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Konstrukce a dekonstrukce složitých objektů s lokálně přehlednou strukturou

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Constructions and deconstructions of locally well-behaved complex objects

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INTRODUCTION

This thesis consists of this introductory text and the following five original papers.

1. J. Šaroch, *On the non-existence of right almost split maps*, Invent. Math. **209** (2017), no. 2, 463–479. DOI: 10.1007/s00222-016-0712-2

2. J. Šaroch, *Approximations and Mittag-Leffler conditions—the tools*, Israel J. Math. **226** (2018), no. 2, 737–756. DOI: 10.1007/s11856-018-1710-4

3. L. Angeleri Hügel, J. Šaroch and J. Trlifaj, *Approximations and Mittag-Leffler conditions—the applications*, Israel J. Math. **226** (2018), 757–780. DOI: 10.1007/s11856-018-1711-3

4. J. Šaroch and J. Šťovíček, *Singular compactness and definability for* Σ*-cotorsion and Gorenstein modules*, Selecta Math. New Ser. **26** (2020), Paper No. 23. DOI: 10.1007/s00029-020-0543-2

5. J. Šaroch and J. Trlifaj, *Test sets for factorization properties of modules*, to appear in Rend. Sem. Mat. Univ. Padova.

The unifying theme of all the works above, and hence of the thesis itself, is (de)construction of objects (usually modules over a ring) which are locally well behaved. Naturally, some of the papers go also in other directions.

In the next few paragraphs, we elaborate a little bit more on what types of (de)constructions we have in our mind here. Along the way, we present some of the main results appearing in the papers. Finally, we conclude this introduction by two new results which nicely illustrate and supplement the theme of this thesis.

1 General notes on (de)constructions

A phenomenon of two principles, one opposing the other, whose competition yields new quality and brings deeper understanding of the subject matter exceeds the boundaries of mathematics. It is pretty tempting to call it a universal truth which is recognized in all sciences, humanities and theology. In this thesis though, we stay firmly in the realm of mathematics. And the two competing principles we are mostly interested in here are *compactness* and *reflection*.

Informally put, a *compactness principle* is a statement which asserts, for some object O (of a certain type) and a given property P, that if enough small subobjects of O satisfy the property P, then O satisfies P as well. On the other hand, a *reflection principle* is a statement which asserts, for some object O (of a certain type) and a given property P, that if O satisfies P, then there exist enough small subobjects of O which satisfy P as well.

Of course, in a particular context, we have to explicate what 'small' and 'enough' mean. Usually, 'small' refers to 'having cardinality bounded by some fixed cardinal (less than $|O|$)' or, in case of locally accessible categories, 'having presentability rank bounded by some fixed cardinal (less than the presentability rank of O). As for the explication of 'enough', it typically refers to some variation of a 'closed and unbounded' subposet of objects, e.g. each small subobject of O is contained in a small subobject satisfying P, and the class of subobjects satisfying P is upward (λ-)directed (for some small λ). Note that even the notion of 'subobject' can be subject to a weakening in some applications: see, for instance, the theory developed in Section 4.2.

Let us illustrate the concept on the concrete example of objects from Mod-R, for a ring R, and the property 'being projective'. For an arbitrary fixed regular uncountable cardinal λ, we define 'small' as 'being $< \lambda$-generated', and 'enough' as 'O is the λ-directed union of a set of its small projective submodules'. It is not hard to see, using a decomposition of a projective O into a direct sum of countably generated projective modules, that the property 'being projective' reflects in this setting (for all right R-modules).

It is a somewhat harder question whether 'being projective' can be compact for some λ as above. It turns out that it depends on the ring R as well as on the model of ZFC we are working in. If R is right perfect, then the compactness holds for every λ since the class of projective modules is closed under direct limits in this case. If R is not right perfect, then [9, Theorem VII.1.4] implies that, assuming V = L, 'being projective' is not compact for any λ for all right R-modules (and, if we restrict only to λ-generated modules, then it is compact for a λ as above if and only if λ is a weakly compact cardinal). On the other hand, Theorem 5.3.3 implies that 'being projective' is compact for λ (for all right R-modules) provided that $\lambda \geq \kappa > |R|$ where κ is a strongly compact cardinal.

Constructions. An object O possessing enough small subobjects satisfying a given property P, which itself, however, does not satisfy P, is sometimes called pathological. Whether such designation is used, heavily depends on the context, and it can easily be the case that objects once deemed pathological are considered abundant and rather common later on. Anyway, the objects O described here are what the author of this thesis refers to as locally well-behaved but complex. Using the terminology from the paragraphs above, we could also say that compactness fails on these objects. Their (possible) existence can have very interesting implications in various kinds of structure theories.

It is often the case that such objects embody limitations of a particular structure theory: for instance, there was a suggestion by Drinfeld, which appeared in an earlier version of [7] and was being discussed for several years after its publication, to use flat Mittag-Leffler modules, also called $\aleph_1$-projective modules, instead of finitely generated projective modules in the definition of an infinite-dimensional vector bundle on a scheme; this promising approach (the class of $\aleph_1$-projective modules has several nice closure properties), however, encountered serious technical difficulties after Herbera and Trlifaj proved in [12] that the class of $\aleph_1$-projective right R-modules is not deconstructible unless R is right perfect. This means that there are arbitrarily large $\aleph_1$-projective modules which cannot be filtered by smaller $\aleph_1$-projective modules. To put it simply, we encountered a failure of compactness. This problem was later reasserted when the author showed that the class of $\aleph_1$-projective modules over a non-right perfect ring is not even a precovering class of modules, see Theorem 2.3.3. This severely disqualifies the discussed class of modules from playing an interesting role in the approximation theory and/or relative homological algebra.

On the other hand, a construction of locally well-behaved but complex objects can also lead to a significant simplification of a theory. For example, the construction of the so-called tree modules was used in the first paper included in this thesis to show that, in any finitely accessible additive category, codomains of right almost split morphisms are necessarily finitely presented objects (with local endomorphism rings), cf. Theorem 1.4.4 and the subsequent remark. This answered, at that time, a 40 years old question posed by Auslander. The dual question, whether the domain of a left almost split morphism has to be pure-injective, remains open though.

Of course, the constructions of locally well-behaved but complex objects depend on an occurence of an instance of incompactness. This can be, in some cases, spoiled by a possible presence of large cardinals, such as the strongly compact ones. In these cases, we have to consider using additional set-theoretic hypotheses which would forbid these large cardinals and bring some useful non-compactness principles into play. When it comes to such principles, perhaps the best known and often employed one is the Jensen's square $\square$; mostly for its implications toward the existence of non-reflecting stationary sets. An additional hypothesis in this spirit is used in Section 2 below to prove an important special case of Enochs' conjecture. Another special case is covered in Section 3.5.

The square principle was also recently employed, together with the Singular cardinal hypothesis (SCH), in the proof of Proposition 5.1.5 to show that, over a ring R which is not right perfect, the category Mod-R does not have enough λ-pure-injective objects whenever λ is regular and uncountable. It turns out that this assertion actually characterizes the rings which are not right pure-semisimple, in the presence of the additional set-theoretic principles mentioned. This result will appear during the next year in a joint paper with Manuel Cortés-Izurdiaga.

