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Exact spacetimes in modified theories of gravity

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Abstract: In the review part of the thesis we summarize various modified theories of gravity, especially those that are characterized by additional curvature invariants in the Lagrangian density. Further, we review non-twisting geometries, especially their Kundt subclass. Finally, from the principle of least action we derive field equations for the case with the Lagrangian density corresponding to an arbitrary function of the curvature invariants. In the original part of the thesis we explicitly express particular components of the field equations for non-gyratonic Kundt geometry in generic quadratic gravity in arbitrary dimension. Then we discuss how this, in general fourth order, field equations restrict the Kundt metric in selected geometrically privileged situations. We also analyse the special case of Gauss–Bonnet theory.

Keywords: General relativity, $f(R, R_{cd}^2, R_{cdef}^2)$ theories, quadratic gravity, Gauss–Bonnet theory, non-twisting and shear-free geometries, Kundt spacetimes.

Název: Přesné prostoročasy v modifikovaných teoriích gravitace

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Abstrakt: V rešeršní části této diplomové práce shrnujeme poznatky o modifikovaných teoriích gravitace, především pak takových, které jsou charakterizovány dodatečnými křivostnímy invarianty v hustotě lagrangiánu. Dále uvádíme přehled o netwistujících geometriích, speciálně s důrazem na jejich Kundtovu podtřídu. Z principu nejmenší akce poté odvozujeme rovnice gravitačního pole v případě, kdy hustota lagrangiánu odpovídá obecné funkci křivostních invariantů. V rámci původní části této práce explicitně nalézáme jednotlivé složky rovnic pole pro negyratonovou Kundtovu geometrii v obecné kvadratické teorii gravitace v obecné dimenzi. Diskutujeme, jak rovnice pole omezují obecnou Kundtovu metriku ve vybraných geometricky privilegovaných případech. Speciální pozornost pak věnujeme případu Gauss-Bonnetovy teorie.

Klíčová slova: Obecná relativita, $f(R, R_{cd}^2, R_{cdef}^2)$ teorie, kvadratická gravitace, Gaussova–Bonnetova teorie, netwistující a bezshearové geometrie, Kundtovy prostoročasy.

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Contents

Introduction

In 1687 sir Isaac Newton published his *Philosophiæ Naturalis Principia Mathematica* where he formulated, among other concepts that revolutionized the way humanity sees nature, the *law of universal gravitation.* This law states that material bodies instantly act on each other by attractive force proportional to their masses and inversely proportional to the square of their distance. Although others, e.g., Robert Hooke, had proposed the inverse square law for gravitation before, Newton was the first who supported this concept with precise astronomical calculations, proving that Kepler's orbits shaped as conic sections result from the attractive force described by this law.

For more than three following centuries Newton's theory became the most widely accepted theory of gravitation, and it repeatedly demonstrated its astonishing predictive power. In fact, existence of planet Neptune was predicted by Urbain Le Verrier using Newton's theory in his attempt to explain specific deviations of planet Uranus from its Keplerian orbit. When it was discovered that Mercury also deviates from its expected trajectory by the precession of 43 arc seconds per century, existence of yet another body of matter was proposed to explain this tiny discrepancy. However, no such object was ever found, and it required Albert Einstein to overcome Newton's theory, inherently including the concepts of absolute space and time, to finally explain the precession of Mercury using his new geometric theory of gravity, i.e., *general relativity*, see [1].

General relativity, which after several years of Einstein's intensive work resulted in 1915 from the need to formulate a theory of gravitation compatible with the framework of special relativity, uses the concept of geodesic motion in curved four-dimensional spacetime instead of gravitational force to explain motions of free particles. Geometric properties of spacetime are encoded in its metric tensor, which is coupled with the matter content of the universe through *Einstein's field equations* [2]. Like many things in the branch of theoretical physics, Einstein's field equations can be derived using the *principle of least action* with suitable Lagrangian, or here more precisely, Lagrangian density. Remarkably simple Lagrangian density for general relativity was discovered already in 1915 by David Hilbert, see classic textbooks [3, 4].

Throughout the last century, Einstein's theory of gravitation became extraordinary successful and still remains a subject of lively research. It deeply extended our understanding of the universe and made various fascinating predictions many of which were already confirmed by very precise experiments, see [5] for the review, from the bending of light rays by presence of matter already measured by Arthur Eddington during the solar eclipse in 1919 to existence of the gravitational waves, which were only recently directly observed on the LIGO experiment [6, 7].

Nonetheless, it seems that not all properties of our universe can be easily explained in the framework of general relativity. There is now strong experimental evidence that universe has gone through the phase of rapid expansion shortly after the big bang, and now once again its expansion is accelerating, measured rotation curves of galaxies significantly deviate from those predicted by the theory, and so on. These phenomena are usually explained by an introduction of new matter forms with exotic properties as sources of the gravitational field in Einstein's general relativity. But even though it should now constitute 96% of the universe content this seemingly invisible matter has never been directly observed, and its physical nature remains probably the most fascinating mystery of astrophysics.

This situation reminds us the similar one with the precession of the Mercury perihelion in the beginning of the last century. Maybe instead of introducing new forms of the matter to explain deviations from the theory, we should modify this theory itself. Indeed many modified theories of gravitation attempted to particularly solve some of these problems with various levels success. Some specific examples of them will be discussed on the following pages. However, efforts to modify general relativity driven by diverse motivations are not restricted to the recent years. Shortly after Einstein's formulation of general relativity, physicists began trying to modify it for example by introducing new terms in Lagrangian, in order to unify gravitation with other forces, to find theory of gravitation susceptible to methods of quantum field theory, or just to satisfy their natural intellectual curiosity, see e.g., [8] for the comprehensive review.

From the mathematical point of view Einstein's field equations represent system of ten non-linear second order partial differential equations. Not surprisingly, field equations of modified theories are usually even more complicated and they can contain fourth or even higher order derivative terms. To solve such equations exactly is in the most of realistic scenarios impossible which is even true in the case of Einstein's theory. Instead, only approximate situations and solutions are often studied either analytically or numerically. Yet, exact solutions, often representing only very idealized spacetimes, play invaluable role in our attempt to understand given theory of gravitation, see [9, 10] for the review of exact solutions in the Einstein theory. Among other things, they allow us to fully explore the non-linearity of equations in the presence of strong fields, to check validity of numerical methods, and they also provide necessary backgrounds for more relevant perturbative approximations.

It is possible to find interesting exact solutions by imposing geometric restrictions on the metric tensor of exact spacetime and introducing such metric ansatz into the field equations. For example the requirement of spherical symmetry thus provides famous Schwarzschild solution, observational cosmological requirements of homogeneity and isotropy give arise to the Friedmann–Lemaître–Robertson–Walker cosmological models, and restrictions employed on the optical properties of null geodesic congruence can lead to the Robinson–Trautman or Kundt spacetimes with exact gravitational waves.

In this thesis, we briefly review some of the modified theories of gravity, see chapter 1, and provide variational derivation of the field equations of broad class of such theories with arbitrary function of linear and quadratic curvature invariants in the Lagrangian density in chapter 2.

After a brief introduction of non-twisting and shear-free geometries in chapter 3, we explore the special case of expansion-free *non-gyratonic Kundt geometries* in the context of quadratic gravity, i.e., interesting subclass of modified theories of gravitation with quadratic curvature terms in the action. The most general form of field equations in this case can be found in chapter 4. Moreover, we discuss specific geometric settings as *pp*-waves (section 4.2.1), VSI spacetimes (section 4.2.2), or the Kundt waves on direct product backgrounds (section 4.2.3). Subsequently, we analyze a specific subclass of quadratic gravity, namely the Gauss–Bonnet theory, in chapter 5 again together with the explicit examples mentioned above.

Finally, in appendices A and B we summarize explicit expressions for geometric objects constructed from the non-gyratonic Kundt metric ansatz.

1. Modified theories of gravity

The main aim of this chapter is to give a brief review of various extensions and modifications of classic Einstein's general relativity, namely its higher-dimensional extension, the Lovelock gravity, or the theories with quadratic corrections in the action. After this summary, the field equations for quadratic gravity are derived in chapter 2.

1.1 General relativity and the least action principle

Einstein's general relativity describes spacetime as a *four*-dimensional pseudo-Riemannian (more precisely Lorentzian) manifold equiped with *symmetric metric tensor field* g_{ab}. Another geometric property of the spacetime manifold used by general relativity is the connection associated with covariant differentiation. Although various connections are possible in the differential geometry, general relativity is defined in terms of the Levi-Civita connection given via metric tensor g_{ab}. This connection can thus be expressed by the *Christoffel symbols*,

$$\Gamma^c_{ab} = \frac{1}{2} g^{cd} \left(g_{da,b} + g_{db,a} - g_{ab,d} \right) . \tag{1.1}$$

Its torsion is obviously vanishing which means

$$\Gamma^c_{[ab]} = 0 . \tag{1.2}$$

The crucial concept standing in the heart of general relativity is *curvature*. It naturally encodes inhomogeneity of the gravitational field, see e.g. classical textbooks [3,4]. In terms of differential geometry it is described by the *Riemann curvature tensor* $R^a{}_{bcd}$ given by

$$R^a{}_{bcd} = \Gamma^a_{bd,c} - \Gamma^a_{bc,d} + \Gamma^a_{ce}\Gamma^e_{bd} - \Gamma^a_{de}\Gamma^e_{bc} . \tag{1.3}$$

Contracting the Riemann curvature tensor we obtain the rank two *Ricci curvature tensor* R_{ab}, and contracting the Ricci tensor we finally get the *scalar curvature* R, respectively,

$$R_{ab} = R^c{}_{acb} , \qquad R = R^a{}_a . \tag{1.4}$$

Moreover, the scalar curvature R is the essential part of the so-called *Einstein–Hilbert action.*

As pointed out already by David Hilbert the field equations of Einstein's theory can be derived from the principle of least action. This can be expressed simply as

$$\delta S = 0 , \tag{1.5}$$

where the action S is defined as Lagrangian density $\mathcal{L}$ integrated over the whole four-dimensional spacetime manifold, namely

$$S = \int \mathcal{L} \, \mathrm{d}^4 \mathrm{x} . \tag{1.6}$$

In the case of general relativity the *Einstein-Hilbert action* can be written as

$$S = \int \left[\frac{1}{2\kappa'} \left(R - 2\Lambda \right) + \mathcal{L}_{\mathrm{M}} \right] \sqrt{-g} \, \mathrm{d}^4 \mathrm{x} , \tag{1.7}$$

where $\mathcal{L}_{\mathrm{M}}$ is Lagrangian density of matter fields representing sources, symbol g stands for determinant of the metric g_{ab}, and parameter κ' is composed of the theory constants[1],

$$\kappa' = \frac{8\pi G}{c^4} . \tag{1.8}$$

[1] For the convenience we distinguish constants κ' and $\kappa = 2\kappa'$

Varying the Einstein–Hilbert action *with respect to the metric* g_{ab} and using the principle of least action ($\delta S = 0$) we obtain Einstein's field equation[2]

$$R_{ab} - \frac{1}{2}R g_{ab} + \Lambda g_{ab} = \frac{8\pi G}{c^4} T_{ab}\,, \tag{1.9}$$

where T_{ab} is the stress–energy tensor representing sources of the field. It is defined as

$$T_{ab} \equiv -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\,\mathcal{L}_M)}{\delta g^{ab}}\,. \tag{1.10}$$

The formalism of Einstein's theory is thoroughly discussed for example in canonical books [3,4]. Systematic review of exact solutions to Einstein's field equations can be found in [9], and physical interpretation of the most important exact spacetimes is given in [10].

Uniqueness of Einstein's gravity governed by the field equation (1.9) can be formulated as *Lovelock's theorem* [11, 12]. Condition of the action extremality, which leads to the field equations, can be generally expressed in the form of Euler–Lagrange equations, namley

$$E^{ab}[\mathcal{L}] \equiv \frac{\mathrm{d}}{\mathrm{d}x^c}\left[\frac{\partial \mathcal{L}}{\partial g_{ab,c}} - \frac{\mathrm{d}}{\mathrm{d}x^d}\left(\frac{\partial \mathcal{L}}{\partial g_{ab,cd}}\right)\right] - \frac{\partial \mathcal{L}}{\partial g_{ab}} = 0\,. \tag{1.11}$$

Lovelock's theorem (Theorem 5 in [11]): If $D = 4$ the only second order Euler–Lagrange equations arising from a scalar density $\mathcal{L} = \mathcal{L}(g_{ab},\, g_{ab,c},\, g_{ab,cd})$ are

$$E^{ab} = \alpha\sqrt{-g}\left[R^{ab} - \frac{1}{2}g^{ab}R\right] + \lambda\sqrt{-g}g^{ab} = 0\,, \tag{1.12}$$

i.e., Einstein's equation with cosmological term, where both α and λ are constants.

The cosmological term can be absorbed into the additional source part of the field equations. This theorem means that if we want to create any theory of gravity in $D = 4$ Lorentzian spacetime arising from the principle of least action with Lagrangian constructed from the metric and its derivatives only, then the only field equations with *no higher that the second order derivatives* are Einstein's field equations. However, this does not mean that the Einstein–Hilbert action is the *unique* one leading to the expression (1.12). In fact any Lagrangian

$$\mathcal{L} = \alpha\sqrt{-g}R - 2\lambda\sqrt{-g} + \beta\epsilon^{abcd}R^{ef}{}_{ab}R_{efcd} + \gamma\sqrt{-g}\left(R^2 - 4R_{cd}R^{cd} + R_{cdef}R^{cdef}\right) \tag{1.13}$$

leads to (1.12), and therefore to Einstein's equations as well.

We can already see several ways in which this scheme can be modified, and alternative theories of gravitation thus obtained. Let us mention the following possibilities (and their various combinations):

- Consider *higher number of spacetime dimensions*, i.e., $D > 4$. This very simple modification leaves the field equations unchanged, but varies the number of tensor components. Surprisingly, it allows interesting solutions which are absent in classic Einstein's gravity.
- Describe gravitational field by *additional scalar, vector, or rank-2 tensor fields.* There is plethora of such theories including various scalar-tensor, Einstein-aether, tensor-vector-scalar, or bivector theories.
- Allow *connection with torsion.* Non-vanishing torsion is used by various $f(T)$ theories or, e.g., Einstein–Cartan–Sciama–Kibble theory.
- Introduce *arbitrary function of the scalar curvature, higher order curvature invariants*, or, e.g., of the Weyl tensor, in the Lagrangian density. This is the case of $f(R)$ theories, quadratic gravity, or conformal gravity.

For summary of various aspects of such modified theories see for example [8, 13–15].

[2] Other approaches than variation with respect to the metric are possible. In the so-called *Palatini formalism* variation is performed with respect to both metric and connection, which is now considered to be independent of the metric. For the Einstein–Hilbert action both of these approaches yield the same result, but for modified actions the results may differ. In what follows, we will only be considering variation with respect to the metric.

1.2 Higher-dimensional general relativity

The simplest modification of Einstein's general relativity is its extension into higher spacetime dimensions $D > 4$. Instead of classic four-dimensional Einstein–Hilbert action (1.7) we can simply consider the action

$$S = \int \left[\frac{1}{2\kappa'} (R - 2\Lambda) + \mathcal{L}_{\mathrm{M}} \right] \sqrt{-g}\, \mathrm{d}^D \mathrm{x} \,, \tag{1.14}$$

where integration is performed over the D-dimensional spacetime manifold. Subsequently, variation of this action with respect to the metric g_{ab} provides field equations

$$R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} = \frac{8\pi G}{c^4} T_{ab} \,, \tag{1.15}$$

which are identical to their four-dimensional counterparts as discussed in the previous section, see the Einstein equations (1.9).

Many interesting exact solutions to higher-dimensional Einstein's field equations had been discovered during last decades, and standard four-dimensional mathematical methods had been adapted to $D > 4$, e.g., the algebraic classification had been developed [16–18].

Let us mention at least the most important spacetimes here. The direct generalization of Schwarzschild solution into higher dimension is called Schwarzschild–Tangherlini metric, see the original work [19]. Usual Kerr–Schild ansatz leads to Myers–Perry black hole [20] which is described by $\lfloor \frac{D-1}{2} \rfloor$ spin parameters. Since black hole uniqueness theorems depend on use of the Gauss–Bonnet theorem, their extension to higher-dimension is non-trivial. This also means that black holes with several different topologies are possible here. In five dimensions so-called black ring solution was found in [21], which has an event horizon with $S^1 \times S^2$ topology. Classic examples of exact non-expanding Kundt gravitational waves [22, 23] were generalised in [24, 25] and analysed in [26, 27]. Surprisingly, their expanding Robinson–Trautman counterpart well-known in $D = 4$, see [28, 29], does not exist in $D > 4$, where this class with standard matter contains only type D spacetimes, see [30–34].

Despite all the efforts of string theorists in the previous decades, at least theoretical justification for higher dimensions remains greatly speculative. Obviously, we only perceive one temporal and three spatial dimensions, and there is no direct observational or experimental evidence for any additional (space) dimensions. Inaccessibility of the higher dimensions to our perception could be explained by their compactification on very small scales. This approach was pioneered by Theodor Kaluza and Oskar Klein, see [35–37], already in the second decade of twentieth century. The Kaluza–Klein theory represents five-dimensional theory with additional four-vector and scalar fields, which aims to unify gravitation with electromagnetism. Ideas used by Kaluza and Klein later led to the string theories with *ten*, *eleven*, or even *twenty-six* spacetime dimensions. Many theorists believe that framework of the string theory is sill plausible candidate for a mystic 'theory of everything'.

Let us remark that besides compactification mentioned above there is another possibility how to explain the apparent absence of observed higher dimensions. This is so-called brane-world scenario, see [38, 39], according to which our accessible universe is restricted to the four-dimensional subspace (*brane*) of a higher-dimensional spacetime.

1.3 Lovelock theory

Here, we would like to discuss a broad class of modified theories of gravitation in arbitrary dimension D with the Lagrangian density which is general function of the Riemann curvature tensor and its covariant derivatives, i.e.,

$$\mathcal{L} = \mathcal{L}(g_{ab}, R_{abcd}, \nabla_{a_1} R_{abcd}, \dots, \nabla_{a_1 \dots a_p} R_{abcd}) \,. \tag{1.16}$$

The principle of the least action (1.5) for such general Lagrangian leads to field equations of the following form, see [40],

$$-T^{ab} = \frac{\partial \mathcal{L}}{\partial g_{ab}} + E^{a}{}_{cde} R^{bcde} + \nabla_{(c} \nabla_{d)} E^{acdb} + \frac{1}{2} g^{ab} \mathcal{L} \,, \tag{1.17}$$

where tensorial quantity $E^a{}_{cde}$ is defined as

$$E^{bcde} = \frac{\partial \mathcal{L}}{\partial R_{bcde}} - \nabla_{a_1} \frac{\partial \mathcal{L}}{\partial (\nabla_{a_1} R_{bcde})} + \cdots + (-1)^p \nabla_{(a_1} \cdots \nabla_{a_p)} \frac{\partial \mathcal{L}}{\partial (\nabla_{(a_1} \cdots \nabla_{a_p)} R_{bcde})} . \tag{1.18}$$

Such field equations can be vastly complicated for the most particular choices of the Lagrangian density, and they usually won't even be the second order differential equations as is the case in Einstein's theory. Therefore, it is natural to ask a question how should we choose the specific Lagrangian density for the field equations to be of the second order. Theories satisfying this requirement are called *Lovelock theories*, or *Lovelock gravity*, see the original papers [11, 12], and their Lagrangian density can be written in a unified way as

$$\mathcal{L} = \sqrt{-g} \sum_{n=0}^{t} \alpha_n \, \mathcal{L}^{(n)}, \qquad \text{with} \qquad \mathcal{L}^{(n)} = \frac{1}{2^n} \delta^{c_1 d_1 \ldots c_n d_n}_{a_1 b_1 \ldots a_n b_n} \prod_{r=1}^{n} R^{a_r b_r}{}_{c_r d_r} , \tag{1.19}$$

where α_n are specific constants of the particular theory, and the generalized Kronecker delta $\delta^{\cdots}_{\cdots}$ is defined as the antisymmetric product

$$\delta^{c_1 d_1 \ldots c_n d_n}_{a_1 b_1 \cdots a_n b_n} = \frac{1}{n!} \delta^{c_1}_{[a_1} \delta^{d_1}_{b_1} \cdots \delta^{c_n}_{a_n} \delta^{d_n}_{b_n]} . \tag{1.20}$$

The integer t, which denotes maximal possible number of terms in Lagrangian (1.19), depends on dimension D of the spacetime. Without any loss of generality, we can say that for even dimensions it holds that $D = 2t + 2$ and for odd dimensions $D = 2t + 1$.

Examining $\mathcal{L}^{(n)}$ for few lowest values of n we see that for $D = 3$ and $D = 4$, the Lovelock gravity must have identical form to Einstein's gravity, as it considers only $n = 0$ and $n = 1$.

- *Zeroth* order, so-called cosmological term, is trivially

$$\mathcal{L}^{(0)} = 1 , \tag{1.21}$$

  which means that there is the usual cosmological term in the field equations.

- *First* order represents the well-known Einstein–Hilbert term

$$\mathcal{L}^{(1)} = R . \tag{1.22}$$

- *Second* order, usually called Gauss–Bonnet term, can be written as

$$\mathcal{L}^{(2)} = R^2 - 4 R_{ab} R^{ab} + R_{abcd} R^{abcd} . \tag{1.23}$$

  This is the only non-trivial combination of quadratic curvature terms which does not cause higher than second derivatives in the field equations. It only contributes in $D > 4$.

- *Third* order. This is the last term we explicitly show here,

$$\begin{aligned} \mathcal{L}^{(3)} = & R^3 - 12 R R_{ab} R^{ab} + 16 R_{ab} R^a{}_c R^{bc} + 24 R_{ab} R_{cd} R^{acbd} + 3 R R_{abcd} R^{abcd} \\ & - 24 R_{ab} R^a{}_{cde} R^{bcde} + 4 R_{abcd} R^{abef} R^{cd}{}_{ef} - 8 R_{acbd} R^a{}_e{}^b{}_f R^{cedf} . \end{aligned} \tag{1.24}$$

It is not surprising that Lagrangian terms are getting more and more complicated as the number of possible contractions of the Riemann tensors grows.

In general, it can be shown that the n-th Lagrangian expression $\mathcal{L}^{(n)}$ gives rise to $E^{(n)}_{ab}$ term in the field equations, namely

$$E^{a\,(n)}_{\;b} = -\frac{1}{2^{n+1}} \delta^{a e_1 \ldots e_n f_1 \ldots f_n}_{b g_1 \cdots g_n h_1 \ldots h_n} R_{g_1 h_1}{}^{e_1 f_1} \cdots R_{g_n h_n}{}^{e_n f_n} . \tag{1.25}$$

The most general form of the Lovelock field equations can thus be written as

$$\sum_{n=0}^{t} \alpha_n E^{(n)}_{ab} = T_{ab} . \tag{1.26}$$

Various exact solutions have been found in the Lovelock theories including black holes, black strings, or black branes [41–43]. The theory with additional Gauss–Bonnet term only will be discussed separately in the following section.

1.4 Gauss–Bonnet gravity

As mentioned in the previous section, the Gauss–Bonnet gravity is a special case of Lovelock's theory with Lagrangian containing the quadratic Gauss–Bonnet term (1.23),

$$\mathcal{L}^{(2)} = R^2 - 4R_{ab}R^{ab} + R_{abcd}R^{abcd} \,,$$

but no other higher order corrections [44]. This means that such a theory is also special case of so called quadratic gravity, which will be discussed later in section 1.5. The appropriate action for the Gauss–Bonnet gravity in vacuum can be written as

$$S = \int \Big[\frac{1}{\kappa} \left(R - 2\Lambda_0 \right) + \gamma \left(R^2_{cdef} - 4R^2_{cd} + R^2 \right) \Big] \sqrt{-g} \, \mathrm{d}^D \mathrm{x} \,, \tag{1.27}$$

where we have replaced the Lovelock constants α_n with more usual $\kappa = \frac{16\pi G}{c^4}$, cosmological-like term Λ_0, and newly introduced Gauss–Bonnet constant γ. As we have already mentioned, this extension of general relativity is only meaningful for dimension $D > 4$, in four dimensions the Gauss–Bonnet term becomes trivial. This action leads to the field equations

$$\begin{aligned} &\frac{1}{\kappa} \left(R_{ab} - \frac{1}{2} R g_{ab} + \Lambda_0 g_{ab} \right) \\ &\quad + 2\gamma \left[R R_{ab} - 2R_{acbd}R^{cd} + R_{acde}{R_b}^{cde} - 2R_{ac}{R_b}^{c} - \frac{1}{4} \, g_{ab} \left(R^2_{cdef} - 4R^2_{cd} + R^2 \right) \right] = 0 \,. \end{aligned} \tag{1.28}$$

We immediately see that the equations (1.28) are of the second order as they should be for any member of the Lovelock family of theories. The Gauss-Bonnet gravity has been studied extensively since its quadratic term appears in the low energy effective action of the heterotic string theory and also in certain compactifications of M-theory, see e.g., [45].

Moreover, another interesting property of this theory is that after quantization it does not lead to any ghost modes [46] on many background spacetimes including flat Minkowski. Therefore, various exact solutions of the Gauss–Bonnet field equations have been discovered and analysed including so-called Boulware–Deser solution [47], which describe certain type of black hole or naked singularity, depending on the specific value of its parameter.

