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Geometry of Symplectic Graded Manifolds and Their Morphisms

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I am incredibly grateful for the help of my advisor, Branislav Jurčo; thank you so much!

Title: Geometry of Symplectic Graded Manifolds and Their Morphisms

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Abstract: Graded manifolds naturally arise in the context of the Batalin-Vilkovisky quantization as one introduces fields of non-trivial ghost degrees. We study structures tied to the dynamics and gauge symmetry of AKSZ models involving the classical master action on symplectic differential non-negatively graded manifolds ($\mathcal{NQP}$ manifolds) in the language of sheaves of graded-commutative algebras. We review the one-to-one correspondence between isomorphism classes of Courant algebroids and $\mathcal{NQP}$ manifolds of degree 2. Applying the construction of Lagrangian correspondences in the spirit of Weinstein's symplectic category, we extend the one-to-one correspondence to an equivalence of categories.

Keywords: graded symplectic BV Lagrangian correspondences Courant

Contents

Introduction

Gauge symmetry is a concept at the very heart of contemporary fundamental physics and provides language for some of the most illuminating ideas and mathematical models. Becchi et al. [1976], Batalin and Vilkovisky [1981] have shown that such symmetries extend and unfurl into a much richer geometric world of graded symplectic geometry. But why would we embark on such a journey in the first place? The answer, precisely formulated by Costello [2007], is that the *BV-BRST* formalism provides a gateway for quantization of gauge theories.

Even though we start with *bosonic* gauge fields, the *BRST formalism* leads us to a theory on a space of bosonic *and fermionic* fields. Moreover, we will see that on top of the $\mathbb{Z}_2$ even/odd grading, auxiliary fields within the BV framework may in general acquire a $\mathbb{Z}$-grading in the form of the so called *ghost degree*. These fields are purely auxiliary; the so called *ghosts*.

$$\underset{\text{fields}}{\overset{\text{bosonic}}{\text{gauge}}}\; A_\mu \rightsquigarrow A_\mu, c \;\; \text{gauge fields, bosonic or fermionic ghosts}$$

This, however, is not the whole story. *BV formalism* relies on the extension of this field-ghost configuration "along cotangent directions". We may introduce fields formally conjugate to gauge fields and ghosts, the *antifields*.

$$\underset{\text{fields}}{\overset{\text{bosonic}}{\text{gauge}}}\; A_\mu \rightsquigarrow A_\mu, c, (A^*)^\mu, c^* \;\; \text{gauge fields, ghosts, antifields, antighosts}$$

This may remind us of the famous solution to the Zeno's paradox within classical mechanics. Zeno wonders:

"What is *movement*, if at every *instant*, an arrow is *motionless*?"

In other words; we may wonder where the information about movement along a path $x(t)$ is lost if we restrict ourselves to a single instant $t = t_0$. The answer seems easy from today's viewpoint: we need to include the whole *phase space* into our model.

$$\text{position coordinates}\;\; x^i \rightsquigarrow x^i, p_i \;\; \text{position \& momentum coordinates}$$

Movement does not reside in the instant $t = t_0$ but in its microscopic neighbourhood. To talk about movement at $t = t_0$, we have to keep the information about some kind of a *germ of movement*, a Taylor expansion of the $x(t)$ path around the $x(t_0)$ point. The tiniest non-trivial information about such germ — its linear part — is precisely what we call the *momentum* p_i. The phase space can thus be thought of as a minimal (linear) model of:

a *space* M and the the *germs of movement* $\mathbb{R} \to M$

Similarly, we might wonder:

"What is (BRST) *gauge symmetry*, if for every *physical field configuration*, the gauge is *fixed*?"

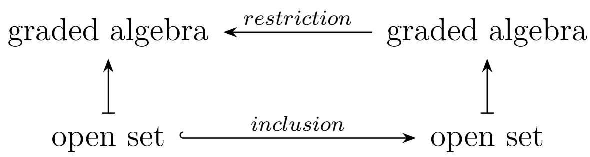
This phrasing may seem rather paradoxical, but remember that this is precisely the case of Zeno's *paradox* as well. We may, once again, try to consider *germs of movement*, but this times, what moves is the gauge itself.

a *field configuration* and the the *germs of gauge change*

These germs are the antifields (and antighosts).

We witness graded manifolds arising from gauge theoretic reasoning as infinite-dimensional "ghost-enriched manifolds". Within the AKSZ framework due to Alexandrov et al. [1997], however, one constructs the relevant geometric structures on two finite-dimensional manifolds and then *lifts* them onto a naturally defined infinite-dimensional manifold of maps betwen them. To follow the AKSZ philosophy of quantization of field theories thus means to study the geometry of the source and the target finite-dimensional graded manifolds.

In the presence of fermionic (and ghost) degrees of freedom, we adapt differential geometry using the simple concept of a *sheaf*: a functorial assingment of an algebra (of fields, ghosts, ...) to every open set of a space.

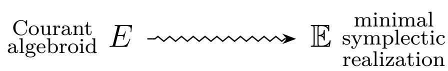


The particularly well-behaved class of *non-negatively* graded manifolds will translate the geometric data familiar from gauge theories to graded symplectic language. The central example is the *classical master equation* encoding gauge symmetry of the dynamics given by the *actional* S (action functional).

$$\{S, S\} = 0$$

Here, the Poisson bracket refers to the precisely the bracket whose conjugate coordinate pairs are fields and antifields, it is the result of our "gauge phase space" extension.

In particular, the low degree cases will provide fruitful examples; the degree 2 case has been found to coincide with *Courant algebroids* known from generalized geometry. We will follow the reasoning of Roytenberg [2002], who shows that in this case, the "cotangent extension" takes the form of a *minimal symplectic realization.*



To extend this correspondence to a proper categorial equivalence, we follow the construction of *Lagrangian correspondences* in the spirit of Weinstein [2010] and Wehrheim and Woodward [2007]. The idea is to consider a morphism not as an element-wise assignment $x \longmapsto f(x)$ subject to algebraic rules, but as a subset L of the cartesian product of the target and source spaces M, N endowed with geometric properties.

$$L \subseteq M \times N$$

We can, for example, represent smooth maps as smooth submanifolds.

$$L \hookrightarrow M \times N$$

We will use the results of Vysoky [2020], who gives a precise description of correspondences of Courant algebroids using *Dirac structures*, the natural "substructures of Courant algebroids". Constructing the appropriate "cotangent extension", we will rely on results of Grützmann [2010]; in particular they will take the form of *conormal subbundles*.

$$\underset{\text{structure}}{\text{Dirac}}\ L \rightsquigarrow \boldsymbol{N}^*L\ \underset{\text{subbundle}}{\text{conormal}}$$

Finally, we will mention how the resulting categorial framework fits into the idea of *functorial quantization* of odd symplectic manifolds put forth by Severa [2002], which will be the end of our journey.

We presuppose basic knowledge of differential geometry, theory of fibre and principal bundles and elementary knowledge of category theory. A familiarity with gauge theories provides an excellent motivation for the topics at hand, but is not needed.

1. Ghosts & Graded Manifolds

Gauge-theoretic structures of quantum field theories[1] may be extended into a larger and richer *graded-commutative geometric world*, built *above* the original concept of a Lie group G-action and a connection 1-form $A_\mu(x)$ corresponding to a **bosonic gauge field**.

Firstly, in section 1.1, we will briefly recall elements of bosonic gauge theories and the geometric objects arising in the Batalin-Vilkovisky (BV) quantization formalism, refering to Becchi et al. [1976], Batalin and Vilkovisky [1981]. Moreover, we will mention the AKSZ framework originally due to Alexandrov et al. [1997] which allows one to talk about the geometry of the BV field theories in terms of finite-dimensional manifolds. We will focus precisely on *(the category of)* those finite dimensional manifolds equipped with bosonic, fermionic and general auxiliary "*ghost coordinates*".

A large part of this chapter, sections 1.2, 1.3, 1.4, is devoted to the basics of the theory of non-negatively graded manifolds ($\mathcal{N}$-manifolds) endowed with geometric structures mirrored in the BV formalism. In section 1.5, we will provide a treatment of "the first non-trivial class" of examples of $\mathcal{N}$-manifolds, the special "$\boldsymbol{deg} = 2$" case, which is known to correspond to familiar structures from *generalized geometry*. We follow mainly Roytenberg [2002] and Cattaneo and Schaetz [2011].

1.1 Gauge Fields & the AKSZ Philosophy

The BRST supersymmetry of a theory, due to Becchi et al. [1976] is one of the first to stand witness to an extension of a gauge theory by enlarging the field space. It will be our first example: let us briefly sketch the relevant ideas as a motivation for the following study of graded manifolds. This chapter may also be seen as the one that sets one of the main goals of this thesis: to review a rigorous geometric framework that embodies the structures found in this chapter.

For further reading, we refer to the extensive textbooks of Weinberg [2013] and Henneaux and Teitelboim [2020] . This section is strongly inspired by lectures on gauge theory led by Jiří Novotný at Charles University in Prague in the winter semester of 2020.

1.1.1 Becchi-Rouet-Stora-Tyutin Cohomology

Let us start with the example of Yang-Mills theories. Gauge symmetry endows a Yang-Mills theory with a set of contraints onto which we may "reduce" the dynamics. To ensure compatibility of the dynamics given by an actional S_0 with the gauge symmetry, it is desirable to find a "pull-back" of the constrained

[1] Or more precisely; a classical field theory we aim to quantize.

generating functional Z_{red} of the form

$$Z_{red}[J] = \int \mathcal{D}\phi \; e^{iS_0[J]+iJ\cdot\phi}$$

along the reduction, into "a general coordinate system", so to speak. Here, we may understand $\mathcal{D}\phi$ as a formal path integration of fields that may have values in (a representation of) a Lie algebra $\mathfrak{g}$.

The solution is the *Faddeev formula*. We simply multiply the integrand by appropriate formal Dirac δ-functionals and certain *volume corrections*.

We are, however, not quite done yet. To obtain a manifestly Lorentz invariant (ie. "covariant") generating functional, we may construct the following extension of the field space:

We smear the delta functionals by introducing the bosonic Nakanishi-Lautrup fields η and construct the volume corrections from Berezinian[2] integration of a new set of *fermionic* fields; the *Faddeev-Popov ghosts* $b(x)$, $c(x)$. The resulting generating functional is described by the *Faddeev-Popov formula*.

$$Z[J] = \int \mathcal{D}A\mathcal{D}\eta\mathcal{D}b\mathcal{D}c \; e^{iS[J]+iJ\cdot\phi}$$

The new extended actional S now exhibits an extended version of the gauge symmetry given by the gauge field $A_\mu(x)$, the *BRST symmetry* given by a $\delta_{\boldsymbol{BRST}}$ operator.

Remark 1 (BRST Symmetry). *On $A_\mu(x)$ and $c(x)$, in particular (and with explicit Lie algebra indeces), the BRST* (Becchi-Rouet-Stora-Tyutin) *symmetry transformation reads as follows.*[3]

$$\begin{aligned} \delta_{\boldsymbol{BRST}} A^a_\mu &= (\boldsymbol{D}_\mu c)^a \\ \delta_{\boldsymbol{BRST}} c^c &= \frac{1}{2} e \boldsymbol{C}_{ab}{}^c c^a c^b \end{aligned}$$

where $\boldsymbol{D}_\mu$ is the covariant derivative associated to the connection A^a_μ, e is a coupling constant and $\boldsymbol{C}_{ab}{}^c$ the structure constants of $\mathfrak{g}$. Note that this is a "supersymmetry" as it switches between bosonic and fermionic fields. Moreover, it is important that this is an ***affine transformation***.

The integral representation of the BRST operator is the ***Slavnov operator***. *Let us write out its form for the restriction on A^a_μ, c fields and by $\frac{\delta}{\delta\phi}$ denote a formal functional derivative.*

$$\delta_{\boldsymbol{BRST}} = \int \mathcal{D}A\mathcal{D}c \; \left((\boldsymbol{D}_\mu c)^a \frac{\delta}{\delta A^a_\mu} + \frac{1}{2} e \boldsymbol{C}_{ab}{}^c c^a c^b \frac{\delta}{\delta c^c} \right)$$

[2] The natural notion of integration on supermanifolds, we refer eg. to Caston and Fioresi [2011].

[3] Here, latin indeces enumerate Lie algebra components, greek indeces enumerate spacetime coordinates. Note that the (x)-dependence is implicit and so is the "(x)-contraction" within the action of the operator $\boldsymbol{D}_\mu \leftrightarrow \boldsymbol{D}^a_{\mu b}$. To be a bit more precise about the contraction, we may think of it as of "integrating over a derivative of a delta function".

The remaining fields η and b serve to define the **gauge fixing fermion** $\mathcal{F}$ terms in the actional. It should not be surprising that its purpose is to effectively fix the gauge upon integration.

It turns out that $\delta^2_{\boldsymbol{BRST}} = 0$, ie. the BRST differential defines a chain complex on the space of fields. Eg. its cohomology classes in degree 0 coincide precisely with gauge invariant functionals depending only on the A_μ gauge fields, independent of the auxiliary fields. In other words:

"*gauge symmetry* $\longmapsto$ *zeroth* $\delta_{\boldsymbol{BRST}}$*-cohomology*"

Note that other cohomology groups do also have physical interpretations, we refer to Jurčo et al. [2019].

Remark 2 (Ward identities)**.** *Within the BRST framework, we can construct a mean value a functional $\mathcal{G}$ roughly via path integration over the space of fields. Given a theory with a gauge fixing fermion $\mathcal{F}$, we denote the mean value by:*

$$\langle \mathcal{G} \rangle_{\mathcal{F}}$$

One of the results that goes by the name of ***Ward identities*** *states that the mean value is invariant wrt. a deformation of the gauge fixing fermion by an infinitesimal transformation δ, provided it is a symmetry of the actional.*

$$\langle \mathcal{G} \rangle_{\mathcal{F}+\delta\mathcal{F}} = \langle \mathcal{G} \rangle_{\mathcal{F}}$$

We will not specify this statement here. Let us just remark that the deformation of $\mathcal{F}$ may be understood in the sense of remark 58, which closes this thesis.

1.1.2 Batalin-Vilkovisky Formalism

Staying with the example of a Yang-Mills theory, one may think of the BV method as an extension of the BRST framework "along cotangent directions". Given a set of fields (including the auxiliary ghosts), for each ϕ we introduce an **antifield** ϕ^*. We extend the BRST actional S by adding terms of the form:

$$\phi^*_a \left(\delta_{\boldsymbol{BRST}}\phi\right)^a$$

In other words, the antifields play the role of the "sources of the BRST transformation". We define a *graded version of a Poisson bracket,* the **antibracket** st. $(\phi^a, \ldots, \phi^*_b, \ldots)$ forms its (formal) Darboux chart.

$$\{\phi^a, \phi^*_b\} = \delta^a_b$$

We will give a precise definition later, within the language of non-negatively graded manifolds.

This construction further intertwines the actional and its gauge symmetry. Now, by construction, the extended actional S is a "Hamiltonian generator" of the BRST transformation on ϕ fields.

$$\delta_{\boldsymbol{BRST}}\phi = \{S, \phi\}$$

A natural generalization of the transformation is to simply let $\{S, \bullet\}$ act on the antifields as well. We will denote it by $\mathcal{Q}$.

$$\mathcal{Q} = \{S, \bullet\}$$

Since $\mathcal{Q}(S) = \{S, S\}$, the extended gauge invariance of the actional now takes on a new form:

$$\{S, S\} = 0$$

This is the **classical master equation**, its solution S is said to be a **classical master actional**.

Definition 1 (Heuristic)**.** A **classical BV-BRST theory** defined for an actional S_0 with a given symmetry is a *field-antifield* configuration with a ghost field for every dimension of the symmetry action equipped with the following structure.

- A graded Poisson structure ("antibracket") $\{\bullet, \bullet\}$.
- An odd differential $\mathcal{Q}$ st. $\mathcal{Q}^2 = 0$.
- A classical master actional S st. $S\big|_{\phi^*=0} = S_0$.

Note that given the actional and one of the other objects, we can define the remaining one.

Remark 3 (Gauge fixing)**.** *We may define a gauge fixing fermion for general BV theories in analogy with the $\mathcal{F}$ field of the Faddeev-Popov actional. The **gauge fixing** is a restriction on a subspace defined by:*

$$S_{\text{fixed}} := S\big|_{\phi^*_a = \frac{\delta \mathcal{F}}{\delta \phi^a}}$$

In the general case of "gauge symmetries having their own symmetries", we may introduce *higher ghosts* with each next level fixing the lower ones. The **ghost degree** $|\phi|$ is an integer asigned to each field ϕ that labels these levels of gauge fixing. The operator $\mathcal{Q}$ is of ghost degree 1. The antighosts acquire negative ghost degrees: $|\phi^*| = -|\phi| - 1$. We refer to Jurčo et al. [2019] for a contemporary summary in the language of L_∞-algebras and higher algebraic structures. For an introduction to higher gauge theory, we refer to Baez and Huerta [2011].

One of our main tasks will be to describe the structures appearing in the "heuristic prescription" 1 in the language of non-negatively graded manifolds (and Courant algebroids) without a priori bounding the ghost degree (from above).

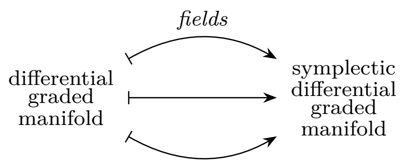
In what follows, we will have good reasons to choose a different grading while the cohomological data of the $\mathcal{Q}$ operator stays the same. We refer to Roytenberg [2002], who mentions a theorem clarifying this statement, due to Kostant and Sternberg [1987].

To quantize a BV field theory with a classical master action and the $\mathcal{Q}$ differential, we must take one last step and add the differential operator $\boldsymbol{\Delta}_{\text{BV}}$ to the mix. This is the so called *BV Laplacian* and it is a shadow of the calculations of mean values in terms of path integrals weighted by the BV action. It is desirable to obtain a theory with a gauge symmetry compatible with both S and $\boldsymbol{\Delta}_{\mathrm{BV}}$. This is what we need to start the general renormalization procedure of a gauge-fixed BV action; we refer to Costello [2007]. We will touch on quantization again in section 2.2.4.

1.1.3 Alexandrov, Kontsevich, Schwarz & Zaboronsky

Finally, let us mention the core *philosophy* of the AKSZ method originally due to Alexandrov et al. [1997], refering to Cattaneo and Schaetz [2011] for a an introductory text in a formalism similar to ours.

Roughly, one models the classical fields as maps between graded manifolds and *lifts* the symplectic and $\mathcal{Q}$ structure from the source and target to $\boldsymbol{Map}(\mathcal{M},\mathcal{N})$ by means of (Berezinian) integration. Here, $\mathcal{M}$ and $\mathcal{N}$ are graded manifolds and $\boldsymbol{Map}(\mathcal{M},\mathcal{N})$ is an appropriate notion of a space of maps between the two manifolds.



Here, "differential" refers to the structure of a $\mathcal{Q}$ operator. We will provide the precise defintions in the next chapter.

Remark 4 (DeWitt notation)**.** *On the target graded manifold, in the case its coordinates are at most of degree* 2*, we will usually choose the following notation for a local chart:*

$$\left(x^i, p_j, \xi^\mu, e_\nu\right)$$

These correspond to the following field-antifield configuration: bosonic (gauge) fields $A^a_\mu(x)$, their antifields $(A^*)^\nu_a(x)$, fermionic ghosts $c^a(x)$ and their antighosts $c^*_b(x)$.

$$\left(A^a_\mu(x), (A^*)^\nu_b(x), c^a(x), c^*_b(x)\right)$$

In terms of indeces, we have the following assignment:

latin, spacetime fields	$(a,(x)),(b,(x)),\ldots$	$\longmapsto$	greek, graded coordinates	$\mu,\nu,\ldots$
greek & latin, spacetime fields	$(\mu,a,(x)),(\nu,b,(x)),\ldots$	$\longmapsto$	latin, graded coordinates	$i,j,\ldots$

Thus the "structure constants" (obviously not constant in general) $\boldsymbol{D}^a_{\mu b}$ will correspond to graded structure constants of the form $\boldsymbol{\varrho}^i_\mu$, the "structure constants" $\boldsymbol{C}_{ab}{}^c$ to $\boldsymbol{C}_{\mu\nu}{}^\rho$.

1.2 Sheaves & the Three Magic Theorems of Differential Geometry

Our task is now to introduce a theory of graded geometry that may serve as a source or target manifold of an AKSZ model. In particular, we have the heuristic prescription of definition 1 in mind; the geometric contents of a BV-BRST theory.

We start with bosonic fields $A_\mu^a(x)$ — those simply correspond to smooth functions x^i on the target manifold, see remark 4. We may, however, shift the attention to the *sheaf of functions*. This provides us with a perspective which can be adapted to fermionic *supercommutative* “coordinates”. Similarly, we will introduce ghosts as *graded-commutative* coordinates. For a comprehensive treatment of sheaves on manifolds, we refer to Kashiwara and Schapira [1990].

