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Variation of Fractional Processes

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Abstract: In this thesis, we study various notions of variation of certain stochastic processes, namely p-variation, pathwise p-th variation along sequence of partitions and p-th variation along sequence of partitions. We study these concepts for fractional Brownian motions and Rosenblatt processes. A fractional Brownian motion is a Gaussian process and it has been intensively developed and studied over the last two decades because of its importance in modeling various phenomena. On the other hand, a Rosenblatt process, which is a non-Gaussian process that can be used for modeling non-Gaussian fluctuations, has not been getting as much attention as fractional Brownian motion. For that reason, we concentrate in this thesis on this process and we present some original results that deal with ergodicity, p-variation, pathwise p-th variation along sequence of partitions and p-th variation along sequence of partitions.

Keywords: p-th variation along a sequence of partitions, fractional Brownian motion, Rosenblatt process, Ergodicity

Contents

List of Symbols

$:=$	... definition
$\mathbb{N}$	... set of natural numbers
$\mathbb{N}_0 := \mathbb{N} \cup \{0\}$	... the union of $\mathbb{N}$ and $\{0\}$
$\mathbb{Z}$	... set of integers
$\mathbb{R}$	... set of real numbers
$\mathbb{R}^+$	... set of non-negative real numbers
$\mathbb{Q}$	... set of rational numbers
$\mathbb{Q}^+$	... set of non-negative rational numbers
$\mathbb{P}$	... probability measure
$\mathbb{E}$	... expected value
Var	... variance
$\mathcal{L}(X)$	... law of random variable X
$N(\mu, \sigma^2)$	... Normal distribution with mean $= \mu$, variance $= \sigma^2$
$\overset{\mathcal{D}}{\sim}$	... equivalence of random variables in distribution
$\mathcal{B}(\mathbb{R})$	... Borel sigma algebra
$\lvert x \rvert$	... absolute value of x
x_+	... $\max(x, 0)$
Γ	... Gamma function
$a \wedge b$	... $\min(a, b)$, for $a, b \in \mathbb{R}$
$\approx$	... is approximately equal to
(a, b)	... open interval from a to b
$[a, b]$	... closed interval from a to b
A^C	... complement of set A
$A \triangle B$	... symmetric difference of sets A and B
$L^p(\Omega) := L^p(\Omega, \mathcal{F}, \mathbb{P})$	... L^p space, where $p \geq 1$
$\xrightarrow[t\to\infty]{\mathbb{P}\text{-a.s.}}$	... convergence $\mathbb{P}$-almost surely, as $t \to \infty$
$\xrightarrow[t\to\infty]{L^1}$	... convergence in L^1, as $t \to \infty$
$\xrightarrow[t\to\infty]{\mathbb{P}}$	... convergence in probability $\mathbb{P}$, as $t \to \infty$
$\xrightarrow[t\to\infty]{\mathcal{D}}$	... convergence in distribution, as $t \to \infty$
$\xrightarrow[n\to\infty]{\mathrm{ucp}}$	... uniform convergence on compacts in probability, as $n \to \infty$
$\lambda(\cdot)$	... Lebesgue measure on $\mathcal{B}(\mathbb{R})$
$f\vert_A$	... mapping f restricted on set A
$C([a, b], \mathbb{R})$	... space of all continuous functions $f : [a, b] \to \mathbb{R}$

Introduction

Fractional Brownian motions (fBm) and Rosenblatt processes are families of processes that are indexed by Hurst parameter $H \in (0,1)$ and $H \in (\frac{1}{2},1)$ respectively. In the past two decades, fBm received a lot of attention, because it was found very useful in modelling various long-range and short-range dependent phenomena. It is a Gaussian, self-similar process and if $H \in (\frac{1}{2},1)$ then it is also long-range dependent. It has strictly stationary increments, and it can be showen that fBm is the only Gaussian process that is self-similar with strictly stationary increments ([1, p. 2]).
A Rosenblatt process, on the other hand, is a non-Gaussian process. It is, similarly as fBm, self-similar, long-range dependent and has strictly stationary increments. Long-range dependence turned out to be a desired mathematical property when modeling certain, for example, economical phenomena. Rosenblatt process is not as widely used as fBm, however, non-Gaussian data with fractal noise had been observed (see [2]). In these cases, using a Rosenblatt process to model these phenomena might be more appropriate than using fBm. Because Rosenblatt processes have not been as intensively studied as fBms, there are still many mathematical properties that are yet to be proved or disproved.

In this thesis, we are interested in different concepts of variations. For the fBm, most of these properties had already been proved and our goal is to prove them for Rosenblatt process.
Namely, we analyze p-variation, pathwise p-th variation along the sequence of partitions, and p-th variation along the sequence of partitions, for $p > 0$.

Thesis is divided into two chapters. In first chapter, we give some necessary definitions. First, we define the Wiener process and fBm. We continue with the Rosenblatt process for which we show how it was discovered and we also comment on its connection with a fBm. After that, we define all various definitions of variations. Finally, we recall some basic definitions and theorems of the ergodic theory.

The second chapter is the author's original work, however, some techniques of proofs are known. We start by proving ergodicity of increments of a Rosenblatt process. The reason why we need ergodicity is that we would like to use Birkhoff's Ergodic Theorem to prove Theorem 4 and consequently, prove that a Rosenblatt process is not a semimartingale. In [4], it has been proved that a fBm is a semimartingale if and only if $H = \frac{1}{2}$. In the case of fBm, it is easier because ergodicity follows directly from the fact that its increments are mixing (see [5, Proposition 5.1.1]). After proving that a Rosenblatt process is not a semimartingale, we continue with proving that there exists some sequence of partitions of some bounded interval for which the Rosenblatt process with Hurst parameter $H \in (\frac{1}{2},1)$ has pathwise finite $\frac{1}{H}$-th variation along this sequence of partitions. In the last section of the second chapter, we prove that $\mathbb{P}$-a.a. paths of the Rosenblatt process are of infinite $\frac{1}{H}$-variation.

1. Preliminaries

1.1 Wiener process

Recall that a Gaussian process is a process with Gaussian finite-dimensional distributions. Gaussian processes are uniquely, up to finite-dimensional distributions, determined by their mean and covariance functions. For that reason, it makes sense to define Gaussian processes only by the mean and covariance functions.

Definition 1. A centered, continuous, Gaussian process $W = (W_t, t \in \mathbb{R}^+)$ with $W_0 = 0$ $\mathbb{P}$-a.s. is called the Wiener process if its covariance function ϕ is of the form

$$\phi(t, s) = \mathbb{E}[W_t W_s] = s \wedge t, \tag{1.1}$$

for every $t, s \in \mathbb{R}^+$.

To show the existence of the Wiener process, we would need to show that the function given by the right-hand side of the (1.1) is positive semidefinite and then appeal to the Daniell-Kolmogorov theorem.
What follows is a characterization of Wiener process.

Lemma 1. ([6, p. 47], [9, p. 24]) *The stochastic process $W = (W_t, t \in \mathbb{R}^+)$ is the Wiener process, if*

1. *its trajectories are continuous,*
2. $W_0 = 0$ $\mathbb{P}$*-a.s.,*
3. *it has independent increments, i.e. for every $n \in \mathbb{N}$, $t_0, \dots, t_n \in \mathbb{R}^+$, $t_0 < \dots < t_n$ it holds that the random variables $W_{t_1} - W_{t_0}, W_{t_2} - W_{t_1}, \dots, W_{t_n} - W_{t_{n-1}}$ are independent,*
4. *for every $s, t \in \mathbb{R}^+$ it holds that $W_t - W_s \overset{\mathcal{D}}{\sim} N(0, |t - s|)$.*

For more properties of Wiener process see, for example, Chapter 2 in [7] or Chapter 11 in [8].
Now we will define two-sided Wiener process (i.e. Wiener process defined on $\mathbb{R}$).

Definition 2. ([9, p. 60]) Let $W^1 = (W_t^1, t \in \mathbb{R}^+)$ and $W^2 = (W_t^2, t \in \mathbb{R}^+)$ are two independent Wiener processes defined on the same probability space. Then the process $W = (W_t, t \in \mathbb{R})$ defined by

$$W_t = \begin{cases} W_t^1, & t \geq 0, \\ W_{-t}^2, & t < 0, \end{cases}$$

is called two-sided Wiener process.

In the rest of this thesis, by Wiener process we will always mean the two-sided Wiener process from Definition 2.

1.2 Fractional Brownian Motion

Fractional Brownian motions are a family of stochastic processes parametrised by the Hurst parameter $H \in (0,1)$.

Definition 3. ([10, p. 273]) A centered, Gaussian process $W^H = (W_t^H, t \in \mathbb{R}^+)$ with $W_0^H = 0$ $\mathbb{P}$-a.s. is called the fractional Brownian motion (fBm) with Hurst parameter $H \in (0,1)$ if its covariance function ϕ_H is of the form

$$\phi_H(t,s) = \mathbb{E}[W_t^H W_s^H] = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right), \tag{1.2}$$

for every $t, s \in \mathbb{R}^+$.

Again, if we would like to prove the existence of the fBm, we would need to show that the function on the right-hand side of the (1.2) is positive semidefinite. Now, we are going to discuss some properties of the fBm which immediately follow from the form of covariance function (1.2).
If $H = \frac{1}{2}$, then it holds that $\phi_{\frac{1}{2}} = \frac{1}{2}(t+s-|t-s|) = s \wedge t$. This is the same form as the covariance function (1.1) and we can conclude that if $H = \frac{1}{2}$, then fBm is in fact the Wiener process. Hence, in this case it holds that its increments are independent.
Next, we have that the fBm is self-similar (or more specifically H-self-similar). It means that it is invariant in distribution under suitable scaling of time (see the exact definition in [11, Definition 1.1.1]). More precisely, for every $\alpha > 0$ it holds that the processes $(\alpha^{-H} W_{\alpha t}^H, t \in \mathbb{R}^+)$ and $(W_t^H, t \in \mathbb{R}^+)$ have the same finite-dimensional distributions. This follows because both processes are centered, Gaussian and for every $s, t \in \mathbb{R}^+$ it holds that

$$\begin{aligned}
\mathbb{E}[(\alpha^{-H} W_{\alpha t}^H)(\alpha^{-H} W_{\alpha s}^H)] &= \alpha^{-2H}\frac{1}{2}((\alpha t)^{2H} + (\alpha s)^{2H} - |\alpha t - \alpha s|^{2H} \\
&= \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right) \\
&= E[W_t^H W_s^H].
\end{aligned}$$

It also holds that fBm has strictly stationary increments (in the sense that it holds $W_t^H - W_s^H \overset{\mathcal{D}}{\sim} W_{t-s}^H$ for every $s, t \in \mathbb{R}^+$, $s < t$). Indeed, let us have $s, t \in \mathbb{R}^+$, $s < t$. Then $W_t^H - W_s^H$ and W_{t-s}^H are both centered, normally distributed random variables and it holds

$$\begin{aligned}
\mathbb{E}[|W_t^H - W_s^H|^2] &= \mathbb{E}[(W_t^H)^2] - 2\,\mathbb{E}[W_t^H W_s^H] + \mathbb{E}[(W_s^H)^2] \\
&= t^{2H} - \left(t^{2H} + s^{2H} - |t-s|^{2H}\right) + s^{2H} \\
&= |t-s|^{2H} \\
&= \mathbb{E}[|W_{t-s}^H|^2].
\end{aligned}$$

Now, we are interested whether the increments of the fBm are dependent or independent. Let us choose $t_1, s_1, t_2, s_2 \in \mathbb{R}^+$, $s_1 < t_1 < s_2 < t_2$ and let us have increments $W^H_{t_1} - W^H_{s_1}$, $W^H_{t_2} - W^H_{s_2}$. It holds that

$$\begin{aligned}\mathbb{E}[(W^H_{t_1} - W^H_{s_1})(W^H_{t_2} - W^H_{s_2})] &= \phi_H(t_1, t_2) - \phi_H(t_1, s_2) - \phi_H(s_1, t_2) + \phi_H(s_1, s_2) \\ &= \frac{1}{2}\Big(-|t_1 - t_2|^{2H} + |t_1 - s_2|^{2H} \\ &\quad +|s_1 - t_2|^{2H} - |s_1 - s_2|^{2H}\Big). \end{aligned} \tag{1.3}$$

If $H \in (0, \frac{1}{2})$, then by concavity of the right-hand side of (1.3), it holds

$$|t_1 - s_2|^{2H} + |s_1 - t_2|^{2H} < |t_1 - t_2|^{2H} + |s_1 - s_2|^{2H}$$

and therefore it holds

$$\mathbb{E}[(W^H_{t_1} - W^H_{s_1})(W^H_{t_2} - W^H_{s_2})] < 0.$$

If $H \in (\frac{1}{2}, 1)$ then

$$|t_1 - s_2|^{2H} + |s_1 - t_2|^{2H} > |t_1 - t_2|^{2H} + |s_1 - s_2|^{2H}$$

and therefore it holds

$$\mathbb{E}[(W^H_{t_1} - W^H_{s_1})(W^H_{t_2} - W^H_{s_2})] > 0.$$

Overall, we can conclude that if $H \in (0, \frac{1}{2})$, then the increments of fBm are negatively correlated and if $H \in (\frac{1}{2}, 1)$, then the increments of fBm are positively correlated. If $H = \frac{1}{2}$, then fBm is the Wiener process and its increments are independent.
We say that a stochastic process $(X_t, t \in \mathbb{N}_0)$ with finite second moments is long-range dependent (definition from [11, p. 21]) if

$$\sum_{n=1}^{\infty} |r(n)| = \infty,$$

where r is the function $r(n) := \mathbb{E}[(X_{n+1} - X_n)(X_1 - X_0)]$, for $n \in \mathbb{N}$. In the case of the fBm with $H \in (0, 1)$, we have that $(W^H_{n+1} - W^H_n)_{n \in \mathbb{N}_0}$ is a strictly stationary, Gaussian process with

$$\begin{aligned} r_H(n) &= \mathbb{E}[(W^H_{n+1} - W^H_n)(W^H_1 - W^H_0)] \\ &= \frac{1}{2}\left(|n+1|^{2H} + |n-1|^{2H} - 2|n|^{2H}\right), \text{ for every } n \in \mathbb{N}. \end{aligned}$$

From there, we obtain that if $H \in (\frac{1}{2}, 1)$, then the fBm is long-range dependent, i.e. it holds $\sum_{n=1}^{\infty} |r(n)| = \infty$. If $H \in (0, \frac{1}{2})$, then $\sum_{n=1}^{\infty} |r_H(n)| < \infty$.

By Kolmogorov's continuity theorem (see [7, Theorem 2.9]), it holds that there exists modification of a fBm with all paths being continuous functions (proof of this can be found in [1, Section 3]).
In [4, Section 2], it was proved that fBm is a semimartingale if and only if $H = \frac{1}{2}$. For that reason, we cannot integrate with respect to the fBm in the sense of Itô stochastic integration theory.

It was actually Mandelbrot and Van Ness who first coined the term fractional Brownian motion in their paper [12]. They obtained an integral representation of fBm in terms of the Wiener process.

