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DIPLOMOVÁ PRÁCE

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Měření eficience a eko-eficience

Katedra pravděpodobnosti a matematické statistiky

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Prehlasujem, že som svoju diplomovú prácu napísala samostatne a výhradne s použitím citovaných prameňov. Súhlasím so zapožičiavaním práce.

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podpis

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Abstrakt

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Abstrakt: Porovnání ekonomických subjektů na mikro- i makroekonomické úrovni jednotným ukazatelem se provádí měřením eficience jako nástroje pro ohodnocení výkonnosti nebo efektivity subjektu. Pro každý ekonomický subjekt může být změřena technická, alokativní, ekonomická a škálová eficience a eko-eficience, jestliže jsou k dispozici všechna potřebná data. Práce se věnuje zkoumání metody lineárního programování - analýze obalů dat. Detailně popsány a porovnány jsou dva modely pro technickou eficienci a jejich odvození pro měření dalších eficiencí - prostý a aditivní model analýzy obalů dat. Na závěr jsou zjištěné teoretické postupy aplikovány na data. Na základě naměřených hodnot eficiencí je provedeno porovnání lesnického průmyslu v jednotlivých krajích České republiky.
Klíčová slova: eficience, eko-eficience, analýza obalů dat, hraniční produkční funkce

Abstract

Title: Measures for efficiency and eco-efficiency
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Abstract: Comparison between the economic subjects on micro and macro economic levels by a unitary indicator is provided by measuring efficiency as an evaluation of the performance of subjects. For each economic subject technical, allocative, economic, scale efficiencies and eco-efficiency can be computed if all data required are available. This paper attends to examine a method based on linear programming - Data Envelopment Analysis (DEA). Two models for technical efficiency and their derivates to measure other efficiencies are presented and compared in detail - a simple DEA model and an additive DEA model. Finally, theoretical procedures are applied on the forest industry in between the counties of Czech Republic and comparison by measurements of efficiencies is provided.
Keywords: efficiency, eco-efficiency, data envelopment analysis, frontier production function

Introduction

In order to make a decisions on both micro and macro economic levels there is a need of comparison between the economic subjects. By economic subjects we mean levels of economic units, e.g. departments, divisions, sections, firms, companies, branches or units by the location, e.g. cities, regions, counties, countries.

An appropriate desired comparison is often complicated or even unreachable because of lack of the clear and unitary indicators. Subjects should be compared by several indicators they have in common and under the performance of each indicator the total comprehensive performance should be set. The aim of comparison is the final evaluation to be uniform, so the results could be uniquely presented in general.

Such a comparison of economic subjects is the evaluation of performance of each unit by the level of its efficiency. This method of comparison is usually attained by evaluating each economic subject with the efficiency value relative to the set of the same types of subjects. Thus we compare the economic units only in between the group of the same type.

This paper attends to examine such comparisons by measuring the values of efficiencies. In first chapter we introduce all types of efficiencies that evaluate the performance of the unit and briefly describe different measurement techniques that can be used. In second chapter we closely examine one method of measurements, where two main models and their derivates are presented and finally, in third chapter we apply introduced theoretical procedures to the data.

Chapter 1

Efficiency and eco-efficiency

There are several different efficiencies that we can measure as an evaluation of the performance of economic subjects. In first section we introduce efficiencies that are attached to the economic meaning of comparison - technical, allocative, economic and scale efficiencies. In second section we introduce efficiency that evaluates economic subjects also in the environmental meaning and that is eco-efficiency. In third section we introduce different techniques used to measure the values of efficiencies, evaluate the performance of a unit and compare the economic units.

1.1 Definitions of efficiencies

Measurements of efficiencies are often divided upon the target of the subject's production. It can be input (conserving) orientation, where the producer's aim is to minimize the inputs or output (augmenting) orientation, where the producer's aim is to maximize the outputs.

The most common of all efficiencies is technical efficiency. If there is no confusion in the measurement, technical efficiency is in literature commonly referred only as efficiency. Technical efficiency refers to the ability to avoid waste by producing as much output as input usage allows (input orientation), or by using as little input as output production allows (output orientation).

Definition of technical efficiency by Koopmans: A producer is technically efficient if an increase in any output requires a reduction in at least one other output or an increase in at least one input, and if a reduction in any input requires an increase in at least one other input or a reduction in at least one output. [in Koopmans (1951)]

Thus technically inefficient producer could produce the same outputs with less of at least one input, or could use the same inputs to produce more of at least one output.

Farrell (1957) proposed an original idea that the efficiency of the economic subject consists of two components: technical efficiency and allocative efficiency. These two measures combined provide a measure of total economic efficiency.

Definition of technical efficiency by Farrell: The technical efficiency measure is defined as one minus the maximum equiproportionate reduction in all inputs that still allows continued production of given outputs. A score of unity indicates technical efficiency because no equiproportionate input reduction is feasible, and a score less than unity indicates the severity of technical inefficiency. [in Farrell (1957)]

Farrell's definition refers to an input orientation. In case of an output orientation the Farrell's measure need to be converted to equiproportionate output expansion with given inputs, where the conversion is straightforward.

Definition of allocative efficiency: Allocative efficiency refers to the ability of producer to combine inputs and outputs in optimal proportions, as given by the production technology, in the light of prevailing prices. [in Farrell (1957)]

Definition of economic efficiency: Economic efficiency is a compound of its two components - technical and allocative efficiencies. It reflects the overall efficiency of the producer, by given production technology, with its technical and allocative abilities.

Definition of scale efficiency: Scale efficiency refers to the potential productivity gain from achieving optimal size of economic subject. Producer is scale efficient if operates on the optimal scale.

In the literature several different names for the efficiencies are used. Technical efficiency is also called X-efficiency, allocative efficiency is also called cost (in input orientation) or price (in output orientation) efficiency and finally economic efficiency is also called overall, cost (in input orientation), profit (in output orientation) or productive efficiency. In the recent years, however, authors has unite on the terms technical, allocative and economic and we use this expressions in this paper as well. Further we use notation as follows: $\boldsymbol{TE}$ for technical efficiency, $\boldsymbol{AE}$ for allocative efficiency, $\boldsymbol{EE}$ for economic efficiency and $\boldsymbol{SE}$ for scale efficiency.

Depending on data available we can evaluate efficiencies. If quantities only data are available we can measure technical efficiency and scale efficiency. If also prices and costs data are available we can measure not only technical and scale efficiencies but allocative and economic efficiencies as well.

1.2 Definition of eco-efficiency

The prefix eco in notation eco-efficiency refers to both economic and ecologic (environmental) performance. Eco-efficiency does not represent economic efficiency, as presented in the previous section, but incorporate both economic welfare and environmental quality. Further in this paper for eco-efficiency we use notation $\boldsymbol{ECO}$.

The World Business Council for Sustainable Development (WBCSD) and Organisation for Economic Cooperation and Development (OECD) incorporated the concept and launched eco-efficiency as a "business link to sustainable development". According to them for producers eco-efficiency in general means saving resources - improving competitiveness.

The WBCSD definition of eco-efficiency: Eco-efficiency is achieved by the delivery of competitively-priced goods and services that satisfy human needs and bring quality of life, while progressively reducing ecological impacts and resource intensity throughout the life cycle, to a level at least in line with the earth's estimated carrying capacity. [in Verfaillie and Bidwell (2000)]

This broad definition of eco-efficiency combines welfare, competitiveness, the products' ecological impacts throughout their life cycle, the use of natural resources and the environmental carrying capacity.

To evaluate eco-efficiency we need a special extension of quantities and prices data available or most likely a whole different data set that includes also the ecologic information.

1.3 Techniques for efficiency measurements

The theory of production in economy is based on efficient subsets of production sets. Is is based on values dual to each other such as minimum cost functions and maximum revenue or profit functions. On envelope properties yielding cost-minimizing input demand functions, revenue-maximizing output supply functions, and profit-maximizing supply functions. Most of the techniques use economic frontiers for the measurements.

There are several approaches to efficiency measurements. The main developed ones are econometric approach and mathematical programming approach. Other approaches that does not fit into either one of the two main and are still known and used are **goal programming** and **non-frontier efficiency measurement method**. Goal programming can be found in e.g. Lovell (1993) and non-frontier efficiency measurement method in e.g. Lawrence and Yotopoulos (1971). Econometric approach we describe briefly in the next section and we introduce mathematical programming approach at the end of this chapter. Our main focus further in this paper is on technique called Data Envelopment Analysis (DEA) from mathematical programming approach.

1.3.1 Econometric approach

Econometric approach of efficiency measurement arises form statistical regression. In order to investigate and estimate the economic frontiers and to measure efficiencies relative to the frontiers the conventional econometric techniques need to be, however, modified.

Econometric models can be categorized in several ways. According to the type of data used it can be cross-sectional data model or panel data model. According to the type of variable used it can measure technical efficiency (availability of quantities data only) or allocative and economic efficiencies (availability of quantities and prices data). Finally according to the number of equations in the model it can be single equation model or multiple equations model. The multiple equations model is, however, still in evolving process and not fully used yet. The most widely used model is the single equation cross-sectional model.

The main idea of econometric approach models in measuring efficiency is decomposition of the residuals into separate estimates of statistical noise and technical inefficiency. We demonstrate briefly this idea on the single equation cross-sectional model using **stochastic frontier analysis** (**SFA**). Assume we have I producers, each one using inputs $\boldsymbol{x} \in R^n_+$ to produce output $y \in R_+$ with the production technology defined as

$$y_i = f(\boldsymbol{x}_i; \boldsymbol{\beta}) \exp\{v_i + u_i\} \qquad i = 1, \ldots, I \tag{1.1}$$

where $\boldsymbol{\beta}$ is a vector of technology parameters to be estimated. The random disturbance term v_i captures the statistical noise, v_i iid. $\sim N(0, \sigma_v^2)$. The disturbance term u_i is distributed independently from v_i, satisfies $u_i \leq 0$ and represents technical inefficiency. $[f(\boldsymbol{x}_i; \boldsymbol{\beta})]$ represents deterministic production frontier and $[f(\boldsymbol{x}_i; \boldsymbol{\beta}) \exp\{v_i\}]$ represents stochastic production frontier.

Stochastic production frontier defines maximum feasible output of the producer and technical efficiency is then measured by the ratio of observed to maximum feasible output. The stochastic production frontier is defined by the structure of production technology (by the deterministic production frontier) and by external events beyond the control of the producer that could be unfavorable as well as favorable. A detailed study on stochastic frontier production functions can be found in e.g. Aigner, Lovell, and Schmidt (1977). Technical efficiency measure of the $i-$th producer is then

$$TE_i = \exp\{u_i\} = \frac{y_i}{[f(\boldsymbol{x}_i; \boldsymbol{\beta}) \exp\{v_i\}]} \tag{1.2}$$

To achieve the efficiency score an estimation of (1.1) and decomposition of the residual (v_i+u_i) is needed. A solution of decomposition was first proposed by Jondrow, Lovell, Materov, and Schmidt (1982) where authors had specified the functional form of the distribution of one-sided inefficiency component and derived conditional distribution $(u_i|v_i + u_i)$. A point estimate of u_i achieved as the mean or mode of this distribution can be then inserted in (1.2) to estimate TE_i.

Parameters of the stochastic production frontier can be estimated by the modified ordinary least squares (MOLS) and maximum likelihood estimation (MLE) techniques. Both methods begin with the assumption on the functional form of the distribution of the non-positive efficiency term u_i. MOLS adjust the ordinary least squares (OLS) intercept by minus the mean of u_i, which is extracted from the moments of the OLS residual. MLE estimates technology and efficiency parameters simultaneously. The resulting residuals obtained from both MOLS and MLE contain statistical noise and technical inefficiency, thus need to be decomposed.

Jondrow, Lovell, Materov, and Schmidt (1982) proposed few functional forms of the distribution of asymmetric error term. A half-normally, an exponentially and a gamma distributed u_i. For a half-normally distributed inefficiency, u_i iid $\sim N_-(0, \sigma_u^2)$, where $N_-(0, \sigma_u^2)$ represents non-positive half of $N(0, \sigma_u)$. The expected value of u_i conditional on the composed error term is

$$E[u_i|\epsilon_i] = \frac{\sigma\lambda}{1+\lambda^2}\left[\frac{\phi(\epsilon_i\lambda/\sigma)}{\Phi(-\epsilon_i\lambda/\sigma)} - \frac{\epsilon_i\lambda}{\sigma}\right] \tag{1.3}$$

where $\phi(.)$ is the density of the standard normal distribution, $\Phi(.)$ is the cumulative density function, $\lambda = \sigma_u/\sigma_v$, $\epsilon_i = v_i + u_i$, and $\sigma = (\sigma_u^2 + \sigma_v^2)^{1/2}$. For the exponentially distributed u_i, expected value of u_i is

$$E[u_i|\epsilon_i] = (\epsilon_i - \theta\sigma_v^2) + \frac{\sigma_v\phi[(\epsilon_i - \theta\sigma_v^2)/\sigma_v]}{\Phi[(\epsilon_i - \theta\sigma_v^2)/\sigma_v]} \tag{1.4}$$

where θ is a parameter from the density function $f(u_i) = \theta \exp\{-\theta u_i\}$. Finally for the gamma distributed u_i the expected value of u_i is

$$E[u_i|\epsilon_i] = \frac{h(P, \epsilon_i)}{h(P-1, \epsilon_i)} \tag{1.5}$$

where $h(P, \epsilon_i) = E[z^P | z > 0, \epsilon_i]$ when $z \sim N\left(-(\epsilon_i + \theta\sigma_v^2), \sigma_v^2\right)$ and parameter P is from $u_i \sim \Gamma(\Theta, P)$, $f(u_i) = \frac{\Theta^P}{\Gamma(P)} u_i^{P-1} \exp\{-\Theta u_i\}$, $\Theta, P > 0$. Detailed procedures how to estimate technical, allocative and economic efficiencies using stochastic frontier analysis (SFA) can be found in e.g. Schmidt and Lovell (1979) and Green (1993).