Let us remark that the most important algebraic properties that provide the right framework for generalizations of “commutative” smooth manifolds are the famous “*three magic theorems of differential geometry.*”

1.2.1 I: Manifolds as Locally Ringed Spaces

We denote by $\mathcal{Open}_M$ the category consisting of open sets of M with arrows given by the natural **subset inclusion** $\subseteq$ relations.

$$U \hookrightarrow V$$

A **presheaf on** M is a (contravariant) functor $\mathcal{F} : \mathcal{Open}_M \to \mathcal{C}^{\text{ op}}$, where we will restrict ourselves to $\mathcal{C}$ being a category of locally free modules or (commutative or graded-commutative) associative algebras[4]. The most common example will be:

$$\mathcal{F} : \mathcal{Open}_M \longrightarrow \mathcal{Alg}^{\text{op}}$$

In other words, here, $\mathcal{F}(U)$ for $U \in \mathcal{Open}_M$ is an **associative algebra** and the morphisms induced by inclusions are called **restriction maps**.

$$U \hookrightarrow V$$
$$\mathcal{F}(U) \leftarrow \mathcal{F}(V)$$

Since we choose target categories where object admit *elements*, we may talk about them as of **local sections** of a presheaf. We denote the restriction by:

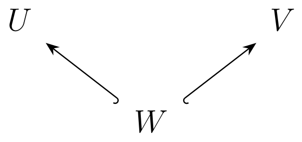
$$f \mapsto f\big|_U$$

Given the objects of $\mathcal{Open}_M$ are sets consisting of points on the manifold M, we can localize this notion of a “neighbourhood algebra” $\mathcal{F}(U)$ even furher; look at the “microscopic” behaviour around $x \in M$. We denote by

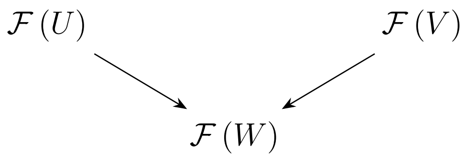
$$\coprod_{U \ni x} \mathcal{F}\,(U)$$

all open sets containing the point $x \in M$. Note that the inclusions induce a partial order on $\coprod \mathcal{F}\,(U)$. For every pair of open sets U, V, there is another one, W, such that:

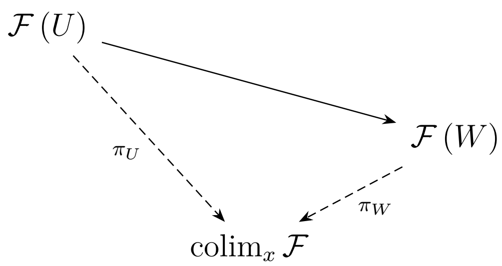
[4] By “algebra” we implicitely mean $\mathbb{R}$-algebra, unless we say otherwise.



eg. $W = U \cap V$. Correspondingly, a presheaf $\mathcal{F}$ yields:



Considering this whole partially ordered set as a diagram, we can take its colimit, which usually goes by the name "stalk of $\mathcal{F}$ at x". Compatibility of the canonical projections $\pi_U, \pi_V, \ldots$ with the restriction maps



implies that in $\operatorname{colim}_x \mathcal{F}$, we have forgotten all the data outside of any intersection of sets containing x. If there existed a set U_x, "the smallest open set containing x", then $\operatorname{colim}_x \mathcal{F}$ would be precisely $\mathcal{F}(U_x)$. That is not the case on a manifold, but given we work with a well-behaved target category $\mathcal{Alg}$, we can define the stalk explicitely as follows.

Definition 2 (Stalk)**.** For a presheaf $\mathcal{F}$ on M we define the **stalk at** $x \in M$ as:

$$\mathcal{F}\big|_x := \left(\coprod_{U \ni x} \mathcal{F}(U) \right) / \sim$$

where two sections $f, g \in \coprod \mathcal{F}(U)$ are glued together, $f \sim g$, if their restrictions coincide on some neigbourhood of x. Such an equivalence class $[f] \in \mathcal{F}\big|_x$ is called a **germ** at x.

It is easy to show that a stalk inherits the mutliplication on $\mathcal{F}(U)$ and thus forms a ring.

Example. *Considering the basic example of a presheaf on a manifold, $\mathcal{C}^\infty(M)$, its stalk at $x \in M$ consists of Taylor expansions of smooth functions. Ie. a germ of a smooth function at x is its Taylor expansion around x, its jet.*

Definition 3 (Sheaf)**.** We say a presheaf $\mathcal{F}$ on M is a **sheaf** if the following axioms hold for any $U \in \mathcal{Open}_M$ and its arbitrary open cover $\mathfrak{U}$.

- *the uniqueness axiom*: If the restrictions of two sections of $\mathcal{F}(U)$ coincide at any elements of the open cover $\mathfrak{U}$, they are the same.

- *the gluing axiom*: If there is a family of sections defined on the whole open cover $\mathfrak{U}$ such that their restrictions onto intersections coincide, there is a section of $\mathcal{F}(U)$ that produces them as its restrictions.

Definition 4 (Locally ringed space)**.** A pair $\mathcal{M} = (M, \mathcal{O}_M)$ with $\mathcal{O}_M$ being a presheaf on M is said to be a **locally ringed space** if the following axioms hold:

- $\mathcal{O}_M$ is a sheaf.
- Every stalk $\mathcal{O}_M\big|_x$ is a **local ring**; it contains a unique maximal ideal.

The manifold M is said to be the **body** of $\mathcal{M}$ and $\mathcal{O}_M$ the **structure sheaf**.

Example. *The sheaf of smooth functions on a manifold* $\mathcal{C}^\infty(M)$ *is a locally ringed space, the maximal ideal at a point* $x \in M$ *is* $\mathfrak{m}_x$ *the germ of functions vanishing at* x*. Similarly for the sheaf of differential forms* $\Omega(M)$*.*

Specifying a **local model** of $(M, \mathcal{O}_M)$ amounts to prescribing what the structure (pre)sheaf locally looks like; $\mathcal{O}_M(U) \simeq \mathcal{A}_U$, for any $U \in \mathcal{Open}_M$.

$$(U, \mathcal{A}_U)$$

The *first magic theorem of differential geometry* states that it is natural to think of manifolds as locally ringed spaces with the local model

$$(U, \mathcal{C}^\infty(U))$$

For a proof, we refer to Kolář et al. [1993].

Theorem 5 (Milnor's Exercise)**.** *The* ***smooth function functor***

$$\mathcal{C}^\infty(\bullet) : \mathcal{Man} \longrightarrow \mathcal{Alg}^{\mathrm{op}}$$

is fully faithful*.*

More specifically, it states that the morphisms of manifolds are completely described by the morphisms of their smooth function algebras.

$$\mathbf{Hom}_{\mathcal{Man}}(M, N) \simeq \mathbf{Hom}_{\mathcal{Alg}}(\mathcal{C}^\infty(N), \mathcal{C}^\infty(M))$$

This is an essential observation as it will guide us when defining a category of manifolds with supercommutative fermionic coordinates: Talk in language of the *"local function algebras"*!

There is, however, one thing we must be careful about if we are to introduce generalizations of manifolds. The statement of the *first magic theorem of differential geometry* non-trivially depends on the structure of the very specific smooth function algebra functor $\mathcal{C}^\infty(\bullet)$. If we are to introduce its generalizations, we shall consider the *morphisms of the structure sheaves as such.* That is, morphisms of the families of local sections, not just the global ones over M. This way, we know we are staying in whatever "category of generalized manifolds" we have constructed if we talk about local sections, their derivations etc.

In other words, we will talk in the language of locally ringed spaces.

1.2.2 II: $\mathcal{C}^\infty(M)$-Modules as Vector Bundles

It is becoming clear that the central object of our interest will be sheaves of algebras on a manifold. They will be naturally accompanied by locally free modules over the ring $\mathcal{C}^\infty(M)$. The *second magic theorem of differential geometry* states that we can always think of them as modules of sections of some vector bundle.

Let $\mathcal{VectB}_M$ denote the category of **smooth vector bundles over** M **of finite rank** and $\mathcal{ProjMod}_{\mathcal{A}}$ the category of **locally free** (ie. projective) **finitely generated modules** over the algebra $\mathcal{A}$. For a proof, we refer to Nestruev [2003].

Theorem 6 (Smooth Serre-Swan). *Let M be a connected smooth manifold. Then the category $\mathcal{VectB}_M$ is equivalent to $\mathcal{ProjMod}_{\mathcal{C}^\infty(\mathcal{M})}$. More precisely,* ***smooth section functor***

$$\Gamma(\bullet) : \mathcal{VectB}_M \longrightarrow \mathcal{ProjMod}_{\mathcal{C}^\infty(M)}$$

that produces locally free finitely generated modules as the modules of smooth sections $\Gamma(M)$ over the algebra of smooth functions $\mathcal{C}^\infty(M)$ is fully faithful *and* essentially surjective*.*

The theorem gives rise to a heuristic; if we specify a locally free $\mathcal{C}^\infty(U)$-module $\mathcal{E}$ we will identify the local generators of $\mathcal{E}$ with some local coordinates in the fibre of a vector bundle E. This is the basic ingredient we need to introduce the notion of a *Courant algebroid*.

Moreover, it inspires the *definition of a vector bundle* over locally ringed spaces. We will just replace the ring defined by the algebra $\mathcal{C}^\infty(U)$ by the ring defined by the local model $\mathcal{O}_{\mathcal{M}}$.

Note that by a "sheaf of $\mathcal{O}_{\mathcal{M}}$-modules", we mean that we have a sheaf $\mathcal{F}$ locally given by a module $\mathcal{F}(U)$ over the $\mathcal{O}_{\mathcal{M}}(U)$ ring. By definition, the local models are isomorphic for any $U \in \mathcal{Open}_M$.

1.2.3 III: Tangent Fields are $\mathcal{O}_M$-Derivations

The *third magic theorem of differential geometry* is perhaps the most well-known. It states that the natural map

$$\text{direction} \longmapsto \text{directional derivative}$$

is an isomorphism. More precisely, if we denote $\mathbf{Der}\left(\mathcal{C}^\infty(M)\right)$ the (finitely generated projective) $\mathcal{C}^\infty(M)$-module of derivations of the algebra $\mathcal{C}^\infty(M)$, the following theorem instantiates the isomorphism in the category $\mathcal{PrMod}_{\mathcal{C}^\infty(M)}$.

Theorem 7. *Tangent fields are the derivations of smooth functions.*

$$\Gamma(TM) \simeq \mathbf{Der}\left(\mathcal{C}^\infty(M)\right)$$

Once again, this property of ordinary manifolds provides a natural way to define a familiar object in a generalized setting given by a local model; local tangent fields *should be* derivations of the local algebras $\mathcal{O}_{\mathcal{M}}(U)$.

1.3 Fermionic Sheaves are Supermanifolds

The next task is to describe fermionic "anticommuting functions" on a smooth manifold in terms of local algebras.

$$\psi\phi = -\phi\psi$$

Recall that the auxiliary ghosts (already in the BRST framework) can in general be fermionic even though the "physical" content is just a set of bosonic fields.

Anticommuting algebras are precisely $\mathbb{Z}_2$-graded supercommutative algebras and supergeometry is just what we are looking for. We will present only the most elementary definitions, for a comprehensive treatment we refer to Caston and Fioresi [2011].

1.3.1 Basics of Supergeometry

A **super vector space** V is a $\mathbb{Z}_2$-graded vector space.

$$V = V_0 \oplus V_1$$

V_0 is said to be the **even** part and V_1 the **odd** part. It is equipped with the **parity**; the map that assigns the numbers 0 or 1 to homogeneous elements.

$$|v| \coloneqq 0 \text{ if } v \in V_0$$
$$|v| \coloneqq 1 \text{ if } v \in V_1$$

Morphisms of super-vector spaces are linear and preserve parity, the category of super vector spaces is denoted by $\mathcal{V}ect_{\mathcal{S}}$. The **parity shift functor**

$$\overset{\Pi}{\circlearrowleft}\ \mathcal{V}ect_{\mathcal{S}}$$

merely switches the parity of homogeneous elements. If we embed the category of ordinary vector spaces into super vector spaces $\mathcal{V}ect \hookrightarrow \mathcal{V}ect_{\mathcal{S}}$ as the purely even ones, the parity functor restricted to $\mathcal{V}ect$ sends vector spaces to purely odd super vector spaces $\Pi\big|_{\mathcal{V}ect} : \mathcal{V}ect \to \mathcal{V}ect_{\mathcal{S}}$.

Definition 5. A **superalgebra** is a $\mathbb{Z}_2$-graded (associative) algebra $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ equipped with the parity map, st. for all homogeneous elements $f, g \in \mathcal{A}$:

$$fg = (-1)^{|f||g|} gf$$
$$|fg| = |f| + |g|$$

The morphisms of superalgebras are morphisms of algebras respecting the parity, the category of superalgebras is denoted by $\mathcal{A}lg_{\mathcal{S}}$.

Any super vector space V can be made into a superalgebra by taking the quotient of its tensor algebra by the ideal consisting of elements of the form:

$$v \otimes w - (-1)^{|v||w|} v \otimes w \qquad \text{for homogeneous } v, w \in V$$

Considering a super vector space consisting only of odd elements (ie. one that can be thought of as a shifted ordinary vector space ΠW) the ideal reduces to the symmetric ideal generated by:

$$v \otimes w + w \otimes v \qquad \text{for } \theta \in \Pi W$$

The resulting superalgebra is the well-known **Grassmann algebra** $\wedge(W)$.

Definition 6. A **supermanifold** of dimension $m|n$ is a locally ringed space $\mathcal{M} = (M, \mathcal{O}_{\mathcal{M}})$ with a local model of the form

$$(U, \mathcal{C}^{\infty}(U) \otimes \wedge(W))$$

where $\dim M = m$ and $\Pi W = \operatorname{Span}(\theta_1, \ldots, \theta_n)$. It is equipped with an **atlas** composed of charts of the form (U, φ, Φ), where $U \in \mathcal{O}pen_M$, φ is a coordinate map of the body M and Φ is an isomorphism of superalgebras which preserves maximal ideals of the stalks in U.

$$\mathcal{C}^{\infty}(U) \otimes \wedge(W) \xrightarrow{\quad \Phi \quad} \mathcal{O}_M(U)$$

Definition 7 (Morphisms of supermanifolds)**.** Let $\mathcal{M}, \mathcal{N}$ be supermanifolds. We denote the category of supermanifolds $\mathcal{M}an_{\mathcal{S}}$, where the morphisms of $\mathcal{M}, \mathcal{N}$ are defined to be the natural transformations of the structure sheaves of superalgebras, ie .

$$\mathbf{Hom}_{\mathcal{M}an_{\mathcal{S}}}(\mathcal{M}, \mathcal{N}) \simeq [\mathcal{O}_{\mathcal{N}}, \mathcal{O}_{\mathcal{M}}]$$

Remark 8. *Vector bundles on supermanifolds are sheaves $\mathcal{E}$ of $\mathcal{O}_{\mathcal{M}}$-modules, their duals are sheaves of graded (ie. degree shifting) natural transformations into the structure sheaf $\mathcal{E} \Longrightarrow \mathcal{O}_{\mathcal{M}}$, tensor products and pullbacks are defined stalkwise.*

1.4 Sheaves of Ghosts & $\mathcal{NQP}$-Manifolds

Now, we will extend the $\mathbb{Z}_2$ grading to a general integer degree. Let us recall and look what structures (in def. 1) we aim to describe and what we know about them (eg. from remark 1 and 4).

- Graded-commutative "ghost" coordinates, subject to *affine transformations.*
- An odd differential operator $\mathcal{Q}$ with a non-trivial chomology.
- A graded Poisson structure ("antibracket") $\{\bullet, \bullet\}$.
- A Hamiltonian generator Θ of $\mathcal{Q}$. In other words, a solution of the classical master equation.

$$\{\Theta, \Theta\} = 0$$

Note that here, Θ, the analogue of the BV actional S, will be constructed simply as a section of (pre)sheaf, a $\mathcal{C}^\infty(U)$-linear combination of local generators of a given local model. It is no longer a functional — an integral operator — investigating only the target space of an AKSZ model instead of the whole "space of fields" reduces the reasoning to finite dimension.

We will mention general $\mathbb{Z}$-grading and some of its sheaf-theoretic issues and then move on to a review of *non-negative grading.* It will provide us with a sheaf-theoretic description of all the objects mentioned above. At the end of this chapter, we will turn out attention to the degree 2 case, ie. the geometry of non-negatively graded manifolds with local coordinates of degree at most 2. We will follow mainly Roytenberg [2002], Cattaneo and Schaetz [2011]. We also often refer to Vysoky [2021], who recently developped a precise theory of general $\mathbb{Z}$-graded manifolds.

1.4.1 $\mathbb{Z}$-Grading

A **graded vector space** is a vector space decomposing into

$$V = \bigoplus_{i \in \mathbb{Z}} V_i$$

equipped with the **degree** map that assigns an integer to every homogeneous element.

$$|v| \coloneqq i \quad \text{for } v \in V_i$$

The **dual** of a graded vector space is the graded vector space with

$$(V^*)_i = (V_{-i})^* .$$

The k**-shifted** graded vector space is defined by shifting the degree of homogeneous elements k steps *down*.

$$V[\boldsymbol{k}] \coloneqq \bigoplus_{i-k \in \mathbb{Z}} V_i$$

This way, the *linear functions* on $V[\boldsymbol{k}]$ are shifted k steps *up*:

$$((V[\boldsymbol{k}])^*)_i = (V^*)_{i+k}$$

Remark 9. *All the definitions can be repeated for a vector bundle E (over an ordinary manifold); fibre-wise. We use the same notation for a shifted vector bundle, $E[\boldsymbol{k}]$. In the case of the tangent bundle, we denote the shifted bundle $T[\boldsymbol{k}]M$. We can think of the k-shift as of a functor in the category of graded vector bundles and define the* ***k-shifted tangent functor****.*

$$T[\boldsymbol{k}] \coloneqq [\boldsymbol{k}] \circ T$$

The same can be repeated for the cotangent functor. We will use these $[\boldsymbol{k}]$-shifted (co)tangent bundles to construct a "total space model" of graded manifolds in what follows.

Definition 8. A **graded algebra** is a $\mathbb{Z}$-graded associative algebra $A = \bigoplus_{i\in\mathbb{Z}} A_i$ equipped with the degree map, st. for all homogeneous elements $f, g \in A$:

$$fg = (-1)^{|f||g|} gf$$
$$|fg| = |f| + |g|$$

The morphisms of graded algebras are morphisms of algebras respecting the degree. The category of graded algebras is denoted by $\mathcal{GrAlg}$.

Given a graded vector space V, we obtain a graded algebra by defining the suitable factoralgebra of the tensor algebra.

Definition 9. The **graded-symmetric algebra** $\mathrm{S_g}(V)$ of polynomials on V is the tensor algebra factored by the ideal composed of:

$$v \otimes w - (-1)^{|f||g|} w \otimes v \qquad \text{for homogeneous } v, w \in V$$

Now we may try to define the appropriate notion of a **graded manifold** with the structure sheaf of graded "ghost algebras" following Cattaneo and Schaetz [2011] by a local model given as:

$$(U, \mathcal{C}^\infty(U) \otimes \mathrm{S_g}(V))$$

Example 10 (BV Ghost-Grading)**.** *First, we refer to the "translation chart" between BV-BRST fields and the graded coordinates on a low degree graded manifold in remark 4.*

In the context of the BV formalism without higher ghosts, we would have just the bosonic body coordinates x^i, their antifields p_j of degree -1, ghosts ξ^μ of degree 1 *and antighosts e_μ of degree -2. The antibracket has degree* 1*.*

There are, however, sheaf-theoretic problems with general $\mathbb{Z}$-graded manifolds defined naively. Let us mention just two of them: one of a global nature and one on the microscopic level of stalks.

1. The gluability condition is problematic for an infinite cover $\mathfrak{U}$, as we may need a global section with unbounded powers of even coordinates. But there is simply no such a global *polynomial* section; its powers must vanish at some points.
2. The rings given by local models fail to be local.

The solution of Vysoky [2021] is to consider formal power series instead of polynomial sections. Then one may define the proper graded incarnations of vector spaces, algebras, sheaves and locally ringed spaces; eg. by requiring them to satisfy familiar categorial properties.

We will not need the general theory, we will use instead restrict ourselves to non-negative grading. That will be just enough to talk about Courant algebroids and the structures familiar from BV-BRST theories from definition 1 and the beginning of this section 1.4 (up to a degree shift). For those, the sheaf theoretic problems are just minor.