1.5 Quadratic gravity

Omitting the requirement for the field equations to be of the second order we can simply allow any combination of the quadratic curvature terms, namely R^2, R^2_{cd}, and R^2_{cdef}, in the Lagrangian density. Preserving the Gauss–Bonnet combination for convenience we can write the action for general quadratic gravity as

$$S = \int \Big[\frac{1}{\kappa} \left(R - 2\Lambda_0 \right) + \alpha R^2 + \beta R^2_{cd} + \gamma \left(R^2_{cdef} - 4R^2_{cd} + R^2 \right) \Big] \sqrt{-g} \, \mathrm{d}^D \mathrm{x} \,, \tag{1.29}$$

where α, β, γ, κ are constants of the theory. Such action leads to the field equations [48]

$$\begin{aligned} &\frac{1}{\kappa} \left(R_{ab} - \frac{1}{2} R g_{ab} + \Lambda_0 g_{ab} \right) + 2\alpha R \left(R_{ab} - \frac{1}{4} R g_{ab} \right) \\ &\quad + (2\alpha + \beta) \big(g_{ab} \Box - \nabla_a \nabla_b \big) R \\ &\quad + 2\gamma \left[R R_{ab} - 2R_{acbd}R^{cd} + R_{acde}{R_b}^{cde} - 2R_{ac}{R_b}^{c} - \frac{1}{4} \, g_{ab} \left(R^2_{cdef} - 4R^2_{cd} + R^2 \right) \right] \\ &\quad + \beta \Box \left(R_{ab} - \frac{1}{2} R g_{ab} \right) + 2\beta \left(R_{acbd} - \frac{1}{4} \, g_{ab} R_{cd} \right) R^{cd} = 0 \,, \end{aligned} \tag{1.30}$$

which are generally of the fourth order as they contain the second covariant derivatives of the Ricci tensor and scalar, respectively.

Higher order theories are usually plagued by the presence of so-called ghosts [46,49]. These are additional degrees of freedom representing particles propagating with negative energy. It is then possible for normal particle and ghost to suddenly appear without violation of the energy conservation law. Such hypothetical phenomenon is called vacuum decay, and it is usually considered to be a pathology of the particular theory.

Much less exact solutions are known in the general quadratic gravity than in, for example, Gauss–Bonnet theory. Yet some wavelike and spherically symmetrical solutions have been found, and various propositions about possible solutions with respect to their algebraical types have been made [50, 51]. In four dimensions it holds that all Einstein spacetimes (i.e., $R_{ab} \sim g_{ab}$) obey the vacuum equations of the quadratic gravity. However, additional non-Einstein solutions are known as well, see e.g., [52]. Most recently, the static spherically symmetric non-Schwarzschild black hole solution has been found in $D = 4$ quadratic gravity [53]. This implies that the Birkhoff theorem does not hold here any more.

Apart from the Gauss–Bonnet gravity, other special cases of quadratic gravity have been studied. One of them is the *Einstein–Weyl gravity* [54] described by the action

$$S = \int \left[\frac{1}{\kappa} (R - 2\Lambda_0) + \gamma \, C_{abcd} C^{abcd} \right] \sqrt{-g} \, \mathrm{d}^D \mathrm{x} \,, \tag{1.31}$$

where quantity C_{abcd} here represents the Weyl tensor defined as

$$\begin{aligned} C_{abcd} =& R_{abcd} + \frac{1}{D-2} \left(R_{ad} g_{bc} - R_{ac} g_{bd} + R_{bc} g_{ad} - R_{bd} g_{ac} \right) \\ &+ \frac{1}{(D-1)(D-2)} R \left(g_{ac} g_{bd} - g_{ad} g_{bc} \right) . \end{aligned} \tag{1.32}$$

We can easily see that the contraction of the Weyl tensor with itself get exactly those quadratic curvature terms that appear in the action for quadratic gravity (1.29). Discarding the Einstein–Hilbert and cosmological terms completely and keeping only the contracted Weyl term, we get a special subcase of the *conformal gravity* [55] with the action

$$S = \int \gamma \, C_{abcd} C^{abcd} \sqrt{-g} \, \mathrm{d}^D \mathrm{x} \,. \tag{1.33}$$

More precisely, the theory is called conformal if the conformal transformation of the metric, namely $g_{ab} \rightarrow \Omega^2(x) g_{ab}$, preserves the action which is obviously true in the case of (1.33). The conformal gravity has many interesting properties. Among them let us emphasize the fact that ghost modes are decoupled from the physical matter.

1.6 $f(R)$-theories

Another class of theories that lately have drawn interest of many physicists wishing to explain recent cosmological discoveries is so-called $f(R)$ gravity, or $f(R)$-theories. As its name indicates, the action can now contain an arbitrary function $f(R)$ of the Ricci scalar curvature instead of the simple linear expression in R, which is the case of the Einstein–Hilbert action, and general relativity, respectively, see [56, 57]. The action for $f(R)$-theories can be simply written as

$$S = \int \left[f(R) + \mathcal{L}_M \right] \sqrt{-g} \, \mathrm{d}^D \mathrm{x} \,. \tag{1.34}$$

Of course, the Einstein theory can be easily recovered by the appropriate choice of function f. Here it would be $f(R) = \frac{1}{\kappa} (R - 2\Lambda)$. In general, the action (1.34) leads to the field equations

$$f_R \, R_{ab} - \frac{1}{2} f \, g_{ab} + \left(g_{ab} \Box - \nabla_a \nabla_b \right) f_R = \frac{1}{2} T_{ab} \,, \tag{1.35}$$

where f_R is derivative of the function $f(R)$ with respect to the scalar curvature R. We can see that due to the presence of the d'Alembert operator these are generally fourth order equations. It can be shown that the covariant divergence of the left hand side is vanishing and therefore also ${T^{ab}}_{;b} = 0$. Conservation laws thus hold in the same form as they do in the Einstein gravity [58]. We can also calculate trace of the field equations to get

$$f_R \, R - \frac{1}{2} D f + (D-1) \Box f_R = \frac{1}{2} T \,, \tag{1.36}$$

where T is simply the trace of stress-energy tensor. In the case of the Einstein gravity we would have simple algebraic relation between R and T. However, equation (1.36) is a differential equation since it contain $\Box f_R$, and it can be interpreted as a field equation describing the dynamics of propagating scalar degree of freedom $\varphi = f_R$ called *scalaron* [59]. The $f(R)$-gravity can also be shown to be equivalent to the special case of the Brans–Dicke theory [60], which is a theory with additional scalar field explicitly appearing in the action [61].

The $f(R)$-theories have been studied extensively over the last two decades both with the general function $f(R)$ and with particular choices like

$$R^n\,, \qquad \ln(\lambda R)\,, \qquad e^{\lambda R}\,, \quad \text{or even} \quad R - a(R-\Lambda_1)^{-m} + b(R-\Lambda_2)^n\,.$$

Many researchers consider $f(R)$-gravity merely an interesting toy model, but some try to find the exact expression for the function $f(R)$ that would satisfy all of the known constraints so that such a theory could faithfully describe real nature. Viable theory should conform to galaxy clustering spectra and CMB anisotropy spectra. It should also allow the existence of matter dominated era and stable stars, but it should not contain ghost or tachyons. Moreover, it should agree with the results of Einstein's gravity on small scales. These requirements impose strong constraints on the function $f(R)$ which are summarized in the following list [14, 15].

- $f_{,R} > 0$ for $R \geq R_0$, where R_0 is today's value of the Ricci scalar. This must hold in order to avoid the presence of ghost states.
- $f_{,RR} > 0$ for $R \geq R_0$. This is necessary to avoid the existence of scalar degree of freedom with negative mass, i.e., tachyons.
- $f(R) \to R - 2\Lambda_0$ for $R \geq R_0$. This needs to be true for the presence of the matter dominated era and for agreement with the local gravity constraints.
- $0 < \frac{Rf_{,RR}}{f_{,R}} < 1$ when $\frac{Rf_{,R}}{f} = 2$. This condition is important for stability and late time de Sitter limit of the universe.

One of the functions specially designed to pass all such requirements was proposed by Starobinsky, see [62], and can be written as

$$f(R) = R + \lambda R_0 \left[\left(1 + \frac{R^2}{R_0^2} \right)^{-n} - 1 \right], \tag{1.37}$$

where n, $\lambda > 0$, and R_0 is of the order of H_0^2, which is a square of today's Hubble constant.

Various exact solutions have been found in $f(R)$-theories. It can be shown that any vacuum solution to Einstein's gravity is also a solution of $f(R)$-gravity except for some pathological choices of the function f. This includes the usual black hole solutions, e.g., Schwarzschild spacetime. However, since the Birkhoff theorem does not hold here, other spherically symmetric solutions exist in the $f(R)$-gravity. Propagating scalar degree of freedom implies the existence of an additional types of longitudinally polarized wave-like solutions. Since the scalaron is massive, these waves would travel with the speed lower than that of light.

2. Generalized field equations

In this chapter we explicitly derive equations of the gravitational field using the principle of least action (1.5),

$$\delta S = 0\,.$$

For the theories summarized in the previous sections the action S can be written in a unified way as

$$S = \int \Big[f\left(R,\, \Psi,\, \Omega\right) + \mathcal{L}_M \Big] \sqrt{-g}\, \mathrm{d}^D \mathrm{x}\,, \tag{2.1}$$

where g denotes the determinant of D-dimensional Lorentzian metric g_{ab}, i.e., $g \equiv \det g_{ab}$, and symbol x represents all spacetime coordinates. The key ingredient f is arbitrary differentiable function which depends on:

R	*scalar curvature*,
$\Psi \equiv R_{cd}R^{cd} \equiv R^2_{cd}$	*square of the Ricci tensor*,
$\Omega \equiv R_{cdef}R^{cdef} \equiv R^2_{cdef}$	*square of the Riemann tensor*.

Finally, the Lagrange density $\mathcal{L}_M$ corresponds to the matter fields which represent sources in the resulting equations of the gravitational field.

2.1 Useful identities

Since our aim is to express the final form of variation δS of the action (2.1) in terms of the contravariant metric variations δg^{ab}, it is natural to begin with several useful relations.

- *Covariant and contravariant metric variations.* The relation between variations of covariant and contravariant metric can be easily derived from the expression of Kronecker's delta as a metric contraction,

$$g_{ac}g^{cb} = \delta^b_a\,. \tag{2.2}$$

Making a variation of this expression,

$$\delta g_{ac}g^{cb} + g_{ac}\delta g^{cb} = 0\,, \tag{2.3}$$

and performing a contraction (with some renaming of indices) we finally obtain

$$\delta g_{ab} = -g_{ac}g_{bd}\delta g^{cd}\,. \tag{2.4}$$

- *Metric determinant variation.* We also need to rewrite the determinant variation δg in terms of the metric variation δg^{ab}. To do so, let us recall the formulae which relate a determinant and an inverse matrix with the minors M_{ab}, i.e., determinants of the matrices obtained from the original one by omission of the a-th row and the b-th column. For the line element g_{ab} it thus holds

$$g = \sum_b g_{ab}M_{ab}(-1)^{a+b}\,, \qquad\qquad g^{ab} = \frac{1}{g}M_{ab}(-1)^{a+b}\,, \tag{2.5}$$

where the only summation is performed over index b in the first equation, and in fact, the index a is fixed here. Taking the derivative of the first equation with respect to the specific covariant metric component g_{ab} we get

$$\frac{\partial g}{\partial g_{ab}} = M_{ab}(-1)^{a+b}\,. \tag{2.6}$$

Combining this result with the second equation in (2.5) we obtain

$$\frac{\partial g}{\partial g_{ab}} = g\, g^{ab}\,. \tag{2.7}$$

Multiplication of the last formula by variation δg_{ab} leads to

$$\delta g \equiv \frac{\partial g}{\partial g_{ab}}\delta g_{ab} = g\, g^{ab}\delta g_{ab}\,, \tag{2.8}$$

and using (2.4) we finally arrive to the expression for the metric determinant variation in terms of the contravariant metric variation,

$$\delta g = -g\, g_{ab}\delta g^{ab}\,. \tag{2.9}$$

Moreover, it can be easily seen that

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\, g_{ab}\, \delta g^{ab}\,. \tag{2.10}$$

- *Variation of the Riemann tensor and connection.* We employ the convention where the Riemann curvature tensor is defined as

$$R^a{}_{bcd} = \partial_c \Gamma^a_{bd} - \partial_d \Gamma^a_{bc} + \Gamma^a_{ce}\Gamma^e_{bd} - \Gamma^a_{de}\Gamma^e_{bc}\,. \tag{2.11}$$

Its variation thus becomes

$$\delta R^a{}_{bcd} = \partial_c\, \delta\Gamma^a_{bd} - \partial_d\, \delta\Gamma^a_{bc} + \delta\Gamma^a_{ce}\Gamma^e_{bd} + \Gamma^a_{ce}\delta\Gamma^e_{bd} - \delta\Gamma^a_{de}\Gamma^e_{bc} - \Gamma^a_{de}\delta\Gamma^e_{bc}\,. \tag{2.12}$$

In fact, the connection variation $\delta\Gamma^a_{bc}$ is a difference of two connections. Since their non-tensorial parts cancel out the result $\delta\Gamma^a_{bc}$ is tensor, and it holds that

$$\delta\Gamma^a_{bc} = \frac{1}{2} g^{ad}\left(\nabla_b\, \delta g_{dc} + \nabla_c\, \delta g_{db} - \nabla_d\, \delta g_{bc}\right)\,. \tag{2.13}$$

This can be simply seen employing the identity $\delta(\nabla_d\, g_{bc}) = 0$, namely

$$0 = \nabla_d\, \delta g_{bc} - g_{ec}\, \delta\Gamma^e_{bd} - g_{be}\, \delta\Gamma^e_{cd}\,. \tag{2.14}$$

The cyclic permutation of its indices followed by appropriate summation prove the relation (2.13). Moreover, we can calculate the covariant derivative of this tensorial expression as

$$\nabla_d\, \delta\Gamma^a_{bc} = \partial_d\, \delta\Gamma^a_{bc} + \Gamma^a_{ed}\delta\Gamma^e_{bc} - \Gamma^e_{cd}\delta\Gamma^a_{be} - \Gamma^e_{bd}\delta\Gamma^a_{ec}\,. \tag{2.15}$$

Examining formulae (2.12) and (2.15) we see that variation of the Riemann tensor can be expressed as

$$\delta R^a{}_{bcd} = \nabla_c\, \delta\Gamma^a_{bd} - \nabla_d\, \delta\Gamma^a_{bc}\,. \tag{2.16}$$

Finally, by a simple contraction of indices in this formula we obtain

$$\delta R_{bd} \equiv \delta R^a{}_{bad} = \nabla_a \delta\Gamma^a_{bd} - \nabla_d \delta\Gamma^a_{ab}\,, \tag{2.17}$$

which is so-called *Palatini identity.*

- *Bianchi identity and its contractions.* We start with the second Bianchi identity,

$$R_{ab[cd;e]} = 0\,, \qquad \text{i.e.} \qquad R_{abcd;e} + R_{abde;c} + R_{abec;d} = 0\,, \tag{2.18}$$

and contract it using g^{ac} to get

$$R_{bd;e} + g^{ac}R_{abde;c} - R_{be;d} = 0\,. \tag{2.19}$$

Contracting this equation furthermore using g^{bd} we obtain

$$R_{;e} - 2R_{ae;c}g^{ac} = 0\,, \qquad \text{i.e.} \qquad R_{ea}{}^{;a} - \frac{1}{2}R_{;e} = 0\,. \tag{2.20}$$

- *Commutators of covariant derivatives.* For the rank 1 tensor we have

$$A_{f;cd} - A_{f;dc} = R^e{}_{fcd}A_e\,, \tag{2.21}$$

and for the rank 2 tensor it holds

$$T_{fg;cd} - T_{fg;dc} = -R_{fecd}T^e{}_g - R_{gecd}T_f{}^e\,. \tag{2.22}$$

2.2 Variation of the action

Now we can proceed to derivation of the field equations. To begin with, let us apply the Leibniz rule for variations to the action (2.1), namely

$$\delta S = \int \Big[\big(f_R\, \delta R + f_\Psi\, \delta \Psi + f_\Omega\, \delta \Omega \big) \sqrt{-g} + f\, \delta \sqrt{-g} + \frac{\delta(\sqrt{-g}\, \mathcal{L}_M)}{\delta g^{ab}}\, \delta g^{ab} \Big] \mathrm{d}^D \mathrm{x}\,, \tag{2.23}$$

where f_R, f_Ψ, and f_Ω are partial derivatives of the function f with respect to R, Ψ, and Ω.

• *Source term.* Considering the definition of *energy–momentum tensor*,

$$T_{ab} \equiv -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\, \mathcal{L}_M)}{\delta g^{ab}}\,, \tag{2.24}$$

we see that the last term in (2.23) can be rewritten as

$$\frac{\delta(\sqrt{-g}\, \mathcal{L}_M)}{\delta g^{ab}}\, \delta g^{ab} = -\frac{1}{2} T_{ab} \sqrt{-g}\, \delta g^{ab}\,. \tag{2.25}$$

• *The Ricci scalar variation δR.* We express the scalar curvature as a contraction of the Ricci tensor and use the Palatini identity (2.17) together with formulae (2.13) and (2.4) to obtain

$$\begin{aligned} \delta R = \delta\left(R_{ab} g^{ab}\right) &= R_{ab} \delta g^{ab} + g^{ab} \delta R_{ab} \\ &= R_{ab} \delta g^{ab} + g^{ab} \left(\nabla_e\, \delta \Gamma^e_{ab} - \nabla_b\, \delta \Gamma^e_{ea} \right) \\ &= R_{ab} \delta g^{ab} + \frac{1}{2} g^{ec} \Big[\nabla_e \big(\nabla_a\, \delta g_{cb} + \nabla_b\, \delta g_{ca} - \nabla_c\, \delta g_{ab} \big) \\ &\qquad - \nabla_b \big(\nabla_e\, \delta g_{ca} + \nabla_a\, \delta g_{ce} - \nabla_c\, \delta g_{ea} \big) \Big] \\ &= R_{ab}\, \delta g^{ab} + g_{ab}\, \Box\, \delta g^{ab} - \nabla_a \nabla_b\, \delta g^{ab}\,. \end{aligned} \tag{2.26}$$

• *Variation of the Ricci tensor square $\delta\Psi$.* Similarly as in the previous case with using symmetries of the curvature tensor we obtain

$$\begin{aligned} \delta \Psi \equiv \delta\left(R_{cd} R^{cd}\right) &= 2 R_{ac} R_b{}^c\, \delta g^{ab} + 2\, \delta R_{cd} R^{cd} \\ &= 2 R_{ac} R_b{}^c\, \delta g^{ab} - 2 \nabla_d \nabla_c \left(\delta^d_a\, \delta g^{ab}\right) R^c{}_b + \Box \left(\delta g^{ab}\right) R_{ab} + \nabla_c \nabla_d \left(g_{ab} \delta g^{ab}\right) R^{cd}\,. \end{aligned} \tag{2.27}$$

• *Variation of the Riemann tensor square $\delta\Omega$.* Finally, employing the uncontraced formula (2.16), instead of the Palatini identity, we get

$$\begin{aligned} \delta \Omega \equiv \delta\left(R_{cdef} R^{cdef}\right) &= 2 R_{cdea} R^{cde}{}_b\, \delta g^{ab} + 2 \delta R^c{}_{def} R_c{}^{def} \\ &= 2 R_{cdea} R^{cde}{}_b\, \delta g^{ab} - 4 \nabla_d \nabla_c \left(\delta g^{ab}\right) R^c{}_{ab}{}^d\,. \end{aligned} \tag{2.28}$$

We substitute all these expanded results, i.e., formulae (2.10), (2.26), (2.27), and (2.28), into variation of the action (2.23) to get

$$\begin{aligned} \delta S = \int \Big\{ & f_R \left(g_{ab}\, \Box\, \delta g^{ab} - \nabla_a \nabla_b\, \delta g^{ab}\right) \\ & + f_\Psi \left[-2 \nabla_c \nabla_d \left(\delta^d_a\, \delta g^{ab}\right) R^c{}_b + \Box \left(\delta g^{ab}\right) R_{ab} + \nabla_c \nabla_d \left(g_{ab}\, \delta g^{ab}\right) R^{cd} \right] \\ & - 4 f_\Omega\, \nabla_c \nabla_d \left(\delta g^{ab}\right) R^c{}_{ab}{}^d \\ & + \Big[f_R\, R_{ab} + 2 f_\Psi\, R_{ac} R_b{}^c + 2 f_\Omega\, R_{cdea} R^{cde}{}_b - \frac{1}{2} f\, g_{ab} - \frac{1}{2} T_{ab} \Big] \delta g^{ab} \Big\} \sqrt{-g}\, \mathrm{d}^D \mathrm{x}\,. \end{aligned} \tag{2.29}$$

As a final step, we rewrite the whole integrand as a specific expression proportional to the metric variation δg^{ab}. Terms containing *derivatives* of the metric variation have to be put into this form using the Leibniz rule and Gauss's theorem together with few useful identities.

For example, let us demonstrate this procedure on the second term in (2.29). We begin with application of the Leibniz rule for covariant derivatives to split the integral into two parts. Then we use the Leibniz rule one more time to expand the second term,

$$\begin{aligned}
&\int f_R \nabla_a \nabla_b \left(\delta g^{ab}\right) \sqrt{-g}\, \mathrm{d}^D\mathrm{x} \\
&\quad = \int \nabla_a \left[f_R \nabla_b \left(\delta g^{ab}\right) \sqrt{-g} \right] \mathrm{d}^D\mathrm{x} - \int \nabla_a \left(f_R \sqrt{-g} \right) \nabla_b \left(\delta g^{ab}\right) \mathrm{d}^D\mathrm{x} \\
&\quad = \int \nabla_a \left[f_R \nabla_b \left(\delta g^{ab}\right) \right] \sqrt{-g}\, \mathrm{d}^D\mathrm{x} - \int \nabla_b \left[\nabla_a \left(f_R\, \delta g^{ab}\right) \right] \sqrt{-g}\, \mathrm{d}^D\mathrm{x} \\
&\qquad + \int \nabla_b \nabla_a \left(f_R \right) \delta g^{ab} \sqrt{-g}\, \mathrm{d}^D\mathrm{x}\,,
\end{aligned} \tag{2.30}$$

where we have also used the identity $\nabla_a \sqrt{-g} = 0$ to pull the tensor density $\sqrt{-g}$ outside the derivatives. Consequently, we use the well-known formula

$$\sqrt{-g}\, \nabla_a V^a = \partial_a \left(\sqrt{-g}\, V^a \right), \tag{2.31}$$

to rewrite the first pair of resulting integrals in (2.30) as integrals of partial divergence. Then we apply Gauss's theorem

$$\int_\Omega \partial_a V^a\, \mathrm{d}^D\mathrm{x} = \int_{\partial\Omega} V^a\, \mathrm{dS}_a\,, \tag{2.32}$$

where $\mathrm{dS}_a \equiv \sqrt{-\gamma}\, n_a\, \mathrm{d}^{D-1}\mathrm{x}$ with n_a being normal to $\partial\Omega$ and γ representing determinant of $(D-1)$-dimensional induced metric, to get surface integrals which we consider to be vanishing. Utilizing symmetry of the metric variation in the third term in (2.30) we thus obtain

$$\int f_R \nabla_a \nabla_b \left(\delta g^{ab}\right) \sqrt{-g}\, \mathrm{d}^D\mathrm{x} = \int \nabla_a \nabla_b \left(f_R \right) \delta g^{ab} \sqrt{-g}\, \mathrm{d}^D\mathrm{x}\,. \tag{2.33}$$

Applying the analogous approach to all remaining terms containing derivatives of δg^{ab} in (2.29) we may express variation of the action (2.1) as

$$\begin{aligned}
\delta S = \int \Big[& g_{ab}\, \Box f_R - \nabla_a \nabla_b f_R - 2 \nabla_c \nabla_d \left(f_\Psi\, \delta^d_a R^c{}_b \right) + \Box \left(f_\Psi\, R_{ab} \right) \\
& + g_{ab} \nabla_c \nabla_d \left(f_\Psi\, R^{cd} \right) - 4 \nabla_c \nabla_d \left(f_\Omega\, R^c{}_{ab}{}^d \right) + f_R\, R_{ab} + 2 f_\Psi\, R_{ac} R_b{}^c \\
& + 2 f_\Omega\, R_{cdea} R^{cde}{}_b - \frac{1}{2} f\, g_{ab} - \frac{1}{2} T_{ab} \Big] \delta g^{ab} \sqrt{-g}\, \mathrm{d}^D\mathrm{x}\,.
\end{aligned} \tag{2.34}$$

2.3 Field equations

Finally, to satisfy the principle of least action (1.5), i.e., $\delta S = 0$, the parentheses must be equal to zero, and we thus obtain the gravitational field equations corresponding to the action (2.1) in the form

$$\begin{aligned}
& f_R\, R_{ab} - \frac{1}{2} f\, g_{ab} + \left(g_{ab}\, \Box - \nabla_a \nabla_b \right) f_R + 2 \left(f_\Psi\, R_{ac} R_b{}^c + f_\Omega\, R_{cdea} R^{cde}{}_b \right) \\
& \quad + \Box \left(f_\Psi\, R_{ab} \right) + g_{ab} \nabla_c \nabla_d \left(f_\Psi\, R^{cd} \right) - 2 \nabla_c \nabla_d \left(f_\Psi\, \delta^d_a R^c{}_b + 2 f_\Omega\, R^c{}_{ab}{}^d \right) = \frac{1}{2} T_{ab}\,.
\end{aligned} \tag{2.35}$$

Here it is possible to write explicitly the symmetrization in a, b indices of last two terms (as some authors do, e.g., [63,64]), since the whole expression in (2.34) is contracted with the metric tensor variation which is obviously symmetric.

Next, we can expand the second derivatives in last four terms on the left hand side and rewrite the specific second derivatives of the curvature tensors using simpler expressions, mostly curvature tensor contractions. This approach will prove to be useful especially in the case of quadratic gravity, where f_Ψ and f_Ω are mere constants and their derivatives thus disappear.