The econometric approach is stochastic and so attempts to differentiate the effects of inefficiency from the effects of statistical noise. It is also parametric and thus unfortunately disturbs inefficiency with the effects of mis-specification of functional form (of both technology and inefficiency).

Because the mostly used SFA is applied on the single-output equation the application of this method to measure eco-efficiency is unrealistic at the present time. Once a multiple-equations SFA model will evolve, it will be probably the most precise measurement for efficiencies, as it will be able to distinguish efficiency from the statistical noise and handle multiple outputs.

1.3.2 Mathematical programming approach

The mathematical programming approach to efficiency measurement leads as well as the econometric approach to the construction of the production frontiers. The measurement of efficiency relative to the constructed frontiers in this approach is represented by the **data envelopment analysis** (**DEA**). This method was developed in public sector, not-for profit environment and has originally not considered the prices. The majority of the DEA models use quantity data only and so calculate technical efficiency only. However, if the prices data are available, models can be easily extended to calculate allocative and economic efficiencies as well.

In case of a panel data available, DEA method can be extended to measure productivity change index that decomposes into technical change and technical efficiency change. This method is called **Malmquist index** and details about this theory can be found in e.g. Jacobs, Smith, and Street (2006) or Lovell (1993).

DEA method is, in comparison to the SFA method of econometric approach, a non-parametric, so there is an advantage of no need to estimate the parameters. On the other hand, DEA does not make an accommodation for statistical noise and so the results of measurements of efficiency might be affected by the random unpredictable events (that can be unfavourable as well as favourable). Despite the noise, subject to certain assumptions about the structure of production technology, DEA does envelop the data as tightly as possible. Another advantage in comparison to SFA is that DEA has no problem dealing with multiple output and so thanks to that, DEA can be extended to measure not only technical, allocative, scale and economic efficiencies but also eco-efficiency.

We present the DEA method closely in the chapter two, as it is our main focus in this paper, where we examine two main models and their derivates for measurements of all efficiencies mentioned.

Chapter 2

Data envelopment analysis

Data envelopment analysis (DEA) is a non-parametric data-oriented approach to efficiency measurement. DEA's empirical orientation and absence of a need for the numerous a priori assumptions that accompany other approaches (such as standard forms of statistical regression analysis) have resulted in its use in a number of studies involving efficiency frontier estimation in many different sectors of the economy (from governmental and non-profit sector, through the regulated sector to the private sector). DEA evaluates the performances of a set of peer entities called **decision making units** (**DMU**s) which convert multiple inputs into multiple outputs, where we use the simple notion that an organization that uses less inputs than another to produce the same amount of outputs is considered as more efficient.

The location and the shape of the efficiency frontier is determined by the data, therefore the so-called data-oriented approach. Basic metric used to evaluate efficiency is the ratio of the inputs and outputs (in the simple DEA model). The efficiency frontier is constructed by those DMUs with the highest ratios that are considered efficient and this construction is based on the best observed practice and its therefore only an approximation of the true efficiency frontier. The efficient DMUs create surface that envelops other DMUs and their inefficiency is calculated within the frontier boundary relative to this surface as a distance from it.

We present two main models for technical efficiency and their derivates to measure allocative, economic efficiencies and eco-efficiency. We also present scale efficiency that can contaminate the scores of other efficiencies, specially the technical one.

2.1 Graphical interpretation and fundamental efficiency

2.1.1 Input-oriented efficiencies

Suppose a decision making unit (DMU) use two inputs x_1 and x_2 to produce a single output y. The production frontier is represented by curve ZZ' and all DMUs lie on or above the frontier. DMUs that lie on the frontier are efficient and DMUs that lie above the frontier are inefficient and could proportionally reduce their inputs x_1, x_2 to produce the (given) same output level y. In the Figure 2.1 the production of DMU A is above the production frontier and is obviously inefficient, thus producer A could proportionally reduce inputs x_1 and x_2 for the given amount of output y and move to a feasible and technically efficient point on the production frontier curve, such as the points B or B'.

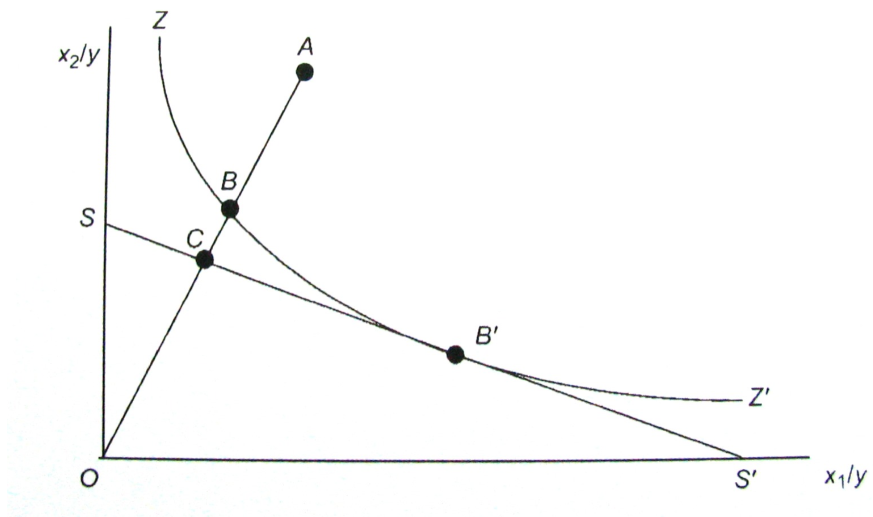

Figure 2.1: Technical and allocative efficiencies under an input orientation. [in Jacobs, Smith, and Street (2006)]

The budget (iso-cost) line that reflects the ratio of the prices of inputs x_1 and x_2 is represented by the line SS'. The point where the marginal rate of the substitution of x_1 for x_2 is equal to the price ratio is allocatively efficient (cost-efficient) point of production - represented by the point B'.

Technical efficiency in the DEA is usually measured by the ratio of inputs and outputs. Therefore the models measuring efficiency by the ratio are often referred as the radial measure of inefficiency. This is achieved by comparing the distances where the producer is located in relation to the production frontier function (BA) with where it is located in relation to the origin (OA). Thus the distance BA is the amount by which all inputs x_1 and x_2 could be proportionally reduced without reduction in producer's admissions. This is expressed in percentage terms by the ratio BA/OA. The input-oriented technical efficiency (TE) of the producer A is expressed as follows

$$TE_{\text{input}} = \frac{OB}{OA} \tag{2.1}$$

where the subscript input denotes the input orientation. The ratio OB/OA from (2.1) is equal to $1 - BA/OA$. Pure technical efficiency shows the deviation from the production frontier ZZ'. This value lies between zero and one, with the value of one indicating full technical efficiency (if the producer A has produced at a point such as B).

If the prices of inputs (costs) are known, the budget (iso-cost) can be specified and used to compute the allocative efficiency (AE). For the producer operating at point A the ratio is as follows

$$AE_{\text{input}} = \frac{OC}{OB} \tag{2.2}$$

where the distance CB is reduction in production costs that would occur if production were to take place at the allocatively (and technically) efficient point B' instead of at the technically efficient, but allocatively inefficient, point B. It thus represents the deviation from the price-efficient point.

The (total) economic efficiency (EE) as a combination of technical efficiency and allocative efficiency is given by

$$EE_{\text{input}} = TE_{\text{input}} \times AE_{\text{input}} = \frac{OB}{OA} \times \frac{OC}{OB} = \frac{OC}{OA} \tag{2.3}$$

2.1.2 Output-oriented efficiencies

An alternative to input-oriented efficiencies measurement is an output-oriented one. Suppose producer produces two outputs y_1 and y_2 from a single input x. The upper bound of all the technically feasible production possibilities is represented by the ZZ', the production possibility curve. All DMUs that lie on the production frontier are efficient and DMUs that lie below the frontier are inefficient and could proportionally expand their output quantities y_1 and y_2 holding their level of input use x constant (such as producer A in the Figure 2.2). The expansion could be done up to a point on the production frontier such as B.

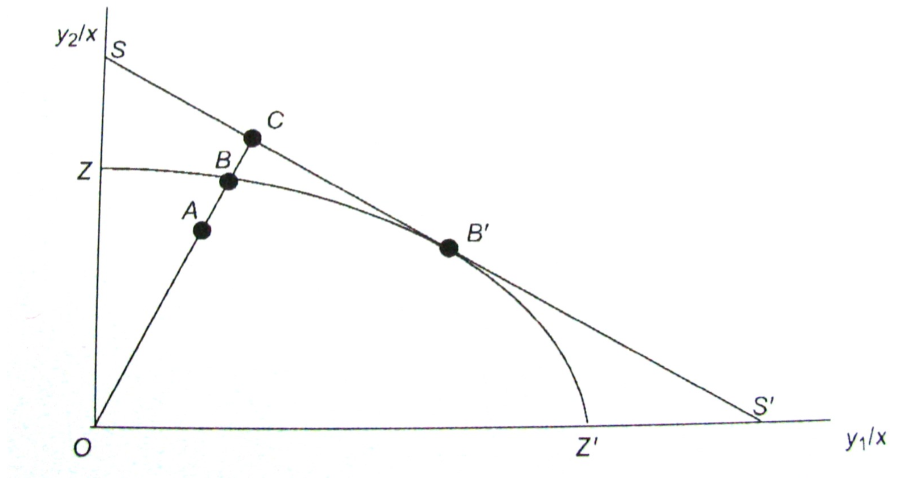

Figure 2.2: Technical and allocative efficiencies under an output orientation. [in Jacobs, Smith, and Street (2006)]

The profit (iso-revenue) line that reflects the market values of outputs y_1 and y_2 is represented by the line SS'. The (technically and allocatively) efficient point of production is B'. ZZ' is tangential to the profit line, thus the technical efficiency of DMU A is represented by

$$TE_{\text{output}} = \frac{OA}{OB} \tag{2.4}$$

where the subscript output denotes the output orientation. If the selling prices of outputs are known, the profit (iso-revenue) can be specified and used to compute allocative efficiency. For the producer A, allocative efficiency is represented by

$$AE_{\text{output}} = \frac{OB}{OC} \tag{2.5}$$

and (total) economic efficiency is then given by

$$EE_{\text{output}} = TE_{\text{output}} \times AE_{\text{output}} = \frac{OA}{OB} \times \frac{OB}{OC} = \frac{OA}{OC} \tag{2.6}$$

Measures of efficiencies are all again bounded by zero and one. Because efficiencies are measured along a ray from the origin to the observed production point, they hold relative proportions of inputs (or outputs) constant. These radial efficiency measures are units-invariant in the sense that changing the units of measurement will not change the value of the efficiency measure.

Note that relationship between the solutions of technical, allocative and economic efficiencies under input and output orientation applies as follows

$$TE_{\text{output}} = \frac{1}{TE_{\text{input}}} \qquad AE_{\text{output}} = \frac{1}{AE_{\text{input}}} \qquad EE_{\text{output}} = \frac{1}{EE_{\text{input}}} \tag{2.7}$$

2.1.3 Fundamental efficiency

The efficiency measures examined in the previous sections assume the production function of the fully efficient organization is known. In practice the efficient isoquant must be estimated from the data. DEA assesses efficiency in two stages, also depending on the orientation - whether input or output. First the frontier is identified. In input-oriented measurement the frontier is based on organizations using the lowest input mix to produce given outputs. In output-oriented measurement frontier is based on organizations achieving the highest output mix given their inputs. Second, an efficiency score is assigned to each DMU by comparing output/input (input/output) ratio to that of efficient DMUs that form a piecewise linear envelop of surfaces in multidimensional space.

Production frontier becomes a surface in the $m + n$ dimensional space, if there are m inputs and n outputs considered. Thus the efficiency of the DMU is a distance it lies from this surface. The distance in input-oriented measurement is a maximum extent by which DMU could improve its outputs at given level of inputs and in the output-oriented it is a maximum reduce of its inputs at the given level of outputs.

Because in most applications (specially in social science) the theoretically possible levels of efficiency will not be known, we need to alter efficiency by emphasizing its uses with only the information that is empirically available. We compare the efficiency of the DMU to other DMUs, so the efficiency we measure is thus relative to the set and its surface. This leads to definition of the relative efficiency.

Efficiency: Full (100 %) efficiency is attained by any DMU if and only if none of its inputs or outputs can be improved without worsening some of its other inputs or outputs. [in Cooper, Lawrence, and Zhu (2004)]

Relative Efficiency: A DMU is to be rated as fully (100 %) efficient on the basis of available evidence if and only if the performance of other DMUs does not show that some of its inputs or outputs can be improved without worsening some of its other inputs or outputs. [in Cooper, Lawrence, and Zhu (2004)]

Note that this definition does not mention the explicitly specific formal relations between the inputs and outputs that suppose to exist, also no mention about any assumption of weights or prices that are suppose to reflect the relative importance of the different inputs and outputs. This efficiency is referred as technical efficiency.

The efficiency presented is the most elemental of the various efficiencies that might be considered (allocative, economic, ecologic, scale efficiencies) in that it requires only minimal information and minimal assumptions for its use. It is also fundamental because other types of efficiency such as allocative, economic and scale efficiencies require technical efficiency to be attained before these can be achieved.

2.2 Simple DEA model

Suppose there are I DMUs to be evaluated. Each DMU uses varying amounts of m different inputs x to produce n different outputs y, where $\boldsymbol{x} \in R_+^m$ and $\boldsymbol{y} \in R_+^n$ and further we assume that each DMU has as least one positive input and one positive output value.

