Point Simpliciality in Choquet's Theory

Doctoral thesis

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Hereby I declare that I have written this thesis on my own and that I cited all used sources of information. I agree with public availability and lending of the thesis.

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M.B.

Contents

Preface

This thesis consists of five research papers which present new results in Choquet's theory of integral representation. Namely,

- M. Bačák: *Representation of Concave Functions by Radon Probability Measures,* WDS'06 Proceedings of Contributed Papers: Part I – Mathematics and Computer Sciences (eds. J. Šafránková and J. Pavlů), Prague, Matfyzpress, pp. 106 – 111.
- M. Bačák: *Point simpliciality in the Choquet theory,* submitted to Extracta Mathematicae.
- M. Bačák: *Minimal measures and nonsimplicial function cones,* submitted to Commentationes Mathematicae Universitatis Carolinae.
- M. Bačák: *Unique decomposition property and extreme points,* to appear in Rocky Mountain Journal of Mathematics.
- M. Bačák, J. Spurný: *Complementability of spaces of affine continuous functions on simplices,* to appear in Bulletin of the Belgian Mathematical Society.

In the first chapter, we employ Choquet's theorem to obtain Radon probability measures that represent given compact convex sets of L_p functions, and we investigate uniqueness of those representing measures. We get an example of a compact convex set which is not a simplex, that is a set where maximal representing measures are not uniquely determined. That provides a natural motivation for a theory developed in Chapter 2. We define the point of simpliciality as a point which has a unique maximal representing measure. The set of all such points is called the set of simpliciality. This is a central term of this work. We describe topological, algebraic, and measure-theoretical properties of the set of simpliciality.

A generalization of point simpliciality for function cones is presented in Chapter 3.

Forth chapter presents a solution to an open problem whether the Unique Decomposition Property of a function space is necessary for obtaining a full characterization of extreme points of the unit ball in the dual space of a quotient of the function space.

The last chapter includes a joint work with Jiří Spurný. We show that complementability of a space of affine continuous functions on a simplex does not depend on topological properties of the set of extreme points. Namely, we have conctructed two simplices X_1 and X_2 with homeomorphic sets of extreme points such that the space of affine continuous functions on X_1 is complemented in $C(X_1)$ whereas the space of affine continuous functions on X_2 is not complemented.

CHAPTER 1

Representation of Concave Functions by Radon Probability Measures

ABSTRACT. The aim of this paper is to represent given sets of concave functions by Radon probability measures. We define sets K_p (for $p \in [1, \infty]$) of concave functions from the spaces $L^p((0,1))$ having some additional properties. These sets of functions are convex and compact so that Choquet's theorem can be used to obtain existence of representing measures. Uniqueness is examined on a case-by-case basis.

1.1. Preliminaries

At first, let us introduce some notation. The symbol χ_A means the characteristic function of a set A, the symbol $\operatorname{ext}(A)$ denotes the set of all extreme points of a convex set A. The set of all positive Radon measures on a compact space K is denoted $\mathcal{M}^+(K)$, the set of all Radon probability measures $\mathcal{M}^1(K)$. By ε_x we denote the Dirac measure at the point $x \in K$. The symbol $C(P)$ stands for the set of all real-valued continuous functions on a topological space P.

We present some basic facts on the Choquet theory. Details can be found in [**10**] or [**24**].

THEOREM 1.1.1. *(Hervé) If C is a nonempty metrizable compact convex set in a locally convex space E, then the set of extreme points* $\operatorname{ext}(C)$ *is Borel.*

THEOREM 1.1.2. *(Choquet) If C is a nonempty metrizable compact convex set in a locally convex space E, then for each $x \in C$ there exists a Radon probability measure $\mu \in \mathcal{M}^1(C)$ such that*

$$f(x) = \int_C f(y)\ \mathrm{d}\mu(y) \tag{1}$$

for all $f \in E^*$ and $\mu(\operatorname{ext}(C)) = 1$.

A Radon probability measure μ satisfying (1) is called a *representing measure.* In general, it is not determined uniquely.

We say that a metrizable compact convex set K in a locally convex space is a *simplex* if for each point $x \in K$ there exists only one representing measure $\mu \in \mathcal{M}^1(K)$ such that $\mu(\operatorname{ext}(K)) = 1$.

Let (X, τ) be a topological vector space. We say that a net $(x_\gamma)_{\gamma\in\Gamma}$ is a *Cauchy net* if for every neighborhood U of 0 there exists $\gamma_0 \in \Gamma$ such that $x_\alpha - x_\beta \in U$ for all α, $\beta \succeq \gamma_0$. A subset M of a topological vector space is called *complete* if every Cauchy net is convergent in M. A subset M of a topological vector space (X, τ) is *precompact* if for every neighborhood U of 0 there exists a finite set $F \subset X$ such that $M \subset U + F$. Analogously as for a metric space we have the following theorem, see [**16**, Theorem 7.6].

THEOREM 1.1.3. *A subset M of a topological vector space (X, τ) is compact if and only if it is complete and precompact.*

1.2. Definition of sets K_p

We consider the space $C((0,1))$ of all continuous real functions on the interval $(0,1)$ equipped with the topology of the pointwise convergence. Then $C((0,1))$ is a locally convex space and its topology is generated by the family of seminorms $\{p_x : x \in (0,1)\}$ defined by

$$p_x(f) = |f(x)| \qquad \text{for all } f \in C((0,1)).$$

Neighborhoods of 0 are indexed by $\varepsilon > 0$ and by finitely many points $x_1, \dots, x_n$ from $(0,1)$:

$$U_0(\varepsilon, x_1, \dots, x_n) = \{f \in C((0,1)) : |f(x_1)| < \varepsilon, \dots, |f(x_n)| < \varepsilon\}.$$

Let us define the following sets of functions

$$K_p = \{f : (0,1) \to [0,\infty),\ f \text{ concave}, \|f\|_{L^p} \leq 1\}, \qquad p \in [1,\infty].$$

Proposition 1.2.1. *The sets K_p for $p \in [1,\infty]$ are convex.*

Proof. We want to show that $\alpha f + (1-\alpha)g \in K_p$, provided f, $g \in K_p$ and $\alpha \in (0,1)$. Since the function $\alpha f + (1-\alpha)g$ is obviously nonnegative and concave, it remains to show that its norm is less or equal 1. It follows from the triangle inequality that $\|\alpha f + (1-\alpha)g\| \leq \|\alpha f\| + \|(1-\alpha)g\| = \alpha\|f\| + (1-\alpha)\|g\| \leq 1$. We conclude that the function $\alpha f + (1-\alpha)g$ belongs to K_p. □

We note that the sets K_p are metrizable, since it suffices to take seminorms p_x for x rational. Consequently, we can deal with sequences instead of nets.

Proposition 1.2.2. *The sets K_1 and K_∞ are compact in $C((0,1))$ equipped with the topology of the pointwise convergence.*

Proof. Since both cases are similar we will proceed to prove compactness of K_1 and K_∞ simultaneously. According to Theorem 1.1.3, we want to prove that these sets are complete and precompact. To show completeness, consider a Cauchy sequence $\{f_n\}$ of K_1 (of K_∞, respectively). Then for every $x \in (0,1)$ and for every $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that $|f_n(x) - f_m(x)| < \varepsilon$, provided n, $m > n_0$. For each $x \in (0,1)$ one gets a Cauchy real-valued sequence. Using completeness of $\mathbb{R}$ we can define the function f by

$$f(x) = \lim_{n\to\infty} f_n(x), \qquad x \in (0,1).$$

It is easy to see that $f \in K_1$ ($f \in K_\infty$, respectively). (The norm inequality in the case of K_1 follows by Fatou's lemma). It remains to show that K_1 (K_∞, respectively) is a precompact set, that is, for each neighborhood $U_0(\varepsilon, x_1, \dots, x_n) = \{f \in C((0,1)) : |f(x_1)| < \varepsilon, \dots, |f(x_n)| < \varepsilon\}$ to find a finite set $\mathcal{G} \subset C((0,1))$ such that for every $h \in K_1$ ($h \in K_\infty$, respectively) there exists $g \in \mathcal{G}$ such that $h \in g + U_0(\varepsilon, x_1, \dots, x_n)$. Let us notice that obviously $h(0,1) \subset [0,2]$ whenever $h \in K_1 \cup K_\infty$. Fix $m \in \mathbb{N}$ such that $m > 1/\varepsilon$ and denote $x_0 = 0$, $x_{n+1} = 1$. Define $\mathcal{F}$ as the family of restrictions to $(0,1)$ of all piecewise linear continuous functions $g : [0,1] \to [0,2]$ such that $g(x_j) \in \{k/m : k = 0, \dots, 2m\}$. Clearly, $\mathcal{G}$ has the desired property. We conclude that the sets K_1 and K_∞ are compact. □

We shall see in section 1.4 that compactness of K_p, for $p \in (1,\infty)$, is useless.

In the following sections, we shall look for Radon probability measures that represent elements of K_p, where $p \in [1,\infty]$. At first, we have to determine sets of extreme points of K_p. Then we use Choquet's theorem which provides representing measures supported by extreme points. Using images of these measures, we obtain explicite integral formulas for elements of K_p, where the integration domain is a unit interval. Finally, we examine uniqueness of representing measures.

1.3. Representation of functions from K_1

Extreme points of K_1. For every $y \in [0,1]$ define the following functions g_y on the interval $(0,1)$ by

$$g_y(x) = \begin{cases} 2\frac{x}{y}, & 0 < x \le y < 1, \\ 2\frac{1-x}{1-y}, & 0 < y \le x < 1, \end{cases}$$

for $y \in (0,1)$ and

$$g_0(x) = 2(1-x), \quad g_1(x) = 2x, \quad x \in (0,1).$$

We claim that

$$\operatorname{ext}(K_1) = \{g_y : \ y \in [0,1]\} \cup \{0\}. \tag{2}$$

It is easy to show that the zero function and every function g_y are extreme points of K_1. Indeed, suppose that $g_y = \frac{1}{2}(f+h)$ for some f, $h \in K_1$. Then from the fact that f, h are concave we immediately get $f = h = g_y$. Now, let us prove the opposite inclusion in (2). Take $f \in \operatorname{ext}(K_1)$. Then clearly either $f \equiv 0$ or $\|f\|_{L^1} = 1$. Suppose that $\|f\|_{L^1} = 1$ and that $\sup f = \beta < 2$. Denote $c \in [0,1]$ a point where the least upper bound is attained. Let us define functions f_0, f_1 on $(0,1)$ by:

$$f_0(x) = \begin{cases} f(x) - \beta x, & x \le c, \\ \beta(1-x), & c \le x, \end{cases}$$

and

$$f_1(x) = \begin{cases} \beta x, & x \le c, \\ f(x) - \beta(1-x), & c \le x. \end{cases}$$

It is straightforward to verify that these functions are concave. Further, denote

$$\lambda_0 = \int_0^1 f_0(x)\,\mathrm{d}x, \quad \lambda_1 = \int_0^1 f_1(x)\,\mathrm{d}x.$$

Then

$$\lambda_0 + \lambda_1 = 1, \qquad \lambda_0 > 0,\ \lambda_1 > 0,$$

since $f = f_0 + f_1$ and $\|f\|_{L^1} = 1$. Now we have f as a convex combination

$$f = \lambda_0 \frac{f_0}{\lambda_0} + \lambda_1 \frac{f_1}{\lambda_1},$$

where f_0/λ_0, $f_1/\lambda_1 \in K_1$. Since $f \in \operatorname{ext}(K_1)$, we get

$$f = \frac{f_0}{\lambda_0} = \frac{f_1}{\lambda_1}.$$

In particular,

$$f(x) = \begin{cases} \lambda_1^{-1}\beta x, & x \in [0,c], \\ \lambda_0^{-1}\beta(1-x), & x \in [c,1], \end{cases}$$

which is a contradiction to the assumption that $\beta < 2$ and simultaneously $\|f\|_{L^1} = 1$. We conclude that $\sup f = 2$. To finish the proof we notice that the only functions in K_1 having least upper bound 2 are exactly the functions g_y, $y \in [0,1]$.

Representation of K_1. Applying Choquet's theorem 1.1.2 to the set K_1 we get that for each $f \in K_1$ there exists $\hat{\hat{\mu}} \in \mathcal{M}^1(K_1)$ such that

$$\varphi(f) = \int_{\operatorname{ext}(K_1)} \varphi(g) \, \mathrm{d}\hat{\hat{\mu}}(g) \qquad \text{for all } \varphi \in C((0,1))^*,$$

and $\hat{\hat{\mu}}(\operatorname{ext}(K_1)) = 1$. In particular, for the evaluation functional $\varphi(f) = f(x)$, $x \in (0,1)$ we have

$$f(x) = \int_{\operatorname{ext}(K_1)} g(x) \, \mathrm{d}\hat{\hat{\mu}}(g),$$

or equivalently, since $\hat{\hat{\mu}} = \hat{\mu} + c \cdot \varepsilon_0$ for some $c \in [0,1]$ and $\hat{\mu} \in \mathcal{M}^+(K_1)$, we have

$$f(x) = \int_{\operatorname{ext}(K_1) \setminus \{0\}} g(x) \, \mathrm{d}\hat{\mu}(g) + c \cdot 0.$$

Furthermore, let us define the homeomorphism $\Phi : [0,1] \to \operatorname{ext}(K_1) \setminus \{0\} : y \mapsto g_y$. Then the image μ of the measure $\hat{\mu}$ defined as $\mu = \Phi^{-1}\hat{\mu}$ is a positive Radon measure on the interval $[0,1]$. Hence

$$\begin{aligned} f(x) &= \int_{[0,1]} g_y(x) \, \mathrm{d}\mu(y), && x \in (0,1) \\ f(x) &= \int_{[0,x]} 2\frac{1-x}{1-y} \, \mathrm{d}\mu(y) + \int_{(x,1]} 2\frac{x}{y} \, \mathrm{d}\mu(y), && x \in (0,1) \end{aligned} \tag{3}$$

and $\mu([0,1]) = 1 - c$.

Choquet's theorem does not assert uniqueness of representing measures. This question is answered in the next paragraph.