Lemma 2. ([12, Definition 2.1], [13, p. 35], [10, Proposition 5.1.2])
Let $H \in (0,1)$ and $(W_t, t \in \mathbb{R})$ be a Wiener process. Then the process $W^H = (W_t^H, t \in \mathbb{R}^+)$ defined for $t \in \mathbb{R}^+$ by

$$W_t^H = C_1(H) \int_{\mathbb{R}} \left((t-s)_+^{H-\frac{1}{2}} - (-s)_+^{H-\frac{1}{2}} \right) \mathrm{d}W_s \tag{1.4}$$

is called the fractional Brownian motion with the Hurst parameter $H \in (0,1)$. Here $x_+ = \max(x,0)$ for $x \in \mathbb{R}$ and

$$C_1(H) = \frac{(\Gamma(2H+1)\sin(\pi H))^{\frac{1}{2}}}{\Gamma(H+\frac{1}{2})}$$

where $\Gamma(z) = \int_0^\infty x^{z-1} e^{-x}\, \mathrm{d}x$, $z > 0$, is the Gamma function.

1.3 Rosenblatt process

1.3.1 History and definition of Rosenblatt distribution

The term Rosenblatt process was first coined by Taqqu in [14]. It is called after the American mathematician Murray Rosenblatt. While studying central limit theorems for mixing sequences of random variables, he gave an interesting counterexample for one of the central limit theorems.
Let us briefly describe the example. In [15, p. 434-435] he found the following process: Let $Y^H = (Y_k^H, k \in \mathbb{Z})$ be a centered, strictly stationary, Gaussian process with unit variance and the the covariance function

$$r_H(k) = \mathbb{E}[Y_0^H Y_k^H] = (1+k^2)^{\frac{1-H}{2}}, \text{ for } k \in \mathbb{Z},$$

where $H \in (\frac{1}{2}, 1)$. From the definition of covariance function r_H we see that Y^H is long-range dependent because it holds $\sum_{k=1}^\infty |r_H(k)| = \infty$. Now let us consider process $X^H = (X_k^H, k \in \mathbb{Z})$, defined by

$$X_k^H = (Y_k^H)^2 - 1, \text{ for every } k \in \mathbb{Z}, \tag{1.5}$$

and set

$$\sigma_H = \left(\frac{1}{2}(2H-1)H \right)^{\frac{1}{2}}.$$

Then there exists a random variable Z_H such that

$$\frac{\sigma_H}{n^H} \sum_{k=1}^n X_k^H \xrightarrow[n\to\infty]{\mathcal{D}} Z_H \tag{1.6}$$

and it holds that random variable Z_H is centered and has unite variance. The characteristic function of Z_H for $\theta \in \mathbb{R}$ is

$$\varphi_{Z_H}(\theta) = \mathbb{E}[e^{i\theta Z_H}] = \exp\left(\frac{1}{2} \sum_{k=2}^\infty (2i\theta\sigma_H)^k \frac{c_k^H}{k} \right), \tag{1.7}$$

where

$$c_k^H = \underbrace{\int_0^1 \cdots \int_0^1}_{k\text{-times}} |x_1 - x_2|^{H-1} |x_2 - x_3|^{H-1} \cdots |x_{k-1} - x_k|^{H-1} |x_k - x_1|^{H-1}\, \mathrm{d}x_k\, \mathrm{d}x_{k-1} \ldots \mathrm{d}x_1.$$

It holds that series on the right-hand side of (1.7) converges for small values of θ (see [13, Appendix] for the proof and exact computation) and that is enough to characterize the distribution function. The distribution of a random variable characterized by (1.7) is called a Rosenblatt distribution. From the form of right-hand side of (1.7), it is easy to see that the Rosenblatt distribution with parameter $H \in (\frac{1}{2}, 1)$ is non-Gaussian. We can see also that it holds

$$\sigma_H \xrightarrow[H\to 1]{} \sqrt{\frac{1}{2}}, \quad c_k^H \xrightarrow[H\to 1]{} 1,$$

for every $k \in \mathbb{N}$ and it holds for $\theta \in \mathbb{R}$ that

$$\begin{aligned}\lim_{H\to 1^-} \varphi_{Z_H}(\theta) &= \exp\left(\frac{1}{2}\sum_{k=2}^{\infty}\frac{(\sqrt{2}i\theta)^k}{k}\right) \\ &= \exp\left(-\frac{1}{2}(\log(1-\sqrt{2}i\theta)+\sqrt{2}i\theta)\right) \\ &= e^{-\frac{\sqrt{2}}{2}i\theta}\frac{1}{\sqrt{1-\sqrt{2}i\theta}}.\end{aligned} \tag{1.8}$$

In the second equality we used definition of natural logarithm via its Taylor series. Function (1.8) is a characteristic function of random variable $\frac{1}{2}(\varepsilon^2 - 1)$, where the distribution of ε is $N(0, 1)$. In other words, if $H = 1$, then the Rosenblatt distribution is a chi-squared distribution ([16, p. 2]).
If $H \to \frac{1}{2}^+$, then the corresponding distribution of random variable Z_H is $N(0, 1)$ (see the proof in [16]).

1.3.2 Definition of Rosenblatt process

Taqqu in [14] defines Rosenblatt process $(Z^H(t), t \in \mathbb{R}^+)$ for $H \in (\frac{1}{2}, 1)$ as the limit

$$Z^H(t) := \lim_{n\to\infty} Z_n^H(t) \tag{1.9}$$

where

$$Z_n^H(t) := \frac{\sigma}{n^H}\sum_{k=1}^{\lfloor nt\rfloor} X_k^H, \quad t \in \mathbb{R}^+,$$

with $X^H = (X_k^H, k \in \mathbb{Z})$ defined in (1.5) and limit on the right-hand side of (1.9) is in the sense of convergence in distribution. Note that the Rosenblatt distribution (1.7) is the same as distribution of Rosenblatt process $Z^H(t)$ in $t = 1$.

Now we continue with the definition of the integral representation of a Rosenblatt process.

Definition 4. ([13, Equations 37-39], [5, Equation 3], [17, Example 2])
Let $H \in (\frac{1}{2}, 1)$. The Rosenblatt process $R^H = (R_t^H, t \in \mathbb{R}^+)$ of the Hurst parameter H is defined by

$$R_t^H := C_H^R \int_{\mathbb{R}^2}' \left(\int_0^t (u-y_1)_+^{\frac{H}{2}-1}(u-y_2)_+^{\frac{H}{2}-1}\,\mathrm{d}u\right)\mathrm{d}W_{y_1}\,\mathrm{d}W_{y_2}, \quad t \in \mathbb{R}^+,$$

where C_H^R is a normalizing positive constant such that $\mathbb{E}[(R_1^H)^2] = 1$, the double stochastic integral is the Wiener-Itô multiple integral of order two with respect to the Wiener process $W = (W_t, t \in \mathbb{R})$ where the prime means that integration excludes the diagonal $y_1 = y_2$.

Remark 1. [18, Remark 1] It holds that

$$C_H^R = \frac{\sqrt{2H(2H-1)}}{2\beta(1-H, \frac{H}{2})}$$

where $\beta(a,b) := \int_0^1 v^{a-1}(1-v)^{b-1}\,\mathrm{d}v$ is the Beta function at $a, b \in \mathbb{R}^+$.

Remark 2. From Definition 4, it immediately follows that $R_0^H = 0$ $\mathbb{P}$-a.s. for every $H \in (\frac{1}{2}, 1)$.

1.3.3 Properties of Rosenblatt process

We start by characterizing the Rosenblatt process via its characteristic function.

Lemma 3. ([14, section 6], [13, equation (12), (13)])
Let $R^H = (R_t^H, t \in \mathbb{R}^+)$ be a Rosenblatt process with Hurst parameter $H \in (\frac{1}{2}, 1)$. For $n \in \mathbb{N}$, $t_1, \ldots, t_n \in \mathbb{R}^+$, the characteristic function $\varphi_{R_{t_1}^H, \ldots, R_{t_n}^H}$ of random vector $R := (R_{t_1}^H, \ldots, R_{t_n}^H)^\top$ at $\theta := (\theta_1, \ldots, \theta_n) \in \mathbb{R}^n$ is

$$\begin{aligned}\varphi_{R_{t_1}^H, \ldots, R_{t_n}^H}(\theta) &= \mathbb{E}[e^{i\langle\theta, R\rangle}] \\ &= \mathbb{E}[e^{i\sum_{j=1}^n \theta_j R_{t_j}}] \\ &= \exp\left(\frac{1}{2}\sum_{k=2}^{\infty}\frac{(2i\sigma_H)^k}{k}\sum_{s_1,\ldots,s_k \in \{1,\ldots,n\}} \theta_{s_1}\cdots\theta_{s_k} S_H^*(t_{s_1}, \ldots, t_{s_k})\right)\end{aligned} \tag{1.10}$$

where $\sigma = \left(\frac{1}{2}(2H-1)H\right)^{\frac{1}{2}}$, $\langle \cdot, \cdot \rangle$ is standard Euclidean inner product and

$$\begin{aligned}S_H^*(t_{s_1}, \ldots, t_{s_k}) := \int_0^{t_{s_1}} \ldots \int_0^{t_{s_k}} &|x_1 - x_2|^{H-1}|x_2 - x_3|^{H-1}\ldots \\ &\ldots |x_{k-1} - x_k|^{H-1}|x_k - x_1|^{H-1}\,\mathrm{d}x_k\,\mathrm{d}x_{k-1}\ldots\mathrm{d}x_1.\end{aligned} \tag{1.11}$$

Remark 3. Note again that the series on the right-hand side of (1.10) converges for small values of $\theta \in \mathbb{R}^n$ i.e. there exists $\varepsilon > 0$ such that the series on right hand side of (1.10) converges for every $\theta \in \mathbb{R}^n$ such that $||\theta|| < \varepsilon$. The exact computation of ε can be found in [13, Appendix].

It has been shown in [16, p. 4] that

$$\begin{aligned}S_H^*(1,1) &= \int_0^1\int_0^1 |x_1 - x_2|^{H-1}|x_2 - x_1|^{H-1}\,\mathrm{d}x_2\,\mathrm{d}x_1 \\ &= \frac{1}{(2H-1)H}\end{aligned}$$

and

$$\begin{aligned}S_H^*(1,1,1) &= \int_0^1\int_0^1\int_0^1 |x_1 - x_2|^{H-1}|x_2 - x_3|^{H-1}|x_3 - x_1|^{H-1}\,\mathrm{d}x_3\,\mathrm{d}x_2\,\mathrm{d}x_1 \\ &= \frac{2}{H(3H-1)}\beta(H,H).\end{aligned}$$

Closed form expression of function S_H^* for $k \geq 4$, $k \in \mathbb{N}$, could not be found which means that it has to be computed numerically. In [16, Section 2], a more sophisticated method for numerical computing of S_H^* has been proposed.

It holds that a Rosenblatt process has strictly stationary increments. This follows from the form of characteristic function (1.10) (or more specifically from (1.11)), or alternatively, the proof of this can be also found in [17, Example 2]. Moreover, it holds that the covariance function of a Rosenblatt process is of the same form as the covariance function (1.2) of fBm (proof can be found in [13, p. 40]). From there it follows that the Rosenblatt process is long-range dependent (which is not surprising because it is defined as a limit of sum of long range dependent random variables). Because the covariance function of a Rosenblatt process is of the same form as the covariance function of the fBm (with the same Hurst parameter $H \in (\frac{1}{2}, 1)$), we can apply Kolmogorov's continuity theorem (see [7, Theorem 2.9]) and we obtain that there exists a modification of the Rosenblatt process with all paths being continuous functions.
Finally, from the the form of characteristic function (1.10) it also follows that Rosenblatt process is self-similar (alternatively this also follows from the finite time interval representation of Rosenblatt process in [13, p. 40 and equation (48)]).

Summary of properties of Rosenblatt process

Rosenblatt process $(R_t^H, t \in \mathbb{R}^+)$ with Hurst parameter $H \in (\frac{1}{2}, 1)$

- is a non-Gaussian process,
- starts at zero (i.e. $R_0^H = 0$ $\mathbb{P}$-a.s.),
- is centered,
- has variance $\mathbb{E}[(R_t^H)^2] = t^{2H}$, for every $t \in \mathbb{R}^+$,
- has covariance function of the same form (1.2) as fBm,
- is self-similar,
- has strictly stationary increments,
- is long-range dependent,
- has modification with all paths being continuous functions.

1.3.4 Connection with Fractional Brownian Motion

We begin with the definition of a Hermite process of order $k \in \mathbb{N}$.

Definition 5. [5, p. 2] We define the Hermite process $Z_H^k = (Z_{H,t}^k, t \in \mathbb{R}^+)$ with Hurst parameter $H \in (\frac{1}{2}, 1)$ of order $k \in \mathbb{N}$ for $t \in \mathbb{R}^+$ by

$$Z_H^k(t) := c(H,k) \int_{\mathbb{R}^k} \int_0^t \left(\prod_{j=1}^k (s - y_j)_+^{-(\frac{1}{2} + \frac{1-H}{k})} \right) \mathrm{d}s \, \mathrm{d}W_{y_1} \ldots \mathrm{d}W_{y_k} \tag{1.12}$$

where the above integral is a multiple Wiener-Itô stochastic integral with respect to Wiener process $W = (W_t, t \in \mathbb{R})$ and $c(H,k)$ is a positive constant such that it holds $\mathbb{E}[(Z_H^k(1))^2] = 1$.

Hermite processes are H-self-similar and have stationary increments (see [5, p. 2]). In Subsection 1.3.1 we showed in (1.6) that the sum of correctly normalized sum of dependent chi-squared random variables converge to Rosenblatt distribution. Now we will show the significance of transformation $x^2 - 1$.

Hermite polynomial of degree $m \in \mathbb{N}_0$ is defined (see [5, p. 2]) for $x \in \mathbb{R}$ by

$$H_m(x) = (-1)^m e^{\frac{x^2}{2}} \frac{\mathrm{d}^m}{\mathrm{d}\, x^m} e^{-\frac{x^2}{2}}.$$

It holds that $H_2(x) = x^2 - 1.$, i.e. the transformation in (1.5) is Hermite polynomial of degree 2. Note that Hermite polynomials play important role in Malliavin calculus because they form a complete orthonormal system in the space L^2 generated by Gaussian sequences (see [10, Theorem 1.1.1]). For more details we refer the reader to the book [10].
Let us again consider centered, strictly stationary, Gaussian process $Y^H = (Y_k^H, k \in \mathbb{Z})$ with unit variance such that its correlation function satisfies

$$r_H(n) := \mathbb{E}[Y_0 Y_n] = n^{\frac{2H-2}{k}} L(n)$$

with $H \in (\frac{1}{2}, 1)$, $k \in \mathbb{N}$ and L is slowly varying function at infinity. Slowly varying function L (see [11, Definition 2.1.1]) is a measurable, positive function such that for every $x \in \mathbb{R}^+$ it holds

$$\lim_{t \to \infty} \frac{L(tx)}{L(t)} = 1.$$

Let us choose function g such that $\mathbb{E}[g(Y_0)] = 0$ and $\mathbb{E}[g(Y_0)^2] < \infty$. Furthermore, let us suppose that if we can rewrite g as

$$g(x) = \sum_{j \in \mathbb{N}_0} c_j H_j(x)$$

where

$$c_j = \frac{1}{j!} \mathbb{E}[g(Y_0 H_j(Y_0))]$$

and it holds

$$k = \min\{j \in \mathbb{N}_0 : c_j \neq 0\}.$$

Finally, by the Non-Central Limit Theorem [19, Theorem 1′, p. 32], it holds for every $t \in \mathbb{R}^+$ that

$$Z_{H,n}^k(t) := \frac{1}{n^H} \sum_{j=1}^{\lfloor nt \rfloor} g(Y_j) \xrightarrow[n \to \infty]{\mathcal{D}} Z_H^k(t)$$

where $Z_H^k = (Z_H^k(t), t \in \mathbb{R}^+)$ is a Hermite process of order k. Hermite processes are therefore limits of normalized sums of long range dependent random variables.