Deconstructions. Sometimes we want to break large objects into small, more easy to understand (or already better understood) pieces. With infinite-length modules, we typically cannot hope for nice decomposition properties. We necessarily encounter indecomposable modules of arbitrarily large cardinality unless we work over a right pure-semisimple ring. To achieve a better understanding of modules inside a given class $\mathcal{C} \subseteq$ Mod-R, it turns out to be useful to *deconstruct*[1] modules from $\mathcal{C}$ rather than try to decompose them.

The deconstruction is a process of presenting large modules from $\mathcal{C}$ as transfinite extensions by small modules from $\mathcal{C}$. To be more precise, given a (regular) cardinal κ, we would like to show that each module in $\mathcal{C}$ has a filtration with consecutive factors $< \kappa$-presented modules from $\mathcal{C}$ (see Section 3 below for more details). If there exists κ with this property, we say that $\mathcal{C}$ is *(κ-)deconstructible*. Deconstructible classes of modules, especially if they are closed under filtrations, play important role in the approximation theory of modules and relative homological algebra. A standard example of a deconstructible class (that is not decomposable unless we work over a right perfect ring R) is the class $\mathcal{F}_0$ of all flat right R-modules. This class is always $|R|^+$-deconstructible as a result of being closed under pure submodules and pure-epimorphic images.

It follows from Hill Lemma, cf. [11, Theorem 7.10], that objects in a deconstructible class $\mathcal{C} \subseteq$ Mod-R reflect, in the sense described above, if we consider the property P as 'belonging to $\mathcal{C}$', and if 'small' means '$< \kappa$-presented'. This reflection property is, however, just a necessary condition for the deconstructibility of $\mathcal{C}$. To illustrate this point, consider the clas $\mathcal{C}$ consisting of all κ-free abelian groups (or, more generally, κ-projective right modules over a non-right perfect ring) for a regular uncountable κ. Unless κ is weakly

[1] This terminology goes back to the Eklof's paper [8]. He reportedly borrowed the term 'deconstruction' from the famous French philosopher Jacques Derrida.

compact, it is consistent that there exists a κ-free group G of cardinality κ that is not free. On the other hand, all κ-free groups of cardinality $< \kappa$ are trivially free, whence G does not possess a filtration with consecutive factors $< \kappa$-presented modules from $\mathcal{C}$.

To sum it up, in order to prove that a given class $\mathcal{C}$ is deconstructible, the two competing principles—reflection and compactness—must coexist in our setting. In some cases, e.g. when the class $\mathcal{C}$ is *a priori* well-behaved and/or structurally simple, this coexistence comes naturally from the algebraic properties of $\mathcal{C}$. In more interesting cases, such as the case of PGF-modules or Gorenstein flat modules studied in Section 4.4, one has to work considerably harder to successfully deconstruct the class. As a reward though, we developed a deeper understanding of modules in these two classes, cf. Theorems 4.4.9, 4.4.11 and Corollary 4.4.12. According to the significant citation response, these results are of much interest among the people working in Gorenstein homological algebra.

Somewhat eluding our deconstructive endeavours so far is the class $\mathcal{GP}$ of Gorenstein projective modules over a general ring R. Unlike with the κ-free abelian groups, the problems here are caused by the apparent absence of the reflection principle: given a large Gorenstein projective module, it is not clear how to reliably find (enough of) its small submodules which are also Gorenstein projective. Viewed from another perspective, the problem here could stem from the incompactness of the property 'not being Gorenstein projective'. And indeed, we show in Section 3 below that, if we help ourselves by assuming the existence of one strongly compact cardinal λ such that $\lambda > |R|$, we are able to prove that $\mathcal{GP}$ is actually λ-deconstructible. It is only apposite that this additional set-theoretic assumption is not consistent with the extra assumption $(*)$ from Section 2.

2 Enochs' conjecture on covering classes an illustrative example—construction

Precovers (and their dual counterpart preenvelopes) belong to the basic tools in the approximation theory of modules. A class $\mathcal{C} \subseteq \text{Mod-}R$ is called *precovering* if each $M \in \text{Mod-}R$ possesses a *$\mathcal{C}$-precover*, i.e. a homomorphism $f : C \to M$ where $C \in \mathcal{C}$ and such that $\text{Hom}_R(C', f)$ is surjective for each $C' \in \mathcal{C}$. Depending on the class $\mathcal{C}$, the $\mathcal{C}$-precovers are often epimorphisms but, in general, they need not be onto. Rada and Saorín in [15] noticed that a class $\mathcal{C}$ is precovering if and only if its closure under direct summands is precovering. In other words, when studying precovering classes, we can concentrate on the ones closed under direct summands. It is an easy exercise to show that, in this case, they are closed under direct sums, too. As a consequence, we can see that, unless $\mathcal{C}$ contains only the zero module, $\mathcal{C}$-precovers are not unique by any means.

Some precovering classes, though, provide us with minimal versions of precovers called covers. A $\mathcal{C}$-precover $f : C \to M$ satisfying that each $g \in \text{End}_R(C)$ such that $fg = f$ is an automorphism of C is called a *$\mathcal{C}$-cover* of M. A precovering class $\mathcal{C}$ is *covering* if all modules have $\mathcal{C}$-covers. The $\mathcal{C}$-covers, if exist, are unique up to isomorphism. However, they still need not be surjective. And also, even if $\mathcal{C}$-covers exist, they need not be functorial.

Precovering and preenveloping classes represent basic tools in relative homological algebra. They allow us to define (relative) resolutions and coresolutions and also to introduce meaningful notions of (relative homological) dimensions in some cases. Their minimal versions, covers and envelopes, are used, for instance, in the definition of Bass' invariants over commutative noetherian rings, or Xu's dual Bass invariants over Gorenstein rings.

The basic examples of precovering classes are $\mathcal{P}_0$ and $\mathcal{F}_0$, i.e. the class of all projective and flat modules, respectively. In a more general fashion, given a set $\mathcal{S} \subseteq \text{Mod-}R$, the class

$\mathrm{Add}(\mathcal{S})$ of all direct summands of arbitrary direct sums of modules from $\mathcal{S}$ is precovering by the Quillen's small object argument. Of course, in this case $\mathrm{Add}(\mathcal{S}) = \mathrm{Add}(\{\bigoplus_{M\in\mathcal{S}} M\})$ where we usually drop the braces in the latter expression and write simply $\mathrm{Add}(\bigoplus_{M\in\mathcal{S}} M)$.

The rings where $\mathcal{P}_0$ is covering are called right perfect. The class $\mathcal{F}_0$ is covering over any ring R as shown in the famous paper [6] by Bican, El Bashir and Enochs. Among other things, Enochs proved that a precovering class closed under direct limits is covering. The question whether the converse implication holds as well is known as the Enochs' conjecture.