Simpliciality of K_1. We want to prove that K_1 is a simplex. Choose arbitrary function $f \in K_1$. We begin with the claim that for each measure μ satisfying (3) we have $h\mu_0 = T_f''$, where $\mu_0 = \mu \restriction (0,1)$, T_f is a distribution defined by

$$T_f(\varphi) = \int_{(0,1)} f(x) \, \varphi(x) \, \mathrm{d}x \qquad \text{for all } \varphi \in C_c^\infty((0,1)),$$

and h is a density defined as $h(x) = \frac{-2}{x(1-x)}$, $x \in (0,1)$. In particular, we want to prove uniqueness of the measure μ_0. Obviously $\mu = \alpha\varepsilon_0 + \mu_0 + \beta\varepsilon_1$ for some nonnegative real numbers α, β. Here $\varepsilon_0, \varepsilon_1$ stand for Dirac measures from $\mathcal{M}^1([0,1])$. Let $\varphi \in C_c^\infty((0,1))$. By definition of the distributional derivative and by integration by parts

$$\begin{aligned} T_f''(\varphi) &= \int_{(0,1)} f(x) \, \varphi''(x) \, \mathrm{d}x = \int_{(0,1)} \int_{[0,1]} g_y(x) \, \mathrm{d}\mu(y) \, \varphi''(x) \, \mathrm{d}x = \\ &= \int_{(0,1)} \alpha 2(1-x) \, \varphi''(x) \, \mathrm{d}x + \int_{(0,1)} \int_{(0,1)} g_y(x) \, \mathrm{d}\mu_0(y) \, \varphi''(x) \, \mathrm{d}x + \\ &+ \int_{(0,1)} \beta 2x \, \varphi''(x) \, \mathrm{d}x = \int_{(0,1)} \int_{(0,1)} g_y(x) \, \mathrm{d}\mu_0(y) \, \varphi''(x) \, \mathrm{d}x = \\ &= \int_{(0,1)} \int_{(0,x]} 2\frac{1-x}{1-y} \, \varphi''(x) \, \mathrm{d}\mu_0(y) \, \mathrm{d}x + \int_{(0,1)} \int_{(x,1)} 2\frac{x}{y} \, \varphi''(x) \, \mathrm{d}\mu_0(y) \, \mathrm{d}x. \end{aligned}$$

Using Fubini's theorem and continuity of Lebesgue measure we get

$$\int_{(0,1)}\int_{(0,x]} 2\frac{1-x}{1-y}\,\varphi''(x)\ \mathrm{d}\mu_0(y)\ \mathrm{d}x + \int_{(0,1)}\int_{(x,1)} 2\frac{x}{y}\,\varphi''(x)\ \mathrm{d}\mu_0(y)\ \mathrm{d}x =$$

$$= \int_{(0,1)}\int_{(y,1)} 2\frac{1-x}{1-y}\,\varphi''(x)\ \mathrm{d}x\ \mathrm{d}\mu_0(y) + \int_{(0,1)}\int_{(0,y)} 2\frac{x}{y}\,\varphi''(x)\ \mathrm{d}x\ \mathrm{d}\mu_0(y) =$$

$$= -\int_{(0,1)} \frac{2}{y(1-y)}\,\varphi(y)\ \mathrm{d}\mu_0(y).$$

The last equality follows by integration by parts. We have proven that the measure $h\mu_0$ equals to the second distributional derivative T_f'', thus it is determined uniquely.

Finally, we will prove uniqueness of the measure μ. It is enough to show uniqueness of numbers α, β. If there is another pair of nonnegative real numbers α_1, β_1, from (3) we get

$$f(x) = \alpha 2(1-x) + \int_{(0,1)} g_y(x)\ \mathrm{d}\mu_0(y) + \beta 2x$$

$$f(x) = \alpha_1 2(1-x) + \int_{(0,1)} g_y(x)\ \mathrm{d}\mu_0(y) + \beta_1 2x.$$

These equalities yield

$$\alpha 2(1-x) + \beta 2x - \alpha_1 2(1-x) - \beta_1 2x = 0$$

for $x \in (0,1)$. Thus $\alpha = \alpha_1$ and $\beta = \beta_1$. We get uniqueness of numbers α, β. The proof of simpliciality of K_1 is finished.

1.4. Representation of functions from K_p, for $p \in (1,\infty)$

Extreme points of K_p for $p \in (1,\infty)$. In this case, we have

$$\operatorname{ext}(K_p) = \{0\} \cup \{f \in K_p :\ \|f\|_{L^p} = 1\}.$$

It is clear that every nonzero extreme point has norm 1. On the other hand, every $f \in K_p$, $\|f\|_{L_p} = 1$, is an extreme point because L^p spaces are strictly convex for $p \in (1,\infty)$. Consequently, this result makes sets K_p for $p \in (1,\infty)$ of no interest from the representation theorem point of view.

1.5. Representation of functions from K_∞

Extreme points of K_∞. We will show that

$$\operatorname{ext}(K_\infty) = \{0\} \cup \{g_{a,b} :\ 0 \le a \le b \le 1\}, \tag{4}$$

where

$$g_{a,b}(x) = \begin{cases} \frac{x}{a}, & 0 < x < a, \\ \frac{1-x}{1-b}, & b < x < 1, \\ 1, & a < x < b. \end{cases}$$

for $a \in (0,1)$, $b \in (0,1)$ and

$$g_{0,b}(x) = \begin{cases} 1, & 0 < x < b, \\ \frac{1-x}{1-b}, & b < x < 1, \end{cases}$$

for $b \in [0, 1)$ and

$$g_{a,1}(x) = \begin{cases} \frac{x}{a}, & 0 < x < a, \\ 1, & a < x < 1, \end{cases}$$

for $a \in (0, 1]$ and

$$g_{0,1}(x) \equiv 1.$$

It can be easily checked that $\mathrm{ext}(K_\infty) \supset \{0\} \cup \{g_{a,b} :\ 0 \le a \le b \le 1\}$. Indeed, suppose that $g_{a,b} = \frac{1}{2}(f + h)$ for some f, $h \in K_\infty$. Then from the fact that f and h are concave we immediately get $f = h = g_{a,b}$. Now, let $f \in \mathrm{ext}(K_\infty)$. Then clearly either $f \equiv 0$ or $\sup f = 1$.
1) If there is $c \in (0, 1)$ such that $f(c) = 1$, denote

$$a = \inf\{x \in (0, 1) :\ f(x) = 1\}, \qquad b = \sup\{x \in (0, 1) :\ f(x) = 1\}.$$

If $a = 0$ and $b = 1$, then $f = g_{0,1}$. Otherwise, without loss of generality, we can assume that $b < 1$. Since f is decreasing on $(b, 1)$, the limit $\lim_{x\to 1-} f(x)$ exists.
1a) If $\lim_{x\to 1-} f(x) > 0$, we can find $\delta > 0$ such that the functions

$$f_1(x) = \begin{cases} f(x), & x \in (0, b], \\ f(x)(1 - \delta) + \delta, & x \in [b, 1), \end{cases}$$

and

$$f_2(x) = \begin{cases} f(x), & x \in [0, b], \\ f(x)(1 + \delta) - \delta, & x \in [b, 1], \end{cases}$$

belong to K_∞ and $f = \frac{1}{2}\left(f_1 + f_2\right)$. Then $f \notin \mathrm{ext}(K_\infty)$.
1b) If $\lim_{x\to 1-} f(x) = 0$, and $f \neq \frac{1-x}{1-b}$ on $(b, 1)$ then there exist s, $t \in (b, 1)$ such that

$$f'_-(s) < f'_-(t). \tag{5}$$

Let $\lambda = f(b) - f(s) + f'_-(s)(1 - s)$. Then $\lambda \in (0, 1)$, because of (5). Define the function f_1 by

$$f_1(x) = \begin{cases} f(x), & x \in (0, b], \\ \lambda^{-1}(f(x) - f(s) + f'_-(s)(1 - s)), & x \in [b, s], \\ \lambda^{-1} f'_-(s)(1 - x), & x \in [s, 1), \end{cases}$$

and function f_2 by

$$f_2(x) = \begin{cases} f(x), & x \in (0, b], \\ 1, & x \in [b, s], \\ (1 - \lambda)^{-1}(f(x) - \lambda f_1(x)), & x \in [s, 1). \end{cases}$$

It is straightforward to verify that these functions are concave and thus f_1, $f_2 \in K_\infty$. Since $f = \lambda f_1 + (1 - \lambda) f_2$, we get $f \notin \mathrm{ext}(K_\infty)$.
1c) If $\lim_{x\to 1-} f(x) = 0$, and $f = \frac{1-x}{1-b}$ on $(b, 1)$, then we can symmetrically prove that $f \notin \mathrm{ext}(K_\infty)$, provided $f \neq \frac{x}{a}$ on $(0, a)$.
2) If $f(c) < 1$ for all $c \in (0, 1)$, then $\lim_{x\to 0+} f(x) = 1$ or $\lim_{x\to 1-} f(x) = 1$. Without loss of generality, assume that $\lim_{x\to 0+} f(x) = 1$. Following the steps 1a) and 1b) with setting $b = 0$, we can prove that $f = g_{0,0}$. Hence (4) holds.

Representation of K_∞. From Choquet's theorem 1.1.2 the following representation for the set K_∞ can be obtained. For each $f \in K_\infty$ there exists $\hat{\hat{\mu}} \in \mathcal{M}^1(K_\infty)$ such that

$$\varphi(f) = \int_{\mathrm{ext}(K_\infty)} \varphi(g)\ \mathrm{d}\hat{\hat{\mu}}(g) \qquad \text{for all } \varphi \in C((0,1))^*,$$

and $\hat{\hat{\mu}}(\mathrm{ext}(K_\infty)) = 1$. In particular, for the evaluation functional $\varphi(f) = f(x)$, $x \in (0,1)$ we have

$$f(x) = \int_{\mathrm{ext}(K_\infty)} g(x)\ \mathrm{d}\hat{\hat{\mu}}(g),$$

or equivalently, since $\hat{\hat{\mu}} = \hat{\mu} + c \cdot \varepsilon_0$ for some $c \in [0,1]$ and $\hat{\mu} \in \mathcal{M}^+(K_\infty)$, we have

$$f(x) = \int_{\mathrm{ext}(K_\infty)\setminus\{0\}} g(x)\ \mathrm{d}\hat{\mu}(g) + c \cdot 0.$$

Furthermore, let us define the homeomorphism $\Phi : \{0 \le a \le b \le 1\} \to \mathrm{ext}(K_\infty) \setminus \{0\} :\ (a,b) \mapsto g_{a,b}$. Then the image μ of the measure $\hat{\mu}$ defined as $\mu = \Phi^{-1}\hat{\mu}$ is a positive Radon measure on the set $\{0 \le a \le b \le 1\}$. Hence

$$f(x) = \int\limits_{\{0\le a\le b\le 1\}} g_{a,b}(x)\ \mathrm{d}\mu(a,b),$$

and $\mu(\{0 \le a \le b \le 1\}) = 1 - c$.

Simpliciality of K_∞. As we will see, the set K_∞ is not a simplex. Consider the function $f \in K_\infty$ defined as

$$f(x) = \begin{cases} \frac{4}{3}\,x, & x \in (0, \frac{1}{4}], \\ \frac{1}{3}, & x \in [\frac{1}{4}, \frac{3}{4}], \\ \frac{4}{3}\,(1-x), & x \in [\frac{3}{4}, 1). \end{cases}$$

This function can be decomposed in two different ways:

$$f = \frac{1}{4}\left(g_{\frac{1}{4},\frac{1}{4}} + g_{\frac{3}{4},\frac{3}{4}}\right) + \frac{1}{2}\,0,$$

and

$$f = \frac{1}{3}\,g_{\frac{1}{4},\frac{3}{4}} + \frac{2}{3}\,0.$$

Then for every continuous linear functional φ on $C((0,1))$ holds

$$\varphi(f) = \frac{1}{4}\left(\varphi(g_{\frac{1}{4},\frac{1}{4}}) + \varphi(g_{\frac{3}{4},\frac{3}{4}})\right) + \frac{1}{2}\,\varphi(0),$$

and

$$\varphi(f) = \frac{1}{3}\,\varphi(g_{\frac{1}{4},\frac{3}{4}}) + \frac{2}{3}\,\varphi(0).$$

In other words, the measures $\frac{1}{4}\,\varepsilon_{g_{\frac{1}{4},\frac{1}{4}}} + \frac{1}{4}\,\varepsilon_{g_{\frac{3}{4},\frac{3}{4}}} + \frac{1}{2}\,\varepsilon_0$ and $\frac{1}{3}\,\varepsilon_{g_{\frac{1}{4},\frac{3}{4}}} + \frac{2}{3}\,\varepsilon_0$ are two different measures representing the function f which are supported by $\mathrm{ext}(K_\infty)$. We conclude that K_∞ is not a simplex.

1.6. Endnotes

(i) We saw that the set K_∞ is not a simplex. In this case it is possible to study simpliciality as a point phenomenon, that is to ask *at which points* there exist unique representing measures supported by extreme points. It is not known to us, which points of K_∞, have unique representing measures supported by extreme points.

(ii) The Krein-Milman theorem can be used instead of the Choquet theorem, because the sets of extreme points are closed.

CHAPTER 2

Point simpliciality in the Choquet theory

ABSTRACT. Let $\mathcal{H}$ be a function space on a compact space K. If $\mathcal{H}$ is not simplicial, we can ask *at which points* of K there exist unique maximal representing measures. We shall call the set of such points the *set of simpliciality*. The aim of this paper is to examine topological, algebraic and measure-theoretic properties of the set of simpliciality. We shall also define and investigate sets of points enjoying other simplicial-like properties.

2.1. Introduction

The conception of the infinite-dimensional simplex in locally convex spaces was introduced by G. Choquet, see [**11**]. Later, it was generalized by means of measure theory for general (nonconvex) compact spaces as the simplicial function space. Several authors (e.g. C.-H. Chu [**12**], J. Köhn [**17**], Å. Lima [**18**]) have studied simpliciality restricted on faces generated by a given point. In Köhn's paper [**17**], there is an implicit definition of point simpliciality and some equivalent conditions for it. We should also mention an abstract framework due to S. Simons, [**28**].

In this paper, we define a point of simpliciality, this enables us to consider simpliciality as a point phenomenon. Then we define the set of simpliciality as the set of all points of simpliciality. Moreover, we use a more general setup of function spaces, than that used in previous work (cited above) concerned with simpliciality of faces. That was limited to compact convex subsets of locally convex spaces.

The main results of this paper (Theorems 2.4.1, 2.4.5, 2.5.6, 2.6.2, 2.6.4) describe properties of the set of simpliciality (and Bauer simpliciality). We also define "generalized" simpliciality in the set of Radon probability measures, this provides a characterization of measures carried by the set of simpliciality (Theorem 2.5.1) and enables us to prove measure extremality of the set of simpliciality (Theorem 2.5.6).

2.2. Preliminaries

At the beginning we introduce some notation and basic facts concerning Choquet's theory, for details, see e.g. [**1**], [**10**], [**20**] or [**24**]. All topological spaces in this paper are supposed to be Hausdorff. Let K be a compact space. The symbol $C(K)$ stands for the Banach space of all real continuous functions on K equipped with the sup-norm. A subspace $\mathcal{H}$ of $C(K)$ is called a *function space* on K provided it separates points of K and contains all constant functions. Notice that the function space $\mathcal{H}$ does not have to be closed. Let us denote the set of all Radon measures, positive Radon measures and probability Radon measures as $\mathcal{M}(K), \mathcal{M}^+(K)$ and $\mathcal{M}^1(K)$, respectively. These sets of measures are equipped with the weak* topology. We say that a measure $\mu \in \mathcal{M}^1(K)$ *represents* a point $x \in K$ if $f(x) = \mu f$ for all $f \in \mathcal{H}$. If a measure μ represents a point $x \in K$, we also say that x is the *barycenter* of μ, and we denote $x = r_\mu$. Since $\mathcal{H}$ separates points of K, the barycenter of μ, if it exits, is determined uniquely. The set of all measures representing a point $x \in K$

will be denoted by $\mathcal{M}_x(\mathcal{H})$. Further, we define an equivalence on $\mathcal{M}^1(K)$ by

$$\mu \sim \nu \quad \text{if} \quad \mu - \nu \in \mathcal{H}^\perp,$$

where $\mathcal{H}^\perp$ stands for the anihilator of $\mathcal{H}$ defined as $\mathcal{H}^\perp = \{\mu \in \mathcal{M}(K) : \mu f = 0 \text{ for all } f \in \mathcal{H}\}$.