1.4.2 $\mathcal{N}$-Manifolds

We will now restrict ourselves to **non-negative grading** in the local model of our graded manifolds. As a shorthand, we will use the letter $\mathcal{N}$ to stand for "non-negatively graded". Here, we follow mainly Roytenberg [2002].

Note that a presheaf $\mathcal{F}$ with values in $\mathcal{N}$-algebras defines a family of presheaves of vector spaces:

$$(\mathcal{F}_i)_{i\in\{0,1,2,\dots\}}$$

The restriction maps of $\mathcal{F}_i$ are well defined since $\mathcal{F}$ preserves the degree thanks to its functoriality.

Definition 10. We say a pair $(M, \mathcal{O}_{\mathcal{M}})$ is a **locally ringed $\mathcal{N}$-space** if the presheaf $\mathcal{O}_{\mathcal{M}}$ is a presheaf of $\mathcal{N}$-alebras st. in each degree $i \in \{0, 1, 2, \dots\}$, $(\mathcal{O}_{\mathcal{M}})_i$ is a sheaf and the stalks of $\mathcal{O}_{\mathcal{M}}$ are local rings. We may call such a "degree-wise sheaf" $\mathcal{O}_{\mathcal{M}}$ a **$\mathcal{N}$-sheaf**. In the case of the "structure presheaf" here, we will refer to $\mathcal{O}_{\mathcal{M}}$ as to the **structure $\mathcal{N}$-sheaf of**$\mathcal{M}$.

Definition 11. A **$\mathcal{N}$-manifold** is a locally ringed $\mathcal{N}$-space $\mathcal{M} = (M, \mathcal{O}_{\mathcal{M}})$ with the local model:

$$(U, \mathcal{C}^\infty(U) \otimes \mathrm{S_g}(\Xi))$$

where $\Xi = \mathrm{Span}\,(\xi^1, \dots, \xi^n)$ is a $\mathcal{N}$-vector space with $|\xi^\mu| \geq 1$ for all μ. It is equipped with an **atlas** composed of charts of the form (U, ϕ, Φ), where $U \in \mathit{Open}_M$, ϕ is a coordinate map of the body M and Φ is an isomorphism of $\mathcal{N}$-algebras which preserves maximal ideals of the stalks in U.

We defer the question of consistency and sheaf-theoretic rigor to later, to the discussion of the Batchelor's theorem 14.

Definition 12 (Morphisms of $\mathcal{N}$-manifolds)**.** Let $\mathcal{M}, \mathcal{N}$ be $\mathcal{N}$-manifolds. We denote the category of $\mathcal{N}$-manifolds $\mathit{Man}_{\mathcal{N}}$, where the morphisms of $\mathcal{M}, \mathcal{N}$ are defined to be the morphisms of the structure $\mathcal{N}$-sheaves of $\mathcal{N}$-algebras.

$$\mathbf{Hom}_{\mathit{Man}_{\mathcal{N}}}(\mathcal{M}, \mathcal{N}) := [\mathcal{O}_{\mathcal{N}}, \mathcal{O}_{\mathcal{M}}]$$

$$\begin{array}{ccc} \varphi^{-1}U & \xrightarrow{\varphi} & U \\ \uparrow & & \uparrow \\ \varphi^{-1}V & \xrightarrow{\varphi} & V \end{array} \qquad \begin{array}{ccc} \mathcal{O}_{\mathcal{M}}(\varphi^{-1}(U)) & \longleftarrow & \mathcal{O}_{\mathcal{N}}(U) \\ \downarrow{\scriptstyle \text{restriction}} & & \downarrow{\scriptstyle \text{restriction}} \\ \mathcal{O}_{\mathcal{M}}(\varphi^{-1}(V)) & \longleftarrow & \mathcal{O}_{\mathcal{N}}(V) \end{array}$$

The **local coordinates** of an $\mathcal{N}$-manifold are the local coordinates of the body together with a homogeneous basis of the $\mathcal{N}$-vector space Ξ generating the local $\mathcal{N}$-algebra.

$$\left(x^1, \dots, x^m, \xi^1, \dots, \xi^n\right)$$

Remark 11. *The* ***degree*** *of $\mathcal{M}$ defines the highest degree of a local coordinate* ***deg***$\mathcal{M}$*. A $\mathcal{N}$-manifold is a tower of affine fibrations.*

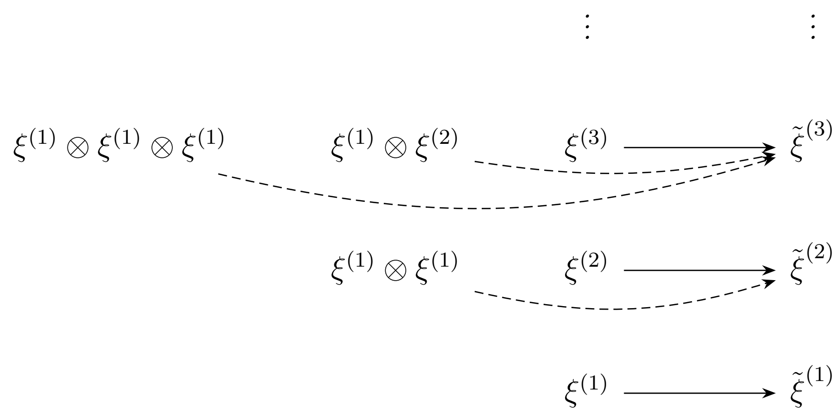
$$\mathcal{M}_0 \longleftarrow \mathcal{M}_1 \longleftarrow \mathcal{M}_2 \longleftarrow \cdots \longleftarrow \mathcal{M}_{\boldsymbol{deg}\mathcal{M}} = \mathcal{M}$$

The arrows in $\mathfrak{Man}_{\mathcal{N}}$ simply denote the morphisms of structure presheaves and the tower translates to a filtration

$$\mathcal{O}_{\mathcal{M}_0} \hookrightarrow \mathcal{O}_{\mathcal{M}_1} \hookrightarrow \mathcal{O}_{\mathcal{M}_2} \hookrightarrow \cdots \hookrightarrow \mathcal{O}_{\mathcal{M}_{\boldsymbol{deg}\mathcal{M}}} = \mathcal{O}_{\mathcal{M}}$$

consisting of inclusions of structure sheaves. Here, $\mathcal{M}_k$ is a $\mathcal{N}$-manifold of degree k and the $\mathcal{M}_0 \longleftarrow \mathcal{M}_1$ can be thought of as a vector bundle projection.

Let us clarify that "affine fibration" refers to the fact that the maps coincide with vector bundle projections, only the structure group is affine. If by $\xi^{(k)}$ we denote a local coordinate of degree k, then a local coordinate change has the following diagrammatic form:



Here, a full line stands for a $Gl(k)$ transformation, which can be thought of as a bundle isomorphism of a vector bundle corresponding to $\mathcal{M}_0 \longleftarrow \mathcal{M}_1$. A dashed line stands for an affine contribution proportional to a (graded-symmetric) tensor product of lower degree coordinates to the new $\tilde{\xi}$ coordinate. Degree 2 is thus the "lowest non-trivially affine degree".

Definition 13. A **graded derivation of degree** $k \equiv |D| \in \mathbb{Z}$ of a graded algebra $\mathcal{A}$ is a linear map D st.

$$D(fg) = D(f)g - (-1)^{k|f|} f D(g) \qquad \text{for all } f, g \in \mathcal{A}$$

A **graded derivation** of $\mathcal{A}$ is a $\mathcal{O}_{\mathcal{M}}$-linear combination of homogeneous graded derivations.

A local **tangent field** at U on a $\mathcal{N}$-manifold $\mathcal{M}$ is a graded derivation of the local $\mathcal{N}$-algebra $\mathcal{O}_{\mathcal{M}}(U)$ in the structure presheaf. Globally, a tangent field is a family of local tangent fields compatible with the restrictions; they form a (co)presheaf we will denote as $\mathfrak{X}_{\mathcal{M}}$.

Similarly to the case of ordinary manifolds (or supermanifolds), every tangent field is uniquely specified by its action on the local coordinates.

$$X^i := X\left(x^i\right), \qquad X^\mu := X\left(\xi^\mu\right)$$

This is captured by the lemma proved in greater generality by Vysoky [2021], Prop 4.14.

Lemma 12. *Local tangent fields are free $\mathcal{O}_{\mathcal{M}}$-modules generated by:*

$$\left\{ \frac{\partial}{\partial x^i} \right\}_{i=1}^{m} \quad \text{and} \quad \left\{ \frac{\partial}{\partial \xi^\mu} \right\}_{\mu=1}^{n}$$

Here, the partial derivative operators $\frac{\partial}{\partial x^i}$ are defined in the usual way and $\frac{\partial}{\partial \xi^\mu}$ analogously.

Every $\mathcal{N}$-manifold comes equipped with the following tangent field.

Definition 14. The **Euler field $\boldsymbol{\varepsilon}$** is locally defined to be:

$$\boldsymbol{\varepsilon} := \sum_{\mu=1}^{n} |\xi^\mu|\, \xi^\mu \frac{\partial}{\partial \xi^\mu}$$

The important property of the Euler field is that the homogeneous elements of the local $\mathcal{N}$-algebras of the structure presheaf are precisely its eigenvectors.

$$\boldsymbol{\varepsilon}(f) = kf \quad \Leftrightarrow \quad |f| = k$$

Thus if we denote the local $\mathcal{C}^\infty(U)$-module of k-eigenvectors of order k by $\mathcal{A}^k(U)$, the definition of a $\mathcal{N}$-manifold ensures it defines a sheaf $\mathcal{A}^k$ and we recover the local $\mathcal{N}$-algebra model as

$$\mathcal{O}_{\mathcal{M}}(U) = \bigoplus_{i=0}^{\infty} \mathcal{A}^i(U)$$

equipped with the original multiplication of sections. Let us stress that while the $\mathcal{N}$-algebras do not form a sheaf, the definition of a $\mathcal{N}$-manifold ensures each $\mathcal{A}^k$ does; this is precisely what we will need when

Remark 13. *Severa [2001] noted that a $\mathcal{N}$-manifold can be seen as a supermanifold with additional structure in the local model. Taking the odd generators ξ^μ as the generators of the local Grassmann algebra, we have a local model consisting of $\mathcal{C}^\infty(U)$ and $\wedge(\xi^{\text{odd}})$ which is now mixed with ξ^{even} by means of the graded-symmetric algebra.*

In the language of the Euler field, extending a supermanifold to a $\mathcal{N}$-manifold amounts to introducing the action of the the multiplicative semigroup $(\mathbb{R}, \times)$ by means of the Euler field;

$$f \longmapsto \lambda^{\boldsymbol{\varepsilon}(f)} f, \qquad \lambda \in \mathbb{R},$$

st. for $\lambda = -1$ it coincides with the parity shift

$$\Pi = (-1)^{\boldsymbol{\varepsilon}}.$$

Essential examples of $\mathcal{N}$-manifolds are shifted vector bundles with the fibre coordinates viewed as a homogeneous basis of the $\mathcal{N}$-space Ξ generating the local graded-symmetric algebra. More precisely, given a shifted vector bundle

$$E[\boldsymbol{k}] \longrightarrow M,$$

we think of $\Gamma(\wedge E^*)$ as of the structure $\mathcal{N}$-sheaf of a $\mathcal{N}$-manifold over the body M.

Definition 15. We say $(\mathcal{M}, E[\bullet])$ is a **split** $\mathcal{N}$-manifold if the collection of vector bundles $E[\bullet] = (E_i)_{i \in \mathcal{N}_0}$ is such that:

$$\mathbb{E}[\bullet] \equiv \bigoplus_i E_i[\boldsymbol{i}] = \mathcal{M}$$

The structure sheaf is given by exterior powers of the dual bundles. We refer to Bonavolontà and Poncin [2013] for more details and for the proof of the following theorem, which is central to the theory of $\mathcal{N}$-manifolds.

Theorem 14 (Batchelor's theorem for $\mathcal{N}$-manifolds)**.** *Every $\mathcal{N}$-manifold is (non-canonically) isomorhpic to a $\mathcal{N}$-manifold associated to graded vector bundle $\mathbb{E}[\bullet]$.*

Remark 15 (Are $\mathcal{N}$-manifolds well defined $\mathcal{N}$-sheaves?)**.** *In an arbitrary splitting, we can see that the sheaf conditions are given by the Serre-Swan theorem 6.*

Let us sketch the proof of locality of the stalk rings, which was brought to our attention at the Prague Mathematical Physics seminar in 2021 by Jan Vysoký.

The locality of the rings defined by the local models can be seen by inspecting the set of invertible germs $\mathfrak{U}$ in a stalk in $x \in M$. This encodes the potential non-locality of the ring. Since there are no coordinates of negative degrees, graded sections have no inversions and the set $\mathfrak{U}$ reduces to the set of germs of $\mathcal{C}^\infty(U)$ functions non-zero at x. One then constructs the ***Jacobson radical*** *$\mathfrak{J}$ composed of $\mathcal{O}_\mathcal{M}\big|_x$ excluding the set $\mathfrak{U}$. In other words, $\mathfrak{J}$ consists of sections* ***either vanishing at x or of degree*** *≥ 1.*

Now, we may use an essential observation: the ring $\mathcal{O}_\mathcal{M}\big|_x$ is local iff $\mathfrak{J}$ is an ideal. Clearly, for $\mathcal{N}$-manifolds, this is an ideal: multiplying sections either vanishing at x or of degree ≥ 1 by any section gives a section vanishing at x of degree ≥ 1. From this we see that any $\mathcal{N}$-manifold is automatically locally ringed. We refer to Vysoky [2021] for more[5] details.

[5] A lot more.

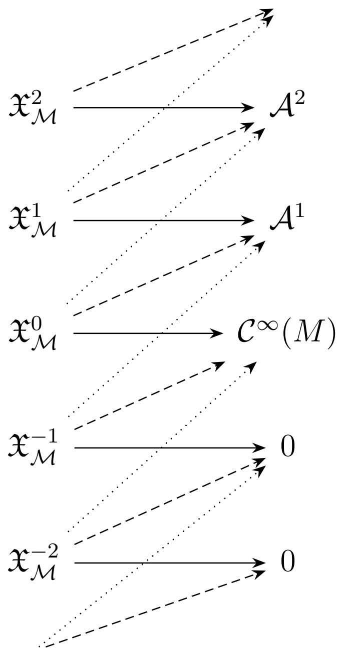
1.4.3 The Antitangent $\mathcal{N}$-Manifold

To talk about differential forms on $\mathcal{N}$-manifolds, we let the mechanisms of ordinary manifolds chart the course. Locally, we have a free $\mathcal{O}_{\mathcal{M}}(U)$-module of tangent fields $\mathfrak{X}_{\mathcal{M}}(U)$. The idea is to consider its *dual*, denoting the dual local frame by:

$$\left(\mathrm{d}x^1, \ldots, \mathrm{d}x^m, \mathrm{d}\xi^1, \ldots, \mathrm{d}\xi^n\right)$$

We need to be careful about what we mean by "the dual" in the graded case. To be able to assign non-trivial grading of the $\mathrm{d}\xi$ coordinates (now **not** in the sense of p in p-forms, but in the sense of grading of graded modules), we need to consider maps that *change* the degree. In other words, we will consider k**-shifting** morphisms, ie. morphisms of graded modules:

$$\mathfrak{X}_{\mathcal{M}}(U)[\boldsymbol{k}] \longrightarrow \mathcal{O}_{\mathcal{M}}(U)$$



A single differential 1-form decomposes into a collection $\left(\alpha_{(i)}\right)_{i\in\mathbb{Z}}$ of maps upon restriction onto $\mathfrak{X}^i_{\mathcal{M}}$, tangent fields of degree i. In the diagrammatic example, the *full* lines symbolize a 0-shifting differential form, the *dashed* lines a 1-shifting form and the *dotted* lines a 2-shifting form. Notice that since we exclude negative degrees in the structure $\mathcal{N}$-sheaf, the only way to make a differential form of degree k "structure-preserving" — in the category $\mathcal{M}od_{\mathcal{O}_{\mathcal{M}}}$ — is to let it vanish on all tangent fields of degree i st. $i + k < 0$.

Notice, however, that this is not a special property of differential forms — by definition, a local partial derivative field $\frac{\partial}{\partial \xi^\mu}$ is $(-|\xi^\mu|)$-shifting.

Definition 16. We define a k**-shifting differential** 1**-form** on $\mathcal{M}$ to be a morphism of functors $\mathcal{O}pen_M \longrightarrow \mathcal{G}r\mathcal{P}roj\mathcal{M}od_{\mathcal{O}_{\mathcal{M}}}$ st.

$$\alpha \in [\mathfrak{X}_{\mathcal{M}}[\boldsymbol{k}], \mathcal{O}_{\mathcal{M}}]$$

More concretely; it sends a local homogeneous tangent field X of degree $|X|$ to a section of $\mathcal{O}_{\mathcal{M}}(U)$ of degree $|X| + k$ for some integer $k \in \mathbb{Z}$ independent of $U \in \mathcal{O}pen_M$.

$$\mathfrak{X}^i_{\mathcal{M}} \longrightarrow \mathcal{A}^{i+k}$$

Naturally (or *naively*), for each coordinate ξ we might be tempted to define a $|\xi|$-shifting 1-form. Then the **naive local duality condition** is well-defined.

$$\mathrm{d}\xi^\mu \left(\frac{\partial}{\partial \xi^\nu}\right) = \delta^\nu_\mu$$

That is; since the degree of $\frac{\partial}{\partial \xi^\mu}$ is $-|\xi^\mu|$ and $\mathrm{d}\xi^\mu$ is $|\xi^\mu|$-shifting and thus eg. $\mathrm{d}\xi^1\left(\frac{\partial}{\partial \xi^1}\right)$ really yields a degree 0 function in $\mathcal{C}^\infty(M)$.

Now, we can define $\mathrm{d}\xi$'s as local coordinates of a new $\mathcal{N}$-manifold. For TM, we have $\mathcal{C}^\infty(TU) = \mathcal{C}^\infty(U) \otimes \mathrm{Span}\,(\mathrm{d}x, \ldots)$; we consider $T[\mathbf{1}]M \in \mathcal{M}an_{\mathcal{N}}$ such that for $\mathcal{M} = M$ an ordinary smooth manifold, they coincide.

To get the most out of this idea, we should ensure compatibility of the natural grading of the exterior algebra of local differential forms and the degree we are adding "artificially" in the context of $\mathcal{N}$-manifolds. The natural notion of a degree of a differential k-shifting 1-form is k (see the *naive* duality condition). Then, the degrees of $(\mathrm{d}x^1, \ldots, \mathrm{d}x^m, \mathrm{d}\xi^1, \ldots, \mathrm{d}\xi^n)$ are opposite to the degrees of:

$$\left(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^m}, \frac{\partial}{\partial \xi^1}, \ldots, \frac{\partial}{\partial \xi^m}\right)$$

Those, in turn, are defined to have the degree opposite to the degrees of the original coordinates $(x^1, \ldots, x^m, \xi^1, \ldots, \xi^n)$. In other words, we have:

$$\left|\mathrm{d}x^i\right| = \left|x^i\right|, \quad |\mathrm{d}\xi^\mu| = |\xi^\mu|$$

This way, we get $|\mathrm{d}x^i| = 0$ for all the differentials of the base coordinates. This is clearly not compatible with the exterior algebra grading: in the local $\mathcal{N}$-algebra, we have $\mathrm{d}x^i\,\mathrm{d}x^j = \mathrm{d}x^j\,\mathrm{d}x^i$, while the exterior product differs by a minus sign: $\mathrm{d}x^i \wedge \mathrm{d}x^j = -\,\mathrm{d}x^j \wedge \mathrm{d}x^i$.

The two products coincide if we *shift* the degree by 1. Therefore, we can capture the idea of differential forms by associating the following $\mathcal{N}$-manifold to a given $\mathcal{N}$-manifold.

Definition 17. The **antitangent $\mathcal{N}$-manifold $\mathcal{T}[\mathbf{1}]\mathcal{M}$** of a $\mathcal{N}$-manifold $\mathcal{M}$ is defined over the same body M, by associating to each set of local coordinates of $\mathcal{M}$; $(x^1, \ldots, x^m, \xi^1, \ldots, \xi^n)$, the local coordinates:

$$\left(x^1, \ldots, x^m, \xi^1, \ldots, \xi^n, \mathrm{d}x^1, \ldots, \mathrm{d}x^m, \mathrm{d}\xi^1, \ldots, \mathrm{d}\xi^n\right)$$

where we define the degrees as

$$\left|\mathrm{d}x^i\right| := 1, \quad |\mathrm{d}\xi^\mu| := |\xi^\mu| + 1$$

and the atlas is such that in the case of $T[\mathbf{1}]M \in \mathcal{M}an_{\mathcal{N}}$ the coincides with.

By **differential p-forms** we refer to sections of $\mathcal{T}[\mathbf{1}]M$ from the p-th symmetric power.

