2g_{AB}h + 2x_B\partial_Ah + 2x_A\partial_Bh + r\partial_A\partial_Bh) = \\ &= 2(n+2)h + 2g^{AB}(x_B\partial_Ah + x_A\partial_Bh) + r\,\tilde{\Delta}\,h = \\ &= 2(n+2)h + 4x^A\partial_Ah + r\,\tilde{\Delta}\,h = \\ &= r\,\tilde{\Delta}\,h + 2(n+2w-2)h\end{aligned}$$

□

It immediately follows that, if f is $(1-n/2)$-homogeneous extension of F, then $\tilde{\Delta}\,f|_{\mathcal{N}}$ depends only on the restriction of f to the null cone and hence it depends only on F. This defines a differential operator on $\mathbb{R}^{p,q}$.

5.14 Theorem *Let F be a smooth function on some open neighbourhood of $0 \in \mathbb{R}^{p,q}$ and let f be the smooth homogeneous function of degree $1-n/2$ that corresponds to F via (13) and is defined on some open neighbourhood of $(1,0,0) \in \mathbb{R}^{p+1,q+1}$. Then the following equality holds*

$$(\tilde{\Delta}\,f)\circ\phi = \Delta\,F.$$

Proof Using lemma 5.7 and equations (9) we obtain

$$\begin{aligned}\left(g^{AB}\partial_A\partial_Bf\right)\circ\phi &= \left(g^{AB}\delta_A{}^C\delta_B{}^D\partial_C\partial_Df\right)\circ\phi \\ &= \Big[g^{AB}\frac{1}{t^2}(X_AZ^C + Z_AX^C + Y_A^qY_q^C - \rho U_A{}^C)\,\cdot \\ &\quad\cdot\frac{1}{t^2}(X_BZ^D + Z_BX^D + Y_B^rY_r^D - \rho U_B{}^D)\partial_C\partial_Df\Big]\circ\phi \\ &= [(Z^CX^D + X^CZ^D)\partial_C\partial_Df + g^{qr}Y_q^CY_r^D\partial_C\partial_Df]\circ\phi\end{aligned}$$

where the last equation holds because we are evaluating the result on the image of ϕ and that means that $\rho = 0$. Because we are working on the flat Riemannian manifold, we have $[\partial_A,\partial_B] = 0$ due to (1), which means that the first term in the last expression equals

$$\begin{aligned}2Z^CX^D\partial_C\partial_D &= 2\left(Z^C\partial_CX^D\partial_D - Z^C(\partial_CX^D)\partial_D\right) \\ &= 2\left(Z^C\partial_CX^D\partial_D - Z^C(\delta_C{}^D)\partial_D\right) \\ &= 2Z^C\partial_C(\mathbb{E}-1)\end{aligned}$$

applied to f and evaluated on the image of ϕ. The second term is

$$\begin{aligned}\Big(g^{qr}Y_q^C Y_r^D \partial_C \partial_D f\Big) \circ \phi &= \Big(g^{qr}[Y_q^C \partial_C Y_r^D \partial_D - Y_q^C (\partial_C Y_r^D)\partial_D] f\Big) \circ \phi \\ &= \Big(g^{qr} Y_q^C \partial_C (Y_r^D \partial_D f)\Big) \circ \phi - \Big(g^{qr} Y_q^C g_{qr} Z^D \partial_D f\Big) \circ \phi \\ &= \Delta\, F - n(Z^D \partial_D f) \circ \phi\end{aligned}$$

Hence

$$(\tilde{\Delta}\, f) \circ \phi = \Delta\, F - \Big(Z^D \partial_D (n + 2\mathbb{E} - 1) f\Big) \circ \phi$$

which concludes the proof. □

6 Symmetries of the Laplace operator

6.1 Conformal Killing tensor fields

6.1 Definition The *conformal Killing tensor* field is a symmetric trace-free tensor field V with s indices satisfying one of these equivalent conditions:

$$\nabla^{(i_0}V^{i_1\cdots i_s)} = g^{(i_0i_1}\lambda^{i_2\cdots i_s)} \quad \text{for some tensor field} \quad \lambda^{i_2\cdots i_s} \tag{15}$$

$$\text{the trace-free part of} \quad \nabla^{(i_0}V^{i_1\cdots i_s)} = 0 \tag{16}$$

$$\nabla^{(i_0}V^{i_1\cdots i_s)} = \frac{s}{n+2s-2}\, g^{(i_0i_1}\nabla_e V^{i_2\cdots i_s)e} \tag{17}$$

Proof Suppose that

$$\nabla^{(i_0}V^{i_1\cdots i_s)} = U^{i_0\cdots i_s} + g^{(i_0i_1}T^{i_2\cdots i_s)} \tag{18}$$

where $U^{i_0\cdots i_s}$ and $T^{i_2\cdots i_s}$ are some symmetric trace-free tensor fields. The equivalence of the first two conditions is clear if we establish this decomposition. The field $U^{i_0\cdots i_s}$ is called the trace-free part of $\nabla^{(i_0}V^{i_1\cdots i_s)}$. In order to verify this decomposition we compute $T^{i_2\cdots i_s}$ explicitly and show that all the imposed conditions are satisfied. Such a computation will also gives us the equivalence with the third condition.

Let's take the trace[15] of on both sides of (18)

$$g_{i_0i_1}\nabla^{(i_0}V^{i_1\cdots i_s)} = 0 + g_{i_0i_1}g^{(i_0i_1}T^{i_2\cdots i_s)}$$

Now the left hand side equals[16] to

$$g_{i_0i_1}\frac{1}{s+1}\left(\nabla^{i_0}V^{i_1\cdots i_s} + \nabla^{i_1}V^{i_0\cdots i_s} + \cdots + \nabla^{i_s}V^{i_1\cdots i_0}\right) = \frac{2}{s+1}\nabla_{i_1}V^{i_1\cdots i_s}$$

while the right hand side is

$$g_{i_0i_1}\frac{1}{\#\{j<k\,|\,j,k\in\{0,\ldots,s\}\}}\sum_{\substack{j<k\\ j,k\in\{0,\ldots,s\}}} g^{i_ji_k}T^{i_2\cdots\widetilde{i_j}\cdots\widetilde{i_k}\cdots i_s}$$

where $\widetilde{i_j} = i_0$ and $\widetilde{i_k} = i_1$. If we expand the sum we have

$$\begin{aligned} g_{i_0i_1}\frac{2}{s(s+1)}\Big(& g^{i_0i_1}T^{i_2\cdots i_s} + g^{i_0i_2}T^{i_1i_3\cdots i_s} + \cdots + g^{i_0i_s}T^{i_2\cdots i_{s-1}i_1} + \\ & + g^{i_1i_2}T^{i_0i_3\cdots i_s} + \cdots + g^{i_1i_s}T^{i_2\cdots i_{s-1}i_0} + \\ & + \sum_{\substack{j<k\\ j,k\in\{2,\ldots,s\}}} g^{i_ji_k}T^{i_2\cdots\widetilde{i_j}\cdots\widetilde{i_k}\cdots i_s}\Big) \end{aligned}$$

[15] Since all the tensor fields involved are symmetric it doesn't matter over which indices are we taking the trace.