We shall denote the set of all measures $\nu \in \mathcal{M}^1(K)$ which are equivalent with a given $\mu \in \mathcal{M}^1(K)$ as $\mathcal{M}_\mu(\mathcal{H})$. Clearly, $\mathcal{M}_{\varepsilon_x}(\mathcal{H}) = \mathcal{M}_x(\mathcal{H})$, where ε_x denotes the Dirac measure at a point $x \in K$. The symbol $\mathrm{Ch}_\mathcal{H}(K)$ stands for the *Choquet boundary,* which is, by definition, the set of all $x \in K$ having only one representing measure ε_x.

We present two examples of function spaces.

EXAMPLE 2.2.1. *Let X be a compact convex subset of a locally convex space. The set of all continuous affine functions $A(X)$ is a function space on X. The Choquet boundary corresponds with the set of extreme points of K. We will refer to this setting as to the "convex case". As we will see later (definition of the state space), every compact space (with a function space defined on it) can be considered to be embedded into a certain compact convex set (which depends on the function space). In this sence, the "convex case" is the most important example.*

EXAMPLE 2.2.2. *Let U be a bounded open subset of the Euclidean space $\mathbb{R}^m$. Then $H(U)$, the family of all continuous functions on $\overline{U}$ which are harmonic on U, is a function space on the compact set $\overline{U}$. The Choquet boundary of $H(U)$ corresponds with the set U_{reg} of regular points of U (see* [**23**, Theorem, p. 625]*).*

Define the *state space* of a function space $\mathcal{H}$ as

$$S(\mathcal{H}) = \{\varphi \in \mathcal{H}^* : \ 0 \le \varphi, \ \|\varphi\| = 1\}.$$

It is well known that $\mathcal{H}^*$ is isometrically isomorphic to the quotient space

$$\mathcal{M}(K)/\mathcal{H}^\perp$$

and that

$$S(\mathcal{H}) = \pi(\mathcal{M}^1(K)).$$

Here π stands for the quotient mapping from $\mathcal{M}(K)$ to $\mathcal{H}^*$. Furthermore, define homeomorphic embedding $\phi : K \to S(\mathcal{H}) : x \mapsto \phi_x$ by $\phi_x = \pi(\varepsilon_x)$.

A Borel bounded function f on K is said to be *$\mathcal{H}$–affine* if $f(x) = \mu f$, for all $x \in K$ and $\mu \in \mathcal{M}_x(\mathcal{H})$. Let us denote the set of all $\mathcal{H}$-affine functions on K as $\mathcal{A}(\mathcal{H})$ and the set of all continuous $\mathcal{H}$–affine functions on K as $\mathcal{A}^c(\mathcal{H})$. It is a closed subspace of $C(K)$ and it contains $\mathcal{H}$. In the "covex case", we have $\mathcal{A}^c(\mathcal{H}) = \mathcal{H} = A(X)$, hence $\mathcal{A}^c(\mathcal{H})$ is the set of all continuous affine functions.

A Borel bounded function f on K is said to be *$\mathcal{H}$–convex* if $f(x) \le \mu f$, for all $x \in K$ and $\mu \in \mathcal{M}_x(\mathcal{H})$. Denote the set of all $\mathcal{H}$–convex functions on K as $\mathcal{K}(\mathcal{H})$ and the set of all continuous $\mathcal{H}$-convex functions on K as $\mathcal{K}^c(\mathcal{H})$. The cone of all continuous $\mathcal{H}$–convex functions induces so-called *Choquet ordering* $\preceq$ on $\mathcal{M}^+(K)$ by

$$\mu \preceq \nu \quad \text{if} \quad \mu f \le \nu f \qquad \text{for all } \ f \in \mathcal{K}^c(\mathcal{H}).$$

LEMMA 2.2.3. *Let f be a semicontinuous $\mathcal{H}$–convex function and $\mu, \nu \in \mathcal{M}^+(K)$. If $\mu \preceq \nu$, then $\mu f \le \nu f$.*

Proof. Can be found in [**21**, Lemma 2.7] □

For each measure $\mu \in \mathcal{M}^1(K)$, there exists a maximal (in the Choquet ordering) measure $\nu \in \mathcal{M}^1(K)$ such that $\mu \preceq \nu$. The set of all maximal measures representing a point $x \in K$ will be denoted by $\mathcal{M}_x^{\max}(\mathcal{H})$. If K is metrizable, then the Choquet boundary is a Borel measurable set and a measure μ is maximal if and only if $\mu(\mathrm{Ch}_{\mathcal{H}}(K)) = 1$.

A function space $\mathcal{H}$ on a compact space K is called *simplicial* if, for every $x \in K$, there exists a unique maximal measure $\mu \in \mathcal{M}_x(\mathcal{H})$. Moreover, if $\mathrm{Ch}_{\mathcal{H}}(K)$ is closed, then $\mathcal{H}$ is called a *Bauer simplicial space.* We shortly say that a compact convex set X is a *Choquet simplex,* if $A(X)$ is simplicial.

For a bounded function f on K, its upper envelope f^* is defined as

$$f^* = \inf\{h : \ h \geq f, \ h \in \mathcal{H}\},$$

and its lower envelope f_* as

$$f_* = \sup\{h : \ h \leq f, \ h \in \mathcal{H}\}.$$

An upper envelope is upper semicontinuous and $-f^* \in \mathcal{K}(\mathcal{H})$.

The following lemma is called the Mokobodzki maximality test ([**21**, Theorem 2.8]).

LEMMA 2.2.4. *A measure $\mu \in \mathcal{M}^1(K)$ is maximal if and only if $\mu f = \mu f^*$ for all $f \in \mathcal{K}^c(\mathcal{H})$.*

Let X be a compact convex subset of a locally convex space, we say that a subset $F \subset X$ is extremal if for any $x, y \in X$, $t \in (0,1)$ is $x, y \in F$, provided $tx + (1-t)y \in F$. If F is extremal and convex, we say that F is a *face.* Let $x \in X$, we can define the *smallest face* face(x) *containing* x as the intersection of all faces containting x.

The following Proposition is due to J. Köhn, [**17**, Proposition 2].

PROPOSITION 2.2.5. *Let X be a compact convex subset of a locally convex space, $x \in X$. Then there exists a unique maximal measure representing the point x if and only if, for every $y \in$ face(x), there exists a unique maximal measure representing the point y.*

A Borel set $B \subset K$ is called *measure convex* if every measure $\mu \in \mathcal{M}^1(K)$ such that $\mu(B) = 1$ has its barycenter in B, provided it has the barycenter. A Borel set $B \subset K$ is called *measure extremal* if for each $x \in B$ and for each $\mu \in \mathcal{M}_x(\mathcal{H})$, it is $\mu(B) = 1$. Define the following functionals

$$\mathrm{Q}^\mu f = \inf\{\mu h : \ h \geq f, \ h \in \mathcal{H}\}, \qquad \text{and} \qquad \mathrm{P}^\mu f = \mu f^*,$$

where f is a bounded Borel function and $\mu \in \mathcal{M}^1(K)$. If $\mu = \varepsilon_x$ for some $x \in K$, then

$$\mathrm{Q}^\mu f = \mathrm{P}^\mu f = f^*(x).$$

Similarly, define functionals $\mathrm{Q}_\mu f = \sup\{\mu h : \ h \leq f, \ h \in \mathcal{H}\}$, and $\mathrm{P}_\mu f = \mu f_*$.

LEMMA 2.2.6. *For each $f \in C(K)$ and $\mu \in \mathcal{M}^1(K)$, we have*

$$\mathrm{Q}^\mu f = \sup\{\nu f : \ \nu \in \mathcal{M}_\mu(\mathcal{H})\}$$

and the supremum is attained.

Proof. See [**21**, Proposition 2.3]. ∎

COROLLARY 2.2.7. (Bauer) *For each $f \in C(K)$ and $x \in K$, we have*

$$f^*(x) = \sup\{\nu f : \nu \in \mathcal{M}_x(\mathcal{H})\}$$

and the supremum is attained.

LEMMA 2.2.8. *For each $f \in C(K)$ and each $\mu \in \mathcal{M}^1(K)$ we have*

$$\mathrm{P}^\mu f = \sup\{\nu f : \nu \in \mathcal{M}^1(K),\ \mu \preceq \nu\}$$

and the supremum is attained.

Proof. This lemma can be easily proved replacing Q^μ by P^μ in the proof of Proposition 2.3 in [**21**]. □

LEMMA 2.2.9. *If a sequence $(f_n) \in C(K)$ converges uniformly on K to a function $f \in C(K)$, then the sequence (f_n^*) converges to f^* uniformly on K.*

Proof. Follows from the inequality $|f_n^* - f_m^*| \leq \|f_n - f_m\|$. □

2.3. Examples

Let us start with the pivotal definition of this paper and present some examples.

DEFINITION 2.3.1. *Let $\mathcal{H}$ be a function space on a compact space K. We say that $x \in K$ is a point of simpliciality if there exists only one maximal measure representing the point x. We denote the set of all points of simpliciality by $\mathcal{P}_\mathrm{S}$, and we call it the set of simpliciality. The complement $K \setminus \mathcal{P}_\mathrm{S}$ is called the set of nonsimpliciality.*

REMARK 2.3.2. *Clearly $\mathrm{Ch}_\mathcal{H}(K) \subset \mathcal{P}_\mathrm{S}$; and $\mathcal{P}_\mathrm{S} = K$ if and only if $\mathcal{H}$ is simplicial.*

EXAMPLE 2.3.3. *Consider a square in $\mathbb{R}^2$. It is a compact convex set which is not a simplex. The set $\mathcal{P}_\mathrm{S}$ consists of its edges.*

EXAMPLE 2.3.4. *Let us introduce "McDonald's nonsimplex" (Example 1.9 in [**22**]). Choose $\mu \in \mathcal{M}([0,1])$ such that the positive and negative variations μ^+, μ^- are in $\mathcal{M}^1([0,1])$ and $\mathrm{spt}(\mu^+) = \mathrm{spt}(\mu^-) = [0,1]$. Define*

$$\mathcal{H} = \mathrm{Ker}\ \mu = \{f \in C([0,1]) : \mu f = 0\}.$$

Obviously,

$$\mathcal{H}^\perp = \{\nu \in \mathcal{M}([0,1]) : \nu = \alpha\mu \text{ for some } \alpha \in \mathbb{R}\}.$$

We claim that $\mathcal{H}$ is a function space on $[0,1]$. Indeed, it contains constant functions since $\mu([0,1]) = 0$, and it separates points of $[0,1]$: choose $x, y \in [0,1]$ and suppose $f(x) = f(y)$ for every $f \in \mathcal{H}$. Then $\varepsilon_x - \varepsilon_y \in \mathcal{H}^\perp$, hence

$$\varepsilon_x - \varepsilon_y = \alpha\mu = \alpha\mu^+ - \alpha\mu^-,$$

for some $\alpha \in \mathbb{R}$. Since $\mathrm{spt}(\mu^+) = \mathrm{spt}(\mu^-) = [0,1]$, we get $\alpha = 0$, and thus $x = y$. Further, we will show that $\mathrm{Ch}_\mathcal{H}([0,1]) = [0,1]$. Choose $x \in [0,1]$ and $\nu \in \mathcal{M}_x(\mathcal{H})$. Then $\nu - \varepsilon_x \in \mathcal{H}^\perp$, and thus $\nu - \varepsilon_x = \alpha\mu$, for some $\alpha \in \mathbb{R}$. Similarly as above, we have $\nu = \varepsilon_x$, and thus $\mathrm{Ch}_\mathcal{H}([0,1]) = [0,1]$. McDonald's nonsimplex is defined as the state space $S(\mathcal{H})$ of $\mathcal{H}$. We show that it is not a simplex. Since $s := \pi(\mu^+) = \pi(\mu^-) \in S(\mathcal{H})$ and

$$r_{\phi\mu^+} = \pi(\mu^+) = \pi(\mu^-) = r_{\phi\mu^-},$$

we see that the point s has two different representing measures $\phi\mu^+, \phi\mu^-$ supported by $\mathrm{Ch}_\mathcal{H}([0,1])$, therefore it is maximal.

Now, we want to find the set of simpliciality $\mathcal{P}_{\mathrm{S}}$. *Suppose that* Λ_1 *and* Λ_2 *are maximal probability measures on* $S(\mathcal{H})$ *representing a point* $x \in S(\mathcal{H})$. *Then there exist maximal measures* $\lambda_1, \lambda_2 \in \mathcal{M}^1([0,1])$ *such that* $\Lambda_1 = \phi\lambda_1$ *and* $\Lambda_2 = \phi\lambda_2$. *Then* $\lambda_1 - \lambda_2 \in \mathcal{H}^\perp$, *and thus*

$$\lambda_1 - \lambda_2 = \alpha\mu = \alpha\mu^+ - \alpha\mu^-,$$

for some $\alpha \in \mathbb{R}$. *Without loss of generality assume that* $\alpha \geq 0$. *Since* $\lambda_1 \geq \alpha\mu^+$, *we get* $\alpha \leq 1$. *Hence*

$$\lambda_1 = \alpha\mu^+ + (1-\alpha)\gamma,$$

and

$$\lambda_2 = \alpha\mu^- + (1-\alpha)\gamma,$$

where γ *is a measure from* $\mathcal{M}^1([0,1])$. *If* $\alpha > 0$, *then we have two different maximal measures* Λ_1, Λ_2 *representing the point*

$$x = \alpha\phi\mu^+ + (1-\alpha)\phi\gamma = \alpha\phi\mu^- + (1-\alpha)\phi\gamma$$

and thus $x \notin \mathcal{P}_{\mathrm{S}}$. *We conclude that*

$$S(\mathcal{H}) \setminus \mathcal{P}_{\mathrm{S}} = \{\alpha\phi\mu^+ + (1-\alpha)\phi\gamma :\ \alpha \in (0,1],\ \gamma \in \mathcal{M}^1([0,1])\}.$$

REMARK 2.3.5. *The previous Example 2.3.4 can be generalized in the following sense. Let* $M \subset \{\mu \in \mathcal{M}([0,1]) :\ \mu^+, \mu^- \in \mathcal{M}^1([0,1]),\ \operatorname{spt}(\mu^+) = \operatorname{spt}(\mu^-) = [0,1]\}$ *and define*

$$\mathcal{H} = M^\perp = \{f \in C([0,1]) :\ \mu f = 0 \text{ for all } \mu \in M\}.$$

In this setting, we were not able to find $\mathcal{P}_{\mathrm{S}}$.