Conjecture 2.1. *(Enochs) Every covering class of modules is closed under direct limits.*

This conjecture has been verified for various special types of classes: from the Bass' famous Theorem P, [3], we know that it holds for $\mathcal{P}_0$ (i.e. the existence of projective covers implies that $\mathcal{P}_0$ is closed under direct limits); recently, Bazzoni and Le Gross in [4] verified the conjecture for the class $\mathcal{P}_1$ of modules of projective dimension at most 1 over a commutative semihereditary ring; in Theorem 3.5.2, we show that it holds for classes $\mathcal{A}$ which appear in a cotorsion pair $(\mathcal{A}, \mathcal{B})$ with $\mathcal{B}$ closed under direct limits. However, the general case still remains open.

In this section, we verify the Enochs' conjecture, under an additional incompactness set-theoretic assumption, for the classes $\mathrm{Add}(M)$ where M is any module. In fact, we will even show a little bit more: the module M has to have a perfect decomposition provided that $\mathrm{Add}(M)$ is covering.

Recall that M *has a perfect decomposition* if every local direct summand in a module from $\mathrm{Add}(M)$ is a direct summand. We say that a submodule $\bigoplus_{i\in I} N_i$ in N is a *local direct summand*[2] in N if the subsum $\bigoplus_{i\in J} N_i$ is a direct summand in N for each finite $J \subseteq I$. This notion was studied, for instance, in [10] and [2]. In particular, it follows from [10, Corollary 2.3] that a module with perfect decomposition has a decomposition in modules with local endomorphism ring.

Example 2.2. Consider the boolean ring $R = \mathcal{P}(\omega)$ where, as usual, the addition is the symmetric difference and the multiplication is the intersection. Then $\bigoplus_{a\in\omega}\{a\}R = \mathrm{Soc}\, R$ is a local direct summand in the regular module R which is not a direct summand in R. So R does not have a perfect decomposition. Of course, this is not surprising: otherwise $\mathrm{Add}(R) = \mathcal{P}_0$ would be closed under direct limits, and R would be semisimple artinian.

We start with a simple, yet very useful proposition which allows us to capitalize on the covering assumption. It goes back to Bazzoni, Positselski and Šťovíček. Recall that a morphism $m : K \to M$ is called *locally split* if $(\forall x \in K)(\exists h \in \mathrm{Hom}_R(M, K))\, h(m(x)) = x$.

Proposition 2.3. *Let $\mathcal{C} \subseteq \text{Mod-}R$ and $f \in \mathrm{Hom}_R(N, L)$ be a surjective $\mathcal{C}$-precover with a locally split kernel. If L has a $\mathcal{C}$-cover, then $L \in \mathcal{C}$ and the epimorphism f splits.*

Proof. See [5, Corollary 4.3] where the authors prove a general version for Ab5 categories. □

Apart from Proposition 2.3, we shall use the following additional set-theoretic assumption.

$$\begin{gathered}\text{There is a proper class of cardinals } \kappa \text{ such that each stationary set } E' \subseteq \kappa^+ \\ \text{has a non-reflecting stationary subset } E.\end{gathered} \tag{$*$}$$

[2] Compare it with the notion of a quasi-split monomorphism from [5].

Recall that a subset $E \subseteq \kappa^+$ is *non-reflecting* if $E \cap \alpha$ is non-stationary in α whenever $\alpha < \kappa^+$ is a limit ordinal. The principle $(*)$ is consistent with ZFC since it holds, for instance, in the constructible universe or, more generally, in absence of $0^\#$. It brings some incompactness to the universe of sets: simply put, it postulates the existence of many 'large' (i.e. stationary) sets which are locally 'small' (non-stationary).

The construction in the proof of the following theorem is inspired by the one from [9, Theorem VII.2.3].

Theorem 2.4. *Assume that* $(*)$ *holds true. Let* $M \in \text{Mod-}R$ *be such that each module in* $\varinjlim \operatorname{Add}(M)$ *has an* $\operatorname{Add}(M)$*-cover. Then* M *has a perfect decomposition. In particular,* $\operatorname{Add}(M)$ *is closed under direct limits.*

Proof. Assume that M is μ-presented, μ infinite, and does not have a perfect decomposition. Then there exists a local direct summand $K = \bigoplus_{i \in I} N_i$ in a module $N \in \operatorname{Add}(M)$ which is not a direct summand there. By the Walker's lemma, we can w.l.o.g. assume that each N_i is μ-presented. Aiming for minimality, we can also assume that each submodule $\bigoplus_{i \in J} N_i$ where $|J| < |I|$ is an actual direct summand in N. If $|I| > \mu$, then the inclusion $\nu : K \hookrightarrow N$ is the directed union of a μ^+-directed system consisting of split inclusions into N. It is thus trivially a locally split morphism. Moreover, since M is μ-presented, any homomorphism from M into N/K factorizes through the μ^+-pure canonical projection $\pi : N \to N/K$. So π is an $\operatorname{Add}(M)$-precover and Proposition 2.3 implies that π splits, a contradiction.

Thus $\aleph_0 \leq |I| \leq \mu$. If $|I|$ is singular, we write I as a disjoint union $\bigcup_{j \in J} I_j$ of subsets I_j where $|J| < |I|$ is regular and $|I_j| < |I|$ for each $j \in J$; then $\bigoplus_{j \in J} \big(\bigoplus_{i \in I_j} N_i \big)$ equals to K, whence it is not a direct summand in N, however, it is still a local direct summand by the minimality of I. Hence we may assume without loss of generality that $I = \lambda \leq \mu$ is an infinite regular cardinal and K is μ-presented. Since N is a direct sum of μ-presented modules (again, by the Walker's lemma), we can also assume that N itself is μ-presented. Using $(*)$, we pick a suitable $\kappa \geq \mu$ and fix a non-reflecting stationary set $E \subseteq \kappa^+$ consisting of ordinal numbers with cofinality λ.

We apply a certain homogenization procedure based on the well-known Eilenberg's trick, i.e. on the fact that, if A is a direct summand in B, then $A \oplus B^{(\nu)} \cong B^{(\nu)}$ whenever ν is infinite. Put $H = (K \oplus N)^{(\kappa)}$. This is a κ-presented module from $\operatorname{Add}(M)$. We define a splitting filtration $\mathfrak{F} = (M_\alpha \mid \alpha \leq \lambda)$ recursively by putting $M_{\alpha+1} = M_\alpha \oplus N_\alpha \oplus H$ and taking unions in the limit steps. Then $M_\lambda = K \oplus H^{(\lambda)} \subseteq N \oplus H^{(\lambda)}$. Notice that we have $N \oplus H^{(\lambda)} \cong H \cong M_{\alpha+1}/M_\alpha$ for each $\alpha < \lambda$, and $M_\alpha \cong H$ for each nonzero $\alpha \leq \lambda$, as a consequence of the Eilenberg's trick. Also observe that

$$M_\alpha \text{ is a direct summand in } N \oplus H^{(\lambda)} \text{ for each } \alpha < \lambda, \qquad (\dagger)$$

whilst M_λ does not split in $N \oplus H^{(\lambda)}$.