[16] Here we are using the fact that the tensor field $V^{i_1\cdots i_s}$ is trace-free and that contraction with metric commutes with the Levi–Civita covariant derivative.

and by taking into consideration that $T^{i_2\cdots i_s}$ is trace-free we have the last sum equal to zero and hence the right hand side is

$$\frac{2}{s(s+1)}\Big(nT^{i_2\cdots i_s} + \delta_{i_1}{}^{i_2}T^{i_1 i_3\cdots i_s} + \cdots + \delta_{i_1}{}^{i_s}T^{i_2\cdots i_{s-1}i_1} + \\ + \delta_{i_0}{}^{i_2}T^{i_0 i_3\cdots i_s} + \cdots + \delta_{i_0}{}^{i_s}T^{i_2\cdots i_{s-1}i_0}\Big)$$

Putting it all together we have

$$\frac{2}{s+1}\nabla_{i_1}V^{i_1\cdots i_s} = \frac{2}{s(s+1)}(n+2s-2)T^{i_2\cdots i_s}$$

So if we take $T^{i_2\cdots i_s} := \frac{s}{n+2s-2}\nabla_{i_1}V^{i_1\cdots i_s}$ and define

$$U^{i_0\cdots i_s} := \nabla^{(i_0}V^{i_1\cdots i_s)} - g^{(i_0 i_1}T^{i_2\cdots i_s)}$$

then we need to show that we indeed have two symmetric trace-free tensor fields.

That these tensor fields are symmetric is obvious. Trace-freeness of $T^{i_2\cdots i_s}$ follows from the fact that contraction with metric commutes with the Levi-Civita covariant derivative whilst the same property of $U^{i_0\cdots i_s}$ is just a consequence of the preceding calculations - i.e. $T^{i_2\cdots i_s}$ was obtained from the proposition that the trace of $\nabla^{(i_0}V^{i_1\cdots i_s)}$ equals to $g_{i_0 i_1}g^{(i_0 i_1}T^{i_2\cdots i_s)}$ □

6.2 Theorem *Any symmetry $\mathcal{D}$ of the Laplacian on a Riemannian manifold is canonically equivalent to one whose symbol is a conformal Killing tensor.*

Proof This proof is taken from the article [Eas05a].

Since

$$g^{(bc}\mu^{d\cdots e)}\nabla_b\nabla_c\nabla_d\cdots\nabla_e = \mu^{d\cdots e}\nabla_d\cdots\nabla_e\Delta + \text{ lower order terms,}$$

any trace in the symbol of $\mathcal{D}$ may be canonically removed by using equivalence. Thus, let us suppose that

$$\mathcal{D} = V^{i_1\cdots i_s}\nabla_{i_1}\cdots\nabla_{i_s} + \text{ lower order terms}$$

is a symmetry of Δ and that $V^{i_1\cdots i_s}$ is trace-free symmetric. Then

$$\Delta\mathcal{D} = V^{i_1\cdots i_s}\nabla_{i_1}\cdots\nabla_{i_s}\Delta + 2\nabla^{(a}V^{i_1\cdots i_s)}\nabla_a\nabla_{i_1}\cdots\nabla_{i_s} \\ + \text{ lower order terms}$$

and the only way that the Laplacian can emerge from the sub-leading term is if (15) holds. □

Now we give the reason for the adjective 'conformal' in conformal Killing tensors. This is the result of Fegan's work on classification of conformally invariant differential operators of first order [Feg76]

6.3 Theorem *Let λ be a dominant weight of $\mathfrak{so}(p,q)$ and let $\mathbb{R}^{n*} \otimes \mathbb{V}_\lambda = \sum_\rho \mathbb{V}_\rho$ be the decomposition into irreducible representations with dominant weights ρ of $\mathfrak{so}(p,q)$. Let us define*

$$\alpha(\lambda,\rho) = \frac{1}{2}((n-1) + \langle\rho, 2\delta+\rho\rangle - \langle\lambda, 2\delta+\lambda\rangle) \tag{19}$$

where δ is half the sum of all positive roots of $\mathfrak{so}(p,q)$ and $\langle\ ,\ \rangle$ is the Killing form.

Let us write $E_\lambda(\alpha)$ for the associated homogeneous vector bundle over $\mathrm{SO}(p+1,q+1)/P$ corresponding to the dominant weight λ and conformal weight α.

Then, beside the zero operators and the constant multiples of the identities, all linear infinitesimally natural[17] first order operators which are defined on the sections of the vector bundles $E_{\lambda,\alpha}$ with some conformal weight α are given by the projections

$$\sigma_\rho \circ \nabla : \Gamma(E_\lambda(\alpha)) \to \Gamma(\mathbb{R}^{n*} \otimes E_\lambda(\alpha)) \to \Gamma(E_\rho(\beta))$$

of the first covariant derivatives with respect to an arbitrary metric from the conformal class onto the irreducible components $\mathbb{V}_\rho$, and the conformal weight of $E_{\lambda,\alpha}$ is then $\alpha = \alpha(\lambda,\rho)$, while the conformal weight of $E_\rho(\beta)$ equals to $\alpha(\lambda,\rho)+1$.

Let e^i, $i = 1,\dots,n$ be a dual basis of the Cartan subalgebra of $\mathfrak{so}(p,q)$. Then each irreducible component $\mathbb{V}_\rho$ has multiplicity one and all the dominant weights ρ are listed below:

1. *If $n = 2l$, then $\rho = \lambda \pm e^i$, $i \in S \subseteq \{1,\dots,l\}$.*
2. *If $n = 2l+1$ and e^l appears in λ with a non-zero coefficient, then $\rho = \lambda \pm e^i$, $i \in S \subseteq \{1,\dots,l\}$, or $\rho = \lambda$.*
3. *If $n = 2l+1$ and e^l does not appear in λ with a non-zero coefficient, then $\rho = \lambda \pm e^i$, $i \in S \subseteq \{1,\dots,l-1\}$, or $\rho = \lambda + e^l$.*

where the subset S is such that the resulting weight ρ is dominant.