EXAMPLE 2.3.6. *The last example deals with convex functions on* $[0,1]$. *A similar set of functions was investigated from the point of view of the noncompact Choquet theory by R.M. Rakestraw ([**25**]). Let us define the following set of functions*

$$Z = \{f : [0,1] \to [0,\infty),\ f \text{ convex},\ f(0) + f(1) = 1\}.$$

This set is convex and compact in $\{f : [0,1] \to \mathbb{R},\ \text{bounded, continous on } (0,1)\}$ *with respect to the topology of pointwise convergence.*

The set of extreme points is

$$\operatorname{ext}(Z) = \{g_y^1,\ g_y^2 :\ y \in [0,1]\},$$

where

$$g_y^1(x) = \begin{cases} 0, & 0 \leq x \leq y < 1, \\ \frac{x-y}{1-y}, & 0 \leq y < x \leq 1, \end{cases}$$

for $y \in [0,1)$,

$$g_y^2(x) = \begin{cases} 1 - \frac{x}{y}, & 0 \leq x < y \leq 1, \\ 0, & 0 < y \leq x \leq 1, \end{cases}$$

for $y \in (0,1]$, *and*

$$g_1^1(x) = \chi_{\{1\}}(x), \qquad g_0^2(x) = \chi_{\{0\}}(x).$$

A function $f \in Z$ *belongs to* $\mathcal{P}_{\mathrm{S}}$ *if and only if it satisfies at least one of these conditions:*

- f *is affine on* $(0,1)$,

- $\inf_{x\in[0,1]} f(x) = 0$.

Verification of these facts is rather technical but elementary.

In the following three sections, we will present the main results concerning the set of simpliciality and the set of Bauer simpliciality. The letter "K" stands for a compact space, whereas we use the letter "X" instead, for a convex compact subset of a locally convex space.

2.4. The set of simpliciality

In the previous section, we introduced examples of nonsimplicial function spaces for which we were able to find the sets of simpliciality. Now we will investigate some general properties of such sets.

THEOREM 2.4.1. *Let X be a compact convex subset of a locally convex space. Then the set of simpliciality $\mathcal{P}_\mathrm{S}$ is extremal and, consequently, the set of nonsimpliciality $X \setminus \mathcal{P}_\mathrm{S}$ is convex.*

Proof. According to Proposition 2.2.5, if the set $\mathcal{P}_\mathrm{S}$ contains a point $x \in X$, it also contains face(x). Hence

$$\mathcal{P}_\mathrm{S} = \bigcup_{x\in\mathcal{P}_\mathrm{S}} \{x\} \subset \bigcup_{x\in\mathcal{P}_\mathrm{S}} \operatorname{face}(x) \subset \mathcal{P}_\mathrm{S}.$$

Then

$$\mathcal{P}_\mathrm{S} = \bigcup_{x\in\mathcal{P}_\mathrm{S}} \operatorname{face}(x).$$

It is straightforward to verify that a union of faces is an extremal set. $\square$

REMARK 2.4.2. *In the "convex case", the set $\mathcal{P}_\mathrm{S}$ is the complementary set to $X \setminus \mathcal{P}_\mathrm{S}$ and we have a disjoin union $X \setminus \mathcal{P}_\mathrm{S} \cup (X \setminus \mathcal{P}_\mathrm{S})' = X$. We recall that, for a subset A of a compact convex set X, the complementary set A' is defined as the union of all faces of X disjoint with A.*

REMARK 2.4.3. *In general, if a compact space K does not have an algebraic structure, we can ask whether the set $\mathcal{P}_\mathrm{S}$ is measure extremal, or equivalently, whether the set $K \setminus \mathcal{P}_\mathrm{S}$ is measure convex. But we do not know yet that these sets are Borel measurable. We recall that in the "convex case" a measure convex set is convex, but a convex set is not necessarily measure convex, and similarly, a measure extremal set is extremal, but an extremal set is not necessarily measure extremal. For counterexamples, see* [**13**]*, and* [**21**, Examples 4.3]*.*

Now, we present some characterizations of point simpliciality for function spaces which will be useful further on. "Global version" of Proposition 2.4.4 for the "convex case" can be found in [**24**, Theorem, p. 56].

PROPOSITION 2.4.4. *Let $\mathcal{H}$ be a function space on a compact space K and M a dense (with respect to the norm topology) subset of $\mathcal{K}^c(\mathcal{H})$. Let $x \in K$. The following assertions are equivalent:*

(i) $x \in \mathcal{P}_\mathrm{S}$,
(ii) $f^*(x) = \mu f^*$ *for all* $f \in M$ *and* $\mu \in \mathcal{M}_x(\mathcal{H})$,
(iii) $f^*(x) = \mu f$ *for all* $f \in M$ *and* $\mu \in \mathcal{M}_x^{\max}(\mathcal{H})$,
(iv) $(f+g)^*(x) = f^*(x) + g^*(x)$ *for all* f, $g \in M$.

Proof. If $M = \mathcal{K}^c(\mathcal{H})$, the proof is similar to the proof of the "global version" in the "convex case", [**24**, Theorem, p. 56]. For an arbitrary dense $M \subset \mathcal{K}^c(\mathcal{H})$, the proof follows from Lemma 2.2.9 and the Lebesgue Dominated Theorem. □

THEOREM 2.4.5. *Let $\mathcal{H}$ be a function space on a metrizable compact space K. Then the set of simpliciality $\mathcal{P}_{\mathrm{S}}$ is a G_δ–set.*

Proof. Since K is metrizable, the space $C(K)$ is separable, and thus $\mathcal{K}^c(\mathcal{H})$ is such. Choose a dense countable set $M \subset \mathcal{K}^c(\mathcal{H})$. According to (iv) in Proposition 2.4.4 we have

$$\mathcal{P}_{\mathrm{S}} = \{x \in K : f^*(x) + g^*(x) = (f+g)^*(x) \text{ for all } f,\ g \in M\}.$$

Hence,

$$\begin{aligned}\mathcal{P}_{\mathrm{S}} &= \bigcap_{k\in\mathbb{N}} \bigcap_{f,g\in M} \bigcap_{(h\in\mathcal{H},\ h\geq f+g)} \left\{x \in K : f^*(x) + g^*(x) - h(x) < \frac{1}{k}\right\} \\ &= \bigcap_{k\in\mathbb{N}} \bigcap_{f,g\in M} \bigcap_{(h\in N,\ h\geq f+g)} \left\{x \in K : f^*(x) + g^*(x) - h(x) < \frac{1}{k}\right\},\end{aligned}$$

where N is a dense countable subset of $\mathcal{H}$. The function $f^* + g^* - h$ is upper semicontinuous, hence the set $\{x \in K : f^*(x) + g^*(x) - h(x) < \frac{1}{k}\}$ is open for each $k \in \mathbb{N}$. We conclude that $\mathcal{P}_{\mathrm{S}}$ is a G_δ–set. □

REMARK 2.4.6. *The set $\mathcal{P}_{\mathrm{S}}$ can be closed in K as we saw in Example 2.3.3. But in the "convex case" it cannot be open in K. Moreover, its interior in K is empty (of course, provided $\mathcal{P}_{\mathrm{S}} \neq K$). Indeed, if there exists a point x in interior of $\mathcal{P}_{\mathrm{S}}$, then for arbitrary point $y \in K \setminus \mathcal{P}_{\mathrm{S}}$ we can find $z \in \mathcal{P}_{\mathrm{S}}$ on the line segment, say $z = \lambda x + (1-\lambda)y$, for some $\lambda \in (0,1)$. Let $\mu_x \in \mathcal{M}_x^{\max}(\mathcal{H})$ and μ_y^1, $\mu_y^2 \in \mathcal{M}_y^{\max}(\mathcal{H})$. Then*

$$\lambda\mu_x + (1-\lambda)\mu_y^1,$$

and

$$\lambda\mu_x + (1-\lambda)\mu_y^2$$

are two different maximal measures representing the point z, which is a contradiction to $z \in \mathcal{P}_{\mathrm{S}}$.

Generally, in a nonconvex case, the set $\mathcal{P}_{\mathrm{S}}$ can be open. To show this, consider the set $K = \{[0,0],[1,1],[1,-1],[-1,-1],[-1,1]\} \subset \mathbb{R}^2$ equipped with the relative topology from $\mathbb{R}^2$. Then K is a compact set and restrictions of affine functions form a function space. Clearly $\mathcal{P}_{\mathrm{S}} = \{[1,1],[1,-1],[-1,-1],[-1,1]\}$, which is an open set in K.

COROLLARY 2.4.7. *Let X be a compact convex subset of a locally convex space. The set $X \setminus \mathcal{P}_{\mathrm{S}}$ of nonsimpliciality is dense in X, provided it is not empty.*

Proof. Follows immediately from Remark 2.4.6 □

2.5. Generalized simpliciality

We know that a point $x \in K$ is a point of simpliciality if there exists only one maximal measure $\nu \in \mathcal{M}^1(K)$ representing x. That is, if there exists a unique maximal measure $\nu \in \mathcal{M}^1(K)$ such that $\varepsilon_x \preceq \nu$. For every $x \in K$ and $\nu \in \mathcal{M}^1(K)$, the following equivalence holds

$$\varepsilon_x \preceq \nu \qquad \text{if and only if} \qquad \varepsilon_x \sim \nu.$$

In general, for measures $\mu, \nu \in \mathcal{M}^1(K)$, we have only implication

$$\text{if} \quad \mu \preceq \nu, \quad \text{then} \quad \mu \sim \nu.$$

We can ask when, for a given measure $\mu \in \mathcal{M}^1(K)$, there exists a unique maximal measure $\nu \in \mathcal{M}^1(K)$ such that $\mu \preceq \nu$, and when there exists a unique maximal measure $\nu \in \mathcal{M}^1(K)$ such that $\mu \sim \nu$. We will see that such measure μ can be characterized by the functionals P^μ and Q^μ, respectively, in a similar way like a point of simpliciality in (iv) in Proposition 2.4.4.

We shall say that a measure $\mu \in \mathcal{M}^1(K)$ belongs to the set $\mathcal{P}_{\mathrm{PS}}$ if

$$\mathrm{P}^\mu f + \mathrm{P}^\mu g = \mathrm{P}^\mu(f+g)$$

for all $f,\ g \in \mathcal{K}^c(\mathcal{H})$. Similarly, we shall say that a measure $\mu \in \mathcal{M}^1(K)$ belongs to the set $\mathcal{P}_{\mathrm{QS}}$ if

$$\mathrm{Q}^\mu f + \mathrm{Q}^\mu g = \mathrm{Q}^\mu(f+g)$$

for all $f,\ g \in \mathcal{K}^c(\mathcal{H})$.

THEOREM 2.5.1. *Let $\mathcal{H}$ be a function space on a metrizable compact space K and $\mu \in \mathcal{M}^1(K)$. The following assertions are equivalent:*

(i) $\mu \in \mathcal{P}_{\mathrm{PS}}$,
(ii) *there exists only one maximal measure $\nu \in \mathcal{M}^1(K)$ such that $\mu \preceq \nu$,*
(iii) *for every maximal measure $\nu \in \mathcal{M}^1(K)$, $\mu \preceq \nu$ and every $f \in \mathcal{K}^c(\mathcal{H})$, we have $\nu f = \mathrm{P}^\mu f$,*
(iv) *μ is supported by the set $\mathcal{P}_{\mathrm{S}}$, that is $\mu(\mathcal{P}_{\mathrm{S}}) = 1$.*

Proof. (i) $\Rightarrow$ (ii) Suppose that

$$\mathrm{P}^\mu f + \mathrm{P}^\mu g = \mathrm{P}^\mu(f+g),$$

for all $f,\ g \in \mathcal{K}^c(\mathcal{H})$. Let us define a linear functional φ on $\mathcal{K}^c(\mathcal{H}) - \mathcal{K}^c(\mathcal{H})$ as

$$\varphi(f-g) = \mathrm{P}^\mu f - \mathrm{P}^\mu g, \qquad \text{for } f,\ g \in \mathcal{K}^c(\mathcal{H}).$$

It is straightforward to verify that the definition does not depend on the choice of $f, g \in \mathcal{K}^c(\mathcal{H})$. Hence the functional φ is well defined. Further, it is bounded and $\|\varphi\| = 1$. Indeed,

$$\varphi(f-g) = \mathrm{P}^\mu f - \mathrm{P}^\mu g \le \mathrm{P}^\mu(f-g) \le \|f-g\| \tag{6}$$

implies after changing f and g that

$$|\varphi(f-g)| \le \|f-g\|,$$

and hence $\|\varphi\| \le 1$. Further, $\varphi(1) = 1$ implies $\|\varphi\| = 1$. The first inequality in (6) follows from

$$\mathrm{P}^\mu f = \mathrm{P}^\mu[(f-g)+g] \le \mathrm{P}^\mu(f-g) + \mathrm{P}^\mu g \qquad \text{for all } f,\ g \in \mathcal{K}^c(\mathcal{H}).$$

Since $\mathcal{K}^c(\mathcal{H}) - \mathcal{K}^c(\mathcal{H})$ is a dense subspace of $C(K)$, there exists a uniquely determined linear extension ν of φ to whole $C(K)$, such that $\|\nu\| = \|\varphi\|$. Using Riesz's representation theorem, we can assume that $\nu \in \mathcal{M}^1(K)$. Take a function $f \in \mathcal{K}^c(\mathcal{H})$, then $\nu f = \mathrm{P}^\mu f \ge \mu f$. That is $\mu \preceq \nu$. According to Lemma 2.2.8, for any measure $\lambda \in \mathcal{M}^1(K)$ such that $\mu \preceq \lambda$ we have $\lambda \preceq \nu$. We conclude that ν is the unique maximal measure such that $\mu \preceq \nu$.

(ii) $\Rightarrow$ (iii) Follows immediately from Lemma 2.2.8.

(iii) $\Rightarrow$ (i) Let $f,\ g \in \mathcal{K}^c(\mathcal{H})$ and $\mu \in \mathcal{M}^1(K)$. Take a maximal measure $\nu \in \mathcal{M}^1(K)$, $\mu \preceq \nu$. Then $\mathrm{P}^\mu(f+g) = \nu(f+g) = \nu f + \nu g = \mathrm{P}^\mu f + \mathrm{P}^\mu g$.

(i) $\Rightarrow$ (iv) Suppose that for $f,\ g \in \mathcal{K}^c(\mathcal{H})$ is

$$\mathrm{P}^\mu f + \mathrm{P}^\mu g = \mathrm{P}^\mu (f+g).$$

Hence

$$\mu f^* + \mu g^* = \mu (f+g)^*,$$

and

$$\mu (f^* + g^* - (f+g)^*) = 0.$$

Since the function $f^*+g^*-(f+g)^*$ is nonnegative, we have $f^*+g^* = (f+g)^*$ μ–almost everywhere. Using characterization (iv) in Proposition 2.4.4 we get $\mu(\mathcal{P}_{\mathrm{S}}) = 1$.