We are going to extend the filtration $\mathfrak{F}$ to a filtration $\mathfrak{H} = (M_\alpha \mid \alpha \leq \kappa^+)$. While defining it, we ensure that:

(i) for each $0 < \alpha < \kappa^+$, $M_\alpha \cong H \in \operatorname{Add}(M)$;

(ii) for each $\beta < \alpha < \kappa^+$ with $\beta \notin E$, $M_\alpha = M_\beta \oplus G$ for some $G \subseteq M_\alpha$ isomorphic to H;

(iii) if $\beta \in E$, then M_β does not split in $M_{\beta+1}$.

These conditions are clearly satisfied for the piece $\mathfrak{F}$ of $\mathfrak{H}$ we have constructed so far. Assume now that $\lambda < \alpha < \kappa^+$ and that M_β is already defined for every $\beta < \alpha$. We distinguish the following three cases.

1. α is limit: then we know that $E \cap \alpha$ is not stationary, thus we can find a closed and unbounded subset S of α such that $S \cap E = \varnothing$. We have to define $M_\alpha = \bigcup_{\gamma<\alpha} M_\gamma = \bigcup_{\gamma\in S} M_\gamma$. Since $\mathfrak{G} = (M_\gamma \mid \gamma \in S)$ is a splitting filtration of M_α where the consecutive factors are isomorphic to H, we conclude that $M_\alpha \cong \bigoplus_{\gamma\in S} H \cong H$. This gives us (i). The condition (ii) then follows easily: given any $\beta < \alpha$ with $\beta \notin E$, we first find $\gamma \in S$ such that $\beta < \gamma$; then we use (ii) for $\beta < \gamma$ and the fact that M_γ is a direct summand in M_α from the splitting filtration $\mathfrak{G}$.

2. $\alpha = \beta + 1$ for $\beta \in E$: then we have a splitting filtration $\mathfrak{G} = (M_\gamma \mid \gamma \in S)$ of M_β from the previous limit step. Since $\beta \in E$, we have $\mathrm{cf}(\beta) = \lambda$, and so we can assume w.l.o.g. that S has order type λ, i.e. we can enumerate $\mathfrak{G} = (M_{\gamma_\delta} \mid \delta < \lambda)$. By our construction, there exists an isomorphism $\iota : M_\lambda \to M_\beta$ such that $\iota \restriction M_\delta$ is an isomorphism onto M_{γ_δ} for each $\delta < \lambda$. Recall that $M_\lambda = K \oplus H^{(\lambda)}$. We define M_α using the pushout of ι and the inclusion $K \oplus H^{(\lambda)} \subseteq N \oplus H^{(\lambda)}$ which is non-split since K is not a direct summand in N. Thus M_β is not a direct summand in $M_\alpha \cong N \oplus H^{(\lambda)} \cong H$ either, and we have (i) and (iii) checked. The condition (ii) follows immediately from the property of ι and (†).

3. $\alpha = \beta + 1$ for $\beta \notin E$: in this case, we simply put $M_\alpha = M_\beta \oplus H$ and immediately check that the conditions (i)–(iii) are, indeed, satisfied for α.

Now, $Z = M_{\kappa^+}$ is the directed union of the κ^+-directed system $(M_\alpha \mid \alpha < \kappa^+)$ consisting of modules in $\mathrm{Add}(M)$. Since M is κ-presented, every $g \in \mathrm{Hom}_R(M, Z)$ factorizes through the canonical epimorphism $z : \bigoplus_{\alpha<\kappa^+} M_\alpha \to Z$ yielding that z is an $\mathrm{Add}(M)$-precover. It is well-known that $\mathrm{Ker}(z)$ is locally split (cf. [10, Lemma 2.1]), whence we deduce that z is a split epimorphism and $Z \in \mathrm{Add}(M)$ by Proposition 2.3. By Walker's lemma, we know that Z is a direct sum of κ-presented modules. This gives us a filtration $\mathfrak{H}' = (M'_\alpha \mid \alpha \le \kappa^+)$ of Z such that M'_α is a κ-presented direct summand in Z for each $\alpha < \kappa^+$. It follows that the set $T = \{\alpha < \kappa^+ \mid M_\alpha = M'_\alpha\}$ is closed and unbounded, whence we can pick a $\beta \in T \cap E$. Then M_β splits in Z, and so it splits in $M_{\beta+1}$, too, in contradiction with (iii).

We have proved that M has a perfect decomposition. Finally, it follows from [2, Theorem 1.4] that $\mathrm{Add}(M)$ is closed under direct limits. □

Let us shortly sum up what we have actually done in the proof above. Assuming that $\mathrm{Add}(M)$ is not closed under direct limits, we know (e.g. from [2, Theorem 1.4]) that M does not have a perfect decomposition. Hence there exists a local direct summand K in a module $N \in \mathrm{Add}(M)$ which is not a direct summand. Using the non-split inclusion $K \subseteq N$ and our assumption (∗), we are able to build arbitrarily large modules $Z \in \varinjlim \mathrm{Add}(M)$ which locally look like elements of $\mathrm{Add}(M)$, i.e. each submodule Y of Z with $|Y| < |Z|$ is contained in a submodule of Z belonging to $\mathrm{Add}(M)$, but which do not decompose as a direct sum of modules of cardinality strictly smaller than Z. On the other hand, Proposition 2.3 implies that this is impossible if Z is larger than M and has an $\mathrm{Add}(M)$-cover.

Remark. If M is countably presented, the proof of Theorem 2.4 does not need the extra assumption (∗). The point is that $\lambda = \mu = \aleph_0$ in this case, and $\kappa = \aleph_0$ works since the set $E \subseteq \aleph_1$ consisting of all limit ordinals is stationary and non-reflecting.

As a corollary of Theorem 2.4, we get the following sufficient (and also necessary) condition for a perfect decomposition of a module M.

Corollary 2.5. *Assume (∗). Let M be an R-module. Then M has a perfect decomposition if and only if the endomorphism ring of each $M^{(\kappa)}$, κ a cardinal, is semiregular,* *i.e. the quotient of* $\mathrm{End}_R(M^{(\kappa)})$ *modulo the Jacobson radical is von Neumann regular and idempotents lift modulo the Jacobson radical.*

Proof. [2, Theorem 1.1] shows the only-if part where the assumption $(*)$ is not needed. The if part follows from [14, Corollary 2.3] , [1, Proposition 4.1] and Theorem 2.4. □

Notice that, if R is the boolean ring from Example 2.2, then $\mathrm{End}_R(R^{(\omega)})$ is not semiregular, cf. [14, Theorem 3.9].

If we do not insist on proving that M has a perfect decomposition, we can verify the Enochs' conjecture in ZFC for $\mathrm{Add}(\mathcal{S})$, where $\mathcal{S}$ is any class of countably generated modules, using results from the first paper included in this thesis.