[17] Proposition 6.7 in [Slo92] states that these operators are conformally invariant and extend to a natural operator on the whole category of oriented conformal manifolds. The adjective 'natural' has a precise meaning in this context – it basically states that such a operator commutes with local morphisms.

6.4 Corollary *The operator*

$$\text{Kill} : V^{i_1 \cdots i_s} \mapsto \nabla^{(i_0} V^{i_1 \cdots i_s)}{}_{\textit{trace-free part}} \tag{20}$$

which defines conformal Killing tensor fields is conformally invariant.

Proof

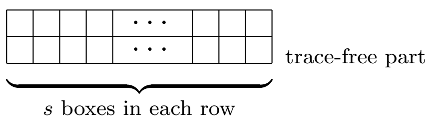
The space of symmetric trace-free tensors with s indices is an irreducible representation of $\mathfrak{so}(p,q)$ and the trace free part of symmetrisation is precisely the projector $\mathbb{R}^n \otimes \bigodot_0^s \mathbb{R}^n \to \bigodot_0^{s+1} \mathbb{R}^n$. The highest weights of these spaces are $(s, 0, \ldots, 0)$ and $(s+1, 0, \ldots, 0)$ respectively [FH91].

Using the formula (19), one can easily compute the conformal weight α which coincides with the conformal weight computed via the BGG machinery [Slo92, Slo93]. □

6.5 Definition Let

$$V^{A_1 B_1 \cdots A_s B_s} \in \bigotimes^{2s} \mathbb{R}^{n+2}$$

be a tensor that is skew in each pair of indices $A_i B_i$, is totally trace-free, and such that skewing over any three indices gives zero. (It follows that $V^{A_1 B_1 \cdots A_s B_s}$ is symmetric in the paired indices and that symmetrising over any $s+1$ indices gives zero.) These symmetries can be expressed as a Young tableau



Vector space of tensors with these symmetries forms an irreducible representation of $\mathfrak{so}(p+1, q+1)$ of dimension

$$\frac{(n+s-3)!(n+s-2)!(n+2s-2)(n+2s-1)(n+2s)}{s!(s+1)!(n-2)!n!}$$

and highest weight $(0, s, 0, \ldots, 0)$. For details see [GW98] or [Kin71].

6.6 Theorem *Let* $V^{A_1 B_1 \cdots A_s B_s}$ *be a tensor with symmetries described in the previous definition 6.5 and let* $X_{A_i}, Y_{B_i}{}^{c_i}$ *be the tensor fields on* $\mathbb{R}^n$ *defined in 5.6. Then the formula*

$$V^{c_1 \cdots c_s} = \left(V^{A_1 B_1 \cdots A_s B_s} X_{A_1} \cdots X_{A_s} Y_{B_1}{}^{c_1} \cdots Y_{B_s}{}^{c_s} \right) \circ \phi, \tag{21}$$

defines conformal Killing tensor field.

Proof The symmetry of $V^{c_1\cdots c_s}$ follows directly from the symmetry of $V^{A_1B_1\cdots A_sB_s}$ in the pairs A_iB_i and from the symmetry of $X_{A_1}\cdots X_{A_s}$. The tensor field in question is also trace-free since

$$\begin{aligned} g_{c_1c_2}V^{c_1\cdots c_s} &= (g_{c_1c_2}V^{A_1B_1\cdots A_sB_s}X_{A_1}\cdots X_{A_s}Y_{B_1}{}^{c_1}\cdots Y_{B_s}{}^{c_s})\circ\phi \\ &= (V^{A_1B_1\cdots A_sB_s}X_{A_1}\cdots X_{A_s}Y_{B_1}{}^{c_1}Y_{B_2\,c_1}Y_{B_3}{}^{c_3}\cdots Y_{B_s}{}^{c_s})\circ\phi \end{aligned}$$

now we use the fact that

$$Y^{c_1}_{B_1}Y_{B_2\,c_1} = (g_{B_1B_2} - X_{B_1}Z_{B_2} - Z_{B_1}X_{B_2})$$

on the image of ϕ which allows us to conclude that

$$\begin{aligned} g_{c_1c_2}V^{c_1\cdots c_s} &= V^{A_1B_1\cdots A_sB_s}X_{A_1}\cdots X_{A_s}g_{B_1B_2}Y_{B_3}{}^{c_3}\cdots Y_{B_s}{}^{c_s})\circ\phi \\ &\quad - V^{A_1B_1\cdots A_sB_s}X_{A_1}\cdots X_{A_s}X_{B_1}Z_{B_2}Y_{B_3}{}^{c_3}\cdots Y_{B_s}{}^{c_s})\circ\phi \\ &\quad - V^{A_1B_1\cdots A_sB_s}X_{A_1}\cdots X_{A_s}Z_{B_1}X_{B_2}Y_{B_3}{}^{c_3}\cdots Y_{B_s}{}^{c_s})\circ\phi \\ &= 0 \end{aligned}$$

because $V^{A_1B_1\cdots A_sB_s}$ is trace free and antisymmetric in A_iB_i whereas $X_{A_i}X_{B_i}$ is symmetric. Let's check if we fulfil also the differential condition (15)

$$\begin{aligned} \partial^{c_0}V^{c_1\cdots c_s} &= \partial^{c_0}\left(V^{A_1B_1\cdots A_sB_s}X_{A_1}\cdots X_{A_s}Y_{B_1}{}^{c_1}\cdots Y_{B_s}{}^{c_s}\right)\circ\phi \\ &= \left(Y^{c_0D}\partial_D V^{A_1B_1\cdots A_sB_s}X_{A_1}\cdots X_{A_s}Y_{B_1}{}^{c_1}\cdots Y_{B_s}{}^{c_s}\right)\circ\phi \end{aligned}$$

now the Leibniz rule applies

$$= \text{ terms containing } Y^{c_0D}\partial_D X_{A_i} + \text{ terms containing } Y^{c_0D}\partial_D Y^{c_i}_{B_i}$$

and because we are evaluating the result on the image of ϕ we can write according to (12)

$$\begin{aligned} &= \text{ terms containing } Y^{c_0D}g_{DA_i} + \text{ terms containing } g^{c_0c_i}Z_{B_i} \\ &= \text{ terms containing } Y^{c_0}_{A_i} + \text{ trace terms} \end{aligned}$$