We thus apply the Leibniz rule to obtain

$$
\begin{aligned}
& f_R\, R_{ab} - \frac{1}{2} f\, g_{ab} + \left(g_{ab}\,\Box - \nabla_a \nabla_b\right) f_R + 2\left(f_\Psi\, R_{ac} R_b{}^c + f_\Omega\, R_{cdea} R^{cde}{}_b\right) \\
& \quad + R_{ab}\,\Box f_\Psi + 2 g^{cd}\,\nabla_c f_\Psi\,\nabla_d R_{ab} + f_\Psi\,\Box R_{ab} \\
& \quad + g_{ab}\left(R^{cd}\,\nabla_c\nabla_d f_\Psi + 2\nabla_c f_\Psi\,\nabla_d R^{cd} + f_\Psi \nabla_c\nabla_d R^{cd}\right) \\
& \quad - 2\left[\delta_a^d R^c{}_b\,\nabla_c\nabla_d f_\Psi + 2\nabla_{(c} f_\Psi\,\nabla_{d)}(\delta_a^d R^c{}_b) + f_\Psi\,\nabla_c\nabla_d\,(\delta_a^d R^c{}_b)\right] \\
& \quad - 4\left(R^c{}_{ab}{}^d\,\nabla_c\nabla_d f_\Omega + 2\nabla_{(c} f_\Omega\,\nabla_{d)} R^c{}_{ab}{}^d + f_\Omega\,\nabla_c\nabla_d R^c{}_{ab}{}^d\right) = \frac{1}{2}\,T_{ab}\,. \qquad (2.36)
\end{aligned}
$$

Now we rewrite terms with the second derivatives of curvature tensors. First we look at $\nabla_c\nabla_d R^{cd}$ term, which can be easily simplified using doubly contracted Bianchi identity (2.20),

$$
\nabla_c\nabla_d R^{cd} \equiv g^{df} g^{ce} R_{fe;cd} = \frac{1}{2} g^{df} R_{;fd} \equiv \frac{1}{2}\Box R\,. \qquad (2.37)
$$

We can simply express term $\nabla_c\nabla_d(\delta_a^d R^c{}_b)$ as

$$
\nabla_c\nabla_d(\delta_a^d R^c{}_b) = g^{ce} R_{be;ac}\,, \qquad (2.38)
$$

and also rewrite term $\nabla_c\nabla_d R^c{}_{ab}{}^d$ using contracted Bianchi identity (2.19)

$$
\nabla_c\nabla_d R^c{}_{ab}{}^d = -g^{ce}\,\nabla_c R_{dbea}{}^{;d} = g^{ce}\,\nabla_c\left(R_{be;a} - R_{ba;e}\right) = g^{ce} R_{be;ac} - \Box R_{ab}\,. \qquad (2.39)
$$

Subsequently, we are left with term $g^{ce}R_{be;ac}$ in two previous equations. We may rewrite this contraction of Ricci tensor derivatives as the second derivative of the Ricci scalar and terms without higher derivatives of metric. To do so, we use the commutator of derivatives of rank 2 tensor, see expression (2.22), and then we utilize doubly contracted Bianchi identities (2.20), namely

$$
g^{ce} R_{be;ac} = g^{ce}\left(R_{be;ca} - R_{bfac} R_e^f - R_{efac} R_b^f\right) = \frac{1}{2}\,R_{;ab} - R_{beaf} R^{ef} + R_{af} R_b{}^f\,. \qquad (2.40)
$$

Finally, putting all these expressions back into the original equations (2.36), we get the *expanded form of the field equations*,

$$
\begin{aligned}
& f_R\, R_{ab} - \frac{1}{2} f\, g_{ab} + \left(g_{ab}\,\Box - \nabla_a\nabla_b\right) f_R + \Box f_\Psi + 2 g^{cd}\nabla_c f_\Psi \nabla_d R_{ab} + (f_\Psi + 4 f_\Omega)\Box R_{ab} \\
& \quad - 4 f_\Omega R_{ac} R_b{}^c + (2 f_\Psi + 4 f_\Omega) R_{acbd} R^{cd} + 2 f_\Omega R_{cdea} R^{cde}{}_b \\
& \quad + g_{ab}\left(R^{cd}\,\nabla_c\nabla_d f_\Psi + 2\nabla_c f_\Psi\,\nabla_d R^{cd} + \frac{1}{2} f_\Psi\,\Box R\right) \\
& \quad - (f_\Psi + 2 f_\Omega)\nabla_a\nabla_b R - 2\Big(\nabla_c\nabla_d f_\Psi\,\delta_a^d R^c{}_b + 2\nabla_{(c} f_\Psi\,\delta_a^d \nabla_{d)} R^c{}_b \\
& \qquad\qquad + 2 R^c{}_{ab}{}^d\,\nabla_c\nabla_d f_\Omega + 4\nabla_{(c} f_\Omega\,\nabla_{d)} R^c{}_{ab}{}^d\Big) = \frac{1}{2}\,T_{ab}\,. \qquad (2.41)
\end{aligned}
$$

2.3.1 Generic quadratic gravity

To derive field equations of the quadratic gravity, see section 1.5, we have to substitute specific function $f(R, \Psi, \Omega)$ into the general form of field equations (2.41) derived above, namely

$$
f = \frac{1}{\kappa}\left(R - 2\Lambda_0\right) + \alpha\, R^2 + \beta\,\Psi + \gamma\left(\Omega - 4\Psi + R^2\right), \qquad (2.42)
$$

which is equivalent to using the action

$$
S = \int \left[\frac{1}{\kappa}\left(R - 2\Lambda_0\right) + \alpha R^2 + \beta R_{cd}^2 + \gamma\left(R_{cdef}^2 - 4 R_{cd}^2 + R^2\right)\right]\sqrt{-g}\,\mathrm{d}^D\mathrm{x}\,. \qquad (2.43)
$$

Here, we also restrict our attention to the vacuum spacetimes, i.e., we set $T_{ab} = 0$. First we calculate derivatives of the function f with respect to quantities R, Ψ and Ω to obtain

$$
\begin{aligned}
f_R &= \frac{1}{\kappa} + 2\alpha R + 2\gamma R\,, \\
f_\Psi &= \beta - 4\gamma\,, \qquad (2.44) \\
f_\Omega &= \gamma\,.
\end{aligned}
$$

Substituting these formulas into the expanded form of the field equations (2.41), terms containing covariant derivatives of f_Ψ and f_Ω disappear, and after few simple rearrangements we finally arrive to the *quadratic gravity field equations*, namely

$$\begin{aligned}
&\frac{1}{\kappa}\left(R_{ab}-\frac{1}{2}Rg_{ab}+\Lambda_0 g_{ab}\right)+2\alpha R\left(R_{ab}-\frac{1}{4}Rg_{ab}\right)\\
&\quad+(2\alpha+\beta)\big(g_{ab}\Box-\nabla_a\nabla_b\big)R\\
&\quad+2\gamma\left[RR_{ab}-2R_{acbd}R^{cd}+R_{acde}R_b{}^{cde}-2R_{ac}R_b{}^c-\frac{1}{4}\,g_{ab}\left(R^2_{cdef}-4R^2_{cd}+R^2\right)\right]\\
&\quad+\beta\Box\left(R_{ab}-\frac{1}{2}Rg_{ab}\right)+2\beta\left(R_{acbd}-\frac{1}{4}\,g_{ab}R_{cd}\right)R^{cd}=0\,,
\end{aligned} \tag{2.45}$$

which are identical to those presented for example in [48].

2.3.2 Generic $f(R)$-gravity

To demonstrate usefulness of the general result (2.41) and complexity of generic quadratic theories discussed in the previous section, we simply derive the field equations of the so-called $f(R)$ gravity, see section 1.6 or [14, 15]. We thus assume that the specific function $f(R,\Psi,\Omega)$ in (2.41) depends only on the scalar curvature R, namely

$$f=f(R)\,,\qquad \text{i.e.}\qquad f_\Psi=0\,,\quad f_\Omega=0\,. \tag{2.46}$$

The action becomes

$$S=\int\Big[f\,(R)+\mathcal{L}_M\Big]\sqrt{-g}\,\mathrm{d}^D\mathrm{x}\,, \tag{2.47}$$

and corresponding *field equations of the $f(R)$ gravity* takes the form

$$f_R\,R_{ab}-\frac{1}{2}f\,g_{ab}+\left(g_{ab}\,\Box-\nabla_a\nabla_b\right)f_R=\frac{1}{2}\,T_{ab}\,. \tag{2.48}$$

3. Non-twisting geometries

In this chapter we explicitly derive completely general metric form of the spacetime that admits *non-twisting null geodesic congruence.* After heuristic geometric construction of the line element we introduce the optical scalars to prove its non-twisting character. Then we discuss special cases of non-twisting, shear-free, *expanding* geometries (i.e. *Robinson–Trautman* class [28, 29]), and non-twisting, shear-free, *non-expanding* geometries (i.e. *Kundt* class [22, 23]). Since the Kundt spacetimes will be investigated in the following chapters as the ansatz for field equations, we briefly mention their geometric properties and introduce some of their subclasses, namely non-gyratonic *pp*-waves, VSI spacetimes, and direct product geometries.

3.1 Null foliation and adapted coordinates

We start with generic *Lorentzian manifold* $\mathcal{M}$ and metric tensor g_{ab} in any dimension $D \geq 4$, which is locally covered by a set of coordinates x^a, where the index a ranges from 0 to $D-1$. We consider the manifold to be foliated by a family of null hypersurfaces implicitly given by the condition $u(x^a) = \text{const.}$ In the opposite more intuitive way, we may consider any worldline $\gamma(u) = \{x^a(u)\}$, where u is the corresponding parameter, and then in every point of γ we construct null hypersurface ('turn the light on'), which will thus be identified by a specific value of u.

Since the parameter u is *unique* for every hypersurface, we use it as a new coordinate on the manifold $\mathcal{M}$. Moreover, we introduce the *normal* and *tangent* to the null hypersurfaces $u(x^a) = \text{const.}$ simply as $\tilde{k}_a = -u_{,a} = -\delta^u_a$. Because its direction has to be null as well, it satisfies

$$0 = g^{ab}\tilde{k}_a\tilde{k}_b = g^{ab}u_{,a}u_{,b} = g^{uu}\,.$$

We have thus obtained restriction on the contravariant metric component g^{uu}.

Next, we construct a null vector field $\tilde{k}^b$ corresponding to $\tilde{k}_a$ via $\tilde{k}^b = g^{ab}\tilde{k}_a$. Since the vector field $\tilde{k}^b$ is null, i.e., $\tilde{k}^b\tilde{k}_b = 0$, it holds that

$$0 = (\tilde{k}^b\tilde{k}_b)_{;a} = 2\,\tilde{k}_{b;a}\tilde{k}^b\,.$$

The covariant derivative $\tilde{k}_{b;a}$ is now symmetric in a and b, and we thus have

$$\tilde{k}_{a;b}\tilde{k}^b = 0\,.$$

Therefore the corresponding integral curves of vector field $\tilde{k}^b$ are affine geodesics. We denote the affine parameter along these geodesics as r, and use it as another coordinate on the manifold $\mathcal{M}$. Geometrically this means that

$$\tilde{\boldsymbol{k}} = \partial_r\,, \tag{3.1}$$

and therefore $\tilde{k}^b = \delta^b_r$. By raising an index of $\tilde{k}_b$ we also see that $\tilde{k}^b = -g^{ub}$ which immediately implies further conditions for the metric components, namely

$$g^{ur} = -1\,, \qquad \text{and} \qquad g^{up} = 0\,,$$

where the index p goes through remaining $D-2$ values, i.e., $p = 2, \ldots, D-1$.

The manifold $\mathcal{M}$ is now locally covered by a set of adapted coordinates $(r,\, u,\, x^p)$, where r is the *affine parameter* along null rays generated by the vector field $\tilde{\boldsymbol{k}}$, see figure 3.1. This vector field is normal to the *null hypersurfaces* labelled by coordinate u. The coordinates x^p represents $D-2$ *spatial coordinates* of the transverse Riemannian space with r and u fixed. Employing the conditions derived above, the line element of such spacetime can be written as

$$\mathrm{d}s^2 = g_{pq}(r,u,x)\,\mathrm{d}x^p\mathrm{d}x^q + 2g_{up}(r,u,x)\,\mathrm{d}x^p\mathrm{d}u - 2\,\mathrm{d}u\mathrm{d}r + g_{uu}(r,u,x)\,\mathrm{d}u^2\,, \tag{3.2}$$

where indices p and q range from 2 to $D-1$, and the metric components of the transverse space g_{pq} are arbitrary functions of all coordinates. Moreover, the covariant and contravariant metric components are related by the following expressions

$$g_{up} = g_{pq}g^{rq}\,, \qquad g_{uu} = -g^{rr} + g_{up}g^{rp}\,, \qquad g_{pn}g^{nq} = \delta^q_p\,. \tag{3.3}$$

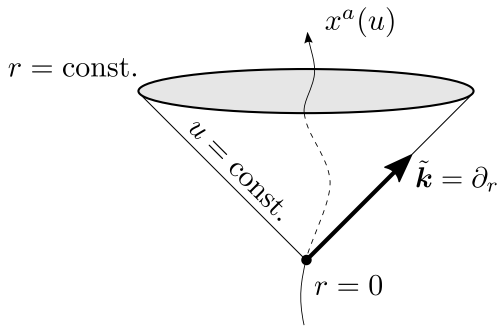

Figure 3.1: Geometric meaning of coordinates $(r,\, u,\, x^p)$ naturally adapted to the presence of non-twisting null geodesic congruence. Spacetime is foliated by null hypersurfaces $u = \text{const.}$ with tangent (and normal) $\tilde{\boldsymbol{k}} = \partial_r$ and r being affine parameter along the congruence. A subspace defined by $u = \text{const.}$ and $r = \text{const.}$ represents $(D-2)$-dimensional Riemannian manifold covered by x^p where $p = 2, \ldots, D-1$.

3.2 Optical scalars

Geometric properties of null geodesic congruences in general D-dimensional spacetimes had been described in [16]. Here we briefly summarized relevant results in unified notation. Let us introduce a frame $(\boldsymbol{k},\, \boldsymbol{l},\, \boldsymbol{m}_i)$, where $\boldsymbol{k}$ and $\boldsymbol{l}$ are null vectors pointing in the future direction, and $\boldsymbol{m}_i$ represent $D-2$ perpendicular spacelike vectors such that

$$\boldsymbol{k}\cdot\boldsymbol{l} = -1\,, \qquad \boldsymbol{m}_i\cdot\boldsymbol{m}_j = \delta_{ij}\,, \qquad \boldsymbol{k}\cdot\boldsymbol{k} = \boldsymbol{l}\cdot\boldsymbol{l} = 0\,, \qquad \boldsymbol{k}\cdot\boldsymbol{m}_i = \boldsymbol{l}\cdot\boldsymbol{m}_i = 0\,. \tag{3.4}$$

The metric tensor can now be decomposed as

$$g_{ab} = -2k_{(a}l_{b)} + \delta_{ij} m^i_a m^j_b\,, \tag{3.5}$$

and the covariant derivative of null vector $\boldsymbol{k}$ can be rewritten in the form

$$k_{a;b} = K_{11}\, k_a k_b + K_{10}\, k_a l_b + K_{1i}\, k_a m^i_b + K_{i1}\, m^i_a k_b + K_{i0}\, m^i_a l_b + K_{ij}\, m^i_a m^j_b\,. \tag{3.6}$$

The components K_{ab} can conversely be expressed as

$$\begin{aligned} K_{11} &= k_{a;b}\, l^a l^b\,, & K_{10} &= k_{a;b}\, l^a k^b\,, & K_{11} &= -k_{a;b}\, l^a m^b_i\,, \\ K_{i1} &= -k_{a;b}\, m^a_i l^b\,, & K_{i0} &= -k_{a;b}\, m^a_i k^b\,, & K_{ij} &= k_{a;b}\, m^a_i m^b_j\,. \end{aligned} \tag{3.7}$$

These components are invariant under null rotation with direction $\boldsymbol{k}$ fixed and simply rescaled under boosts in the $\boldsymbol{k}$-$\boldsymbol{l}$ plane.

To analyze specific meaning of the coefficients (3.7) let us multiply (3.6) by the vector k^b. We obtain $k_{a;b}k^b = -K_{10}\, k_a - K_{i0}\, m^i_a$, which implies that if the vector field $\boldsymbol{k}$ should be geodesic, i.e., $k_{a;b}k^b \sim k_a$, it must hold that $K_{i0} = 0$. Moreover, it will be affine geodesic, i.e., $k_{a;b}k^b = 0$, if, and only if, $K_{10} = 0$. The matrix K_{ij} describes geometric properties of the integral curves congruence generated by the null vector field $\boldsymbol{k}$. To further examine these geometric and optical properties we shall decompose the K_{ij} matrix into its *trace*, *traceless symmetric* part, and *antisymmetric* part, namely

$$K_{ij} = \Theta\delta_{ij} + \sigma_{ij} + A_{ij}\,, \tag{3.8}$$

where

$$\Theta = \frac{\operatorname{Tr} K_{ij}}{D-2}\,, \qquad \sigma_{ij} = K_{(ij)} - \frac{\operatorname{Tr} K_{ij}}{D-2}\delta_{ij}\,, \qquad A_{ij} = K_{[ij]}\,. \tag{3.9}$$

These quantities are usually called *expansion*, *shear matrix*, and *twist matrix*, respectively. They are preserved under null rotation with $\boldsymbol{k}$ direction fixed. For affine geodesic the classic expansion, shear, and twist scalars can be constructed out of them. The optical scalars can be expressed in terms of the vector $\boldsymbol{k}$ covariant derivatives only,

$$\Theta = \frac{1}{D-2} k^a{}_{;a}\,, \qquad \sigma^2 = k_{(a;b)}k^{a;b} - \frac{1}{D-2}(k^a{}_{;a})^2\,, \qquad A^2 = -k_{[a;b]}k^{a;b}\,. \tag{3.10}$$

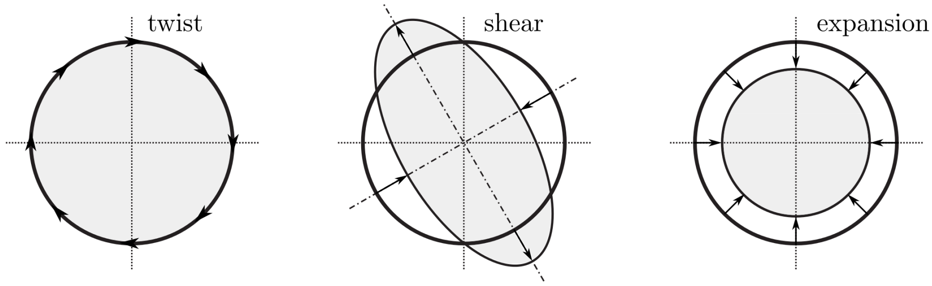

Figure 3.2: Schematic transverse deformation of the test particles congruence caused be the effect of optical scalars.

Vanishing of these optical quantities is fully equivalent to vanishing of the respective original quantities in (3.9).

Non-twisting geometries

Now we calculate optical scalars for the affine geodesic associated with the null vector field $\tilde{\boldsymbol{k}}$, see (3.1), introduced during our construction of the metric (3.2) in previous section. We simply prove that $\tilde{\boldsymbol{k}}$ is *non-twisting*. Since the null vector covariant derivative is now $\tilde{k}_{a;b} = \frac{1}{2} g_{ab,r}$, we obtain

$$\Theta = \frac{1}{2(D-2)} g^{ab} g_{ab,r} = \frac{1}{2(D-2)} g^{pq} g_{pq,r} = \frac{1}{2(D-2)} \left(\ln \det g_{pq}\right)_{,r} \,, \tag{3.11}$$

$$\sigma^2 = \frac{1}{4}\left(g^{ac}g^{bd} - \frac{1}{D-2} g^{ab}g^{cd}\right) g_{ab,r}g_{cd,r} = \frac{1}{4} g^{mp}g^{nq}g_{mn,r}g_{pq,r} - (D-2)\Theta^2 \,, \tag{3.12}$$

$$A^2 = 0 \,, \tag{3.13}$$

where $a,\, b,\, c,\, d = 0, ..., D-1$ and $m,\, n,\, p,\, q = 2, ..., D-1$. Since $A = 0$, we see that the geometry given by metric (3.2) indeed admits *non-twisting null geodesic congruence* generated by $\tilde{\boldsymbol{k}}$.

Finally notice that the construction presented in the previous section is fully general and covers all possible non-twisting geometries. The well-know *Frobenius theorem* generally tells, when the manifold can be foliated by hypersurfaces locally orthogonal to some vector field $\boldsymbol{k}$. This condition can be equivalently written as

$$k_{[a;b}k_{c]} = 0 \,. \tag{3.14}$$

We can see that this holds true for our tangent field $\tilde{k}_a = -u_{,a} = -\delta^u_a$. In an opposite way, it can also be shown, using (3.7), that this condition can be rewritten in terms of the twist matrix A_{ij},

$$k_{[a;b}k_{c]} = \frac{1}{3} A_{ij} m^i_{[a} m^j_b k_{c]} \,, \tag{3.15}$$

and multiplying this expression by *arbitrary* null vector l^c gives

$$k_{[a;b}k_{c]} l^c = -\frac{1}{3} A_{ij} m^i_a m^j_b \,, \tag{3.16}$$

which implies that the condition of vanishing twist matrix $A_{ij} = 0$ is equivalent to (3.14).

Non-twisting and shear-free geometries

In the following sections we mention more particular subclasses of the general non-twisting metrics (3.2), namely

- the *Robinson–Trautman* class: shear-free and expanding geometries,
- the *Kundt* class: shear-free and non-expanding geometries.

To employ the additional shear-free condition, without loss of generality, it is convenient to decompose spatial metric tensor g_{pq} in (3.2), see e.g., [30, 31],

$$g_{pq} = \mathrm{p}^{-2}\gamma_{pq}\,, \tag{3.17}$$

where γ_{pq} is some unimodular matrix, which means that $\det\gamma_{pq} = 1$. Now we simply obtain an expression for the determinant of complete spacetime metric (3.2),

$$\det g_{ab} = -\det g_{pq} = \mathrm{p}^{-2(D-2)}\,. \tag{3.18}$$

For the optical scalars (3.10) we thus have

$$\Theta = -(\ln \mathrm{p})_{,r}\,, \qquad \sigma^2 = \frac{1}{4}\gamma^{mp}\gamma^{nq}\gamma_{mn,r}\gamma_{pq,r}\,. \tag{3.19}$$

These expressions provide restrictions on the r-dependence of spatial part of metric in the specific Robinson–Trautman and Kundt subcases of the non-twisting geometries, respectively.

Finally, let us notice an alternative expression of the shear-free condition which directly uses the shear matrix σ_{ij}, see [34]. We start with the K_{ij} coefficient in (3.7), usually denoted as the optical matrix ρ_{ij}, for the metric (3.2) and vector field $\tilde{\boldsymbol{k}} = \partial_r$, namely

$$K_{ij} \equiv k_{a;b}\, m_i^a m_j^b = k_{p;q}\, m_i^p m_j^q = \frac{1}{2} g_{pq,r}\, m_i^p m_j^q\,, \tag{3.20}$$

where we used $\boldsymbol{m}_i \cdot \boldsymbol{k} = 0$ implying $m_i^u = 0$. It is obviously symmetric which again confirms the non-twisting character of spacetime (3.2), i.e., $A_{ij} = 0$. Imposing the additional shear-free condition $\sigma_{ij} = 0$ in (3.8) thus gives

$$\Theta\delta_{ij} = \frac{1}{2}\, g_{pq,r} m_i^p m_j^q\,, \tag{3.21}$$

which using the orthonormality relation $\delta_{ij} = g_{pq} m_i^p m_j^q$ for the transverse spatial vectors immediately implies

$$g_{pq,r} = 2\Theta g_{pq}\,. \tag{3.22}$$

This relation can be simply integrated to obtain

$$g_{pq} = \mathrm{R}^2(r,u,x)\, h_{pq}(u,x)\,, \qquad \text{with} \qquad \frac{\mathrm{R}_{,r}}{\mathrm{R}} = \Theta \quad \Leftrightarrow \quad \mathrm{R} = \exp\left(\textstyle\int \Theta(r,u,x)\,\mathrm{d}r\right). \tag{3.23}$$

Two subclasses thus naturally arise

- the *Robinson–Trautman* class with $\Theta = \Theta(r,u,x)$, see e.g., [9, 10, 28–34, 65],
- the *Kundt* class with $\Theta = 0$, see e.g., [9, 10, 22–27, 34, 65, 66].

3.3 Geometry of the Kundt class

As mentioned above, the Kundt class is defined in a purely geometric way, i.e., without any relation to specific field equations and particular theory of gravity, as Lorentzian manifolds admitting non-twisting, shear-free, and non-expanding null geodesic congruence. The shear-free ($\sigma = 0$) and non-expanding ($\Theta = 0$) conditions, through the equations (3.19), imply that both factor p and unimodular tensor γ_{pq} are independent of the r coordinate. In other words, this means that the spatial metric g_{pq} is r-independent. The line element of any Kundt spacetime can thus be written as

$$\mathrm{d}s^2 = g_{pq}(u,x)\mathrm{d}x^p\mathrm{d}x^q + 2g_{up}(r,u,x)\mathrm{d}x^p\mathrm{d}u - 2\mathrm{d}u\mathrm{d}r + g_{uu}(r,u,x)\mathrm{d}u^2\,. \tag{3.24}$$

Moreover, by setting

$$g_{up} = 0\,,$$

we restrict our investigation to the *non-gyratonic Kundt spacetimes*, see e.g. [67] for the physical interpretation of the gyratonic terms g_{up}.