If $k = 1$ then from Definition 5 we obtain that Z_H^k is the fBm with Hurst parameter H from interval $(\frac{1}{2}, 1)$ (compare (1.4) with (1.12) for $k = 1$). Remember that fBm is long-range dependent only for Hurst parameter $H \in (\frac{1}{2}, 1)$. That being said, fractional Brownian motion with Hurst parameter $H \in (\frac{1}{2}, 1)$ is the simplest Hermite process and the only Gaussian Hermite process. Hermite processes are non-Gaussian for every order $k \in \mathbb{N}$, $k \geq 2$. The simplest non-Gaussian Hermite process, with $k = 2$, is the Rosenblatt process with the Hurst parameter $H \in (\frac{1}{2}, 1)$.

1.4 Various Variations

In this section we introduce three different concepts of variation:

- p-variation (Definition 6),
- pathwise p-th variation along the sequence of partitions (Definition 7),
- p-th variation along the sequence of partitions (Definition 8).

We start with definition of p-variation. Recall that path (or trajectory) of real-valued stochastic process $(X_t, t \in I)$ indexed on non-empty set I is (deterministic) function $X_\cdot(\omega) : I \to \mathbb{R}$ for $\omega \in \Omega$.

Definition 6. ([20, Definition 5.1], [21, p. 163]) Let $p > 0$ and $T > 0$. A function $f : [0, T] \to \mathbb{R}$ is said to be of finite p-variation if

$$||f||_{p\text{-var},[0,T]} := \left(\sup_{\pi \in \mathcal{D}([0,T])} \sum_{[t_j, t_{j+1}] \in \pi} |f(t_{j+1}) - f(t_j)|^p \right)^{\frac{1}{p}} < \infty$$

where the supremum is taken over the set $\mathcal{D}([0, T])$ of all partitions π of the interval $[0, T]$.

Note that the notion p-variation is usually defined more generally on metric space (E, d), but for our purposes, taking real-valued functions will be sufficient. The notion of p-variation is of interest only for $p \geq 1$, because all continuous functions with finite p-variation, where $p \in (0, 1)$, are constant (for the proof see [20, Proposition 5.2]). It also holds that if f is a real continuous function on $[0, T]$ with finite p-variation and finite $\hat{p}$-variation on $[0, T]$, where $0 < p < \hat{p}$, then $||f||_{\hat{p}\text{-var},[0,T]} \leq ||f||_{p\text{-var},[0,T]}$ ([20, Proposition 5.3]). From there, it follows that p-variation as a function of p is non-increasing. If $p = 1$, then 1-variation is sometimes also called total variation. Functions with finite total variation are called bounded variation functions. Bounded variation functions can be used as integrators (for details see for example [22, Subsection 1.7.3]). It holds that $\mathbb{P}$-a.a. paths of Wiener process have infinite total variation on any non-trivial interval (see [7, Corollary 2.17]). For that reason we cannot use Lebesgue-Stieltjes integral if we want to integrate with respect to Wiener process, but we need some additional theory.
We continue with the definition of the pathwise p-th variation along a sequence

of partitions. The concept of pathwise quadratic (case when $p = 2$) variation along the sequence of partitions was first proposed by Föllmer in [3] and later generalized by Cont and Perkowski in [21] for $p > 2$.

Definition 7. ([21, Definition 1.1]) Let $p > 0$ and $T > 0$. A continuous function $f : [0, T] \to \mathbb{R}$ is said to have finite pathwise p-th variation along a sequence of partitions $\pi = (\pi_n)_{n\in\mathbb{N}}$ of interval $[0, T]$, where $\pi_n = \{t_0^n, \ldots, t_{N(\pi_n)}^n\}$ is such that $0 = t_0^n < t_1^n < \ldots < t_{N(\pi_n)}^n = T$ and has vanishing mesh $|\pi_n| = \sup_{i\in\{1,\ldots,N(\pi_n)\}} |t_i^n - t_{i-1}^n| \xrightarrow[n\to\infty]{} 0$, if the sequence of measures $\{\mu^n\}_{n\in\mathbb{N}}$ where

$$\mu^n := \sum_{[t_j,t_{j+1}]\in\pi_n} \delta_{t_j} |f(t_{j+1}) - f(t_j)|^p, \text{ for every } n \in \mathbb{N},$$

converges weakly to a non-atomic measure μ (here, δ_u denotes Dirac measure at the point $u \in \mathbb{R}$). In that case we write $f \in V_p(\pi)$ and $[f]_\pi^p(t) := \mu([0, t])$ for $t \in [0, T]$, and we call $[f]_\pi^p$ the pathwise p-th variation of f along sequence of partitions π.

Remark 4. Authors of [21, Definition 1.1] define p-th variation along sequence of partitions (without the word "pathwise"). We added the this word because we need to differentiate between concepts in Definition 7 and Definition 8. And since we will be interested in applying Definition 7 to paths of stochastic processes, we added the word "pathwise".

Note that pathwise p-th variation along sequence of partitions is purely deterministic notion. In Section 2.4, we will prove that $\mathbb{P}$-a.a. trajectories of Rosenblatt process with Hurst parameter $H \in (\frac{1}{2}, 1)$ have finite pathwise $\frac{1}{H}$-th variation along sequence of dyadic partitions.

Definition 8. Choose $p > 0$ and let us suppose that $X = (X_t, t \in \mathbb{R}^+)$ is a real valued stochastic process defined on probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Furthermore, choose $t \in \mathbb{R}^+$ and $\pi = (\pi_n)_{n\in\mathbb{N}}$, where π is the sequence of partitions of the interval $[0, t]$ with $\pi_n = \{t_0^n, \ldots, t_{N(\pi_n)}^n\}$ such that $0 = t_0^n < t_1^n < \ldots < t_{N(\pi_n)}^n = t$ and has vanishing mesh $|\pi_n| = \sup_{i\in\{1,\ldots,N(\pi_n)\}} |t_i^n - t_{i-1}^n| \xrightarrow[n\to\infty]{} 0$. We say that process X has p-th variation along the sequence of partitions π at t, if the limit

$$\lim_{n\to\infty} \sum_{[t_j,t_{j+1}]\in\pi_n} |X_{t_j} - X_{t_{j-1}}|^p \tag{1.13}$$

which is defined in the sense of convergence in probability, exists. In that case we write $\langle X\rangle_\pi^p(t) := \lim_{n\to\infty} \sum_{[t_j,t_{j+1}]\in\pi_n} |X_{t_j} - X_{t_{j-1}}|^p$ (where the limit is again in the sense of convergence in probability).

1.5 Ergodic Theory

This section serves as a quick introduction to Ergodic Theory. We present some essential definitions and we prove few lemmas because we will need some steps in proofs later in Section 2.1. Finally we will present fundamental result - Birkhoff Ergodic Theorem. Most of the definitions and lemmas are taken from the excellent book by Seidler [24] (book is in czech only).
We begin by defining endomorphism, one of the essential concepts in Ergodic Theory.

Definition 9 ([24, p. 19])**.** Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. We say that a transformation $T : \Omega \to \Omega$ is an endomorphism of space $(\Omega, \mathcal{F}, \mu)$ if it holds that

(i) T is measurable, i.e. it holds $T^{-1}F \in \mathcal{F}$ for every $F \in \mathcal{F}$,

(ii) T is measure-preserving, i.e. it holds $\mu(T^{-1}F) = \mu(F)$ for every $F \in \mathcal{F}$.

We say that T is an automorphism of $(\Omega, \mathcal{F}, \mu)$ if T is a bijection and both T and T^{-1} are endomorphisms.

Note that endomorphisms and automorphisms are sometimes called (for example in [25]) measure-preserving transformation and invertible measure-preserving transformation, respectively. The quadruplet $(\Omega, \mathcal{F}, \mu, T)$ is sometimes called a dynamical system. Let T be an endomorphism of probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then we define T^0 as the identity function, $T^1 = T$, and for every $n \in \mathbb{N}$ we define $T^{n+1} = T \circ T^n$.

Dynamical systems have a natural probabilistic interpretation.

Lemma 4. [24, Section 1.2] *If T is an endomorphism of probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $f : \Omega \to \mathbb{R}$ is a Borel measurable function, then $(f \circ T^n, n \in \mathbb{N})$ is a strictly stationary process.*

Proof. Let T be an endomorphism of probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $f : \Omega \to \mathbb{R}$ be a measurable function. Choose $n \in \mathbb{N}$, $t_1, \dots, t_n \in I$ and $h > 0$ such that $t_i + h \in I$ for every $i \in \{1, \dots, n\}$ and choose $x_1, \dots x_n \in \mathbb{R}$. Then, it holds

$$\begin{aligned}
&\mathbb{P}\left[\{\omega \in \Omega : (f \circ T^{t_1+h})(\omega) \le x_1, \dots, (f \circ T^{t_n+h})(\omega) \le x_n\}\right] \\
&= \mathbb{P}\left[(f \circ T^{t_1+h})^{-1}((-\infty, x_1]) \cap \dots \cap (f \circ T^{t_n+h})^{-1}((-\infty, x_n])\right] \\
&= \mathbb{P}\left[(f \circ T^{t_1} \circ T^h)^{-1}((-\infty, x_1]) \cap \dots \cap (f \circ T^{t_n} \circ T^h)^{-1}((-\infty, x_n])\right] \\
&= \mathbb{P}\left[((T^h)^{-1} \circ (f \circ T^{t_1})^{-1})((-\infty, x_1]) \cap \dots \cap ((T^h)^{-1} \circ (f \circ T^{t_n})^{-1})((-\infty, x_n])\right] \\
&= \mathbb{P}\left[(T^h)^{-1}\left((f \circ T^{t_1})^{-1}((-\infty, x_1]) \cap \dots \cap (f \circ T^{t_n})^{-1}((-\infty, x_n])\right)\right] \\
&\stackrel{*}{=} \mathbb{P}\left[(f \circ T^{t_1})^{-1}((-\infty, x_1]) \cap \dots \cap (f \circ T^{t_n})^{-1}((-\infty, x_n])\right] \\
&= \mathbb{P}\left[\{\omega \in \Omega : (f \circ T^{t_1})(\omega) \le x_1, \dots, (f \circ T^{t_n})(\omega) \le x_n\}\right].
\end{aligned}$$

In $\stackrel{*}{=}$ we used the measure-preserving property (ii) from Definition 9. □

Opposite implication also holds. Remember that canonical version of the process is defined as process of projections on space of trajectories that have the same finite-dimensional distributions as the original process ([24, p. 9]).

Lemma 5. [24, Section 1.2] *Let $(Z_n, n \in \mathbb{N})$ be a strictly stationary stochastic process. Then there exists process $(f \circ T^n, n \in \mathbb{N})$, which is a canonical version of process $(Z_n, n \in \mathbb{N})$, where f is Borel measurable function and T is endomorphism, both defined on the space of trajectories of $(Z_n, n \in \mathbb{N})$.*

Proof. Let us have a strictly stationary stochastic process $(Z_t, t \in \mathbb{N})$ on probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let us define

$$\begin{aligned}
&\tilde{\Omega} = \mathbb{R}^{\mathbb{N}} = \{\tilde{\omega} : \mathbb{N} \to \mathbb{R}\}, \\
&G_t : \tilde{\Omega} \to \mathbb{R},\, \tilde{\omega} \to \tilde{\omega}(t), t \in \mathbb{N}, \\
&\tilde{\mathcal{F}} = \bigotimes_{\mathbb{N}} \mathcal{B}(\mathbb{R}) = \sigma(R_t, t \in \mathbb{N}).
\end{aligned} \tag{1.14}$$

Sigma algebra $\tilde{\mathcal{F}}$ is generated by measurable cylinders of the form $\{\tilde{\omega} \in \tilde{\Omega} : \tilde{\omega}(t_i) \in \Gamma_i, i \in \{1, \ldots, k\}\}$, where $k \in \mathbb{N}$, $t_1, \ldots, t_k \in \mathbb{N}$ and $\Gamma_1, \ldots, \Gamma_k \in \mathcal{B}(\mathbb{R})$. We can see that the mapping $\Lambda : \Omega \to \tilde{\Omega}$ defined as $\omega \to Z_\cdot(\omega) = (Z_t(\omega), t \in \mathbb{N})$ is $\mathcal{F}$-measurable because we have that for every measurable cylinder $A \in \tilde{\mathcal{F}}$ of the form $A = \{\tilde{\omega} \in \tilde{\Omega} : \tilde{\omega}(t_i) \in \Gamma_i, i \in \{1, \ldots, k\}\}$, where $k \in \mathbb{N}$, $t_1, \ldots, t_k \in \mathbb{N}$ and $\Gamma_1, \ldots, \Gamma_k \in \mathcal{B}(\mathbb{R})$, it holds $\Lambda^{-1}(A) = \{\omega \in \Omega : Z_{t_i}(\omega) \in \Gamma_i, i \in \{1, \ldots, k\}\} \in \mathcal{F}$. We define probability measure $\tilde{\mathbb{P}}$ as the image of measure $\mathbb{P}$ under transformation Λ, i.e. $\tilde{\mathbb{P}}[\cdot] = \mathbb{P}[\Lambda(\cdot)^{-1}]$.
We have thus constructed probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ with stochastic process $(G_i, i \in \mathbb{N})$. From the construction, it follows that the finite-dimensional distributions of $(Z_i, i \in \mathbb{N})$ on probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and those of $(G_i, i \in \mathbb{N})$ on probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ are identical. Indeed, let us choose $n \in \mathbb{N}$, $0 \le t_1 \le \ldots \le t_n < \infty$ and $A_1, \ldots, A_n \in \mathcal{B}(\mathbb{R})$. Then it holds

$$\begin{aligned}
\mathbb{P}[\{\omega \in \Omega : Z_{t_1}(\omega) \in A_1, \ldots, Z_{t_n}(\omega) \in A_n\}] &= \\
&= \tilde{\mathbb{P}}[\Lambda(\{\omega \in \Omega : Z_{t_1}(\omega) \in A_1, \ldots, Z_{t_n}(\omega) \in A_n\})] \\
&= \tilde{\mathbb{P}}[\{\tilde{\omega} \in \tilde{\Omega} : \tilde{\omega}(t_1) \in A_1, \ldots, \tilde{\omega}(t_n) \in A_n\}] \\
&= \tilde{\mathbb{P}}[\{\tilde{\omega} \in \tilde{\Omega} : G_{t_1}(\tilde{\omega}) \in A_1, \ldots, G_{t_n}(\tilde{\omega}) \in A_n\}].
\end{aligned}$$

Now, we define the shift operator $T : \tilde{\Omega} \to \tilde{\Omega}$, by $\tilde{\omega}(\cdot) \to \tilde{\omega}(\cdot + 1)$. Then it holds $(T^2\tilde{\omega})(n) = (T \circ T)(\tilde{\omega})(n) = (T \circ \tilde{\omega})(n+1) = \tilde{\omega}(n+2)$, for every $n \in \mathbb{N}$. Therefore, it can be easily shown by induction that for any $k \in \mathbb{N}$ it holds $(T^k\tilde{\omega})(n) = \tilde{\omega}(n+k)$.
Now, for $t \in \mathbb{N}$, we have $G_t(\tilde{\omega}) = \tilde{\omega}(t) = (T^0\tilde{\omega})(t) = (T^{t-1}\tilde{\omega})(1) = G_1(T^{t-1}\tilde{\omega}) = (G_1 \circ T^{t-1})(\tilde{\omega})$ for $\tilde{\omega} \in \tilde{\Omega}$. For that reason, we can see that $(G_1 \circ T^{t-1}, t \in \mathbb{N})$ and process $(Z_t, t \in \mathbb{N})$ have the same finite dimensional distributions or in other words $(G_1 \circ T^{t-1}, t \in \mathbb{N})$ is a canonical version of $(Z_t, t \in \mathbb{N})$.
Lastly, we will show that the shift operator T is an endomorphism. Operator T is measurable because for any measurable cylinder $A \in \tilde{\mathcal{F}}$, we have $T^{-1}(A) = \{\tilde{\omega} \in \tilde{\Omega} : \tilde{\omega}(t_i + 1) \in \Gamma_i, i \in \{1, \ldots, k\}\}$ is again a measurable cylinder and therefore an element of $\tilde{\mathcal{F}}$. Now, we will show that operator T is measure invariant. By [24, Lemma 1.1] it is enough to show that for every cylindrical set $A \in \tilde{\mathcal{F}}$ it holds