Notice, however, that the "naive local duality condition" cannot be satisfied now: the *differentials of coordinates* always shift the corresponding *partial derivative tangent fields* into $\mathcal{A}^1$ instead of $\mathcal{A}^0$, where the constant function 1 hidden in δ^i_j resides. This is a small price to pay for the model that unifies the graded-symmetric algebra with the "Cartan algebra". The solution is simple: We define the **contraction** $\iota_{\frac{\partial}{\partial \xi^\mu}}$ to be a derivation of $\mathcal{O}_{\mathcal{T}[\mathbf{1}]\mathcal{M}}$ of degree $(\xi^\mu - 1)$ st:

$$\iota_{\frac{\partial}{\partial \xi^\mu}}\,\mathrm{d}\xi^\nu = \delta^\mu_\nu$$

We may call this relation the **shifted local duality condition**.

Every antitangent $\mathcal{N}$-manifold $\mathcal{T}[\mathbf{1}]\mathcal{M}$ is equipped with the **de Rham tangent field** $\mathrm{d} \in \mathfrak{X}_{\mathcal{T}[\mathbf{1}]\mathcal{M}}$ defined locally as:

$$\mathrm{d} := \mathrm{d}x^i \frac{\partial}{\partial x^i} + \mathrm{d}\xi^\mu \frac{\partial}{\partial \xi^\mu}$$

The **graded commutator** of homogeneous tangent fields X, Y is defined as:

$$[X, Y] := X \circ Y - (-1)^{|X||Y|} Y \circ X$$

Now, we can define the **Lie derivative** $\mathcal{L}_X$ of a differential form.

$$\mathcal{L}_X := [\iota_X, \mathrm{d}] \equiv \iota_X \circ \mathrm{d} + (-1)^{|X|} \mathrm{d} \circ \iota_X$$

This way, we can easily recover the following identity of Cartan calculus; the Lie derivative is a de Rham chain map.

$$[\mathcal{L}_X, \mathrm{d}] = 0$$

Now, we can see that the $\boldsymbol{\varepsilon}$-eigenvalues of coordinate 1-forms:

$$\mathcal{L}_{\boldsymbol{\varepsilon}} (\mathrm{d}\xi^\mu) = \mathrm{d}\mathcal{L}_{\boldsymbol{\varepsilon}} \xi^\mu = |\xi^\mu| \, \mathrm{d}\xi^\mu$$

Now it makes sense to define *the degree k of a differential p-form* such that it coincides with both the $\boldsymbol{\varepsilon}$-eigenvalue and the original "naive" intuition for the grading of the coordinate 1-forms.

Definition 18. The **degree of a differential p-form** $\alpha \in \boldsymbol{\Omega}_{\mathcal{M}} := \mathcal{O}_{\mathcal{T}[\mathbf{1}]\mathcal{M}}$ is defined to be

$$k := |\alpha| - p,$$

where $|\alpha|$ is computed in the sense of degrees of sections of $\mathcal{O}_{\mathcal{T}[\mathbf{1}]\mathcal{M}}$.

1.4.4 $\mathcal{NQ}$-Manifolds are Lie n-Algebroids

We have built the theory of $\mathcal{N}$-manifolds as such, now we can start to construct additional structure. First, let us turn our attention to BRST-like differentials: we require $\mathcal{Q}$ to be odd, to square to zero and define an a priori non-trivial cohomology.

Remark 16 (Graded Frobenius Theorem)**.** *Let us first remark on the notion of* integrability *on $\mathcal{N}$-manifolds. Let $X \in \mathfrak{X}_{\mathcal{M}}$ be an odd tangent field. Then*

$$[X, X] = 0$$

if and only if X integrates into a "graded curve" with a graded "time parameter". For more details, we refer to Cattaneo and Schaetz [2011].

Definition 19. A tangent field $\mathcal{Q}$ on a $\mathcal{N}$-manifold is said to be **cohomological** if it satisfies the following axioms:

- *integrability:* $[\mathcal{Q}, \mathcal{Q}] = 0$
- *degree($\mathcal{Q}$) is 1:* $[\boldsymbol{\varepsilon}, \mathcal{Q}] = \mathcal{Q}$

Note that since in degree 1 $\mathcal{Q}$ is odd, a cohomological tangent field is always nilpotent:

$$0 = [\mathcal{Q}, \mathcal{Q}] = \mathcal{Q} \circ \mathcal{Q} - (-1)^1 \mathcal{Q} \circ \mathcal{Q} = 2\mathcal{Q} \circ \mathcal{Q}.$$

We conclude that any $\mathcal{NQ}$-manifold defines a cochain complex $\left(\mathcal{A}^k, \mathcal{Q}\right)$.

$$\mathcal{A}^0 \xrightarrow{\mathcal{Q}} \mathcal{A}^1 \xrightarrow{\mathcal{Q}} \mathcal{A}^2 \xrightarrow{\mathcal{Q}} \cdots$$

Definition 20. Morphisms of $\mathcal{NQ}$-manifolds are defined as morphisms of $\mathcal{N}$-manifolds which are also chain maps.

$$\begin{array}{ccccccc} \mathcal{A}^0 & \xrightarrow{\mathcal{Q}} & \mathcal{A}^1 & \xrightarrow{\mathcal{Q}} & \mathcal{A}^2 & \xrightarrow{\mathcal{Q}} & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \\ \mathcal{B}^0 & \xrightarrow{\mathcal{Q}} & \mathcal{B}^1 & \xrightarrow{\mathcal{Q}} & \mathcal{B}^2 & \xrightarrow{\mathcal{Q}} & \cdots \end{array}$$

Example 17. $(T[\mathbf{1}]M, \mathrm{d}\,)$ *has local coordinates* $(x^1, \ldots, x^n, \mathrm{d}x^1, \ldots, \mathrm{d}x^n)$ *and the canonical de Rham field* $\mathrm{d} = \mathrm{d}x^i \frac{\partial}{\partial x^i}$ *has degree 1 and is integrable:*

$$\mathrm{d}^2 = 0$$

Its cohomology coincides with the ordinary de Rham cohomology of M*.* $\mathcal{NQ}$*-maps* $T[\mathbf{1}]M \to T[\mathbf{1}]N$ *are such that they induce* 0 *maps in de Rham cohomology. These are the "small transformations".*

There is a generalization of the canonical $\mathcal{NQ}$-structure on TM given as:

$$(TM, \mathrm{d}\,\text{and}\,[\bullet, \bullet]_{Lie})$$

Namely, we consider a sheaf of Lie algebras $\mathcal{L}$ together with structure that models the "de Rham-closure / Lie-involutiveness" on M.

$$(\mathcal{L}, \text{models of integrability})$$

A **Lie algebroid** is a vector bundle $L \to M$ equipped with a sheaf of Lie algebras $\mathcal{L} \equiv \Gamma(L)$ equipped with the **anchor** map $\boldsymbol{\varrho}\left(\bullet\right) : \mathcal{L} \longrightarrow \Gamma(TM)$ st.

- the anchor is a Lie algebra homomorphism

$$\boldsymbol{\varrho}\left([X, Y]\right) = \left[\boldsymbol{\varrho}\left(X\right), \boldsymbol{\varrho}\left(X\right)\right],$$

- the anchor *extends the action of* $[\mathcal{L}, \bullet]$ *onto* $\mathcal{C}^\infty(M) \otimes \mathcal{L}$.

$$[X, fY] = f\,[X, Y] + \left(\boldsymbol{\varrho}\left(X\right) f\right) Y$$

The Frobenius's theorem in the relates Lie algebroid structures on a manifold and (generalized) foliations into which they integrate. We could say that Lie algebroids, since they turn out to be "tangent-like" sheaves that disjoint the underyling manifold into leaves of a (generalized) foliation, are *"systems of local models of integrability"*. Here, "systems" simply refers to *sheaves.*

Now we will sketch the construction of a *one-to-one* correspondence between (isomorphism classes of) **Lie algebroids** and (isomorphism classes of) **$\mathcal{NQ}$-manifolds of degree 1**. The theorem is the following one, we follow the proof sketched by Roytenberg [2002].

Theorem 18. *A split $\mathcal{N}$-Manifold of degree 1 defines a Lie algebroid iff it is an $\mathcal{NQ}$-manifold.*

Proof (Sketch). For $\boldsymbol{deg}\mathcal{M} = 1$, local homogeneous coordinates are of the form:

$$\left(x^1, \dots, x^n, \xi^1, \dots, \xi^m\right)$$

The complete tower of fibrations truncates after 2-terms.

$$\mathcal{M}_0 \longleftarrow \mathcal{M}_1$$

In the "odd fibre" the transformations are just a $Gl(n)$-action on the fibre coordinates (see remark 11). Therefore a splitting that the Batchelor's theorem 14 provides is of the form $\mathcal{M} = E[\mathbf{1}]$ and we recover a vector bundle over the body $M = \mathcal{M}_0$.

$$M \longleftarrow E[\mathbf{1}]$$

Note that $\mathcal{O}_{\mathcal{M}}$ coincides with the sheaf of "Grassman algebras of odd functionals" on E, ie.

$$\mathcal{O}_{\mathcal{M}} \overset{\text{splitting}}{=} \wedge(E[\mathbf{1}]^*)$$

Now, we can write down a general form of $\mathcal{Q} \in \mathfrak{X}^1_{\mathcal{M}}$ in the chosen homogeneous coordinates:

$$\mathcal{Q} = \boldsymbol{\varrho}^i_\mu \xi^\mu \frac{\partial}{\partial x^i} + \boldsymbol{C}_{\mu\nu}{}^\sigma \xi^\mu \xi^\nu \frac{\partial}{\partial \xi^\sigma}$$

$\mathcal{Q}^2 = 0$ implies that for $(\boldsymbol{\varrho}(e_\mu)\, x^i) \coloneqq \boldsymbol{\varrho}^i_\mu$ and $[e_\mu, e_\nu]_{\boldsymbol{C}} \coloneqq \boldsymbol{C}_{\mu\nu}{}^\sigma e_\sigma$ we have a Lie algebra on $\mathcal{A}^1$ and the anchor $\boldsymbol{\varrho}(\bullet) : \mathcal{A}^1 \longrightarrow \Gamma(TM)$ satisfies the Lie algebroid axioms.

Conversely; given a Lie algebroid, we construct the $\mathcal{N}$-manifold $L[-\mathbf{1}] \in \mathcal{M}an_{\mathcal{N}}$. The structure constants $\boldsymbol{\varrho}^i_\mu$ and $\boldsymbol{C}_{\mu\nu}{}^\sigma$ define $\mathcal{Q}$ uniquely (in a chosen chart) by the non-degeneracy of the Poisson bracket. [6] □

Remark 19. *Compare $\mathcal{Q}$ to the Slavnov operator in remark 1, see remark 4 for a "translation" between the BV-BRST indeces and the $\mathcal{N}$-indeces. Note that the structure constants satisfy the axioms of a NQ manifold of degree 1 iff they define a principal action of a Lie group with $\boldsymbol{C}$ being the structure constants of the corresponding Lie algebra $\mathfrak{g}$.*

Remark 20. *$\mathcal{NQ}$-manifolds over a single point body*

$$\{*\} \longleftarrow \mathcal{M}_1$$

*reduce to L_∞ algebras. Thus $\mathcal{NQ}$-manifolds are really to L_∞-algebras what Lie algebroids are to Lie algebras, which corresponds to the case $\boldsymbol{deg}\mathcal{M} = 1$. This demonstrates that "**Lie n-algebroids**" is indeed a pretty good name for $\mathcal{NQ}$-manifolds. We refer to Bonavolontà and Poncin [2013] for more details.*

[6] Note that the anchor $\boldsymbol{\varrho}(\bullet)$ extends the new Lie bracket $\left[\mathcal{A}^1, \bullet\right]_C$ from $\mathcal{A}^1$ to $\mathcal{A}^0 \otimes \mathcal{A}^1$. But this space contains the whole homogeneous basis of $\mathcal{O}_{\mathcal{M}}$, thus the tangent field $\mathcal{Q}$ is specified uniquely.

1.4.5 $\mathcal{NQP}$-Manifolds are Poisson-Lie n-Algebroids

Now we aim to introduce the antibracket or equivalently a graded symplectic structure. Morevoer, we will arrive at the notion of antighost and the Hamiltonian generator Θ (corresponding to the BV actional S) satisfying the classical master equation.

$$\{\Theta, \Theta\} = 0$$

A (homogeneous) **symplectic structure of degree** k is a 2-form $\Omega \in \mathcal{O}_{\mathcal{T}[\mathbf{1}]\mathcal{M}}$ of degree k, non-degenerate with respect to the contraction[7].

Darboux charts exist via Batchelor's theorem. In a Darboux chart,

$$\Omega = \mathrm{d}p_i \, \mathrm{d}x^i + \mathrm{d}e_\mu \, \mathrm{d}\xi^\mu,$$

where the degrees of the conjugate coordinates add up precisely to $|\Omega|$ in each term.

Lemma 21. *The degree of an $\mathcal{NP}$-manifold cannot exceed the degree of Ω.*

Proof. In a Darboux chart, it is clear from non-degeneracy. □

Recall that a Darboux chart is defined precisely so that the data of a Poisson structure reduce to the following relations.

$$\left\{x^i, p_j\right\} = \delta^i_j$$

We define the **graded Poisson bracket** folowing Cattaneo and Schaetz [2011].

$$\{A, B\} \coloneqq (-1)^{|A|+1} X_A(B)$$

Here, X_A is the unique tangent fields st. $\iota_{X_A}\Omega = \mathrm{d}A$ holds, we say A is a **Hamiltonian** function of X_A. A symplectic structure of degree k defines a Poisson structure $\{\bullet, \bullet\}$ of degree $-k$. This means that $\{f, \bullet\}$ is a derivation of $\mathcal{O}_\mathcal{M}$ of degree $|f| - k$. In a Darboux chart $(x^i, p_j, \xi^\mu, e_\nu)$, we have:

$$\left\{x^i, p_j\right\} = \delta^i_j$$
$$\{\xi^\mu, e_\nu\} = \delta^\mu_\nu$$

Those relations are extended by definition above onto *polynomial* sections of $\mathcal{O}_\mathcal{M}$ by a graded Leibniz rule. It is straightforward to show that the graded Poisson bracket of degree $-k$ satisfies graded analogues of the properties of ordinary Poisson algebras.

Lemma 22 (Properties of a graded Poisson bracket)**.**

$$\{A, BC\} = \{A, B\}\, C + (-1)^{|A|(|B|-k)} B\, \{A, C\}$$
$$\{A, B\} = -(-1)^{(|A|-k)(|B|-k)}\, \{B, A\}$$
$$0 = (-1)^{(|A|-k)(|B|-k)}\, \{A, \{B, C\}\} + \text{c.p.}$$

[7] Contraction with a tangent section $X \in \mathfrak{X}_\mathcal{M}$, ie. non-degeneracy wrt. the map $\iota_X \in \mathfrak{X}_{\mathcal{T}[\mathbf{1}]\mathcal{M}}$.

Example 23. *For $T^*[\mathbf{1}]M$ with a Darboux chart (x^i, X_j), the properties of the antibracket define precisely the* ***Schouten bracket*** *on tangent multivector fields.*

Theorem 24. *A $\mathcal{N}$-manifold equipped with a symplectic form of degree 1 is symplectomorphic to $T^*[\mathbf{1}]M$ with the canonical* ***Schouten bracket*** *on $\mathcal{O}_{T^*[\mathbf{1}]M}$.*

Proof. By lemma 21, the manifold is at most of degree 1. We decompose the action of the Poisson bracket on the $\mathcal{N}$-sheaf of $\mathcal{N}$-algebras $\mathcal{O}_{\mathcal{M}} = \mathcal{A}^0 \oplus \mathcal{A}^1$.

- $\{\mathcal{A}^0, \mathcal{A}^0\} = 0$, on $\mathcal{A}^0$ the Poisson structure coincides with the zero map.
- $\{\mathcal{A}^1, \mathcal{A}^1\} \subset \mathcal{A}^1$, on $\mathcal{A}^1$ the Poisson structure defines a Lie algebra structure.
- $\{\mathcal{A}^1, \mathcal{A}^0\} \subset \mathcal{A}^0$ provides an action of the $\mathcal{A}^1$ on $\mathcal{A}^0 = \mathcal{C}^\infty(M)$. Since $\{\mathcal{A}^1, \bullet\}$ is a derivation, it gives rise to a map $\mathcal{A}^1 \longrightarrow \Gamma(TM)$. Non-degeneracy and the properties of the Poisson bracket ensure it is an isomorphism of (sheaves of) Lie algebras.

Now for $T^*[\mathbf{1}]M \in \mathcal{M}an_{\mathcal{N}}$ we have $\mathcal{A}^1 \simeq \Gamma(TM)$ and the Schouten bracket is the unique extension of the canonical Lie bracket on $\Gamma(TM)$ to $\mathcal{O}_{T^*[\mathbf{1}]M}$. □

Definition 21. A $\mathcal{NQP}$**-manifold** of degree k is a $\mathcal{N}$-manifold equipped with a symplectic structure Ω of degree k and a cohomological field $\mathcal{Q}$ st. $\mathcal{L}_{\mathcal{Q}}\Omega = 0$.

We present the following calculations of Cattaneo and Schaetz [2011].

Lemma 25. *A symplectic 2-form Ω homogeneous of degree $k \geq 1$ is exact.*

Proof. The statement follows from the following usage of the Cartan identity.

$$k\Omega = \mathcal{L}_\varepsilon \Omega = \mathrm{d}\iota_\varepsilon \Omega$$

Now we see that $\Omega = \mathrm{d}\big(\frac{1}{k}\iota_\varepsilon\Omega\big)$. □

Lemma 26. *On a $\mathcal{NQP}$-manifold, $\mathcal{Q}$ is Hamiltonian.*

Proof. Define a candidate for a Hamiltonian as $H \coloneqq \iota_\varepsilon \iota_{\mathcal{Q}}\Omega$. Now:

$$\mathrm{d}H = \mathrm{d}\iota_\varepsilon\iota_{\mathcal{Q}}\Omega = \mathcal{L}_\varepsilon\iota_{\mathcal{Q}}\Omega - \iota_\varepsilon\,\mathrm{d}\iota_{\mathcal{Q}}\Omega = \mathcal{L}_\varepsilon\iota_{\mathcal{Q}}\Omega - \iota_\varepsilon\mathcal{L}_{\mathcal{Q}}\Omega = \mathcal{L}_\varepsilon\iota_{\mathcal{Q}}\Omega = (k + |\mathcal{Q}|)\iota_{\mathcal{Q}}\Omega$$

Thus the function $\frac{1}{k+1}H$ is a Hamiltonian function of $\mathcal{Q}$. □

This lemma provides us with a "Hamiltonian generator" Θ of the transformation given by the $\mathcal{Q}$-structure, analogously to the case of ordinary Hamiltonian geometry.

$$\mathcal{Q}(A) = \{\Theta, A\}$$

In other words, we have a candidate for the finite-dimensional analogue of the classical master actional.

Lemma 27. *The integrability condition of $\mathcal{Q}$ translates to the classical master equation for Θ.*

$$[\mathcal{Q}, \mathcal{Q}] = 0 \quad \Longleftrightarrow \quad \{\Theta, \Theta\} = 0$$

Proof. We have seen earlier that

$$[\mathcal{Q}, \mathcal{Q}](f) = 2\mathcal{Q}^2(f),$$

while the graded Jacobi identity gives

$$\mathcal{Q}^2(f) = \{\Theta, \{\Theta, f\}\} = \pm\{f, \{\Theta, \Theta\}\} - \{\Theta, \{\Theta, f\}\},$$

thus we have $[\mathcal{Q}, \mathcal{Q}](f) = \pm\{f, \{\Theta, \Theta\}\}$ and the statement follows. □

For the next theorem, we refer to theorem 4.11. of Cattaneo and Schaetz [2011], originally due to A. Schwartz.

Theorem 28. *Isomorphism classes of $\mathcal{NQP}$-Manifolds of degree 1 are in a 1-to-1 correspondence with isomorphism classes of Poisson Manifolds.*

Proof. The Hamiltonian is of the form $\Pi = \frac{1}{2}\Pi^{ij}(x)X_iX_j$, the master equation on $\mathcal{O}_\mathcal{M} \simeq \mathcal{O}_{T^*[\mathbf{1}]M}$ (see theorem 30).

$$\{\Pi, \Pi\} = [\Pi, \Pi]_{\text{Schouten}} = 0$$

ensures $\Pi^{ij}(x)$ defines a Poisson tensor on M. □

1.5 Degree 2

In degree 2, $\mathcal{N}$-manifolds start to transform under affine transformations; they thus present, in some sense, the first non-trivial class of examples of $\mathcal{N}$-manifolds (see remark 11). We will show how the additional BV-like structure corresponds to pseudo-Euclidean vector bundles and Courant algebroids.