Implication (iv) $\Rightarrow$ (i) can be proven by following the previous lines backwards. $\square$

Remark 2.5.2. *The equivalences (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) in the previous Theorem 2.5.1 were proven in* [**17**, Proposition 1] *for a compact convex set in a locally convex space.*

Theorem 2.5.3. *The following assertions are equivalent for a measure $\mu \in \mathcal{M}^1(K)$:*

(i) $\mu \in \mathcal{P}_{\mathrm{QS}}$,
(ii) *there exists only one maximal measure $\nu \in \mathcal{M}^1(K)$ such that $\mu \sim \nu$,*
(iii) *for every maximal measure $\nu \in \mathcal{M}_\mu(\mathcal{H})$ and every $f \in \mathcal{K}^c(\mathcal{H})$, we have $\nu f = \mathrm{Q}^\mu f$.*

Proof. Analogous to the proof of equivalences (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) in Theorem 2.5.1. $\square$

Remark 2.5.4. *The following statements are easy to verify.*

- $\{\varepsilon_x :\ x \in \mathcal{P}_{\mathrm{S}}\} \subset \mathcal{P}_{\mathrm{QS}} \subset \mathcal{P}_{\mathrm{PS}}$.

In convex case we have:

- $\mathcal{P}_{\mathrm{QS}} = \{\mu \in \mathcal{M}^1(X) :\ r_\mu \in \mathcal{P}_{\mathrm{S}}\}$,
- *X is a simplex if and only if, for every $\mu \in \mathcal{M}^1(X)$, there exists a unique maximal measure $\nu \in \mathcal{M}^1(X)$ such that $\mu \preceq \nu$, and this is the case if and only if, for every $\mu \in \mathcal{M}^1(X)$, there exists a unique maximal measure $\nu \in \mathcal{M}^1(X)$ such that $\mu \sim \nu$.*

Remark 2.5.5. *According to the Bauer characterization, we know that $x \in \mathrm{Ch}_{\mathcal{H}}(K)$ if and only if $f^*(x) = f_*(x)$, for all $f \in C(K)$, cf. Corollary 2.2.7. In the same way, we can define the "generalized boundaries"*

$$\partial_{\mathrm{P}} = \{\mu \in \mathcal{M}^1(K) :\ \mathrm{P}^\mu f = \mathrm{P}_\mu f \text{ for all } f \in C(K)\},$$

and

$$\partial_{\mathrm{Q}} = \{\mu \in \mathcal{M}^1(K) :\ \mathrm{Q}^\mu f = \mathrm{Q}_\mu f \text{ for all } f \in C(K)\}.$$

Let $\mathcal{H}$ be a function space on a compact space K. The Mokobodzki test immediately yields that

$$\mu \in \partial_{\mathrm{P}} \quad \text{if and only if } \mu \text{ is maximal},$$

and

$$\mu \in \partial_{\mathrm{Q}} \quad \text{if and only if } \mathcal{M}_\mu(\mathcal{H}) = \{\mu\}.$$

Clearly, $\partial_{\mathrm{Q}} \subset \partial_{\mathrm{P}} \subset \mathcal{P}_{\mathrm{PS}}$.

THEOREM 2.5.6. *Let $\mathcal{H}$ be a function space on a metrizable compact space K. Then the set $\mathcal{P}_{\mathrm{S}}$ of simpliciality is measure extremal and the set $K \setminus \mathcal{P}_{\mathrm{S}}$ of nonsimpliciality is measure convex.*

Proof. From Theorem 2.4.5 we know that the set $\mathcal{P}_{\mathrm{S}}$ is Borel measurable. Choose $x \in \mathcal{P}_{\mathrm{S}}$ and $\mu \in \mathcal{M}_x(\mathcal{H})$. Then there is a unique maximal measure $\nu \in \mathcal{M}^1(K)$ such that $\mu \preceq \nu$. According to Theorem 2.5.1 we have $\mu(\mathcal{P}_{\mathrm{S}}) = 1$. It means that $\mathcal{P}_{\mathrm{S}}$ is measure extremal. Now choose a measure $\lambda \in \mathcal{M}^1(K)$ having its barycenter r_λ in K and suppose $\lambda(K \setminus \mathcal{P}_{\mathrm{S}}) = 1$. If $r_\lambda \in \mathcal{P}_{\mathrm{S}}$, we get a contradiction to measure extremality of $\mathcal{P}_{\mathrm{S}}$. We conclude that $K \setminus \mathcal{P}_{\mathrm{S}}$ is measure convex. $\square$

REMARK 2.5.7. *We can reformulate the previous result as follows. If $x \in K$ is a point of simpliciality, then μ–almost all points in K are points of simpliciality, provided $\mu \in \mathcal{M}_x(\mathcal{H})$.*

2.6. The set of Bauer simpliciality

To define point Bauer simpliciality, we need some characterization which enables us to localize the conception of closed Choquet boundary of a simplicial space. For this purpose we introduce the *point CE–property.* We recall that a function space is called a *CE–space,* if the upper envelopes of continuous functions are continuous. For the proof of the following Proposition 2.6.1 in the "convex case", see [**18**, Theorem 7].

PROPOSITION 2.6.1. *Let $\mathcal{H}$ be a function space on a compact space K. Then $\mathcal{H}$ is a Bauer simplicial space if and only if it is simplicial and a CE–space.*

Let $\mathcal{H}$ be a function space on a compact space K. We say that an $x \in K$ is a *point of continuity of envelopes* (or a *point of CE–property*) if the upper envelopes f^* are continuous at the point x for all $f \in C(K)$. We denote the set of all such points from K with $\mathcal{P}_{\mathrm{CE}}$.

THEOREM 2.6.2. *Let $\mathcal{H}$ be a function space on a metrizable compact space K. The Choquet boundary $\mathrm{Ch}_{\mathcal{H}}(K)$ is contained in the set $\mathcal{P}_{\mathrm{CE}}$, in particular, $\mathcal{P}_{\mathrm{CE}}$ is nonempty.*

Proof. Consider $x \in \mathrm{Ch}_{\mathcal{H}}(K)$ and a sequence $x_n \in K$ such that $x_n \to x$. We want to show $f^*(x_n) \to f^*(x)$ for an arbitrary $f \in C(K)$. By Corollary 2.2.7, there exists $\mu_n \in \mathcal{M}_{x_n}(\mathcal{H})$, for every $n \in \mathbb{N}$, such that $f^*(x_n) = \mu_n f$. We claim that the sequence (μ_n) converges to ε_x. If it be to the contrary, there exists a neighborhood U of ε_x such that $\mu_n \notin U$, for infinitely many $n \in \mathbb{N}$. By compactness of $\mathcal{M}^1(K)$, we can find a subsequence (μ_{n_k}), $\mu_{n_k} \notin U$ and a measure $\mu \in \mathcal{M}^1(K)$ such that $\mu_{n_k} \to \mu$. Especially, for $h \in \mathcal{H}$, we have $\mu_{n_k} h \to \mu h$. Since $\mu_{n_k} h = h(x_{n_k})$, for every $k \in \mathbb{N}$, and $h(x_{n_k}) \to h(x)$, we get $\mu h = h(x)$. Hence $\mu \in \mathcal{M}_x(K)$, which, together with $x \in \mathrm{Ch}_{\mathcal{H}}(K)$, yields $\mu = \varepsilon_x$. Contradiction.

Thus we get $f^*(x_n) = \mu_n f \to f(x)$. Since $x \in \mathrm{Ch}_{\mathcal{H}}(K)$, we have $f^*(x) = f(x)$, which finishes the proof. $\square$

Now, we are able to define the *set $\mathcal{P}_{\mathrm{BS}}$ of Bauer simpliciality* as

$$\mathcal{P}_{\mathrm{BS}} = \mathcal{P}_{\mathrm{S}} \cap \mathcal{P}_{\mathrm{CE}}.$$

REMARK 2.6.3. *According to Remark 2.3.2 and Proposition 2.6.2 we see that*

$$\mathrm{Ch}_{\mathcal{H}}(K) \subset \mathcal{P}_{\mathrm{BS}}.$$

THEOREM 2.6.4. *Let $\mathcal{H}$ be a function space on a metrizable compact space K. Then the set $\mathcal{P}_{\mathrm{CE}}$ is a dense G_δ–set subset of K.*

Proof. Since K is metrizable, $C(K)$ is separable. Let M be a countable dense subset of $C(K)$. Using Lemma 2.2.9, it is easy to verify that $x \in \mathcal{P}_{\mathrm{CE}}$ if (and only if) f^* is continuous at the point x for every $f \in M$. Consider a function $g \in M$. As g^* is upper semicontinuous, the set of points of continuity of g^* is a dense G_δ–set. Since M is countable, the set $\mathcal{P}_{\mathrm{CE}}$ is also a G_δ–set and the Baire Category Theorem yields that $\mathcal{P}_{\mathrm{CE}}$ is dense in K.

□

COROLLARY 2.6.5. *Let $\mathcal{H}$ be a function space on a metrizable compact space K. Then the set $\mathcal{P}_{\mathrm{CE}}$ is residual, that is, its complement is of the first cathegory.*

COROLLARY 2.6.6. *Let $\mathcal{H}$ be a function space on a metrizable compact space K. Then the set $\mathcal{P}_{\mathrm{BS}}$ is a G_δ–set.*

Proof. Follows immediately from Theorems 2.4.5 and 2.6.4. □

CHAPTER 3

Minimal measures and nonsimplicial function cones

ABSTRACT. The set of simpliciality of the nonsimplicial function cone is defined in order to distinguish which points have *unique* minimal representing measures, and at which points this uniqueness fails. We investigate some general properties of the set of simpliciality and we also provide a characterization of Radon measures supported by the set of simpliciality.

3.1. Introduction

The framework of function cones, which plays an important role in potential theory, has been investigated for decades. A special attention was paid to simplicial function cones, since they appear in many general situations. Main results in this direction were achieved by J. Bliedtner and W. Hansen in [**7**], [**8**].

In this work, we focus on nonsimplicial function cones and ask, which points cause nonsimpliciality, that is, which points have more than one minimal representing measures). We continue in investigation of point simpliciality, which was introduced for function spaces on compact spaces in [**3**]. Namely, we extend the previous results from [**3**] to a noncompact setting, that is of function cones on locally compact spaces with countable bases. We refer the reader to the next section for terminology and notation not explained here. Similarly to the above mentioned "compact case", we define the set of simpliciality $\mathrm{Sim}_T(X) \subset X$, for a function cone T on a locally compact space X with a countable base, as the set of all points of X which have unique minimal representing measures. To prevent the reader from confusion we note that dealing with minimal measures instead of maximal measures (like in [**3**]) is just a matter of convetion. We prefer "minimal measures" in this paper in order to be consistent with classical texts on function cones [**6**],[**7**], and [**8**].

We obtain an analogy to the following well known fact, [**7**, Corollary 1.2]:

Let T be a function cone on a locally compact space X with a countable base. Then the Choquet boundary $\mathrm{Ch}_T(X)$ is a G_δ–set, and a measure $\mu \in \mathcal{M}(T)$ is minimal if and only if $\mu(X \setminus \mathrm{Ch}_T(X)) = 0$.

The promised analogy, which should be compared with the above fact, is contained in Theorem 3.3.6. Let us mention the statement here:

Let T be a function cone on a locally compact space X with a countable base. Then the set of simpliciality $\mathrm{Sim}_T(X)$ is a G_δ–set, and, for a measure $\mu \in \mathcal{M}(T)$, there exists a unique minimal measure $\nu \in \mathcal{M}_\mu(T)$ if and only if $\mu(X \setminus \mathrm{Sim}_T(X)) = 0$.

The main results of this paper are: Theorem 3.3.6 (Borel measurability of the set of simpliciality and characterization of measures supported by the set of simpliciality), and Proposition 3.3.7 (measure extremality of the set of simpliciality).

Since the notion of function cones on a locally compact space with a countable base includes function spaces on a metrizable compact space, we get a generalization of some results from [**3**].

3.2. Preliminaries

Let us briefly outline the conception of function cones. We refer the reader to [**6**] or [**7**] for a more detailed description. Let X be a locally compact space with a countable base. We denote $C(X)$ the space of real continuous functions on X. We say that a convex cone $T \subset C(X)$ is a *function cone* on X if

- there exists a strictly positive function $\tilde{t} \in T$,
- for every disjoint $x, y \in X$ and every $\lambda \geq 0$, there exists a nonnegative function $t \in T$ such that $t(x) \neq \lambda t(y)$,
- for every $t \in T$ there exists a nonnegative function $t_0 \in T$ such that the sets $\{|t| > \varepsilon t_0\}$ are relatively compact for all $\varepsilon > 0$.

Further, define the space of *T−bounded continuous functions* as

$$C_T(X) = \{f \in C(X) : \ |f| \leq t \text{ for some nonnegative } t \in T\},$$

and the *wedge* of T as

$$W(T) = \{h_1 \wedge \cdots \wedge h_n : \ n \in \mathbb{N}, \ h_1, \ldots, h_n \in T\}.$$

A nonnegative Radon measure μ on X belongs to $\mathcal{M}(T)$ if all functions from T are integrable with respect to μ.

The *Choquet ordering* on $\mathcal{M}(T)$ is defined as follows:

$$\mu \preceq \nu \text{ if } \mu f \leq \nu f \text{ for all } f \in W(T).$$

Whenever we say that a measure is *minimal,* we mean that it is minimal with respect to this ordering. Further, for a measure $\mu \in \mathcal{M}(T)$, define the sets

$$\mathcal{M}_\mu(T) = \{\nu \in \mathcal{M}(T) : \ \nu \preceq \mu\},$$

and

$$\mathcal{M}_\mu^{\min}(T) = \{\nu \in \mathcal{M}(T) : \ \nu \preceq \mu, \ \nu \text{ minimal}\}.$$

In a particular case, when $\mu = \varepsilon_x$, for an $x \in X$, we use more simple symbols $\mathcal{M}_x(T)$, and $\mathcal{M}_x^{\min}(T)$, respectively. Here, the symbol ε_x stands for the Dirac measure at a point $x \in X$.

The *Choquet boundary* of T is a subset of X defined as

$$\mathrm{Ch}_T(X) = \{x \in X : \ \mathcal{M}_x(T) = \{\varepsilon_x\}\}.$$

Let $f \in C_T(X)$. Its *lower envelope* is the function

$$f_* = \sup\{t \in -T : \ t \leq f\},$$

and its *upper envelope* is the function

$$f^* = \inf\{t \in T : \ t \geq f\}.$$

For $f \in C_T(X)$ and $\mu \in \mathcal{M}(T)$, denote $\mathrm{P}_\mu f = \mu f_*$. Then P_μ is a positive functional which, for $\mu = \varepsilon_x$, coincides with the lower envelope:

$$\mathrm{P}_\mu f = f_*(x).$$

Similarly, define $\mathrm{P}^{\mu} f = \mu f^*$, for $f \in C_T(X)$.

The following lemma belongs to the folklore of potential theory, [**15**]. Its proof is similar to a version for function cones on compact spaces, see [**9**, Lema 1.3].