In this non-gyratonic case the general Kundt line element (3.24) simplifies into

$$\mathrm{d}s^2 = g_{pq}(u,x)\,\mathrm{d}x^p\,\mathrm{d}x^q - 2\,\mathrm{d}u\,\mathrm{d}r + g_{uu}(r,u,x)\,\mathrm{d}u^2\,, \tag{3.25}$$

and for the contravariant metric components we immediately get

$$g^{pq}\,, \qquad g^{ru} = -1\,, \qquad g^{rr} = -g_{uu}\,. \tag{3.26}$$

The curvature tensors for the metric (3.25), their quadratic combinations, and covariant derivatives are summarized in appendices A and B.

Finally, the natural null frame for the metric (3.25) is given by

$$\boldsymbol{k} = \partial_r\,, \qquad \boldsymbol{l} = \frac{1}{2}\,g_{uu}\partial_r + \partial_u\,, \qquad \boldsymbol{m}_i = m_i^p\,\partial_p\,, \tag{3.27}$$

which satisfies the normalization conditions

$$\boldsymbol{k}\cdot\boldsymbol{l} = -1\,, \qquad \boldsymbol{m}_i\cdot\boldsymbol{m}_j = \delta_{ij}\,, \tag{3.28}$$

which means $g_{pq}m_i^p m_j^q = \delta_{ij}$.

3.3.1 Algebraic structure

For the algebraic classification of Kundt spacetimes in an arbitrary dimension see [26, 34]. Following the general classification scheme of the review [18], let us introduce the Weyl tensor components with respect to the null frame (3.27) sorted by their boost weights as

$$\begin{aligned}
\Psi_{0^{ij}} &= C_{abcd}\,k^a\,m_i^b\,k^c\,m_j^d\,, & & \\
\Psi_{1^{ijk}} &= C_{abcd}\,k^a\,m_i^b\,m_j^c\,m_k^d\,, & \Psi_{1T^i} &= C_{abcd}\,k^a\,l^b\,k^c\,m_i^d \\
\Psi_{2^{ijkl}} &= C_{abcd}\,m_i^a\,m_j^b\,m_k^c\,m_l^d\,, & \Psi_{2S} &= C_{abcd}\,k^a\,l^b\,l^c\,k^d\,, \\
\Psi_{2^{ij}} &= C_{abcd}\,k^a\,l^b\,m_i^c\,m_j^d\,, & \Psi_{2T^{ij}} &= C_{abcd}\,k^a\,m_i^b\,l^c\,m_j^d\,, \\
\Psi_{3^{ijk}} &= C_{abcd}\,l^a\,m_i^b\,m_j^c\,m_k^d\,, & \Psi_{3T^i} &= C_{abcd}\,l^a\,k^b\,l^c\,m_i^d\,, \\
\Psi_{4^{ij}} &= C_{abcd}\,l^a\,m_i^b\,l^c\,m_j^d\,. & &
\end{aligned} \tag{3.29}$$

Moreover, the irreducible components of these scalars, needed for invariant (sub)classification of the Weyl tensor algebraic structure [18], can be written as

$$\begin{aligned}
\tilde{\Psi}_{1^{ijk}} &= \Psi_{1^{ijk}} - \frac{2}{D-3}\delta_{i[j}\Psi_{1T^{k]}}\,, \\
\tilde{\Psi}_{2T^{(ij)}} &= \Psi_{2T^{(ij)}} - \frac{1}{D-2}\delta_{ij}\Psi_{2S}\,, \\
\tilde{\Psi}_{2^{ijkl}} &= \Psi_{2^{ijkl}} - \frac{2}{D-4}\big(\delta_{ik}\tilde{\Psi}_{2T^{(jl)}} + \delta_{jl}\tilde{\Psi}_{2T^{(ik)}} - \delta_{il}\tilde{\Psi}_{2T^{(jk)}} - \delta_{jk}\tilde{\Psi}_{2T^{(il)}}\big) \\
&\quad - \frac{4\,\delta_{i[k}\delta_{l]j}}{(D-2)(D-3)}\Psi_{2S}\,, \\
\tilde{\Psi}_{3^{ijk}} &= \Psi_{3^{ijk}} - \frac{2}{D-3}\delta_{i[j}\Psi_{3T^{k]}}\,.
\end{aligned} \tag{3.30}$$

Projecting the coordinate components of the Weyl tensor, see [26, 34] or (A.21)–(A.30) in appendix A, of a non-gyratonic Kundt geometry on the null frame (3.27) yields the following non-trivial Weyl scalars

$$\Psi_{2S} = \frac{D-3}{D-1}\left[\frac{1}{2}\,g_{uu,rr} + \frac{1}{(D-2)(D-3)}\,{}^{S}\!R\right], \tag{3.31}$$

$$\tilde{\Psi}_{2T^{(ij)}} = m_i^p m_j^q\,\frac{1}{D-2}\left[{}^{S}\!R_{pq} - \frac{1}{D-2}\,g_{pq}\,{}^{S}\!R\right], \tag{3.32}$$

$$\tilde{\Psi}_{2^{ijkl}} = m_i^m m_j^p m_k^n m_l^q\;{}^{S}C_{mpnq}\,, \tag{3.33}$$

$$\Psi_{3T^i} = m_i^p\,\frac{D-3}{D-2}\left[-\frac{1}{2}\,g_{uu,rp} + \frac{1}{D-3}\,g^{mn} g_{m[n,u||p]}\right], \tag{3.34}$$

$$\tilde{\Psi}_{3^{ijk}} = m_i^p m_j^m m_k^q\left[g_{p[m,u||q]} - \frac{1}{D-3}\,g^{os}\left(g_{pm}g_{o[s,u||q]} - g_{pq}g_{o[s,u||m]}\right)\right], \tag{3.35}$$

$$\begin{aligned}\Psi_{4^{ij}} = m_i^p m_j^q \Bigg[&-\frac{1}{2}g_{uu||pq} - \frac{1}{2}g_{pq,uu} + \frac{1}{4}g^{os}g_{op,u}g_{sq,u} + \frac{1}{4}g_{pq,u}g_{uu,r} \\ &-\frac{g_{pq}}{D-2}\,g^{mn}\left(-\frac{1}{2}g_{uu||mn} - \frac{1}{2}g_{mn,uu} + \frac{1}{4}g^{os}g_{om,u}g_{sn,u} + \frac{1}{4}g_{mn,u}g_{uu,r}\right)\Bigg], \end{aligned} \tag{3.36}$$

where ${}^{S}C_{mpnq}$, ${}^{S}R_{pq}$, ${}^{S}R$ are the Weyl tensor, the Ricci tensor, and the Ricci scalar with respect to the transverse space metric g_{pq}, respectively. The symbol $_{||}$ represents the trasverse space covariant derivative.

We can thus conclude that the non-gyratonic Kundt spacetimes (3.25) are at least of the Weyl type II(d). The specific conditions under which this geometry becomes more algebraically special with respect to the null frame (3.27) are summarized in table 3.1.

type	necessary and sufficient conditions
II(ad)	$g_{uu} = -\frac{{}^{S}R(u,x)}{(D-2)(D-3)}\,r^2 + b(u,x)\,r + c(u,x)$
II(bd)	${}^{S}R_{pq} = \frac{1}{D-2}\,g_{pq}\,{}^{S}R$
II(cd)	${}^{S}C_{mpnq} = 0$
III	II(abcd)
III(a)	${}^{S}R_{,p} = 0$ and $b_{,p} = \frac{2}{D-3}\,g^{mn}g_{m[n,u\|\|p]}$
III(b)	$g_{p[m,u\|\|q]} = \frac{1}{D-3}\,g^{os}\left(g_{pm}g_{o[s,u\|\|q]} - g_{pq}g_{o[s,u\|\|m]}\right)$
N	III(ab)
O	$g^{os}g_{o[s,u\|\|p]\|\|q} + \frac{1}{2(D-2)}\,{}^{S}R\,g_{pq,u} = \frac{g_{pq}}{D-2}\,g^{mn}\left(g^{os}g_{o[s,u\|\|m]\|\|n} + \frac{1}{2(D-2)}\,{}^{S}R\,g_{mn,u}\right)$
	$c_{\|\|pq} + g_{pq,uu} - \frac{1}{2}g^{os}g_{op,u}g_{sq,u} - \frac{1}{2}\,b\,g_{pq,u} = \frac{g_{pq}}{D-2}\,g^{mn}\left(c_{\|\|mn} + g_{mn,uu} - \frac{1}{2}g^{os}g_{om,u}g_{sn,u} - \frac{1}{2}\,b\,g_{mn,u}\right)$

Table 3.1: The scheme for algebraic classification of the Weyl tensor in the case of arbitrary dimensional non-gyratonic Kundt geometries (3.25). The null vector $\boldsymbol{k} = \partial_r$ represents multiple Weyl aligned direction (WAND). Moreover, the type D subclass can be directly identified with respect to the the null vector $\boldsymbol{l} = \frac{1}{2}g_{uu}\,\partial_r + \partial_u$ using (3.34)–(3.36).

3.3.2 Special cases of the Kundt geometry

Here, we briefly present several special, geometrically restricted, cases of the Kundt class. Further discussion, geometric and physical interpretation together with algebraic classification of the Weyl tensor can be found, e.g., in [9,10,24–27].

pp-waves

Defining property of the *pp*-wave geometries, i.e., plane-fronted waves with parallel rays, is existence of a *covariantly constant null vector field* $\boldsymbol{k}$. This means that all the optical scalars associated with this field, see (3.9), have to vanish by the definition, and *pp*-waves thus belong to the Kundt class of geometries. In terms of coordinates naturally adapted to the non-twisting geometries (3.2) with $\boldsymbol{k} = \partial_r$ the defining condition for *pp*-waves can be written as

$$k_{a;b} = \frac{1}{2} g_{ab,r} = 0\,,$$

and all the metric functions have to be necessarily independent of the coordinate r. The line element (3.25) for non-gyratonic *pp*-waves thus becomes

$$\mathrm{d}s^2 = g_{pq}(u,x)\,\mathrm{d}x^p\,\mathrm{d}x^q - 2\,\mathrm{d}u\,\mathrm{d}r + g_{uu}(u,x)\,\mathrm{d}u^2\,, \tag{3.37}$$

where the metric component g_{uu} is r-independent. The algebraic type is II(d) or more special.

VSI spacetimes

The VSI spacetimes are defined by the property that their *scalar curvature invariants of all orders vanish* identically. They belong to the Kundt class and their Riemann tensor is of algebraic type III or more special with respect to the vector field $\boldsymbol{k}$. Their transverse space must be flat, i.e., $g_{pq} = \delta_{pq}$, and the metric component g_{uu} is at most quadratic in r. In the general case coefficient proportional to r^2 in g_{uu} depends only on the off-diagonal gyratonic terms. This implies that it has to be at most linear in r in the non-gyratonic case (3.25) which thus becomes

$$\mathrm{d}s^2 = \delta_{pq}\,\mathrm{d}x^p\,\mathrm{d}x^q - 2\,\mathrm{d}u\,\mathrm{d}r + [\,c(u,x)\,r + d(u,x)\,]\,\mathrm{d}u^2\,. \tag{3.38}$$

More details can be found in [68–70].

Direct product spacetimes

Here we consider a special case of the non-gyratonic Kundt metrics, namely

$$\mathrm{d}s^2 = g_{pq}(x)\,\mathrm{d}x^p\mathrm{d}x^q - 2\,\mathrm{d}u\mathrm{d}r + b\,r^2\mathrm{d}u^2\,, \tag{3.39}$$

where b is a constant, and the transverse metric g_{pq} is u-independent, i.e., $g_{pq,u} = 0$. Such spacetime must be of algebraic type D or O, respectively, see table 3.1. In fact it corresponds to *direct product geometry* which can be understood as a composition of $(D-2)$-dimensional Riemannian space with metric $g_{pq}(x)$ and 2-dimensional Lorentzian spacetime of constant curvature. Indeed, using the transformation

$$u = \frac{1}{b\,U}\,, \qquad\qquad r = \frac{-2U}{1-\frac{1}{2}b\,UV}\,, \tag{3.40}$$

the metric (3.39) can be written in the canonical form

$$\mathrm{d}s^2 = g_{pq}(x)\mathrm{d}x^p\mathrm{d}x^q - \frac{2\,\mathrm{d}U\mathrm{d}V}{(1-\frac{1}{2}b\,UV)^2}\,, \tag{3.41}$$

where the sign of constant parameter b determines curvature of the 2-dimensional temporal surface $x = const.$, namely

- for $b > 0$: 2-dimensional *de Sitter* space dS_2,
- for $b = 0$: 2-dimensional flat *Minkowski* space M_2,
- for $b < 0$: 2-dimensional *anti-de Sitter* space AdS_2.

In the generic case of algebraic type D, the transverse metric g_{pq} does not have to be of constant curvature. For the more special subtype D(a), Ricci scalar ${}^{S}\!R$ of the transverse $(D-2)$-dimensional space must be constant. Moreover, ${}^{S}\!R$ is then uniquely related to the Gaussian curvature b. In such a case, the spacetime is higher-dimensional analogue of either Bertotti–Robinson, (anti-)Nariai, or Plebański–Hacyan spacetime in four dimensions [10].

Constant spatial curvature

The aditional purely geometric constraint which can be imposed on the Kundt ansazt is a specific *restriction on the transverse space*. In particular, we may require that the scalar curvature ${}^S R$ of the transverse $(D-2)$-dimensional space with Riemannian metric $g_{pq}(u,x)$ is *constant*. This is assumed with respect to the spatial coordinates x^p, i.e., it may still depend on the coordinate u. In such a case the Riemann tensor becomes

$$ {}^S R_{pqmn} = \frac{{}^S R}{(D-3)(D-2)} \left(g_{pm} g_{qn} - g_{pn} g_{qm} \right) , \tag{3.42} $$

and the Ricci tensor then takes the form

$$ {}^S R_{pq} = \frac{{}^S R}{D-2} \, g_{pq} \, . \tag{3.43} $$

The spatial metric $g_{pq} = g_{pq}(x,u)$ can be written in a conformally flat form as

$$ g_{pq} = P^{-2} \, \delta_{pq} \, , \qquad \text{with} \qquad P = 1 + \frac{{}^S R}{4(D-3)(D-2)} \left[\left(x^2\right)^2 + \cdots + \left(x^{D-1}\right)^2 \right] . \tag{3.44} $$

4. Non-gyratonic Kundt spacetimes in quadratic gravity

In this chapter we formulate field equations for non-gyratonic Kundt geometries, i.e., those with off-diagonal terms vanishing, in a generic quadratic gravity, see section 4.1. Our aim is to observe qualitative differences between the fourth order equations of quadratic gravity and the second order Einstein's equations of general relativity. It is crucial to identify such physically defined situations which can be directly compared in both theories. Subsequently, we thus investigate several geometrically privileged settings of the Kundt metric within these generic field equations. The simplest and also most illustrative case corresponds to the famous *pp*-wave spacetime with flat transverse metric, see section 4.2.1.

As a special case of the quadratic gravity we will discuss the non-gyratonic Kundt geometries in the Gauss–Bonnet theory. These results are summarized in the independent chapter 5.

4.1 General form of the field equations

Here we present completely general form of *field equations* (1.30) for the *non-gyratonic Kundt spacetimes* (3.25) in *generic quadratic gravity* (1.29). To obtain this final result we started with components of the curvature tensors, see appendix A. Then we calculated required quadratic terms and corresponding derivatives in advance for each coordinate component, see appendix B. Finally, we substituted these partial results into the field equations of quadratic gravity (1.30) and simplified them. The obtained results are presented here and can serve as a starting point for the further discussion of non-gyratonic Kundt geometries with arbitrary matter fields. In chapters that follow, we investigate several particular geometrically privileged subcases of the Kundt class within full quadratic gravity and the Gauss–Bonnet theory, respectively.

For the convenience we write the field equations (1.30) in the shortened form including possible matter fields,

$$J_{ab} = \frac{1}{2}\, T_{ab}\,, \tag{4.1}$$

where the geometric left hand side is defined as

$$\begin{aligned} J_{ab} \equiv & \frac{1}{\kappa}\left(R_{ab} - \frac{1}{2} R g_{ab} + \Lambda_0 g_{ab}\right) + 2\alpha R\left(R_{ab} - \frac{1}{4} R g_{ab}\right) \\ & + (2\alpha+\beta)\big(g_{ab}\Box - \nabla_a\nabla_b\big) R \\ & + 2\gamma\left[R R_{ab} - 2R_{acbd}R^{cd} + R_{acde}{R_b}^{cde} - 2R_{ac}{R_b}^{c} - \frac{1}{4}\, g_{ab}\left(R^2_{cdef} - 4R^2_{cd} + R^2\right)\right] \\ & + \beta\Box\left(R_{ab} - \frac{1}{2} R g_{ab}\right) + 2\beta\left(R_{acbd} - \frac{1}{4}\, g_{ab} R_{cd}\right) R^{cd}\,. \end{aligned} \tag{4.2}$$

Let us also introduce symbol ${}^S\Box$ for the spatial d'Alembert operator acting on any tensorial quantity $\mathbf{T}$, which can as well be called the Laplacian since the spatial metric is positive definite,

$${}^S\Box\, \mathbf{T} \equiv g^{pq}\, \mathbf{T}_{||pq}\,, \tag{4.3}$$

where $_{||}$ represents the covariant derivative with respect to the transverse space metric g_{pq}. In genral, the upper left index ^{S} denotes quantities associated with this $(D-2)$-dimensional Riemannien space.

Now, we can finally present the most general left hand side J_{ab} of the field equations (4.1):

• *rr-component*

$$J_{rr} = (2\alpha + \beta)\, g_{uu,rrrr}\,, \tag{4.4}$$

• *rp-component*

$$J_{rp} = (2\alpha + \beta)\, g_{uu,rrrp}\,, \tag{4.5}$$

• *ru-component*

$$\begin{aligned} J_{ru} =& \frac{1}{2\kappa}\left({}^S\!R - 2\Lambda_0\right) + \frac{\alpha}{2}\left({}^S\!R^2 - {g_{uu,rr}}^2\right) \\ &+ (2\alpha+\beta)\left(g_{uu}g_{uu,rrrr} + g_{uu,rrru} + \frac{1}{2}g_{uu,r}g_{uu,rrr} + \frac{1}{2}g^{mn}g_{mn,u}g_{uu,rrr} - {}^S\Box\, g_{uu,rr} - {}^S\Box\, {}^S\!R\right) \\ &+ \frac{\gamma}{2}\left({}^S\!R^2_{klmn} - 4\,{}^S\!R^2_{mn} + {}^S\!R^2\right) + \frac{\beta}{4}\left(2\,{}^S\Box\, {}^S\!R - {g_{uu,rr}}^2 + 2\,{}^S\!R^2_{mn}\right)\,, \end{aligned} \tag{4.6}$$

• *pq-component*

$$\begin{aligned} J_{pq} =& \frac{1}{\kappa}\left[{}^S\!R_{pq} - \frac{1}{2}\,g_{pq}\left({}^S\!R + g_{uu,rr} - 2\Lambda_0\right)\right] + 2\alpha\left[{}^S\!R_{pq}\left({}^S\!R + g_{uu,rr}\right) - \frac{1}{4}\,g_{pq}\left({}^S\!R + g_{uu,rr}\right)^2\right] \\ &+ (2\alpha+\beta)\left[g_{pq}\left({}^S\Box\, {}^S\!R + {}^S\Box\, g_{uu,rr} - g_{uu}g_{uu,rrrr} - 2g_{uu,rrru} - g_{uu,r}g_{uu,rrr} - \frac{1}{2}g^{mn}g_{mn,u}g_{uu,rrr}\right)\right. \\ &\qquad\qquad \left. -g_{uu,rr||pq} + \frac{1}{2}g_{pq,u}g_{uu,rrr} - {}^S\!R_{||pq}\right] \\ &+ 2\gamma\left[{}^S\!R_{pq}\left({}^S\!R + g_{uu,rr}\right) - 2\,{}^S\!R_{pmqn}\,{}^S\!R^{mn} + {}^S\!R_{pklm}\,{}^S\!R_q{}^{klm} - 2\,{}^S\!R_{pm}\,{}^S\!R_q{}^m\right. \\ &\qquad \left. -\frac{1}{4}\,g_{pq}\left(2g_{uu,rr}\,{}^S\!R + {}^S\!R^2_{klmn} - 4\,{}^S\!R^2_{mn} + {}^S\!R^2\right)\right] \\ &+ \beta\left[{}^S\Box\, {}^S\!R_{pq} + 2\,{}^S\!R_{pmqn}\,{}^S\!R^{mn}\right. \\ &\qquad - \frac{1}{2}g_{pq}\left({}^S\Box\, {}^S\!R + {}^S\Box\, g_{uu,rr} + {}^S\!R^2_{mn} - g_{uu}g_{uu,rrrr} - 2g_{uu,rrru}\right. \\ &\qquad\qquad \left.\left. -g_{uu,r}g_{uu,rrr} + \frac{1}{2}{g_{uu,rr}}^2 - \frac{1}{2}g^{mn}g_{mn,u}g_{uu,rrr}\right)\right]\,, \end{aligned} \tag{4.7}$$

and *pq-component* trace

$$\begin{aligned} g^{pq}J_{pq} =& \frac{1}{\kappa}\left[{}^S\!R - \frac{1}{2}\,(D-2)\left({}^S\!R + g_{uu,rr} - 2\Lambda_0\right)\right] + 2\alpha\left[{}^S\!R\left({}^S\!R + g_{uu,rr}\right) - \frac{1}{4}\,(D-2)\left({}^S\!R + g_{uu,rr}\right)^2\right] \\ &+ (2\alpha+\beta)\left[(D-2)\left({}^S\Box\, {}^S\!R + {}^S\Box\, g_{uu,rr} - g_{uu}g_{uu,rrrr} - 2g_{uu,rrru} - g_{uu,r}g_{uu,rrr} - \frac{1}{2}g^{mn}g_{mn,u}g_{uu,rrr}\right)\right. \\ &\qquad\qquad \left. - {}^S\Box\, g_{uu,rr} + \frac{1}{2}g^{mn}g_{mn,u}g_{uu,rrr} - {}^S\Box\, {}^S\!R\right] \\ &+ 2\gamma\left[g_{uu,rr}\,{}^S\!R + {}^S\!R^2_{klmn} - 4\,{}^S\!R^2_{mn} + {}^S\!R^2 - \frac{1}{4}\,(D-2)\left(2g_{uu,rr}\,{}^S\!R + {}^S\!R^2_{klmn} - 4\,{}^S\!R^2_{mn} + {}^S\!R^2\right)\right] \\ &+ \beta\left[{}^S\Box\, {}^S\!R + 2\,{}^S\!R^2_{mn}\right. \\ &\qquad - \frac{1}{2}\,(D-2)\left({}^S\Box\, {}^S\!R + {}^S\Box\, g_{uu,rr} + {}^S\!R^2_{mn} - g_{uu}g_{uu,rrrr} - 2g_{uu,rrru}\right. \\ &\qquad\qquad \left.\left. -g_{uu,r}g_{uu,rrr} + \frac{1}{2}{g_{uu,rr}}^2 - \frac{1}{2}g^{mn}g_{mn,u}g_{uu,rrr}\right)\right]\,, \end{aligned} \tag{4.8}$$

• *up-component*

$$
\begin{aligned}
J_{up} = &\frac{1}{\kappa}\left(-\frac{1}{2}g_{uu,rp} + g^{mn}g_{m[p,u||n]}\right) + 2\alpha\left({}^{S}\!R + g_{uu,rr}\right)\left(-\frac{1}{2}g_{uu,rp} + g^{mn}g_{m[p,u||n]}\right)\\
&-(2\alpha+\beta)\left[g_{uu,rrup} + \frac{1}{2}g_{uu,p}g_{uu,rrr} + {}^{S}\!R_{,up} - \frac{1}{2}g^{mn}g_{mp,u}\left({}^{S}\!R_{,n} + g_{uu,rrn}\right)\right]\\
&+2\gamma\left[\left(-\frac{1}{2}g_{uu,rp} + g^{mn}g_{m[p,u||n]}\right){}^{S}\!R - 2g_{m[p,u||n]}\,{}^{S}\!R^{mn}\right.\\
&\qquad\left. + g_{k[l,u||m]}\,{}^{S}\!R_{p}{}^{klm} + \left(g_{uu,rm} - 2g^{kl}g_{k[m,u||l]}\right){}^{S}\!R_{p}{}^{m}\right]\\
&+\beta\left[\frac{1}{2}g_{uu}g_{uu,rrrp} + g_{uu,rrup} + \frac{1}{2}g_{uu,p}g_{uu,rrr} - \frac{1}{2}g_{uu,rr}g_{uu,rp} + 2g_{m[p,u||n]}\,{}^{S}\!R^{mn} - \frac{1}{2}\,{}^{S}\Box\, g_{uu,rp}\right.\\
&\qquad + g^{mn}\left(\frac{1}{2}g_{m[p,u||n]}g_{uu,rr} - \frac{1}{2}g_{uu,rm}\,{}^{S}\!R_{np} + {}^{S}\Box\, g_{m[p,u||n]} + \frac{1}{4}g_{uu,rr}g_{pm,u||n}\right.\\
&\qquad\qquad\left.\left. - \frac{1}{2}\,{}^{S}\!R^{k}{}_{p}\,g_{km,u||n} - {}^{S}\!R^{k}{}_{p||n}\,g_{km,u} + \frac{1}{4}g_{mn,u}g_{uu,rrp}\right)\right], \qquad (4.9)
\end{aligned}
$$