1.5.1 Pseudo-Euclidean Vector Bundles & ghosts⊕antighosts

Locally, in a chosen chart of an $\mathcal{NP}$-manifold, a Poisson structure of degree -2 defines a non-degenerate symmetric matrix of $\mathcal{O}_\mathcal{M}$ functions by its restriction on $\mathcal{A}^1$.

$$\mathrm{g}^{\mu\nu}(x) \coloneqq \{\xi^\mu, \xi^\nu\}$$

We may call it the **ghost matrix**. The matrix has degree $2 - k$, thus for $k = 2$, $\mathrm{g}^{\mu\nu}(x)$ is a matrix of $\mathcal{C}^\infty(U)$ functions.

Note that if we consider the inverse matrix and denote it by $\mathrm{g}_{\mu\nu}(x)$ we have an "index-lifting mechanism" on the non-trivially graded variables. We may call the local sections $e_\nu \coloneqq \mathrm{g}_{\nu\mu}\xi^\mu$ the "**antighosts**". Note that in the following, we will find it useful to consider the following scaling of antighosts: $\theta_\nu \coloneqq \frac{1}{2}\mathrm{g}_{\nu\mu}\xi^\mu$. We may call $\mathrm{g}_{\mu\nu}(x)$ the **antighost matrix**.

Remark 29 (Minimal Symplectic Realization)**.** *Let us consider a ghost matrix on a split $\mathcal{NP}$-manifold $E[\mathbf{1}]$ (of degree 1) equipped with a symplectic structure of degree 2.*

Firstly, note that on $(E \oplus E^*)[\mathbf{1}]$, a ***ghost-antighost***[8] *coordinate system defines a presymplectic*[9] *structure of degree 2.*

$$\frac{1}{2}\mathrm{g}_{\mu\nu}\,\mathrm{d}\xi^{\nu}\,\mathrm{d}\xi^{\mu}$$

The embedding $E[\mathbf{1}] \hookrightarrow (E \oplus E^*)[\mathbf{1}]$ by $\xi \longmapsto \xi \oplus \frac{1}{2}\mathrm{g}(\xi)$ is an isometry wrt. the natural pairing $\langle \xi \oplus \theta, \eta \oplus \phi \rangle = \xi(\phi) + \eta(\theta)$ on $(E \oplus E^*)[\mathbf{1}]$.

Furthermore, we can see that for $\mathrm{g}_{\mu\nu}(x)$ constant wrt. every chart $\{x^i\}_i$, the "ghost-antighost generating set" extends into a Darboux chart on $T^*[\mathbf{2}]E[\mathbf{1}]$.

$$\Omega = \mathrm{d}p_i\,\mathrm{d}x^j + \mathrm{d}\theta_\mu\,\mathrm{d}\xi^\mu,$$

which than reduces to:

$$\Omega = \mathrm{d}p_i\,\mathrm{d}x^j + \frac{1}{2}\mathrm{g}_{\mu\nu}\,\mathrm{d}\xi^{\nu}\,\mathrm{d}\xi^{\mu}$$

The Darboux chart then comprises only of the three types of variables — (x^i, p_j, ξ^μ) — since θ_μ's (or e's) can be obtained from ξ^μ's. Given x^i and ξ^μ come from $E[\mathbf{1}]$, the coordinates can only have the following degrees so that $T^*[\mathbf{2}]E[\mathbf{1}]$ is a $\mathcal{NP}$-manifold: $|x^i| = 0$, $|p_j| = 2$, $|\xi^\mu| = 1$, $|e_\mu| = |\theta_\mu| = 1$. The nonzero brackets are:

$$\left\{x^i, p_j\right\} = \delta^i_j$$
$$\{e_\mu, e_\nu\} = \mathrm{g}_{\mu\nu} \qquad \{\xi^\mu, e_\nu\} = \delta^\mu_\nu \qquad \{\xi^\mu, \xi^\nu\} = \mathrm{g}^{\mu\nu}$$

It is an important observation that $T^*[\mathbf{2}]E[\mathbf{1}]$ defines a ***minimal symplectic realization*** *of the Poisson manifold $(E \oplus E^*)[\mathbf{1}]$. That is, it is a symplectic manifold with a Poisson map into $(E \oplus E^*)[\mathbf{1}]$ with the minimal possible dimension.*

We say a vector bundle is **pseudo-Euclidean** if it is equipped with a non-degenerate fibre-wise pairing $\langle \bullet, \bullet \rangle$. We will sketch the proof of the following theorem due to Roytenberg [2002].

Theorem 30. *A split $\mathcal{N}$-manifold of degree 2 defines a unique pseudo-Euclidean vector bundle iff it is a $\mathcal{NP}$-manifold.*

Proof. First, assume we are given a split $\mathcal{NP}$-manifold $\mathcal{M}$.
By lemma 21, the degree of $\mathcal{M}$ is at most 2. We decompose the action of the Poisson bracket on the $\mathcal{N}$-sheaf of $\mathcal{N}$-algebras $\mathcal{O}_\mathcal{M} = \mathcal{A}^0 \oplus \mathcal{A}^1 \oplus \mathcal{A}^2$.

- $\{\mathcal{A}^0, \mathcal{A}^0\} = 0$.
- $\{\mathcal{A}^1, \mathcal{A}^0\} = 0$.

[8] We can think of $E \oplus E^*$ as a $\mathfrak{ghosts} \oplus \mathfrak{antighosts}$ bundle, more in example 38.

[9] A presymplectic structure is a degenerate symplectic structure — we revoke the non-degeneracy axiom.

- $\{\mathcal{A}^1, \mathcal{A}^1\} \subset \mathcal{A}^0$, on $\mathcal{A}^1$ the Poisson bracket locally defines the ghost matrix $\mathbb{g}^{\mu\nu} \equiv \{\xi^\mu, \xi^\nu\}$. This corresponds to a pseudo-Euclidean structure on E for a splitting of $\mathcal{M}$ st. $\mathcal{A}^1 = \Gamma(E[\mathbf{1}]^*)$. $\mathcal{M}_2 \to \mathcal{M}_1$ is a morphism of $\mathcal{N}$-manifolds, it is degree preserving and it defines a Poisson map from a symplectic manifold $(\mathcal{M}_2, \{\bullet, \bullet\})$ to a Poisson manifold $(\mathcal{M}_1, \{\bullet, \bullet\}\big|_{\mathcal{O}_{\mathcal{M}_1}})$. By dimensionality, it is a *minimal symplectic realization.*

- $\{\mathcal{A}^2, \mathcal{A}^0\} \subset \mathcal{A}^0$ provides an action of $\mathcal{A}^2$ on $\mathcal{A}^0$. Note that by the Leibniz rule, $\{\xi^\mu\xi^\nu, f\} = 0 + 0 = 0$. Ie.

$$\left\{\mathcal{A}^1\mathcal{A}^1, \mathcal{A}^0\right\} = 0.$$

So the action of $\mathcal{A}^2$ on $\mathcal{A}^0$ is given completely by the canonical relations $\{x^i, p_j\} = \delta^i_j$. We get this exact sequence of $\mathcal{N}$-sheaves of graded modules.

$$0 \longrightarrow \mathcal{A}^1\mathcal{A}^1 \longrightarrow \mathcal{A}^2 \longrightarrow \Gamma(TM) \longrightarrow 0$$

- $\{\mathcal{A}^2, \mathcal{A}^1\} \subset \mathcal{A}^1$ provides a Hamiltonian action of $\mathcal{A}^2$ on $\mathcal{A}^1$.

- $\{\mathcal{A}^2, \mathcal{A}^2\} \subset \mathcal{A}^2$ defines a Lie algebra structure on $\mathcal{A}^2$.

Now, the Serre-Swan theorem provides a vector bundle $\mathbb{A}$ st. $\Gamma(\mathbb{A}) = \mathcal{A}^2$. The map $\mathbb{A} \to TM$ is a morphism of Lie algebroids. By the above, $\mathbb{A}$ fits into the following exact sequence; the **Atiyah exact sequence of** $(E, \langle\bullet, \bullet\rangle)$.

$$0 \longrightarrow \wedge^2(E^*) \longrightarrow \mathbb{A} \longrightarrow TM \longrightarrow 0$$

$\mathbb{A}$ can be thought of as the bundle of symmetries of $(E, \langle\bullet, \bullet\rangle)$ with $\wedge^2(E^*)$ acting trivially on the base manifold. Thus all the structure of $\mathcal{M}$ corresponds to the structure of a pseudo-Euclidean vector bundle and its natural symmetries.

$$(M, E, \mathbb{A})$$

Now, conversely, assume we are given a pseudo-Euclidean vector bundle $E \to M$. We define $\mathcal{M}_1 := E[\mathbf{1}]$ and the pairing defines a graded Poisson bracket on its sections. Now we just define $\mathcal{M}_2 = \mathcal{M} = \mathbb{E}[\bullet]$, as a minimal symplectic realization by pulling back the minimal symplectic realization along the isometric embedding from remark 29.

$$\begin{array}{ccc} \mathbb{E}[\bullet] & \dashrightarrow & T^*[\mathbf{2}]E[\mathbf{1}] \\ \downarrow & & \downarrow \\ E[\mathbf{1}] & \longrightarrow & (E \oplus E^*)[\mathbf{1}] \end{array}$$

We have the split $\mathcal{NP}$-manifold:

$$M \longleftarrow E[\mathbf{1}] \longleftarrow \mathbb{E}[\bullet]$$

□

Remark 31 (Affine transformations of the momenta)**.** *On a $\mathcal{N}$-manifold, coordinates of degree 2 transform non-linearly, see remark 11. In the case of $\mathcal{NP}$-manifolds of degree 2, the p coordinates acquire an affine transformation term. We refer to Roytenberg [2002], specifically the text after example 3.4. Consider a canonical transformation of the form $f + \{H, f\}$, where $H \in \mathcal{A}^2$ can be thought of as a section of the Atiyah bundle $\mathbb{A}$, see proof of theorem 30. Is is straightforward to show that for*

$$H = v^i(x)p_i + \frac{1}{2}\xi^\mu h_{\mu\nu}\xi^\nu,$$

a general canonical transformation reads:

$$\begin{aligned} q^i &= q^i(q') \\ \xi^\mu &= T^\mu_{\mu'}(q')\xi^{\mu'} \\ p_j &= \frac{\partial q^{i'}}{\partial q^i}p_{i'} + \frac{1}{2}\xi^{\mu'}\frac{\partial T^\mu_{\mu'}}{\partial x^i}\mathrm{g}_{\mu\nu}T^\nu_{\nu'}\xi^\nu \end{aligned}$$

Note that for $\mathrm{g}_{\mu\nu} = const.$, we have $\mathrm{g}_{\mu'\nu'} = const.$

1.5.2 Poisson-Lie 2-Algebroids are Courant-Lie 2-Algebroids

It is time to add the *differential* structure and move to $\mathcal{NQP}$-manifolds of degree 2, ie. Poisson-Lie 2-algebroids. In degree 2, the Hamiltonian generator Θ is necessarily of degree 3. Locally, it has the following form.

$$\Theta = \boldsymbol{\varrho}^i_\mu\xi^\mu p_i - \frac{1}{6}\boldsymbol{C}_{\mu\nu\sigma}\xi^\mu\xi^\nu\xi^\sigma$$

The $-\frac{1}{6}$ coefficient is an arbitrary choice which will be useful later. Notice the analogy with the structure constants of a $\boldsymbol{deg} = 1$ $\mathcal{NQP}$-manifold and. With the structure constants of a BRST operator, see remark 1 and 4.

Now, we follow Roytenberg [1999] and introduce the *derived brackets*.

Definition 22. Given a Θ and $\{,\}$, the **derived bracket** $[\![,]\!]$ is defined as follows.

$$[\![f, g]\!] \coloneqq \{\{f, \Theta\}, g\}$$

Derived brackets satisfy some very useful properties, namely a version of a *graded Leibniz-Jacobi identity* and their *graded-symmetric part is "infinitesimal"*. See lemma 3.5.1. of Roytenberg [1999]. These properties resemble some of the properties of *Courant algebroids*, which we will find useful in the next section.

Remark 32. *Note that in degree 1, the derived bracket*

$$[\![f, g]\!]_\Pi = \{\{f, \Pi\}, g\}$$

on T^*M defines precisely the Poisson bracket associated to the canonical symplectic structure on T^*M.

Definition 23. A **Courant algebroid** is given by the data

$$(E, \langle \bullet, \bullet \rangle, [\![\bullet, \bullet]\!], \boldsymbol{\varrho}(\bullet))$$

Here, $E \to M$ is vector bundle, $\langle \bullet, \bullet \rangle$ is a non-degenerate fibre-wise pairing on E, $[\![\bullet, \bullet]\!]$ is a linear bracket on the sections of E, $\boldsymbol{\varrho}(\bullet)$ is a bundle map $E \to TM$. They are required to satisfy the following axioms for $\forall e, e_1, e_2 \in \Gamma(E)$.

1. $[\![e, [\![e_1, e_2]\!]]\!] = [\![[\![e, e_1]\!], e_2]\!] + [\![e_1, [\![e, e_2]\!]]\!]$
2. $\boldsymbol{\varrho}([\![e_1, e_2]\!]) = [\boldsymbol{\varrho}(e_1), \boldsymbol{\varrho}(e_2)]$
3. $[\![e_1, fe_2]\!] = f[\![e_1, e_2]\!] + (\boldsymbol{\varrho}(e_1) \cdot f) e_2$
4. $\langle e, [\![e_1, e_2]\!] + [\![e_2, e_1]\!] \rangle = \boldsymbol{\varrho}(e) \cdot \langle e_1, e_2 \rangle$
5. $\boldsymbol{\varrho}(e) \cdot \langle e_1, e_2 \rangle = \langle [\![e, e_1]\!], e_2 \rangle + \langle e_1, [\![e, e_2]\!] \rangle$

Remark 33. *Note that we can define the **coanchor** $\boldsymbol{\varrho}^* : T^*M \longrightarrow E$ st.*

$$\boldsymbol{\varrho}(e_\mu) \cdot x^i = \left\langle e_\mu, \boldsymbol{\varrho}^* \left(\mathrm{d}x^i\right)\right\rangle$$

It can easily be shown that $\boldsymbol{\varrho} \circ \boldsymbol{\varrho}^* = 0$. Note that this makes a Courant algebroid into a chain complex:

$$0 \longrightarrow T^*M \xrightarrow{\boldsymbol{\varrho}^*} E \xrightarrow{\boldsymbol{\varrho}} TM \longrightarrow 0$$

*If the chain reduces to an exact sequence, the Courant algebroid E is said to be **exact**.*

Example 34. *Given a smooth manifold $M \in \mathcal{Man}$, one defines the **standard Courant algebroid** as the vector bundle given by $TM \oplus T^*M$ equipped with the natural fibre-wise pairing of split signature;*

$$\langle X \oplus \alpha, Y \oplus \beta \rangle \coloneqq \alpha(Y) + \beta(X),$$

*an anchor given by the canonical projection $TM \oplus T^*M \longrightarrow TM$ and the **Dorfmann bracket** defined as:*

$$[\![X \oplus \alpha, Y \oplus \beta]\!] \coloneqq [X, Y] \oplus (\mathcal{L}_X \beta - \iota_Y(\mathrm{d}\alpha))$$

It can be checked this satisfied the axioms of a Courant algebroid. Moreover, it is an exact Courant algebroid.

A **Courant-Lie 2-algebroid** is a $\mathcal{N}$-manifold of degree 2 equipped with the maps

$$\begin{aligned} \langle \bullet, \bullet \rangle &: \mathcal{A}^1 \times \mathcal{A}^1 \longrightarrow \mathcal{A}^0 \\ [\![\bullet, \bullet]\!] &: \mathcal{A}^1 \times \mathcal{A}^1 \longrightarrow \mathcal{A}^1 \\ \boldsymbol{\varrho}(\bullet) &: \mathcal{A}^1 \times \mathcal{A}^0 \longrightarrow \mathcal{A}^0 \end{aligned}$$

defined on local sections of the structure sheaf, satisfying the same axioms as the operations of a Courant algebroid above. We fix this terminology to make it clear we are formally always working within the $\mathcal{N}$-world in the following calculations. We state the following lemma due to Roytenberg [2002] and sketch its proof[10] to give at least a rough idea of how the structure constants recombine into the Courant algebroid axioms.

[10] We have found no detailed proof in the literature.

Lemma 35. *Let* $(\mathcal{M}, \Theta, \{\bullet, \bullet\})$ *be a split Poisson-Lie 2-algebroid and* $[\![\bullet, \bullet]\!]$, $\boldsymbol{\varrho}(\bullet) \cdot \bullet$ *the derived brackets associated restricted onto* $\mathcal{A}^1 \times \mathcal{A}^1$ *and* $\mathcal{A}^1 \times \mathcal{A}^0$ *respectively. Then:*

$$\begin{aligned}\{\Theta, \Theta\} &= \left(\left(\boldsymbol{\varrho} \circ \boldsymbol{\varrho}^* \,\mathrm{d}x^j\right) \cdot x^i\right) p_i p_j \\ &\quad + \left(\left[\boldsymbol{\varrho}(e_\mu), \boldsymbol{\varrho}(e_\nu)\right] - \boldsymbol{\varrho}\left([\![e_\mu, e_\nu]\!]\right)\right) \cdot x^i p_i \xi^\mu \xi^\nu \\ &\quad + \frac{1}{12} \left\langle [\![[\![e_\mu, e_\nu]\!], e_\rho]\!] + [\![e_\nu, [\![e_\mu, e_\rho]\!]]\!] - [\![e_\mu, [\![e_\nu, e_\rho]\!]]\!], e_\sigma \right\rangle \xi^\mu \xi^\nu \xi^\rho \xi^\sigma\end{aligned}$$

Proof (Sketch). Locally, the master equation (which takes place in $\mathcal{A}^4$) decomposes into three independent equations proportionate to either pp, $p\xi\xi$ or $\xi\xi\xi\xi$; the three ways one can construct a term of degree 4 in the given chart.

$$\{\Theta, \Theta\} = \{\Theta, \Theta\}\big|_{pp} + \{\Theta, \Theta\}\big|_{p\xi\xi} + \{\Theta, \Theta\}\big|_{\xi\xi\xi\xi}$$

The $\{\Theta, \Theta\}$ expression reads:

$$\begin{aligned}\{\Theta, \Theta\} &= \left\{\boldsymbol{\varrho}^i_\mu \xi^\mu p_i - \frac{1}{6}\boldsymbol{C}_{\mu\nu\sigma}\xi^\mu\xi^\nu\xi^\sigma, \boldsymbol{\varrho}^j_\alpha \xi^\alpha p_j - \frac{1}{6}\boldsymbol{C}_{\alpha\beta\gamma}\xi^\alpha\xi^\beta\xi^\gamma\right\} = \\ &= \left\{\boldsymbol{\varrho}^i_\mu \xi^\mu p_i, \boldsymbol{\varrho}^j_\alpha \xi^\alpha p_j\right\} - \frac{1}{6}\left\{\boldsymbol{\varrho}^i_\mu \xi^\mu p_i, \boldsymbol{C}_{\alpha\beta\gamma}\xi^\alpha\xi^\beta\xi^\gamma\right\} \\ &\quad - \frac{1}{6}\left\{\boldsymbol{C}_{\mu\nu\sigma}\xi^\mu\xi^\nu\xi^\sigma, \boldsymbol{\varrho}^j_\alpha \xi^\alpha p_j\right\} + \frac{1}{36}\left\{\boldsymbol{C}_{\mu\nu\sigma}\xi^\mu\xi^\nu\xi^\sigma, \boldsymbol{C}_{\alpha\beta\gamma}\xi^\alpha\xi^\beta\xi^\gamma\right\}\end{aligned}$$

- **The** pp **term**
  Only the term with two anchors which contains two p's contributes. From it, the graded Leibniz rule then picks out only the single term containing the bracket $\{\xi^\mu, \xi^\alpha\}$.

  $$\{\Theta, \Theta\}\big|_{pp} = \{\xi^\mu, \xi^\alpha\}\, \boldsymbol{\varrho}^i_\mu p_i \boldsymbol{\varrho}^j_\alpha p_j$$

  The term can easily be seen to equal to $\left(\left(\boldsymbol{\varrho} \circ \boldsymbol{\varrho}^* \,\mathrm{d}x^j\right) \cdot x^i\right) p_i p_j$.