$\tilde{\mathbb{P}}[T^{-1}A] = \tilde{\mathbb{P}}[A]$. We have

$$\begin{aligned}
\tilde{\mathbb{P}}[T^{-1}A] &= \tilde{\mathbb{P}}[\{\tilde{\omega} \in \tilde{\Omega} : \tilde{\omega}(t_i+1) \in \Gamma_i, i \in \{1,\dots,k\}\}] \\
&= \tilde{\mathbb{P}}[\{\tilde{\omega} \in \tilde{\Omega} : R_{t_i+1}(\tilde{\omega}) \in \Gamma_i, i \in \{1,\dots,k\}\}] \\
&= \mathbb{P}[\Lambda^{-1}(\{\tilde{\omega} \in \tilde{\Omega} : R_{t_i+1}(\tilde{\omega}) \in \Gamma_i, i \in \{1,\dots,k\}\})] \\
&= \mathbb{P}[\{\omega \in \Omega : Z_{t_i+1} \in \Gamma_i, i \in \{1,\dots,k\}\}] \\
&= \mathbb{P}[\{\omega \in \Omega : Z_{t_i} \in \Gamma_i, i \in \{1,\dots,k\}\}] \\
&= \mathbb{P}[\Lambda^{-1}(\{\tilde{\omega} \in \tilde{\Omega} : R_{t_i}(\tilde{\omega}) \in \Gamma_i, i \in \{1,\dots,k\}\})] \\
&= \tilde{\mathbb{P}}[\{\tilde{\omega} \in \tilde{\Omega} : R_{t_i}(\tilde{\omega}) \in \Gamma_i, i \in \{1,\dots,k\}\}] \\
&= \tilde{\mathbb{P}}[\{\tilde{\omega} \in \tilde{\Omega} : \tilde{\omega}(t_i) \in \Gamma_i, i \in \{1,\dots,k\}\}] \\
&= \tilde{\mathbb{P}}[A],
\end{aligned}$$

where we used strict stationarity of process $(Z_i, i \in \mathbb{N})$. Thus operator T is shown to be measure-invariant and therefore an endomorphism. □

Overall, we have showen that for every strictly stationary stochastic process $(Z_n, n \in \mathbb{N})$, there exists an endomorphism T and a measurable function f such that the process $(f \circ T^n, n \in \mathbb{N})$ has the same finite-dimensional distributions as $(Z_n, n \in \mathbb{N})$. On the other hand, we have showen that every process of the form $(f \circ T^n, n \in \mathbb{N})$, where f is measurable function and T is endomorphism, is strictly stationary.

Let us assume that T is an endomorphism of probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We say that set $A \in \mathcal{F}$ is T-invariant if $T^{-1}A = A$. We say that function $f : \Omega \to \Omega$ is T-invariant if $f \circ T = f$ on Ω. Let us denote $\mathcal{S} = \{A \in \mathcal{F} : T^{-1}A = A\}$, a system of all T-invariant sets. It is not difficult to verify that $\mathcal{S}$ is a σ-algebra. It is also not difficult to show that a measurable function f on Ω is T-invariant if and only if f is $\mathcal{S}$-measurable (the proof can be found in [24, p. 21]).
Now we will state a fundamental result - Birkhoff Pointwise Ergodic Theorem. Note that there is also Mean Ergodic Theorem proved by von Neumann (see for example [26, p. 23]) which we do not discuss in this work.

Theorem 1 ([26, Birkhoff Ergodic Theorem, 1931, p. 30])**.** *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, T be an endomorphism on this space, and $f \in L^1(\Omega, \mathcal{F}, \mathbb{P})$. Then it holds that*

1. *the limit $\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} f(T^k\omega) = f^*(\omega)$ exists for $\mathbb{P}$-a.a. $\omega \in \Omega$,*
2. *$f^*(T\omega) = f^*(\omega)$ for $\mathbb{P}$-a.a. $\omega \in \Omega$, i.e. f^* is T-invariant,*
3. *$f^* \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ and $||f^*||_1 \leq ||f||_1$,*
4. *if $A \in \mathcal{F}$ with $T^{-1}A = A$, then $\int_A f \,\mathrm{d}\mathbb{P} = \int_A f^* \,\mathrm{d}\mathbb{P}$,*
5. *$\frac{1}{n}\sum_{k=0}^{n-1} fT^k \to f^*$, in $L^1(\Omega, \mathcal{F}, \mathbb{P})$.*

Note that if we denote $\mathcal{S}$ to be the σ-algebra of T-invariant sets, then statement (4) in the above theorem, says that $f^* = \mathbb{E}[f|\mathcal{S}]$, $\mathbb{P}$-a.e. Next we would like to know some criterion when the function $f^* = \mathbb{E}[f|\mathcal{S}]$ in Birkhoff Ergodic Theorem is constant. This leads us to the definition of an ergodic operator.

Definition 10. Let T be an endomorphism of probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let us denote $\mathcal{S} = \{A \in \mathcal{F} : T^{-1}A = A\}$ the σ-algebra of T-invariant sets. We say that dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, T)$ is ergodic if $\mathbb{P}(A) \in \{0, 1\}$ for every $A \in \mathcal{S}$.

If the probability space is fixed, we usually talk about ergodicity of operator T. We can think of ergodicity as "in-decomposability condition". If $A \in \mathcal{F}$ is T-invariant set and $\mathbb{P}[A] \in (0, 1)$, then we can study T on probability space $(A, \mathcal{F}|_A, \mathbb{P}|_A)$ and $(\Omega \setminus A, \mathcal{F}|_{\Omega\setminus A}, \mathbb{P}|_{\Omega\setminus A})$ separately. If $(\Omega, \mathcal{F}, \mathbb{P}, T)$ is ergodic than this separation cannot be done. For that reson, ergodicity of dynamical system gives us property that we cannot reduce or factor the dynamical system into smaller components in the sense as described above.
An important characterization of an the ergodic operator is the following lemma.

Lemma 6. ([26, Proposition 4.1, p. 42]) *Operator T on $(\Omega, \mathcal{F}, \mathbb{P})$ is ergodic if and only if every T-invariant measurable function on Ω is constant $\mathbb{P}$-a.e.*

This leads us to the following theorem which is usually called the law of large numbers for strictly stationary sequences.

Theorem 2 ([26, p. 44])**.** *T is an ergodic operator on probability space $(\Omega, \mathcal{F}, \mathbb{P})$ if and only if for every $f \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ it holds that*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k\omega) = \int_\Omega f \, \mathrm{d}\mathbb{P}, \quad \textit{for } \mathbb{P}\textit{-a.a. } \omega \in \Omega.$$

Proof. Let us assume that dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, T)$ is ergodic. Birkhoff Ergodic Theorem statement (2) then tells us that if the limit f^* exists, then it is T-invariant. Therefore, because T is ergodic, by Lemma 6 it holds that f^* from Birkhoff Ergodic Theorem is constant $\mathbb{P}$-a.e. and this constant value is $\int_\Omega f \, \mathrm{d}\mathbb{P}$.

On the other hand, let us suppose that for each $f \in L^1(\Omega, \mathcal{F}, \mathbb{P})$, it holds that f^* is constant $\mathbb{P}$-a.e. We would like to show that T is ergodic. For that it is enough, by Lemma 6, to show that every T-invariant measurable function on Ω is constant $\mathbb{P}$-a.s. Let f be a T-invariant measurable function on Ω. Then, from T-invariance, we have $f \circ T = f$ on Ω and from there, we have $\frac{1}{n}\sum_{k=0}^{n-1} f(T^k\omega) = \frac{1}{n}\sum_{k=0}^{n-1} f(\omega) = f(\omega) = f^*(\omega)$, $\mathbb{P}$-a.e.. We showed that f is constant on Ω, for $\mathbb{P}$-a.a. $\omega \in \Omega$.

□

2. Variations of Rosenblatt process

2.1 Ergodicity of increments of Rosenblatt process

Our goal in this section is to show that the increments of Rosenblatt process are ergodic. We first prove the following lemma which, as it turns out, is a key step in proving ergodicity of increments of Rosenblatt process.

Lemma 7. *Let $(R_t^H, t \in \mathbb{R}^+)$ be a Rosenblatt process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with Hurst parameter $H \in (\frac{1}{2}, 1)$. Denote $Y^H = (Y_k^H, k \in \mathbb{N})$ by $Y_k = R_k^H - R_{k-1}^H$, for every $k \in \mathbb{N}$. Furthermore, denote characteristic function of random vector $(Y_{t_1}^H, \ldots, Y_{t_p}^H)^\top$ by $\varphi_{Y_{t_1}^H, \ldots, Y_{t_p}^H}$, where $p \in \mathbb{N}$ and $t_1, \ldots, t_p \in \mathbb{R}^+$, $t_1 < \ldots < t_p$. Then, for every $\theta_1, \ldots, \theta_p, \theta_{p+1}, \ldots \theta_{2p} \in \mathbb{R}$ and $\tau \in \mathbb{N}$, it holds*

$$\begin{aligned}
&|\varphi_{Y_{t_1}^H, \ldots, Y_{t_p}^H, Y_{t_1+\tau}^H, \ldots, Y_{t_p+\tau}^H}(\theta_1, \ldots, \theta_p, \theta_{p+1}, \ldots, \theta_{2p}) \\
&\quad - \varphi_{Y_{t_1}^H, \ldots, Y_{t_p}^H}(\theta_1, \ldots, \theta_p)\varphi_{Y_{t_1+\tau}^H, \ldots, Y_{t_p+\tau}^H}(\theta_{p+1}, \ldots, \theta_{2p})| \xrightarrow[\tau \to \infty]{} 0.
\end{aligned}$$

Proof. Let us fix $p \in \mathbb{N}$, $\tau \in \mathbb{N}$ and choose $t_1, \ldots, t_p \in \mathbb{R}^+$. Let $(R_t^H, t \in \mathbb{R}^+)$ be a Rosenblatt process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with Hurst parameter $H \in (\frac{1}{2}, 1)$.
In order to ease notation in this proof, we denote $t_{i+p} := t_i + \tau$ for $i \in \{1, \ldots, p\}$ and we will also write $\varphi(\theta_1, \ldots, \theta_p)$, $\varphi(\theta_{p+1}, \ldots, \theta_{2p})$ and $\varphi(\theta_1, \ldots, \theta_{2p})$ instead of $\varphi_{Y_{t_1}^H, \ldots, Y_{t_p}^H}(\theta_1, \ldots, \theta_p)$, $\varphi_{Y_{t_{p+1}}^H, \ldots, Y_{t_{2p}}^H}(\theta_{p+1}, \ldots, \theta_{2p})$ and $\varphi_{Y_{t_1}^H, \ldots, Y_{t_{2p}}^H}(\theta_1, \ldots, \theta_{2p})$.
By Lemma 3, we have

$$\varphi(\theta_1, \ldots, \theta_p)\varphi(\theta_{p+1}, \ldots, \theta_{2p}) = \exp\left(\frac{1}{2}\sum_{k=2}^{\infty}\frac{(2i\sigma_H)^k}{k}F_k(\tau)\right)$$

where

$$F_k(\tau) = \sum_{r_1, \ldots, r_k \in \{1, \ldots, p\}} \theta_{r_1} \cdots \theta_{r_k} S_H^*(t_{r_1}, \ldots, t_{r_k}) + \sum_{l_1, \ldots, l_k \in \{p+1, \ldots, 2p\}} \theta_{l_1} \cdots \theta_{l_k} S_H^*(t_{l_1}, \ldots, t_{l_k}) \tag{2.1}$$

and function S_H^* is from (1.11). Again, by Lemma 3, we have

$$\varphi(\theta_1, \ldots, \theta_{2p}) = \exp\left(\frac{1}{2}\sum_{k=2}^{\infty}\frac{(2i\sigma)^k}{k}\sum_{j_1, \ldots, j_k \in \{1, \ldots, 2p\}} \theta_{j_1} \cdots \theta_{j_k} S_H^*(t_{j_1}, \ldots, t_{j_k})\right) \tag{2.2}$$

We can rewrite the series on the right-hand side of (2.2) as

$$\sum_{j_1, \ldots, j_k \in \{1, \ldots, 2p\}} \theta_{j_1} \cdots \theta_{j_k} S_H^*(t_{j_1}, \ldots, t_{j_k}) = F_k(\tau) + L_k(\tau)$$

where $F_k(\tau)$ is defined in (2.1) and

$$L_k(\tau) = \sum_{\substack{j_1, \ldots, j_k \in \{1, \ldots, 2p\} \\ \exists h \in \{1, \ldots, k\}: j_h \in \{1, \ldots, p\} \\ \exists g \in \{1, \ldots, k\}: j_g \in \{p+1, \ldots, 2p\}}} \theta_{j_1} \cdots \theta_{j_k} S_H^*(t_{j_1}, \ldots, t_{j_k}). \tag{2.3}$$

Now we would like to show that $L_k(\tau) \xrightarrow[\tau\to\infty]{} 0$. In order to do that, we need to show on the right-hand side of (2.3) that it holds $S_H^*(t_{j_1}, \dots, t_{j_k}) \xrightarrow[\tau\to\infty]{} 0$.
We suppose that $j_1, \dots, j_k \in \{1, \dots, 2p\}$ and we fix $h \in \{1, \dots, k\}$ such that $j_h \in \{1, \dots, p\}$ and we fix $g \in \{1, \dots, k\}$ such that $j_g \in \{p+1, \dots, 2p\}$. We can also suppose, without loss of generality, that $h < g$. Note that it holds $t_{j_g} = t_{j_g - p} + \tau$ because we used notation $t_j + \tau = t_{j+p}$ for every $j \in \{1, \dots, p\}$. Then it holds, for function S_H^* from (2.3), that

$$\begin{aligned}
&S_H^*(t_{j_1}, \dots, t_{j_k}) = S_H^*(t_{j_1}, \dots, t_{j_h}, \dots, t_{j_g}, \dots, t_{j_k}) \\
&= S_H^*(t_{j_1}, \dots, t_{j_h}, \dots, t_{j_g-p} + \tau, \dots, t_{j_k}) \\
&= \int_{t_{j_1}-1}^{t_{j_1}} \cdots \int_{t_{j_h}-1}^{t_{j_h}} \cdots \int_{t_{j_g-p}+\tau-1}^{t_{j_g-p}+\tau} \cdots \int_{t_{j_k}-1}^{t_{j_k}} \\
&\qquad |x_1 - x_2|^{H-1} \cdots |x_{g-1} - x_g|^{H-1} \cdot |x_g - x_{g+1}|^{H-1} \cdots |x_k - x_1|^{H-1} \, \mathrm{d}x_k \dots \mathrm{d}x_1 \\
&= \int_{t_{j_1}-1}^{t_{j_1}} \cdots \int_{t_{j_h}-1}^{t_{j_h}} \cdots \int_{t_{j_g-p}-1}^{t_{j_g-p}} \cdots \int_{t_{j_k}-1}^{t_{j_k}} \\
&\qquad |x_1 - x_2|^{H-1} \cdots |x_{g-1} - u + \tau|^{H-1} \cdot |u - \tau - x_{g+1}|^{H-1} \cdots |x_k - x_1|^{H-1} \\
&\qquad\qquad \mathrm{d}x_k \dots \mathrm{d}x_{g+1} \, \mathrm{d}u \, \mathrm{d}x_{g-1} \dots \mathrm{d}x_1
\end{aligned} \tag{2.4}$$

where in the last equality we substituted $u = x_g + \tau$.
Let us denote, for g and h being still fixed, the function

$$\begin{aligned}
f_\tau(x_1, \dots, x_k) := |x_1 - x_2|^{H-1} \cdots |x_{g-2} - x_{g-1}|^{-H-1} \cdot |x_{g-1} - x_g + \tau|^{H-1} \\
\cdot |x_g - \tau - x_{g+1}|^{H-1} \cdot |x_{g+1} - x_{g+2}|^{H-1} \cdots |x_k - x_1|^{H-1}.
\end{aligned}$$