Proposition 2.6. *Let* $\mathcal{S}$ *be a class of countably generated modules such that each module in* $\varinjlim \mathrm{Add}(\mathcal{S})$ *has an* $\mathrm{Add}(\mathcal{S})$*-cover. Then* $\mathrm{Add}(\mathcal{S})$ *is closed under direct limits.*

Proof. It is enough to show that $\mathrm{Add}(\mathcal{S})$ is closed under direct limits of well-ordered systems of the form $\mathcal{C} = (C_\alpha, f_{\beta\alpha} \mid \alpha \leq \beta < \mu)$ where μ is an infinite regular cardinal. Let $C = \varinjlim \mathcal{C}$. Fix an $\mathrm{Add}(\mathcal{S})$-precover $f : B \to C$; it exists since $\mathcal{S}$ is skeletally small. Furthermore, it is an epimorphism, and we claim that it splits.

Let θ be a cardinal number with $\mathrm{cf}(\theta) = \mu$ and such that $\mathcal{C}$ consists of $< \theta$-presented modules. Following the proof of Theorem 1.4.2(1) from the second paragraph on, we obtain a tree module L such that f splits if and only if $\mathrm{Hom}_R(L, f)$ is surjective. To prove that the latter is the case, it is enough to check that $L \in \mathrm{Add}(\mathcal{S})$. This follows from Lemma 1.3.4(3): indeed, the lemma says that L is the directed union of its direct summands $L_S = \sum_{\eta\in S} \mathrm{Im}(\rho\nu_\eta) \in \mathrm{Add}(\mathcal{S})$ where S runs through finite subsets of T. It follows that L is the directed union of the $\aleph_1$-directed system of submodules $L_U = \sum_{\eta\in U} \mathrm{Im}(\rho\nu_\eta) \in \mathrm{Add}(\mathcal{S})$ where U runs through countable subsets of (the uncountable set) T. In particular, if we denote by $h : \bigoplus_{U\subseteq T, |U|=\aleph_0} L_U \to L$ the canonical epimorphism, then $\mathrm{Hom}_R(M, h)$ is surjective for each (countably generated) $M \in \mathcal{S}$. Thus h is an $\mathrm{Add}(\mathcal{S})$-precover which splits by [5, Theorem 4.4], implying that $L \in \mathrm{Add}(\mathcal{S})$. □

With some extra effort, Proposition 2.6 can be generalized to cover classes $\mathcal{S}$ consisting of $\aleph_1$-generated modules. The author finds it plausible that even the version with $\aleph_n$-generated modules ($n \in \omega$) can be proven in ZFC. Note however the following limitation: the assumption that $\mathrm{Add}(\mathcal{S})$ is covering is used here to show that, if a module M is the directed union of a κ^+-directed system of submodules belonging to $\mathrm{Add}(\mathcal{S})$ and $\mathcal{S}$ consists of κ-generated modules, then $M \in \mathrm{Add}(\mathcal{S})$ and the canonical κ^+-pure epimorphism from the direct sum onto the direct limit splits; from this consequent assertion, we deduce that $\mathrm{Add}(\mathcal{S}) = \varinjlim \mathrm{Add}(\mathcal{S})$. This will not work if a strongly compact cardinal $\kappa > |R|$ is present: the consequent assertion holds true for $\mathcal{S} = \{R\}$ and this κ by Theorem 5.3.3 but the class $\mathrm{Add}(\mathcal{S})$ of projective modules is not closed under direct limits unless R is right perfect.

3 Gorenstein projective modules an illustrative example—deconstruction

Two of the long-standing open problems in the Gorenstein homological algebra are 1) the clarification of the relation between Gorenstein flat and Gorenstein projective modules over a general ring; 2) the question whether the class $\mathcal{GP}$ of Gorenstein projective modules provides for (special) precovers. The main conjecture in 1) is that every Gorenstein projective module is Gorenstein flat, in analogy with the corresponding statement about projective and flat modules from the standard homological algebra. Results achieved in the fourth paper included in this thesis imply that the positive answer to 1) would immediately yield the positive answer to 2). In particular, the positive answer to 1) yields that the class of Gorenstein projective modules coincides with the class $\mathcal{PGF}$ studied in the Section 4 of the paper. Theorem 4.4.9 states that $\mathcal{PGF}$ is a special precovering class.

In this section, we give a positive answer to 2) under an additional large-cardinal assumption. To be more precise, we show that the class $\mathcal{SGP}$ of all strongly Gorenstein projective (right R-)modules is λ-deconstructible whenever λ is a strongly compact cardinal number such that $|R| < \lambda$. Before we start, let us recall the relevant definitions.

Let $\mathcal{C} \subseteq \text{Mod-}R$ and σ be an ordinal number. An ascending chain $(M_\alpha \mid \alpha \leq \sigma)$ of modules is called a *$\mathcal{C}$-filtration* of the module M_σ if $M_0 = 0$, $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$ for all limit ordinals $\alpha \leq \sigma$, and $M_{\alpha+1}/M_\alpha$ is isomorphic to an element from $\mathcal{C}$ for each $\alpha < \sigma$. We say that a module M is *$\mathcal{C}$-filtered* if there exists a $\mathcal{C}$-filtration as above, for some σ, and $M = M_\sigma$. If the parameter $\mathcal{C}$ is omitted, i.e. if we speak about filtrations, we do not put any restrictions on the consecutive factors. The notion of filtration naturally extends from modules to short exact sequences.

Given a cardinal κ, we say that a class $\mathcal{C}$ of modules is *κ-deconstructible* if each module $M \in \mathcal{C}$ possesses a $\mathcal{C}$-filtration with $< \kappa$-presented consecutive factors.

We know already from the paper [13] by Holm that the class $\mathcal{GP}$ of Gorenstein projective modules is closed under filtrations, i.e. if a module M is $\mathcal{GP}$-filtered, then $M \in \mathcal{GP}$. Furthermore, each module $M \in \mathcal{GP}$ is a direct summand in a *strongly Gorenstein projective* module, i.e. in a module N which fits into a short exact sequence $\mathcal{S} : 0 \to N \to Q \to N \to 0$ where Q is a projective module and $\text{Ext}^1_R(N, P) = 0$ for each P projective. (In fact, $\mathcal{GP}$ is precisely the class of all direct summands of strongly Gorenstein projective modules.) It is clear that if N is κ-generated, where κ is infinite, then it is enough to test the latter property on κ-generated modules P. Let us denote the class of all strongly Gorenstein projective modules by $\mathcal{SGP}$. Since, for each cardinal λ, there is a representative set of $< \lambda$-presented modules, [11, Theorem 6.11 and Corollary 6.14] show that $\mathcal{GP}$ is a special precovering class, i.e. we have the positive answer to 2), provided that $\mathcal{SGP}$ is λ-deconstructible for some λ.

To show the deconstructibility of $\mathcal{SGP}$, we need the notion of λ-purity. Given an infinite regular cardinal λ, an embedding $M \subseteq N$ of R-modules is called *λ-pure* if each system consisting of R-linear equations with parameters in M and having cardinality $< \lambda$ has a solution in M provided that it has a solution in N. A module F is called *λ-pure-injective* if it is injective relative to all λ-pure embeddings. This is equivalent to saying that each system consisting of R-linear equations with parameters in F has a solution in F provided that each its subsystem of cardinality $< \lambda$ has one.