The remaining terms are of the form

$$\left(V^{A_1B_1\cdots A_sB_s}X_{A_1}\cdots\widehat{X_{A_i}}\cdots X_{A_s}Y_{B_1}{}^{c_1}\cdots\widehat{Y^{c_i}_{B_i}}\cdots Y_{B_s}{}^{c_s}Y^{c_i}_{B_i}Y^{c_0}_{A_i}\right)\circ\phi$$

which means that their symmetrisation over small indices contain pairs

$$Y^{c_j}_{B_i}Y^{c_k}_{A_i} + Y^{c_j}_{A_i}Y^{c_k}_{B_i}$$

that are clearly symmetric in A_iB_i. Thus these terms contract to zero due to the antisymmetry of $V^{A_1B_1\cdots A_sB_s}$ in the pair A_iB_i. □

6.7 Lemma *The mapping* (21) *is a monomorphism*

$$\bigodot^{s}(\Lambda^2\mathbb{R})^{n+2}{}_{\circ} \to \Gamma(\mathbb{R}^n, \bigotimes^{s}(\mathbb{R}^n))$$

Proof First of all, it is sufficient to check only the injectivity of

$$f : V_{AB} \mapsto V_{AB}X^A Y^{cB} \circ \phi = V^c,$$

since the mapping (21) is linear and it is just the sth tensor product of the linear mapping f. Substituting from the definition 5.6 we get this condition on the kernel of f:

$$\begin{aligned} 0 = V_c &= X^A \left(V_{Ab}\delta_c{}^b + V_{A\infty}(-x_c) \right) \\ &= V_{0c} + V_{ac}x^a - \frac{x_d x^d}{2} V_{\infty c} - V_{0\infty}x_c - V_{a\infty}x_c x^a, \end{aligned} \tag{22}$$

for V_{AB} is antisymmetric. We see that the components of V^c are polynomials in n variables of degree at most two. Let $x^a \in \mathbb{R}^n$ be te^a where e^a is the i^{th} vector of the orthonormal basis of $\mathbb{R}^{p,q}$ and t is a real number. The ith component of $V^c(x^a)$ with respect to the orthonormal basis is equal to

$$V_{0\mathbf{i}} + tV_{\mathbf{ii}} - \frac{t^2}{2} V_{\infty\mathbf{i}} - tV_{0\infty} - t^2 V_{\mathbf{i}\infty}.$$

Using again the antisymmetry of V_{AB}, this can be further simplified to

$$\frac{t^2}{2} V_{\infty\mathbf{i}} - tV_{0\infty} + V_{0\mathbf{i}}.$$

It follows that the kernel of f consists of the antisymmetric matrices such that $V_{\infty\mathbf{i}} = V_{0\infty} = V_{0\mathbf{i}} = 0$ which means that (22) simplifies to

$$V_{ac}x^a = 0,$$

for every $x^a \in \mathbb{R}^n$. Hence also $V_{ac} = 0$ and the conclusion follows. □

6.8 Theorem *Every conformal Killing tensor field on* $\mathbb{R}^{p,q}$ *is given by the formula* (21).

Proof

In view of the preceding lemma it is sufficient to show, that the space of conformal Killing tensor field with s indices has the same dimension as $\bigotimes^{2s} \mathbb{R}^{n+2}$. Proof by elementary representation theory is sketched in [Eas05b]. Another approach, which we will elaborate more on here, is suggested in [Eas05a]. It uses rather general theorems of parabolic geometry concerning existence and properties of invariant differential between associated bundles.

We already know that the conformal Killing tensor fields are the kernel of conformally invariant operator of first order 6.4. This operator can be found in the so called generalised Bernstein–Gelfand–Gelfand resolution [ČSS01] of the sheaf of locally constant sections of the associated bundle corresponding to $\lambda = (0, s, 0, \dots, 0)$:

$$0 \to \mathbb{V}_\lambda \to \Gamma(G \times_P V_\lambda) \xrightarrow{\text{Kill}} \Gamma(G \times_P V_\rho) \to \cdots$$

, by G invariant differential operators. Since it is a resolution of sheaves, it is exact (at least locally), which implies that the kernel of Kill is isomorphic to the image of the inclusion $\mathbb{V}_\lambda \hookrightarrow \Gamma(G \times_P V_\lambda)$ and hence these spaces have the same dimension.

Using the techniques of [Slo92, Slo93] one easily checks that the weight ρ is $(-2|s+1, o, \dots, 0)$, where -2 is the weight of the center and $(s+1, 0, \dots, 0)$ is the weight of $\mathfrak{g}_0^{ss}$.

Using the Fegan's formula for conformal weight (19) one must get the same result because of the operator's uniqueness up to a multiple. □

6.2 Symmetry operators of the Laplace

6.9 Definition Let $V^{A_1 B_1 \cdots A_s B_s}$ be tensor with symmetries described in the previous section and define an ambient differential operator

$$\mathcal{D}_V := V^{A_1 B_1 \cdots A_s B_s} x_{A_1} \cdots x_{A_s} \partial_{B_1} \cdots \partial_{B_s}$$

Now we compute its crucial properties which allows us to use $\mathcal{D}_V$ for construction of the desired symmetry operators.

6.10 Lemma *For $r = x^A x_A$ the defining function of null-cone we have $\mathcal{D}_V\, rf = r\, \mathcal{D}_V\, f$ and moreover this operator commutes with the ambient Laplace operator $\tilde{\Delta}\, \mathcal{D}_V = \mathcal{D}_V\, \tilde{\Delta}$ and preserves w-homogeneous functions.*

Proof If we calculate two commutators

$$[\partial_B, r]f = \partial_B rf - r\partial_B f = (2x_B f + r\partial_B f) - r\partial_B f = 2x_B f$$
$$[\partial_A, x_B]f = \partial_A x_B f - x_B \partial_A f = (g_{AB} f + x_B \partial_A f) - x_B \partial_A f = g_{AB} f$$

we easily see that after commuting r with two derivatives we get trace of V which is zero by definition. What's left is to consider the case of odd s:

$$\begin{aligned}\mathcal{D}_V\, rf &= V^{A_1 B_1 \cdots A_s B_s} x_{A_1} \cdots x_{A_s} \partial_{B_1} r \partial_{B_2} \cdots \partial_{B_s} f = \\ &= V^{A_1 B_1 \cdots A_s B_s} x_{A_1} \cdots x_{A_s} (r \partial_{B_1} \cdots \partial_{B_s} f + 2x_{B_1} \partial_{B_2} \cdots \partial_{B_s} f)\end{aligned}$$

The second term is zero, because $x_{A_1} \cdots x_{A_s} x_{B_1}$ is completely symmetric tensor and we contract over indices in which V is antisymmetric.