Let us denote

$$\begin{aligned}
S := (t_{j_1} - 1, t_{j_1}) \times \dots \times (t_{j_{g-1}} - 1, t_{j_{g-1}}) \times (t_{j_g-p} - 1, t_{j_g-p}) \\
\times (t_{j_{g+1}} - 1, t_{j_{g+1}}) \times \dots \times (t_{j_k} - 1, t_{j_k}).
\end{aligned}$$

Then, for fixed τ, we can rewrite (2.4) as

$$\int_S f_\tau(x) \, \mathrm{d}\lambda(x),$$

where λ is the Lebesgue measure.
Now we will verify assumptions of Dominated convergence theorem ([22, Theorem 1.4.49]) for sequence of functions $\{f_\tau\}_{\tau\in\mathbb{N}}$. Let us define function ϕ as

$$\phi(x_1, \dots, x_k) = f_\tau(x_1, \dots, x_{g-1}, x_g + \tau, x_{g+1}, \dots, x_k).$$

Then it holds that $\phi \in L^1(S, \mathcal{B}(S), \lambda)$. From the way functions f and ϕ are defined, we can see that $|f_\tau(x)| \le \phi(x)$ for λ-a.a. $x \in S$ and for every $\tau \in \mathbb{N}$. From definition of function f_τ, it holds that $f_\tau \xrightarrow[\tau\to\infty]{} 0$ λ-a.s. on S.
All assumptions of Dominated convergence theorem ([22, Theorem 1.4.49]) have been fulfilled, therefore we have

$$\int_S f_\tau(x) \, \mathrm{d}\lambda(x) \xrightarrow[\tau\to\infty]{} 0.$$

From there it follows that $L_k(\tau) \xrightarrow[\tau\to\infty]{} 0$ and the proof of our lemma is complete.
$\square$

Now we are ready to prove the main theorem of this section.

Theorem 3. *Rosenblatt process* $(R_t^H, t \in \mathbb{R}^+)$ *with Hurst parameter* $H \in (\frac{1}{2}, 1)$ *has ergodic increments, i.e. it holds that the process*

$$(R_k^H - R_{k-1}^H, k \in \mathbb{N})$$

is ergodic.

Proof. Let $(R_t^H, t \in \mathbb{R}^+)$ be a Rosenblatt process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with Hurst parameter $H \in (\frac{1}{2}, 1)$. Let us define, similarly as in the proof of Lemma 5, the following notions

$$\begin{aligned}
&\tilde{\Omega} = \{\tilde{\omega} : \mathbb{N} \to \mathbb{R}\}, \\
&\tilde{G}_t \colon \tilde{\Omega} \to \mathbb{R},\, \tilde{\omega} \to \tilde{\omega}(t), t \in \mathbb{N}, \\
&\tilde{\mathcal{F}} = \bigotimes_{\mathbb{N}} \mathcal{B}(\mathbb{R}).
\end{aligned}$$

Next, let us denote $Y^H = (Y_k^H, k \in \mathbb{N})$ by $Y_k^H = R_k^H - R_{k-1}^H$ for every $k \in \mathbb{N}$. Furthermore, let us have a mapping $\Lambda : \Omega \to \tilde{\Omega}$ defined by $\omega \to Y_\bullet^H(\omega)$ where $Y_\bullet^H(\omega) = (Y_k^H(\omega), k \in \mathbb{N})$ is a trajectory of process Y^H. Then Λ is $\mathcal{F}$-measurable (as we argued in the proof of Lemma 5). We define a probability measure $\tilde{\mathbb{P}}$ by $\tilde{\mathbb{P}}[\cdot] = \mathbb{P}[\Lambda^{-1}(\cdot)]$.
Finally, we define a shift operator $T : \tilde{\Omega} \to \tilde{\Omega}$ by $\tilde{\omega}(\cdot) \to \tilde{\omega}(\cdot + 1)$. Operator T is an endomorphism as we showed in the proof of Lemma 5 and it also holds that Y^H on $(\Omega, \mathcal{F}, \mathbb{P})$ and $(\tilde{G}_1 \circ T^k, k \in \mathbb{N}_0)$ on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ have the same finite dimensional distributions.
Our goal is to show that T is an ergodic operator. Let us have $\mathcal{S} = \{A \in \tilde{\mathcal{F}} : T^{-1}A = A\}$. Then it holds that $\mathcal{S}$ is a σ-algebra. Let us fix $A \in \mathcal{S}$. We would like to show that $\tilde{\mathbb{P}}[A] \in \{0, 1\}$. We know that every event (i.e. every set) in a σ-algebra can be approximated by a finite dimensional event (for the proof of a slightly more general claim, see for example [27, Theorem D., p. 56]). Therefore, for every $\varepsilon > 0$ there exists $n \in \mathbb{N}$, $t_1, \ldots, t_n \in \mathbb{N}$, $t_1 < \ldots < t_n$, and $\xi_1, \ldots, \xi_n \in \mathcal{B}(\mathbb{R})$ such that if we denote

$$B = \left\{\tilde{\omega} \in \tilde{\Omega} : \tilde{\omega}(t_i) \in \xi_i, \forall i \in \{1, \ldots, n\}\right\},$$

then it holds that $B \in \tilde{\mathcal{F}}$ and

$$\tilde{\mathbb{P}}[A \triangle B] < \varepsilon,$$

where $\triangle$ is the symmetric difference of two sets. From there we obtain

$$|\,\tilde{\mathbb{P}}[A] - \tilde{\mathbb{P}}[B]| < \varepsilon. \tag{2.5}$$

It holds that B is generated by random vector $(Y_{t_1}^H, \ldots, Y_{t_n}^H)^\top$ because we have

$$\begin{aligned}
B &= \{\tilde{\omega} \in \tilde{\Omega} : \tilde{\omega}(t_i) \in \xi_i, \forall i \in \{1, \ldots, n\} \\
&= \{\tilde{\omega} \in \tilde{\Omega} : \tilde{G}_{t_i}(\tilde{\omega}) \in \xi_i, \forall i \in \{1, \ldots, n\} \\
&= \Lambda(\{\omega \in \Omega : Y_{t_i}^H(\omega) \in \xi_i, \forall i \in \{1, \ldots, n\}\}).
\end{aligned}$$

Note that for the inverse mapping T^{-1} it holds

$$T^{-1}B = \Big\{\tilde{\omega} \in \tilde{\Omega} : \tilde{\omega}(t_i + 1) \in \xi_i, \forall i \in \{1, \ldots, n\}\Big\}$$

and more generally

$$T^{-k}B = (\underbrace{T^{-1} \circ \ldots \circ T^{-1}}_{k\text{-times}})B = \Big\{\tilde{\omega} \in \tilde{\Omega} : \tilde{\omega}(t_i + k) \in \xi_i, \forall i \in \{1, \ldots, n\}\Big\}.$$

for every $k \in \mathbb{N}$.

Now, let us denote for every $\tau \in \mathbb{N} : A_\tau := T^{-\tau}A$ and $B_\tau := T^{-\tau}B$. We have $A = A_\tau$ because $A \in \mathcal{S}$ (this holds in the sense that $A \subseteq A_\tau$ & $A_\tau \subseteq A$). We also have that B_τ approximates A_τ for every $\tau \in \mathbb{N}$ because B_τ is generated by the random vector $(Y^H_{t_1+\tau}, \ldots, Y^H_{t_n+\tau})^\top$ and because Y^H is strictly stationarity. In other words, we have

$$|\,\tilde{\mathbb{P}}\,[A_\tau] - \tilde{\mathbb{P}}\,[B_\tau]| < \varepsilon.$$

From there we have

$$\tilde{\mathbb{P}}\,[(A \cap A_\tau)\triangle(B \cap B_\tau)] \leq \tilde{\mathbb{P}}\,[(A\triangle B)] + \tilde{\mathbb{P}}\,[(A_\tau \triangle B_\tau)] < 2\varepsilon$$

because $(A \cap A_\tau)\triangle(B \cap B_\tau) \subseteq (A\triangle B) \cup (A_\tau \triangle B_\tau)$. So it holds

$$|\,\tilde{\mathbb{P}}\,[A \cap A_\tau] - \tilde{\mathbb{P}}\,[B \cap B_\tau]| < 2\varepsilon.$$

Because $A = A_\tau$, we have $\tilde{\mathbb{P}}\,[A \cap A_\tau] = \tilde{\mathbb{P}}\,[A]$ and we can rewrite the last inequality as

$$|\,\tilde{\mathbb{P}}\,[A] - \tilde{\mathbb{P}}\,[B \cap B_\tau]| < 2\varepsilon. \tag{2.6}$$

We know that because T is an endomorphism, it holds $\tilde{\mathbb{P}}\,[B] = \tilde{\mathbb{P}}\,[B_\tau]$.

Next we have by Lemma 7 the following convergence

$$|\,\tilde{\mathbb{P}}\,[B \cap B_\tau] - \tilde{\mathbb{P}}\,[B]\,\tilde{\mathbb{P}}\,[B_\tau]| \xrightarrow[\tau\to\infty]{} 0. \tag{2.7}$$

Because T is endomorphism (and therefore $\tilde{\mathbb{P}}$-measure preserving mapping), it holds $\tilde{\mathbb{P}}\,[B] = \tilde{\mathbb{P}}\,[B_\tau]$. But this means that (2.7) could be rewritten as

$$|\,\tilde{\mathbb{P}}\,[B \cap B_\tau] - \tilde{\mathbb{P}}\,[B]^2| \xrightarrow[\tau\to\infty]{} 0.$$

But from (2.5) it follows that $\tilde{\mathbb{P}}\,[A]$ can be approximated by $\tilde{\mathbb{P}}\,[B]$ and from (2.6) it follows that $\tilde{\mathbb{P}}\,[A]$ can be approximated by $\tilde{\mathbb{P}}\,[B \cap B_\tau]$ which converges to $\tilde{\mathbb{P}}\,[B]^2$ as $\tau \to \infty$. Thus, both $\tilde{\mathbb{P}}\,[B]$ and $\tilde{\mathbb{P}}\,[B]^2$ approximate $\tilde{\mathbb{P}}\,[A]$ which means that either $\tilde{\mathbb{P}}\,[A] = 0$ or $\tilde{\mathbb{P}}\,[A] = 1$. In other words, T is ergodic operator. Therefore Y^H is ergodic stochastic process. $\square$

2.2 p-th variation along the sequence of partitions of Rosenblatt process

The following theorem is key step in proving that Rosenblatt process is not a semimartingale in the next section. We will prove that Rosenblatt process has finite p-th variation along the sequence of dyadic partitions in the sense of Definition 8 for $p \geq \frac{1}{H}$.

Theorem 4. *Let* $(R_t^H, t \in \mathbb{R}^+)$ *be a Rosenblatt process with Hurst parameter* $H \in (\frac{1}{2}, 1)$ *defined on probability space* $(\Omega, \mathcal{F}, \mathbb{P})$*. Then it holds*

$$\sum_{j=1}^{2^n} |R^H_{\frac{j}{2^n}} - R^H_{\frac{j-1}{2^n}}|^p \xrightarrow[n\to\infty]{\mathbb{P}} \begin{cases} 0, & \text{if } p > \frac{1}{H}, \\ \mathbb{E}[|R_1^H|^{\frac{1}{H}}], & \text{if } p = \frac{1}{H}, \\ \infty, & \text{if } 0 < p < \frac{1}{H}. \end{cases}$$

Proof. Let us fix parameters $H \in (\frac{1}{2}, 1)$ and $p > 0$. Let us denote Rosenblatt process with Hurst parameter H by $(R_t^H, t \in \mathbb{R}^+)$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let us assume we have a sequence of dyadic partitions $\{\frac{j}{2^n}, j = 0, \dots, 2^n\}_{n\in\mathbb{N}}$ of the interval $[0, 1]$. We define

$$Q_{n,p} = \sum_{j=1}^{2^n} |R^H_{j2^{-n}} - R^H_{(j-1)2^{-n}}|^p (2^n)^{pH-1}.$$

Let us now consider

$$\hat{Q}_{n,p} = \frac{1}{2^n} \sum_{j=1}^{2^n} |R_j^H - R_{j-1}^H|^p.$$

From self-similarity of Rosenblatt process, we obtain that for every $n \in \mathbb{N}$, $Q_{n,p}$ has the same distribution as $\hat{Q}_{n,p}$. As we already proved, the sequence $(R_k^H - R_{k-1}^H, k \in \mathbb{N})$ is ergodic (Theorem 3) and strictly stationary (Subsection 1.3.3). Let us have the same definitions of $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, Y^H, $(\tilde{G}_k, k \in \mathbb{N})$ and the shift operator T as in the proof of Theorem 3. Then we define the function $f_p : \tilde{\Omega} \to \mathbb{R}$ by $\tilde{\omega} \to |\tilde{\omega}(1)|^p$. We then have $f_p \in L^1(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ because $p > 0$. Therefore, we can use the Birkhoff Ergodic Theorem and we obtain the following convergence $\tilde{\mathbb{P}}$-a.s. and in $L^1(\tilde{\Omega})$

$$\frac{1}{2^n} \sum_{k=0}^{2^n-1} f_p(T^k \tilde{\omega}) \xrightarrow[n\to\infty]{\tilde{\mathbb{P}}\text{ -a.s.}, L^1(\tilde{\Omega})} \int_{\tilde{\Omega}} f_p(\tilde{\omega}) \, \mathrm{d}\tilde{\mathbb{P}}(\tilde{\omega}).$$

By the similar argument as in the proof of Theorem 3, it holds that $(|R_k^H - R_{k-1}^H|^p, k \in \mathbb{N})$ has the same finite dimensional distributions as $(f_p \circ T^k, k \in \mathbb{N}_0) = (|(T^k(\bullet))(1)|^p, k \in \mathbb{N}_0)$. Now we have that

$$\begin{aligned} \int_{\tilde{\Omega}} f_p(\tilde{\omega}) \, \mathrm{d}\tilde{\mathbb{P}}(\tilde{\omega}) &= \int_{\tilde{\Omega}} |\tilde{\omega}(1)|^p \, \mathrm{d}\tilde{\mathbb{P}}(\tilde{\omega}) \\ &= \int_{\tilde{\Omega}} |\tilde{G}_1(\tilde{\omega})|^p \, \mathrm{d}\tilde{\mathbb{P}}(\tilde{\omega}) \\ &= \int_{\Omega} |Y_1^H(\omega)|^p \, \mathrm{d}\mathbb{P}(\omega) \\ &= \int_{\Omega} |R_1^H(\omega) - R_0^H(\omega)|^p \, \mathrm{d}\mathbb{P}(\omega) \\ &= \mathbb{E}[|R_1^H - R_0^H|^p] \\ &= \mathbb{E}[|R_1^H|^p]. \end{aligned}$$