- **The** $p\xi\xi$ **term**
  This terms is linear in p, therefore the only the following terms contribute.

  $$\begin{aligned}\{\Theta, \Theta\}\big|_{p\xi\xi} &= \boldsymbol{\varrho}^i_\mu \left\{p_i, \boldsymbol{\varrho}^j_\alpha\right\} \xi^\mu \xi^\alpha p_j + \boldsymbol{\varrho}^j_\alpha \left\{\boldsymbol{\varrho}^i_\mu, p_j\right\} \xi^\mu \xi^\alpha p_i \\ &\quad - \frac{1}{6}\boldsymbol{\varrho}^i_\mu p_i \left\{\xi^\mu, \xi^\alpha\xi^\beta\xi^\gamma\right\} \boldsymbol{C}_{\alpha\beta\gamma} - \frac{1}{6}\boldsymbol{C}_{\mu\nu\rho}\left\{\xi^\mu\xi^\nu\xi^\rho, \xi^\alpha\right\} \boldsymbol{\varrho}^i_\alpha p_j\end{aligned}$$

  The first two terms give a Lie derivative of tangent fields, each one given by a single $\boldsymbol{\varrho}$ symbol, this gives $\left[\boldsymbol{\varrho}(e_\mu), \boldsymbol{\varrho}(e_\nu)\right] \cdot x^i p_i \xi^\mu \xi^\nu$. The rest gives a term with the anchor of a Courant bracket; it corresponds to $-\boldsymbol{\varrho}\left([\![e_\mu, e_\nu]\!]\right) \cdot x^i p_i \xi^\mu \xi^\nu$.

- **The** $\xi\xi\xi\xi$ **term**
  Here, we get contributions of terms where p's are either absent or "consumed" by the Poisson bracket.

  $$\begin{aligned}\{\Theta, \Theta\}\big|_{\xi\xi\xi\xi} &= -\frac{1}{6}\boldsymbol{\varrho}^i_\mu \left\{p_i, \boldsymbol{C}_{\alpha\beta\gamma}\right\} \xi^\mu\xi^\alpha\xi^\beta\xi^\gamma - \frac{1}{6}\left\{\boldsymbol{C}_{\mu\nu\rho}, p_j\right\} \boldsymbol{\varrho}^j_\alpha \xi^\mu\xi^\nu\xi^\rho\xi^\alpha \\ &\quad + \frac{1}{36}\boldsymbol{C}_{\mu\nu\rho}\boldsymbol{C}_{\alpha\beta\gamma}\left\{\xi^\mu\xi^\nu\xi^\rho, \xi^\alpha\xi^\beta\xi^\gamma\right\}\end{aligned}$$

The last term breaks into a combination of the structure constants $\boldsymbol{C}$ corresponding precisely to the Leibniz-Jacobi identity as it is familiar from the case of a Lie algebra. The first two terms correspond to the action of the derived bracket on the structure constants, which is implicitely present in terms like this one: $[\![[\![e_\mu, e_\nu]\!], e_\rho]\!]$.

□

We continue with Roytenberg [2002], stating the central theorem of this chapter.

Theorem 36. *A split $\mathcal{NQP}$-manifold of degree 2 is a Poisson-Lie 2-algebroid iff it is a Courant-Lie 2-algebroid.*

Proof. Given either $\Theta = \boldsymbol{\varrho}^i_\mu \xi^\mu p_i - \frac{1}{6}\boldsymbol{C}_{\mu\nu\sigma}\xi^\mu\xi^\nu\xi^\sigma$ or the derived brackets $[\![\bullet, \bullet]\!]$, $\boldsymbol{\varrho}(\bullet)\cdot\bullet$, we require:

$$[\![e_\mu, e_\nu]\!] \overset{!}{=} \boldsymbol{C}_{\mu\nu}{}^{\rho} e_\rho$$
$$\boldsymbol{\varrho}(e_\mu)\cdot x^i \overset{!}{=} \boldsymbol{\varrho}^i_\mu$$

- **Poisson-Lie ⇒ Courant-Lie**
  We define the Courant bracket to be the derived bracket wrt. Θ restricted on $\mathcal{A}^1 \times \mathcal{A}^1$ and the anchor to be the derived bracket restricted onto $\mathcal{A}^1 \times \mathcal{A}^0$. We refer to calculations carried out explicitely by Roytenberg [1999]: Axioms 1. and 4. of a Courant Algebroid (see definition 23) are provided by the general properties of derived brackets in lemma 3.7.1. Proof of theorem 3.7.3. contains the derivation of axioms 2., 3. and 5. from the classical master equation.

- **Courant-Lie ⇒ Poisson-Lie**
  By the non-degeneracy of the Poisson bracket, locally, the relations defining the derived brackets fix Θ uniquely st. $\boldsymbol{\varrho}(e_\mu)\cdot x^i = \boldsymbol{\varrho}^i_\mu$ and $\boldsymbol{C}_{\mu\nu\rho} = \langle [\![e_\mu, e_\nu]\!], e_\rho\rangle$. It can be directly checked that Θ is defined globally using canonical transformations from remark 31. Note that the affine term vanishes thanks to the fact that $\mathrm{g}_{\mu\nu} = const. \Rightarrow \mathrm{g}_{\mu'\nu'} = const.$, using the chain rule. The classical master equation is satisfied by lemma 35 and remark 33.

□

Corollary 37. *Isomorphism classes of Poisson-Lie 2-algebroids are in a 1-to-1 correspondence with isomorphism classes of Courant algebroids.*

Example 38 (BRST Charge)**.** *Following example 4.9 of Roytenberg [2002], we recover a BV-BRST model (see definition 1) given by a Lie algebra $\mathfrak{g} \equiv \mathfrak{ghost}$ acting on a manifold M. Consider the following trivial bundle:*

$$\mathbb{E}[\bullet] = (\mathfrak{ghost} \oplus \mathfrak{ghost}^*)[\mathbf{1}] \times T^*[\mathbf{2}]M$$

We have the following "BRST charge"

$$\Theta = \xi^\mu \boldsymbol{\varrho}^i_\mu p_i - \frac{1}{2}\xi^\mu\xi^\nu \boldsymbol{C}_{\mu\nu}{}^{\sigma}\theta_\sigma$$

where $\boldsymbol{\varrho}_\mu = \boldsymbol{\varrho}^i_\mu(x)\frac{\partial}{\partial x^i}$ is a generator of the Lie algebra $\mathfrak{ghost}$-action and $\boldsymbol{C}$ its structure constants.

2. Gauge Fixing & The Wehrheim-Woodward Category

It was noted by Weinstein [1971] that the rather restrictive class of "the obvious morphisms" of symplectic manifolds — symplectomorphisms — can be extended by introducing the so called *Lagrangian relations*. A symplectomorphism is a smooth map that preserves the non-degenerate symplectic form and thus must be non-degenerate itself. There are, however, relevant relationships between symplectic manifolds that cannot be described by diffeomorphisms. Those include *symplectic reductions*, originally described by Marsden and Weinstein [1974].

This construction has been adapted to the case of Courant algebroids, in most detail by Vysoky [2020]. We will introduce the appropriate notion of relations of $\mathcal{NP}$ and $\mathcal{NQP}$-manifolds that coincides with relations of Courant algebroids in degree 2, following the ideas sketched by Severa [2001]. Furthermore, to obtain a proper categorial picture, we will extend relations into *generalized correspondences* following the ideas of Wehrheim and Woodward [2007] summarized by Weinstein [2010].

Example 39 (Gauge fixing)**.** *First, let us mention a motivating example of a Lagrangian relation. The equation*

$$\frac{\partial \mathcal{F}}{\partial \phi} = \phi^*$$

familiar from the BV formalism, see reamrk 3, fixes a submanifold that is clearly Lagrangian. We may thus view the choice of a Lagrangian submanifold as a generalization of the gauge fixing process.

2.1 The Courant Algebroid Category

We will follow Vysoky [2020], who provides a very precise treatment of relations of Courant algebroids. Let us recall the data of a Courant algebroid;

$$(E, \langle \bullet, \bullet \rangle, [\![\bullet, \bullet]\!], \varrho(\bullet))$$

a vector bundle $E \longrightarrow M$, a non-degenerate pairing $\langle \bullet, \bullet \rangle$ of arbitrary signature and the anchor defining a "bracket homomorphism" from E to TM satisfying the Lie algebroid anchor axioms. We fix the following terminology.

A subbundle $L \subseteq E$ is a **subbundle of E supported on a submanifold** $S \subseteq M$ if L is a vector subbundle of the restricted vector bundle $E\big|_S$. We denote the $\langle \bullet, \bullet \rangle$-orthogonal complement by $L^\perp$. We say L is **isotropic** if for any two sections $f, g \in \Gamma(E)$ the pairing vanishes, $\langle \psi, \phi \rangle = 0$. A subbundle L is **compatible with the anchor** if $\varrho(L) \subseteq TS$.

Lemma 40. *Let a subbundle $L \subseteq E$ supported on $S \hookrightarrow M$ satisfy $L \neq E\big|_S$ and $L \neq 0$. Then L and $\mathcal{L}^\perp$ are compatible with the anchor.*

To talk about "subbundles over a submanifold", we denote the submodule of sections that restrict to L over S.

$$\Gamma(E; L) \coloneqq \left\{ A \in \Gamma(E) \ \mid A\big|_e S \in \Gamma(L) \right\}$$

A subbundle L is **involutive** if

$$[\![\Gamma(E; L) \,, \Gamma(E; L)]\!] \subseteq \Gamma(E; L) \,.$$

It is a very pleasant fact[1] that L is involutive *iff it is locally involutive* on every element of an arbitrary open cover. This means we need only to check involutivity locally.

Definition 24. A **Dirac structure** of a Courant algebroid $E \to M$ **supported on** $S \hookrightarrow M$ is a subbundle L st:

- L is isotropic,
- L is involutive,
- $L^\perp$ is compatible with the anchor.

Example 41. *We refer to Gualtieri [2004] for examples of Dirac structures on the standard courant algebroid $TM \oplus T^*M$ which correspond to symplectic, Poisson or complex structures on M as well as their generalizatins and interpolations.*

2.1.1 Dirac Relations

Remark 42. *In the rest of this text, we will adapt the convention suggested eg. by Baez and Huerta [2011]. In diagrams, we will draw arrows **from the right to the left**. Now the composition notation $f \circ g$ has the same order of f and g as the diagram.*

$$\bullet \xleftarrow[f]{} \bullet \xleftarrow[g]{} \bullet$$

Moreover, we denote:

$$\mathbf{Hom}_{\mathcal{C}}(A, A') \equiv \mathcal{C}(A', A)$$

We follow Vysoky [2020].

Remark 43. *Take a pair of Courant algebroids, $E \to M$ and $E' \to M'$. Now, if $L \subseteq E$ and $L' \subseteq E'$ are involutive structures over S and S' respectively,*

$$L' \times L \subseteq E' \times E$$

is an involutive structure supported on $S' \times S$. This is the content of proposition XXX of Vysoky [2020].

The prove this, we only need to check that the Dirac structure naturally constructed "pair-wise" truly defines a Dirac structure. Note that this is generally not true for maximally isotropic subbundles usually considered in the literature.

[1] See Vysoky [2020].

Definition 25 (Dirac Relations)**.** Let $E \to M$ and $E' \to M'$ be Courant algebroids. A **Dirac relation** from E to E' denoted

$$E' \overset{}{\underset{R}{\dashleftarrow}} E$$

is a Dirac structure $R \subseteq \overline{E'} \times E$ supported on $S \subseteq M' \times M$.

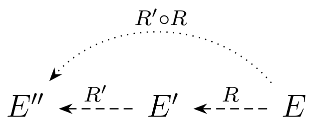
Remark 44. *The* ***diagonal relation*** $\Delta_E \subseteq \overline{E} \times E$ *is defined by the diagonal embedding of* E *into the Cartesian product. It is easy to check that it is indeed a Dirac structure: the sign flip in* $\overline{E}$ *provides isotropy, compatibility with the anchor can be seen eg. by from the fact that* $\Delta_\bullet$ *and* $\varrho(\bullet)$ *"commute" and involutivity is, similarly, inherited pair-wise.*

The composition $R' \circ R$ of Dirac relations $R' : E'' \dashleftarrow E'$ and $R : E' \dashleftarrow E$ is defined as the set:

$$R' \circ R := \left\{(e'', e) \in \overline{E''} \times E \mid (e'', e') \in R', \; (e', e) \in R \text{ for some } e' \in E'\right\}$$

Clearly, $R' \circ R \in \mathcal{Rel}(E'', E)$ The hard question reads:

"*Is a composition of Dirac relations a Dirac relation?*"



The question is adressed in great detail by Vysoky [2020]. In general, the answer is *no*, the resulting set might even fail to have the structure of a smooth submanifold.

To find the right "composability conditions", it is useful to decompose the definition of composition into two sequential geometric constructions.

1. We define the **diagonal concatenation** of relations.

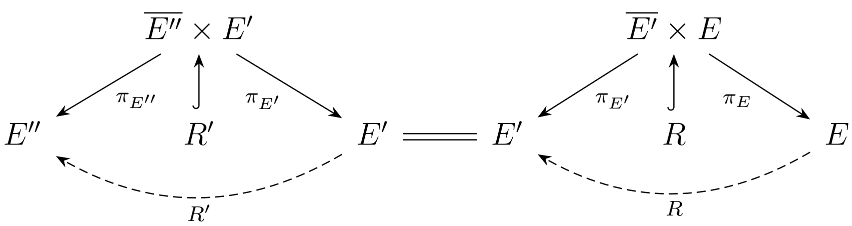
   $$R' \diamond R = (R' \times R) \cap \left(\overline{E''} \times \Delta_{E'} \times E\right)$$

   Ie. we take the fibre product $R' \diamond R = R' \times_{\Delta_{E'}} R$ It consists of the data of R' and R in the form of the pairs of pairs:

   $$((e'', e'), (e', e))$$

   $$E'' \dashleftarrow E' \qquad\qquad E' \dashleftarrow E$$

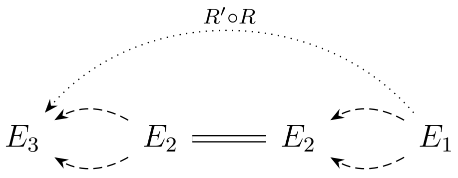
   encoded as a composition of morphisms *all the way* from E to E'', glued along E' by the diagonal relation (the identity).



2. A composition is then defined to be the projection of a diagonal concatenation forgetting the gluing data (the choice of the element $e_2 = e_2'$).

$$R' \circ R = p_{\diamond \to \circ} \left(R' \diamond R\right)$$

where $p_{\diamond \to \circ}$ is the projection onto the first and the fourth element of the quadruples in $\left(\overline{E''} \times \Delta_{E'} \times E\right)$.



This setting provides one with geometric objects to carry the right kind of "composability conditions". Similarly to the original works of Weinstein [1971], these will be some incarnation of the *transversality* conditions.

Definition 26. Let $S, S' \subseteq M$ be submanifolds. We say S and S' **intersect cleanly** in M and write

$$S' \overset{*}{\cap} S$$

if, for $\forall x \in S \cap S'$, we have:

$$T_x \left(S' \cap S\right) = T_x S' \cap T_x S$$

In other words, the two submanifolds locally look like intersecting vector spaces.

Definition 27. We say R' and R

$$E'' \overset{}{\underset{R'}{\longleftarrow}} E' \underset{R}{\longleftarrow} E$$

compose cleanly and write

$$R' \circ R = R' \circledast R$$

if the following conditions are satisfied.

1. $(R' \times R) \overset{*}{\cap} \left(\overline{E''} \times \Delta_{E'} \times E\right)$

2. $p_{\diamond \to \circ}$ is a *surjective submersion.*

Remark 45. *Let us say a bit more on those two conditions.*

- *Condition* 1. *ensures that the diagonal concatenation of* R' *and* R *— the fibred product* $R' \diamond R = (R' \times R) \cap \left(\overline{E''} \times \Delta_{E'} \times E\right)$ *— forms a vector bundle.*

- *Provided condition* 1. *holds, condition* 2. *ensures that* $R' \circ R \subseteq R' \diamond R$ *is a vector subbundle.*

Again, for more details, one may consult Vysoky [2020].

The conditions are thus purely on the level of vector bundles. We need not mention the Courant algebroid structure; neither the pairing, nor the bracket and the anchor. The following theorem due to Vysoky [2020], however, shows that no other obstructions arise if we are asking the “composition vector bundle” to carry induced “Dirac data”.

Theorem 46. *If two Dirac relations compose cleanly, their composition defines a Dirac relation.*

$$E_3 \xleftarrow[R']{} E_2 \xleftarrow[R]{} E_1, \qquad E_3 \xleftarrow{R' \circledast R} E_1$$

The diagonal relation can be shown to compose cleanly with every Dirac relation and to act like an identity morphism:

$$R \circledast \Delta_\bullet = R$$
$$\Delta_\bullet \circledast R = R$$

Note that the $\circledast$ operation is also associative whenever a double clean composition is defined.

We have thus arrived at a “category” with a weakened composition axiom: not all morphisms (going *into* and *out of* the same object, respectively) can be composed. We have a “category” of Courant algebroids with morphisms being Dirac relations which compose into another Dirac relation if they compose cleanly. We will denote this collection of objects and morphisms as:

$$\mathscr{CourAlgRel}$$

Example 47 (Dirac points). *Note that every Dirac structure $L \subseteq E$ supported on $S \subseteq M$ defines a Dirac correspondence from the singleton.*

$$E \xleftarrow[L]{} \{*\}$$

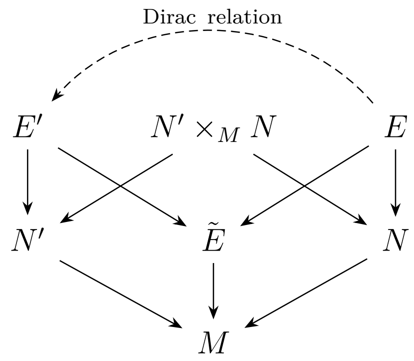
We may borrow the terminology from supergeometry (and algebraic geometry), Caston and Fioresi [2011] and call such relation a ***Dirac point****.*

Remark 48. *The most natural class of Dirac relations is given by graphs of bundle maps. Restricting the “***Hom** *sets” to graphs of bundle maps, we force the cleannes conditions of definition and obtain a well-defined category $\mathscr{CourAlgGr}$.* [2]

An elementary example of a functor with values in $\mathscr{CourAlgGr}$ is the ***Dorfmann functor****. It takes an ordinary manifold M into the category of vector bundles by the tangent functor $T : M \mapsto TM$ and then to the standard Courant algebroid $TM \oplus T^*M$.*

[2] Note that what we denote by $\mathscr{CourAlgRel}$ is by Vysoky [2020] denoted by $\overline{\mathscr{CAlg}}$, the category given by graphs of bundle maps $\mathscr{CourAlgGr}$ is denoted by $\mathscr{CAlg}$.

Example 49. *Let us mention only in a diagrammatic manner the example of a Dirac relation given by the so called* Poisson-Lie T-duality*, we refer to Bugden [2019] for a thorough treatment. In the diagram, we have the following special case.*



N and N' are total spaces of principal bundles with M being a quotient they share, the bundle maps to $\tilde{E}$ are given by pullbacks. If we are able to lift the Lie group action to Courant algebroids, given appropriate technical conditions, E' and E can be seen to be non-trivially Dirac-related. This relation cannot be described by a bundle map.

2.1.2 Generalized Dirac Correspondences

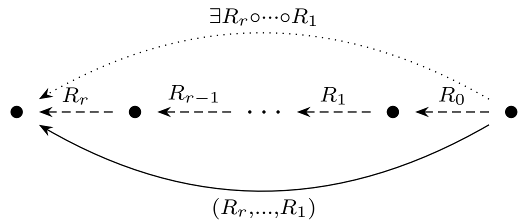
We follow Weinstein [2010] inspired by Wehrheim and Woodward [2007].

$$(R_r, \ldots, R_1)$$

Definition 28. A **(generalized) Dirac correspondence** is an equivalence class of sequences of *Dirac* relations of the form

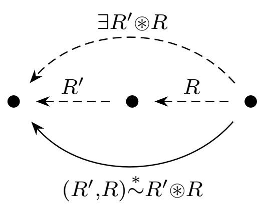
$$(R_r, \ldots, R_1)$$

composable in $\mathcal{Rel}$ wrt. the equivalence relation $\stackrel{*}{\sim}$. We will denote such sequences by a full line.



The equivalence $\stackrel{*}{\sim}$ is defined as follows.

$$(R', R) \stackrel{*}{\sim} R' \circ R \quad \text{iff} \quad R' \circ R = R' \circledast R.$$



That is, a 2-term sequence (R', R) is identified with the 1-term sequence $(R' \circledast R)$.