Let's deal with the second equality:

$$\begin{aligned}
\tilde{\Delta}\,\mathcal{D}_V &= g^{AB}\partial_A\partial_B V^{A_1B_1\cdots A_sB_s} x_{A_1}\cdots x_{A_s}\partial_{B_1}\cdots\partial_{B_s} = \\
&= g^{AB} V^{A_1B_1\cdots A_sB_s} x_{A_1}\cdots x_{A_s}\partial_A\partial_B\partial_{B_1}\cdots\partial_{B_s} + \\
&+ \sum_{\substack{i,j=1\\ i\neq j}}^{s} g^{AB} g_{AA_i} g_{BA_j} V^{A_1B_1\cdots A_sB_s} x_{A_1}\cdots\widehat{x_{A_i}}\cdots\widehat{x_{A_j}}\cdots x_{A_s}\partial_{B_1}\cdots\partial_{B_s} +
\end{aligned} \tag{23}$$

$$+\ 2\sum_{i=1}^{s} g^{AB} g_{AA_i} V^{A_1B_1\cdots A_sB_s} x_{A_1}\cdots\widehat{x_{A_i}}\cdots x_{A_s}\partial_B\partial_{B_1}\cdots\partial_{B_s} \tag{24}$$

Because $g^{AB} g_{AA_i} g_{BA_j} = \delta^B_{A_i} g_{BA_j} = g_{A_iA_j}$ we see that (23) is zero as we are again taking the trace of trace-free tensor. In (24) we have $g^{AB} g_{AA_i} = \delta^B_{A_i}$ and so this term equals $V^{A_1B_1\cdots A_sB_s} x_{A_1}\cdots\widehat{x_{A_i}}\cdots x_{A_s}\partial_{A_i}\partial_{B_1}\cdots\partial_{B_s}$ which is zero as all our derivatives commute.[18].

Since differentiation lowers the homogeneity by one and multiplying by x raises the homogeneity by one the proof is complete. □

The operator $\mathcal{D}_V$ commutes with the defining function of the null-cone $\mathcal{N}$ and hence it defines some differential operator on $\mathbb{R}^{p,q}$ in a similar way how the ambient Laplace operator defines the Laplace operator on $\mathbb{R}^{p,q}$.

6.11 Theorem *Let $V^{A_1B_1\cdots A_sB_s}$ be a tensor with symmetries as in 6.5 and let $V^{c_1\cdots c_s}$ be given by (21). Let F be a smooth function on $\mathbb{R}^n$ and let f be its w-homogeneous extension on some open neighbourhood of $\phi(\mathbb{R}^n)$. The operator on $\mathbb{R}^n$ defined by*

$$\mathrm{D}^w_V\, F = (\mathcal{D}_V\, f)\circ\phi$$

equals to

$$\mathrm{D}^w_V\, F = \sum_{k=0}^{s} A(s,k,w)(\partial_{c_1}\cdots\partial_{c_k} V^{c_1\cdots c_s})\partial_{c_{k+1}}\cdots\partial_{c_s} F, \tag{25}$$

where

$$A(s,k,w) = (-1)^k \frac{(w-1)\cdots(w-k+1)}{(n+2s-1)(n+2s-2)\cdots(n+2s-k)} \tag{26}$$

[18] Note that this fails in the curved setting.

Proof From definitions we have

$$\mathrm{D}^w_V F = (\mathcal{D}_V f) \circ \phi = (V^{A_1 B_1 \cdots A_s B_s} x_{A_1} \cdots x_{A_s} \partial_{B_1} \cdots \partial_{B_s} f) \circ \phi$$

and we can replace x_{A_i} by X_{A_i} because this amounts, due to the remark 5.8, to the change of coordinates on an open subset of $\mathbb{R}^{n+2}$. Because the right hand side is evaluated at the image of ϕ where $\rho = 0$ and $t = 1$ we can substitute $\delta_{B_i}{}^{D_i}$ by $X_{B_i} Z^{D_i} + Z_{B_i} X^{D_i} + Y^{c_i}_{B_i} Y^{D_i}_{c_i}$ because of the (8). Hence we can proceed as follows bearing in mind that the equalities hold only on the image of ϕ.

$$\begin{aligned}
\mathcal{D}_V f &= V^{A_1 B_1 \cdots A_s B_s} X_{A_1} \cdots X_{A_s} \delta_{B_1}{}^{D_1} \cdots \delta_{B_s}{}^{D_s} \partial_{D_1} \cdots \partial_{D_s} f \\
&= V^{A_1 B_1 \cdots A_s B_s} X_{A_1} \cdots X_{A_s} (X_{B_1} Z^{D_1} + Z_{B_1} X^{D_1} + Y^{c_1}_{B_1} Y^{D_1}_{c_1}) \cdots \\
&\qquad \cdots (X_{B_s} Z^{D_s} + Z_{B_s} X^{D_s} + Y^{c_s}_{B_s} Y^{D_s}_{c_s}) \partial_{D_1} \cdots \partial_{D_s} f
\end{aligned}$$

Because the tensor $V^{A_1 B_1 \cdots A_s B_s}$ is antisymmetric in $A_i B_i$ whereas $X_{A_i} X_{B_i}$ is symmetric the appropriate contractions will be zero and we do not need to consider the terms $X_{B_i} Z^{D_i}$.

$$\begin{aligned}
\mathcal{D}_V f &= V^{A_1 B_1 \cdots A_s B_s} X_{A_1} \cdots X_{A_s} \prod_{i=1}^{s} (Z_{B_i} X^{D_i} + Y^{c_i}_{B_i} Y^{D_i}_{c_i}) \partial_{D_1} \cdots \partial_{D_s} f \\
&= V^{A_1 B_1 \cdots A_s B_s} X_{A_1} \cdots X_{A_s} \sum_{\substack{I, J \subseteq \{1, \ldots, s\} \\ I \cap J = \emptyset}} \left(\prod_{i \in I} Z_{B_i} X^{D_i} \prod_{j \in J} Y^{c_j}_{B_j} Y^{D_j}_{c_j} \right) \partial_{D_1} \cdots \partial_{D_s} f
\end{aligned}$$