A ratio of a weighted sum of outputs to a weighted sum of inputs is used to measure efficiency of the DMU_o to be evaluated relative to the ratios of all the $k = 1, \ldots, I$ DMU_k. Because of this ratio the simple DEA is often referred as the radial measure of the efficiency. Note that the data we use are quantities only, thus the efficiency we measure with this model is technical efficiency and it is evaluated by solving for each DMU following mathematical programming problem

$$TE_o = \max_{\boldsymbol{u},\boldsymbol{v}} \frac{\sum_{j=1}^{n} u_j y_{jo}}{\sum_{i=1}^{m} v_i x_{io}} \tag{2.8}$$

subject to

$$\begin{aligned} \frac{\sum_{j=1}^{n} u_j y_{jk}}{\sum_{i=1}^{m} v_i x_{ik}} &\leq 1 \qquad k = 1, \ldots, I \\ v_i, u_j &\geq 0 \qquad \forall i, j \end{aligned}$$

where x_{ik} is the quantity of input i consumed to produce y_{jk} the quantity of output j by DMU_k. v_i, u_j are weights attached to input i and output j, respectively. The objective function seeks for the set of output u_j and input v_i weights that maximizes the efficiency of DMU_o (in the sense that no other set of weights will yield a higher level of efficiency), subject to the set of constraints that reflects the condition that the output to input ratio of every DMU can not have an efficiency greater than unity.

The equation above, however, has an infinite number of solutions; if $(\boldsymbol{u}^*, \boldsymbol{v}^*)$ is optimal, then also $(\alpha\boldsymbol{u}^*, \alpha\boldsymbol{v}^*)$ is optimal for $\alpha > 0$. To achieve the transformation of this problem to a linear problem we select a representative solution by adding an additional constraint: the denominator of efficiency ratio to be equal to one [i.e. solution $(\boldsymbol{u}, \boldsymbol{v})$ for which $\sum_{i=1}^{m} v_i x_{io} = 1$]. Now we can rewrite (2.8) into the linear programming multiplier problem

$$TE_o = \max_{\boldsymbol{\mu},\boldsymbol{\nu}} \sum_{j=1}^{n} \mu_j y_{jo} \tag{2.9}$$

subject to

$$\begin{aligned}
\sum_{j=1}^{n} \mu_j y_{jk} - \sum_{i=1}^{m} \nu_i x_{ik} &\leq 0 \qquad k = 1, \dots, I \\
\sum_{i=1}^{m} \nu_i x_{io} &= 1 \\
\nu_i, \mu_j &\geq 0 \qquad \forall i, j
\end{aligned}$$

where change of variables $(\boldsymbol{u}, \boldsymbol{v})$ to $(\boldsymbol{\mu}, \boldsymbol{\nu})$ is a result of the transformation we have provided. This maximization problem has its equivalent minimization dual problem which is the linear programming envelopment problem

$$TE_o = \min_{\theta_o, \boldsymbol{\lambda}} \theta_o \tag{2.10}$$

subject to

$$\begin{aligned}
\sum_{k=1}^{I} x_{ik} \lambda_k &\leq \theta_o x_{io} \qquad i = 1, \dots, m \\
\sum_{k=1}^{I} y_{jk} \lambda_k &\geq y_{jo} \qquad j = 1, \dots, n \\
\lambda_k &\geq 0 \qquad k = 1, \dots, I
\end{aligned}$$

Problem (2.10) is solved I times, once for each DMU$_o$ being evaluated ($o = 1, \dots, I$), to generate I optimal values of $(\theta_o, \boldsymbol{\lambda})$. The objective of linear envelopment program seeks for the minimum θ_o that reduces the input vector x_{io} to $\theta_o x_{io}$ while guaranteeing at least the y_{jo} output level. Lets mark $\theta_o^* = \min \theta_o$. The value of technical efficiency $TE_o = \theta_o^*$ obtained is the efficiency score for the evaluated DMU$_o$ and satisfies $\theta_o^* \leq 1$. The DMUs for which $TE = \theta^* < 1$ are inefficient, while the DMUs for which $TE = \theta^* = 1$ are boundary points that indicate a points on the frontier (these DMUs are efficient).

Program (2.10) also generates weights $\boldsymbol{\lambda}$ that are specific for each DMU. Value of λ_k is a weight to be attached to DMU$_k$ in forming the efficient benchmark for DMU$_o$. DMU$_o$ is compared to the point on the frontier that is created by composite peer DMU that is formed from a linear combination of other DMUs with the weights λ_k. Only the efficient DMUs have non-zero weight in the peer group and inefficient DMUs do not appear in any peer group. DMUs with a non-zero weight are referred to as efficient peers or comparators of DMU$_o$. This approach preserves the input-output mix of DMU$_o$, which is therefore compared to a set of efficient peers that use similar or identical input-output ratios, but at more efficient levels.

The simple DEA model we have examined is input-oriented with the assumptions of constant returns to scale and strong disposability of inputs and outputs (suppression of the slacks). Technical efficiency obtained from this model satisfies Farell's definition but not Koopman's one because of the strong disposability assumption. How to relax the assumptions, consider the possible presence of the non-zero slacks in the model, achieve the efficiency that satisfies also Koopman's definition and convert an input-oriented problem to an output-oriented one we solve in the section Considerations in DEA.

2.2.1 Simple DEA model for economic efficiency

To extend the simple DEA model to measure economic efficiency the data available have to be quantities and costs. We present an input-oriented problem.

Assume there are I DMUs to be evaluated that use inputs $\boldsymbol{x} \in R_+^m$, for the costs $\boldsymbol{c} \in R_{++}^m$, to produce outputs $\boldsymbol{y} \in R_+^n$. Objective is the minimization of the costs, subject to the constraints imposed by input costs, output demand, and the structure of the production technology. Assuming constant return to scale, strong disposability of inputs and outputs the linear programming envelopment problem is

$$\min_{\boldsymbol{x},\boldsymbol{\lambda}} \sum_{i=1}^{m} c_{io} x_i \tag{2.11}$$

subject to

$$\begin{aligned} \sum_{k=1}^{I} x_{ik}\lambda_k &\le x_i \qquad & i &= 1,\dots,m \\ \sum_{k=1}^{I} y_{ik}\lambda_k &\ge y_{jo} \qquad & j &= 1,\dots,n \\ \lambda_k &\ge 0 \qquad & k &= 1,\dots,I \end{aligned}$$

where the objective is to choose x_i and λ_k values to minimize the total cost as satisfying the output constraints. Note that c_{io} represents unit costs for DMU_o and thus are allowed to vary from one DMU to another.

A measure of economic (cost) efficiency for each DMU_o evaluated is obtained by utilizing the ratio

$$EE_o \quad = \quad \frac{\sum_{i=1}^{m} c_{io}x_i^*}{\sum_{i=1}^{m} c_{io}x_{io}} \tag{2.12}$$

where x_i^* are the optimal values obtained from (2.11) and x_{io} are the observed values for DMU_o. Efficiency score is again bounded by zero and one, thus EE satisfies $0 \leq EE \leq 1$.

According to Cooper, Lawrence, and Zhu (2004) the use of a ratio like (2.12) is standard and yields an easily understood measure. It has shortcomings, however, as witness the following example: let EE_a and EE_b represent economic (cost) efficiency, as determined from (2.12), for DMUs a and b. Now suppose $x_{ia}^* = x_{ib}^*$ and $x_{ia} = x_{ib}$, $\forall i$, but $c_{ia} = 2c_{ib}$ so that the unit costs for a are twice as high as for b in every input. We then have

$$EE_a = \frac{\sum_{i=1}^{m} c_{ia}x_{ia}^*}{\sum_{i=1}^{m} c_{ia}x_{ia}} = \frac{\sum_{i=1}^{m} 2c_{ib}x_{ib}^*}{\sum_{i=1}^{m} 2c_{ib}x_{ib}} = \frac{\sum_{i=1}^{m} c_{ib}x_{ib}^*}{\sum_{i=1}^{m} c_{ib}x_{ib}} = EE_b \tag{2.13}$$

Thus, as might be expected with a use of ratios, important information may be lost since $EE_a = EE_b$ suppress the fact that a is twice as costly as b.

The costs minimization problem needs to be solved separately for each DMU_o evaluated. Economic (cost) efficiency, the ratio of observed to a minimum possible cost, decomposes into the product of input-oriented technical efficiency and input mix allocative efficiency.

Economic efficiency obtained as the solution to equations (2.11) and (2.12) and input-oriented technical efficiency obtained as the solution to equation (2.10) are applied to evaluate allocative efficiency, that is then for each DMU separately obtained residually from the formula

$$AE = \frac{EE}{TE} \tag{2.14}$$

2.3 Considerations in DEA

There are several considerations in determining the DEA models. We have to make a decision which statements we select for model to be the best fit. Depending on that an appropriate assumptions for constraints, orientation or form of the model is required. In this section we examine considerations and describe how the change of the assumption will change the simple DEA model (2.10) presented to the one more appropriate regarding the circumstances.

2.3.1 Constant, variable or non-increasing returns to scale

The simple DEA model (2.10) assumes constant returns to scale. Constant returns to scale DEA model expects that all DMUs can be considered to be operating at an optimal scale. Also other options than **constant returns to scale** (**CRS**) need to be considered - **variable returns to scale** (**VRS**) and **non-increasing returns to scale** (**NIRS**).

More flexible than constant returns to scale DEA model is variable returns to scale DEA model and may be appropriate when not all DMUs can be considered to be operating at an optimal scale. For example imperfect competition, constraints on finance, regulatory constraints on entry or mergers may often result in organisations operating at an inefficient scale. The simple DEA linear programming problem (2.10), and by its assumption also CRS model, can be adjusted to an VRS model by adding a (convex) constraint $\sum_{k=1}^{I} \lambda_k = 1$ to equation (2.10). This added constraint introduces an additional variable ω_o into the multiplier problem placed in the objective function as well as in constraints [i.e., in (2.9) the objective function changes to $\max_{\mu,\nu,\omega_o} \sum_{j=1}^{n} \mu_j y_{jo} + \omega_o$ and constraints change to $\sum_{j=1}^{n} \mu_j y_{jk} - \sum_{i=1}^{m} \nu_i x_{ik} + \omega_o \quad \forall k$]. Exact formulation can be found e.g. in Agha and Lawrence (1993).

The choice of CRS or VRS is an important decision and relies on the analyst's understanding of market constraints facing firms within a particular sector. If the CRS technology is inappropriately applied (when e.g. all organizations are operating at a sub-optimal scale) then the estimates of technical efficiency will be confounded by scale efficiency effects.

To calculate scale inefficiency, both CRS and VRS DEA models should be applied on the same data, and any change in measured efficiency can be attributed to the presence of scale inefficiency.

See Figure 2.3, where we assume that DMU A uses single input x to produce single output y. Figure captures the difference between CRS, VRS and NIRS production frontiers. The CRS frontier is illustrated by line OE and VRS frontier by fragmented line $FGHIJ$. Assuming an input orientation, where we imply a reduction of input x in the horizontal plane, technical efficiency of DMU A with respect to the CRS technology is

$$TE_{\text{input, CRS}} = \frac{CB_C}{CA} \tag{2.15}$$

where the subscribes stand for input orientation and constant returns to scale technology. Technical efficiency of input orientation with the variable returns

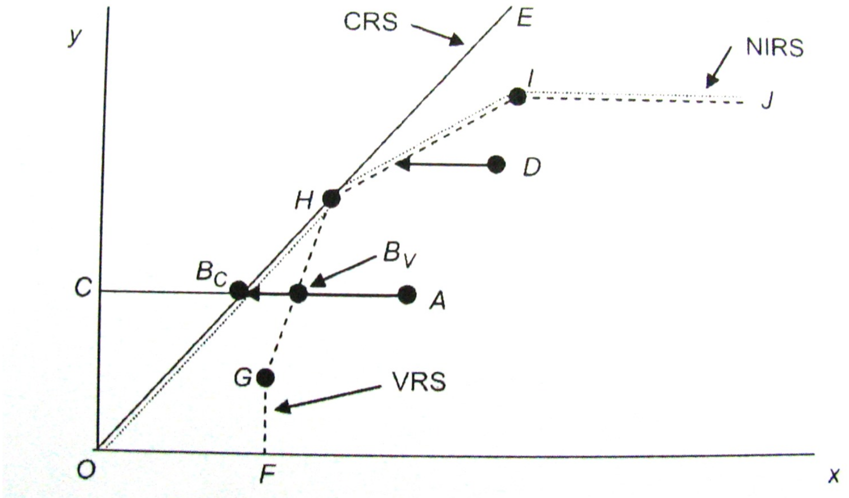

Figure 2.3: Constant, variable and non-increasing returns to scale. [in Jacobs, Smith, and Street (2006)]

to scale technology is expressed on the other hand as

$$TE_{\text{input, VRS}} = \frac{CB_V}{CA} \tag{2.16}$$

where the subscript denotes the variable returns to scale technology. Scale efficiency is then measured as the distance between CRS and VRS technologies as follows

$$SE_{\text{input}} = \frac{CB_C}{CB_V} \tag{2.17}$$

and therefore

$$TE_{\text{input, CRS}} = TE_{\text{input, VRS}} \times SE_{\text{input}}$$

$$SE_{\text{input}} = \frac{TE_{\text{input, CRS}}}{TE_{\text{input, VRS}}} \tag{2.18}$$

All efficiency measures are again bounded by zero and one. If we imagine the same situation in multidimensional space, the VRS technology forms a convex hull of intersecting planes which envelop data points, such as A and D, more tightly than CRS technology, where the frontier would extend from the origin. Thus, by introducing an additional constraint, VRS produces technical efficiency scores which are greater than or equal to those obtained by using CRS.

The convexity constraint $\sum_{k=1}^{I} \lambda_k = 1$ ensures that an inefficient DMU will be compared only with DMUs of a roughly similar size. Thus the projected point for DMU A on the DEA frontier will be a convex combination of other DMUs, such as G and H. This convexity restriction implies that the efficient frontier is formed only by interpolation between DMUs, and precludes extrapolation of performance at one scale to a different scale. In contrast, the CRS case permits extrapolation, with the result that DMUs may be compared with others operating at substantially different scales. Thus under CRS the λ_k weights may sum up to a value greater than (or less than) one.