Notice that λ-pure embeddings, and correlatively λ-pure short exact sequences, are rather frequent. For instance, for the direct limit L of any λ-directed system $\mathcal{M}$ of modules, the canonical pure short exact sequence $0 \to K \to \bigoplus_{M\in\mathcal{M}} M \to L \to 0$ is actually λ-pure.

On the other hand, only a little is known about (non-pure-injective) λ-pure-injective modules for uncountable λ. A few results in this direction appear in the fifth paper. In what follows, we shall use that many modules suddenly become λ-pure-injective if λ is a sufficiently large cardinal. However, for small cardinals such as $\lambda = \aleph_1$, it can happen that all λ-pure-injective modules are, in fact, ($\aleph_0$-)pure-injective, whilst there still exist pure embeddings which are not $\aleph_1$-pure, cf. Example 5.1.7.

The following lemma serves as an important building block on the way to the deconstruction of $\mathcal{SGP}$. In the sequel, we shall freely use that, if $\mu \geq |R|$ for an infinite cardinal μ, then a module M is μ-presented if and only if M is μ-generated, if and only if $|M| \leq \mu$.

Lemma 3.1. *Assume that λ is an infinite regular cardinal and μ is an infinite cardinal such that $|R| \leq \mu = \mu^{<\lambda}$. Let $\mathcal{D} : 0 \to N \overset{\subseteq}{\to} \bigoplus_{\delta \in \Delta} P_\delta \overset{\pi}{\to} N \to 0$ be a short exact sequence of R-modules where P_δ is μ-generated for each $\delta \in \Delta$. Let X be a subset of N of cardinality $\leq \mu$. Then there is a subobject $\mathcal{F} : 0 \to A \overset{\subseteq}{\to} \bigoplus_{\delta \in E} P_\delta \to A \to 0$ of $\mathcal{D}$ such that $|E| \leq \mu$, $X \subseteq A$ and A is λ-pure in N.*

Proof. We construct $\mathcal{F}$ as the union of an increasing chain $(\mathcal{F}_\alpha : 0 \to A_\alpha \overset{\subseteq}{\to} \bigoplus_{\delta \in E_\alpha} P_\delta \to A_\alpha \to 0 \mid 0 < \alpha < \lambda)$ of subobjects of $\mathcal{D}$. We start by letting A_0 be the submodule of N generated by X and $E_0 = \varnothing$.

Let $0 < \alpha < \lambda$ and assume that A_β and E_β is defined for all $\beta < \alpha$ in such a way that $|E_\beta| \leq \mu$ holds; then $|A_\beta| \leq \mu$ holds as well. Suppose first that $\alpha = \gamma + 1$ for some ordinal γ, i.e. α is non-limit. We consider all the systems $\mathcal{S}$ of R-linear equations with parameters in A_γ such that $|\mathcal{S}| < \lambda$. Since $|R| \leq \mu = \mu^{<\lambda}$, there is at most μ such systems. Thus we can find $A'_\gamma \supseteq A_\gamma$ of cardinality $\leq \mu$ such that A'_γ contains a solution of each of these systems $\mathcal{S}$ which has solution in N.

Next, we find $D_0 \subseteq \Delta$ containing E_γ, of cardinality $\leq \mu$ and such that $A'_\gamma \subseteq \operatorname{Im} \pi \restriction \bigoplus_{\delta \in D_0} P_\delta =: A^0_\gamma$. Further, we find $D_1 \subseteq \Delta$ containing D_0, of cardinality $\leq \mu$ and such that $A^0_\gamma \subseteq \bigoplus_{\delta \in D_1} P_\delta$ and $A^1_\gamma := \operatorname{Im} \pi \restriction \bigoplus_{\delta \in D_1} P_\delta \supseteq N \cap \bigoplus_{\delta \in D_0} P_\delta$. We continue by finding D_2 containing D_1, of cardinality $\leq \mu$ and such that $A^1_\gamma \subseteq \bigoplus_{\delta \in D_2} P_\delta$ and $A^2_\gamma := \operatorname{Im} \pi \restriction \bigoplus_{\delta \in D_2} P_\delta \supseteq N \cap \bigoplus_{\delta \in D_1} P_\delta$, and so on. Finally, we put $E_\alpha = \bigcup_{n<\omega} D_n$ and $A_\alpha = \bigcup_{n<\omega} A^n_\gamma$, thus obtaining a short exact sequence $\mathcal{F}_\alpha : 0 \to A_\alpha \overset{\subseteq}{\to} \bigoplus_{\delta \in E_\alpha} P_\delta \to A_\alpha \to 0$ which is a subobject of $\mathcal{D}$.

For limit $\alpha < \lambda$, we define $\mathcal{F}_\alpha$ as the directed union of all the $\mathcal{F}_\beta$ where $\beta < \alpha$. Finally, we put $\mathcal{F} = \bigcup_{\alpha < \lambda} \mathcal{F}_\alpha$.

It is clear that $\mathcal{F}$ has the form $0 \to A \overset{\subseteq}{\to} \bigoplus_{\delta \in E} P_\delta \to A \to 0$ where $|E| \leq \mu$. It remains to check that A is λ-pure in N. So let $\mathcal{S}$ be a system of cardinality $< \lambda$ consisting of R-linear equations with parameters in A and assume that $\mathcal{S}$ has a solution in N. Since λ is a regular cardinal, there exists $\alpha < \lambda$ such that A_α contains all the parameters from $\mathcal{S}$. By the construction, $\mathcal{S}$ has a solution in $A_{\alpha+1} \subseteq A$. It follows that A is λ-pure in N. □

The proposition below provides the first half of the deconstruction of $\mathcal{SGP}$. In a sense, it says that $\mathcal{SGP}$ is almost λ^+-deconstructible if λ is a strongly compact cardinal greater than $|R|$. Recall that an uncountable cardinal λ is called *strongly compact* if each λ-complete filter on any set can be extended to a λ-complete ultrafilter. Such λ is necessarily a measurable cardinal, in particular $\lambda^{<\lambda} = \lambda$ holds. More importantly, it follows from [16, Proposition 2.1] that all modules of cardinality $< \lambda$ are λ-pure-injective.[3]

Proposition 3.2. *Let $\lambda > |R|$ be a strongly compact cardinal and $\mathcal{E} : 0 \to M \overset{\subseteq}{\to} Q \to M \to 0$ a short exact sequence with Q projective. Assume that $\operatorname{Ext}^1_R(M, P) = 0$ for each ($< \lambda$-generated) projective module P.*

Then there is a filtration $(\mathcal{E}_\alpha : 0 \to M_\alpha \to Q_\alpha \to M_\alpha \to 0 \mid \alpha \leq \sigma)$ of $\mathcal{E}$ such that, for each $\alpha < \sigma$, Q_α is a (projective) direct summand in Q, $M_{\alpha+1}/M_\alpha$ is λ-presented and $\operatorname{Ext}^1_R(M_{\alpha+1}/M_\alpha, P) = 0$ for each $< \lambda$-generated projective module P.