Because the tensor $X_{A_1} \cdots X_{A_s}$ is symmetric and $V^{A_1 B_1 \cdots A_s B_s}$ is symmetric in pairs $A_i B_i$ we can write

$$\begin{aligned}
&V^{A_1 B_1 \cdots A_s B_s} X_{A_1} \cdots X_{A_s} \left(\prod_{i \in I} Z_{B_i} X^{D_i} \prod_{j \in J} Y^{c_j}_{B_j} Y^{D_j}_{c_j} \right) = \\
&\qquad = V^{A_1 B_1 \cdots A_s B_s} X_{A_1} \cdots X_{A_s} \left(\prod_{i=1}^{k} Z_{B_i} X^{D_i} \prod_{j=k+1}^{s} Y^{c_j}_{B_j} Y^{D_j}_{c_j} \right) \qquad (27)
\end{aligned}$$

for any two disjoint subsets I, J of $\{1, \ldots, s\}$ whose union is the whole set and where I has cardinality k. Lets elaborate more the case of four indices to see what steps are involved.

$$\begin{aligned}
&V^{A_1 B_1 A_2 B_2} X_{A_1} X_{A_2} Z_{B_2} X^{D_2} Y^{c_1}_{B_1} Y^{D_1}_{c_1} = \\
&\quad = V^{A_2 B_2 A_1 B_1} X_{A_2} X_{A_1} Z_{B_1} X^{D_1} Y^{c_1}_{B_2} Y^{D_2}_{c_1} && \text{relabelling } A_1 \leftrightarrow A_2, B_1 \leftrightarrow B_2 \\
&\quad = V^{A_1 B_1 A_2 B_2} X_{A_2} X_{A_1} Z_{B_1} X^{D_1} Y^{c_1}_{B_2} Y^{D_2}_{c_1} && \text{by the symmetry of } V \\
&\quad = V^{A_1 B_1 A_2 B_2} X_{A_2} X_{A_1} Z_{B_1} X^{D_1} Y^{c_2}_{B_2} Y^{D_2}_{c_2} && \text{relabelling } c_1 \to c_2
\end{aligned}$$

What remains is to analyse the individual terms

$$V^{A_1 B_1 \cdots A_s B_s} X_{A_1} \cdots X_{A_s} Z_{B_1} \cdots Z_{B_k} Y^{c_{k+1}}_{B_{k+1}} \cdots Y^{c_s}_{B_s} \cdot \\ \cdot X^{D_1} \cdots X^{D_k} Y^{D_{k+1}}_{c_{k+1}} \cdots Y^{D_s}_{c_s} \partial_{D_1} \cdots \partial_{D_s} f$$

where it is understood that for $k = 0$ we have

$$V^{A_1 B_1 \cdots A_s B_s} X_{A_1} \cdots X_{A_s} Y^{c_1}_{B_1} \cdots Y^{c_s}_{B_s} X^{D_1} \cdots X^{D_k} Y^{D_{k+1}}_{c_{k+1}} \cdots Y^{D_s}_{c_s} \partial_{D_1} \cdots \partial_{D_s} f$$

and analogously there are only 'Z terms' in the case of $k = s$.

For fixed k we have $\binom{s}{k}$ such a terms in the equation because $\binom{s}{k}$ is the number of subsets of $\{1, \ldots, s\}$ of the cardinality k. These terms can be splitted into the symbol part

$$S(k) = V^{A_1 B_1 \cdots A_s B_s} X_{A_1} \cdots X_{A_s} Z_{B_1} \cdots Z_{B_k} Y^{c_{k+1}}_{B_{k+1}} \cdots Y^{c_s}_{B_s} \tag{28}$$

and the differentiation part

$$D(k)f = X^{D_1} \cdots X^{D_k} Y^{D_{k+1}}_{c_{k+1}} \cdots Y^{D_s}_{c_s} \partial_{D_1} \cdots \partial_{D_s} f \tag{29}$$

Lets start with the latter one. First of all we must realise that the tensor $\partial_{D_1} \cdots \partial_{D_s} f$ is symmetric. That follows directly from the fact that $\partial_A f$ is covariant derivative with respect to the flat ambient metric and hence both the torsion and curvature vanishes which yields $\partial_A \partial_B - \partial_B \partial_A = 0$ on vector fields. Since the result of ∂_A applied on arbitrary tensor field is determined by its action on vector fields the commutativity follows.

Now let $T(k)$ denote the function $X^{D_1} \cdots X^{D_k} \partial_{D_1} \cdots \partial_{D_k} f$.

$$\begin{aligned}
T(k) &= X^{D_1} \cdots X^{D_k} \partial_{D_1} \cdots \partial_{D_k} f \\
&= X^{D_1} \cdots X^{D^{k-1}} (\partial_{D_1} X^{D_k} - \delta_{D_1}{}^{D_k}) \partial_{D_2} \cdots \partial_{D_k} f \\
&= X^{D_1} \cdots X^{D^{k-1}} \partial_{D_1} X^{D_k} \partial_{D_2} \cdots \partial_{D_k} f - T(k-1) \\
&= X^{D_1} \cdots X^{D^{k-2}} (\partial_{D_1} X^{D_{k-1}} - \delta_{D_1}{}^{D_{k-1}}) X^{D_k} \partial_{D_2} \cdots \partial_{D_k} f - T(k-1) \\
&= X^{D_1} \cdots X^{D^{k-2}} \partial_{D_1} X^{D_{k-1}} X^{D_k} \partial_{D_2} \cdots \partial_{D_k} f - 2T(k-1) \\
&\;\;\vdots \\
&= X^{D_1} \partial_{D_1} X^{D_2} \cdots X^{D_k} \partial_{D_2} \cdots \partial_{D_k} f - (k-1)T(k-1) \\
&= \mathbb{E}\, T(k-1) - (k-1)T(k-1) = (\mathbb{E} - k + 1)T(k-1)
\end{aligned}$$

Since $T(1) = X^{D_1} \partial_{D_1} f = \mathbb{E}\, f$ we see that

$$X^{D_1} \cdots X^{D_k} \partial_{D_1} \cdots \partial_{D_k} f = (\mathbb{E} - k + 1)(\mathbb{E} - k + 2) \cdots (\mathbb{E} - 1)\, \mathbb{E}\, f \tag{30}$$

We can view Y_b^A as a 1-homogeneous function on $\mathbb{R}^n$ with values in $\mathbb{R}^{n+2}$ because the homogeneity in the standard coordinates translates to homogeneity in t by the remark 5.8. Therefore the Euler operator acts as the identity on Y_c^D due to (11). That implies $[\mathbb{E}, Y_{c_i}^{D_i}] = 0$.