To obtain an indication whether a DMU is operating in the area of increasing or decreasing returns to scale a non-increasing returns to scale constraint $\sum_{k=1}^{I} \lambda_k \leq 1$ need to be added to (2.10) as alternate to the convexity constraint. In Figure 2.3 the NIRS frontier is illustrated as the fragmented line that runs from O to H and then follows the VRS frontier HIJ.

Scale inefficiencies can then be determined (whether increasing or decreasing returns to scale) by comparing the DMU's technical efficiency score under VRS constraint $\sum_{k=1}^{I} \lambda_k = 1$ to their technical efficiency score under NIRS constraint $\sum_{k=1}^{I} \lambda_k \leq 1$. If they are not equal, increasing returns to scale exist; if they are equal, then decreasing returns to scale apply. DMUs between F and H such as DMU A have increasing returns to scale whereas DMUs between H and J such as DMU D have decreasing returns to scale. A DMU at point C is scale-efficient under both CRS and VRS. More DMUs are therefore likely to be found efficient under VRS than CRS.

The incorrect scale assumption is likely to have an effect on efficiency score specially when sample size is small. With the large sample size there is a greater probability of being able to form a peer group which exhibit efficiency which is close to that of the unconstrained peer group. A complication to the choice of returns to scale (whether CRS or VRS) will occur when data have the form of ratios instead of absolute numbers of inputs and outputs measures. Creation of ratios removes the information about the size of the organization and so using such a data automatically implies an assumption of constant returns to scale.

A measurement of efficiency score in a simple DEA model assuming CRS is often referred as the global technical efficiency. A measurement of efficiency score in a simple DEA model assuming VRS is on the other hand often referred as the local pure technical efficiency.

NIRS assumption is not used commonly as CRS or VRS and the most often used models assume VRS. Hence, further in the paper and in application on the data in third chapter we consider only models with CRS and/or VRS.

2.3.2 Input or output oriented model

In the simple DEA model we can examine efficiency of DMUs using either an input or an output orientation. Input-oriented technical efficiency measure keeps outputs fixed and explore the proportional reduction in inputs usage that is possible, while output-oriented technical efficiency measure keeps inputs constant and explore the proportional expansion in outputs quantities that are possible.

With CRS constraint, the DEA results are the same whether an input orientation or an output orientation is specified. However, under VRS assumption the two are not equivalent in general. The difference is illustrated in Figure 2.4, using one input x and one output y with an inefficient DMU operating at point C.

With VRS constraint, technical efficiency measure for DMU C in the input-orientation specification depends on the horizontal distance from the frontier, and in the output-orientation specification on the vertical distance from the frontier. Therefore from the Figure 2.4:

$$TE_{\text{input, CRS}} = \frac{AB}{AC} = \frac{DC}{DE} = TE_{\text{output, CRS}} \tag{2.19}$$

but

$$TE_{\text{input, VRS}} = \frac{AB}{AC} \neq \frac{DC}{DF} = TE_{\text{output, VRS}} \tag{2.20}$$

The choice of orientation does not affect which observations are identified as fully efficient, since the models will estimate exactly the same frontier. The difference lies in the part of the frontier to which the inefficient DMU is projected.

Therefore, under VRS assumption, the choice of input or output orientation may be an important consideration that will be affected by the analyst's view on which parameters managers are able to control. For instance, firm's specialities may face a fixed quantity of inputs in any given period. Subject to this resource constraint, managers must decide the amount of outputs. This would imply that technical efficiency is measured by considering the extent to which outputs can be expanded proportionately without altering the quantity of inputs. This suggests an output-oriented measure of efficiency. On the other hand, contractual arrangements with a firm may be specified in terms of a target number of outputs. The managerial problem might then be better formulated by considering how much input quantities could be reduced while still maintaining the output target. This would imply an input-orientation to the problem.

The simple DEA problems (2.8), (2.9) and (2.10) are input-oriented. To obtain an analogous output-oriented problems it is needed to replace the

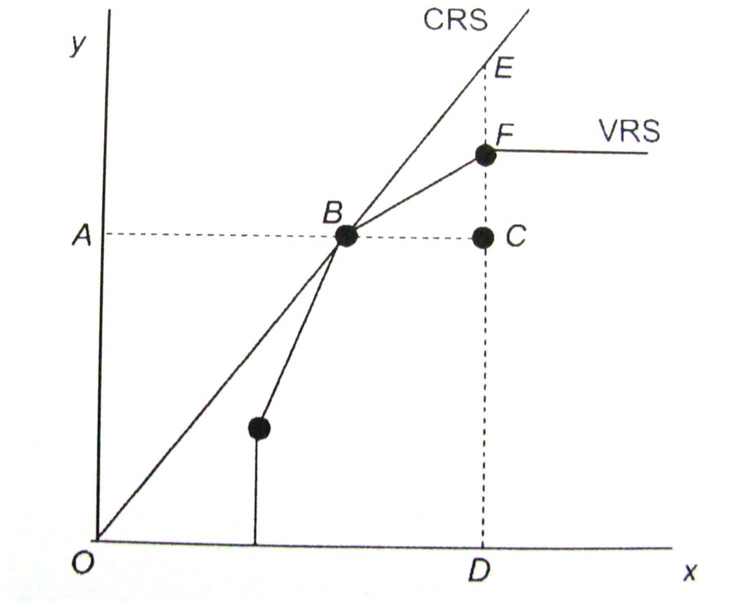

Figure 2.4: Input and output orientation. [in Jacobs, Smith, and Street (2006)]

equation (2.8) with a minimization problem, equation (2.9) with a minimization multiplier problem, and equation (2.10) with a maximization envelopment problem. Exact formulations of such output-oriented simple DEA models can be found in e.g. Lovell (1993) or Agha and Lawrence (1993).

2.3.3 Dealing with the slacks

Slacks can occur on both input or output side. They represent excesses in inputs and shortfalls in outputs. In the matter of the efficient production the goal is thus all the slacks being equal to zero. Lets mark $\boldsymbol{s}^- \in R_+^m$ as slacks on the input side and $\boldsymbol{s}^+ \in R_+^n$ as slacks on the output side.

The simple DEA model (2.10) does not consider the slacks in neither input nor output constraints what is expressed by the strong disposal assumption. Thus in the model (2.10) the slacks are ignored what results in the efficiency measure to satisfied Farrell's definition of the efficiency but not the Koopmans' definition.

In order to satisfied also Koopmans' definition of efficiency the optimal solution of linear programme (2.10) have to include slacks in input and output constraints. For constraints with non-zero slacks, the performance of the peer group suggests that the DMU under scrutiny can improve beyond the level implied by the overall efficiency estimate. For such inputs (or outputs) the estimated frontier effectively runs parallel to the relevant input or output axis in multidimensional space. To illustrate, suppose we have two inputs x_1 and x_2 to produce a single output y. Figure 2.5 shows four DMUs, where A and B

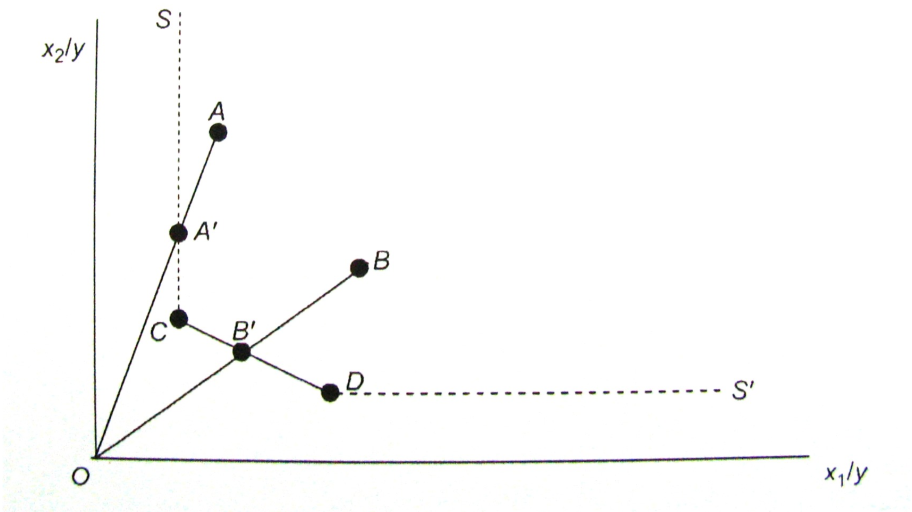

Figure 2.5: Efficiency measurement and input slacks. [in Jacobs, Smith, and Street (2006)]

represent inefficient production units and C and D represent efficient units, forming the frontier. Thus the inefficiency of DMUs A and B is calculated as OB'/OB and OA'/OA, respectively. However, the radial projection for A does not encounter the frontier interpolated between C and D, so is not naturally enveloped because the frontier is incomplete.

Whether point A' is an efficient point is questionable, because input x_2 could still be reduced by the amount $A'C$. Any point along such artificial frontier extensions (the broken lines in Figure 2.5) is always dominated by a point on the edge of the frontier. For inefficient DMU A, the difference in input x_2 between these two points (A' on the extension and C on the frontier) is the slack associated with that input.

We can imagine a similar situation in Figure 2.6 where we have two outputs y_1 and y_2 and a single input x. DMUs A and B represent inefficient production units and C, D and E represent efficient units, forming the frontier. Thus the inefficiency of DMUs A and B is calculated as OA/OA' and OB/OB', respectively. $A'C$ represents the output slack or the amount by which output y_1 can still be expanded.

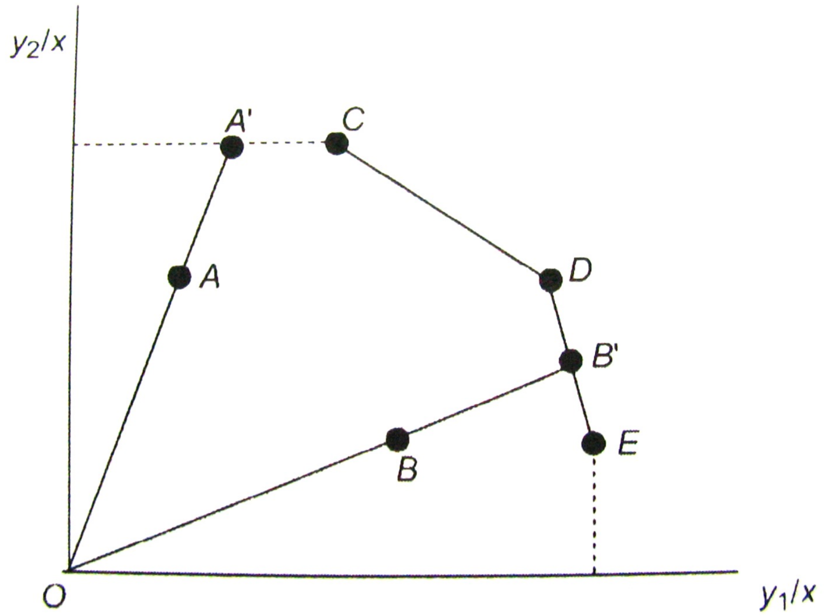

Figure 2.6: Efficiency measurement and output slacks. [in Jacobs, Smith, and Street (2006)]

Point A' in both Figures 2.5 and 2.6 represents the Farrell's definition of efficiency, the radial reduction in inputs (Figure 2.5) or radial expansion in outputs (Figure 2.6) that is possible. Farrell's technical efficiency is necessary, but not sufficient, for Koopmans' technical efficiency definition that imply that points such as A' are not efficient. According to the Koopmans' definition, DMUs are technically efficient only if they operate on the frontier (such as DMUs C and D) and all associated slacks are zero. Failure to account for slacks will result in an overestimation of technical efficiency (using Farrell estimates) for those DMUs operating with slack (such as DMU A). Furthermore, if the targets calculated for inefficient DMUs (such as A) include the slack values, they may imply a significant change in the input/output mix (moving from a point A' to C) which may then result in the targets not being helpful or practicable.

Dealing with the slacks, keeping the structure of the simple DEA model (2.10) and satisfying both Farrell's and Koopmans' efficiency definitions can be accomplished by using a two-stages DEA linear programming problem which can be run to move from a point such as A' (Farrell's efficient) to

C (Koopmans' efficient) on the frontier. This is accomplished by taking the θ value from the first-stage linear programming problem then running a second-stage linear programming problem and setting the input and output slacks to zero. First we examine the second-stage and the two-stages simple DEA envelopment problem we introduce in the next section.

Simple DEA envelopment problem (2.10) suppress the slacks and as a result of that some boundary points from the solution may be "weakly efficient" because of the presence of non-zero slacks. This is a concern, because some alternate optimum may have non-zero slacks in some solutions but not in others. This can be solved by invoking the linear program in which the slacks are taken into their maximal values

$$\max_{\boldsymbol{\lambda},\boldsymbol{s}^-,\boldsymbol{s}^+} \sum_{i=1}^{m} s_i^- + \sum_{j=1}^{n} s_j^+ \tag{2.21}$$

subject to

$$\begin{aligned} \sum_{k=1}^{I} x_{ik}\lambda_k + s_i^- &= \theta_o^* x_{io} & i &= 1,\ldots,m \\ \sum_{k=1}^{I} y_{jk}\lambda_k - s_j^+ &= y_{jo} & j &= 1,\ldots,n \\ s_i^-, s_j^+, \lambda_k &\geq 0 & &\forall i,j,k \end{aligned}$$

where optimal solution is $(\boldsymbol{s}^{-*}, \boldsymbol{s}^{+*})$. Note that the choices of $\boldsymbol{s}^-$ and $\boldsymbol{s}^+$ do not affect the optimal value of technical efficiency θ_o^* which is determined from the model (2.10). This leads to DEA definitions of efficiency based on the definition of relative efficiency.