Let us denote $C_p := \mathbb{E}[|R_1^H|^p]$.
Because it holds

$$\frac{1}{2^n} \sum_{k=0}^{2^n-1} f_p(T^k \tilde{\omega}) \overset{\mathcal{D}}{\sim} \hat{Q}_{n,p},$$

for every $n \in \mathbb{N}$ and because $Q_{n,p} \overset{\mathcal{D}}{\sim} \hat{Q}_{n,p}$ for every $n \in \mathbb{N}$, we have that

$$Q_{n,p} \xrightarrow[n\to\infty]{\mathcal{D}} C_p.$$

And because C_p is deterministic, we have

$$Q_{n,p} \xrightarrow[n\to\infty]{\mathbb{P}} C_p.$$

From there it follows that

$$(2^n)^{1-pH} Q_{n,p} = \sum_{j=1}^{2^n} |R^H_{\frac{j}{2^n}} - R^H_{\frac{j-1}{2^n}}|^p \xrightarrow[n\to\infty]{\mathbb{P}} \begin{cases} 0, & \text{if } p > \frac{1}{H}, \\ \mathbb{E}[|R^H_1|^{\frac{1}{H}}], & \text{if } p = \frac{1}{H}, \\ \infty, & \text{if } 0 < p < \frac{1}{H}. \end{cases}$$

□

Remark 5. In the statement of Theorem 4, we considered sequence of dyadic partitions $\{\frac{j}{2^n}, j = 0, \ldots, 2^n\}_{n\in\mathbb{N}}$ of the interval $[0, 1]$. If we take sequence of dyadic partitions of the interval $[0, t]$ for $t > 0$, i.e. we take sequence of partitions $\{\frac{j}{2^n}t, j = 0, \ldots, 2^n\}_{n\in\mathbb{N}}$, then all the calculations can be done in similar way as in proof of Theorem 4 and we obtain the following convergence

$$\sum_{j=1}^{2^n} |R^H_{\frac{j}{2^n}t} - R^H_{\frac{(j-1)}{2^n}t}|^p \xrightarrow[n\to\infty]{\mathbb{P}} \begin{cases} 0, & \text{if } p > \frac{1}{H}, \\ t\,\mathbb{E}[|R^H_1|^{\frac{1}{H}}], & \text{if } p = \frac{1}{H}, \\ \infty, & \text{if } 0 < p < \frac{1}{H}. \end{cases}$$

Similarly, if we take sequence of uniform partitions $\{\frac{i}{n}t, i = 0, \ldots, n\}_{n\in\mathbb{N}}$ of the interval $[0, t]$ for $t > 0$, then all the calculations will pass in the same way as in the dyadic partitions case and we obtain the following convergence

$$\sum_{j=1}^{n} |R^H_{\frac{j}{n}t} - R^H_{\frac{j-1}{n}t}|^p \xrightarrow[n\to\infty]{\mathbb{P}} \begin{cases} 0, & \text{if } 1 > \frac{1}{H}, \\ t\,\mathbb{E}[|R^H_1|^{\frac{1}{H}}], & \text{if } p = \frac{1}{H}, \\ \infty, & \text{if } 0 < p < \frac{1}{H}. \end{cases} \tag{2.8}$$

2.3 Rosenblatt process is not a semimartingale

In [4, Section 2], Rogers has proved that fracional Brownian motion is semimartingale if and only if $H = \frac{1}{2}$ (i.e. only if fBm is the Wiener process). In this subsection we will show that Rosenblatt process is not a semimartingale for every $H \in (\frac{1}{2}, 1)$. Proof for this is depends substantially on Theorem 4.

Theorem 5. *Rosenblatt process with Hurst parameter $H \in (\frac{1}{2}, 1)$ is not a semimartingale.*

Proof. Let us fix $H \in (\frac{1}{2}, 1)$, and let $R^H = (R^H_t, t \in \mathbb{R}^+)$ be a Rosenblatt process with Hurst parameter H defined on probability space $(\Omega, \mathcal{F}, \mathbb{P})$.
Let us denote for every $p > 0$ and $n \in \mathbb{N}$

$$V_{n,p} := \sum_{j=1}^{2^n} |R^H_{\frac{j}{2^n}} - R^H_{\frac{j-1}{2^n}}|^p.$$

Because $H \in (\frac{1}{2}, 1)$, it holds that $\frac{1}{H} \in (1, 2)$.
If $p \in (\frac{1}{H}, 2)$, then by Theorem 4 we have $V_{n,p} \xrightarrow[n\to\infty]{\mathbb{P}} 0$. Therefore there exists increasing subsequence $\{n_i\}_{i\in\mathbb{N}} \subseteq \mathbb{N}$, $n_i \xrightarrow[i\to\infty]{} \infty$ such that $V_{n_i,p} \xrightarrow[i\to\infty]{\mathbb{P}\text{-a.s.}} 0$. But that means that quadratic variation of R^H is zero and therefore, if R^H is to be a semimartingale, then R^H must be a process of a finite variation.

If $p \in (1, \frac{1}{H})$, then by Theorem 4 it holds $V_{n,p} \xrightarrow[n\to\infty]{\mathbb{P}} \infty$. Therefore, there exists some increasing subsequence $\{m_j\}_{j\in\mathbb{N}} \subseteq \mathbb{N}$, $m_j \xrightarrow[j\to\infty]{} \infty$ such that $V_{m_j,p} \xrightarrow[j\to\infty]{\mathbb{P}\text{-a.s.}} \infty$ and therefore R^H cannot be a process of finite variation because $p > 1$.
Therefore, we have to conclude, Rosenblatt process with Hurst parameter $H \in (\frac{1}{2}, 1)$ is not a semimartingale. $\square$

2.4 Pathwise $\frac{1}{H}$-th variation along the sequence of partitions of Rosenblatt process

In this section, we would like to show that the Rosenblatt process with Hurst parameter $H \in (\frac{1}{2}, 1)$ has a finite pathwise $\frac{1}{H}$-th variation along a sequence of partitions in the sense of Definition 7. It means that we need to find some sequence of partitions π or at least show that there exists some sequence of partitions π of interval $[0, T]$, such that for every continuous path S on interval $[0, T]$ of Rosenblatt process, it holds that $S \in V_{\frac{1}{H}}(\pi)$, in the sense of Definition 7. We first define sequence of dyadic partitions because of its extensive use in this section.

Definition 11. Let $T > 0$. We define the sequence of dyadic partitions of interval $[0, T]$, $E_T = \{E_{T,n}\}_{n\in\mathbb{N}}$, by $E_{T,n} = \{t_i^n, i = 0, \dots, n\}$ with $t_i^n = \frac{i}{2^n}T$, for $i \in \{0, \dots, 2^n\}$.

Now, we state the main theorem of this section. Throughout this section we suppose, without loss of generality, that all paths of Rosenblatt process are continuous.

Theorem 6. *Let us fix $T > 0$ and let $(R_t^H, t \in [0, T])$ be a Rosenblatt process defined on $(\Omega, \mathcal{F}, \mathbb{P})$ with Hurst parameter $H \in (\frac{1}{2}, 1)$. Let $E_T = \{E_{T,n}\}_{n\in\mathbb{N}}$ be the sequence of dyadic partitions of interval $[0, T]$. Then there exists a subsequence $\tilde{E}_T$ of E_T such that*

$$R_{\bullet}^H(\omega) \in V_{\frac{1}{H}}(\tilde{E}_T), \ \textit{for} \ \mathbb{P}\text{-a.a. } \omega \in \Omega,$$

where $V_{\frac{1}{H}}(\bullet)$ denotes the set of all continuous paths with finite pathwise $\frac{1}{H}$-th variation along the sequence of partitions $\tilde{E}_T$, from Definition 7

Before we begin with the proof of Theorem 6, we will first prove the following lemma.

Lemma 8. *Let us fix $T > 0$ and let $(R_t^H, t \in [0, T])$ be a Rosenblatt process defined on $(\Omega, \mathcal{F}, \mathbb{P})$ with Hurst parameter $H \in (\frac{1}{2}, 1)$. Then, for every $t \in [0, T]$*

it holds

$$\sum_{\substack{[t_j^n, t_{j+1}^n] \in E_{T,n} \\ t_j \le t}} |R^H_{t^n_{j+1}} - R^H_{t^n_j}|^{\frac{1}{H}} \xrightarrow[n\to\infty]{\mathbb{P}} t\,\mathbb{E}[|R^H_1|^{\frac{1}{H}}],$$

where $E_T = \{E_{T,n}\}_{n\in\mathbb{N}}$ *is a sequence of dyadic partitions of interval* $[0,T]$.

Proof. Let us fix $T > 0$ and $t \in [0,T]$ and consider the sequence of dyadic partitions E_T. In Theorem 4 we have already proved the statement of the lemma for the case when $t = T$. For that reason we consider $t \in [0,T)$.
Firstly, let us suppose that there exists $m \in \mathbb{N}$ such that $t \in E_{T,m}$, or in other words, there exists $m \in \mathbb{N}$ and $j \in \{0,\dots,2^m-1\}$ such that $t = \frac{j}{2^m}T$. Then we have for $n \in \mathbb{N}$, $n > m$ that

$$\begin{aligned}
\sum_{\substack{[t_i, t_{i+1}] \in E_{T,n} \\ t_i \le t}} |R^H_{t_{i+1}} - R^H_{t_i}|^{\frac{1}{H}} &= \sum_{i=0}^{2^{n-m}j} |R^H_{\frac{i+1}{2^n}T} - R^H_{\frac{i}{2^n}T}|^{\frac{1}{H}} \\
&\overset{\mathcal{D}}{\sim} \frac{T}{2^n} \sum_{i=0}^{2^{n-m}j} |R^H_{i+1} - R^H_i|^{\frac{1}{H}} \\
&= \left(\frac{T}{2^n}(2^{n-m}j+1)\right)\left(\frac{1}{2^{n-m}j+1}\sum_{i=0}^{2^{n-m}j} |R^H_{i+1} - R^H_i|^{\frac{1}{H}}\right) \\
&= \left(\frac{j}{2^m}T + \frac{T}{2^n}\right)\left(\frac{1}{2^{n-m}j+1}\sum_{i=0}^{2^{n-m}j} |R^H_{i+1} - R^H_i|^{\frac{1}{H}}\right).
\end{aligned} \tag{2.9}$$

In the second row, we used self-similarity of Rosenblatt process. Now, similarly as in the proof of Theorem 4, by Birkhoff Ergodic Theorem, we obtain the following convergence $\mathbb{P}$-a.s. and in $L^1(\Omega)$:

$$\frac{1}{2^{n-m}j+1}\sum_{i=0}^{2^{n-m}j} |R^H_{i+1} - R^H_i|^{\frac{1}{H}} \xrightarrow[n\to\infty]{\mathbb{P}\text{-a.s.},\, L^1(\Omega)} \mathbb{E}[|R^H_1|^{\frac{1}{H}}]. \tag{2.10}$$

Combining (2.9), (2.10) and the fact that $t = \frac{j}{2^m}T$ we obtain the convergence

$$\sum_{\substack{[t_i, t_{i+1}] \in E_{T,n} \\ t_i \le t}} |R^H_{t_{i+1}} - R^H_{t_i}|^{\frac{1}{H}} \xrightarrow[n\to\infty]{\mathcal{D}} t\,\mathbb{E}[|R^H_1|^{\frac{1}{H}}].$$

From there, we obtain the desired convergence in probability

$$\sum_{\substack{[t_i, t_{i+1}] \in E_{T,n} \\ t_i \le t}} |R^H_{t_{i+1}} - R^H_{t_i}|^{\frac{1}{H}} \xrightarrow[n\to\infty]{\mathbb{P}} t\,\mathbb{E}[|R^H_1|^{\frac{1}{H}}]. \tag{2.11}$$

because $t\,\mathbb{E}[|R^H_1|^{\frac{1}{H}}]$ is deterministic.
Now, let us suppose that t is not dyadic rational, i.e. we suppose that there does not exist any $m \in \mathbb{N}$ and $j \in \mathbb{N}$ such that $t = \frac{j}{2^m}T$, or equivalently, for every $m \in \mathbb{N}$ it holds $t \notin E_{T,m}$. It means that for every $n \in \mathbb{N}$, there exists index $x_n \in \{0,\dots,2^n-1\}$ such that $t \in (\frac{x_n}{2^n}T, \frac{x_n+1}{2^n}T)$. That gives us a sequence

$\{x_n\}_{n\in\mathbb{N}}$ for which it holds $\frac{x_n}{2^n}T < t < \frac{x_n+1}{2^n}T$ for every $n \in \mathbb{N}$. We know that set $\{\frac{j}{2^n}, j \in \mathbb{N}_0, n \in \mathbb{N}\}$ is dense on $\mathbb{R}^+$. From there it follows that

$$\frac{x_n}{2^n}T \xrightarrow[n\to\infty]{} t \tag{2.12}$$

and

$$\frac{x_n+1}{2^n}T \xrightarrow[n\to\infty]{} t. \tag{2.13}$$

As we already mentioned, for every fixed $m \in \mathbb{N}$ it holds that $\frac{x_m}{2^m}T \in E_{T,m}$ and $\frac{x_m+1}{2^m}T \in E_{T,m}$. For that reason, as we already proved in (2.11), it holds that

$$\sum_{\substack{[t_i,t_{i+1}]\in E_{T,n}\\ t_i\le \frac{x_m}{2^m}T}} |R^H_{t_{i+1}} - R^H_{t_i}|^{\frac{1}{H}} \xrightarrow[n\to\infty]{\mathbb{P}} \frac{x_m}{2^m}T\,\mathbb{E}[|R^H_1|^{\frac{1}{H}}]$$

and

$$\sum_{\substack{[t_i,t_{i+1}]\in E_{T,n}\\ t_i\le \frac{x_m+1}{2^m}T}} |R^H_{t_{i+1}} - R^H_{t_i}|^{\frac{1}{H}} \xrightarrow[n\to\infty]{\mathbb{P}} \frac{x_m+1}{2^m}T\,\mathbb{E}[|R^H_1|^{\frac{1}{H}}].$$

It also holds on Ω and for every $n \in \mathbb{N}$, $n > m$ that

$$\sum_{\substack{[t_i,t_{i+1}]\in E_{T,n}\\ t_i\le \frac{x_m}{2^m}T}} |R^H_{t_{i+1}} - R^H_{t_i}|^{\frac{1}{H}} \le \sum_{\substack{[t_i,t_{i+1}]\in E_{T,n}\\ t_i\le t}} |R^H_{t_{i+1}} - R^H_{t_i}|^{\frac{1}{H}} \le \sum_{\substack{[t_i,t_{i+1}]\in E_{T,n}\\ t_i\le \frac{x_m+1}{2^m}T}} |R^H_{t_{i+1}} - R^H_{t_i}|^{\frac{1}{H}}.$$

Therefore, if we send $n \to \infty$, we obtain (for m being still fixed)

$$\frac{x_m}{2^m}T\,\mathbb{E}[|R^H_1|^{\frac{1}{H}}] \le \sum_{\substack{[t_i,t_{i+1}]\in E_{T,n}\\ t_i\le t}} |R^H_{t_{i+1}} - R^H_{t_i}|^{\frac{1}{H}} \le \frac{x_m+1}{2^m}T\,\mathbb{E}[|R^H_1|^{\frac{1}{H}}] \tag{2.14}$$

Finally, (2.14) with $\frac{x_m}{2^m}T \xrightarrow[m\to\infty]{} t$ and $\frac{x_m+1}{2^m}T \xrightarrow[m\to\infty]{} t$, completes the proof. □

Remark 6. In the statement of Lemma 8, we assumed E_T to be the sequence of dyadic partitions (because dyadic rationals is a dense set on $\mathbb{R}$) because we needed to show that (2.12) and (2.13) hold. The proof of Lemma 8 would pass in the same way for any sequence of partitions for which the convergences (2.12) and (2.13) hold.