[3] Compare it with the well-known result that finite modules are pure-injective (in fact, they are even Σ-pure-injective).

Proof. For the sake of non-triviality, assume that $\kappa = |M| > \lambda$. Using the Kaplansky Theorem, we can fix a decomposition $Q = \bigoplus_{\gamma\in\Gamma} P_\gamma$ where each P_γ is countably generated. Let us also fix a generating set $\{m_\alpha \mid \alpha < \kappa\}$ of M. We recursively build a filtration $(\mathcal{E}_\alpha : 0 \to M_\alpha \to Q_\alpha \to M_\alpha \to 0 \mid \alpha \leq \kappa)$ of $\mathcal{E}$ with the following properties for each $\alpha < \kappa$:

(i) $Q_\alpha = \bigoplus_{\gamma\in\Gamma_\alpha} P_\gamma$ for some $\Gamma_\alpha \subseteq \Gamma$;

(ii) $M_{\alpha+1}/M_\alpha$ is λ-presented and $\mathrm{Ext}^1_R(M_{\alpha+1}/M_\alpha, P) = 0$ for each $< \lambda$-generated projective module P;

(iii) $\mathrm{Ext}^1_R(M/M_\alpha, P) = 0$ for each $< \lambda$-generated projective module P;

(iv) $m_\alpha \in M_{\alpha+1}$.

We start with $\mathcal{E}_0$ consisting of trivial modules. Assume that $\mathcal{E}_\beta$ is already constructed for all $\beta < \alpha$. If α is a limit ordinal, we simply put $\mathcal{E}_\alpha = \bigcup_{\beta<\alpha} \mathcal{E}_\beta$. Using (ii), we get $\mathrm{Ext}^1_R(M_\alpha, P) = 0$ for each $< \lambda$-generated projective module P by the Eklof Lemma. It follows that each homomorphism from M_α to a $< \lambda$-generated projective module can be extended to Q_α and subsequently to Q, in particular to M. Using the assumption on M, we infer that (iii) holds, whence (i)–(iv) are satisfied for α.

Now let $\alpha = \delta + 1$ for some δ. Consider the short exact sequence $\mathcal{D} : 0 \to M/M_\delta \to \bigoplus_{\gamma\in\Gamma\setminus\Gamma_\delta} P_\gamma \to M/M_\delta \to 0$ and put $N = M/M_\delta$. Using Lemma 3.1 for $\mu = \lambda$, we find a subobject $\mathcal{F} : 0 \to A \overset{\subseteq}{\to} \bigoplus_{\gamma\in E} P_\gamma \to A \to 0$ of $\mathcal{D}$ such that $|E| \leq \lambda$, $m_\delta + M_\delta \in A$ and A is λ-pure in N.

Since λ is strongly compact, it follows from [16, Proposition 2.1] that all modules of cardinality $< \lambda$ are λ-pure-injective. Using (iii) for δ, we thus infer that $\mathrm{Ext}^1_R(N/A, P) = 0$ for each $< \lambda$-generated projective module P. The 3×3 lemma gives us also that $\mathrm{Ext}^1_R(A, P) = 0$ for each $< \lambda$-generated projective module P.

We define the short exact sequence $\mathcal{E}_\alpha$ as the preimage of $\mathcal{F}$ in the canonical projection from $\mathcal{E}$ onto $\mathcal{D}$. The conditions (i)–(iv) then immediately follow: indeed, we have $\Gamma_\alpha = \Gamma_\delta \cup E$ and $M/M_\alpha \cong N/A$. Finally, $M_\alpha/M_\delta \cong A$ is a λ-presented module since $|E| \leq \lambda$. □

We have already mentioned that a strongly compact cardinal λ is measurable. A well-known result by Scott says that there exists a normal ultrafilter on λ, i.e. a λ-complete uniform ultrafilter $\mathcal{U}$ satisfying that each regressive function $r : \lambda \to \lambda$ is constant on a set from $\mathcal{U}$. This property will allow us to finally carry out the deconstruction of $\mathcal{SGP}$.

Theorem 3.3. *Let R be a ring and $\lambda > |R|$ a strongly compact cardinal. Then the class $\mathcal{SGP}$ of strongly Gorenstein projective modules is λ-deconstructible. As a consequence, the class $\mathcal{GP}$ is λ-deconstructible as well.*

Proof. Using Proposition 3.2, we can find, for each short exact sequence $\mathcal{E} : 0 \to M \to Q \to M \to 0$ with Q projective and $M \in \mathcal{SGP}$, a filtration of $\mathcal{E}$ consisting of short exact sequences $\mathcal{E}_\alpha : 0 \to M_\alpha \to Q_\alpha \to M_\alpha \to 0$ where, for each α, Q_α is a direct summand in Q, $M_{\alpha+1}/M_\alpha$ is λ-presented and $\mathrm{Ext}^1_R(M_{\alpha+1}/M_\alpha, P) = 0$ for each $< \lambda$-generated projective module P.

We shall deconstruct the consecutive factors $\mathcal{E}_{\alpha+1}/\mathcal{E}_\alpha$ one step further. So let

$$\mathcal{B} : 0 \to B \overset{\subseteq}{\to} C \overset{f}{\to} B \to 0$$

be a short exact sequence where B is λ-presented, C is projective and $\mathrm{Ext}^1_R(B,P)=0$ for any $<\lambda$-generated projective module P. Recall that if B is actually $<\lambda$-generated, then $B\in\mathcal{SGP}$: indeed, each homomorphism $g:B\to P'$ where P' is any projective module has its image contained in a $<\lambda$-generated projective submodule P of P', and as such, g extends to a homomorphism from C into $P\subseteq P'$ by the assumption on $\mathcal{B}$.

In the interesting case when $|B|=|C|=\lambda$, we start with any filtration

$$\mathfrak{D}=(\mathcal{D}_\alpha : 0\to B_\alpha \overset{\subseteq}{\to} C_\alpha \overset{f_\alpha}{\to} B_\alpha \to 0 \mid \alpha<\lambda)$$

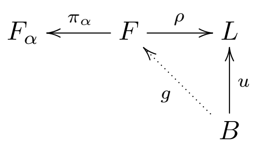
of $\mathcal{B}$ where C_α is $<\lambda$-generated projective direct summand in C and $f_\alpha = f\restriction C_\alpha$ for each $\alpha<\lambda$. It is easy to obtain such a filtration: we can either work as in the proof of Lemma 3.1 disregarding the part of the second paragraph which treats λ-purity, or we use a more elegant approach and consider a filtration $\mathfrak{F}$ built using $<\lambda$-generated elementary submodels of the expanded structure $(C;f)$. The fact that they are elementary ensures that the sentence $(\forall x)\,(f(x)=0\leftrightarrow(\exists y)\,x=f(y))$ remains valid in the submodels. The R-module reducts of such submodels will not necessarily be projective direct summands in C, however, since C has a filtration consisting of $<\lambda$-generated projective direct summands and λ is regular, we can find a subfiltration of $\mathfrak{F}$ with the desired property.