DEA efficiency: The performance of DMU$_o$ is fully (100 %) efficient if and only if both (i) $\theta_o^* = 1$ and (ii) slacks $s_i^{-*} = s_j^{+*} = 0$ $\forall i, j$. [in Cooper, Lawrence, and Zhu (2004)]

Weak DEA efficiency: The performance of DMU$_o$ is weakly efficient if and only if both (i) $\theta_o^* = 1$ and (ii) $s_i^{-*} \neq 0$ and/or $s_j^{+*} \neq 0$ for some i and j in some alternate optimum. [in Cooper, Lawrence, and Zhu (2004)]

2.4 Two-stages simple DEA model

A simple DEA model considering the presence of slacks can be solved in two stages. First stage is minimizing θ_o in (2.10) by ignoring slacks and second stage is optimizing slacks without altering the previously determined value of θ_o by fixing $\min \theta_o = \theta_o^*$ as in (2.21). Which is equivalent as solving

$$TE_o = \min_{\theta_o, \boldsymbol{\lambda}, \boldsymbol{s}^-, \boldsymbol{s}^+} \theta_o - \epsilon \left(\sum_{i=1}^{m} s_i^- + \sum_{j=1}^{n} s_j^+ \right) \tag{2.22}$$

subject to

$$\begin{aligned} \sum_{k=1}^{I} x_{ik}\lambda_k + s_i^- &= \theta_o x_{io} \qquad & i = 1, \ldots, m \\ \sum_{k=1}^{I} y_{jk}\lambda_k - s_j^+ &= y_{jo} \qquad & j = 1, \ldots, n \\ s_i^-, s_j^+, \lambda_k &\geq 0 \qquad & \forall i, j, k \end{aligned}$$

where the slack variables s_i^-, s_j^+ convert the inequalities in (2.10) to equivalent equations and $\epsilon > 0$ is smaller than any positive real number. ϵ also appear in the non-linear problem (2.8) and in the multiplier problem (2.9), where constraints $v_i, u_j \geq 0$ and $\nu_i, \mu_j \geq 0$ are substituted for $v_i, u_j \geq \epsilon (> 0)$ and $\nu_i, \mu_j \geq \epsilon (> 0)$.

For converting two stages input-oriented simple DEA model to output-oriented one we use the same procedure as described in the previous section. Exact formulation of such output-oriented two-stages simple DEA model, can be found in e.g. Cooper, Lawrence, and Zhu (2004) or Agha and Lawrence (1993).

The simple DEA model requires to be distinguished as an input-oriented or an output-oriented model. We can avoid making the decision about orientation together with the consideration of slacks and use a different model - additive DEA model. This model does not require orientation and include the slacks problem so in addition we does not need to worry about that . We examine this model in the next section.

2.5 Additive DEA model

The simple DEA model requires to be distinguished as an input-oriented or an output-oriented. We can, however, combine both orientations in a single model called additive DEA model. This model uses a different metric to evaluate technical efficiency that the one used in the radial measure model (2.10). It is important thought to still decide whether VRS or CRS model. We examine the VRS model because of its more common use, and we describe the CRS model briefly.

In multi-dimensional space (R^{m+n}) the envelopment surface consists of portions of supporting hyperplanes that form particular facets of the convex hull of the points $(\boldsymbol{x}_k, \boldsymbol{y}_k), k = 1, \ldots, I$, where $\boldsymbol{x}_k \in R_+^m$ and $\boldsymbol{y}_k \in R_+^n$ represents the vectors of input and output values for DMU_k, respectively. The general equation for a hyperplane in R^{m+n} is given by $\sum_{i=1}^m \nu_i x_i - \sum_{j=1}^n \mu_j y_j + \omega$, where $(\nu_1, \ldots, \nu_m, -\mu_1, \ldots, -\mu_n)$ are normals. According to Agha and Lawrence (1993) such a hyperplane is a supporting hyperplane (and forms a facet of the envelopment surface) if and only if all the points $(\boldsymbol{y}_k, \boldsymbol{x}_k)$ lie on or above the hyperplane and, additionally, the hyperplane passes through at least one of the points. These conditions identifies the portion of convex hull that demonstrates high-output and/or low-input and may be stated as

$$\sum_{i=1}^{m} \nu_i x_{ik} - \sum_{j=1}^{n} \mu_j y_{jk} + \omega \geq 0 \qquad \forall k \tag{2.23}$$

where (2.23) requires the inequalities to be equal to zero for some k, which assures that the hyperplane passes through at least one of the points.

The preceding conditions are entering in the following linear programming problem. Note that x_{ik} and y_{jk} are the values of input i and output j for the DMU_k, respectively; ν_i, μ_j, and ω are variables. For a feasible solution (ν_i, μ_j, ω) DMU always lies on or above the hyperplane. The objective function measures the distance from hyperplane $\sum_{i=1}^m \nu_i x_i - \sum_{j=1}^n \mu_j y_j + \omega_o = 0$ to DMU_o being evaluated. Minimization of the objective function selects a hyperplane with the minimal distance. Note that objective function value is non-negative; hence an optimal value of zero indicates that DMU_o lies on this hyperplane and is thus efficient. Inefficient DMU lie above the closest supporting hyperplane and thus correspond to a non-zero objective function value.

$$TE_o \quad = \quad \min_{\boldsymbol{\mu},\boldsymbol{\nu},\omega_o} \sum_{i=1}^{m} \nu_i x_{io} - \sum_{j=1}^{n} \mu_j y_{jo} + \omega_o \tag{2.24}$$

subject to

$$\begin{aligned} \sum_{i=1}^{m} \nu_i x_{ik} - \sum_{j=1}^{n} \mu_j y_{jk} + \omega_o \geq 0 \qquad & k = 1, \ldots, I \\ \nu_i, \mu_i \geq 1 \qquad & \forall i, j \end{aligned}$$

Problem (2.24) need to be solved I times, once for each DMU. The values of optimal solution $(\boldsymbol{\nu}_o, \boldsymbol{\mu}_o, \omega_o)$ are in literature interpreted as virtual multipliers. As such, problem (2.24) is referred as multiplier linear programming problem. These values represent normals for supporting hyperplanes that define facets of envelopment surface. Note that each of the I programs does not yield a different hyperplane; in fact, supporting hyperplanes for efficient DMUs serve as the closest supporting hyperplane for inefficient DMU.

The following linear programming problem dual to the multiplier problem presents, as a convex combination of efficient DMUs, more accessible representation of facet. Dual linear problem consist of $m + n + 1$ equality constraints since the variables in (2.24) are unconstrained and associated dual variables $\boldsymbol{\lambda}, \boldsymbol{s}^-, \boldsymbol{s}^+$, where $\boldsymbol{\lambda}$ is associated with variable ω and $\boldsymbol{s}^-$, $\boldsymbol{s}^+$ are associated to m inputs and n outputs, respectively. Since the primal constraint is "$\geq$", all variables $\lambda_k, k = 1, \ldots, I$; $s_i^-, i = 1, \ldots, m$; $s_j^+, j = 1, \ldots, n$ are non-negative $\forall i, j, k$. The linear programming envelopment problem dual to (2.24) is given by

$$TE_o \quad = \quad \max_{\boldsymbol{\lambda},\boldsymbol{s}^-,\boldsymbol{s}^+} \sum_{i=1}^{m} s_i^- + \sum_{j=1}^{n} s_j^+ \tag{2.25}$$

subject to

$$\begin{aligned} \sum_{k=1}^{I} x_{ik} \lambda_k + s_i^- = x_{io} \qquad & i = 1, \ldots, m \\ \sum_{k=1}^{I} y_{jk} \lambda_k - s_j^+ = y_{jo} \qquad & j = 1, \ldots, n \\ \sum_{k=1}^{I} \lambda_k = 1 & \\ s_i^-, s_j^+, \lambda_k \geq 0 \qquad & \forall i, j, k \end{aligned}$$

where s_i^- represents slack on input i and s_j^+ represents slack on output j. The objective function sums up the slacks, where the value equal to zero indicate a DMU_o being evaluated as technically efficient, i.e. $TE_o = 0$. The bigger the value of objective function the less efficient DMU.

The relationship between the values of input and output slacks and the multipliers is given for each DMU separately by the following complementary slackness conditions

$$\begin{aligned} s_i^- > 0 &=> \nu_i = 1 \qquad i = 1, \ldots, m \\ s_j^+ > 0 &=> \mu_j = 1 \qquad j = 1, \ldots, n \end{aligned} \tag{2.26}$$

that arises from linear programming duality theory.

If we consider CRS assumption, an additive DEA multiplier problem omits the variable ω_o in (2.24) and additive DEA envelopment problem omits the convexity constraint $\sum_{k=1}^{I} \lambda_k = 1$ in (2.25).

Scale efficiency can be computed by the same procedure as for the simple DEA model. Model (2.25) runs on data twice, once with CRS and once with VRS assumption. Any difference between the efficiencies is attributed to scale efficiency as follows

$$SE = TE_{\text{CRS}} - TE_{\text{VRS}} \tag{2.27}$$

For further use we introduce new variables $\hat{y}_{jo}$ and $\hat{x}_{io}$ defined as follows

$$\begin{aligned} \hat{x}_{io} &= x_{io} - s_i^{-*} \leq x_{io} \qquad i = 1, \ldots, m \\ \hat{y}_{jo} &= y_{jo} + s_j^{+*} \geq y_{jo} \qquad j = 1, \ldots, n \end{aligned} \tag{2.28}$$

where s_i^{-*} and s_j^{+*} are solutions from (2.25). Hence, an optimum is not reached until it is not possible to increase any output $\hat{y}_{jo}$ or reduce any input $\hat{x}_{io}$ without decreasing some other output or increasing some other input. For efficient DMU_o applies $(\hat{\boldsymbol{x}}_o, \hat{\boldsymbol{y}}_o) = (\boldsymbol{x}_o, \boldsymbol{y}_o)$. For a DMU_o that is inefficient, the point $(\hat{\boldsymbol{x}}_o, \hat{\boldsymbol{y}}_o)$ that lies on the envelopment surface is referred as the projected point of $(\boldsymbol{x}_o, \boldsymbol{y}_o)$. Vectors $\boldsymbol{s}^-$ and $\boldsymbol{s}^+$ measure the distance between an inefficient DMU $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ and its projected point $(\hat{\boldsymbol{x}}_o, \hat{\boldsymbol{y}}_o)$ on the envelopment surface.

2.5.1 Additive DEA model for allocative efficiency

According to Cooper, Lawrence, and Zhu (2004) first we observe that we can multiply input slacks by unit costs $\boldsymbol{c} \in R^m_{++}$ and output slacks by unit prices $\boldsymbol{p} \in R^n_{++}$ after we have solved (2.25) and thereby accord a monetary value to this solution. Then we can use (2.28) to write

$$\begin{aligned}\sum_{i=1}^{m} c_{io} s_i^{-*} + \sum_{j=1}^{n} p_{jo} s_j^{+*} &= \left(\sum_{i=1}^{m} c_{io} x_{io} - \sum_{i=1}^{m} c_{io} \hat{x}_{io} \right) + \left(\sum_{j=1}^{n} p_{jo} \hat{y}_{jo} - \sum_{j=1}^{n} p_{jo} y_{jo} \right) = \\ &= \left(\sum_{j=1}^{n} p_{jo} \hat{y}_{jo} - \sum_{i=1}^{m} c_{io} \hat{x}_{io} \right) - \left(\sum_{j=1}^{n} p_{jo} y_{jo} - \sum_{i=1}^{m} c_{io} x_{io} \right)\end{aligned} \tag{2.29}$$

After multiplication of (2.25) by unit costs and unit prices we can see from the last pair of parenthesized expressions in (2.29) that at an optimum, the objective in (2.25) is equal to the profit available when production is technically efficient minus the profit obtained from the observed performance. Thus, when multiplied by unit costs and unit prices the solution of (2.25) provides a measure in the form of the amount of profits lost by not performing in a technically efficient manner. We can then similarly develop a measure of allocative efficiency by the following additive envelopment model

$$AE_o = \max_{\hat{\boldsymbol{\lambda}}, \hat{\boldsymbol{s}}^-, \hat{\boldsymbol{s}}^+} \sum_{i=1}^{m} c_{io} \hat{s}_i^- + \sum_{j=1}^{n} p_{jo} \hat{s}_j^+ \tag{2.30}$$

subject to

$$\begin{aligned}\sum_{k=1}^{I} x_{ik} \hat{\lambda}_k + \hat{s}_i^- &= \hat{x}_{io} & i &= 1, \dots, m \\ \sum_{k=1}^{I} y_{jk} \hat{\lambda}_k - \hat{s}_j^+ &= \hat{y}_{jo} & j &= 1, \dots, n \\ \sum_{k=1}^{I} \hat{\lambda}_k &= 1 \\ \hat{s}_i^-, \hat{s}_j^+, \hat{\lambda}_k &\geq 0 & \forall i&, j, k\end{aligned}$$

where DMU$_o$ being evaluated is allocativelly efficient if objective is equal to zero, i.e. $AE_o = 0$. Comparing (2.31) with (2.25) we can see following differences: the objective in (2.25) is replaced by multiplied objective in (2.30); y_{jo} and x_{io} in (2.25) are replaced by $\hat{y}_{jo}$ and $\hat{x}_{io}$ in (2.30) as obtained from (2.28).