We will continue with citations of two lemmas. The first lemma gives us sufficient conditions for uniform convergence in probability of family of continuous processes.

Lemma 9. ([28, Lemma 3.1]) *Let $(Z_\varepsilon)_{\varepsilon>0}$ be a family of continuous processes. Let us suppose that*

1. *for every $\varepsilon > 0$, $t \to Z_\varepsilon(t)$ is non-decreasing $\mathbb{P}$-a.s.,*

2. *there exists a continuous process $(Z(t))_{t\ge0}$ such that $Z_\varepsilon(t) \xrightarrow[\varepsilon\to0^+]{\mathbb{P}} Z(t)$.*

Then $(Z_\varepsilon)_{\varepsilon>0}$ converges to Z uniformly in probability $\mathbb{P}$.

The next lemma gives us necessary and sufficient conditions for continuous path to have a p-th variation along a sequence of partitions.

Lemma 10. ([21, Lemma 1.3]) *Let $T > 0$, $S \in C([0,T],\mathbb{R})$ and $\pi = \{\pi_n\}_{n\in\mathbb{N}}$ be a sequence of partitions of $[0,T]$. It holds that $S \in V_p(\pi)$ if and only if there exists a continuous function $[S]^p$ such that*

$$\forall t \in [0,T] : \sum_{\substack{[t_j,t_{j+1}]\in\pi_n \\ t_j\leq t}} |S(t_{j+1}) - S(t_j)|^p \xrightarrow[n\to\infty]{} [S]^p(t). \tag{2.15}$$

If this property holds, then the convergence in (2.15) is uniform.

Now we have enough theory to prove the desired Theorem 6.

Proof of Theorem 6. Our goal is to show that there exists some sequence of partitions and continnuous function such that (2.15) holds.
Let us fix $T > 0$ and let E_T be a sequence of dyadic partitions of interval $[0,T]$. Furthermore, let us denote for every $n \in \mathbb{N}$ and $t \in [0,T]$

$$Z_n(t) := \sum_{\substack{[t_j^n,t_{j+1}^n]\in E_{T,n} \\ t_j\leq t}} |R^H_{t^n_{j+1}} - R^H_{t^n_j}|^{\frac{1}{H}}, \qquad Z(t) := t\,\mathbb{E}[|R^H_1|^{\frac{1}{H}}].$$

Then, from Lemma 8, it holds that

$$Z_n(t) \xrightarrow[n\to\infty]{\mathbb{P}} Z(t), \text{ for every } t \in [0,T]. \tag{2.16}$$

From there, it follows that for every $t \in [0,T]$, there exists some increasing subsequence of indexes $\{n_{t,k}\}_{k\in\mathbb{N}} \subseteq \mathbb{N}$ with $n_{t,k} \xrightarrow[k\to\infty]{} \infty$, and there exists $\Omega_t \in \mathcal{F}$ with $\mathbb{P}[\Omega_t] = 1$ such that it holds, for every $\omega \in \Omega_t$, that

$$Z_{n_{t,k}}(t,\omega) \xrightarrow[k\to\infty]{} Z(t,\omega).$$

In order to verify (2.15), we need to show that $\{n_{t,k}\}_{k\in\mathbb{N}}$ and Ω_t can be chosen independently of t. In other words, we would like to show that there exists $\tilde{\Omega} \in \mathcal{F}$ with $\mathbb{P}[\tilde{\Omega}] = 1$ and an increasing sequence of indexes $\{n_k\}_{k\in\mathbb{N}} \subseteq \mathbb{N}$, $n_k \xrightarrow[k\to\infty]{} \infty$ such that for every $t \in [0,T]$ and for every $\omega \in \tilde{\Omega}$ it holds

$$Z_{n_k}(t,\omega) \xrightarrow[k\to\infty]{} Z(t,\omega).$$

We do that by verifying sufficient assumptions of uniform convergence in probability in Lemma 9.
We have $(Z_n)_{n\in\mathbb{N}}$ family of continuous processes. Firstly, we are interested whether it holds that for every $n \in \mathbb{N}$, the mapping $t \to Z_n(t)$ is non-decreasing $\mathbb{P}$-a.s. on $[0,T]$.
Let us choose $0 \leq s < t \leq T$. Then it trivially holds that $Z_n(s) \leq Z_n(t)$ because

$$Z_n(t) = Z_n(s) + \sum_{\substack{[t_j^n,t_{j+1}^n]\in E_{T,n} \\ s<t^n_j\leq t}} |R^H_{t^n_{j+1}} - R^H_{t^n_j}|^{\frac{1}{H}}$$

and therefore $t \to Z_n(t)$ is non-decreasing $\mathbb{P}$-a.s. on $[0,T]$ for every $n \in \mathbb{N}$. It also holds that $Z_n(t) \xrightarrow[n\to\infty]{\mathbb{P}} Z(t)$, for every $t \in [0,T]$ as we showed in (2.16).
Therefore, by Lemma 9, it holds that $Z_n \xrightarrow[n\to\infty]{\text{ucp}} Z$, or in other words, it holds that

$$\forall \varepsilon > 0 : \mathbb{P}\left[\sup_{t\in[0,T]} |Z_n(t) - Z(t)| > \varepsilon\right] \xrightarrow[n\to\infty]{} 0.$$

From there, it follows that there exists some increasing subsequence of indexes $\{n_k\}_{k\in\mathbb{N}} \subseteq \mathbb{N}$ with $n_k \xrightarrow[k\to\infty]{} \infty$ and there exists $\tilde{\Omega} \in \mathcal{F}$ with $\mathbb{P}[\tilde{\Omega}] = 1$, such that for every $\omega \in \tilde{\Omega}$ it holds

$$\sup_{t\in[0,T]} |Z_{n_k}(t,\omega) - Z(t,\omega)| \xrightarrow[k\to\infty]{} 0.$$

But it means that the following holds:

$$\forall \omega \in \tilde{\Omega},\, \forall t \in [0,T] : \sum_{\substack{[t_j^{n_k}, t_{j+1}^{n_k}] \in E_{T,n_k} \\ t_j^{n_k} \leq t}} |R^H_{t_{j+1}^{n_k}}(\omega) - R^H_{t_j^{n_k}}(\omega)|^{\frac{1}{H}} \xrightarrow[k\to\infty]{} t\,\mathbb{E}[|R_1^H|^{\frac{1}{H}}].$$

Finally, by Lemma 10, we have for every $\omega \in \tilde{\Omega}$ that $R^H_{\bullet}(\omega) \in V_{\frac{1}{H}}(\tilde{E}_T)$, where $\tilde{E}_T = \{E_{T,n_k}\}_{k\in\mathbb{N}}$. In other words, we say that all continuous paths of Rosenblatt process R^H have finite pathwisie $1/H$-th variation along a sequence of partitions $\tilde{E}_T$, i.e. it holds that

$$\mu^n := \sum_{[t_j^{n_k}, t_{j+1}^{n_k}] \in E_{T,n_k}} \delta_{t_j^{n_k}} |R^H_{t_{j+1}^{n_k}} - R^H_{t_j^{n_k}}|^{\frac{1}{H}}$$

converges weakly to a measure μ without atoms for every $t \in [0,T]$ and it holds that $\mu([0,t]) = t\,\mathbb{E}[|R_1^H|^{\frac{1}{H}}]$. □

2.5 $\frac{1}{H}$-Variation of Rosenblatt process

In this subsection we will suppose, without loss of generality, that Rosenblatt process $(R_t^H, t \in \mathbb{R}^+)$ with fixed Hurst parameter $H \in (\frac{1}{2}, 1)$ is defined on complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with all paths being continuous functions.
The next question we might ask is whether paths of Rosenblatt process have finite p-variation in the sense of Definition 6. Clearly, if $p \in (0, \frac{1}{H})$, then the paths of Rosenblatt process will not be of finite variation because, by Theorem 4, it holds that $\sum_{j=1}^{2^n} |R^H_{\frac{j}{2^n}} - R^H_{\frac{j-1}{2^n}}|^p \xrightarrow[n\to\infty]{\mathbb{P}} \infty$.
In this section we will prove, that $\mathbb{P}$-a.a. paths of Rosenblatt process are not of finite p-variation even if $p = \frac{1}{H}$. Results proved in this section could not be found anywhere in the literature, however, the techniques of the proofs are known (see [29] where claims from this section are proved for fractional Brownian motion).

Firstly, let us define some notions. For $a, b \in \mathbb{R}^+, a < b$, we define random variable

$$\Upsilon_{[a,b]} = \sup_{n\in\mathbb{N}, a\leq t_1 < \ldots < t_n \leq b} \sum_{i=1}^{n-1} |R^H_{t_{i+1}} - R^H_{t_i}|^{\frac{1}{H}}.$$

Note that $\Upsilon_{[a,b]}$ is a measurable mapping because Rosenblatt process has continuous sample paths and therefore we can take the supremum over rational numbers $t_1, \ldots, t_n \in \mathbb{Q}$.
Let $\lambda(\bullet)$ be Lebesgue measure on $\mathcal{B}(\mathbb{R}^+)$, U be a finite union of disjoint open intervals $(s_i, t_i) \subset \mathbb{R}^+$ with $s_i, t_i \in \mathbb{Q}$ for every $i \in \{1, \ldots, n\}$ and some $n \in \mathbb{N}$, and let $\mathcal{U}$ be the collection of such subsets of $\mathbb{R}^+$. For every $U \in \mathcal{U}$ of the form $U = \bigcup_{i=1}^n (s_i, t_i)$, where $n \in \mathbb{N}$, $0 \leq s_1 < t_1 \leq s_2 < t_2 \leq \ldots \leq s_n < t_n \leq b$, we define the random variable

$$\zeta_U = \sum_{i=1}^n |R_{t_i}^H - R_{s_i}^H|^{\frac{1}{H}}. \tag{2.17}$$

Note that it holds

$$\Upsilon_{[a,b]}(\omega) = \sup_{U \in \mathcal{U}, U \subseteq [a,b]} \zeta_U(\omega), \quad \text{for every } \omega \in \Omega.$$

Firstly, we prove the following simple lemma which follows from the Birkhoff Ergodic Theorem.

Lemma 11. *Let us fix $m \in \mathbb{N}$ and denote $p_m = \mathbb{P}[|R_1^H|^{\frac{1}{H}} \geq m]$.*
Let us denote $Z_n^{(m)} = \mathbb{1}_{\left[|R_n^H - R_{n-1}^H|^{\frac{1}{H}} \geq m\right]}$ for every $n \in \mathbb{N}$. Then it holds

$$\frac{1}{n} \sum_{i=1}^n Z_i^{(m)} \xrightarrow[n \to \infty]{\mathbb{P}\text{-a.s.}} p_m.$$

Proof. We know that increments of Rosenblatt process are strictly stationary and they are also ergodic, as we proved in Theorem 3. Consequently, the function $\mathbb{1}_{\left[|\cdot|^{\frac{1}{H}} \geq m\right]}$ is measurable and therefore also the process $(Z_n^{(m)}, n \in \mathbb{N})$ is ergodic. For that reason, we can use Birkhoff Ergodic Theorem and we obtain

$$\frac{1}{n} \sum_{i=1}^n Z_n^{(m)} \xrightarrow[n \to \infty]{\mathbb{P}\text{-a.s.}} \mathbb{E}[Z_1^{(m)}].$$

And we use the fact that $E[Z_1^{(m)}] = p_m$. $\square$

Note that we can see from the proof of Lemma 11 that the convergence in the statement of Lemma 11 holds, by Birkhoff Ergodic Theorem, also in the sense of $L^1(\Omega)$, but we do not need this type of convergence in further proofs.
The following lemma lies at the heart of the proof of the main result.

Lemma 12. *Let us choose $s, t \in \mathbb{Q}^+, s < t$, denote $I = (s,t)$ and fix $m > 0$. Denote $p_m = \mathbb{P}[|R_1^H|^{\frac{1}{H}} \geq m]$ and choose $r \in (0, p_m)$. Then there exists $A_m \in \mathcal{F}$ such that $\mathbb{P}[A_m] = 1$ and for every $\omega \in A_m$, there exists $U_\omega \in \mathcal{U}$ such that it holds*

- *$U_\omega \subset I$,*
- *$\lambda(U_\omega) > r\lambda(I)$,*
- *$\zeta_{U_\omega}(\omega) \geq m\lambda(U_\omega)$.*

Proof. Let us fix $m > 0$. For every $n \in \mathbb{N}$ and for every $i \in \{0, \dots, n\}$ we define $t_i^n = s + \frac{i}{n}(t - s)$ and $J_k^n = (t_{k-1}^n, t_k^n)$ for $k \in \{1, \dots, n\}$. Let us also denote

$$S_n^{(m)} = \sum_{i=0}^{n-1} \mathbb{1}_{\left[\left|R^H_{t^n_{i+1}} - R^H_{t^n_i}\right|^{\frac{1}{H}} \geq m\frac{t-s}{n}\right]}.$$

By (2.17), we can see that for every $\omega \in \Omega$ and $k \in \{1 \dots, n\}$ we have $\zeta_{J_k^n}(\omega) = |R^H_{t_k^n}(\omega) - R^H_{t^n_{k-1}}(\omega)|^{\frac{1}{H}}$ and $\lambda(J_k^n) = \frac{t-s}{n}$. For that reason, $S_n^{(m)}(\omega)$ counts the number of subintervals J_k^n for which it holds $\zeta_{J_k^n}(\omega) \geq m\lambda(J_k^n)$. By definition of $\{t_i^n, i \in \{0, \dots, n\}\}$, strict stationarity of increments and self-similarity of the Rosenblatt process, we have for every $i \in \{0, \dots, n-1\}$ that

$$\begin{aligned}
\left|R^H_{t^n_{i+1}} - R^H_{t^n_i}\right|^{\frac{1}{H}} &= \left|R^H_{s+\frac{i+1}{n}(t-s)} - R^H_{s+\frac{i}{n}(t-s)}\right|^{\frac{1}{H}} \\
&\overset{\mathcal{D}}{\sim} \left|R^H_{\frac{i+1}{n}(t-s)} - R^H_{\frac{i}{n}(t-s)}\right|^{\frac{1}{H}} \\
&\overset{\mathcal{D}}{\sim} \left||\frac{t-s}{n}|^H (R^H_{i+1} - R^H_i)\right|^{\frac{1}{H}} \\
&\overset{\mathcal{D}}{\sim} \frac{t-s}{n}\left|R^H_{i+1} - R^H_i\right|^{\frac{1}{H}}.
\end{aligned}$$

For that reason, it holds that $S_n^{(m)}$ has the same distribution as $Z_n^{(m)}$, where $Z_n^{(m)}$ is defined for $n \in \mathbb{N}$ by

$$Z_n^{(m)} = \sum_{i=0}^{n-1} \mathbb{1}_{\left[\left|R^H_{i+1} - R^H_i\right|^{\frac{1}{H}} \geq m\right]}.$$