The **composition of correspondences** is defined by formal concatenation:

$$(R_s, \ldots, R_{r+1})\,(R_r, \ldots, R_1) \coloneqq (R_s, \ldots, R_1)$$

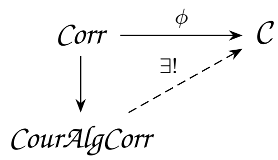
The empty sequence ( ) is equivalent to the identity relation $\Delta_\bullet$. We call the resulting category the **Wehrheim-Woodward category of Courant Algebroids**.

$$\mathcal{C}our\mathcal{A}lg\mathcal{C}orr$$

Remark 50 (Universal property of $\mathcal{C}our\mathcal{A}lg\mathcal{C}orr$)**.** *By construction, the $\mathcal{C}our\mathcal{A}lg\mathcal{C}orr$ is the minimal one with the given objects such that for any functor ϕ defined on all correspondences st.*

$$\phi\,(\mathcal{R}' \circledast \mathcal{R}) = \phi(\mathcal{R}') \circ \phi(\mathcal{R})$$

factors uniquely through $\mathcal{C}our\mathcal{A}lg\mathcal{C}orr$. In other words, we have the following diagram in the (quasi)category of categories $\mathcal{C}at$.



Here, $\mathcal{C}orr$ denotes the Wehrheim-Woodward category built (over the same class of objects) from the relations in $\mathcal{R}el$ (here without taking any equivalence classes).

2.2 The $\mathcal{NQP}$-Category

Again, the obvious morphisms of $\mathcal{NQP}$-manifolds being diffeomorphisms are too restrictive; the do not desribe Courant algebroid relations or quasi-isomorphisms (isomorphisms on cohomology) of the $\mathcal{NQ}$-chain complexes. We will adapt the constructions of Weinstein [2010] and Vysoky [2020] to the case of $\mathcal{NP}$-manifolds. Furthermore, we will show that in degree 2, we can define appropriate relations that extend the relationship of Courant algebroids and Poisson-Lie 2-Algebroids into a categorial equivalence.

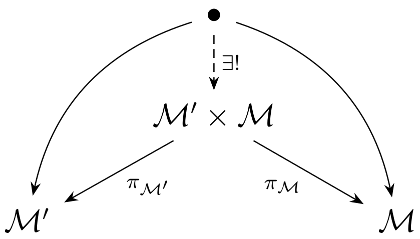
2.2.1 Lagrangian $\mathcal{N}$-Relations

A **$\mathcal{N}$-submanifold** of $\mathcal{M}$ is a $\mathcal{N}$-manifold $\mathcal{L}$ equipped with a monomorphism i embedding $\mathcal{L}$ into $\mathcal{M}$.

$$(i : \mathcal{L} \hookrightarrow \mathcal{M}) \in \mathcal{M}an_{\mathcal{N}}(\mathcal{L}, \mathcal{M})$$

In a splitting, a $\mathcal{N}$-submanifold is given by a subbundle $L \subseteq E$ supported on a submanifold $S \hookrightarrow M$ st. the inclusion map $L \hookrightarrow E$ is a degree preserving map.

We consider the **product** in $\mathcal{M}an_{\mathcal{N}}$ over the product manifold body $M' \times M$. $\mathcal{O}_{\mathcal{M}'\times\mathcal{M}}$ is a $\mathcal{N}$-sheaf over $\mathcal{C}^\infty(M') \otimes \mathcal{C}^\infty(M)$, the needed products of sheaves $\mathcal{A}^i$ are defined stalk-wise. It satisfies the usual universal property described by a diagram in $\mathcal{M}an_{\mathcal{N}}$.



Note that here, the canonical projection maps (such as all the others) are *degree preserving*. The existence of a product of graded manifolds was proved by Vysoky [2021].

A **Lagrangian $\mathcal{N}$-submanifold** of a $\mathcal{NP}$-manifold $(\mathcal{M}, \Omega)$ is a $\mathcal{N}$-submanifold $j : \mathcal{L} \hookrightarrow \mathcal{M}$ st.

$$\Omega\big|_{\mathcal{L}} = 0$$

with the *maximal dimension* st. this condition might be satisfied.

Here, the restriction is to be understood as the pull-back $j^*(\Omega)$. Moreover, we fix the following notation for a "symplectic sign flip":

$$\big(\,\overline{\mathcal{M}}\;,\;\Omega\,\big) \coloneqq (\;\mathcal{M}\;,\;-\Omega\;)$$

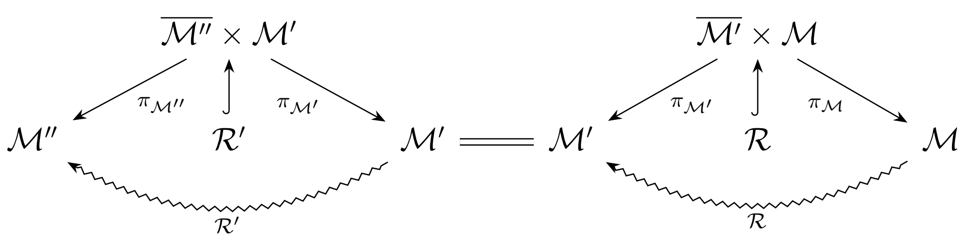
Definition 29. Let $\mathcal{M} = (M, \Omega)$ and $\mathcal{M}' = (M', \Omega')$ be two $\mathcal{NP}$-manifolds. A **(Lagrangian) $\mathcal{N}$-relation** from $\mathcal{M}$ to $\mathcal{M}'$ is a (Lagrangian) $\mathcal{N}$-submanifold

$$\mathcal{R} \hookrightarrow \overline{\mathcal{M}'} \times \mathcal{M}$$

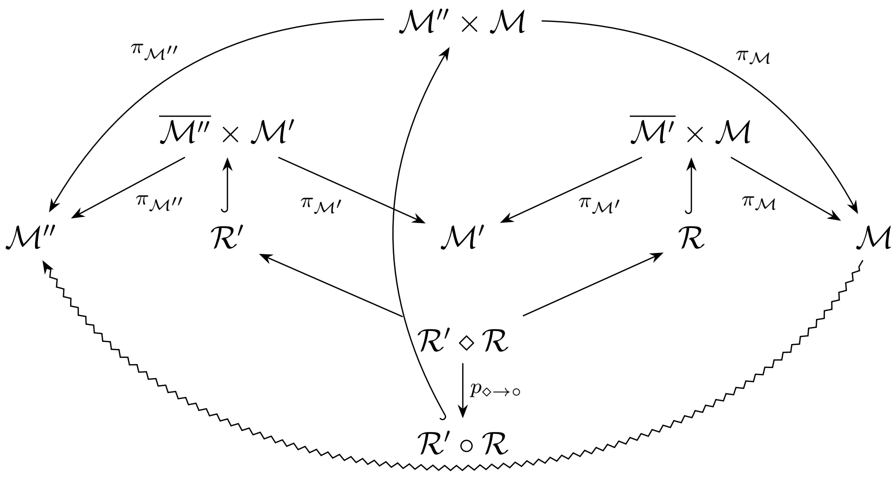
over a body $S \hookrightarrow M' \times M$. We denote Lagrangian $\mathcal{N}$-relations by "squiggly " lines.

$$\mathcal{M}' \overset{\mathcal{R}}{\leftsquigarrow} \mathcal{M}$$

The **diagonal concatenation** $\mathcal{R}' \diamond \mathcal{R}$ of $\mathcal{N}$-relations is defined as a fibre product $\mathcal{M}'' \times_{\mathcal{M}'} \mathcal{M}$ in a setting analoguous to the constuction of Dirac relations.

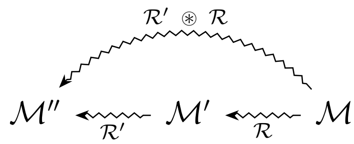


The projection $p_{\diamond \to \circ}$ is precisely the map in $\mathcal{M}an_{\mathcal{N}}$ that makes the maps going from $\mathcal{R}' \diamond \mathcal{R}$ to $\mathcal{M}''$ and $\mathcal{M}$ in the following diagram commute.



Note that the "inner butterfly" subdiagram commutes automatically from definition of $\mathcal{R} \diamond \mathcal{R}$. The conditions of **clean composition** $\mathcal{R}' \circledast \mathcal{R}$ of $\mathcal{N}$-relations can always be checked in a chosen splitting $\mathbb{E}[\bullet] = \bigoplus_i E_i[\boldsymbol{i}]$ of the correspondence space. The task reduces to defining a composition of a collection of the $E_i[\boldsymbol{i}]$ shifted bundles st. the resulting relations are degree-preserving (here, in the sense of graded vector bundles). Ie. we translate the problem to the language of graded components of the relation, $\mathcal{L}_k$ For $\mathcal{L}_k$, we may repeat the construction of Dirac relations. (Recall that we only needed to view the Dirac structure as a vector subbundle, forgetting about the extra structure.) Moreover, the Lagrangian property is completely independent of the grading and works just as in the case of ordinary symplectic manifolds, see Weinstein [2010]. Therefore, we may use theorem 46 for $\mathcal{L}_k$'s and obtain its analogue for $\mathcal{NP}$-manifolds.

Theorem 51. *A clean composition of Lagrangian $\mathcal{N}$-relations is a Lagrangian $\mathcal{N}$-relation.*



We denote the collection of $\mathcal{N}$-manifolds together with $\mathcal{N}$-relations $\mathfrak{Man}_{\mathcal{N}}\mathfrak{Rel}$ and the collection of $\mathcal{NP}$-manifolds together with Lagrangian $\mathcal{N}$-relations by $\mathfrak{Man}_{\mathcal{NP}}\mathfrak{Rel}$.

2.2.2 Λ-Relations of $\mathcal{NQP}$-Manifolds

In this chapter, we elaborate on som ideas of Severa [2001]. The following definition and this entire section is inspired by Grützmann [2010].

Definition 30. A **Λ-structure** of a $\mathcal{NQP}$-manifold[3] $(\mathcal{M}, \Theta)$ is a Lagrangian $\mathcal{N}$-submanifold $\mathcal{L} \hookrightarrow \mathcal{M}$ st.

$$\Theta\big|_{\mathcal{L}} = 0$$

[3]Here, we will not write out the symbols for the symplectic structures explicitely.

Remark 52 (Degree 1 & Coisotropic Submanifolds)**.** *A submanifold $S \hookrightarrow M$ of a Poisson manifold $(M, \{\bullet, \bullet\}_\Pi)$ is* ***coisotropic*** *if the* ***ideal of functions vanishing on S*** *denoted by $\mathcal{I}(S)$ forms a Lie subalgebra:*

$$\{\mathcal{I}(S), \mathcal{I}(S)\}_\Pi \subseteq \mathcal{I}(S)$$

following Severa [2001], we find that $\mathbf{\Lambda}$-structures in $\mathcal{NQP}$-manifolds of degree 1 (which can be split as $\mathcal{M} = T^*[\mathbf{1}]M$) correspond to conormal bundles of coisotropic submanifolds $S \hookrightarrow M$. For more on such structures, we refer to Schätz [2009]. In what follows, will use the same strategy in degree 2.

For a submanifold $\iota : S \hookrightarrow M$, we define the **conormal bundle** $\boldsymbol{N}^*S \subseteq T^*M$ of a submanifold to be the (fibre-wise) dual space to:

$$\boldsymbol{N}_{\iota(x)}S := \frac{T_{\iota(x)}M}{T_xS}$$

In other words, it is a submanifold locally generated by the coordinates of L and the **normal momenta**, which do not vanish under the quotient.

For a split $\mathcal{NQP}$-manifold of degree 2, ie. a Courant-Lie 2-algebroid (see theorem 36), we define the **conormal $\mathcal{N}$-submanifold $\boldsymbol{\mathcal{N}}^*L$** associated to a subbundle $L \subseteq E$ over $S \hookrightarrow M$ by:

$$\boldsymbol{\mathcal{N}}^*L := \boldsymbol{N}^*[\mathbf{2}]L[\mathbf{1}]$$

$$(L \subseteq E) \longmapsto (\boldsymbol{\mathcal{N}}^*L \subseteq \mathcal{M})$$

Generally, in $\mathcal{M}an_{\mathcal{N}}$, (fibre-wise) quotient bundles translate to (stalk-wise) quotient structure $\mathcal{N}$-sheaves.

Theorem 53 (Grützmann [2010])**.** *L is a Dirac structure iff $\boldsymbol{\mathcal{N}}^*(L)$ is a $\mathbf{\Lambda}$-structure.*

Proof. We need to show $\boldsymbol{\mathcal{N}}^*L$ is Lagrangian and that $\Theta\big|_{\boldsymbol{\mathcal{N}}^*L} = 0$.

- Locally, we can choose a trivialization st:

  $$\boldsymbol{\mathcal{N}}^*L = L[\mathbf{1}] \oplus \boldsymbol{N}^*_{\text{base}}[\mathbf{2}]S \oplus \boldsymbol{N}^*_{\text{fibre}}[\mathbf{2}]L[\mathbf{1}]$$

  $\boldsymbol{N}^*_{\text{base}}[\mathbf{2}]S$ and $\boldsymbol{N}^*_{\text{fibre}}[\mathbf{2}]L[\mathbf{1}]$ are by construction Lagrangian in $T^*[\mathbf{2}]M$ and in a fibre of $T^*[\mathbf{2}]L[\mathbf{1}]$ respectively. Now since all the "momenta" in the $\boldsymbol{N}^*$ subspaces are are normal to L and the dimensions are just right, the triple sum $\boldsymbol{\mathcal{N}}^*L$ is then Lagrangian iff L is isotropic.

- To show that

  $$[\![\Gamma(E; L)\,, \Gamma(E; L)]\!] \subseteq \Gamma(E; L) \quad \Leftrightarrow \quad \Theta\big|_{\boldsymbol{\mathcal{N}}^*L} = 0,$$

  one turns sections of $E\big|_S$ into functions in $\mathbb{E}[\bullet]$ by "lifting the index" by the pairing $\psi \mapsto \mathrm{g}(\psi)$ and understanding the resulting sections as functions on $\mathbb{E}[\bullet]$ by pulling back from $E[\mathbf{1}]$. Then, we may prove and use the following equivalence.

$$\psi \in \Gamma(E;L) \Leftrightarrow X_{\mathrm{g}(\psi)} \text{ is tangent to } \boldsymbol{\mathcal{N}}^{*}L.$$

We will not repeat the details, let us just mention that we may the fact that the construction of the Hamiltonian field is a Lie algebra homomorphism;

$$X_{\{f,g\}} = \pm [X_f, X_g],$$

and tranlates the derived (Courant) bracket to a nested graded commutator of tangent fields.

$$X_{\{\{f,\Theta\},g\}} = \pm [[X_f, \mathcal{Q}], X_g]$$

For more details, see proof of theorem 2.62. of Grützmann [2010].

□

Now it is easy to

Lemma 54. *Let* $(\mathcal{M}, \mathbb{E}[\bullet])$ *be a split Poisson-Lie 2-algebroid, where* $\mathbb{E}[\bullet]$ *fits into the following pull-back square (see theorem 30).*

$$\begin{array}{ccc} \mathbb{E}[\bullet] & \longrightarrow & T^*[\mathbf{2}]E[\mathbf{1}] \\ \downarrow & & \downarrow \\ E[\mathbf{1}] & \longrightarrow & (E \oplus E^*)[\mathbf{1}] \end{array}$$

Then the construction of a conormal $\mathcal{N}$*-submanifold*

$$\begin{aligned} \{\text{Dirac structures in } E\} &\longrightarrow \{\boldsymbol{\Lambda}\text{−structures in } \mathcal{M} = \mathbb{E}[\bullet]\} \\ L &\longmapsto \boldsymbol{\mathcal{N}}^{*}L \end{aligned}$$

is a bijection.

Proof. Injectivity is trivial, thoerem 53 ensures that $\boldsymbol{\mathcal{N}}^{*}$ indeed takes values in the set of $\boldsymbol{\Lambda}$-structures. As to surjectivity: for a given $\boldsymbol{\Lambda}$-structure $\mathcal{L}$, we can construct a subbundle $L \subseteq E$ by taking the image of the composition

$$\begin{array}{ccc} \mathcal{L} & \hookrightarrow & \mathbb{E}[\bullet] \\ & & \downarrow \\ & & E[\mathbf{1}] \end{array}$$

and identifying the functions of the graded bundle $f \in \mathcal{O}_{E[\mathbf{1}]}$ with sections of L via the inner product:

$$\psi \longleftrightarrow \mathrm{g}(\psi)$$

Now consider a local trivialization such that $\mathcal{L}$ is defined by setting a certain subset of the local $\mathcal{N}$-algebra generators to zero. Let us collectively denote by $(\boldsymbol{x}, \hat{x})$ the base coordinates, by $(\boldsymbol{\eta}, \hat{\eta})$ the degree 1 coordinates and by $(\boldsymbol{p}, \hat{p})$ the degree 2 coordinates. Locally, $\mathcal{L}$ is defined by fixing:

$$\hat{x} = 0, \ \hat{\eta} = 0, \ \hat{p} = 0$$

In other words, $\mathcal{L}$ is locally spanned by $\boldsymbol{x}, \boldsymbol{\eta}, \boldsymbol{p}$. Now L is an isotropic subbundle defined by the set of $\boldsymbol{x}$ variables and a set of $\boldsymbol{\xi} \longleftrightarrow \mathbb{g}(\boldsymbol{\psi})$ "ghost" coordinates by setting

$$\hat{x} = 0, \; \hat{\xi} = 0$$

in $E[\mathbf{1}]$. Here, ξ's are a subset of η's; $\boldsymbol{\eta} = (\boldsymbol{\xi}, \boldsymbol{e})$, $\hat{\eta} = (\hat{\xi}, \hat{e})$. Note that the "antighosts" $\boldsymbol{e}, \hat{e}$ are absent in $E[\mathbf{1}]$, just as the momenta $(\boldsymbol{p}, \hat{p})$.

Now since $\mathcal{L}$ is Lagrangian, it does not contain any pair of conjugate coordinates. Moreover, by maximality, those coordinates contained in $\mathcal{L}$ while missing in L, that is $\boldsymbol{e}$ and $\boldsymbol{p}$, constitute *a maximal set of momenta that are not conjugate to any of the coordinates* $\boldsymbol{x}, \boldsymbol{\xi}$ *in* L. These are precisely the "normal momenta"; $\boldsymbol{p}$ and $\boldsymbol{e}$. If we take take the conormal $\mathcal{N}$-manifold $\boldsymbol{\mathcal{N}}^*L$, it can be locally generated by the same coordinates as $\mathcal{L}$. Note that the local reasoning was based solely on degree-wise dimension counting and it is thus justified since coordinate changes preserve the degrees.

This shows that every Lagrangian $\mathcal{N}$-submanifold of $\mathbb{E}[\bullet]$ can be constructed as a conormal submanifold of an isotropic subbundle. Then, theorem 53 ensures that if we start with $\mathcal{L}$ being Lagrangian, the isotropic subbundle is also a Dirac structure. This concludes the proof of surjectivity. □

Definition 31. Let $\mathcal{M}', \mathcal{M}$ be $\mathcal{NQP}$-manifolds. A **$\boldsymbol{\Lambda}$-relation** from $\mathcal{M}$ to $\mathcal{M}'$ is a $\boldsymbol{\Lambda}$-structure $\mathcal{R} \in \mathcal{M}' \times \mathcal{M}$.

Now for $R \in \mathcal{CourAlgRel}(E', E)$, the associated conormal $\mathcal{N}$-manifold defines a $\mathcal{N}$-relation.

$$\mathcal{CourAlgRel}(E', E') \xrightarrow{\quad \boldsymbol{\mathcal{N}}^* \quad} \mathcal{Man}_{\mathcal{N}}\mathcal{Rel}(\mathcal{M}', \mathcal{M})$$

$$\big(R \subseteq \overline{E'} \times E\big) \longmapsto \big(\mathcal{R} \equiv \boldsymbol{\mathcal{N}}^*L \subseteq \overline{\mathcal{M}'} \times \mathcal{M}\big)$$

Note that for $\boldsymbol{\Lambda}(R) \hookrightarrow \mathcal{M}' \times \mathcal{M}$, the proof of lemma 54 carries over, since we can always take the natural induced splitting on $\mathcal{M}' \times \mathcal{M}$ given splittings of $\mathcal{M}$ and $\mathcal{M}'$. That means the target category is, in fact, $\mathcal{Man}_{\mathcal{NQP}}\mathcal{Rel}$.

$$\mathcal{CourAlgRel}(E', E) \xrightarrow{\quad \boldsymbol{\mathcal{N}}^* \quad} \mathcal{Man}_{\mathcal{NQP}}\mathcal{Rel}(\mathcal{M}', \mathcal{M})$$

By lemma 54, we obtain the following corollary.