2.5.2 Additive DEA model for economic efficiency

We assume using inputs $\boldsymbol{x} \in R_+^m$, for costs $\boldsymbol{c} \in R_{++}^m$, to produce outputs $\boldsymbol{y} \in R_+^n$, for sale at prices $\boldsymbol{p} \in R_{++}^n$. Extension of envelopment problem (2.25) is made by multiplying the input slacks by unit costs and output slacks by unit prices

$$EE_o \quad = \quad \max_{\boldsymbol{\lambda},\boldsymbol{s}^-,\boldsymbol{s}^+} \sum_{i=1}^{m} c_{io} s_i^- + \sum_{j=1}^{n} p_{jo} s_j^+ \tag{2.31}$$

subject to

$$\begin{aligned}
\sum_{k=1}^{I} x_{ik}\lambda_k + s_i^- &= x_{io} \qquad && i = 1,\dots,m \\
\sum_{k=1}^{I} y_{jk}\lambda_k - s_j^+ &= y_{jo} \qquad && j = 1,\dots,n \\
\sum_{k=1}^{I} \lambda_k &= 1 && \\
s_i^-, s_j^+, \lambda_k &\geq 0 && \forall i,j,k
\end{aligned}$$

where c_i is cost of an input i and p_j is a price for sale of output j. Problem is solved I times for each DMU$_o$ being avaluated. The dual multiplier problem has form

$$EE_o \quad = \quad \min_{\boldsymbol{\mu},\boldsymbol{\nu},\omega_o} \sum_{i=1}^{m} \nu_i x_{io} - \sum_{j=1}^{n} \mu_j y_{jo} + \omega_o \tag{2.32}$$

subject to

$$\begin{aligned}
\sum_{i=1}^{m} \nu_i x_{ik} - \sum_{j=1}^{n} \mu_j y_{jk} + \omega_o \geq 0 \qquad && k = 1,\dots,I \\
\nu_i \geq c_i \qquad && i = 1,\dots,m \\
\mu_j \geq p_j \qquad && j = 1,\dots,n
\end{aligned}$$

where (2.31) measures the (overall) economic efficiency score of the DMU$_o$ being evaluated and as in (2.25) the optimal value the DMU$_o$ to be efficient is equal to zero, i.e. $EE_o = 0$.

Relation of the solutions of this model (2.31) to the previous one (2.30) is as follows

$$\left(\sum_{i=1}^{m} c_{io} s_i^- + \sum_{j=1}^{n} p_{jo} s_j^+\right) = \left(\sum_{i=1}^{m} c_{io} s_i^{-*} + \sum_{j=1}^{n} p_{jo} s_j^{+*}\right) + \left(\sum_{i=1}^{m} c_{io} \hat{s}_i^{-*} + \sum_{j=1}^{n} p_{jo} \hat{s}_j^{+*}\right) \tag{2.33}$$

The value of (overall) economic efficiency for DMU$_o$ obtained from (2.31) is equal to the value of technical efficiency obtained from (2.25) plus the value of allocative efficiency obtained from (2.30)

$$EE_o = TE_o + AE_o \tag{2.34}$$

Thus to avoid to compute to many problems, we can simply solve only (2.31) and (2.25) to obtain the economic efficiency and technical efficiency, respectively. The value of allocative efficiency is then obtained residually from the formula

$$AE = EE - TE \tag{2.35}$$

for each DMU separately.

2.5.3 Additive DEA model for eco-efficiency

An extension of additive DEA model to measure eco-efficiency is based on separation of inputs and outputs to a several categories. Eco-efficiency is usually measured by comparing environmental performance indicators. DEA supports such comparisons, as it does not require the explicit weights to aggregate the indicators. DEA model in general assumes that all inputs and outputs are "goods". This, however, is not valid in the ecological context and we have to include also "bads", i.e. ecologically undesirable outputs (e.g. waste or emissions) and ecologically desirable inputs (e.g. waste in a waste-burning power plant or in a recycling plant).

To yield a higher degree of efficiency in DEA there is a fundamental assumption that for a DMU, to achieve efficiency, ceteris paribus larger quantities of outputs and smaller quantities of inputs are preferable. This assumption, however, causes a problem in the ecologic context, where undesirable outputs are prominent. In the eco-efficiency context these could be waste or emissions, but it may also appear e.g. in business applications (tax payments) or health care (complications of medical operations).

Dyckhoff and Allen (2001) introduced the case in which it is assumed that decision maker is able to assign each input and output considered to be one of the three disjunctive classes: "goods", "bads" and "neutrals". Where "goods" are objects with a positive value, "bads" are objects with a negative value, and "neutrals" are objects without any kind of value with respect to the decision problem available. This case is so-called standard case.

It is important to note, that standard case classification reflects the decision maker's subjective ecological, economical or social judgement on the inputs and outputs considered. The categorization of objects can be difficult and may change. Imagine the cases such as that the output by-product (e.g.

residual heat) could be sold to a nearby firm or that there is a penalty for emitting more than a given limit. Such cases might change the categorization significantly.

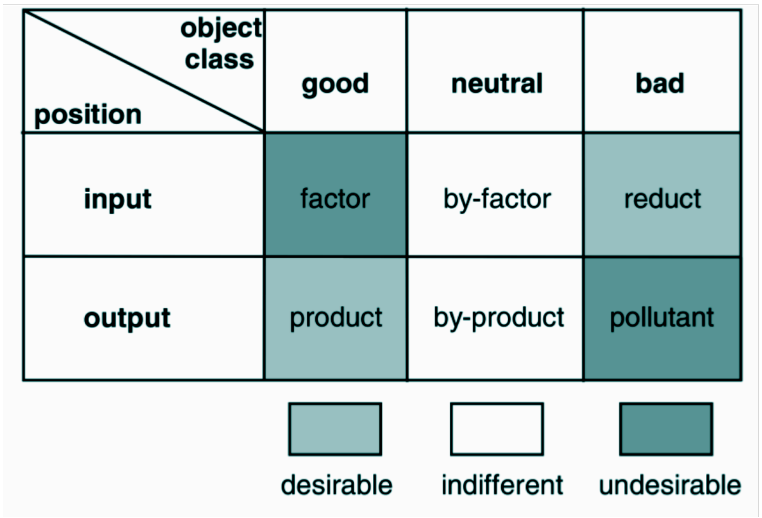

Figure 2.7: Categorization of ecologically relevant objects according to the standard case. [in Dyckhoff and Allen (2001)]

Combination of three classes with inputs and outputs of the production process leads into six categories of objects, all with different desirability. The six categories as an example of the waste-burning power plant are illustrated in Figure 2.7, where the "goods" on the input side are factors (F) and on the output side products (P). In the ecological context, factors may represent the use of the natural resources (in case of waste-burning power plant the use of water should be kept to a minimum). On the other hand the products (such as electric power and heat) that are the aims of the transformation process should be maximized.

The quantity of "bads" as an undesirable objects on the output side (A) should be minimized. The representation of the pollutants are e.g. waste gas, waste water, scrap and cinder. The case of "bads" can rarely appear also on the input side, which are named "reducts" (R). The undesirable object is waste to be burned at the power plant, which destruction is, however, desired and this input should be therefore maximized.

Typical by-factor and by-product are objects without particular "good" or "bad" value, therefore the decision maker is indifferent towards them. In case of a waste-burning power plant it is air on the input side and residual heat on the output side. The point of interest of this category can, however, rapidly change if some new facts to be considered will occur. The output by-product can be sold to other company for further use, simultaneously bringing income, may lead the decision maker to increase its quantity. On the other hand penalties for emitting more by-products than a given limit may lead the decision maker to lower its quantity.

We can see the difference in the ecologically extended model comparing to the basic one. Unlike in the basic model all the inputs are minimized, in the eco-extended model there are some inputs that we want to maximize. The same situation is on the output side that unlike in the basic model where all the outputs are maximizes, in the eco-extended model there are some outputs that we want to minimize.

The additive envelopment problem extended according to the standard case has form

$$ECO_o = \max_{\boldsymbol{\lambda},\boldsymbol{s}} \sum_{l \in F \cup A \cup P \cup R} g_l s_l \qquad (2.36)$$

subject to

$$\begin{aligned}
\sum_{k=1}^{I} x_{ik}\lambda_k + s_i = x_{io} & \qquad (i \in F) \\
\sum_{k=1}^{I} y_{jk}\lambda_k - s_j = y_{jo} & \qquad (j \in P) \\
\sum_{k=1}^{I} y_{jk}\lambda_k + s_j = y_{jo} & \qquad (j \in A) \\
\sum_{k=1}^{I} x_{ik}\lambda_k - s_i = x_{io} & \qquad (i \in R) \\
\sum_{k=1}^{I} \lambda_k = 1 & \\
\lambda_k \geq 0 & \qquad (k = 1, \ldots, I) \\
s_i, s_j \geq 0 & \qquad (i \in F \cup R, j \in P \cup A)
\end{aligned}$$

and the dual multiplier problem corresponding to the model (2.36) has form

$$ECO_o \quad = \quad \min_{\boldsymbol{\nu},\boldsymbol{\mu},\omega_o} \sum_{i\in F} \nu_i x_{io} + \sum_{j\in A} \mu_j y_{jo} - \sum_{j\in P} \mu_j y_{jo} - \sum_{i\in R} \nu_i x_{io} + \omega_o \qquad (2.37)$$

subject to

$$\begin{aligned} \sum_{i\in F} \nu_i x_{ik} + \sum_{j\in A} \mu_j y_{jk} - \sum_{j\in P} \mu_j y_{jk} - \sum_{i\in R} \nu_i x_{ik} + \omega_o \geq 0 & \qquad (k = 1, \ldots, I) \\ \nu_i \geq g_i & \qquad (i \in F \cup R) \\ \mu_j \geq g_j & \qquad (j \in A \cup P) \end{aligned}$$

where x_{ik} and y_{jk} are input and output quantities for each of the $k = 1, \ldots, I$ DMUs, and x_{io} and y_{jo} are input and output quantities for the DMU$_o$ being evaluated. g_l are user-specified weights. The optimal value of (2.36) depends on the scales of inputs and outputs. Influence of the scales can be reduced by an appropriate choice of weights g_l, e.g. means of the sample standard deviation of output or input l or the range of output or input l in the sample. Scale factors ν_i and μ_i in (2.37) are only bounded from below by user specified weights g_i and g_j, respectively and ω_o in (2.37) is attached to the VRS constraint $\sum_{k=1}^{I} \lambda_k = 1$ in (2.36).

The efficiency measure in (2.36) sums up the slacks in each factor, product, pollutant and reduct. A DMU is efficient in the optimal value equal to zero, it is inefficient if at least one component s_l of the slack variables is not zero. Efficiency is measured non-radialy as a distance from DMU$_o$ to a point on the efficient frontier defined by the values of λ_k. This point may be an observed efficient DMU or a convex combination of observed efficient DMUs. The dual model (2.37) seeks for the closest supporting hyperplane with the minimal ECO score.

Additive DEA model (2.24) considers factors ($i \in F$) and products ($j \in P$). In the ecological extension also pollutants ($j \in A$) and reducts ($i \in R$) are considered. Thus in the special case where $A = \emptyset$, $R = \emptyset$ and $g_l = 1$ $\forall l$, (2.36) and (2.37) are equal to additive DEA (2.25) and (2.24) models, respectively.

In (2.36) and (2.37) reduct quantities are maximized by treating them as classical DEA outputs whereas pollutant quantities are minimized by treating them as classical DEA inputs. Though, by using separate indices and constraints, the position of reducts and pollutants as input and output respectively is still apparent. Just the categories "good" and "bad" are considered for the efficiency analysis while "neutrals" are neglected as they are not relevant.

According to Dyckhoff and Allen (2001) there are some practical limitations on using DEA in the ecological context. DEA compares a "large" number of DMUs regarding a "small" number of inputs and outputs. In practice, due to lack of standardization and data availability, usually just a "small" number of DMUs is available. Furthermore usually a "large" number of inputs and outputs is relevant.

2.6 A DEA model based on eco-efficiency ratio

This DEA model is based on a different kind of ratio than the simple DEA models. According to Kuosmanen and Kortelainen (2005) ratio for measurement is made upon the standard definition of eco-efficiency as follows

$$\text{eco-efficiency} = \frac{\text{economic value added}}{\text{environmental damage}} \tag{2.38}$$

Model is focused explicitly on the economic value added and environmental pressures without direct recourse to physical inputs and outputs.

We again separate inputs and outputs into three categories: "goods", "bads" and "neutrals". Upon the categorization and new measurement ratio, inputs and outputs are located as follows: "goods" (objects that have influence on the economic value) appear in the numerator of eco-efficiency ratio, "bads" (objects that have ecological impacts) appear in the denominator of ratio, and finally "neutrals" (objects that influence neither economic value nor environment impact) are not of direct interest in our measurement, thus they are omitted.

Deviation of this model from the previous one (2.36) is that we focus on environmental pressures instead of specific undesirable outputs. Environmental pressures can be climate change, smog, acidification etc.

We assume that the economic value added v is known or can be calculated directly from the available data. Production induces n different environmental pressures, all of them are assumed to be harmful, and their severity is thus measured by variables $\boldsymbol{p} \in R^n_+$.

When generating economic value added we have to consider that some DMUs can be more harmful to the environment than others. Thus comprehensive eco-efficiency measure should consider the possibilities of substitution between environmental pressures because reducing one pollutant can cause the increase of another one.

Substitution possibilities are characterized by the pollution generating technology set $T = \{(v, \boldsymbol{p}) \in R^{n+1}_+ \mid$ value added v can be generated with the damage $\boldsymbol{p}\}$, which includes all possible technically and economically feasible combinations of value added v and environmental damage $\boldsymbol{p}$. We assume the set T of all DMUs to be known.

DEA eco-efficiency: production unit is eco-efficient if and only if it is impossible to decrease any environmental pressure without simultaneously increasing another pressure or decreasing the economic value added. [in Kuosmanen and Kortelainen (2005)]

Efficient frontier is formed by a subset of T (by efficient DMUs only). The concern is directed to the greenhouse effect rather than the amounts of greenhouse gases in the atmosphere. The problem occurs when emissions exceed the carrying capacity of the ecosystem that will result into unpredictable changes in the climate conditions.

In the terms of environmental pressures the carrying capacity of ecosystem represents the concept of sustainability. Analogously to the technology set T we apply a sustainability set $S = \{\boldsymbol{p} \in R_+^n \mid$ ecosystem can sustain damage $\boldsymbol{p}\}$ that lists sustainable levels of damages. While the definition of eco-efficiency depends on the levels of environmental pressures relative to the economic value added, sustainability depends essentially on the absolute levels of pressure. Thus the high levels of environmental pressure can be considered as eco-efficient DMU in compensated for by sufficiently high economic value added. Unfortunately notion of sustainability does not consider such compensation. So eco-efficiency represents the quality of sustainable production but for overall sustainability also quantity is significant.