By Lemma 11, we have that

$$\frac{1}{n} Z_n^{(m)} \xrightarrow[n\to\infty]{\mathbb{P}\text{-a.s.}} p_m.$$

From there, we have

$$\frac{1}{n} S_n^{(m)} \xrightarrow[n\to\infty]{\mathbb{P}} p_m$$

because p_m is a deterministic. Therefore, there exists some increasing subsequence $\{n_i\}_{i\in\mathbb{N}} \subseteq \mathbb{N}$, with $n_i \xrightarrow[i\to\infty]{} \infty$, such that

$$\frac{1}{n_i} S_{n_i}^{(m)} \xrightarrow[i\to\infty]{\mathbb{P}\text{-a.s.}} p_m.$$

Let us denote

$$A_m = \{\omega \in \Omega : \frac{1}{n_i} S_{n_i}^{(m)}(\omega) \xrightarrow[i\to\infty]{} p_m\}.$$

Then $\mathbb{P}[A_m] = 1$. Let us fix $\omega \in A_m$. Then there exists $i_\omega \in \mathbb{N}$ such that

$$\frac{1}{n_i} S_{n_i}^{(m)}(\omega) > r, \text{ for every } i \in \mathbb{N},\ i > i_\omega.$$

Let us choose $i \in \mathbb{N}$, $i > i_\omega$, and let

$$J_\omega = \left\{ J_k^{n_i} : \zeta_{J_k^{n_i}}(\omega) \geq m\frac{t-s}{n_i}, k \in \{1, \dots, n_i\} \right\}.$$

Then $J_\omega \neq \emptyset$ because if $J_\omega = \emptyset$, then $S_{n_i}^{(m)}(\omega) = 0$ would hold but we have $\frac{1}{n_i} S_{n_i}^{(m)}(\omega) > r$ which is a contradiction. In fact, J_ω has exactly $S_{n_i}^{(m)}(\omega)$ elements. Finally, let us denote

$$U_\omega := \bigcup_{V \in J_\omega} V.$$

Then it holds that

$$\lambda(U_\omega) = S_{n_i}^{(m)}(\omega)\frac{t-s}{n_i} > r\lambda(I)$$

and

$$\begin{aligned}
\zeta_{U_\omega}(\omega) &= \sum_{V \in J_\omega} \zeta_V(\omega) \\
&\geq S_{n_i}(\omega) m \frac{t-s}{n_i} \\
&= m\lambda(U_\omega).
\end{aligned}$$

That completes the proof. $\square$

Corollary 1. *Let us fix $m \in \mathbb{N}$, denote $p_m = \mathbb{P}[|R_1^H|^{\frac{1}{H}} \geq m]$ and set $r_m = \frac{p_m}{2}$. Then there exists $C_m \in \mathcal{F}$ with $\mathbb{P}[C_m] = 1$ such that if $\omega \in C_m$ and $V \in \mathcal{U}$, then there exists $U_\omega \in \mathcal{U}$ with $U_\omega \subset V$ for which it holds $\lambda(U_\omega) > r_m \lambda(V)$ and $\zeta_{U_\omega}(\omega) \geq m\lambda(U_\omega)$.*

Proof. The statement follows from Lemma 12. Indeed, by using Lemma 12, we obtain for every $U \in \mathcal{U}$ set $A_m^U \in \mathcal{F}$, $\mathbb{P}[A_m^U] = 1$ and for every $\omega \in A_m^U$ there exists $U_\omega \in \mathcal{U}$ such that $U_\omega \subseteq U$, $\lambda(U_\omega) > r\lambda(U)$ and $\zeta_{U_\omega}(\omega) \geq m\lambda(U_\omega)$. Now it is enough to set

$$C_m = \bigcap_{U \in \mathcal{U}} A_m^U.$$

$\square$

Lemma 13. *Fix $0 \leq a < b$, $a, b \in \mathbb{Q}$. Then $\Upsilon_{[a,b]}(\omega) = \infty$ for $\mathbb{P}$-a.a. $\omega \in \Omega$.*

Proof. Let us choose $m \in \mathbb{N}$ arbitrarily, fix $0 \leq a < b$, $a, b \in \mathbb{Q}$, denote $p_m = \mathbb{P}[|R_1|^{\frac{1}{H}} \geq m]$ and choose $r \in (0, p_m)$. Then, by applying Lemma 12 to the open interval $I = (a, b)$, we obtain that there exists $C_m^1 \in \mathcal{F}$ such that $\mathbb{P}[C_m^1] = 1$ and for every $\omega \in C_m^1$ there exists $U_\omega^1 \in \mathcal{U}$ such that

$$\begin{aligned}
U_\omega^1 &\subset I, \\
\lambda(U_\omega^1) &> r\lambda(I), \\
\zeta_{U_\omega^1}(\omega) &\geq m\lambda(U_\omega^1).
\end{aligned}$$

Let us denote

$$\mathcal{W}^1 = \{U_\omega^1, \omega \in C_m^1\}.$$

Now, by Corollary 1 (with m being already fixed), there exists $C_m^2 \in \mathcal{F}$ such that $\mathbb{P}[C_m^2] = 1$ such that for every $\omega \in C_m^2 \cap C_m^1$ (the following holds by Corollary 1 for every $\omega \in C_m^2$, but we will need it only for every $\omega \in C_m^2 \cap C_m^1$) and every $U_\omega^1 \in \mathcal{W}^1$, there exists $U_\omega^2 \in \mathcal{U}$ such that

$$
\begin{aligned}
U_\omega^2 &\subset I \setminus \overline{U_\omega^1}, \\
\lambda(U_\omega^2) &> r\lambda(I \setminus \overline{U_\omega^1}), \\
\zeta_{U_\omega^2}(\omega) &\geq m\lambda(U_\omega^2).
\end{aligned}
$$

Let us denote

$$\mathcal{W}^2 = \{U_\omega^2, \omega \in C_m^2 \cap C_m^1\}.$$

We continue in the same manner as before. At the k-th step (where $k \in \mathbb{N}$), we have that there exists $C_m^k \in \mathcal{F}$ with $\mathbb{P}[C_m^k] = 1$ such that for every $\omega \in \bigcap_{i=1}^k C_m^i$ and every $U_\omega^1 \in \mathcal{W}^1, \ldots, U_\omega^{k-1} \in \mathcal{W}^{k-1}$ there exists $U_\omega^k \in \mathcal{U}$ for which it holds

$$
\begin{aligned}
U_\omega^k &\subset I \setminus (\overline{U_\omega^1} \cup \ldots \cup \overline{U_\omega^{k-1}}), \\
\lambda(U_\omega^k) &> r\lambda(I \setminus (\overline{U_\omega^1} \cup \ldots \cup \overline{U_\omega^{k-1}})), \\
\zeta_{U_\omega^k}(\omega) &\geq m\lambda(U_\omega^k).
\end{aligned}
$$

We denote

$$\mathcal{W}^k = \{U_\omega^k, \omega \in \bigcap_{i=1}^k C_m^i\}.$$

Let us denote

$$\mathcal{C}_m^k = \bigcap_{i=1}^k C_m^i$$

for every $k \in \mathbb{N}$, and

$$\mathcal{C}_m^\infty = \bigcap_{i=1}^\infty C_m^i.$$

Then it holds $\mathcal{C}_m^k \in \mathcal{F}$ and $\mathbb{P}[\mathcal{C}_m^k] = 1$ for every $k \in \mathbb{N} \cup \{\infty\}$. Let us fix $k \in \mathbb{N}$. Then, for every $\omega \in \mathcal{C}_m^k$, we have $U_\omega^1 \in \mathcal{W}^1, \ldots, U_\omega^k \in \mathcal{W}^k$ and $\bigcap_{i=1}^k U_\omega^i = \emptyset$ (this also holds for $k \in \mathbb{N} \cup \{\infty\}$).
Let us choose $\omega \in \mathcal{C}_m^k$ and denote

$$V_\omega^k = \bigcup_{i=1}^k U_\omega^i.$$

Then we have $V_\omega^k \in \mathcal{U}$. Furthermore, from (2.17) and from properties of $U_\omega^1, \ldots, U_\omega^k$, we have

$$\zeta_{V_\omega^k}(\omega) = \sum_{i=1}^k \zeta_{U_\omega^i}(\omega) \geq \sum_{i=1}^k m\lambda(U_\omega^i) = m\lambda(V_\omega^k). \tag{2.18}$$

Now, we will prove by induction that

$$\lambda(I \setminus \overline{V_\omega^k}) \leq (1-r)^k(b-a).$$

If $k = 1$, we have

$$\begin{aligned}
\lambda(U_\omega^1) &> r\lambda(I)\\
-\lambda(U_\omega^1) &\leq -r\lambda(I)\\
\lambda(I) - \lambda(U_\omega^1) &\leq \lambda(I) - r\lambda(I)\\
\lambda(I \setminus U_\omega^1) &\leq (1-r)\lambda(I)\\
\lambda(I \setminus V_\omega^1) &\leq (1-r)(b-a).
\end{aligned}$$

Now, let us suppose that this inequality holds for $k-1$ and we will prove it for k, where $k \in \mathbb{N}$. In other words, we suppose that it holds

$$\lambda(I \setminus \overline{V_\omega^{k-1}}) \leq (1-r)^{k-1}(b-a).$$

Then,

$$\begin{aligned}
\lambda(I \setminus \overline{V_\omega^k}) &= \lambda(I \setminus (\overline{V_\omega^{k-1}} \cup \overline{U_\omega^k}))\\
&= \lambda(I \setminus \overline{V_\omega^{k-1}}) - \lambda(\overline{U_\omega^k}).
\end{aligned} \tag{2.19}$$

Now, from

$$\begin{aligned}
\lambda(\overline{U_\omega^k}) &> r\lambda(I \setminus (\overline{U_\omega^1} \cup \ldots \cup \overline{U_\omega^{k-1}}))\\
&= r\lambda(I \setminus \overline{V_\omega^{k-1}}),
\end{aligned}$$

we have that

$$-\lambda(\overline{U_\omega^k}) \leq -r\lambda(I \setminus \overline{V_\omega^{k-1}}).$$

Therefore, if we continue in (2.19), we obtain

$$\begin{aligned}
\lambda(I \setminus \overline{V_\omega^k}) &\leq \lambda(I \setminus \overline{V_\omega^{k-1}}) - r\lambda(I \setminus \overline{V_\omega^{k-1}})\\
&= (1-r)\lambda(I \setminus \overline{V_\omega^{k-1}})\\
&\leq (1-r)^k(b-a).
\end{aligned}$$

We proved that $\lambda(I \setminus \overline{V_\omega^k}) \leq (1-r)^k(b-a)$ for every $k \in \mathbb{N}$. From the fact that $1-r \in (0,1)$, we see that

$$\lambda(I \setminus \overline{V_\omega^k}) \xrightarrow[k\to\infty]{} 0$$

and

$$\lambda(V_\omega^k) \xrightarrow[k\to\infty]{} \lambda(I) = b-a.$$

For this reason and from (2.18), it holds for every $\omega \in \mathcal{C}_m^\infty$ that

$$\begin{aligned}
\sup_{k\in\mathbb{N}} \zeta_{V_\omega^k}(\omega) \geq \sup_{k\in\mathbb{N}} m\lambda(V_\omega^k)&\\
= m \lim_{k\to\infty} \lambda(V_\omega^k)&\\
= m(b-a).&
\end{aligned}$$

And from there, it follows that for every $\omega \in \mathcal{C}_m^\infty$ it holds

$$\sup_{U\in\mathcal{U}, U\subset[a,b]} \zeta_U(\omega) \geq \sup_{k\in\mathbb{N}} \zeta_{V_\omega^k}(\omega),$$

and therefore

$$\begin{aligned}\Upsilon_{[a,b]}(\omega) &= \sup_{U\in\mathcal{U}, U\subset[a,b]} \zeta_U(\omega)\\ &\geq m(b-a)\end{aligned}$$

for every $\omega \in \mathcal{C}_m^\infty$.
We proved that for fixed $m \in \mathbb{N}$ and for every $\omega \in \mathcal{C}_m^\infty$ it holds

$$\Upsilon_{[a,b]}(\omega) \geq m(b-a).$$

Now, we denote

$$\mathcal{C} = \bigcap_{m\in\mathbb{N}} \mathcal{C}_m^k.$$

Then it holds $\mathcal{C} \in \mathcal{F}$ and $\mathbb{P}[\mathcal{C}] = 1$, and furthermore, for every $\omega \in \mathcal{C}$, we have that

$$\Upsilon_{[a,b]}(\omega) \geq m(b-a)$$

holds for every $m \in \mathbb{N}$, and therefore

$$\Upsilon_{[a,b]}(\omega) = \infty$$

holds for every $\omega \in \mathcal{C}$ and $\mathbb{P}[\mathcal{C}] = 1$. □

Theorem 7. *There exists $N \in \mathcal{F}$, $\mathbb{P}[N] = 0$, such that for every $a, b \in \mathbb{R}^+, a < b$, it holds that*

$$\sup_{n\in\mathbb{N}, a\leq t_1<\ldots<t_n\leq b} \sum_{i=0}^{n-1} |R_{t_{i+1}}^H(\omega) - R_{t_i}^H(\omega)|^{\frac{1}{H}} = \infty$$

holds for every $\omega \in \Omega \setminus N$.

Proof. The claim follows from Lemma 13. Indeed, for every $0 \leq a < b$, $a, b, \in \mathbb{Q}$, we obtain set $N_{a,b} \in \mathcal{F}$ with $\mathbb{P}[N_{a,b}] = 0$ such that $\Upsilon_{[a,b]}(\omega) = \infty$ for all $\omega \in \Omega \setminus N_{a,b}$. It is enough to set $N = \bigcup_{a,b\in\mathbb{Q}^+, a<b} N_{a,b}$ and we obtain the statement of the theorem. □

Conclusion

In this thesis we have proved numerous properties of Rosenblatt processes that could not be found in the literature. First, we proved that the increments of Rosenblatt processes are ergodic. We continued with proving that Rosenblatt processes have finite p-th variation along sequence of dyadic partitions for $p \geq \frac{1}{H}$ and consequently we proved that Rosenblatt process is not a semimartingale.

After that, in Section 2.4, we showed that Rosenblatt process has finite pathwise $\frac{1}{H}$-th variation along sequence of dyadic partitions. We proved Lemma 8 for only one specific sequence of partitions (for sequence of dyadic partitions). One possibility for further research is to find the class of sequences of partitions for which Lemma 8 holds. The problem may be approached by investigating assumptions on how fast the mesh

$$\sup_{j=0,\ldots,N(n)} \{|t_j^n - t_{j-1}^n|\}$$

will converge to 0 as $n \to \infty$, where $\{t_j^n, j = 0, \ldots, N(n)\}_{n\in\mathbb{N}}$ is a sequence of partitions and $N : \mathbb{N} \to \mathbb{N}$ is some increasing function. For sequence of dyadic partitions, this mesh converges exponentially fast i.e. $\frac{1}{2^n} \xrightarrow[n\to\infty]{} 0$.
Lastly, in Section 2.5, we showed that $\mathbb{P}$-a.a. trajectories of the Rosenblatt process with Hurst parameter $H \in (\frac{1}{2}, 1)$ have infinite p-variation if $p = \frac{1}{H}$. We also argued that this holds for $0 < p < \frac{1}{H}$. The interesting open question remains what happens if $p > \frac{1}{H}$? We could not find answers in the case of $p > \frac{1}{H}$ in literature for fractional Brownian motion either.

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