Iterating the formula (14) we get

$$\partial_{c_1} \cdots \partial_{c_k} F = (Y_{c_1}^{D_1} \partial_{D_1} (Y_{c_2}^{D_2} \partial_{D_2} (\cdots \partial_{D_{k-1}} (Y_{c_k}^{D_k} \partial_{D_k} f)) \cdots) \circ \phi \tag{31}$$

Since $Y_c^D \partial_D Y_b^A = -t g_{cb} Z^A$ by an easy calculation (12), we see that the difference between the above expression and the formula $Y_{c_1}^{D_1} \cdots Y_{c_k}^{D_k} \partial_{c_1} \cdots \partial_{c_k}$ yields a differentiation of f in the direction transversal to the embedding ϕ that clearly depends on the chosen ambient extension and not only on its homogeneity. We already know that the final result does not depend on the ambient extension of F due to the lemma 6.10. Thus we can discard these terms.

To summarise: (29) composed with ϕ equals to

$$(D(k)f) \circ \phi = \prod_{i=1}^{k-1} (w + i - k) \partial_{c_{k+1}} \cdots \partial_{c_s} F \tag{32}$$

for any w-homogeneous extension f of F.

The symbol part (28) gives by the theorem 6.8 the conformal Killing tensor field in the case of $k = 0$. Now we need to compute its divergence. Lets use $X_{A_1 \cdots A_s}$ as a shorthand for $X_{A_1} \cdots X_{A_s}$. By the same formula for the chain rule as above we get

$$\begin{aligned} \partial_{c_1} V^{c_1 \cdots c_s} &= \partial_{c_1} (V^{A_1 B_1 \cdots A_s B_s} X_{A_1 \cdots A_s} Y_{B_1}^{c_1} \cdots Y_{B_s}^{c_s} \circ \phi) \\ &= \partial_{c_1} (S(0) \circ \phi) \\ &= (Y_{c_1}^D \partial_D S(0)) \circ \phi \end{aligned}$$

The next step is the Leibniz rule and since $V^{A_1 B_1 \cdots A_s B_s}$ is a constant tensor we only need to evaluate the contractions of

$$Y_{c_1}^D \partial_D X_{A_i} = Y_{c_1}^D g_{D A_i} = Y_{c_1 A_i} \tag{33}$$

$$Y_{c_1}^D \partial_D Y_{B_i}^{c_i} = -t \delta_{c_1}{}^{c_i} Z_{B_i} \tag{34}$$

on the image of ϕ where we can use the simplified version of (8). The first expression (33) is contracted with $Y_{B_1}^{c_1}$ and hence the result equals $g_{A_i B_1} -$

$X_{A_i}Z_{B_1} - Z_{A_i}X_{B_1}$. If we plug it into the whole expression we get

$$\begin{aligned}
\partial_{c_1}V^{c_1\cdots c_s} &= (Y_{c_1}^D\partial_D S(0))\circ\phi \\
&= (Y_{c_1}^D\partial_D V^{A_1B_1\cdots A_sB_s}X_{A_1\cdots A_s}Y_{B_1}^{c_1}\cdots Y_{B_s}^{c_s})\circ\phi \\
&= (V^{A_1B_1\cdots A_sB_s}X_{A_1\cdots\widehat{A_i}\cdots A_s}(Y_{c_1}^D\partial_D X_{A_i})Y_{B_1}^{c_1}\cdots Y_{B_s}^{c_s} + \sum\ldots)\circ\phi \\
&= (V^{A_1B_1\cdots A_sB_s}X_{A_1\cdots\widehat{A_i}\cdots A_s}(-X_{A_i}Z_{B_1})Y_{B_1}^{c_1}\cdots Y_{B_s}^{c_s} + \sum\ldots)\circ\phi \\
&= -S(1)\circ\phi + (\sum\ldots)\circ\phi
\end{aligned}$$

where the last but one equality holds because $V^{A_1B_1\cdots A_sB_s}$ is trace-free (and hence we can discard $g_{A_iB_1}$) and symmetric in A_1B_1 which allows us to forget the term $Z_{A_i}X_{B_1}$. Therefore the divergence of a conformal Killing tensor field equals

$$\partial_{c_1}V^{c_1\cdots c_s} = [-sS(1) + V^{A_1B_1\cdots A_sB_s}X_{A_1\cdots A_s}Y_{c_1}^D\partial_D(Y_{B_1}^{c_1}\cdots Y_{B_s}^{c_s})]\circ\phi$$

where the second part is due to Leibniz rule and (34)

$$\left(\sum_{i=1}^{s} Y_{B_1}^{c_1}\cdots\widehat{Y_{B_i}^{c_i}}\cdots Y_{B_s}^{c_s}(-\delta_{c_1}{}^{c_i}Z_{B_i})\right)\circ\phi$$

For $i \neq 1$ this gives $-S(1)$ whereas for $i = 1$ we have $-nS(1)$. Putting it all together we see that

$$\partial_{c_1}V^{c_1\cdots c_s} = -(n+s+s-1)S(1)\circ\phi$$

Therefore the second divergence

$$\partial_{c_2}\partial_{c_1}V^{c_1\cdots c_s} = \partial_{c_2}(-(n+2s-1)S(1))\circ\phi$$

and by the same computations as above we get

$$\partial_{c_2}\partial_{c_1}V^{c_1\cdots c_s} = (n+2s-1)(n+s+s-2)S(2)\circ\phi$$

because in $S(1)$ there is Z_{B_1} instead of $Y_{B_1}^{c_1}$ and $Y_{c_2}^D\partial_D Z_{B_1}$ equals zero on the image of ϕ. It readily follows that

$$S(k)\circ\phi = \frac{-1^k}{(n+2s-1)(n+2s-2)\cdots(n+2s-k)}\partial_{c_k}\cdots\partial_{c_1}V^{c_1\cdots c_k} \quad (35)$$