According to Kuosmanen and Kortelainen (2005) sustainable carrying capacities are contingent on the ever-changing interactions between the physical and biotic environment and human production technology and consumption patterns. This makes empirical estimation of sustainable levels of environmental pressures extremely difficult in practice.

We measure eco-efficiency of a DMU under evaluation relative to a sample of I comparable DMUs. Let v_k denote the economic value added and $\boldsymbol{p}_k$ environmental pressures of the DMU$_k$ ($k = 1, \ldots, I$). In the terms of our definition, the eco-efficiency of the DMU$_k$ is formulated as $\frac{v_k}{D(\boldsymbol{p}_k)}$, where D is the damage function that aggregates n environmental pressures into a single environmental damage score.

Data requirements are different than for the previous models, where quantities of inputs and outputs are required. If total value added is known for all DMUs in the sample we don't need detailed data such as quantities of inputs and outputs or their prices. Although the primary production factors are important cost factors for technical, allocative and economic efficiency measurements, they do not appear in this eco-efficiency measurement.

To construct the environmental damage index $D(\boldsymbol{p}_k)$ we use a linear approximation as weighted sum of the various environmental pressures, $D(\boldsymbol{p}) = \sum_{i=1}^{n} \omega_i p_i$, where ω_i $(i = 1, \ldots, n)$ represents the weight attached to environmental pressure i.

Weights of damage index are identified by DEA as they maximize the eco-efficiency score of the evaluated DMU in comparison with a group of similar DMUs. So the eco-efficiency score of the DMU_o being evaluated is calculated as

$$ECO_o = \max_{\boldsymbol{\omega}} \frac{v_o}{\sum_{i=1}^{n} \omega_i p_{io}} \tag{2.39}$$

subject to

$$\begin{aligned} \frac{v_k}{\sum_{i=1}^{n} \omega_i p_{ik}} \leq 1 & \qquad k = 1, \ldots, I \\ \omega_i \geq 0 & \qquad i = 1, \ldots, n \end{aligned}$$

where constraints $\frac{v_k}{\sum_{i=1}^{n} \omega_i p_{ik}} \leq 1$, $\forall k$ mean that any activity in the sample can not exceed the value of one. Because of the constraint of non-negative weights, eco-efficiency score is bounded by zero. Thus from constraints ECO satisfies $0 \leq ECO \leq 1$. Evaluated DMU is considered as eco-efficienct if its score is equal to one ($ECO = 1$), otherwise it is inefficient. The DEA problem (2.39) has both non-linear objective function and non-linear constraints. By inversing the eco-efficiency ratio we achieve a linear problem

$$ECO_o^{-1} = \min_{\boldsymbol{\omega}} \frac{1}{v_o} \sum_{i=1}^{n} \omega_i p_{io} \tag{2.40}$$

subject to

$$\begin{aligned} \frac{1}{v_k} \sum_{i=1}^{n} \omega_i p_{ik} \geq 1 & \qquad k = 1, \ldots, I \\ \omega_i \geq 0 & \qquad i = 1, \ldots, n \end{aligned}$$

and the eco-efficiency ratio is obtained by taking inverse value of the optimal solution of (2.40). Again the same as in previous models this model measures efficiency relative to the best practice in the sample, which is not necessarily the same as best available technology.

According to Kuosmanen and Kortelainen (2005) the weights identified in (2.40) need to be, however, restricted in the common sense, as they can take unreasonable values. In the ecologic context the assigned weights of the environmental impacts of secondary importance may be large, on the other hand the weights of impacts that are generally important may have be zero or left negligible. Thus some activities may appear as efficient even though they perform well only on a single, relatively unimportant criterion. This leads to a need of additional information that captures the relative importance of different environmental impacts.

The restrictions upon the additional information need to be included into the model in the form of constraints. For example, if a certain environmental effect i is certainly at least twice as detrimental as an other effect j, we can enforce this fact by imposing the restriction $\omega_i \geq 2\omega_j$ in the model. The discussion about different approaches of weight restriction and application of this model to road transportation in Finland can be found in Kuosmanen and Kortelainen (2005).

Chapter 3

Application to the data

3.1 Data set

The data available are quantities of inputs and outputs, costs for inputs and prices for outputs. Thus we can provide a measurement of technical, allocative, economic and scale efficiencies. Measurement for eco-efficiency, however, cannot be obtained because of missing ecological information.

The data set contains real data although the data available are mixed from two different years so the data set by itself is illustrative only. Hence, the results of the measurements are not precise and likely skewed from the reality. From the closer look on data we can see that the mixture, however, does not interfere the reality severely. Importance for the comparison is that each variable for all DMUs is not mixed which our data set fulfill because the data for all DMUs (for each variable) are always from the same year.

The measurement of the efficiencies is provided by the comparison of forest industry in between the comparable DMUs which in our case are the counties of Czech Republic. The quantities data are available for four inputs and four outputs. The costs data are available only for one of the inputs and prices data are available for all outputs.

Inputs for each county are: number of industry subjects in the county (with 100 or more employees only), area of soil for the industry, forest area and number of employees. Outputs for each county are: logging amounts of four different types of wood - spruce, pine, oak and beech. The costs of the input number of employees are average salaries for each county and prices of the outputs are prices of each wood type for each county.

Data are available for the 12 counties of Czech Republic: Capital City Prague, Middle-Czech County, South-Czech County, Plzeň County, Karlovy Vary County, Hradec Králové County, Pardubice County, Vysočina County, South-Moravia County, Olomouc County, Zlín County and Moravia-Silesia County.

The number of industry subjects in the county, number of employees and their average salaries are from the year 2003 and all the other data are from the year 2007. The prices of wood were primarily taken from the last quarter of the year 2007 for a certain same quality of wood - III. C class of quality for each wood type. Missing prices were completed from the other quarters of the year 2007 and first quarter of the year 2008 and the rest of the values that were not available in any quarter were estimated by the value overall in Czech Republic which was the weighted mean of all prices in the certain quality and type of wood for the year 2007. A Synoptic Table 3.1 shows all available variables.

Variables for each county	
INPUT quantities	**COSTS**
number of industry subjects	—
area of soil for the industry in hectares	—
forest areas in hectares	—
number of employees	**average salary** in CZK
OUTPUT quantities	**PRICES**
logging amount of spruce wood in m^3	**price for spruce wood** in CZK per m^3
logging amount of pine wood in m^3	**price for pine wood** in CZK per m^3
logging amount of oak wood in m^3	**price for oak wood** in CZK per m^3
logging amount of beech wood in m^3	**price for beech wood** in CZK per m^3

Table 3.1: Inputs, costs, outputs and prices entering the measurements.

The measurements and results are achieved by using software Mathematica 5.2 and DEAP 2.1 [from Centre for Efficiency and Productivity Analysis (CEPA) at the University of Queensland, Australia]. The models used in measurements are Two-stages simple DEA model (2.22), Simple DEA model for economic efficiency (2.11), Additive DEA model (2.25) and Additive DEA model for economic efficiency (2.31); all with both CRS and VRS assumptions. For input-oriented computations on Two-stages simple DEA (2.22) and Simple DEA for economic efficiency (2.11) we use software DEAP 2.1. For computations on Additive DEA model (2.25) and Additive DEA model for economic efficiency (2.31) we use software Mathematica 5.2 and the source code made for the computations can be found at the end of this paper in Appendix.

3.2 Results of measurements for technical efficiency

Results from computations are listed in Tables 3.2 and 3.3 for Two-stages simple DEA and Additive DEA models, respectively. Because of availability of the data we use all inputs and outputs.

Tables are organized the same, where the notations stand for as follows:

#	– sequence number of the county
CRS, VRS	– constant and variable returns to scale assumptions, respectively
TE	– the value of technical efficiency
SE	– the value of scale efficiency
Peer counts	– number of times each county is a peer for another
Peer group	– sequence numbers of the counties that form a peer group for each county
Peer weights	– weights (λ_k values) attached to DMUs in the peer group in the same order as sequence numbers

First we look at the results for model (2.22), Table 3.2. We can see that values of technical efficiency under CRS and VRS assumptions are not equal which indicates that some of the counties are not operating at the optimal scale and thus are scale inefficient. Technically efficient counties under both CRS and VRS assumptions are Middle-Czech County, South-Czech County, South-Moravia County, Olomouc County, Zlín County and Moravia-Silesia County. Technically inefficient counties under both CRS and

Two-stages input-oriented simple DEA model

#	Counties	CRS TE	VRS TE	SE	Peer counts CRS	Peer counts VRS
1	**Capital City Prague**	0.547	1.000	0.543	0	2
2	**Middle-Czech County**	1.000	1.000	1.000	1	2
3	**South-Czech County**	1.000	1.000	1.000	6	2
4	**Plzeň County**	0.667	0.781	0.855	0	0
5	**Karlovy Vary County**	0.639	1.000	0.639	0	0
6	**Hradec Králové County**	0.773	1.000	0.733	0	0
7	**Pardubice County**	0.809	1.000	0.809	0	0
8	**Vysočina County**	0.538	0.690	0.780	0	0
9	**South-Moravia County**	1.000	1.000	1.000	5	1
10	**Olomouc County**	1.000	1.000	1.000	0	2
11	**Zlín County**	1.000	1.000	1.000	2	0
12	**Moravia-Silesia County**	1.000	1.000	1.000	1	2

#	Peer group CRS	Peer group VRS	Peer weights CRS	Peer weights VRS
1	3 9	1	0.002 0.009	1.000
2	2	2	1.000	1.000
3	3	3	1.000	1.000
4	3 9	1 2 3 10 12	0.505 0.034	0.118 0.060 0.369 0.072 0.311
5	3 12	5	0.181 0.123	1.000
6	2 3 9	6	0.140 0.117 0.127	1.000
7	3 9 11	7	0.230 0.075 0.031	1.000
8	3 9 11	1 2 3 9 10 12	0.250 0.030 0.062	0.393 0.044 0.094 0.023 0.018 0.428
9	9	9	1.000	1.000
10	10	10	1.000	1.000
11	11	11	1.000	1.000
12	12	12	1.000	1.000

Table 3.2: Results for model (2.22), both CRS and VRS assumptions.

Additive DEA model				
#	**Counties**	**CRS TE**	**VRS TE**	**SE**
1	**Capital City Prague**	44 747.8	0.0	44 747.8
2	**Middle-Czech County**	0.0	0.0	0.0
3	**South-Czech County**	0.0	0.0	0.0
4	**Plzeň County**	1 347 460.0	1 211 400.0	136 060.0
5	**Karlovy Vary County**	641 676.0	0.0	641 676.0
6	**Hradec Králové County**	581 641.0	0.0	581 641.0
7	**Pardubice County**	898 868.0	0.0	898 868.0
8	**Vysočina County**	1 324 710.0	1 127 724.0	196 986.0
9	**South-Moravia County**	0.0	0.0	0.0
10	**Olomouc County**	0.0	0.0	0.0
11	**Zlín County**	0.0	0.0	0.0
12	**Moravia-Silesia County**	0.0	0.0	0.0

	Peer counts		**Peer group**		**Peer weights**	
#	**CRS**	**VRS**	**CRS**	**VRS**	**CRS**	**VRS**
1	0	2	3 9	1	0.009 0.009	1.000
2	1	0	2	2	1.000	1.000
3	6	2	3	3	1.000	1.000
4	0	0	3 9	1 3 9 12	0.776 0.016	0.137 0.699 0.016 0.148
5	0	0	3 12	5	0.282 0.192	1.000
6	0	0	2 3 9	6	0.115 0.223 0.131	1.000
7	0	0	3 9 11	7	0.304 0.076 0.015	1.000
8	0	0	3 9 11	1 3 9 11 12	0.523 0.031 0.002	0.437 0.504 0.027 0.002 0.030
9	5	2	9	9	1.000	1.000
10	0	0	10	10	1.000	1.000
11	2	1	11	11	1.000	1.000
12	1	2	12	12	1.000	1.000

Table 3.3: Results for model (2.25), both CRS and VRS assumptions.

VRS assumptions are Plzeň County and Vysočina County. All other counties are inefficient under CRS assumption but efficient under VRS assumption.

The results for model (2.25) we can see in Table 3.3. There is also difference between the results under CRS and VRS assumptions and thus some of the counties have non-zero scale efficiency. Lists of the counties that are efficient, inefficient under CRS but efficient under VRS and inefficient under both CRS and VRS are the same as in the (2.22) model results.

The results from computations assign the same set of technically efficient counties for both models (2.22) and (2.25). Under CRS assumption also the peer groups are identical for both models for each county, they differ only in λ_k values which is due to a different measure technique used in the models. Under VRS assumption the peer groups for the Plzeň County and Vysočina County are slightly different in each model. This is due to a different measure technique used and probably also due to a wider range of efficient DMUs assigned than under CRS assumption. A greater set of efficient DMUs creates more possibilities of linear combinations and thus more combinations of efficient DMUs can achieve the same point on the frontier that inefficient DMU is compared to.

3.3 Results of measurements for allocative and economic efficiencies

Results from computations are listed in Table 3.4 for Simple DEA model for economic efficiency (2.11) and for Additive DEA model for economic efficiency (2.31) in Tables 3.5 and 3.6. Because the costs data are not available for all inputs we use only one input (number of employees) and all outputs data. Change in data set might result in different scores of technical efficiency, so we compute it again for one input and all outputs data set for Additive model. The program results of Simple DEA model for economic efficiency already contains computations of technical efficiency so there is no need for additional computation here.