• *uu-component*

$$
\begin{aligned}
J_{uu} = &\frac{1}{\kappa}\left[\frac{1}{4}g^{mn}\left(g_{mn,u}g_{uu,r} - 2g_{mn,uu} + g^{pq}g_{pm,u}g_{qn,u}\right) - \frac{1}{2}\,{}^{S}\Box\, g_{uu} - \frac{1}{2}g_{uu}\left({}^{S}\!R - 2\Lambda_0\right)\right]\\
&+\frac{\alpha}{2}\left({}^{S}\!R + g_{uu,rr}\right)\left[g^{mn}\left(g_{mn,u}g_{uu,r} - 2g_{mn,uu} + g^{pq}g_{pm,u}g_{qn,u}\right) - 2\,{}^{S}\Box\, g_{uu} - g_{uu}\left({}^{S}\!R - g_{uu,rr}\right)\right]\\
&+(2\alpha+\beta)\left[-g_{uu}{}^{2}g_{uu,rrrr} - 2g_{uu}g_{uu,rrru} + g_{uu}\,{}^{S}\Box\left({}^{S}\!R + g_{uu,rr}\right) - {}^{S}\!R_{,uu} - g_{uu,rruu}\right.\\
&\qquad - \frac{1}{2}g_{uu,rrr}\left(g_{uu}g_{uu,r} + g^{mn}g_{mn,u}g_{uu} + g_{uu,u}\right)\\
&\qquad\left. + \frac{1}{2}g_{uu,r}\left({}^{S}\!R_{,u} + g_{uu,rru}\right) - \frac{1}{2}g^{mn}g_{uu,m}\left({}^{S}\!R_{,n} + g_{uu,rrn}\right)\right]\\
&+2\gamma\left[\left({}^{S}\!R^{mn} - \frac{1}{2}\,{}^{S}\!R\,g^{mn}\right)\left(g_{uu||mn} + g_{mn,uu} - \frac{1}{2}g_{uu,r}g_{mn,u} - \frac{1}{2}g^{pq}g_{mp,u}g_{nq,u}\right)\right.\\
&\qquad\left. + \left(g^{mo}g^{ns} - 2\,g^{mn}g^{os}\right)g^{pq}g_{m[p,u||n]}g_{o[q,u||s]} - \frac{1}{4}g_{uu}\left({}^{S}\!R^{2}_{klmn} - 4\,{}^{S}\!R^{2}_{mn} + {}^{S}\!R^{2}\right)\right]\\
&+\beta\left[g_{uu}\left(\frac{1}{4}g_{uu,rr}{}^{2} - \frac{1}{4}g^{mn}g_{mn,u}g_{uu,rrr} + \frac{1}{2}\,{}^{S}\Box\, g_{uu,rr} - \frac{1}{2}\,{}^{S}\Box\,{}^{S}\!R - \frac{1}{2}\,{}^{S}\!R^{2}_{mn}\right)\right.\\
&\qquad - \frac{1}{2}g^{mn}{}_{,u}\,g_{mn,u}g_{uu,rr} - \frac{1}{2}g^{mn}g_{mn,uu}g_{uu,rr} - \frac{1}{2}g^{mn}g_{mn,u}g_{uu,rru} + \left({}^{S}\Box\, g_{uu,r}\right)_{,u}\\
&\qquad + g_{uu,r}\left(\frac{1}{4}g^{mn}g_{mn,u}g_{uu,rr} - \frac{1}{2}\,{}^{S}\Box\, g_{uu,r}\right) - \frac{1}{2}g^{pq}g_{uu,rp}\left(g_{uu,rq} - g^{mn}g_{mq,u||n}\right)\\
&\qquad + g^{mn}g_{uu,m}g_{uu,rrn} + \frac{1}{4}g_{uu,r}g^{mn}\,{}^{S}\Box\, g_{mn,u} + \frac{1}{2}g^{mn}g_{mn,u}\,{}^{S}\Box\, g_{uu,r}\\
&\qquad - \frac{1}{2}g^{mn}\,{}^{S}\Box\, g_{mn,uu} - \frac{1}{2}\,{}^{S}\Box\,{}^{S}\Box\, g_{uu} + {}^{S}\!R^{mn}\left(\frac{1}{2}g_{mn,u}g_{uu,r} - g_{uu||mn} - g_{mn,uu}\right)\\
&\qquad + \frac{1}{2}g^{mn}g^{pq}\left(g_{pm,u}\,{}^{S}\Box\, g_{qn,u} + g^{os}g_{pm,u||o}g_{qn,u||s}\right)\\
&\qquad + g^{pq}\left(g^{mn}g_{mp,u||q}\left(\frac{1}{2}g_{uu,rn} - g^{os}g_{o[n,u||s]}\right) + 2g^{mn}g_{mp,u}\left(\frac{1}{2}g_{uu,rn||q} - g^{os}g_{o[n,u||s]||q}\right)\right.\\
&\qquad\qquad\left.\left. - \frac{1}{8}g_{pq,u}g^{mn}g_{mn,u}g_{uu,rr} + \left({}^{S}\!R^{mn} - \frac{1}{4}g_{uu,rr}g^{mn}\right)g_{mp,u}g_{nq,u}\right)\right], \qquad (4.10)
\end{aligned}
$$

Even in the geometrically restricted case of *vacuum* non-gyratonic Kundt class the general field equation (4.1), namely $J_{ab} = 0$, represent very complicated system of the fourth order non-linear PDEs. At this moment we will not be interested in such a general discussion, see only its sketch at the end of this section. We concentrate our attention on the geometrically privileged cases within non-gyratonic Kundt class in the generic quadratic gravity, i.e., without any assumption on constants α, β, γ, κ, and Λ_0. See section 4.2.1 for the *pp*-wave case, section 4.2.2

analysing subclass of VSI spacetimes, and section 4.2.3 interested in exact gravitational waves on type D or O backgrounds. Moreover, the non-gyratonic Kundt spacetimes in the important subclass of quadratic theories, namely the Gauss–Bonnet gravity, are discussed in chapter 5.

Qualitative discussion of generic vacuum case

As a final point of this part, let us mention at least qualitative steps in discussion of a generic *vacuum* case $J_{ab} = 0$. In particular, the first two equations, i.e., with J_{ab} given by (4.4) and (4.5), provide simple restrictions on the metric component g_{uu}. The remaining equations which tie together metric components g_{uu}, g_{pq}, and the spatial curvature ${}^{S}R_{mpnq}$ are highly non-trivial and will be in detail subject to further analyses.

• vacuum rr-equation can be written as

$$(2\alpha + \beta)\, g_{uu,rrrr} = 0\,, \tag{4.11}$$

which means that we have to discuss two separate cases according to value of $(2\alpha + \beta)$. For the particular choice constants $(2\alpha + \beta) = 0$ the rr-equation is trivially satisfied and provides no restriction on the Kundt metric (3.25).

However, for $(2\alpha+\beta) \neq 0$ we have found out that the r-dependence of the metric component g_{uu} can be integrated. It has to be at most of the third order in r. We can thus write

$$g_{uu} = a(u, x)r^3 + b(u, x)r^2 + c(u, x)r + d(u, x)\,. \tag{4.12}$$

Subsequently, in this generic case, i.e., $(2\alpha+\beta) \neq 0$, we can substitute (4.12) into the remaining field equations and try to find restrictions on functions a, b, c, and d, respectively.

• vacuum rp-equation becomes

$$(2\alpha + \beta)\, g_{uu,rrrp} = 0\,, \tag{4.13}$$

which in the case $(2\alpha + \beta) = 0$ puts no nrestriction on the metric.

On the other hand, in the generic case $(2\alpha + \beta) \neq 0$ we substitute (4.12) to obtain

$$(2\alpha + \beta)\, a_{,p} = 0\,, \tag{4.14}$$

which implies $a_{,p} = 0$, and the function a is thus independent of the spatial coordinates x^p.

• vacuum ru-equation given by (4.6) relates specific components contained in metric function g_{uu}, see (4.12), and u-derivative of the function a proportional to the highest order in r with the transverse space geometry in the case $(2\alpha + \beta) \neq 0$, or it constraints just this transverse geometry in the complementary case $(2\alpha + \beta) = 0$.

• vacuum pq-equation together with its trace, see (4.7) and (4.8), respectively, give primarily restriction on the spatial dependence of b function in g_{uu}, see (4.12), in terms of transverse geometry in the case $(2\alpha + \beta) \neq 0$. For $(2\alpha + \beta) = 0$ it represents very non-trivial condition for g_{uu} metric term which makes it hopeless to continue in this qualitative discussion here.

• vacuum up-equation (4.9) can be interpreted as constraint on the spatial dependence of c coefficient in (4.12) in the case $(2\alpha + \beta) \neq 0$.

• vacuum uu-equation $J_{uu} = 0$ with (4.9) finally gives condition for the spatial dependence of r-independent d term in (4.12) in the case $(2\alpha + \beta) \neq 0$.

4.2 Geometrically privileged situations

Here we employ our general form of the field equations to derive specific constraints for several geometrically privileged members of the Kundt family of spacetimes.

4.2.1 *pp*-waves

These geometries are defined by admitting a covariantly constant null vector field which leads to restriction on the non-gyratonic Kundt metric (3.25) to be r-independent, see section 3.3.2,

$$\mathrm{d}s^2 = g_{pq}(u,x)\,\mathrm{d}x^p\,\mathrm{d}x^q - 2\,\mathrm{d}u\,\mathrm{d}r + g_{uu}(u,x)\,\mathrm{d}u^2\,. \tag{4.15}$$

Due to the r-independence of g_{uu} function *rr-component* and *rp-component* of the *vacuum field equations* are satisfied identically. The remaining components can be expressed as

- *ru-component*

$$\begin{aligned}&\frac{1}{2\kappa}\left({}^S\!R - 2\Lambda_0\right) + \frac{\alpha}{2}\,{}^S\!R^2 - (2\alpha+\beta)\,{}^S\Box\,{}^S\!R\\ &\quad + \frac{\gamma}{2}\left({}^S\!R^2_{klmn} - 4\,{}^S\!R^2_{mn} + {}^S\!R^2\right) + \frac{\beta}{2}\left({}^S\Box\,{}^S\!R + {}^S\!R^2_{mn}\right) = 0\,,\end{aligned} \tag{4.16}$$

- *pq-component*

$$\begin{aligned}&\frac{1}{\kappa}\left[{}^S\!R_{pq} - \frac{1}{2}\,g_{pq}\left({}^S\!R - 2\Lambda_0\right)\right] + 2\alpha\,{}^S\!R\left({}^S\!R_{pq} - \frac{1}{4}\,g_{pq}\,{}^S\!R\right) + (2\alpha+\beta)\left(g_{pq}\,{}^S\Box\,{}^S\!R - {}^S\!R_{||pq}\right)\\ &+ 2\gamma\left[{}^S\!R\,{}^S\!R_{pq} - 2\,{}^S\!R_{pmqn}\,{}^S\!R^{mn} + {}^S\!R_{pklm}\,{}^S\!R_q{}^{klm} - 2\,{}^S\!R_{pm}\,{}^S\!R_q{}^m\right.\\ &\qquad\left. -\frac{1}{4}\,g_{pq}\left({}^S\!R^2_{klmn} - 4\,{}^S\!R^2_{mn} + {}^S\!R^2\right)\right]\\ &+ \beta\left[{}^S\Box\,{}^S\!R_{pq} + 2\,{}^S\!R_{pmqn}\,{}^S\!R^{mn} - \frac{1}{2}g_{pq}\left({}^S\Box\,{}^S\!R + {}^S\!R^2_{mn}\right)\right] = 0\,,\end{aligned} \tag{4.17}$$

- *up-component*

$$\begin{aligned}&\frac{1}{\kappa}g^{mn}g_{m[p,u||n]} + 2\alpha\,{}^S\!R g^{mn}g_{m[p,u||n]} - (2\alpha+\beta)\left({}^S\!R_{,up} - \frac{1}{2}g^{mn}g_{mp,u}\,{}^S\!R_{,n}\right)\\ &+ 2\gamma\left(g^{mn}g_{m[p,u||n]}\,{}^S\!R - 2g_{m[p,u||n]}\,{}^S\!R^{mn} + g_{k[l,u||m]}\,{}^S\!R_p{}^{klm} - 2g^{kl}g_{k[m,u||l]}\,{}^S\!R_p{}^m\right)\\ &+ \beta\left[2g_{m[p,u||n]}\,{}^S\!R^{mn} + g^{mn}\left({}^S\Box\,g_{m[p,u||n]} - \frac{1}{2}\,{}^S\!R^k{}_p\,g_{km,u||n} - {}^S\!R^k{}_{p||n}\,g_{km,u}\right)\right] = 0\,,\end{aligned} \tag{4.18}$$

- *uu-component*

$$\begin{aligned}&\frac{1}{\kappa}\left[\frac{1}{4}g^{mn}\left(-2g_{mn,uu} + g^{pq}g_{pm,u}g_{qn,u}\right) - \frac{1}{2}\,{}^S\Box\,g_{uu} - \frac{1}{2}g_{uu}\left({}^S\!R - 2\Lambda_0\right)\right]\\ &+ \frac{\alpha}{2}\,{}^S\!R\left[g^{mn}\left(-2g_{mn,uu} + g^{pq}g_{pm,u}g_{qn,u}\right) - 2\,{}^S\Box\,g_{uu} - g_{uu}\,{}^S\!R\right]\\ &+ (2\alpha+\beta)\left(g_{uu}\,{}^S\Box\,{}^S\!R - {}^S\!R_{,uu} - \frac{1}{2}g^{mn}g_{uu,m}\,{}^S\!R_{,n}\right)\\ &+ 2\gamma\left[\left({}^S\!R^{mn} - \frac{1}{2}\,{}^S\!R\,g^{mn}\right)\left(g_{uu||mn} + g_{mn,uu} - \frac{1}{2}g^{pq}g_{mp,u}g_{nq,u}\right)\right.\\ &\qquad\left. + \left(g^{mo}g^{ns} - 2\,g^{mn}g^{os}\right)g^{pq}g_{m[p,u||n]}g_{o[q,u||s]} - \frac{1}{4}g_{uu}\left({}^S\!R^2_{klmn} - 4\,{}^S\!R^2_{mn} + {}^S\!R^2\right)\right]\\ &+ \beta\left[-\frac{1}{2}g_{uu}\left({}^S\Box\,{}^S\!R + {}^S\!R^2_{mn}\right) - \frac{1}{2}g^{mn}\,{}^S\Box\,g_{mn,uu} - \frac{1}{2}\,{}^S\Box\,{}^S\Box\,g_{uu}\right.\\ &\quad - {}^S\!R^{mn}\left(g_{uu||mn} + g_{mn,uu}\right) + \frac{1}{2}g^{mn}g^{pq}\left(g_{pm,u}\,{}^S\Box\,g_{qn,u} + g^{os}g_{pm,u||o}g_{qn,u||s}\right)\\ &\quad\left. + g^{pq}\left({}^S\!R^{mn}g_{mp,u}g_{nq,u} - g^{mn}g^{os}g_{mp,u||q}g_{o[n,u||s]} - 2g^{mn}g^{os}g_{mp,u}g_{o[n,u||s]||q}\right)\right] = 0\,.\end{aligned} \tag{4.19}$$

Even in this simplified *pp*-wave setting it seems to be very difficult to make any progress in solving these equations without some other assumption on the metric ansatz.

pp-waves with the transverse space of constant curvature

In addition to general non-gyratonic pp-wave case, we can assume transverse space being of constant curvature, see the last paragraph in subsection 3.3.2. Then the ru and pq-component of the field equations relate parameters of the theory with the scalar curvature, namely

• *ru-component*

$$\frac{1}{2\kappa}\left({}^S\!R-2\Lambda_0\right)+\frac{\alpha}{2}\,{}^S\!R^2+\frac{\gamma\,(D-4)(D-5)}{2\,(D-2)(D-3)}\,{}^S\!R^2+\frac{\beta}{2\,(D-2)}\,{}^S\!R^2=0\,, \tag{4.20}$$

• *pq-component*

$$\begin{aligned}&\frac{1}{\kappa}\left[-\frac{D-4}{2\,(D-2)}\,{}^S\!R+\Lambda_0\right]g_{pq}-\alpha\frac{D-6}{2\,(D-2)}\,{}^S\!R^2 g_{pq}\\&\qquad-\gamma\frac{(D-4)(D-5)(D-6)}{2\,(D-2)^2(D-3)}\,{}^S\!R^2 g_{pq}-\beta\frac{D-6}{2\,(D-2)^2}\,{}^S\!R^2 g_{pq}=0\,.\end{aligned} \tag{4.21}$$

Moreover, we can trivially compute trace of this equation, divide it by factor $(D-2)$, and finally combine it with the ru-equation (4.20) to get quadratic equation for the transverse space scalar curvature ${}^S\!R$,

$$\left[2\alpha+2\gamma\frac{(D-4)(D-5)}{(D-2)(D-3)}+\frac{2\beta}{D-2}\right]{}^S\!R^2+\frac{1}{\kappa}\,{}^S\!R=0\,. \tag{4.22}$$

This equation obviously has two distinct solutions:

(a) The *trivial solution*

$${}^S\!R=0\,,\qquad\text{which implies}\qquad g_{pq}=\delta_{pq}\,. \tag{4.23}$$

Substituting this solution into (4.20) we immediately see that in this case the cosmological constant must be vanishing as well,

$$\Lambda_0=0\,. \tag{4.24}$$

(b) The *non-trivial solution* can be written as

$${}^S\!R=-\frac{1}{2\kappa}\,\frac{(D-2)(D-3)}{(D-2)(D-3)\alpha+(D-4)(D-5)\gamma+(D-3)\beta}\,. \tag{4.25}$$

The crucial observation is that the scalar curvature ${}^S\!R$ takes the form *expressed in terms of the theory constants* only, and therefore it is a constant itself. This implies that the constant curvature transverse space metric has to be u-independent, i.e.,

$$g_{pq,u}=0\,,$$

see the relation (3.44) for the metric of constant curvature manifolds.

Finally, we can also substitute this solution back into the original equation (4.20) to get a condition which relates constants of the theory,

$$\Lambda_0=-\frac{1}{8\kappa}\,\frac{(D-2)(D-3)}{(D-2)(D-3)\alpha+(D-4)(D-5)\gamma+(D-3)\beta}=\frac{1}{4}\,{}^S\!R\,. \tag{4.26}$$

• *up-component*: Since the transverse space metric g_{pq} has to necessarily be u-independent in both cases (a) and (b), the up-component of field equations is *satisfied identically.*

• *uu-component*: Employing u-independence of the transverse metric $g_{pq,u}=0$, this component explicitly becomes

$$\begin{aligned}&-\frac{1}{2\kappa}\left[{}^S\Box\,g_{uu}+g_{uu}\left({}^S\!R-2\Lambda_0\right)\right]-\alpha\,{}^S\!R\left[{}^S\Box\,g_{uu}+\frac{1}{2}g_{uu}\,{}^S\!R\right]\\&-\gamma\frac{D-4}{D-2}\left[{}^S\Box\,g_{uu}+\frac{1}{2}\frac{D-5}{D-3}\,g_{uu}\,{}^S\!R\right]{}^S\!R\\&-\frac{\beta}{2}\left[{}^S\Box\,{}^S\Box\,g_{uu}+\frac{2}{D-2}\,{}^S\!R^2\,{}^S\Box\,g_{uu}+\frac{1}{D-2}\,{}^S\!R^2 g_{uu}\right]=0\,.\end{aligned} \tag{4.27}$$

Now we can follow the discussion of two cases introduced above:

(a) *Flat transverse space*: in this case, i.e., $g_{pq} = \delta_{pq}$ and $\Lambda_0 = 0$, the only non-trivial condition given by the uu-component (4.27) is

$$\frac{1}{\kappa}\,{}^S\Box\, g_{uu} + \beta\, {}^S\Box\, {}^S\Box\, g_{uu} = 0\,. \tag{4.28}$$

which can be rewritten

$${}^S\Box\, f = 0\,, \qquad \text{with } f \text{ given by} \qquad {}^S\Box\, g_{uu} + \frac{1}{\kappa\,\beta}\, g_{uu} = f\,. \tag{4.29}$$

In fact, this transcription represents two conditions which have to be satisfied simultaneously. The first condition for f corresponds simply to the Laplace equation. The second one is non-homogeneous Helmholtz-like equation, where the right hand side is constrained to be a solution to the Laplace equation. To obtain an explicit solution of (4.29) we thus have to solve the Helmholtz-like equation with right hand side, which is solution to the Laplace equation itself. Detailed discussion of *pp*-waves with flat transverse metric can be found for example in [52].

As an *simple example* let us consider here four-dimensional spacetime corresponding to the function f given by

$$f = \left(x^2 - y^2 + 2xy\right)\chi(u)\,, \tag{4.30}$$

which is obvious solution to the two-dimensional flat Laplace equation. Solving Helmholtz-like equation (4.29) with such f on the right hand side gives us metric function $g_{uu}(u,x,y)$,

$$\begin{aligned} g_{uu} = \Bigg[& A(u)\sin\left(\frac{1}{\sqrt{\kappa\beta}}\,x\right) + B(u)\cos\left(\frac{1}{\sqrt{\kappa\beta}}\,x\right) \\ & + C(u)\sin\left(\frac{1}{\sqrt{\kappa\beta}}\,y\right) + D(u)\cos\left(\frac{1}{\sqrt{\kappa\beta}}\,y\right) + \kappa\beta\left(x^2 - y^2 + 2xy\right)\Bigg]\chi(u)\,, \end{aligned} \tag{4.31}$$

where A, B, C, D, and χ are arbitrary profile functions of the coordinate u.

Interestingly, in general this expression *does not solve* the Laplace equation ${}^S\Box\, g_{uu} = 0$ which means that (4.31) represents exact solution in quadratic gravity with $\beta \neq 0$, but it is *not allowed* in Einstein's theory. For physical interpretation it would be interesting to explicitly evaluate $\Psi_{4^{ij}}$, and discuss its 'measurable' difference with respect to classic solution in general relativity.

(b) Solution corresponding to the *non-trivial transverse space* is restricted by (4.27) with (4.25) and (4.26) substituted, namely

$$\beta\, {}^S\Box\, {}^S\Box\, g_{uu} + \left[\frac{(D-3)\beta - 2(D-4)\gamma}{\omega\kappa} + \frac{(D-2)(D-3)^2\beta}{2\omega^2\kappa^2}\right] {}^S\Box\, g_{uu} = 0\,, \tag{4.32}$$

with

$$\omega \equiv (D-2)(D-3)\alpha + (D-4)(D-5)\gamma + (D-3)\beta\,. \tag{4.33}$$

Surprisingly this constrain has the same form as condition (4.28) in the flat case and thus specific procedure leading explicit solution would be similar.

4.2.2 VSI spacetimes

Let us begin with slightly *more general line element*

$$\mathrm{d}s^2 = \delta_{pq}\,\mathrm{d}x^p\,\mathrm{d}x^q - 2\,\mathrm{d}u\,\mathrm{d}r + g_{uu}(r,u,x)\,\mathrm{d}u^2\,, \tag{4.34}$$

which under additional restriction represents non-gyratonic VSI geometry, see subsection 3.3.2. In this case the general quadratic gravity vacuum field equations explicitly become:

- *rr-component* takes the form

$$(2\alpha+\beta)\,g_{uu,rrrr} = 0\,. \tag{4.35}$$

Here we focus on the *generic case*

$$2\alpha+\beta \neq 0\,.$$

This assumption together with rr-equation (4.35) imply that the metric component g_{uu} has to be polynomial in r, namely

$$g_{uu} = a(u,x)r^3 + b(u,x)r^2 + c(u,x)r + d(u,x)\,. \tag{4.36}$$

Since the r-dependence of g_{uu} is fully determined now, we can substitute (4.36) into the following field equations to find specific restrictions on factors a, b, c, and d, respectively.

- *rp-component* of the field equations becomes

$$(2\alpha+\beta)\,g_{uu,rrrp} = 0\,, \tag{4.37}$$

which together with (4.36) implies that the function a does not depend on spatial coordinates, that is

$$a_{,p} = 0\,. \tag{4.38}$$

- *ru-component* can be written as

$$-\frac{\Lambda_0}{\kappa} + (2\alpha+\beta)\left(g_{uu}g_{uu,rrrr} + g_{uu,rrru} + \frac{1}{2}g_{uu,r}g_{uu,rrr} - {}^S\Box\, g_{uu,rr} - \frac{1}{4}{g_{uu,rr}}^2\right) = 0\,. \tag{4.39}$$

Now, we can substitute (4.36) into this general ru-equation and decompose the resulting condition into the separate equations with respect to powers of coordinate r.

In the *linear* order we thus get

$$-6(2\alpha+\beta)\,{}^S\Box\, a = 0\,, \tag{4.40}$$

which is trivially satisfied using (4.38).

In the *zeroth* order in r we obtain the equation

$$-\frac{\Lambda_0}{\kappa} + (2\alpha+\beta)\left(3ac - b^2 + 6a_{,u} - 2\,{}^S\Box\, b\right) = 0\,, \tag{4.41}$$

which can be understood as a specific restriction on possible u-dependence of the function $a(u)$, or as we shall see later, equation for the spatial dependence of factor $b(u,x)$.