Now we can finally conclude that

$$\begin{aligned}
\mathrm{D}_V^w F &= (\mathcal{D}_V f) \circ \phi \\
&= \left(\sum_{k=0}^{s} \binom{s}{k} S(k) D(k) f \right) \circ \phi \quad \text{by (28) and (29)} \\
&= \sum_{k=0}^{s} \binom{s}{k} S(k) \circ \phi \cdot (D(k) f) \circ \phi \\
&= \sum_{k=0}^{s} \binom{s}{k} A(s,k,w) (\partial_{c_k} \cdots \partial_{c_1} V^{c_1 \cdots c_k}) \partial_{c_{k+1}} \cdots \partial_{c_s} F
\end{aligned}$$

where

$$A(s,k,w) = (-1)^k \frac{\prod_{i=1}^{k-1} (w + i - k)}{(n + 2s - 1)(n + 2s - 2) \cdots (n + 2s - k)}$$

by (35) and (32). □

6.12 Remark According to the theorem 5.14 the ambient Laplace operator acting on $(1-n/2)$-homogeneous ambient extensions gives rise to the Laplacian on $\mathbb{R}^{p,q}$. It also lowers the homogeneity by two. Bearing in mind the result 6.8 we can conclude that all symmetry operators of Δ are canonically equivalent to the one of the form

$$\mathrm{D}_V = \sum_{k=0}^{s} \binom{s}{k} \frac{\prod_{i=1}^{k-1} [-2(n + 2k - 2i - 2)]}{\prod_{i=1}^{k} n + 2s - i} (\partial_{c_k} \cdots \partial_{c_1} V^{c_1 \cdots c_k}) \partial_{c_{k+1}} \cdots \partial_{c_s}$$

$$\text{and } \Delta \, \mathrm{D}_V = \delta_V \, \Delta \text{ for}$$

$$\delta_V = \sum_{k=0}^{s} \binom{s}{k} \frac{\prod_{i=1}^{k-1} [-2(n + 2k - 2i + 2)]}{\prod_{i=1}^{k} n + 2s - i} (\partial_{c_k} \cdots \partial_{c_1} V^{c_1 \cdots c_k}) \partial_{c_{k+1}} \cdots \partial_{c_s}$$

7 Final remarks

In the preceding section we have determined all the symmetry operators for the Laplace operator. A sort of complementary result can be found in [Kos75]. There the author obtained (as a corollary of his general theorem in parabolic setting) that there exists unique multiplier representation of $\mathfrak{so}(4,2)$ on $\mathbb{R}^4$ consisting of first order symmetry operators of the wave operator.

The form of the ambient operator $\mathcal{D}_V$ as well as the appropriate symmetries of the tensor V is relatively easy to guess. In the simplest case when $s = 1$ the operator $\mathcal{D}_V$ is precisely the infinitesimal generator of rotations in $\mathfrak{so}(p+1, q+1)$. Calculation of composition of two first order ambient symmetry operators then suggest what should be the symmetries of the tensor V.

The presence of conformal Killing tensors in physical literature is vast due to their connection with the solutions of interesting physical equations.

Separation of variables for the geodesic equation was discovered in the Kerr solution by Carter [Car68] and an explanation of this phenomenon in terms of (conformal) Killing tensors was provided by Walker and Penrose [WP70] (see also [Woo75]). In particular, there are space-times with conformal Killing tensors not arising from conformal Killing vectors.

Now we will state the main result of [KM82] concerning separation of variables.

7.1 Definition A particular set of orthogonal coordinates $\{x_l\}$ on a pseudo-Riemannian manifold M is *R-separable* for

$$\Delta f = 0$$

if this equation admits a solution of the form

$$\Psi = \exp(R(x)) \prod_{i=1}^{n} \Psi_i(x^i),$$

where $R(x)$ is a fixed function independent of parameters.

7.2 Definition Let (M, g) be n dimensional Riemannian manifold and let B be a differential operator on M with symbol $b \in \Gamma(\bigodot^2 TM)$. The *eigenform* of B is any non-zero one-form ω such that

$$(b^{ij} - \rho g^{ij})\omega_j = 0$$

is satisfied for some smooth function $\rho \in \mathcal{C}^\infty(M)$.

7.3 Theorem *Necessary and sufficient conditions for the existence of an orthogonal R-separable coordinate system $\{x^i\}$ for the Laplace equation on a pseudo-Riemannian manifold M of dimension n are that there exist $n-1$ second-order symmetry operators $B_2, \dots, B_n$ on M such that*

1. $[B_i, B_j] = 0, \quad 2 \leq i, \ j \leq n$
2. *the set* $\{\Delta = B_1, B_2, \dots, B_n\}$ *is linearly independent*
3. *there is a basis* $\{\omega_{\mathbf{j}} : 1 \leq \mathbf{j} \leq n\}$ *of simultaneous eigenforms for the* B_k.

If conditions (1)-(4) are satisfied then there exist functions $f^{\mathbf{i}}(x)$ *such that* $\omega_{\mathbf{i}} = f^{\mathbf{i}}(x)dx^{\mathbf{i}}, \quad \mathbf{i} = 1, \dots n$

The symmetry algebra of the Laplacian is also interesting from the algebraic point of view.

7.4 Theorem *Let $V \circledcirc W$ denote the trace-free part of $V \odot W - V \wedge W$ for $V, W \in \mathfrak{so}(p+1, q+1)$. The symmetry algebra $\mathscr{A}_\Delta$ of the Laplacian on $\mathbb{R}^{p,q}$ is isomorphic the universal enveloping algebra of $\mathfrak{U}\,(\mathfrak{so}(p+1, q+1))$ modulo the two-sided ideal generated by the elements*

$$V \otimes W - V \circledcirc W - \frac{1}{2}[V, W] + \frac{n-2}{4(n+1)}\langle V, W \rangle$$

Proof See [Eas05a]. □

This two sided ideal has rather special properties – it can be found under the name Joseph ideal in the literature [ESS07, BJ98, BG96, Jos76].

The filtration of $\mathscr{A}_\Delta$ by degree is induced by the usual filtration on $\mathfrak{U}(\mathfrak{so}(p+1, q+1))$ and, for any Lie algebra $\mathfrak{g}$, there is a canonical vector space isomorphism with the corresponding graded algebra $\operatorname{gr}\mathfrak{U}(\mathfrak{g}) \cong \bigodot(\mathfrak{g})$, namely the symmetric tensor algebra of $\mathfrak{g}$. It is possible to transfer the algebra structure to $\bigodot(\mathfrak{g})$ and the result is Kontsevich's $\star$-product [ADS02, Kon03]. An analogous description of $\mathscr{A}_\Delta$ has recently been given by Vasiliev [Vas03] in terms of the Weyl (or Moyal) $\star$-product.

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