LEMMA 3.2.1. *Let $\mu \in \mathcal{M}(T)$ and $f \in C_T(X)$. Then*

$$\{\nu f : \nu \in \mathcal{M}_\mu(T)\} = [\mathrm{P}_\mu f, \mathrm{P}^\mu f].$$

We shall need this lemma due to G. Choquet, see [**6**, Proposition 1.4]:

LEMMA 3.2.2. *Let T be a function cone on a locally compact space X with a countably base. For every additive, positively homogenous, increasing functional $\varphi : W(T) \to \mathbb{R}$, there exists a unique measure $\mu \in \mathcal{M}(T)$ such that $\varphi(f) = \mu f$ for every $f \in W(T)$.*

Let $B \subset X$ be a Borel measurable set. We say that a measure $\mu \in \mathcal{M}(T)$ is *supported* by B if $\mu(X \setminus B) = 0$. We say that B is *measure extremal* if μ is supported by B whenever $\mu \in \mathcal{M}_x(T)$ and $x \in X$.

3.3. Point simpliciality of function cones

Let X be a locally compact space with a countable base and T be a function cone on X. Define the set

$$\mathcal{P}(T) = \{\mu \in \mathcal{M}(T) : \mathrm{P}_\mu f + \mathrm{P}_\mu g = \mathrm{P}_\mu(f+g) \text{ for all } f, g \in W(T)\}.$$

We begin with a useful lemma which will be repeatedly used in the sequel. It is in fact just a variation of known theorems, e.g. [**17**, Proposition 1].

LEMMA 3.3.1. *The following assertions are equivalent.*

(i) $\mu \in \mathcal{P}(T)$,
(ii) *there exists a unique measure $\nu \in \mathcal{M}_\mu^{\min}(T)$,*
(iii) $\nu f = \mathrm{P}_\mu f$, *for all $f \in W(T)$ and $\nu \in \mathcal{M}_\mu^{\min}(T)$.*

Proof. (i) $\Rightarrow$ (ii) We assume that for every $f \in W(T)$

$$\mathrm{P}_\mu f + \mathrm{P}_\mu g = \mathrm{P}_\mu(f+g).$$

Then the functional P_μ satisfies the assumptions of Lemma 3.2.2 and we obtain a unique measure $\nu \in \mathcal{M}(T)$ such that $\mathrm{P}_\mu f = \nu f$, for every $f \in W(T)$. Since

$$\nu f = \mathrm{P}_\mu f = \mu f_* \leq \mu f \text{ for all } f \in W(T),$$

we have $\nu \preceq \mu$. To prove that ν is minimal, assume that $\lambda \preceq \nu$, for some $\lambda \in \mathcal{M}(T)$. From Lemma 3.2.1, it immediately follows that $\lambda f = \mathrm{P}_\mu f = \nu f$, for all $f \in W(T)$. From the uniqueness, we get $\lambda = \nu$. We conclude that $\nu \in \mathcal{M}_\mu^{\min}(T)$.

(ii) $\Rightarrow$ (iii) It follows from Lemma 3.2.1.

(iii) $\Rightarrow$ (i) Find a measure $\nu \in \mathcal{M}_\mu^{\min}(T)$ such that $\nu(f+g) = \mathrm{P}_\mu(f+g)$ for all $f, g \in W(T)$. Then

$$\mathrm{P}_\mu(f+g) = \nu(f+g) = \nu f + \nu g = \mathrm{P}_\mu f + \mathrm{P}_\mu g.$$

This concludes the proof. $\square$

DEFINITION 3.3.2. *A point $x \in X$ is called a point of simpliciality if there exists a unique measure $\nu \in \mathcal{M}_x^{\min}(T)$. The set of all such points of X will be called the set of simpliciality and denoted* $\mathrm{Sim}_T(X)$.

REMARK 3.3.3. *It follows that* $\mathrm{Ch}_T(X) \subset \mathrm{Sim}_T(X) \subset X$, *and that* $\mathrm{Sim}_T(X) = X$ *if and only if the function cone* T *is simplicial.*

COROLLARY 3.3.4. *For the set of simpliciality we have*

$$\mathrm{Sim}_T(X) = \{x \in X : f_*(x) + g_*(x) = (f+g)_*(x) \text{ for all } f, g \in W(T)\}.$$

Proof. It follows immediately from Lemma 3.3.1 when applied to Dirac measures ε_x, $x \in X$. □

REMARK 3.3.5. *Since* X *has a countable base, the space* $C_T(X)$ *is separable and for any norm dense countable set* $M \subset W(T)$ *holds*

$$\mathrm{Sim}_T(X) = \{x \in X : f_*(x) + g_*(x) = (f+g)_*(x) \text{ for all } f, g \in M\}. \tag{7}$$

Indeed, from the definition of the lower envelope we have, for every $x \in X$, *that* $(f_n)_*(x) \to f_*(x)$, *whenever* $f_n, f \in W(T)$ *and* $f_n \to f$, *uniformly on* X. *This yields (7).*

THEOREM 3.3.6. *Let* T *be a function cone on a locally compact space* X *with a countable base. Then the set of simpliciality* $\mathrm{Sim}_T(X)$ *is a* G_δ*–set. Further, let* $\mu \in \mathcal{M}(T)$. *Then* μ *is supported by* $\mathrm{Sim}_T(X)$ *if and only if there exists a unique measure* $\nu \in \mathcal{M}_\mu^{\min}(T)$.

Proof. By Corollary 3.3.4, we have

$$\begin{aligned}
\mathrm{Sim}_T(X) &= \{x \in X : f_*(x) + g_*(x) = (f+g)_*(x) \text{ for all } f, g \in W(T)\} \\
&= \bigcap_{k \in \mathbb{N}} \bigcap_{f,g \in W(T)} \left\{x \in X : f_*(x) + g_*(x) - (f+g)_*(x) > -\frac{1}{k}\right\} \\
&= \bigcap_{k \in \mathbb{N}} \bigcap_{f,g \in W(T)} \bigcap_{(h \le f+g,\, h \in -T)} \left\{x \in X : f_*(x) + g_*(x) - h(x) > -\frac{1}{k}\right\}.
\end{aligned}$$

According to Remark 3.3.5, we can consider only countable intersections in the above equalities. Since the lower envelopes are lower semicontinuous, the set

$$\left\{x \in X : f_*(x) + g_*(x) - h(x) > -\frac{1}{k}\right\}.$$

is open. We conclude that $\mathrm{Sim}_T(X)$ is a G_δ–set.

To prove the remainder, suppose that μ is supported by $\mathrm{Sim}_T(X)$, that is, $\mu(X \setminus \mathrm{Sim}_T(X)) = 0$. Using Corollary 3.3.4, we get

$$\mathrm{P}_\mu f + \mathrm{P}_\mu g = \mathrm{P}_\mu(f+g) \text{ for all } f, g \in W(T),$$

hence $\mu \in \mathcal{P}(T)$. By Lemma 3.3.1, there exists a unique measure $\nu \in \mathcal{M}_\mu^{\min}(T)$.

To prove the reverse inclusion, assume that there exists a unique measure $\nu \in \mathcal{M}_\mu^{\min}(T)$. According to Lemma 3.3.1, $\mu \in \mathcal{P}(T)$. Using a similar argument as above, we get that μ is supported by $\mathrm{Sim}_T(X)$. □

In the last proposition, we shall see that the set of simpliciality of function cones is measure extremal. This generalize the fact, that the set of simpliciality of a compact convex set is extremal, see [**3**, Theorem 4.3]. Measure extremal sets were in general studied e.g. in [**13**]. It is shown in [**3**, Theorem 5.6], that the set of simpliciality of a function space in measure extremal.

PROPOSITION 3.3.7. *Let T be a function cone on a locally compact space X with a countable base. Then the set of simpliciality is measure extremal.*

Proof. We already know that $\mathrm{Sim}_T(X)$ is Borel measurable. Let $x \in \mathrm{Sim}_T(X)$ be a point of simpliciality and $\mu \in \mathcal{M}_x(T)$. Then there exists a unique measure $\nu \in \mathcal{M}_\mu^{\min}(T)$. Indeed, if there were two measures $\nu_1, \nu_2 \in \mathcal{M}_\mu^{\min}(T)$, we would have $\nu_1, \nu_2 \in \mathcal{M}_x^{\min}(T)$, which contradicts the simpliciality of x. By Proposition 3.3.6, $\mu(X \setminus \mathrm{Sim}_T(X)) = 0$. Hence μ is supported by $\mathrm{Sim}_T(X)$, which finishes the proof. $\square$

Proposition 3.3.7 claims that, if $x \in X$ is a point of simpliciality, than μ–almost every point of X is a point of simpliciality, for every $\mu \in \mathcal{M}_x(T)$.

CHAPTER 4

Unique Decomposition Property and Extreme Points

ABSTRACT. This paper presents a solution to an open problem posed by J.J. Font and M. Sanchis in [**14**]. We will show that the Unique Decomposition Property of a function space is necessary to obtain a full characterization of extreme points of the unit ball in the dual space of some quotient of a function space.

4.1. Introduction

At the beginning we introduce some notation and basic facts concerning Choquet's theory. We refer the reader to [**24**] for details. Let X be a Hausdorff compact space, the symbol $\mathcal{C}$ denotes the set of constant functions on X. A subspace $\mathcal{H}$ of the space of continuous functions $C(X)$ is called a *function space* on X provided it separates points of X and $\mathcal{C} \subset \mathcal{H}$. Notice that the function space $\mathcal{H}$ is not supposed to be closed. The dual space $(C(X))^*$ is according to the Riesz Representation Theorem considered to be the set of Radon measures on X, denoted $\mathcal{M}(X)$. The set of probability Radon measures will be denoted $\mathcal{M}^1(X)$. We define the positive part of the closed unit ball in the dual space $\mathcal{H}^*$ as

$$B^+_{\mathcal{H}^*} = \{\psi \in \mathcal{H}^* : \ 0 \le \psi, \ \|\psi\| \le 1\}$$

and the *state space* of $\mathcal{H}$ as

$$S(\mathcal{H}) = \{\psi \in B^+_{\mathcal{H}^*} : \ \|\psi\| = 1\}.$$

It is well known that $\mathcal{H}^*$ is isometrically isomorphic to the quotient space

$$\mathcal{M}(X)/\mathcal{H}^\perp$$

and that

$$S(\mathcal{H}) = \pi(\mathcal{M}^1(X)).$$

Here π stands for the quotient mapping from $\mathcal{M}(X)$ to $\mathcal{H}^*$. Further, define a homeomorphic embedding $\phi : X \to S(\mathcal{H})$ mapping to every $x \in X$ a functional ϕ_x by $\phi_x = \pi(\varepsilon_x)$, where ε_x is the Dirac measure at the point x. The Choquet boundary of $\mathcal{H}$ is denoted $\mathrm{Ch}_{\mathcal{H}}(K)$. The set of extreme points of a convex set M is denoted $\mathrm{ext}(M)$. For the state space we have

$$\mathrm{ext}(S(\mathcal{H})) = \{\phi_x \in S(\mathcal{H}) : \ x \in \mathrm{Ch}_{\mathcal{H}}(K)\}.$$

The symbol $\mathbf{1_X}$ stands for the function identically equal to 1 on X. We denote $\mathcal{H}_d$ the quotient space $\mathcal{H}/\mathcal{C}$ and let $\hat{\pi}$ be the quotient mapping

$$\hat{\pi} : \mathcal{H} \to \mathcal{H}/\mathcal{C}.$$

Define the *diameter norm* for functions in $\mathcal{H}_d$ as

$$\|\hat{\pi}(f)\|_d = \mathrm{diam}(\mathcal{R}(f)),$$

where $\mathcal{R}(f)$ stands for the range of the function f. It is easy to see that

$$\mathrm{diam}(\mathcal{R}(f)) = 2 \inf_{\alpha \in \mathbb{R}} \{\|f - \alpha \cdot \mathbf{1_X}\|\} = 2\|f\|$$

for every $f \in \mathcal{H}$.

Now, we introduce a lemma, which follows immediately from Theorem 4.9 in [**27**]. This lemma enables us to identify the space $(\mathcal{H}_d)^*$ with a subspace of $\mathcal{H}^*$. In sequel, we use this convention without explicit mentioning.

Lemma 4.1.1. *The space $(\mathcal{H}_d)^*$ is isometrically isomorphic to*

$$\{\psi \in \mathcal{H}^* : \ \psi(\mathbf{1_X}) = 0\}.$$

We have $2\|\psi\|_{d^*} = \|\psi\|$ for every $\psi \in (\mathcal{H}_d)^*$, where $\|\cdot\|_{d^*}$ is the norm in $\mathcal{H}^*$ defined as

$$\|\psi\|_{d^*} = \sup_{\|f\|_d \leq 1} \frac{|\psi(f)|}{\|f\|_d}.$$

Then the closed unit ball in $((\mathcal{H}_d)^*, \|\cdot\|_{d^*})$ is denoted by $B_{\mathcal{H}_d^*}$.

The main aim of this paper is to provide a full characterization of the extreme points $\operatorname{ext}(B_{\mathcal{H}_d^*})$. In general situation, when the function space $\mathcal{H}$ is not supposed to have any other properties, we have the following assertion.

Proposition 4.1.2. *Let $\mathcal{H}$ be a function space on a compact space X. Then*

$$\operatorname{ext}(B_{\mathcal{H}_d^*}) \subset \{\phi_x - \phi_y : x,\, y \in \operatorname{Ch}_{\mathcal{H}}(K),\ x \neq y\}.$$

We refer the reader to [**14**] for a proof. There is also an example showing that the inclusion can be strict. Now, it is clear that we have to impose some tacite assumptions on $\mathcal{H}$ in order to obtain a full characterization of $\operatorname{ext}(B_{\mathcal{H}_d^*})$.

Definition 4.1.3. *We say that $\mathcal{H}$ satisfies the unique decomposition property (UDP) if for every $x,\, y \in \operatorname{Ch}_{\mathcal{H}}(K)$, $x \neq y$ and $\psi_1,\, \psi_2 \in B^+_{\mathcal{H}^*}$ such that*

$$\phi_x - \phi_y = \psi_1 - \psi_2 \quad \text{and} \quad \|\phi_x - \phi_y\| = \|\psi_1\| + \|\psi_2\|,$$

there exist $z,\, t \in \operatorname{Ch}_{\mathcal{H}}(K)$ such that $\psi_1 = \phi_z$, $\psi_2 = \phi_t$.

Remark 4.1.4. *Let us note two important things about (UDP).*

- *Such decomposition of $\phi_x - \phi_y$ always exists: For a Hahn-Banach extension $\Phi \in (C(X))^*$ of the functional $\phi_x - \phi_y \in \mathcal{H}^*$ we can consider its positive and negative variations Φ^+, Φ^-. Let us denote the restrictions of these two functionals on $\mathcal{H}$ by ψ^+, ψ^-. Then we have*

$$\phi_x - \phi_y = \psi^+ - \psi^-$$

and

$$\|\phi_x - \phi_y\| = \|\psi^+\| + \|\psi^-\|$$

- *If $\mathcal{H}$ has (UDP), then $\|\phi_x - \phi_y\| = 2$: Indeed, there are $z,\, t \in \operatorname{Ch}_{\mathcal{H}}(K)$ such that $\|\phi_x - \phi_y\| = \|\phi_z\| + \|\phi_t\| = 1 + 1$.*

4.2. Main Theorem

The following Theorem contains the main result of this paper. The first proved implication is due to J. J. Font and M. Sanchis ([**14**]), we present a little different proof for the sake of completeness. The second proved implication answers the question, posed in [**14**], whether the unique decomposition property is necessary for (8) to hold.