- *pq-component* can be analysed similarly as the case of ru-equation. It takes the explicit form

$$\begin{aligned}
&-\frac{1}{2\kappa}\,\delta_{pq}\left(g_{uu,rr} - 2\Lambda_0\right)\\
&+(2\alpha+\beta)\left[\delta_{pq}\left({}^S\Box\, g_{uu,rr} - 2g_{uu,rrru} - g_{uu,r}g_{uu,rrr} - \frac{1}{4}{g_{uu,rr}}^2\right) - g_{uu,rr||pq}\right]\\
&-\frac{\beta}{2}\,\delta_{pq}\left({}^S\Box\, g_{uu,rr} - g_{uu}g_{uu,rrrr} - 2g_{uu,rrru} - g_{uu,r}g_{uu,rrr}\right) = 0\,,
\end{aligned} \tag{4.42}$$

where condition (4.35) is already partly employed. Substituting the above constraints we obtain:

The *second* order in r gives

$$-18(3\alpha+\beta)a^2 = 0\,, \tag{4.43}$$

which can be obviously satisfied by two independent choices, namely

$$(3\alpha+\beta) = 0\,, \qquad\qquad \text{or} \qquad\qquad a = 0\,. \tag{4.44}$$

In the *first* order in r we get

$$-6(\alpha+\beta)a_{||pq} - 3\delta_{pq}\left[4(3\alpha+\beta)ab + \frac{1}{\kappa}a\right] = 0\,. \tag{4.45}$$

Finally, the *zeroth* order in r leads to

$$-2(2\alpha+\beta)b_{||pq} + \delta_{pq}\left[(4\alpha+\beta)(-3ac - 6a_{,u} + {}^S\Box\, b) - (2\alpha+\beta)b^2 - \frac{1}{\kappa}b + \frac{\Lambda_0}{\kappa}\right] = 0\,. \tag{4.46}$$

Now we compute trace of the equation (4.45) and use constraint $a = a(u)$, see (4.38), to get

$$a\left[4(3\alpha+\beta)b + \frac{1}{\kappa}\right] = 0\,, \tag{4.47}$$

where we have to discuss both cases (4.44). In particular, if we take $a = 0$, this condition is satisfied identically. On the other hand, setting $(3\alpha+\beta) = 0$ we find out that function $a(u)$ has to vanish again since $\kappa^{-1} \neq 0$. Therefore, we can conclude that coefficient proportional to r^3 in g_{uu}, see (4.36), has to be zero in both cases of (4.44),

$$a(u) = 0\,. \tag{4.48}$$

Using this result we can proceed to the analysis of the *zeroth* order equation (4.46). Calculating its trace we obtain condition

$$[4\alpha(D-3) + \beta(D-4)]\ {}^S\Box\, b - (D-2)\left[(2\alpha+\beta)b^2 + \frac{1}{\kappa}b - \frac{\Lambda_0}{\kappa}\right] = 0\,, \tag{4.49}$$

which is partial differential equation for $b(u,x)$ containing its square and d'Alembert derivative. The d'Alembert operator can thus be expressed as

$${}^S\Box\, b = \frac{D-2}{4\alpha(D-3)+\beta(D-4)}\left[(2\alpha+\beta)b^2 + \frac{1}{\kappa}b - \frac{\Lambda_0}{\kappa}\right]. \tag{4.50}$$

In fact, this is also the case of equation (4.41) with $a = 0$. Similarly, we can express ${}^S\Box\, b$,

$${}^S\Box\, b = -\frac{1}{2}b^2 - \frac{\Lambda_0}{2\kappa(2\alpha+\beta)}\,. \tag{4.51}$$

We compare both equations for d'Alembert operator above to eliminate it. Finally, the resulting condition is mere quadratic algebraic expression for b with constant coefficients, which means that its solutions have to be constants as well, and therefore we obtain ${}^S\Box\, b = 0$. Now we can thus solve (4.50) and (4.51) separately,

$$(2\alpha+\beta)b^2 + \frac{1}{\kappa}b - \frac{\Lambda_0}{\kappa} = 0\,, \qquad\qquad (2\alpha+\beta)b^2 + \frac{\Lambda_0}{\kappa} = 0\,. \tag{4.52}$$

Subtracting the second equation from the first one we find out that

$$b = 2\Lambda_0\,. \tag{4.53}$$

Substituting this condition for $\Lambda_0 \neq 0$ back into the equation (4.52) we get a constraint for coupling constants of the specific quadratic theory, namely

$$2\alpha+\beta = -\frac{1}{4\,\kappa\,\Lambda_0}\,. \tag{4.54}$$

There is also a trivial solution $b = 0$ and $\Lambda_0 = 0$ leading to the VSI spacetimes, see next section.

• *up-component* takes the form

$$
\begin{aligned}
&-\frac{1}{2\kappa}g_{uu,rp}-\alpha\,g_{uu,rr}g_{uu,rp}-(2\alpha+\beta)\left[g_{uu,rrup}+\frac{1}{2}g_{uu,p}g_{uu,rrr}\right]\\
&+\beta\left(\frac{1}{2}g_{uu}g_{uu,rrrp}+g_{uu,rrup}+\frac{1}{2}g_{uu,p}g_{uu,rrr}-\frac{1}{2}g_{uu,rr}g_{uu,rp}-\frac{1}{2}\,{}^{S}\Box\,g_{uu,rp}\right)=0\,, \qquad (4.55)
\end{aligned}
$$

which can be simplified using explicit form of the g_{uu} metric function and related constraints,

$$
(2\alpha+\beta)\,b\,c_{,p}+\frac{1}{2\kappa}c_{,p}+\frac{1}{2}\beta\,{}^{S}\Box\,c_{,p}=0\,. \qquad (4.56)
$$

Substituting (4.53) and (4.54) to eliminate parameter b we get

$$
\beta\,{}^{S}\Box\,c_{,p}=0\,. \qquad (4.57)
$$

This means that either $\beta=0$, corresponding to special setting the theory, or ${}^{S}\Box\,c_{,p}=0$, which represents generic condition for the spacetime geometry.

• *uu-component* of the field equation explicitly becomes

$$
\begin{aligned}
&\frac{1}{\kappa}\left(-\frac{1}{2}\,{}^{S}\Box\,g_{uu}+\Lambda_0\,g_{uu}\right)+\frac{\alpha}{2}\,g_{uu,rr}\left(-2\,{}^{S}\Box\,g_{uu}+g_{uu}g_{uu,rr}\right)\\
&+(2\alpha+\beta)\Big[-{g_{uu}}^2g_{uu,rrrr}-2g_{uu}g_{uu,rrru}+g_{uu}\,{}^{S}\Box\,g_{uu,rr}-g_{uu,rruu}\\
&\qquad-\frac{1}{2}g_{uu,rrr}\left(g_{uu}g_{uu,r}+g_{uu,u}\right)+\frac{1}{2}g_{uu,r}g_{uu,rru}-\frac{1}{2}\delta^{mn}g_{uu,m}g_{uu,rrn}\Big]\\
&+\beta\Big[g_{uu}\left(\frac{1}{4}{g_{uu,rr}}^2+\frac{1}{2}\,{}^{S}\Box\,g_{uu,rr}\right)+\left({}^{S}\Box\,g_{uu,r}\right)_{,u}\\
&\qquad-\frac{1}{2}g_{uu,r}\,{}^{S}\Box\,g_{uu,r}-\frac{1}{2}\delta^{pq}g_{uu,rp}g_{uu,rq}+\delta^{mn}g_{uu,m}g_{uu,rrn}-\frac{1}{2}\,{}^{S}\Box\,{}^{S}\Box\,g_{uu}\Big]=0\,. \qquad (4.58)
\end{aligned}
$$

In the *second* order in r it provides

$$
b\left[(2\alpha+\beta)b^2+\frac{\Lambda_0}{\kappa}\right]=0\,, \qquad (4.59)
$$

which is already satisfied because of condition (4.52).

In the *first* order we obtain

$$
c\left[(2\alpha+\beta)b^2+\frac{\Lambda_0}{\kappa}\right]-{}^{S}\Box\left[(\alpha+\beta)\,b\,c+\frac{1}{2\kappa}c+\frac{1}{2}\beta\,{}^{S}\Box\,c\right]=0\,. \qquad (4.60)
$$

Employing constrains (4.52) and (4.56) we find out that this condition is also already satisfied.

Finally, in the *zeroth* order in r we get

$$
\begin{aligned}
&\frac{1}{\kappa}\left(-\,{}^{S}\Box\,d+2\Lambda_0 d\right)+4\,\alpha\,b\left(-\,{}^{S}\Box\,d+b\,d\right)\\
&+\beta\left[\,2\,b^2d+2\,{}^{S}\Box\,c_{,u}-c\,{}^{S}\Box\,c-\delta^{pq}c_{,p}c_{,q}-{}^{S}\Box\,{}^{S}\Box\,d\,\right]=0\,, \qquad (4.61)
\end{aligned}
$$

which can be rewritten using (4.52), (4.53) and (4.54) as

$$
\beta\,{}^{S}\Box\left[-\frac{1}{\kappa(2\alpha+\beta)}\,d+2\,c_{,u}-\frac{1}{2}\,c^2-{}^{S}\Box\,d\right]=0\,. \qquad (4.62)
$$

Restriction to VSI spacetimes

Now we would like to focus our attention to the VSI subclass of general line element (4.34) corresponding to g_{uu} which is at most linear in r,

$$g_{uu}(r,u,x) = c(u,x)r + d(u,x)\,. \tag{4.63}$$

In fact, we consider trivial solution to constraint (4.52), namely

$$b = 0\,, \qquad \Lambda_0 = 0\,. \tag{4.64}$$

Then for the field equations we observe that rr, rp, ru, and pq-component, respectively, are satisfied identically.

- *up-component* simplifies to to the form

$$\frac{1}{\kappa} c_{,p} + \beta\, {}^S\Box\, c_{,p} = 0\,. \tag{4.65}$$

- *uu-component* has to be again discussed with respet to different power in r.

  The *second* order of r is trivially satisfied.

  In the *linear* order we have

$$^S\Box \left(\frac{1}{\kappa}\, c + \beta\, {}^S\Box\, c \right) = 0\,. \tag{4.66}$$

  The *zeroth* order implies

$$^S\Box \left[-\frac{1}{\kappa}\, d + \beta \left(2\, c_{,u} - \frac{1}{2}\, c^2 - {}^S\Box\, d \right) \right] = 0\,. \tag{4.67}$$

4.2.3 Geometries related to direct product backgrounds

In this subsection we only formulate field equations of generic quadratic gravity for those vacuum geometries which would represent exact type II (or N) spacetimes corresponding to gravitational waves propagating on type D (or O) backgrounds, see [66]. Detailed discussion is subject for further work. The first step here is to restrict u-dependence of the spatial metric g_{pq}, namely

$$g_{mn,u} = 0\,. \tag{4.68}$$

Employing general expressions (4.4)–(4.10) the field equations then become

- *rr-component*

$$(2\alpha + \beta)\, g_{uu,rrrr} = 0\,, \tag{4.69}$$

- *rp-component*

$$(2\alpha + \beta)\, g_{uu,rrrp} = 0\,, \tag{4.70}$$

- *ru-component*

$$\begin{aligned} &\frac{1}{2\kappa} \left({}^S R - 2\Lambda_0 \right) + \frac{\alpha}{2} \left({}^S R^2 - {g_{uu,rr}}^2 \right) \\ &+ (2\alpha + \beta) \left(g_{uu} g_{uu,rrrr} + g_{uu,rrru} + \frac{1}{2} g_{uu,r} g_{uu,rrr} - {}^S\Box\, g_{uu,rr} - {}^S\Box\, {}^S R \right) \\ &+ \frac{\gamma}{2} \left({}^S R^2_{klmn} - 4\, {}^S R^2_{mn} + {}^S R^2 \right) + \frac{\beta}{4} \left(2\, {}^S\Box\, {}^S R - {g_{uu,rr}}^2 + 2\, {}^S R^2_{mn} \right) = 0\,, \end{aligned} \tag{4.71}$$

• *pq-component*

$$\frac{1}{\kappa}\left[{}^{S}\!R_{pq}-\frac{1}{2}\,g_{pq}\left({}^{S}\!R+g_{uu,rr}-2\Lambda_0\right)\right]+2\alpha\left[{}^{S}\!R_{pq}\left({}^{S}\!R+g_{uu,rr}\right)-\frac{1}{4}\,g_{pq}\left({}^{S}\!R+g_{uu,rr}\right)^2\right]$$
$$+(2\alpha+\beta)\left[g_{pq}\left({}^{S}\Box\,{}^{S}\!R+{}^{S}\Box\,g_{uu,rr}-g_{uu}g_{uu,rrrr}-2g_{uu,rrru}-g_{uu,r}g_{uu,rrr}\right)-g_{uu,rr||pq}-{}^{S}\!R_{||pq}\right]$$
$$+2\gamma\left[{}^{S}\!R_{pq}\left({}^{S}\!R+g_{uu,rr}\right)-2\,{}^{S}\!R_{pmqn}\,{}^{S}\!R^{mn}+{}^{S}\!R_{pklm}\,{}^{S}\!R_q{}^{klm}-2\,{}^{S}\!R_{pm}\,{}^{S}\!R_q{}^{m}\right.$$
$$\left.-\frac{1}{4}\,g_{pq}\left(2g_{uu,rr}\,{}^{S}\!R+{}^{S}\!R^2_{klmn}-4\,{}^{S}\!R^2_{mn}+{}^{S}\!R^2\right)\right]$$
$$+\beta\left[{}^{S}\Box\,{}^{S}\!R_{pq}+2\,{}^{S}\!R_{pmqn}\,{}^{S}\!R^{mn}-\frac{1}{2}g_{pq}\left({}^{S}\Box\,{}^{S}\!R+{}^{S}\Box\,g_{uu,rr}+{}^{S}\!R^2_{mn}-g_{uu}g_{uu,rrrr}-2g_{uu,rrru}\right.\right.$$
$$\left.\left.-g_{uu,r}g_{uu,rrr}+\frac{1}{2}g_{uu,rr}{}^2\right)\right]=0\,, \tag{4.72}$$

• *up-component*

$$-\frac{1}{2\kappa}g_{uu,rp}-\alpha\left({}^{S}\!R+g_{uu,rr}\right)g_{uu,rp}-(2\alpha+\beta)\left[g_{uu,rrup}+\frac{1}{2}g_{uu,p}g_{uu,rrr}+{}^{S}\!R_{,up}\right]$$
$$+2\gamma\left[-\frac{1}{2}g_{uu,rp}\,{}^{S}\!R+g_{uu,rm}\,{}^{S}\!R_p{}^{m}\right]+\beta\left[\frac{1}{2}g_{uu}g_{uu,rrrp}+g_{uu,rrup}+\frac{1}{2}g_{uu,p}g_{uu,rrr}\right.$$
$$\left.-\frac{1}{2}g_{uu,rr}g_{uu,rp}-\frac{1}{2}\,{}^{S}\Box\,g_{uu,rp}-\frac{1}{2}g^{mn}g_{uu,rm}\,{}^{S}\!R_{np}\right]=0\,, \tag{4.73}$$

• *uu-component*

$$\frac{1}{\kappa}\left[-\frac{1}{2}\,{}^{S}\Box\,g_{uu}-\frac{1}{2}g_{uu}\left({}^{S}\!R-2\Lambda_0\right)\right]+\frac{\alpha}{2}\left({}^{S}\!R+g_{uu,rr}\right)\left[-2\,{}^{S}\Box\,g_{uu}-g_{uu}\left({}^{S}\!R-g_{uu,rr}\right)\right]$$
$$+(2\alpha+\beta)\left[-g_{uu}{}^2g_{uu,rrrr}-2g_{uu}g_{uu,rrru}+g_{uu}\,{}^{S}\Box\left({}^{S}\!R+g_{uu,rr}\right)-{}^{S}\!R_{,uu}-g_{uu,rruu}\right.$$
$$\left.-\frac{1}{2}g_{uu,rrr}\left(g_{uu}g_{uu,r}+g_{uu,u}\right)+\frac{1}{2}g_{uu,r}\left({}^{S}\!R_{,u}+g_{uu,rru}\right)-\frac{1}{2}g^{mn}g_{uu,m}\left({}^{S}\!R_{,n}+g_{uu,rrn}\right)\right]$$
$$+2\gamma\left[\left({}^{S}\!R^{mn}-\frac{1}{2}\,{}^{S}\!R\,g^{mn}\right)g_{uu||mn}-\frac{1}{4}g_{uu}\left({}^{S}\!R^2_{klmn}-4\,{}^{S}\!R^2_{mn}+{}^{S}\!R^2\right)\right]$$
$$+\beta\left[g_{uu}\left(\frac{1}{4}g_{uu,rr}{}^2+\frac{1}{2}\,{}^{S}\Box\,g_{uu,rr}-\frac{1}{2}\,{}^{S}\Box\,{}^{S}\!R-\frac{1}{2}\,{}^{S}\!R^2_{mn}\right)+\left({}^{S}\Box\,g_{uu,r}\right)_{,u}-\frac{1}{2}g_{uu,r}\,{}^{S}\Box\,g_{uu,r}\right.$$
$$\left.-\frac{1}{2}g^{pq}g_{uu,rp}g_{uu,rq}+g^{mn}g_{uu,m}g_{uu,rrn}-\frac{1}{2}\,{}^{S}\Box\,{}^{S}\Box\,g_{uu}-{}^{S}\!R^{mn}g_{uu||mn}\right]=0\,. \tag{4.74}$$

Even with substitution of g_{uu} metric function as a solution to the rr-component (4.69), i.e., g_{uu} being cubic polynom in r, constrained by rp-component (4.69), i.e., term propotrional to r^3 in g_{uu} does not depend on spatial coordinates, these equations represent very complicated system. We thus impose additional condition on the transverse space to be of constant curvature, see subsection 3.3.2.

Constant curvature transverse space

For the constant curvature transverse space the rr and rp-component, see (4.69) and (4.70), respectively, remain unchanged, and the following equations explicitly become:

• *ru-component*

$$\frac{1}{2\kappa}\left({}^{S}\!R-2\Lambda_0\right)+\frac{\alpha}{2}\left({}^{S}\!R^2-g_{uu,rr}{}^2\right)$$
$$+(2\alpha+\beta)\left(g_{uu}g_{uu,rrrr}+g_{uu,rrru}+\frac{1}{2}g_{uu,r}g_{uu,rrr}-{}^{S}\Box\,g_{uu,rr}\right)$$
$$+\frac{\gamma}{2}\frac{(D-4)(D-5)}{(D-2)(D-3)}\,{}^{S}\!R^2+\frac{\beta}{4}\left(-g_{uu,rr}{}^2+\frac{2}{D-2}\,{}^{S}\!R^2\right)=0\,, \tag{4.75}$$

- *pq-component*

$$\frac{1}{\kappa}\left[-\frac{D-4}{2(D-2)}\,{}^S\!R-\frac{1}{2}g_{uu,rr}+\Lambda_0\right]g_{pq}+2\alpha\left({}^S\!R+g_{uu,rr}\right)\left[\frac{{}^S\!R}{D-2}-\frac{1}{4}\left({}^S\!R+g_{uu,rr}\right)\right]g_{pq}$$
$$+(2\alpha+\beta)\left[\left({}^S\Box\, g_{uu,rr}-g_{uu}g_{uu,rrrr}-2g_{uu,rrru}-g_{uu,r}g_{uu,rrr}\right)g_{pq}-g_{uu,rr||pq}\right]$$
$$-\gamma\left[\frac{(D-4)(D-5)(D-6)}{2(D-2)^2(D-3)}\,{}^S\!R^2-\frac{D-4}{D-2}\,{}^S\!Rg_{uu,rr}\right]g_{pq}$$
$$+\frac{\beta}{2}\left[-\frac{D-6}{(D-2)^2}\,{}^S\!R^2-{}^S\Box\, g_{uu,rr}+g_{uu}g_{uu,rrrr}+2g_{uu,rrru}\right.$$
$$\left.+g_{uu,r}g_{uu,rrr}-\frac{1}{2}{g_{uu,rr}}^2\right]g_{pq}=0\,, \tag{4.76}$$

- *up-component*

$$-\frac{1}{2\kappa}\,g_{uu,rp}-\alpha\left({}^S\!R+g_{uu,rr}\right)g_{uu,rp}-(2\alpha+\beta)\left[g_{uu,rrup}+\frac{1}{2}g_{uu,p}g_{uu,rrr}\right]$$
$$-\gamma\,\frac{D-4}{D-2}\,{}^S\!Rg_{uu,rp}+\beta\left[\frac{1}{2}g_{uu}g_{uu,rrrp}+g_{uu,rrup}+\frac{1}{2}g_{uu,p}g_{uu,rrr}\right.$$
$$\left.-\frac{1}{2}g_{uu,rr}g_{uu,rp}-\frac{1}{2}\,{}^S\Box\, g_{uu,rp}-\frac{{}^S\!R}{2(D-2)}g_{uu,rp}\right]=0\,, \tag{4.77}$$

- *uu-component*

$$\frac{1}{\kappa}\left[-\frac{1}{2}\,{}^S\Box\, g_{uu}-\frac{1}{2}g_{uu}\left({}^S\!R-2\Lambda_0\right)\right]+\frac{\alpha}{2}\left({}^S\!R+g_{uu,rr}\right)\left[-2\,{}^S\Box\, g_{uu}-g_{uu}\left({}^S\!R-g_{uu,rr}\right)\right]$$
$$+(2\alpha+\beta)\left[-{g_{uu}}^2g_{uu,rrrr}-2g_{uu}g_{uu,rrru}+g_{uu}\,{}^S\Box\, g_{uu,rr}-{}^S\!R_{,uu}-g_{uu,rruu}\right.$$
$$\left.-\frac{1}{2}g_{uu,rrr}\left(g_{uu}g_{uu,r}+g_{uu,u}\right)+\frac{1}{2}g_{uu,r}\left({}^S\!R_{,u}+g_{uu,rru}\right)-\frac{1}{2}g^{mn}g_{uu,m}g_{uu,rrn}\right]$$
$$+2\gamma\left[-\frac{D-4}{2(D-2)}\,{}^S\!R\,{}^S\Box\, g_{uu}-\frac{1}{4}\frac{(D-4)(D-5)}{(D-2)(D-3)}\,{}^S\!R^2g_{uu}\right]$$
$$+\beta\left[g_{uu}\left(\frac{1}{4}{g_{uu,rr}}^2+\frac{1}{2}\,{}^S\Box\, g_{uu,rr}-\frac{{}^S\!R^2}{2(D-2)}\right)+{}^S\Box\, g_{uu,ru}-\frac{1}{2}g_{uu,r}\,{}^S\Box\, g_{uu,r}\right.$$
$$\left.-\frac{1}{2}g^{pq}g_{uu,rp}g_{uu,rq}+g^{mn}g_{uu,m}g_{uu,rrn}-\frac{1}{2}\,{}^S\Box\,{}^S\Box\, g_{uu}-\frac{{}^S\!R}{D-2}\,{}^S\Box\, g_{uu}\right]=0\,. \tag{4.78}$$

These equation will be analysed in near future to obtain specific restriction on the non-gyratonic Kundt geometries of this type.

5. Kundt spacetimes in Gauss–Bonnet theory

In this chapter we investigate field equations of the Gauss–Bonnet theory, see section 1.4, and we try to find the specific restrictions they impose on the Kundt metric (3.25). We start with the most general case, where we find that it is necessary to distinguish three particular subcases. Then we analyse some special geometries, which naturally appear after imposing some additional restrictions. Complementary discussion can be found in [71].

5.1 Generic non-gyratonic Kundt case

To obtain the Gauss–Bonnet field equations for the Kundt geometry we set

$$\alpha = 0\,, \qquad \text{and} \qquad \beta = 0\,, \tag{5.1}$$

in the most general results (4.4 through 4.10) of chapter 4.

• *rr-component* is satisfied identically, see (4.4) with (5.1).

• *rp-component* is satisfied identically, see (4.5) with (5.1).

• *ru-component* (4.5) becomes

$$\frac{1}{2\kappa}\left({}^S\!R - 2\Lambda_0\right) + \frac{\gamma}{2}\left({}^S\!R^2_{klmn} - 4\,{}^S\!R^2_{mn} + {}^S\!R^2\right) = 0\,, \tag{5.2}$$

which does not provide a *direct* restriction on the metric components. But it gives a relation between the transverse space Riemann tensor contractions and the theory parameters. We use this relation to simplify the following field equations.

• *pq-component* (4.7) in the Gauss–Bonnet case takes the form

$$\begin{aligned} &\frac{1}{\kappa}\left[{}^S\!R_{pq} - \frac{1}{2}\,g_{pq}\left({}^S\!R + g_{uu,rr} - 2\Lambda_0\right)\right] \\ &\quad + 2\gamma\Big[{}^S\!R_{pq}\left({}^S\!R + g_{uu,rr}\right) - 2\,{}^S\!R_{pmqn}\,{}^S\!R^{mn} + {}^S\!R_{pklm}\,{}^S\!R_q{}^{klm} - 2\,{}^S\!R_{pm}\,{}^S\!R_q{}^m \\ &\qquad -\frac{1}{4}\,g_{pq}\left(2g_{uu,rr}\,{}^S\!R + {}^S\!R^2_{klmn} - 4\,{}^S\!R^2_{mn} + {}^S\!R^2\right)\Big] = 0\,, \end{aligned} \tag{5.3}$$

with the trace

$$\begin{aligned} &\frac{1}{\kappa}\left[{}^S\!R - \frac{1}{2}\,(D-2)\left({}^S\!R + g_{uu,rr} - 2\Lambda_0\right)\right] \\ &\quad + 2\gamma\Big[g_{uu,rr}\,{}^S\!R + {}^S\!R^2_{klmn} - 4\,{}^S\!R^2_{mn} + {}^S\!R^2 \\ &\qquad -\frac{1}{4}\,(D-2)\left(2g_{uu,rr}\,{}^S\!R + {}^S\!R^2_{klmn} - 4\,{}^S\!R^2_{mn} + {}^S\!R^2\right)\Big] = 0\,. \end{aligned} \tag{5.4}$$

In general, this equation ties together $g_{uu,rr}$, spatial metric g_{pq}, and the spatial curvature corresponding to various contraction of the Riemann tensor. Using the equation (5.2) and collecting terms proportional to $g_{uu,rr}$ we get

$$\begin{aligned} &\left[-\frac{1}{2}g_{pq} + \kappa\gamma\left(2\,{}^S\!R_{pq} - {}^S\!R g_{pq}\right)\right] g_{uu,rr} + {}^S\!R_{pq} \\ &\quad + 2\kappa\gamma\left({}^S\!R_{pq}\,{}^S\!R - 2\,{}^S\!R_{pmqn}\,{}^S\!R^{mn} + {}^S\!R_{pklm}\,{}^S\!R_q{}^{klm} - 2\,{}^S\!R_{pm}\,{}^S\!R_q{}^m\right) = 0\,. \end{aligned} \tag{5.5}$$

Moreover, doing the same for the trace (5.4) and using (5.2) once more leads to

$$\left[\frac{1}{2}(D-2) + \kappa\gamma\,{}^S\!R(D-4)\right] g_{uu,rr} + {}^S\!R - 4\Lambda_0 = 0\,. \tag{5.6}$$

Here we have to distinguish *three qualitatively different situations* with respect to the coefficient near $g_{uu,rr}$ term in (5.5),

$$S_{pq} \equiv -\frac{1}{2} g_{pq} + \kappa\gamma \left(2\, {}^{S}\!R_{pq} - {}^{S}\!R g_{pq} \right) . \tag{5.7}$$

(i) $S_{pq} \neq 0$ and $g^{pq} S_{pq} \neq 0$: in the case of *non-vanishing* square bracket in (5.5), and also its trace, we can use the equation (5.6) to express $g_{uu,rr}$ explicitly. Subsequently, simple integration leads to a general form of the metric component g_{uu}, namely

$$g_{uu}(r,u,x) = b(u,x)\, r^2 + c(u,x)\, r + d(u,x)\,, \tag{5.8}$$

with

$$b = \frac{4\Lambda_0 - {}^{S}\!R}{D - 2 + 2\kappa\gamma\,(D-4)\, {}^{S}\!R} . \tag{5.9}$$

Here it is obvious that the special case $D = 4$ is equivalent to the case $\gamma = 0$, i.e., to the Einstein theory. This corresponds to the fact that in four dimensions the Gauss–Bonnet term in the action (2.43) does not contribute to the field equations.