Corollary 55. *Let* $\mathcal{M}, \mathcal{M}' \in \mathcal{Man}_{\mathcal{NQP}}\mathcal{Rel}$ *be split* $\mathcal{NQP}$*-manifolds of degree 2 given as minimal symplectic realizations of Courant algebroids* $E, E' \in \mathcal{CourAlgRel}$*.*

$$\mathcal{CourAlgRel}(E', E) \simeq_{\text{Bij}} \mathcal{Man}_{\mathcal{NQP}}\mathcal{Rel}(\mathcal{M}', \mathcal{M})$$

Example 56 ($\boldsymbol{\Lambda}$ points)**.** *Repeating the terminology of example 47, we may call a correspondence from a point to* $\mathcal{M}$ *a **$\boldsymbol{\Lambda}$-point**, or a "gauge fixing".*

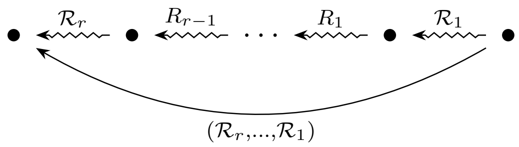
$$\mathcal{M} \xleftarrow[\mathcal{L}]{\quad} \{*\}$$

2.2.3 Generalized $\boldsymbol{\Lambda}$-Correspondences

Here, we repeat the construction of the Wehrheim-Woodward category for Lagrangian $\mathcal{N}$-relations and finally arrive at a categorification of the relationship:

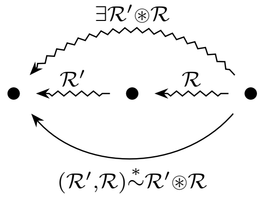
$$\text{Courant algebroids} \quad \longleftrightarrow \quad \mathcal{NQP}\text{−manifolds of degree 2}$$

Definition 32. A **(generalized) Lagrangian $\mathcal{N}$-correspondence** is an equivalence class of sequences of Lagrangian $\mathcal{N}$-relations of the form



wrt. the equivalence relation $\overset{*}{\sim}$, which is defined once again by:

$$(\mathcal{R}', \mathcal{R}) \overset{*}{\sim} \mathcal{R}' \circ \mathcal{R} \quad \text{iff} \quad \mathcal{R}' \circ \mathcal{R} = \mathcal{R}' \circledast \mathcal{R}.$$



The **composition of correspondences** is, again, defined by formal concatenation.

$$(\mathcal{R}_s, \ldots, \mathcal{R}_{r+1})\,(\mathcal{R}_r, \ldots, \mathcal{R}_1) \coloneqq (\mathcal{R}_s, \ldots, \mathcal{R}_1)$$

We denote the resulting **Wehrheim-Woodward $\mathcal{NP}$-category** $\mathcal{M}an_{\mathcal{NP}}\mathcal{C}orr$. Note that $\mathcal{M}an^{n}_{\mathcal{NP}}\mathcal{C}orr$, the category consisting only of $\mathcal{N}$-manifolds of degree n, is a full subcategory of $\mathcal{M}an_{\mathcal{NP}}\mathcal{C}orr$.

In the same fashion, we may define **(generalized) $\boldsymbol{\Lambda}$-correspondences** of $\mathcal{NQP}$-manifolds. The complication is that in general, we do not know[4] whether $\mathcal{R}' \circledast \mathcal{R}$ defines a $\boldsymbol{\Lambda}$-relation. We leave this question to further endeavours; at this point, we may only *conjecture* that the above construction defines a **Wehrheim-Woodward $\mathcal{NQP}$-category** $\mathcal{M}an_{\mathcal{NQP}}\mathcal{C}orr$ under the same conditions as in the case of $\mathcal{M}an_{\mathcal{NP}}\mathcal{C}orr$.

However, in the case $n = 2$, lemma 54 enables one to think of any $\boldsymbol{\Lambda}$-correspondence in $\mathcal{M}an^{\mathbf{2}}_{\mathcal{NQP}}\mathcal{R}el$ in the form $\boldsymbol{\mathcal{N}}^*R$ for some Dirac relation R. But for Courant relations, theorem 46 states that a clean composition $R' \circledast R$ indeed defines a Dirac relation. Then, lemma 54 provides $\boldsymbol{\mathcal{N}}^*(R' \circledast R)$ with a $\boldsymbol{\Lambda}$-structure.

Since this can be done for each element of the sequence $(\mathcal{R}_r, \ldots, \mathcal{R}_1)$ of any $\boldsymbol{\Lambda}$-correspondence, the **Werheim-Woodward $\mathcal{NQP}$-category in degree 2** is a well-defined category.

$$\mathcal{M}an^{\mathbf{2}}_{\mathcal{NQP}}\mathcal{C}orr$$

We can summarize the results in the following theorem. Recall that two categories $\mathcal{A}, \mathcal{B}$ are said to be **equivalent** if there is a functor $\mathcal{A} \longrightarrow \mathcal{B}$ which is **fully faithfull** (a bijection on morphisms between any two given objects) and **essentially surjective** (every object in the target category $\mathcal{B}$ is isomorphic to an object in the image of the functor).

[4] At least we have not found a proof in the literature.

Theorem 57. *The category $\mathfrak{Man}^{\mathbf{2}}_{\mathcal{NQP}}\mathfrak{Corr}$ is equivalent to the category $\mathfrak{CourAlgCorr}$.*

Proof. The equivalence is given by functor defined on objects of $\mathfrak{CourAlgCorr}$ by constructing the minimal symplectic realization.

$$\mathfrak{CourAlgCorr} \ni (M \leftarrow E) \;\longmapsto\; (M \leftarrow E[\mathbf{1}] \leftarrow \mathbb{E}[\bullet]) \in \mathfrak{Man}^{\mathbf{2}}_{\mathcal{NQP}}\mathfrak{Corr}$$

Batchelor's theorem 14 states that the functor is *essentially surjective*, since any $\mathcal{NQP}$-manifold of degree 2 is isomorphic to a split $\mathcal{N}$-manifold of the form $\mathcal{M} = (M \leftarrow E[\mathbf{1}] \leftarrow \mathbb{E}[\bullet])$.

On morphisms, corollary 55 of lemma 54 states that the construction of a conormal $\mathcal{N}$-submanifold

$$\mathfrak{CourAlgCorr}(E', E) \ni R \;\longmapsto\; \boldsymbol{\mathcal{N}}^{*}R \in \mathfrak{Man}^{\mathbf{2}}_{\mathcal{NQP}}\mathfrak{Corr}(\mathcal{M}', \mathcal{M})$$

is a bijection. In other words, it extends the map on objects into a *fully faithfull* functor. □

More loosely, we might say something along the lines:

> *Poisson-Lie 2-algebroids are essentially **minimal symplectic realizations** of Courant algebroids and their* $\mathbf{\Lambda}$*-correspondences are "**conormal Lagrangian realizations**" of Dirac correspondences.*

2.2.4 Quantization of Odd $\mathcal{NP}$-Manifolds

As a final remark, we show how the idea of *functorial quantization of odd $\mathcal{NP}$-manifolds* put forth by Severa [2002] fits into Weinstein's constructions. We consider the **odd symplectic category**

$$\mathcal{OSC} \coloneqq \mathfrak{Man}^{\mathbf{Odd}}_{\mathcal{NP}}\mathfrak{Rel}$$

consisting of $\mathcal{NP}$-manifolds in odd degrees (a full sub"category" of $\mathfrak{Man}_{\mathcal{NP}}\mathfrak{Rel}$). Those are precisely the $\mathcal{NP}$-manifolds equipped with odd symplectic structures.

Let us introduce the necessary ingredients, following Severa [2002]. Let $\mathcal{L} \hookrightarrow \mathcal{M} \in \mathcal{OSC}$ be a Lagrangian $\mathcal{N}$-submanifold. Locally (around $\mathcal{L}$) there exists a Darboux chart $(q^1, \ldots, q^n, \eta_1, \ldots, \eta_n)$ with q's even and η's odd, st. $\mathcal{L}$ is described by setting the η coordinates to 0.

We will consider formal **Dirac δ-distributions supported on a Lagrangian $\mathcal{N}$-submanifold $\mathcal{L}$**. For more details, we refer to Khudaverdian [2004]. In the Darboux chart above, we can write:

$$\delta_{\mathcal{L}} = \delta_{\eta_1} \otimes \cdots \otimes \delta_{\eta_n}$$

Here, the tensor product is over the ring $\mathbb{R}$, in the sense of linear functionals on suitable test functions. We will, further, require $\delta_{\mathcal{L}}$ to behave like a **semi-density** under the coordinate transformations ϕ on $\mathcal{M}$.

$$\phi^* \delta_{\mathcal{L}} = (\det J_\phi)^{1/2}\, \delta_{\mathcal{L}}$$

This ensures the following product of semidensities given by formal integration (over a $\mathcal{N}$-submanifold) is chart independent[5].

$$\int \alpha \otimes \beta$$

We denote the vector space of semidensities on $\mathcal{M}$ by $\mathbf{Dens}^{\frac{1}{2}}(\mathcal{M})$. Note that this space inherits grading as well and thus we have the odd-even decomposition:

$$\mathbf{Dens}^{\frac{1}{2}}(\mathcal{M}) = \mathbf{Dens}^{\frac{1}{2}}_{\text{Even}}(\mathcal{M}) \oplus \mathbf{Dens}^{\frac{1}{2}}_{\text{Odd}}(\mathcal{M})$$

Definition 33. We define the **quantum**[6] **odd symplectic category** $\mathcal{QOSC}$ to have the same objects as $\mathcal{OSC}$ and the morphisms given by semidensities on correspondence spaces:

$$\mathbf{Hom}_{\mathcal{QOSC}}(\mathcal{M}', \mathcal{M}) := \mathbf{Dens}^{\frac{1}{2}}(\overline{\mathcal{M}'} \times \mathcal{M})$$

The **composition** of morphisms

$$\mathcal{M}'' \xleftarrow[\delta_{\mathcal{R}'}]{} \mathcal{M}' \xleftarrow[\delta_{\mathcal{R}}]{} \mathcal{M}$$

is defined by "integration of $\delta_{\mathcal{R}'} \otimes \delta_{\mathcal{R}} \in \mathbf{Dens}^{\frac{1}{2}}(\mathcal{R}' \times \mathcal{R})$ over $\Delta_{\mathcal{M}'}$".

$$\delta_{\mathcal{R}'} \circ \delta_{\mathcal{R}} := \int_{\Delta_{\mathcal{M}'}} \delta_{\mathcal{R}'} \otimes \delta_{\mathcal{R}}$$

Let us specify this prescription a bit further: given $\delta_{\mathcal{R}}$ and $\delta_{\mathcal{R}'}$, we define an induced δ-semidensity on $\mathcal{R}' \diamond \mathcal{R}$. Then, we integrate the coordinates in $\mathcal{R}' \diamond \mathcal{R}$ originating from $\Delta_{\mathcal{M}'}$.

The composition is not always defined. However, for $\mathcal{R}' \circledast \mathcal{R}$, we have a good chance that the integral makes sense. That is:

1. The first condition of clean composition provides $\mathcal{R}' \diamond \mathcal{R}$ with a structure of a smooth ($\mathcal{N}$-)manifold — now it makes sense to define semidensities on $\mathcal{R}' \diamond \mathcal{R}$ in the first place. Moreover, cleanness

   "T after $\cap = \cap$ after T"

   ensures we do not lose any tangent directions in $\mathcal{R}'$ and $\mathcal{R}$ provided they coincide on the diagonal $\Delta_{\mathcal{M}'}$. We can thus take the pullback of the semidensity on $\mathcal{R}' \times \mathcal{R}$ onto $\mathcal{R}' \diamond \mathcal{R}$ (which corresponds to integration of δ-functionals).

2. The second condition says that the projection $p_{\diamond \to \circ}$ is a surjective submersion (we may understand this in terms of an arbitrary splitting). The implicit function theorem states that locally, there exists such a chart in $\mathcal{R}' \diamond \mathcal{R}$ such that $\mathcal{R}' \circ \mathcal{R} = p_{\diamond \to \circ}(\mathcal{R}' \diamond \mathcal{R})$ is be defined by fixing some of the coordinates on $\mathcal{R}' \diamond \mathcal{R}$ to be zero. Now (up to a canonical transformation)

[5] Provided the integral is well-defined in the first place.
[6] We will clarify why it is reasonable to use the word "quantum" a little later.

integration over $\Delta_{\mathcal{M}'}$ does precisely that. This means that for $\mathcal{R}' \circledast \mathcal{R}$, we indeed have

$$\int_{\Delta_{\mathcal{M}'}} \delta_{\mathcal{R}'} \otimes \delta_{\mathcal{R}} \in \mathbf{Dens}^{\frac{1}{2}}(\overline{\mathcal{M}''} \times \mathcal{M})$$

In other words:

$$\delta_{\mathcal{R}'} \circ \delta_{\mathcal{R}} \in \mathcal{QOSC}(\mathcal{M}'', \mathcal{M})$$

Of course, the integrals might still be ill-defined; the argumentation above merely suggests that composition of relations is not be to blamed. Provided the integration indeed makes sense, Weinstein [2010] shows that the resulting semidensity on $\mathcal{M}'' \times \mathcal{M}$ coincides with the δ-distribution supported on $\mathcal{R}' \circledast \mathcal{R}$ defined above.

$$\delta_{\mathcal{R}'} \circ \delta_{\mathcal{R}} = \delta_{\mathcal{R}' \circledast \mathcal{R}}$$

Ie. the assignment $\mathcal{L} \mapsto \delta_{\mathcal{L}}$ satisfies the condition of remark 50 and it always factors through the Wehrheim-Woodward category $\mathcal{OSC}orr := \mathcal{M}an^{\mathrm{Odd}}_{\mathcal{NP}}\mathcal{C}orr$.

$$\begin{array}{ccc} \mathcal{C}orr & \xrightarrow{\;\mathcal{Q}\;} & \mathcal{QOSC}orr \\ \downarrow & \overset{\exists !}{\nearrow} & \\ \mathcal{OSC}orr & & \end{array}$$

Therefore we may define the **Wehrheim-Woodward quantum odd symplectic category** $\mathcal{QOSC}orr$ by considering sequences of $\delta_{\mathcal{L}}$ relations and the **quantization functor** $\mathcal{Q} : \mathcal{OSC}orr \longrightarrow \mathcal{QOSC}orr$ as an identity on objects and by the following assignment on relations

$$\mathcal{Q} : (\mathcal{R} \hookrightarrow \mathcal{M}' \times \mathcal{M}) \longmapsto \Big(\delta_{\mathcal{R}} \in \mathbf{Dens}^{\frac{1}{2}}(\mathcal{M}' \times \mathcal{M})\Big)$$

and the induced element-wise assignment of sequences of relations. Using the terminology of Weinstein [2010], $\mathcal{QOSC}orr$ can be said to be the "universal quantization category" defined for the proper class of odd $\mathcal{NP}$-manifolds.

Thus, with the help of the Wehrheim-Woodward categories, we have arrived to a notion of *functorial quantization of odd $\mathcal{NP}$-manifolds* but we have yet to clarify its "quantum" nature. We will only provide a very brief remark following Severa [2002] and refer to the works of Costello [2007] and Khudaverdian [2004].

Remark 58 (The BV Laplacian)**.** *The classical master equation describes an incarnation of gauge symmetry of the BV action functional. However, to ensure that quantum mean values calculated in terms of path integrals stay gauge invariant, one has to introduce the so called BV Laplacian $\mathbf{\Delta}_{\mathrm{BV}}$ and replace the classical master equation with the* ***quantum master equation****.*

$$\{\Theta, \Theta\} + \hbar \mathbf{\Delta}_{\mathrm{BV}}(\Theta) = 0$$

It is thus desirable to translate our "graded symplectic system" into a "system compatible with the BV Laplacian". The statement of Severa [2002] is that the quantization functor provides such translation from the world of odd $\mathcal{NP}$-manifolds.

Definition 34. The **BV Laplacian** is defined in a Darboux chart

$$\left(q^1, \ldots, q^n, \eta_1, \ldots, \eta_n\right)$$

of $\mathcal{M} \in \mathcal{OSCorr}$ as a differential operator on semidensities.

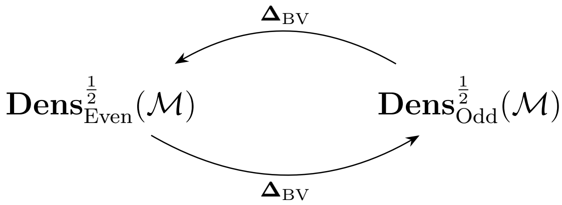
$$\mathbf{\Delta}_{\mathrm{BV}} \coloneqq \frac{\partial^2}{\partial q_a \partial \eta^a}$$

Let us summarize some of its properties mentioned by Severa [2002].

Lemma 59 (Properties of $\mathbf{\Delta}_{\mathrm{BV}}$)**.**

1. $\mathbf{\Delta}_{\mathrm{BV}}$ *is odd*
2. $\mathbf{\Delta}^2_{\mathrm{BV}} = 0$
3. $\left[\mathcal{L}_{X_f}, \mathbf{\Delta}_{\mathrm{BV}}\right] = 0$*, ie.* $\mathbf{\Delta}_{\mathrm{BV}}$ *is invariant under Hamiltonian diffeomorphisms.*

The property 3. ensures $\mathbf{\Delta}_{\mathrm{BV}}$ *is independent of the choice of a Darboux chart. Properties 1. and 2. say that* $\mathbf{\Delta}_{\mathrm{BV}}$ *can be understood as the differential of a* $\mathbb{Z}_2$*-graded chain complex defined on* $\mathbf{Dens}^{\frac{1}{2}}(\mathcal{M})$*.*



Now this is, according to Severa [2002], a suitable kind of "quantum system" in the context of Batalin-Vilkovisky quantization. We repeat his remark on the following symmetry: If we denote by $\phi(\mathcal{L})$ *a deformation of a Lagrangian* $\mathcal{N}$*-submanifold* $\mathcal{L}$ *by a Hamiltonian diffeomorphism (ie. a change of gauge fixing), it can be shown that for* $\alpha \in \mathbf{Dens}^{\frac{1}{2}}(\mathcal{M})$ *being* $\mathbf{\Delta}_{\mathrm{BV}}$*-closed, we have the following kind of gauge invariance:*

$$\int \delta_{\phi(\mathcal{L})} \otimes \alpha = \int \delta_{\mathcal{L}} \otimes \alpha$$

Notice the analogy with the Ward identities in remark 2.

Finally, just as a Lagrangian $\mathcal{N}$*-submanifold* $\mathcal{L}$ *can be seen as a choice of gauge fixing (as in example 39 and remark 3) or a* $\mathbf{\Lambda}$*-point (example 56), the set of* $\mathcal{QOSCorr}$*-points*

$$\{\ \mathcal{M} \longleftarrow \{*\}\ \}$$

can be interpreted as the ***quantization of*** $\mathcal{M}$*.*

$$\mathcal{QOSCorr}(\mathcal{M}, \{*\})$$

Conclusion

Let us summarize our journey.

First, in section 1.1 we recalled the geometric objects arising in BV-BRST theories as a motivation for studying graded symplectic geometry, mentioning the elementary example of the Yang-Mills theories, refering to Becchi et al. [1976], Batalin and Vilkovisky [1981], Weinberg [2013], Henneaux and Teitelboim [2020] and Jurčo et al. [2019]. In particular, we saw the AKSZ framework mentioned in 1.1.3 due to Alexandrov et al. [1997] enables us to forge the needed geometric structures on finite-dimensional differential graded symplectic manifolds instead of the infinite-dimensional field configurations.

In sections 1.2, 1.3, 1.4, 1.5, we reviewed the basics of the theory of $\mathcal{NQP}$-manifolds following mainly Roytenberg [2002] including the example $\boldsymbol{deg} = 2$ corresponding to Courant algebroids known from generalized geometry. We saw that the sheaf-theoretic language naturally extends from supermanifolds to $\mathcal{N}$-manifolds *degree-wise*. It may be an interesting question whether the differential symplectic structure translates into the general theory of graded manifolds in the spirit of Vysoky [2021].

In the second chapter, we introduced Lagrangian $\mathcal{N}$-submanifolds as a generalization of the concept of gauge fixing following the philosophy of Weinstein [2010]. In section 2.2.1, we extended the notion of Lagrangian relations to $\mathcal{NP}$-manifolds using the known results for Courant algebroid relations seen in section 2.1 due to Vysoky [2020]. Furthermore, in sections 2.2.2 and 2.2.3 we showed that the 1-to-1 correspondence between isomorphism classes of Poisson-Lie 2-algebroids and Courant algebroids seen in theorem 36 can be extended into an equvialence of Wehrheim-Woodward categories (introduced by Wehrheim and Woodward [2007]) . We relied on the results of Grützmann [2010].

The following question is left open: Is the $\mathcal{NQP}$-Wehrheim-Woodward category well-defined under the same cleannes conditions for general degrees of $\mathcal{N}$-manifolds? What about the $\mathbb{Z}$-graded setting?

Finally, in section 2.2.4, we mentioned the idea of functorial quantization of odd symplectic manifolds due to Severa [2002] as an application of the Wehrheim-Woodward construction for $\mathcal{NP}$-manifolds.

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