Let $\mathcal{S}=\{\alpha<\lambda\mid B_\alpha\in\mathcal{SGP}\}$. This set is closed in λ: for any two elements $\alpha<\beta$ in $\mathcal{S}$, we have $B_\beta/B_\alpha\in\mathcal{SGP}$ by the 3×3 lemma; the rest follows by the Eklof Lemma. It remains to show that $\mathcal{S}$ is unbounded. Suppose, for the sake of contradiction, that it is not. We can without loss of generality assume that $\mathcal{S}=\{0\}$. The rest of the proof is based on the approach from the proof of [17, Lemma 2.3]. The utilization of a normal ultrafilter is novel though.

Fix, for each nonzero $\alpha<\lambda$, a homomorphism $g_\alpha : B_\alpha\to F_\alpha$ where F_α is $<\lambda$-generated projective module, and such that g_α cannot be extended to a homomorphism $C_\alpha\to F_\alpha$. Since C_α is a direct summand in C and $\mathrm{Hom}_R(\mathcal{B},F_\alpha)$ is right exact, this is equivalent to saying that g_α cannot be extended to a homomorphism $g:B\to F_\alpha$. Also, we can clearly w.l.o.g. assume that F_α is actually a free module of rank $<\lambda$. For technical reasons, put $F_0=\{0\}$ and let $g_0:B_0\to F_0$ be the zero map.

Put $F=\prod_{\alpha<\lambda}F_\alpha$. Let $L=F/\mathcal{U}$ where $\mathcal{U}$ is a normal ultrafilter on λ. We define a homomorphism $u:B\to L$ as follows: for each $b\in B$, we pick a $\beta<\lambda$ such that $b\in B_\beta$ and set $u(b)=[(a_\alpha)_{\alpha<\lambda}]_{\mathcal{U}}$ where $a_\alpha=g_\alpha(b)$ if $\beta\le\alpha$ and $a_\alpha=0$ otherwise. The correctness of the definition of u stems from the uniformity of $\mathcal{U}$.

Since $|R|<\lambda$, L is a free module by [9, Theorem II.3.8]. It follows that the canonical projection $\rho:F\to L$ splits, whence there exists $g\in\mathrm{Hom}_R(B,F)$ such that $\rho g=u$. For each $\alpha<\lambda$, let $\pi_\alpha:F\to F_\alpha$ denote the canonical projection.



By the assumption on the morphisms g_α, we have $\pi_\alpha g\restriction B_\alpha\neq g_\alpha$ for each nonzero $\alpha<\lambda$. This allows us to define a regressive function $r:\lambda\to\lambda$ by the assignment $r(\beta+1)=\beta$ and, for α limit, $r(\alpha)<\alpha$ be arbitrary such that $\pi_\alpha g\restriction B_{r(\alpha)}\neq g_\alpha\restriction B_{r(\alpha)}$. The normality of $\mathcal{U}$ provides us with a $\gamma<\lambda$ such that $r^{-1}\{\gamma\}\in\mathcal{U}$.

On the other hand, since B_γ is $< \lambda$-generated and $\mathcal{U}$ is uniform and λ-complete, we see that $\{\alpha < \lambda \mid \gamma + 1 < \alpha \ \& \ \pi_\alpha g \restriction B_\gamma = g_\alpha \restriction B_\gamma\} \in \mathcal{U}$ (recall that $\rho g = u$). This is a contradiction since $r^{-1}\{\gamma\}$ is disjoint with this latter set from $\mathcal{U}$.

We have obtained a desired subfiltration of $\mathfrak{D}$ witnessing that $B \in \mathcal{SGP}$. Consequently, each $M \in \mathcal{SGP}$ is filtered by $< \lambda$-presented modules from $\mathcal{SGP}$; otherwise put: $\mathcal{SGP}$ is λ-deconstructible. Moreover, [11, Corollary 6.14] implies that a representative set of $< \lambda$-presented modules from $\mathcal{SGP}$ generates the cotorsion pair $(\mathcal{GP}, \mathcal{GP}^\perp)$. This allows us to use [11, Theorem 7.13] to conclude that $\mathcal{GP}$ is λ-deconstructible as well. $\square$

BIBLIOGRAPHY

[1] L. Angeleri Hügel, *Covers and envelopes via endoproperties of modules*, Proc. London Math. Soc. **86** (2003), 649–665.

[2] L. Angeleri Hügel, M. Saorín, *Modules with perfect decompositions*, Math. Scand. **98** (2006), 19–43.

[3] H. Bass, *Finitistic dimension and homological generalization of semi-primary rings* Trans. Amer. Math. Soc. **95** (1960), 466–488.

[4] S. Bazzoni, G. Le Gros, *$\mathcal{P}_1(R)$-covers over commutative rings*, accepted to Rend. Sem. Mat. Univ. Padova.

[5] S. Bazzoni, L. Positselski, J. Šťovíček, *Projective covers of flat contramodules*, preprint, arXiv:1911.11720 [math.RA].

[6] L. Bican, R. El Bashir, E. Enochs, *All modules have flat covers*, Bull. London Math. Soc. **33** (2001), 385–390.

[7] V. Drinfeld, *Infinite–dimensional vector bundles in algebraic geometry: an introduction*, in The Unity of Mathematics, Birkhäuser, Boston 2006, 263–304.

[8] P. C. Eklof, *Shelah's singular compactness theorem*, Publ. Math. **52** (2008), 3–18.

[9] P. C. Eklof, A. H. Mekler, *Almost Free Modules*, Revised ed., North–Holland, New York 2002.

[10] J. L. Gómez Pardo, P. A. Guil Asensio, *Big Direct Sums of Copies of a Module Have Well Behaved Indecomposable Decompositions*, J. Algebra **232** (2000), 86–93.

[11] R. Göbel, J. Trlifaj, *Approximations and Endomorphism Algebras of Modules*, de Gruyter Expositions in Mathematics **41**, 2nd revised and extended edition, Berlin-Boston 2012.

[12] D. Herbera, J. Trlifaj, *Almost free modules and Mittag-Leffler conditions*, Advances in Math. **229** (2012) 3436–3467.

[13] H. Holm, *Gorenstein homological dimensions*, J. Pure Appl. Alg. **189** (2004), 167–193.

[14] W. K. Nicholson, *Semiregular modules and rings*, Canad. J. Math. **28** (1976), 1105–1120.

[15] J. Rada, M. Saorín, *Rings characterized by (pre)envelopes and (pre)covers of their modules*, Comm. Algebra **26** (1998), no. 3, 899–912.

[16] J. Šaroch, *On the nontrivial solvability of systems of homogeneous linear equations over $\mathbb{Z}$ in ZFC*, Comment. Math. Univ. Carolin. **61** (2020), no. 2, 155–164.

[17] J. Šaroch, J. Šťovíček, *The countable Telescope Conjecture for module categories*, Adv. Math. **219** (2008), 1002–1036.