Finally, substituting $g_{uu,rr}$ from (5.6) back into the original equation (5.5) we get rather complicated condition that ties spatial metric g_{pq} with the corresponding curvature,

$$\begin{aligned}
&\left[D - 2 + 4\kappa\gamma(D-4)\left(1 + \kappa\gamma\, {}^{S}\!R\right) {}^{S}\!R + 16\kappa\gamma\Lambda_0\right] {}^{S}\!R_{pq} \\
&- \left(1 + 2\kappa\gamma\, {}^{S}\!R\right)\left(4\Lambda_0 - {}^{S}\!R\right) g_{pq} \\
&- 2\kappa\gamma \left[D - 2 + 2\kappa\gamma(D-4)\, {}^{S}\!R\right] \\
&\qquad \times \left(2\, {}^{S}\!R_{pmqn}\, {}^{S}\!R^{mn} - {}^{S}\!R_{pklm}\, {}^{S}\!R_q{}^{klm} + 2\, {}^{S}\!R_{pm}\, {}^{S}\!R_q{}^{m}\right) = 0 .
\end{aligned} \tag{5.10}$$

(ii) $S_{pq} \neq 0$ and $g^{pq} S_{pq} = 0$: in this case the square bracket S_{pq} in (5.5) is *non-vanishing* but its trace is assumed to be zero now. This condition, namely $g^{pq} S_{pq} = 0$, thus implies that

$${}^{S}\!R = -\frac{D-2}{2\kappa\gamma(D-4)} . \tag{5.11}$$

Because the square bracket in the traced equation (5.6) is now vanishing as well, we also get a coupling between the Ricci scalar and cosmological constant,

$${}^{S}\!R = 4\Lambda_0 . \tag{5.12}$$

We immediately see that the scalar curvature ${}^{S}\!R$ depends only on parameters of the theory which are now tied together. Subsequently, we can conclude from the equation (5.5) that the function g_{uu} can be at most quadratic in r, namely

$$g_{uu}(r,u,x) = b(u,x)\, r^2 + c(u,x)\, r + d(u,x) . \tag{5.13}$$

(iii) $S_{pq} = 0$: in this special case, where the content of the square bracket in front of $g_{uu,rr}$ term in (5.5) is zero, and thus also its trace vanishes, we get *no restriction* on $g_{uu,rr}$. Subsequently, we have to discuss the equation (5.5) together with the explicit condition $S_{pq} = 0$, namely

$${}^{S}\!R_{pq} + 2\kappa\gamma \left({}^{S}\!R_{pq}\, {}^{S}\!R - 2\, {}^{S}\!R_{pmqn}\, {}^{S}\!R^{mn} + {}^{S}\!R_{pklm}\, {}^{S}\!R_q{}^{klm} - 2\, {}^{S}\!R_{pm}\, {}^{S}\!R_q{}^{m} \right) = 0 , \tag{5.14}$$

$$-\frac{1}{2} g_{pq} + \kappa\gamma \left(2\, {}^{S}\!R_{pq} - {}^{S}\!R g_{pq} \right) = 0 . \tag{5.15}$$

We can calculate trace of the second equation, i.e., $g^{pq} S_{pq} = 0$, and solve it for ${}^{S}\!R$ to obtain

$${}^{S}\!R = -\frac{D-2}{2\kappa\gamma(D-4)} . \tag{5.16}$$

Now we substitute this expression for the transverse space scalar curvature back to the original equation (5.15) to get

$$ {}^S\!R_{pq} = -\frac{1}{2\kappa\gamma(D-4)}\, g_{pq}\,. \tag{5.17} $$

The Ricci tensor is thus proportional to the metric, which is the defining condition for the co-called *Einstein space*. We observe that in this step we have not obtained any restriction on the metric component g_{uu}. Also notice that current (iii) case is not possible in Einstein theory of gravity because $\gamma = 0$ would here imply that $g_{pq} = 0$.

Moreover, for the Einstein space it holds that

$$ {}^S\!R_{pq}{}^2 = \frac{{}^S\!R^2}{D-2}\,. \tag{5.18} $$

Substituting all these relations into the equation (5.14) we have

$$ {}^S\!R_{pklm}\, {}^S\!R_q{}^{klm} = \frac{2}{\left[2\kappa\gamma\,(D-4)\right]^2}\, g_{pq}\,. \tag{5.19} $$

Finally, if we calculate the trace of the equation above and combine it with other curvature terms to substitute into the ru-equation (5.2), we find

$$ \Lambda_0 = -\frac{D-2}{8\,\kappa\gamma\,(D-4)}\,, \tag{5.20} $$

which ties *parameters of the theory* with the *number of spacetime dimensions*. This also implies that the cosmological constant Λ_0 must have the *opposite sign* to the product $\kappa\gamma$ since the Gauss–Bonnet gravity is relevant only for $D > 4$. Notice also that combining (5.16) and (5.20), or equivalently (5.6) with $g^{pq} S_{pq} = 0$, gives

$$ {}^S\!R = 4\Lambda_0\,. \tag{5.21} $$

- *up-component* (4.9), with the Gauss–Bonnet constraint (5.1) directly applied, gives

$$ \begin{aligned} &\frac{1}{\kappa}\left(-\frac{1}{2} g_{uu,rp} + g^{mn} g_{m[p,u||n]}\right) \\ &\quad + 2\gamma\left[\left(-\frac{1}{2} g_{uu,rp} + g^{mn} g_{m[p,u||n]}\right) {}^S\!R - 2 g_{m[p,u||n]}\, {}^S\!R^{mn} \right. \\ &\qquad \left. + g_{k[l,u||m]}\, {}^S\!R_p{}^{klm} + \left(g_{uu,rm} - 2 g^{kl} g_{k[m,u||l]}\right) {}^S\!R_p{}^m\right] = 0\,. \end{aligned} \tag{5.22} $$

We start with this general form (5.22) and collect terms in round brackets. Rearranging indices and straightforward simplification gives

$$ \begin{aligned} &\left[-\frac{1}{2} g_{pn} + \kappa\gamma\left(2\,{}^S\!R_{pn} - {}^S\!R g_{pn}\right)\right] g^{mn}\left(g_{uu,rm} - 2 g^{kl} g_{k[m,u||l]}\right) \\ &\qquad + 2\kappa\gamma\left(-2\,{}^S\!R^{kl}\,\delta^m_p + {}^S\!R_p{}^{kml}\right) g_{k[m,u||l]} = 0\,, \end{aligned} \tag{5.23} $$

where the term inside the square brackets is exactly S_{pn}, i.e., the crucial expression (5.7) in the previous analyses of pq-equation (5.5).

Now, we follow discussion of distinct subcases introduced in the previous pq-step:

(i) $S_{pn} \neq 0$ and $g^{pn} S_{pn} \neq 0$: in this most general case, where the first term is non-zero, we substitute for $g_{uu,rm}$ from (5.8) to get

$$ \begin{aligned} &\left[-\frac{1}{2} g_{pn} + \kappa\gamma\left(2\,{}^S\!R_{pn} - {}^S\!R g_{pn}\right)\right] g^{mn}\left(2\,r\,b_{,m} + c_{,m} - 2 g^{kl} g_{k[m,u||l]}\right) \\ &\qquad + 2\kappa\gamma\left(-2\,{}^S\!R^{kl}\delta^m_p + {}^S\!R_p{}^{kml}\right) g_{k[m,u||l]} = 0\,. \end{aligned} \tag{5.24} $$

We can break this equation into two conditions with respect to terms of the first and zeroth order in r, respectively. For the *first* order in r we get

$$\left[-\frac{1}{2}g_{pn}+\kappa\gamma\left(2\,{}^{S}\!R_{pn}-{}^{S}\!R g_{pn}\right)\right]g^{mn}b_{,m}=0, \tag{5.25}$$

which can be, using (5.9), explicitly rewritten as

$$\left[-\frac{1}{2}g_{pn}+\kappa\gamma\left(2\,{}^{S}\!R_{pn}-{}^{S}\!R g_{pn}\right)\right]g^{mn}\frac{D-2+8\kappa\gamma\Lambda_0(D-4)}{(D-2+2\kappa\gamma(D-4)\,{}^{S}\!R)^2}\,{}^{S}\!R_{,m}=0\,. \tag{5.26}$$

According to the assumption of the case (i), the denominator is non-vanishing, and the equation (5.26) holds either if

$$\left[-\frac{1}{2}g_{pn}+\kappa\gamma\left(2\,{}^{S}\!R_{pn}-{}^{S}\!R g_{pn}\right)\right]g^{mn}\,{}^{S}\!R_{,m}=0\,, \tag{5.27}$$

or if

$$\Lambda_0=-\frac{D-2}{8\kappa\gamma\,(D-4)}\,,\qquad \text{with necessarily}\qquad b=-\frac{1}{2\kappa\gamma(D-4)}\,, \tag{5.28}$$

which represents a very special setting of the theory. It should be noted that the equation (5.25) is equivalent to the equation (5.5), as it can be obtained as its covariant divergence while using Bianchi identities and its contractions.

From the *zeroth* order in r of (5.24) we obtain the following condition

$$\left[-\frac{1}{2}g_{pn}+\kappa\gamma\left(2\,{}^{S}\!R_{pn}-{}^{S}\!R g_{pn}\right)\right]g^{mn}\left(c_{,m}-2g^{kl}g_{k[m,u||l]}\right)$$
$$+2\kappa\gamma\left(-2\,{}^{S}\!R^{kl}\,\delta^m_p+{}^{S}\!R_p{}^{kml}\right)g_{k[m,u||l]}=0\,. \tag{5.29}$$

(ii) $S_{pn}\neq 0$ and $g^{pn}S_{pn}=0$: here we could substitute the explicit expression for the constant Ricci scalar together with polynomial form of g_{uu}, see (5.12) and (5.13), respectively, into the general equation (5.23). In this peculiar case the equation remains very similar to the case (i), but without function b being determined.

(iii) $S_{pn}=0$: this case corresponds to setting the square bracket in general equation (5.23) equal to zero. We are thus left with the condition

$$2\kappa\gamma\left(-2\,{}^{S}\!R^{kl}\delta^m_p+{}^{S}\!R_p{}^{kml}\right)g_{k[m,u||l]}=0\,, \tag{5.30}$$

which can be simplified using the explicit expression for the Ricci tensor (5.17) to the form

$$2\left(\frac{1}{D-4}g_{kl}\delta^m_p+\kappa\gamma\,{}^{S}\!R_p{}^{kml}\right)g_{k[m,u||l]}=0\,, \tag{5.31}$$

where the round bracket cannot be equal to zero as the most naive possibility, since the resulting Ricci tensor would be *inconsistent* with the equation (5.17).

Surprisingly, in the case (iii) even the *up*-component of the field equations gives *no restriction* on the metric function g_{uu}, which thus still remains completely unconstrained.

- *uu-component* (4.10) arranged in the case of Gauss–Bonnet theory using (5.1) becomes

$$\frac{1}{\kappa}\left[\frac{1}{4}g^{mn}\left(g_{mn,u}g_{uu,r}-2g_{mn,uu}+g^{pq}g_{pm,u}g_{qn,u}\right)-\frac{1}{2}\,{}^{S}\Box\, g_{uu}-\frac{1}{2}g_{uu}\left({}^{S}\!R-2\Lambda_0\right)\right]$$
$$+2\gamma\left[\left({}^{S}\!R^{mn}-\frac{1}{2}\,{}^{S}\!R\,g^{mn}\right)\left(g_{uu||mn}+g_{mn,uu}-\frac{1}{2}g_{uu,r}g_{mn,u}-\frac{1}{2}g^{pq}g_{mp,u}g_{nq,u}\right)\right.$$
$$\left.+\left(g^{mo}g^{ns}-2\,g^{mn}g^{os}\right)g^{pq}g_{m[p,u||n]}g_{o[q,u||s]}-\frac{1}{4}g_{uu}\left({}^{S}\!R^2_{klmn}-4\,{}^{S}\!R^2_{mn}+{}^{S}\!R^2\right)\right]=0\,. \tag{5.32}$$

Moreover, we can simplify this uu-equation (5.32) employing condition following from the ru-component (5.2) and collecting terms that repeat. Finally, we get

$$\left[-\frac{1}{2}g^{mn}+\kappa\gamma\left(2\,{}^S\!R^{mn}-\,{}^S\!Rg^{mn}\right)\right]\left(g_{uu||mn}+g_{mn,uu}-\frac{1}{2}g_{uu,r}g_{mn,u}-\frac{1}{2}g^{pq}g_{pm,u}g_{qn,u}\right)$$
$$+\,2\kappa\gamma\left(g^{mo}g^{ns}-2\,g^{mn}g^{os}\right)g^{pq}g_{m[p,u||n]}g_{o[q,u||s]}=0\,, \qquad (5.33)$$

where the first square bracket corresponds to $S^{mn}\equiv g^{mp}g^{nq}S_{pq}$ introduced by (5.7). We can thus naturally discuss three distinct possibilities with respect to particular properties of S_{pq} as in the case of pq and up component, respectively.

(i) $S_{pq}\neq 0$ and $g^{pq}S_{pq}\neq 0$: substituting g_{uu} from equation (5.8) we obtain

$$\left[-\frac{1}{2}g^{mn}+\kappa\gamma\left(2\,{}^S\!R^{mn}-\,{}^S\!Rg^{mn}\right)\right]$$
$$\times\left[r^2\,b_{||mn}+r\left(c_{||mn}-b\,g_{mn,u}\right)+d_{||mn}-\frac{1}{2}\,c\,g_{mn,u}+g_{mn,uu}-\frac{1}{2}g^{pq}g_{pm,u}g_{qn,u}\right]$$
$$+\,2\kappa\gamma\left(g^{mo}g^{ns}-2\,g^{mn}g^{os}\right)g^{pq}g_{m[p,u||n]}g_{o[q,u||s]}=0\,. \qquad (5.34)$$

In the *second* order of r we thus have

$$\left[-\frac{1}{2}g^{mn}+\kappa\gamma\left(2\,{}^S\!R^{mn}-\,{}^S\!Rg^{mn}\right)\right]b_{||mn}=0\,, \qquad (5.35)$$

which is equivalent to the particular up-equation (5.25), because if we take (5.25), rearrange indices, perform a covariant derivative, and use the Leibniz rule we get

$$\kappa\gamma\left(2\,{}^S\!R^{mn}{}_{||n}-\,{}^S\!R_{,n}g^{mn}\right)b_{,m}+\left[-\frac{1}{2}g^{mn}+\kappa\gamma\left(2\,{}^S\!R^{mn}-\,{}^S\!Rg^{mn}\right)\right]b_{||mn}=0\,, \qquad (5.36)$$

where the first term is identically equal to zero due to the contracted Bianchi identity (2.20). This means that equations (5.5), (5.25), and (5.35) are all equivalent.

The condition given by the *first* order in r becomes

$$\left[-\frac{1}{2}g^{mn}+\kappa\gamma\left(2\,{}^S\!R^{mn}-\,{}^S\!Rg^{mn}\right)\right]\left(c_{||mn}-b\,g_{mn,u}\right)=0\,. \qquad (5.37)$$

We can combine this equation with the covariant divergence of (5.29) to eliminate c and substitute for b from (5.9) to get condition

$$\frac{4\Lambda_0-\,{}^S\!R}{D-2+2\kappa\gamma\,(D-4)\,{}^S\!R}\,S^{mn}g_{mn,u}+2\kappa\gamma\left(-3\,{}^S\!R^{kl||m}+\,{}^S\!R^{km||l}\right)g_{k[m,u||l]}$$
$$+2\kappa\gamma\left(-2\,{}^S\!R^{kl}g^{mn}+\,{}^S\!R^{nklm}\right)g_{k[m,u||l]||n}-2g^{kl}S^{mn}g_{k[m,u||l]}=0\,, \qquad (5.38)$$

which ties together spatial metric and spatial curvature.

Finally in the *zeroth* order we are left with the equation

$$\left[-\frac{1}{2}g^{mn}+\kappa\gamma\left(2\,{}^S\!R^{mn}-\,{}^S\!Rg^{mn}\right)\right]\left(d_{||mn}-\frac{1}{2}\,c\,g_{mn,u}+g_{mn,uu}-\frac{1}{2}g^{pq}g_{pm,u}g_{qn,u}\right)$$
$$+\,2\kappa\gamma\left(g^{mo}g^{ns}-2\,g^{mn}g^{os}\right)g^{pq}g_{m[p,u||n]}g_{o[q,u||s]}=0\,. \qquad (5.39)$$

(ii) $S_{pq}\neq 0$ and $g^{pq}S_{pq}=0$: the general equation (5.33) could be again rewritten in terms of the explicit expression for the constant Ricci scalar and g_{uu} function, see (5.12) and (5.13) which will lead to constraints similar to those in case (i) without explicit function b.

(iii) $S_{pq}=0$: setting the first square bracket in (5.33) equal to zero we simply get

$$2\kappa\gamma\left(g^{mo}g^{ns}-2\,g^{mn}g^{os}\right)g^{pq}g_{m[p,u||n]}g_{o[q,u||s]}=0\,, \qquad (5.40)$$

which again is only the *constraint for the spatial part of the metric.* We can conclude that in the case of specific transverse Einstein space (5.17) neither equation provides restriction on the metric component $g_{uu}(r,u,x)$, which thus *remain arbitrary function.*

5.2 Geometrically privileged situations

In this section we present several important subclasses of the Kundt family. For their brief description and list of references see the subsection 3.3.2.

5.2.1 Spacetimes with transverse space of constant curvature

Here we naturally employ general equations derived in the previous section in the generic case of Kundt spacetime in the Gauss–Bonnet theory and impose an additional *restriction on the transverse space*. Namely, we assume that the scalar curvature ${}^S\!R$ of the $(D-2)$-dimensional Riemannian space with metric $g_{pq}(u,x)$ is *constant* with respect to the spatial coordinates x^p, i.e., it may only depend on the coordinate u. In this case, for the Riemann tensor and its contractions, it holds

$$
\begin{aligned}
{}^S\!R_{pqmn} &= \frac{{}^S\!R}{(D-3)(D-2)}\left(g_{pm}g_{qn}-g_{pn}g_{qm}\right), &\qquad {}^S\!R^2_{pqmn} &= \frac{2\,{}^S\!R^2}{(D-3)(D-2)}\,,\\
{}^S\!R_{pq} &= \frac{{}^S\!R}{D-2}\,g_{pq}\,, & {}^S\!R^2_{pq} &= \frac{{}^S\!R^2}{D-2}\,.
\end{aligned}
\tag{5.41}
$$

The spatial metric $g_{pq}=g_{pq}(u)$ necessarily has the form

$$
g_{pq}=P^{-2}\,\delta_{pq}\,,\qquad \text{where}\qquad P=1+\frac{{}^S\!R}{4(D-3)(D-2)}\left[\left(x^2\right)^2+\cdots+\left(x^{D-1}\right)^2\right]. \tag{5.42}
$$

Finally, since the quantity S_{pq} defined by (5.7) during the general discussion is

$$
S_{pq}=\left[-\frac{1}{2}-\kappa\gamma\frac{D-4}{D-2}\,{}^S\!R\right]g_{pq}\,, \tag{5.43}
$$

we deal with the subcase (i) of section 5.1.

- *ru-component*: we simply substitute contractions of the Riemann tensor (5.41) into the general equation (5.2) to obtain

$$
\kappa\gamma\frac{(D-5)(D-4)}{(D-3)(D-2)}\,{}^S\!R^2+{}^S\!R-2\Lambda_0=0\,, \tag{5.44}
$$

  which is a *quadratic equation* with *constant coefficients* for the scalar curvature ${}^S\!R$. It immediately implies that ${}^S\!R$ has to be constant with respect to coordinate u as well, and through expression (5.42) the transverse metric g_{pq} is also u-independent.

  Moreover, from (5.44) we can explicitly express the Ricci scalar[1],

$$
{}^S\!R=\frac{(D-2)(D-3)}{2\kappa\gamma(D-4)(D-5)}\left[-1\pm\sqrt{1+8\kappa\gamma\Lambda_0\frac{(D-4)(D-5)}{(D-2)(D-3)}}\,\right], \tag{5.45}
$$

  which ties together geometry of the transverse space ${}^S\!R$ with specific parameters of the theory κ, γ, and Λ_0.

- *pq-component*: although the present case of constant spatial curvature belongs to the class of Einstein spaces, the cutrvature ${}^S\!R$ given by (5.45) is different from (5.16), and the key term in (5.5), namely (5.43), remains non-vanishing. We thus get

$$
g_{uu}=b\,r^2+c\,r+d=\frac{4\Lambda_0-{}^S\!R}{D-2+2\kappa\gamma\,(D-4)\,{}^S\!R}\,r^2+c(u,x)\,r+d(u,x)\,, \tag{5.46}
$$

  where b is now constant due to (5.45).

[1] In the exceptional case $D=5$ we have ${}^S\!R=2\Lambda_0$, which corresponds to the constraint in Einstein's theory.

- *up-component*: since the spatial metric g_{pq} does not depend on the u coordinate, equation (5.23) simplifies considerably to the form

$$\left(1 + 2\kappa\gamma \frac{D-4}{D-2}\,{}^{S}\!R\right) g_{uu,rm} = 0\,. \tag{5.47}$$

As in the previous case, the bracket cannot be zero because of inconsistency with the solution of quadratic equation resulting from the ru-component. Through the equation (5.46) this leaves us with the simple restriction

$$c_{,m} = 0\,, \qquad \text{which gives} \qquad c = c(u)\,. \tag{5.48}$$

- *uu-component*: since the spatial metric is constant, the equation (5.33) simplifies considerably,

$$\left(1 + 2\kappa\gamma \frac{D-4}{D-2}\,{}^{S}\!R\right) {}^{S}\Box\, g_{uu} = 0\,, \tag{5.49}$$

and because of the non-vanishing bracket, b being constant, and c depending on coordinate u only, we arrive to its final form, namely

$${}^{S}\Box\, d = 0\,. \tag{5.50}$$

5.2.2 *pp*-waves

To discuss the *pp*-wave case we demand all the metric functions to be r-independent, see subsection 3.3.2, which considerably simplifies discussion of the Gauss–Bonnet field equations.

• *ru-component* remains the same as in the generic case, namely

$$\frac{1}{2\kappa}\left({}^{S}\!R - 2\Lambda_0\right) + \frac{\gamma}{2}\left({}^{S}\!R^2_{klmn} - 4\,{}^{S}\!R^2_{mn} + {}^{S}\!R^2\right) = 0\,, \tag{5.51}$$

and it can be used to eliminate the Gauss–Bonnet term from the following equations.
• *pq-component* together with $g_{uu,r} = 0$ gives a condition for the spatial curvature only,

$${}^{S}\!R_{pq} + 2\kappa\gamma\left({}^{S}\!R_{pq}\,{}^{S}\!R - 2\,{}^{S}\!R_{pmqn}\,{}^{S}\!R^{mn} + {}^{S}\!R_{pklm}\,{}^{S}\!R_q{}^{klm} - 2\,{}^{S}\!R_{pm}\,{}^{S}\!R_q{}^{m}\right) = 0\,. \tag{5.52}$$

Employing (5.51) its trace directly ties the spatial scalar curvature to the cosmological term

$${}^{S}\!R = 4\Lambda_0\,, \tag{5.53}$$

which thus implies that the Ricci scalar of transverse space is *constant*.
• *up-component* of the vacuum field equations now becomes

$$\begin{aligned} 2\,g^{mn}\left[-\frac{1}{2}g_{pn} + \kappa\gamma\left(2\,{}^{S}\!R_{pn} - {}^{S}\!R g_{pn}\right)\right] g^{kl} g_{k[m,u||l]} \\ + 2\kappa\gamma\left(-2\,{}^{S}\!R^{kl}\,\delta^m_p + {}^{S}\!R_p{}^{kml}\right) g_{k[m,u||l]} = 0\,. \end{aligned} \tag{5.54}$$

• *uu-component* then takes the form

$$\begin{aligned} \left[-\frac{1}{2}g^{mn} + \kappa\gamma\left(2\,{}^{S}\!R^{mn} - {}^{S}\!R g^{mn}\right)\right]\left(g_{uu||mn} + g_{mn,uu} - \frac{1}{2}g^{pq}g_{pm,u}g_{qn,u}\right) \\ + 2\kappa\gamma\left(g^{mo}g^{ns} - 2\,g^{mn}g^{os}\right) g^{pq} g_{m[p,u||n]} g_{o[q,u||s]} = 0\,. \end{aligned} \tag{5.55}$$

5.2.3 VSI spacetimes

In fact, we initially start with slightly more general metric ansatz then just VSI case, see section 3.3.2. Here we only assume non-gyratonic Kundt line element with *flat transverse space*, i.e., we put $g_{pq} = \delta_{pq}$, without other VSI restrictions. Since the transverse space has vanishing curvature we may apply results of subsection 5.2.1.