Tables are organized the same as in previous section and additional notions stand for as follows:
EE – the value of economic efficiency
AE – the value of the allocative efficiency

Economic efficiencies are computed from (2.11) and (2.31) models and allocative efficiencies are computed residually using previously computed economic efficiency scores [obtained from models (2.11) and (2.31)] and techni-

Input-oriented simple DEA model for economic efficiency

#	Counties	CRS EE	VRS EE
1	**Capital City Prague**	0.011	0.308
2	**Middle-Czech County**	1.000	1.000
3	**South-Czech County**	0.886	1.000
4	**Plzeň County**	0.477	0.511
5	**Karlovy Vary County**	0.489	0.705
6	**Hradec Králové County**	0.371	0.602
7	**Pardubice County**	0.331	0.537
8	**Vysočina County**	0.177	0.216
9	**South-Moravia County**	0.581	1.000
10	**Olomouc County**	0.815	0.909
11	**Zlín County**	1.000	1.000
12	**Moravia-Silesia County**	1.000	1.000

#	CRS TE	VRS TE	SE	CRS AE	VRS AE
1	0.011	0.308	0.036	1.000	1.000
2	1.000	1.000	1.000	1.000	1.000
3	0.886	1.000	0.886	1.000	1.000
4	0.477	0.511	0.933	1.000	1.000
5	0.489	0.705	0.694	1.000	1.000
6	0.317	0.602	0.527	1.000	1.000
7	0.331	0.537	0.616	1.000	1.000
8	0.177	0.216	0.819	1.000	1.000
9	0.581	1.000	0.581	1.000	1.000
10	0.815	0.909	0.897	1.000	1.000
11	1.000	1.000	1.000	1.000	1.000
12	1.000	1.000	1.000	1.000	1.000

Table 3.4: Results for model (2.11), both CRS and VRS assumptions.

Additive DEA model for economic efficiency

#	Counties	CRS EE	VRS EE
1	**Capital City Prague**	7.564×10^9	5.25274×10^9
2	**Middle-Czech County**	0.000	0.000
3	**South-Czech County**	1.56267×10^9	0.000
4	**Plzeň County**	5.19012×10^9	2.7389×10^9
5	**Karlovy Vary County**	1.89338×10^9	1.45657×10^9
6	**Hradec Králové County**	3.15057×10^9	2.24514×10^9
7	**Pardubice County**	3.89371×10^9	2.89079×10^9
8	**Vysočina County**	10.5966×10^9	5.37765×10^9
9	**South-Moravia County**	11.2838×10^9	0.000
10	**Olomouc County**	1.0639×10^9	0.825766×10^9
11	**Zlín County**	0.000	0.000
12	**Moravia-Silesia County**	0.000	0.000

#	CRS TE	VRS TE	SE
1	5.21856×10^6	3.65049×10^6	1.56807×10^9
2	0.000	0.000	0.000
3	1.03767×10^6	0.000	1.03767×10^6
4	3.68931×10^6	1.797608×10^6	1.891702×10^6
5	1.30194×10^6	1.00657×10^6	0.29537×10^6
6	2.20294×10^6	1.57996×10^6	0.62298×10^6
7	2.75531×10^6	2.06033×10^6	0.69498×10^6
8	7.04396×10^6	3.61956×10^6	3.4244×10^6
9	7.84963×10^6	0.000	7.84963×10^6
10	0.680968×10^6	0.529355×10^6	0.151613×10^6
11	0.000	0.000	0.000
12	0.000	0.000	0.000

Table 3.5: Results for model (2.31), both CRS and VRS assumptions.

Allocative efficiency in Additive DEA model

#	Counties	CRS AE	VRS AE
1	**Capital City Prague**	7.55878144×10^9	5.24908951×10^9
2	**Middle-Czech County**	0.000	0.000
3	**South-Czech County**	1.56163233×10^9	0.000
4	**Plzeň County**	$5.18643069\ 2 \times 10^9$	2.737102392×10^9
5	**Karlovy Vary County**	1.89207806×10^9	1.4556343×10^9
6	**Hradec Králové County**	3.14836706×10^9	2.24356004×10^9
7	**Pardubice County**	3.89095469×10^9	2.88872967×10^9
8	**Vysočina County**	10.58955604×10^9	5.37403044×10^9
9	**South-Moravia County**	11.27595037×10^9	0.000
10	**Olomouc County**	1.063219032×10^9	0.825236645×10^9
11	**Zlín County**	0.000	0.000
12	**Moravia-Silesia County**	0.000	0.000

Table 3.6: Computed residually for Additive DEA model from results of (2.25) and (2.31).

cal efficiency scores [obtained from models (2.22) and (2.25)] by the formulas $AE = EE/TE$ and $AE = EE - TE$, respectively.

First we look at the result of Simple DEA model for economic efficiency (2.11), Table 3.4. We can see the differences between the results of technical efficiency under CRS and VRS assumptions so some of the counties are scale inefficient. The results from Table 3.4 for technical efficiency are different than the results from Table 3.2. The inequalities arise from the use of different data set, specially in this case of input-oriented model where change from four inputs to one input is strongly significant. From the original set of technically efficient counties under both CRS and VRS assumptions in Table 3.2 only Middle-Czech County, Zlín County and Moravia-Silesia County remained technically efficient also with this measurement. Counties technically efficient under VRS assumption only are South-Czech County and South-Moravia County. All other counties are inefficient under both CRS and VRS assumptions. The results of measurements for economic efficiency indicate the same sets of efficient, efficient under VRS assumption only or inefficient under both CRS and VRS counties, also with the same scores as for technical efficiency. Results show all DMUs to be allocativelly efficient under both CRS and VRS assumptions. These results are partially unexpected for although we used one input only, the average salaries were rather different in between the counties and thus expected some counties to be allocativelly inefficient.

Results of Additive DEA model for economic efficiency (2.31) we can see in Table 3.5. Comparing the results for technical efficiency between measurements in Tables 3.3 and 3.5 we can see the differences caused by using a different data set as in the case of (2.11) model. The same sets of counties as in model (2.11) are assigned as technically and economically efficient, unlike in model (2.11) technical efficiency scores are not identical with economic efficiency scores. Although models use a different techniques, the results indicate that sets of the efficient DMUs are the same for both models. In this model, however, Table 3.6 results indicate that not all DMUs are allocativelly efficient. This difference comparing to results in Table 3.4 is probably caused by the fact that we included one input and four outputs quantities and prices to the (2.31) model instead of one input quantity and cost and four output quantities to the (2.11) model.

3.4 Conclusion of measurements

Technically, allocativelly, economically and also scale efficient counties in each measurement under both CRS and VRS assumptions are Middle-Czech County, Zlín County and Moravia-Silesia County. This leads us to a strong confidence about their evaluated performance as the most stabile counties from the set.

Because of the results in Table 3.4 where all DMUs are designated as allocativelly efficient, there is no contradictory set of counties that are always inefficient under any measurement, assumption and efficiency type.

The rest of the counties are at least once in any measurement, assumption or efficiency type referred as inefficient. Plzeň County and Vysočina County are designated as efficient only in allocative score measurement for (2.22) model with both CRS and VRS, otherwise they are always inefficient.

CRS frontier allows small DMUs to be compared to big DMUs and vice versa. VRS frontier, on the other hand, tends to only compare DMUs of similar sizes. The differences in our measurements under CRS and VRS reveal that counties are not all of similar size and some of them are thus scale inefficient. In this case a VRS assumption is more appropriate choice.

Comparable results for technical, economic and scale efficiencies measured with different models indicate the same DMUs to be efficient, what is very satisfying. Only in case of allocative efficiency the results were ambiguous what was most likely caused by different measurement technique used and different variables entering the evaluation measurement (as previously mentioned). Even unpredictably large deviations between values of each DMU in data might had an affect on the results.

Several large deviations of quantities between the counties can be detected by detailed look on the data set. Large deviations in costs or prices occured rarely. Some of these deviations might have been caused by errors in collecting and filling the data values. Some of the deviations naturally arise from different conditions for logging wood or different expenses for level of living or even from different market demand in each county. Technology and its expenses differs significantly if the wood is logged e.g. from the forests with rugged topography than from the forests on ground. Expenses for level of living are higher in e.g. capital city than in countryside. Greater demand affects selling prices. Nonetheless, these facts affect every economic subject and market in general and thus expecting narrow deviations is not realistic. Some of deviations in our data, though, might have been larger than commonly or normally assumed or expected.

Detailed program outputs containing values of slacks and weights for each county in (2.22), (2.25), (2.31) models and values of optimal amounts of inputs for each county in (2.11) model, together with the data set can be found on CD attached to this paper.

We consider the provided measurements of efficiencies to be successful and conclude the results as interesting and satisfying.

Conclusion

In first chapter we implemented efficiencies that can be measured to evaluate the performances of economic subjects. Computed evaluations then allow comparisons in between the subjects of the same kind. Also several measurement techniques were introduced from which we decided to study the data envelopment analysis from mathematical programming approach.

In second chapter we examined graphical interpretation and studied closely two different data envelopment analysis models for technical efficiency and their derivates for measuring other efficiencies. From the theory we have seen the differences between models by their requirements and usage.

In third chapter we applied the theoretical procedures of models to the data and seeing the results we can compare advantages and disadvantages of simple DEA models and additive DEA models as follows.

Simple DEA models need to be distinguished as input or output oriented what can have affect on results as happened in our measurement (because of different input and output amounts of variables). Simple DEA model also needs to be supplied by slacks consideration in order to satisfy both Farrell's and Koopmans' efficiency, otherwise the results might be skewed. Two stages simple DEA extended by slacks constraints need to be used. On the other hand, the results, as they are bounded by zero and one and can be presented as percentages, are synoptical and easy to read, comprehend and compare.

Additive DEA models combine both orientations in one problem, so the choice of orientation is not a concern and off the duties for decision maker. The results, on the other hand, are bounded only from below, thus although it is easy to recognize efficient DMUs having values equal to zero, the values of inefficient DMUs are not easily comparable and is hard to conclude how greatly inefficient each DMU is.

The best option is to made measurement on both models, if it is possible, and combine the advantages of each model to interpret the results precisely. Otherwise, the results from only one model are sufficient enough and it is only on decision maker's ability how reliabily they are interpreted.

Appendix

The presented source code from the software Mathematica 5.2 contains initiative commands and commands for four Additive DEA models. Additive DEA model (2.25) with CRS and VRS assumptions and Additive DEA model for economic efficiency (2.31) with CRS and VRS assumptions.

First we need initiative commands for reading the data set, setting the sizes of vectors of variables and defining variables of slacks and weights that are generated by the models.

```
x = { ...}

y = { ...}

c = { ...}

p = { ...}

m = Dimensions[x][[1]]

n = Dimensions[y][[1]]

l = Dimensions[x][[2]]

s1 = Table[sx[i], {i, 1, m}]

s2 = Table[sy[j], {j, 1, n}]

lam = Table[λ[k], {k, 1, l}]
```

Where **x, y** stand for matrices of input and output quantities for all counties, respectively; **c, p** stand for matrices of costs and prices for all counties, respectively; **m, n** stand for the dimensions of inputs and outputs, respectively and **l** stands for dimension of subjects, i.e. number of counties in our case. Generated values of slacks are defined by vectors **s1, s2** for input slacks and output slacks, respectively and finally generated values of weights (λ_k's) are defined by vector **lam**.

Program generates as the results value of the the objective function [TE for model (2.25) and EE for model (2.31)], values of input and output slacks ($s_i^-, s_j^+, \ \ \forall i, j$) and values of weights ($\lambda_k, \ \ \forall k$).

Commands for Additive DEA model for technical efficiency (2.25) for both CRS and VRS assumptions.

```
(*Additive for TE with CRS*)
For[o = 1, o ≤ l, Print[
  N[Maximize[{∑_{i=1}^{m} s1[[i]] + ∑_{j=1}^{n} s2[[j]], Table[x[[i]].lam + s1[[i]] == x[[i, o]], {i, 1, m}],
     Table[y[[j]].lam - s2[[j]] == y[[j, o]], {j, 1, n}],
     Table[lam[[k]] ≥ 0, {k, 1, l}], Table[s1[[i]] ≥ 0, {i, 1, m}],
     Table[s2[[j]] ≥ 0, {j, 1, n}]}, Join[s1, s2, lam]]]]; ++o]
```

```
(*Additive for TE with VRS*)
For[o = 1, o ≤ l, Print[
  N[Maximize[{∑_{i=1}^{m} s1[[i]] + ∑_{j=1}^{n} s2[[j]], Table[x[[i]].lam + s1[[i]] == x[[i, o]], {i, 1, m}],
     Table[y[[j]].lam - s2[[j]] == y[[j, o]], {j, 1, n}], ∑_{k=1}^{l} lam[[k]] == 1,
     Table[lam[[k]] ≥ 0, {k, 1, l}], Table[s1[[i]] ≥ 0, {i, 1, m}],
     Table[s2[[j]] ≥ 0, {j, 1, n}]}, Join[s1, s2, lam]]]]; ++o]
```

Commands for Additive DEA model for economic efficiency (2.31) for both CRS and VRS assumptions.

```
(*Additive for EE with CRS*)
For[o = 1, o ≤ l, Print[
  N[Maximize[{∑_{i=1}^{m} s1[[i]] * c[[i, o]] + ∑_{j=1}^{n} s2[[j]] * p[[j, o]], Table[x[[i]].lam + s1[[i]] ==
       x[[i, o]], {i, 1, m}], Table[y[[j]].lam - s2[[j]] == y[[j, o]], {j, 1, n}],
     Table[lam[[k]] ≥ 0, {k, 1, l}], Table[s1[[i]] ≥ 0, {i, 1, m}],
     Table[s2[[j]] ≥ 0, {j, 1, n}]}, Join[s1, s2, lam]]]]; ++o]
```

```
(*Additive for EE with VRS*)
For[o = 1, o ≤ l, Print[
  N[Maximize[{∑_{i=1}^{m} s1[[i]] * c[[i, o]] + ∑_{j=1}^{n} s2[[j]] * p[[j, o]], Table[x[[i]].lam + s1[[i]] ==
       x[[i, o]], {i, 1, m}], Table[y[[j]].lam - s2[[j]] == y[[j, o]], {j, 1, n}],
     ∑_{k=1}^{l} lam[[k]] == 1, Table[lam[[k]] ≥ 0, {k, 1, l}], Table[s1[[i]] ≥ 0, {i, 1, m}],
     Table[s2[[j]] ≥ 0, {j, 1, n}]}, Join[s1, s2, lam]]]]; ++o]
```

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