THEOREM 4.2.1. *Let $\mathcal{H}$ be a function space on a compact space X. Then*

$$\operatorname{ext}(B_{\mathcal{H}_d^*}) = \{\phi_x - \phi_y : x,\ y \in \operatorname{Ch}_{\mathcal{H}}(K),\ x \neq y\}, \tag{8}$$

if and only if the function space $\mathcal{H}$ enjoys (UDP).

Proof. (i) Firstly, we suppose that $\mathcal{H}$ has (UDP). According to the above mentioned Proposition it remains to show that for every $x,\ y \in \operatorname{Ch}_{\mathcal{H}}(K)$, $x \neq y$, the functional $\phi_x - \phi_y$ is an extreme point of the closed unit ball of $(\mathcal{H}_d)^*$. From the previous Remarks it follows that the diameter norm of $\phi_x - \phi_y$ is equal to 1. Let us write

$$\phi_x - \phi_y = \frac{1}{2}(\omega + \psi),$$

where ω and ψ are from the closed unit ball of $(\mathcal{H}_d)^*$, $\|\omega\|_{d^*} = \|\psi\|_{d^*} = 1$. Then for Hahn-Banach extensions $\Omega,\ \Psi \in (C(X))^*$ of functionals $\omega,\ \psi$ we can take their positive and negative variations $\Omega^+,\ \Psi^+,\ \Omega^-,\ \Psi^-$. Then

$$\phi_x - \phi_y = \frac{1}{2}\left[(\omega^+ + \psi^+) - (\omega^- + \psi^-)\right],$$

where ω^+, ψ^+, ω^-, ψ^- stand for the restrictions of Ω^+, Ψ^+, Ω^-, Ψ^- on $\mathcal{H}$. We have

$$\begin{aligned} 2 &= \|\phi_x - \phi_y\| = \frac{1}{2}\left\|(\omega^+ + \psi^+) - (\omega^- + \psi^-)\right\| \leq \\ &\leq \frac{1}{2}\left(\|\omega^+ + \psi^+\| + \|\omega^- + \psi^-\|\right) \leq \\ &\leq \frac{1}{2}\left(\|\omega^+\| + \|\psi^+\| + \|\omega^-\| + \|\psi^-\|\right) = \frac{1}{2}\left(\|\omega\| + \|\psi\|\right) = 2, \end{aligned}$$

which yields

$$\|\phi_x - \phi_y\| = \frac{1}{2}\left(\|\omega^+ + \psi^+\| + \|\omega^- + \psi^-\|\right).$$

Using (UDP) we get

$$\frac{1}{2}\left(\omega^+ + \psi^+\right) = \phi_z, \qquad \frac{1}{2}\left(\omega^- + \psi^-\right) = \phi_t,$$

for some $z,\ t \in \operatorname{Ch}_{\mathcal{H}}(K)$. Further

$$\omega^+ = \psi^+ = \phi_z, \qquad \omega^- = \psi^- = \phi_t,$$

because ϕ_z and ϕ_t are extreme points of the state space. Then $\omega = \psi = \phi_z - \phi_t$ and we see that $\phi_x - \phi_y$ is an extreme point of $B_{\mathcal{H}_d^*}$. This finishes the proof of the first implication.

(ii) If we suppose that $\mathcal{H}$ lacks (UDP), we can find $x,\ y \in \operatorname{Ch}_{\mathcal{H}}(K)$, $x \neq y$, and $\psi_1,\ \psi_2 \in B^+_{\mathcal{H}^*}$ such that

$$\phi_x - \phi_y = \psi_1 - \psi_2, \qquad \|\phi_x - \phi_y\| = \|\psi_1\| + \|\psi_2\|, \tag{9}$$

and $\psi_1 \neq \phi_z$ for every $z \in \mathrm{Ch}_{\mathcal{H}}(K)$ or $\psi_2 \neq \phi_z$ for every $z \in \mathrm{Ch}_{\mathcal{H}}(K)$. Without loss of generality let $\psi_1 \neq \phi_z$ for every $z \in \mathrm{Ch}_{\mathcal{H}}(K)$. Notice that from (9) it follows

$$\|\psi_1\| = \psi_1 \mathbf{1_X} = \psi_2 \mathbf{1_X} = \|\psi_2\|. \tag{10}$$

If $\|\phi_x - \phi_y\| < 2$, then, clearly, the functional $\phi_x - \phi_y$ is not an extreme point of $B_{\mathcal{H}_d^*}$, so we can assume $\|\phi_x - \phi_y\| = 2$. Now (9) and (10) yield

$$\psi_1,\ \psi_2 \in S(\mathcal{H}).$$

Further, ψ_1 is not an extreme point of $S(\mathcal{H})$. Therefore there exist $\xi_1,\ \xi_2 \in S(\mathcal{H})$, $\xi_1 \neq \xi_2$ such that

$$\psi_1 = \frac{1}{2}(\xi_1 + \xi_2).$$

Hence

$$\phi_x - \phi_y = \psi_1 - \psi_2 = \frac{1}{2}\left(\xi_1 + \xi_2\right) - \psi_2 = \frac{1}{2}\left[(\xi_1 - \psi_2) + (\xi_2 - \psi_2)\right]. \tag{11}$$

Since

$$(\xi_1 - \psi_2)\mathbf{1_X} = 0, \qquad (\xi_2 - \psi_2)\mathbf{1_X} = 0,$$

and

$$\|\xi_1 - \psi_2\| \leq (\xi_1 + \psi_2)\mathbf{1_X} = 2, \qquad \|\xi_2 - \psi_2\| \leq (\xi_2 + \psi_2)\mathbf{1_X} = 2,$$

the functionals $\xi_1 - \psi_2$ and $\xi_2 - \psi_2$ are (different) elements of the closed unit ball of $(\mathcal{H}_d)^*$. Having a nontrivial combination in (11), we can conclude that $\phi_x - \phi_y$ is not an extreme point of $B_{\mathcal{H}_d^*}$. $\square$

CHAPTER 5

Complementability of spaces of affine continuous functions on simplices

(joint work with Jiří Spurný)

Abstract. We construct metrizable simplices X_1 and X_2 and a homeomorphism $\varphi : \overline{\operatorname{ext} X_1} \to \overline{\operatorname{ext} X_2}$ such that $\varphi(\operatorname{ext} X_1) = \operatorname{ext} X_2$, the space $A(X_1)$ of all affine continuous functions on X_1 is complemented in $C(X_1)$ and $A(X_2)$ is not complemented in any $C(K)$ space. This shows that complementability of the space $A(X)$ cannot be determined by topological properties of the couple $(\operatorname{ext} X, \overline{\operatorname{ext} X})$.

5.1. Introduction

A Banach space X is called an L^1*–predual* if X^* is isometric to some $L^1(\mu)$ space. A particular example of an L^1–predual is the space $C(K)$ of all continuous functions on a compact space K. There was a question how "different" an L^1–predual can be from $C(K)$–spaces which was answered by Y. Benyamini and J. Lindenstrauss in [**5**] where they constructed an ℓ^1–predual that is not complemented in any $C(K)$–space.

The method of their construction was to find a suitable compact convex subset X of a locally convex space such that X is a simplex and the space $A(X)$ of all continuous affine functions on X is not complemented in any $C(K)$–space (we refer reader to the next section for the notions not explained here). As it is known, the space $A(X)$ on a simplex X is an example of an L^1–predual space (see [**19**, Proposition 3.23]).

Since some properties of $A(X)$ on a simplex X can be characterized by topological properties of the set $\operatorname{ext} X$ of all extreme points of X (see e.g. [**19**, Proposition 3.15] or [**30**, Theorem 1]), it seems natural to ask a similar question for the problem of complementability of $A(X)$ in a $C(K)$–space. The aim of this note is to show that this is not the case.

We prove even more, namely that complementability of $A(X)$ on a simplex X cannot be determined by topological properties of the pair $(\operatorname{ext} X, \overline{\operatorname{ext} X})$. By a modification of the method of [**5**] we get the following theorem.

Theorem 5.1.1. *There exist metrizable simplices X_1 and X_2 and a homeomorphic mapping $\varphi : \overline{\operatorname{ext} X_1} \to \overline{\operatorname{ext} X_2}$ such that the sets $\operatorname{ext} X_1$, $\operatorname{ext} X_2$ are countable, $\varphi(\operatorname{ext} X_1) = \operatorname{ext} X_2$, $A(X_1)$ is complemented in $C(X_1)$ and $A(X_2)$ is not complemented in any $C(K)$ space.*

We remark that the simplices X_1, X_2 are constructed in such a way that the sets of extreme points are of type F_σ (i.e., it is a countable union of closed sets). This might be of some interest since the structure of simplices with extreme points being F_σ–set is more transparent (see e.g. [**26**, Théorème 80] or [**29**, Corollary 3.5]).

5.2. Preliminaries

All topological space will be considered as Hausdorff. If K is a compact space, we denote by $C(K)$ the space of all continuous real–valued functions on K. We will identify the dual of $C(K)$ with the space $\mathcal{M}(K)$ of all Radon measures on K. Let $\mathcal{M}^1(K)$ denote the set of all probability Radon measures on K and let ε_x stand for the Dirac measure at $x \in K$.

Function spaces. Throughout the paper we will consider a *function space* $\mathcal{H}$ on a compact space K. By this we mean a (not necessarily closed) linear subspace of $C(K)$ containing the constant functions and separating the points of K. Let $\mathcal{M}_x(\mathcal{H})$ be the set of all *$\mathcal{H}$–representing measures* for $x \in K$, i.e.,

$$\mathcal{M}_x(\mathcal{H}) = \{\mu \in \mathcal{M}^1(K) : f(x) = \int_K f \, d\mu \text{ for any } f \in \mathcal{H}\}.$$

If $\mu \in \mathcal{M}_x(\mathcal{H})$, we say that x is a *barycenter of* μ and denote $x = r(\mu)$. Where no confusion can arise we simply say that μ represents x.

The set

$$\operatorname{Ch}_{\mathcal{H}} K = \{x \in K : \ \mathcal{M}_x(\mathcal{H}) = \{\varepsilon_x\}\}$$

is called the *Choquet boundary* of $\mathcal{H}$. It may be highly irregular from the topological point of view but it is a G_δ–set if K is metrizable (see [**19**, Proposition 2.9]).

Given a function space $\mathcal{H}$ on a compact space K we can define the set of *$\mathcal{H}$–affine continuous functions* as follows

$$\mathcal{A}^c(\mathcal{H}) = \{f \in C(K) : \ f(x) = \int_K f \, d\mu \text{ for any } x \in K \text{ and } \mu \in \mathcal{M}_x(\mathcal{H})\}.$$

Clearly, $\mathcal{H} \subset \mathcal{A}^c(\mathcal{H})$.

We say that a function $h \in \mathcal{H}$ is *$\mathcal{H}$–exposing* for $x \in K$ if h attains its extremal value precisely at x. Obviously, any $\mathcal{H}$–exposed point is contained in the Choquet boundary of $\mathcal{H}$.

Examples of function spaces. We introduce the following main examples of function spaces.

In the *"convex case"*, the function space $\mathcal{H}$ is the linear space $A(X)$ of all continuous affine functions on a compact convex subset X of a locally convex space. In this example, the Choquet boundary of $A(X)$ coincides with the set of all extreme points of X (see [**2**, Theorem 6.3]) and is denoted by $\operatorname{ext} X$.

Further, the barycenter of a probability measure μ on X is the unique point $r(\mu) \in X$ for which $f(r(\mu)) = \mu(f)$ for any $f \in A(X)$, in other words, $r(\mu)$ is $A(X)$–represented by μ.

In the *"harmonic case"*, U is a bounded open subset of the Euclidean space $\mathbb{R}^m$ and the corresponding function space $\mathcal{H}$ is $\mathbf{H}(U)$, i.e., the family of all continuous functions on $\overline{U}$ which are harmonic on U. In the "harmonic case", the Choquet boundary of $\mathbf{H}(U)$ coincides with the set $\partial_{\text{reg}} U$ of all regular points of U (see [**23**, Theorem]).

Simplicial functions spaces. If $\mathcal{H}$ is a function space on a metrizable compact space K, for any $x \in K$ there exists a measure $\mu \in \mathcal{M}_x(\mathcal{H})$ such that $\mu(K \setminus \operatorname{Ch}_{\mathcal{H}} K) = 0$ (see e.g. [**19**, Theorem 2.10]).

If this measure is uniquely determined for every $x \in K$, we say that $\mathcal{H}$ is a *simplicial function space.* In the "convex case" it is equivalent to say that X is a *Choquet simplex*, briefly *simplex* (see [**1**, Theorem II.3.6], [**2**, Section 2, Theorem 7.3] or [**19**]).

As another example of a simplicial function space serves the space $\mathbf{H}(U)$ from the "harmonic case" (see e.g. [**23**, Theorem]).

State space. By a standard technique briefly described below any function space can be viewed as the space $A(X)$ of affine continuous functions on a suitable compact convex set X. Details can be found in [**1**, Chapter 2, § 2], [**2**, Chapter 1, § 4] or [**24**, Section 6].

If $\mathcal{H}$ is a function space on a compact space K, we set

$$\mathbf{S}(\mathcal{H}) = \{\varphi \in \mathcal{H}^* : \|\varphi\| = \varphi(1) = 1\} .$$

Then $\mathbf{S}(\mathcal{H})$ endowed with the weak* topology is a compact convex set which is metrizable if K is metrizable. Let $\phi : K \to \mathbf{S}(\mathcal{H})$ be the evaluation mapping defined as $\phi(x) = s_x$, $x \in K$, where $s_x(h) = h(x)$ for $h \in \mathcal{H}$. Then ϕ is a homeomorphic embedding of K onto $\phi(K)$ and $\phi(\mathrm{Ch}_{\mathcal{H}} K) = \operatorname{ext} \mathbf{S}(\mathcal{H})$.

Let $\Phi : \mathcal{H} \to A(\mathbf{S}(\mathcal{H}))$ be the mapping defined for $h \in \mathcal{H}$ by $\Phi(h)(s) = s(h)$, $s \in \mathbf{S}(\mathcal{H})$. Then Φ serves as an isometric isomorphism of $\mathcal{H}$ into $A(\mathbf{S}(\mathcal{H}))$. Further, Φ is onto if and only if the function space $\mathcal{H}$ is uniformly closed in $C(K)$. In this case the inverse mapping is realized by

$$\Phi^{-1}(F) = F \circ \phi, \quad F \in A(\mathbf{S}(\mathcal{H})) .$$

In the sequel we will need the following theorem.