• *ru-component* now requires the cosmological constant to be vanishing

$$\Lambda_0 = 0\,. \tag{5.56}$$

• *pq-component* then considerably simplifies into the form

$$g_{pq} g_{uu,rr} = 0\,, \tag{5.57}$$

and its trace becomes

$$(D-2)\, g_{uu,rr} = 0\,. \tag{5.58}$$

In any case it is reasonable to employ dimension $D > 2$ which means that the metric component g_{uu} has to be at most linear in r and can thus be written as

$$g_{uu}(r,u,x) = c(u,x)r + d(u,x)\,, \tag{5.59}$$

which brings us necessarily to the VSI family.
• *up-component* implies that the metric function c is independent of the spatial coordinates

$$c_{,p} = 0\,. \tag{5.60}$$

• *uu-component* restricts the function d to be solution to the Laplace equation with respect to the spatial coordinates as in the case of classic general relativity

$${}^S\Box\, d = 0\,. \tag{5.61}$$

To summarize, it is not surprising that in the case of non-gyratonic VSI spacetimes there is *no difference* between Einstein's gravity and the Guass–Bonnet theory.

5.2.4 Geometries related to direct product backgrounds

Here, we consider specific subcase of non-gyratonic Kundt geometries which would follow discussion of paper [66] in Einstein's theory. Our aim is to find exact solution representing type II (or N) gravitational wave propagating on type D (or conformally flat type O) direct product background. We start with general metric ansatz (3.25) together with an additional assumption

$$g_{pq,u} = 0\,.$$

For ru and pq components of the Gauss–Bonnet field equations we can exactly employ our general results of section 5.1 since these components do not contain u-derivatives of the metric. According to the pq component we have to distinguish three different subcases (i), (ii) and (iii) following the previous section 5.1.

Let us start with *subcase* (i) defined as $S_{pn} \neq 0$ and $g^{pn} S_{pn} \neq 0$, see (5.7):

• *up-component*: The *first* order in r gives again

$$\left[-\frac{1}{2} g_{pn} + \kappa\gamma \left(2\, {}^S R_{pn} - {}^S R g_{pn}\right)\right] g^{mn} b_{,m} = 0, \tag{5.62}$$

which can also be obtained as a consequence of the pq-equation, see the previous section 5.1.

The condition following from the *zeroth* order in r simplifies to

$$\left[-\frac{1}{2} g_{pn} + \kappa\gamma \left(2\, {}^S R_{pn} - {}^S R g_{pn}\right)\right] g^{mn} c_{,m} = 0\,, \tag{5.63}$$

and restricts thus possible spatial dependence of function $c(u,x)$ in the metric component g_{uu}.

• *uu-component*: In the *second* order in r we obtain the equation

$$\left[-\frac{1}{2} g^{mn} + \kappa\gamma \left(2\, {}^S R^{mn} - {}^S R g^{mn}\right)\right] b_{||mn} = 0\,, \tag{5.64}$$

which is again already satisfied by the condition (5.62).

The *first* order in r leads to the equation

$$\left[-\frac{1}{2}g^{mn} + \kappa\gamma\left(2\,{}^{S}\!R^{mn} - {}^{S}\!R g^{mn}\right)\right] c_{||mn} = 0\,, \tag{5.65}$$

which is satisfied by (5.63).

Finally, the *zeroth* order in r restricts spatial dependence of the metric coefficient $d(u,x)$

$$\left[-\frac{1}{2}g^{mn} + \kappa\gamma\left(2\,{}^{S}\!R^{mn} - {}^{S}\!R g^{mn}\right)\right] d_{||mn} = 0\,. \tag{5.66}$$

This constraint represents generalisation of standard Laplace equation for function $d(u,x)$ in the case of Einstain's theory. Obviously the additional term vanishes in four dimensions.

Detailed discussion of the peculiar *subcase* (ii) defined by $S_{pn} \neq 0$ and $g^{pn}S_{pn} = 0$, see (5.7), will be postponed for further work.

The degenerated *subcase* (iii), corresponding to $S_{pn} = 0$ and thus $g^{pn}S_{pn} = 0$, represents an interesting difference from the well-known Kundt spacetimes in Einstein's theory. The geometry is restricted just by ru and pq component of the field equations, respectively, which put specific constraints on the transverse space and coupling constants. Since the up and uu equations non-trivially depend on the u-derivatives of the transverse space metric, which are assumed to be vanishing here, these equations are *satisfied identically*, see case (iii) in section 5.1. The metric function $g_{uu}(r,u,x)$ remains *unconstrained*.

Conclusion

In this master thesis, first we have reviewed modifications of Einstein's theory of gravity, which are supposed to solve some of its observational and theoretical issues, see chapter 1 for more details. We have paid special attention to modifications of Lagrangian density which correspond to presence of additional curvature invariants, and functions of these. This includes Lovelock theories, containing famous Gauss–Bonnet theory, which are characterized by field equations of the second order, quadratic gravity, and general $f(R)$-gravity, respectively. We have considered these modified theories in arbitrary dimension. We have also mentioned some of interesting exact solutions to their respective field equations.

After this review, we have explicitly derived completely general field equations for theories with arbitrary function of the form $f(R^2,\, R^2_{cd},\, R^2_{cdef})$ in the Lagrangian density, see chapter 2. Most of the aforementioned modified theories are special cases of this general class. We also present various equivalent forms of these field equations, which are not common in the literature.

Further, we have reviewed geometries admitting non-twisting null geodesic congruences, especially the Kundt class of non-twisting, shear-free, and non-expanding geometries, see chapter 3. We have introduced the non-gyratonic subclass and summarized its geometric properties together with important members of this family.

Subsequently in the original part of the thesis, after intensive calculations, we have explicitly expressed particular components of the field equations of the general quadratic gravity with metric ansatz corresponding to the non-gyratonic Kundt geometry, see chapter 4. After that, we have begun the discussion of specific restrictions, which these (in general fourth order) field equations impose on the metric functions of the non-gyratonic Kundt line element. This was done in geometrically privileged subcases, namely *pp*-waves, characterized by the existence of covariantly constant vector field, VSI spacetimes with vanishing curvature invariants, and spacetimes corresponding to the direct product of other lower dimensional geometries.

In detail we have discussed Kundt spacetimes in the Gauss–Bonnet theory, i.e., the special case of quadratic gravity, which usually has to be treated separately, see chapter 5. We have performed this discussion for a completely general non-gyratonic metric ansatz. As a special case we have analysed Kundt geometries with transverse space of constant curvature, *pp*-waves, VSI spacetimes, and spacetimes related to the direct product geometries.

The very important class of Kundt spacetimes in generic quadratic gravity with arbitrary number of spacetime dimensions still provides many opportunities for further investigation and discussion of various special cases, which we would like to do in the near future. We would also like to discussed physical interpretation of obtained solutions and their difference from those well-known in Einstein's general relativity.

Appendices

A. Curvature tensors for the Kundt geometry

In this appendix we summarize the conventions and collect the expressions for components of the *geometric objects*[1] for the non-gyratonic Kundt line element (3.25). These can be found in [26, 34] as a special case with metric terms g_{up} vanishing.

We begin with the *Christoffel symbols* defined as $\Gamma^c_{ab} = \frac{1}{2} g^{cd}(2g_{d(a,b)} - g_{ab,d})$,

$$\Gamma^r_{rr} = 0\,, \qquad \Gamma^r_{ru} = -\frac{1}{2} g_{uu,r}\,, \qquad \Gamma^r_{rp} = 0\,,$$
$$\Gamma^r_{uu} = \frac{1}{2} g_{uu} g_{uu,r} - \frac{1}{2} g_{uu,u}\,, \qquad \Gamma^r_{up} = -\frac{1}{2} g_{uu,p}\,, \qquad \Gamma^r_{pq} = \frac{1}{2} g_{pq,u}\,, \tag{A.1}$$

$$\Gamma^u_{rr} = 0\,, \qquad \Gamma^u_{ru} = 0\,, \qquad \Gamma^u_{rp} = 0\,,$$
$$\Gamma^u_{uu} = \frac{1}{2} g_{uu,r}\,, \qquad \Gamma^u_{up} = 0\,, \qquad \Gamma^u_{pq} = 0\,, \tag{A.2}$$

$$\Gamma^m_{rr} = 0\,, \qquad \Gamma^m_{ru} = 0\,, \qquad \Gamma^m_{rp} = 0\,,$$
$$\Gamma^m_{uu} = -\frac{1}{2} g^{mn} g_{uu,n}\,, \qquad \Gamma^m_{up} = \frac{1}{2} g^{mn} g_{np,u}\,, \qquad \Gamma^m_{pq} = {}^S\Gamma^m_{pq}\,. \tag{A.3}$$

The components of Riemann tensor $R_{abcd} = g_{ae}\left(\Gamma^e_{bd,c} - \Gamma^e_{bc,d} + \Gamma^f_{bd}\Gamma^e_{fc} - \Gamma^f_{bc}\Gamma^e_{fd}\right)$ become

$$R_{rprq} = 0\,, \tag{A.4}$$
$$R_{rpru} = 0\,, \tag{A.5}$$
$$R_{rpmq} = 0\,, \tag{A.6}$$
$$R_{ruru} = -\frac{1}{2} g_{uu,rr}\,, \tag{A.7}$$
$$R_{rpuq} = 0\,, \tag{A.8}$$
$$R_{rupq} = 0\,, \tag{A.9}$$
$$R_{mpnq} = {}^S R_{mpnq}\,, \tag{A.10}$$
$$R_{ruup} = \frac{1}{2}\, g_{uu,rp}\,, \tag{A.11}$$
$$R_{upmq} = g_{p[m,u||q]}\,, \tag{A.12}$$
$$R_{upuq} = -\frac{1}{2} g_{uu||pq} - \frac{1}{2} g_{pq,uu} + \frac{1}{4} g_{uu,r} g_{pq,u} + \frac{1}{4} g^{mn} g_{mp,u} g_{nq,u}\,. \tag{A.13}$$

For the Ricci tensor $R_{ab} = g^{cd} R_{acbd}$ we get

$$R_{rr} = 0\,, \tag{A.14}$$
$$R_{rp} = 0\,, \tag{A.15}$$
$$R_{ru} = -\frac{1}{2} g_{uu,rr}\,, \tag{A.16}$$
$$R_{pq} = {}^S R_{pq}\,, \tag{A.17}$$
$$R_{up} = -\frac{1}{2} g_{uu,rp} + g^{mn} g_{m[p,u||n]}\,, \tag{A.18}$$
$$R_{uu} = \frac{1}{2} g_{uu} g_{uu,rr} + \frac{1}{4} g^{mn} g_{mn,u} g_{uu,r} - \frac{1}{2} g^{mn} g_{mn,uu} - \frac{1}{2} g^{mn} g_{uu||mn} + \frac{1}{4} g^{mn} g^{pq} g_{pm,u} g_{qn,u}\,, \tag{A.19}$$

[1] Let's emphasize that all the results of appendices A and B are purely geometric, i.e., *no field equations* have been employed, and can thus be applied even in the case of any other theory admitting Kundt spacetimes.

and the Ricci scalar defined as $R = g^{ab} R_{ab}$ is

$$R = {}^{S}\!R + g_{uu,rr} \,. \tag{A.20}$$

Finally, the Weyl tensor defined as a traceless part of the Riemann tensor,

$$C_{abcd} = R_{abcd} - \frac{1}{D-2}\left(g_{ac}R_{bd} - g_{ad}R_{bc} + g_{bd}R_{ac} - g_{bc}R_{ad}\right) + \frac{R\left(g_{ac}g_{bd} - g_{ad}g_{bc}\right)}{(D-1)(D-2)} \,,$$

can be expressed via its components as

$$C_{rprq} = 0 \,, \tag{A.21}$$

$$C_{rpru} = 0 \,, \tag{A.22}$$

$$C_{rpmq} = 0 \,, \tag{A.23}$$

$$C_{ruru} = -\frac{D-3}{D-1}\left[\frac{1}{2}g_{uu,rr} + \frac{1}{(D-2)(D-3)}\,{}^{S}\!R\right] , \tag{A.24}$$

$$C_{rpuq} = \frac{1}{D-2}\left[{}^{S}\!R_{pq} - \frac{1}{D-1}\,g_{pq}\,{}^{S}\!R + \frac{1}{2}\frac{D-3}{D-1}g_{pq}g_{uu,rr}\right] , \tag{A.25}$$

$$C_{rupq} = 0 \,, \tag{A.26}$$

$$\begin{aligned} C_{mpnq} = {}& {}^{S}C_{mpnq} + \frac{2}{(D-2)(D-4)}\left[g_{mn}\,{}^{S}\!R_{pq} + g_{pq}\,{}^{S}\!R_{mn} - g_{mq}\,{}^{S}\!R_{pn} - g_{pn}\,{}^{S}\!R_{mq}\right] \\ & + \frac{1}{(D-1)(D-2)}\left(g_{mn}g_{pq} - g_{mq}g_{pn}\right)\left[g_{uu,rr} - \frac{2(2D-5)}{(D-3)(D-4)}\,{}^{S}\!R\right] , \end{aligned} \tag{A.27}$$

$$C_{ruup} = \frac{1}{2}\frac{D-3}{D-2}g_{uu,rp} + \frac{1}{D-2}g^{mn}g_{m[p,u||n]} \,, \tag{A.28}$$

$$C_{upmq} = g_{p[m,u||q]} + \frac{1}{D-2}\left[-g_{uu,r[q}g_{m]p} + g^{ns}\left(g_{pm}g_{n[q,u||s]} - g_{pq}g_{n[m,u||s]}\right)\right] , \tag{A.29}$$

$$\begin{aligned} C_{upuq} = {}& -\frac{1}{2}g_{uu||pq} - \frac{1}{2}g_{pq,uu} + \frac{1}{4}g_{uu,r}g_{pq,u} + \frac{1}{4}g^{os}g_{op,u}g_{sq,u} \\ & - \frac{1}{D-2}\,g_{pq}g^{mn}\left[-\frac{1}{2}g_{uu||mn} - \frac{1}{2}g_{mn,uu} + \frac{1}{4}g_{uu,r}g_{mn,u} + \frac{1}{4}g^{os}g_{om,u}g_{sn,u}\right] \\ & + \frac{1}{(D-1)(D-2)}\,g_{uu}g_{pq}\left({}^{S}\!R + g_{uu,rr}\right) \\ & - \frac{1}{2(D-2)}\,g_{uu}g_{pq}g_{uu,rr} - \frac{1}{D-2}\,g_{uu}\,{}^{S}\!R_{pq} + \frac{1}{D-2}\,g_{pq}g^{rn}g_{uu,rn} \,, \end{aligned} \tag{A.30}$$

which are projected onto the natural null frame in section 3.3.

B. Quadratic quantities for the Kundt geometry

Here, we collect the expressions representing various curvature contractions and their derivatives. These relations serve as partial results, and we use them to explicitly express the field equations (1.30), (4.4)–(4.10), respectively, of the quadratic gravity (1.29) in the case of non-gyratonic Kundt spacetimes (3.25).

We begin with the scalar quantities, namely *squares of the curvature components*

$$R^2 = \left({}^S\!R + g_{uu,rr}\right)^2 , \tag{B.1}$$

$$R_{cd}^2 = {}^S\!R_{mn}^2 + \frac{1}{2} {g_{uu,rr}}^2 , \tag{B.2}$$

$$R_{cdef}^2 = {}^S\!R_{klmn}^2 + {g_{uu,rr}}^2 . \tag{B.3}$$

and the *d'Alembert of the Ricci scalar*

$$\Box R = -g_{uu} g_{uu,rrrr} - 2 g_{uu,rrru} - g_{uu,r} g_{uu,rrr} + {}^S\Box\, {}^S\!R + {}^S\Box\, g_{uu,rr} - \frac{1}{2} g^{mn} g_{mn,u} g_{uu,rrr} . \tag{B.4}$$

Subsequently we list components of specific tensorial expressions in the field equations (1.30):

- *rr-component*

$$R_{rc} {R_r}^c = 0 , \tag{B.5}$$

$$R_{rcrd} R^{cd} = 0 , \tag{B.6}$$

$$R_{rcde} {R_r}^{cde} = 0 , \tag{B.7}$$

$$\nabla_r \nabla_r R = g_{uu,rrrr} , \tag{B.8}$$

$$\Box R_{rr} = 0 , \tag{B.9}$$

- *rp-component*

$$R_{rc} {R_p}^c = 0 , \tag{B.10}$$

$$R_{rcpd} R^{cd} = 0 , \tag{B.11}$$

$$R_{rcde} {R_p}^{cde} = 0 , \tag{B.12}$$

$$\nabla_r \nabla_p R = g_{uu,rrrp} , \tag{B.13}$$

$$\Box R_{rp} = 0 , \tag{B.14}$$

- *ru-component*

$$R_{rc} {R_u}^c = -\frac{1}{4} {g_{uu,rr}}^2 , \tag{B.15}$$

$$R_{rcud} R^{cd} = -\frac{1}{4} {g_{uu,rr}}^2 , \tag{B.16}$$

$$R_{rcde} {R_u}^{cde} = -\frac{1}{2} {g_{uu,rr}}^2 , \tag{B.17}$$

$$\nabla_r \nabla_u R = g_{uu,rrru} + \frac{1}{2} g_{uu,r} g_{uu,rrr} , \tag{B.18}$$

$$\Box R_{ru} = \frac{1}{2} g_{uu} g_{uu,rrrr} + \frac{1}{2} g_{uu,r} g_{uu,rrr} + g_{uu,rrru} - \frac{1}{2}\, {}^S\Box\, g_{uu,rr} + \frac{1}{4} g^{mn} g_{mn,u} g_{uu,rrr} , \tag{B.19}$$

- *pq-component*

$$R_{pc} {R_q}^c = {}^S\!R_{pm}\, {}^S\!{R_q}^m , \tag{B.20}$$

$$R_{pcqd} R^{cd} = {}^S\!R_{pmqn}\, {}^S\!R^{mn} , \tag{B.21}$$

$$R_{pcde} {R_q}^{cde} = {}^S\!R_{pklm}\, {}^S\!{R_q}^{klm} , \tag{B.22}$$

$$\nabla_p \nabla_q R = {}^S\!R_{||pq} + g_{uu,rr||pq} - \frac{1}{2} g_{pq,u} g_{uu,rrr} , \tag{B.23}$$

$$\Box R_{pq} = {}^S\Box\, {}^S\!R_{pq} , \tag{B.24}$$

- *up-component*

$$R_{uc}R_p{}^c = \frac{1}{2}g_{uu,rr}\left(-\frac{1}{2}g_{uu,rp} + g^{mn}g_{m[p,u||n]}\right) + g^{mn}\,{}^S\!R_{mp}\left(-\frac{1}{2}g_{uu,rn} + g^{kl}g_{k[n,u||l]}\right), \tag{B.25}$$

$$R_{ucpd}R^{cd} = -\frac{1}{4}g_{uu,rr}g_{uu,rp} + g_{m[p,u||n]}\,{}^S\!R^{mn}, \tag{B.26}$$

$$R_{ucde}R_p{}^{cde} = -\frac{1}{2}g_{uu,rr}g_{uu,rp} + g_{k[l,u||m]}\,{}^S\!R_p{}^{klm}, \tag{B.27}$$

$$\nabla_u\nabla_p R = {}^S\!R_{,up} + g_{uu,rrup} + \frac{1}{2}g_{uu,p}g_{uu,rrr} - \frac{1}{2}g^{mn}g_{mp,u}\left({}^S\!R_{,n} + g_{uu,rrn}\right), \tag{B.28}$$

$$\begin{aligned}\Box R_{up} = {}& \frac{1}{2}g_{uu}g_{uu,rrrp} + g_{uu,rrup} + \frac{1}{2}g_{uu,p}g_{uu,rrr} - \frac{1}{2}\,{}^S\Box\, g_{uu,rp} \\ & + g^{mn}\left(\frac{1}{2}g_{m[p,u||n]}g_{uu,rr} - \frac{1}{2}\,{}^S\!R_{mp}\,g_{uu,rn} + {}^S\Box\, g_{m[p,u||n]} + \frac{1}{4}g_{uu,rr}g_{pm,u||n}\right. \\ & \left. - \frac{1}{2}g^{kl}g_{km,u||n}\,{}^S\!R_{lp} - g^{kl}g_{km,u}\,{}^S\!R_{lp||n} + \frac{1}{4}g_{mn,u}g_{uu,rrp}\right),\end{aligned} \tag{B.29}$$

- *uu-component*

$$\begin{aligned}R_{uc}R_u{}^c = {}& g^{pq}\left(-\frac{1}{2}g_{uu,rp} + g^{mn}g_{m[p,u||n]}\right)\left(-\frac{1}{2}g_{uu,rq} + g^{os}g_{o[q,u||s]}\right) \\ & + \frac{1}{4}g_{uu,rr}\left(g_{uu}g_{uu,rr} + g^{mn}g_{mn,u}g_{uu,r} - 2g^{mn}g_{mn,uu}\right. \\ & \left. -2\,{}^S\Box\, g_{uu} + g^{mn}g^{pq}g_{mp,u}g_{nq,u}\right),\end{aligned} \tag{B.30}$$

$$\begin{aligned}R_{ucud}R^{cd} = {}& -\frac{1}{8}g_{uu,rr}\left(-2g_{uu}g_{uu,rr} + g^{mn}g_{mn,u}g_{uu,r} - 2g^{mn}g_{mn,uu}\right. \\ & \left. -2\,{}^S\Box\, g_{uu} + g^{mn}g^{pq}g_{mp,u}\,g_{nq,u}\right) \\ & + g^{pq}g_{uu,rp}\left(-\frac{1}{2}g_{uu,rq} + g^{mn}g_{m[q,u||n]}\right) \\ & + \frac{1}{4}\,{}^S\!R^{pq}\left(-2g_{uu||pq} - 2g_{pq,uu} + g_{uu,r}g_{pq,u} + g^{mn}g_{mp,u}g_{nq,u}\right),\end{aligned} \tag{B.31}$$

$$R_{ucde}R_u{}^{cde} = \frac{1}{2}g_{uu}g_{uu,rr}{}^2 - \frac{1}{2}g^{pq}g_{uu,rp}g_{uu,rq} + g^{os}g^{mn}g^{pq}g_{o[m,u||p]}g_{s[n,u||q]}, \tag{B.32}$$

$$\begin{aligned}\nabla_u\nabla_u R = {}& {}^S\!R_{,uu} + g_{uu,rruu} - \frac{1}{2}g_{uu,rrr}\left(g_{uu}g_{uu,r} - g_{uu,u}\right) \\ & - \frac{1}{2}g_{uu,r}\left({}^S\!R_{,u} + g_{uu,rru}\right) + \frac{1}{2}g^{mn}g_{uu,m}\left({}^S\!R_{,n} + g_{uu,rrn}\right),\end{aligned} \tag{B.33}$$

$$\begin{aligned}\Box R_{uu} = {}& -g_{uu}\left(\frac{1}{2}g_{uu,r}g_{uu,rrr} + \frac{1}{2}g_{uu}g_{uu,rrrr} + \frac{1}{2}g^{mn}g_{mn,u}g_{uu,rrr} - {}^S\Box\, g_{uu,rr} + g_{uu,rrru}\right) \\ & - \frac{1}{2}g^{mn}{}_{,u}\,g_{mn,u}g_{uu,rr} - g^{mn}g_{mn,uu}g_{uu,rr} - \frac{1}{2}g^{mn}g_{mn,u}g_{uu,rru} + \left({}^S\Box\, g_{uu,r}\right)_{,u} \\ & - \frac{1}{2}g_{uu,rr}\,{}^S\Box\, g_{uu} + \frac{1}{2}g_{uu,r}\left(g^{mn}g_{mn,u}g_{uu,rr} - {}^S\Box\, g_{uu,r}\right) \\ & + g^{pq}g_{uu,rp}\left(\frac{1}{2}g_{uu,rq} - g^{mn}g_{m[q,u||n]}\right) + g^{mn}g_{uu,m}g_{uu,rrn} + \frac{1}{4}g_{uu,r}g^{mn}\,{}^S\Box\, g_{mn,u} \\ & + \frac{1}{2}g^{mn}g^{pq}g_{mn,u||p}\,g_{uu,rq} + \frac{1}{2}g^{mn}g_{mn,u}\,{}^S\Box\, g_{uu,r} - \frac{1}{2}g^{mn}\,{}^S\Box\, g_{mn,uu} - \frac{1}{2}\,{}^S\Box\,{}^S\Box\, g_{uu} \\ & + \frac{1}{2}g^{mn}g^{pq}\left(g_{pm,u}\,{}^S\Box\, g_{qn,u} + g^{os}g_{pm,u||o}g_{qn,u||s}\right) \\ & + g^{pq}\left[g^{mn}g_{mp,u||q}\left(\frac{1}{2}g_{uu,rn} - g^{os}g_{o[n,u||s]}\right) + 2\,g^{mn}g_{mp,u}\left(\frac{1}{2}g_{uu,rn||q} - g^{os}g_{o[n,u||s]||q}\right)\right. \\ & \left. - \frac{1}{8}g^{mn}g_{mn,u}g_{pq,u}g_{uu,rr} + \frac{1}{2}\,{}^S\!R^{mn}g_{mp,u}g_{nq,u}\right].\end{aligned} \tag{B.34}$$

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