THEOREM 5.2.1. *Let $\mathcal{H}$ be a closed function space on a metrizable compact space K. Then the following assertions are equivalent:*

(i) *$\mathcal{H}$ is simplicial;*
(ii) *the state space $\mathbf{S}(\mathcal{A}^c(\mathcal{H}))$ is a simplex.*

Proof. See [**4**, Theorem]. □

By a projection, we always mean a bounded linear operator P on a Banach space such that $P = P^2$.

Without explicit mentioning, every Banach space is assumed to be a subspace of its second dual via its canonical embedding.

5.3. Construction

DEFINITION 5.3.1. *For a Banach space X we define*

$$\lambda(X) = \inf \|T\| \, \|T^{-1}\| \, \|P\|,$$

where the infimum is taken over all isomorphisms T from X into a $C(K)$ space and all projections $P : C(K) \to TX$. If X is not isomorphic to a complemented subspace of any $C(K)$–space, we put $\lambda(X) = \infty$.

LEMMA 5.3.2. *Let X be a Banach space and B_{X^*} be its dual unit ball endowed with the weak* topology. Then*

$$\lambda(X) = \inf\{\|P\| : P \text{ is a projection of } C(B_{X^*}) \text{ onto } X\} .$$

Proof. See [**5**, Lemma]. □

LEMMA 5.3.3. *Let Y be a 1–complemented subspace of a Banach space X. Then $\lambda(Y) \le \lambda(X)$.*

Proof. Let $T : X \to Y$ be a projection of norm 1. We will show that for every projection $P : C(B_{X^*}) \to X$ we can find a projection $Q : C(B_{Y^*}) \to Y$ such that $\|P\| = \|Q\|$. Then, by Lemma 5.3.2, $\lambda(Y) \le \lambda(X)$. If $\pi : X^* \to Y^*$ denotes the restriction operator, then $Q : C(B_{Y^*}) \to Y$ defined as

$$Q : f \mapsto TP(f \circ \pi) , \quad f \in C(B_{Y^*}) ,$$

is a projection of norm $\|P\|$. This finishes the proof.

$\square$

Construction Let $\mathcal{H}$ be a simplicial function space on a compact space K such that $\mathcal{H} = \mathcal{A}^c(\mathcal{H})$. Let

$$L = \left\{ \bigcup_{i\in\mathbb{N}, j=1,2,3} K_{ij} \right\} \cup \{p, q\} \cup \{r_{ij} : i = -1, 0, 1, j = 1, 2, 3\} ,$$

where each K_{ij} is a copy of K. The points p, q are chosen so that the above union was disjoint. The topology on L is defined as follows: a basis of the neighborhoods of r_{0j}, $j = 1, 2, 3$, is given by the sets $\{r_{0j}\} \cup \bigcup_{i=n}^{\infty} K_{ij}$, $n \in \mathbb{N}$, each K_{ij} is both closed and open in L and all the remaining points are isolated.

Let

$$\mathcal{H}_1 = \{f \in C(L) : f\restriction_{K_{ij}} \in \mathcal{H}, i \in \mathbb{N}, j = 1, 2, 3, \\ 2f(r_{0j}) = f(r_{-1j}) + f(r_{1j}), j = 1, 2, 3\} ,$$

and

$$\mathcal{H}_2 = \{f \in C(L) : f\restriction_{K_{ij}} \in \mathcal{H}, i \in \mathbb{N}, j = 1, 2, 3, 2f(r_{01}) = f(p) + f(q), \\ 3f(r_{02}) = 2f(p) + f(q), 3f(r_{03}) = f(p) + 2f(q)\} .$$

It is straightforward to verify that $\mathcal{H}_1$, $\mathcal{H}_2$ are function spaces on L.

LEMMA 5.3.4. *Let $\mathcal{H}$ be a simplicial function space on a compact space K such that $\mathcal{H} = \mathcal{A}^c(\mathcal{H})$ and let $\mathcal{H}_1, \mathcal{H}_2$ be the function spaces on a compact space L constructed above. Then*

(a) $\mathrm{Ch}_{\mathcal{H}_1} L = \mathrm{Ch}_{\mathcal{H}_2} L$, *and if* $\mathrm{Ch}_{\mathcal{H}} K$ *is of type* F_σ, *then* $\mathrm{Ch}_{\mathcal{H}_1} L$ *is an* F_σ*–set as well;*
(b) *both* $\mathcal{H}_1$ *and* $\mathcal{H}_2$ *are simplicial;*
(c) $\mathcal{A}^c(\mathcal{H}_1) = \mathcal{H}_1$, $\mathcal{A}^c(\mathcal{H}_2) = \mathcal{H}_2$;
(d) *if* $\mathcal{H}$ *is* C*–complemented in* $C(K)$, *then* $\mathcal{H}_1$ *is* $\max\{C, 3\}$*–complemented in* $C(L)$;
(e) $\lambda(\mathcal{H}_2) \ge \lambda(\mathcal{H}) + (500\lambda(\mathcal{H}))^{-1}$.

Proof. For the proof of (a) it is enough to show that both the sets $\mathrm{Ch}_{\mathcal{H}_1} L$ and $\mathrm{Ch}_{\mathcal{H}_2} L$ equal

$$\{r_{ij} : i = -1, 1, j = 1, 2, 3\} \cup \{p, q\} \cup \bigcup_{i\in\mathbb{N}, j=1,2,3} \mathrm{Ch}_{\mathcal{H}} K_{ij}.$$

Indeed, for a point $x \in K_{ij}$ we have

$$x \in \mathrm{Ch}_{\mathcal{H}_1} L \Leftrightarrow x \in \mathrm{Ch}_{\mathcal{H}_2} L \Leftrightarrow x \in \mathrm{Ch}_{\mathcal{H}} K_{ij} ,$$

as the characteristic function $\chi_{K_{ij}} \in \mathcal{H}_1 \cap \mathcal{H}_2$, and hence every measure $\mu \in \mathcal{M}_x(\mathcal{H}_1) \cup \mathcal{M}_x(\mathcal{H}_2)$ is supported by K_{ij}. For the points

$$\{r_{ij} : i = -1, 1, j = 1, 2, 3\} \cup \{p, q\},$$

it is easy to find $\mathcal{H}_1$–exposing and $\mathcal{H}_2$–exposing functions and thus all these points belong to $\operatorname{Ch}_{\mathcal{H}_1} L \cap \operatorname{Ch}_{\mathcal{H}_2} L$.

On the other hand, the points $\{r_{0j} : j = 1,2,3\}$ have $\mathcal{H}_1$–representing measures

$$\frac{1}{2}\left(\varepsilon_{r_{-1,1}} + \varepsilon_{r_{1,1}}\right), \quad \frac{1}{2}\left(\varepsilon_{r_{-1,2}} + \varepsilon_{r_{1,2}}\right), \quad \frac{1}{2}\left(\varepsilon_{r_{-1,3}} + \varepsilon_{r_{1,3}}\right), \tag{12}$$

respectively, and $\mathcal{H}_2$–representing measures

$$\frac{1}{2}\left(\varepsilon_p + \varepsilon_q\right), \quad \frac{1}{3}\left(2\varepsilon_p + \varepsilon_q\right), \quad \frac{1}{3}\left(\varepsilon_p + 2\varepsilon_q\right), \tag{13}$$

respectively, and hence they do not belong to the Choquet boundaries $\operatorname{Ch}_{\mathcal{H}_1} L$ and $\operatorname{Ch}_{\mathcal{H}_2} L$.

To show (b), let x be a point of L. If $x \in K_{ij}$ for some i, j, then x has a unique $\mathcal{H}_1$–representing measure and a unique $\mathcal{H}_2$–representing measure, both supported by the Choquet boundary of L, since $\mathcal{H}$ is simplicial and every $\mathcal{H}_1$ or $\mathcal{H}_2$–representing measure is supported by K_{ij}.

To finish the reasoning it is enough to notice that the points r_{0j}, $j = 1,2,3$, have uniquely determined $\mathcal{H}_1$ and $\mathcal{H}_2$–representing measures carried by the Choquet boundary of L (see (12) and (13)).

For the proof of (c), let f be a function from $\mathcal{A}^c(\mathcal{H}_1)$. By the assumption, $f\restriction_{K_{ij}} \in \mathcal{H}$ for each K_{ij} and, obviously, f satisfies $2f(r_{0j}) = f(r_{-1j}) + f(r_{1j})$, $j = 1,2,3$. Hence $f \in \mathcal{H}_1$.

Analogously, $\mathcal{A}^c(\mathcal{H}_2) = \mathcal{H}_2$.

To verify (d), we assume that $P : C(K) \to \mathcal{H}$ is a projection of the norm C. We define an operator $Q : C(L) \to \mathcal{H}_1$ as

$$(Qf)(x) = \begin{cases} P(f\restriction_{K_{ij}})(x), & x \in K_{ij}, \\ f(x), & x = p, q, r_{ij},\ i = 0, -1, j = 1,2,3, \\ 2f(r_{0j}) - f(r_{-1j}), & x = r_{1j},\ j = 1,2,3. \end{cases}$$

It can be easily verified that Q is a projection of $C(L)$ onto $\mathcal{H}_1$ and $\|Q\| = \max\{C, 3\}$.

For the proof of (e), we define a compact space $\widetilde{L} = L \setminus \{r_{ij}; i = -1, 1, j = 1,2,3\}$ and a function space $\widetilde{\mathcal{H}_2} = \{f\restriction_{\widetilde{L}} \colon f \in \mathcal{H}_2\}$. Then $\widetilde{\mathcal{H}_2}$ can be considered to be a subspace of $\mathcal{H}_2$ via the isometric isomorphism $E : \widetilde{\mathcal{H}_2} \to \mathcal{H}_2$ defined as

$$(Ef)(x) = \begin{cases} f(r_{0j}), & x = r_{ij},\ i = 1, -1, j = 1,2,3, \\ f(x), & \text{elsewhere.} \end{cases}$$

By [**5**, Theorem], $\lambda(\widetilde{\mathcal{H}_2}) \geq \lambda(\mathcal{H}) + (500\lambda(\mathcal{H}))^{-1}$.

Since the operator $T : \mathcal{H}_2 \to \widetilde{\mathcal{H}_2}$ defined as

$$Tf = E(f\restriction_{\widetilde{L}}), \quad f \in \mathcal{H}_2,$$

is a projection of norm 1, we get from Lemma 5.3.3 that $\lambda(\widetilde{\mathcal{H}_2}) \leq \lambda(\mathcal{H}_2)$. Hence $\lambda(\mathcal{H}_2) \geq \lambda(\mathcal{H}) + (500\lambda(\mathcal{H}))^{-1}$, which completes the proof. □

5.4. Proof of the theorem

We start with a simplicial function space $\mathcal{H}$ on a metrizable compact space L such that $\mathcal{H} = \mathcal{A}^c(\mathcal{H})$, $\mathcal{H}$ is 1–complemented in $C(L)$ and $\operatorname{Ch}_{\mathcal{H}} L$ is of type F_σ (the simplest choice is to take L as a singleton and $\mathcal{H} = C(L)$). We define two sequences $\{(L^n, \mathcal{H}_1^n)\}$, $\{(L^n, \mathcal{H}_2^n)\}$ of function spaces as follows: $(L^1, \mathcal{H}_1^1) = (L^1, \mathcal{H}_2^1) = (L, \mathcal{H})$, and for $n \in \mathbb{N}$, the space $(L^{n+1}, \mathcal{H}_1^{n+1})$ is the space $\mathcal{H}_1$ from Lemma 5.3.4 constructed from $(L^n, \mathcal{H}_1^n)$ and $(L^{n+1}, \mathcal{H}_2^{n+1})$ is the space $\mathcal{H}_2$ constructed from $(L^n, \mathcal{H}_2^n)$.

Finally, let

$$L_\infty = \bigcup_{n=1}^{\infty} L_n \cup \{x_\infty\}$$

be the one–point compactification of the topological sum of L^n's and

$$\mathcal{H}_i = \{f \in C(L_\infty) :\ f \restriction_{L^n} \in \mathcal{H}_i^n,\ n \in \mathbb{N}\}\ , \quad i = 1, 2\ .$$

Given $i \in \{1, 2\}$, it is easy to realize that $\mathcal{H}_i$ is a simplicial function space, $\mathcal{A}^c(\mathcal{H}_i) = \mathcal{H}_i$ and

$$\operatorname{Ch}_{\mathcal{H}_i} L_\infty = \{x_\infty\} \cup \bigcup_{n=1}^{\infty} \operatorname{Ch}_{\mathcal{H}_i^n} L^n\ .$$

In particular, $\operatorname{Ch}_{\mathcal{H}_1} L = \operatorname{Ch}_{\mathcal{H}_2} L$ and it is an F_σ–set (see Lemma 5.3.4(a)).

According to Lemma 5.3.4(d), $\mathcal{H}_1^n$ is 3–complemented in $C(L^n)$ for each $n \in \mathbb{N}$. It follows that $\mathcal{H}_1$ is 3–complemented in $C(L_\infty)$.

Indeed, if $P_n : C(L^n) \to \mathcal{H}_1^n$ is a projection with $\|P_n\| \leq 3$, the mapping $Q : C(L_\infty) \to \mathcal{H}_1$ defined as

$$Qf(x) = \begin{cases} (P_n f)(x)\ , & x \in L^n, n \in \mathbb{N}\ , \\ f(x_\infty)\ , & x = x_\infty\ , \end{cases} \tag{14}$$

is a projection of $C(L_\infty)$ onto $\mathcal{H}_1$.

On the other hand, by Lemma 5.3.4(e), $\lambda(\mathcal{H}_2^n) \to \infty$. Since each $\mathcal{H}_2^n$ is 1–complemented in $\mathcal{H}_2$, $\mathcal{H}_2$ is not complemented in any $C(K)$ space (see Lemma 5.3.3).

The desired simplices X_1, X_2 will be the state spaces $\mathbf{S}(\mathcal{H}_1)$ and $\mathbf{S}(\mathcal{H}_2)$ (use Theorem 5.2.1). Let $\phi_i : L_\infty \to \mathbf{S}(\mathcal{H}_i)$, $i = 1, 2$, be the respective homeomorphic embeddings. Then $\phi = \phi_2 \circ \phi_1^{-1}$ is a homeomorphism of $\overline{\operatorname{ext} X_1}$ onto $\overline{\operatorname{ext} X_2}$ such that

$$\phi(\operatorname{ext} X_1) = \phi_2(\operatorname{Ch}_{\mathcal{H}_1} L_\infty) = \phi_2(\operatorname{Ch}_{\mathcal{H}_2} L_\infty) = \operatorname{ext} X_2\ .$$

Since $\mathcal{H}_1$ is complemented in $C(L_\infty)$, $A(X_1)$ is complemented in $C(X_1)$ as well. Indeed, using (14) we can define the mapping

$$\widetilde{Q} f = \Phi_1 Q(f \circ \phi_1)\ , \quad f \in C(X_1)\ ,$$

to get a projection of $C(X_1)$ onto $A(X_1)$ (we recall that Φ_1 is the isometric isomorphism of $\mathcal{H}_1$ onto $A(X_1)$).

As $A(X_2)$ is isometric with $\mathcal{H}_2$, $A(X_2)$ is not complemented in any $C(K)$ space. This finishes the proof.

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