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DIPLOMOVÁ PRÁCE

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Interpolation of Operators on Function Spaces

Katedra matematické analýzy

Vedoucí diplomové práce: Doc. RNDr. Luboš Pick, CSc., DSc.
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Rád bych zde poděkoval vedoucímu mé diplomové práce, doc. Luboši Pickovi, za příhodné nasměrování ve výběru témat týkajících se interpolace operátorů i za cenné rady při vlastním vypracování. Dále bych chtěl vyjádřit svůj vděk prof. Janu Malému, neboť v jeho přednáškách jsem prvně zakusil kouzlo teorie rearrangement-invariant prostorů.

Prohlašuji, že jsem svoji diplomovou práci napsal samostatně a výhradně s použitím citovaných pramenů. Souhlasím se zapůjčováním práce.

V Praze dne 8. dubna 2008 Bc. Lukáš Malý

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Abstrakt: *Calderónova věta patří mezi základní pilíře moderní teorie interpolace operátorů. Tvrdí, že veškeré kvazilineární operátory určitého okrajového chování jsou omezené na daném r.i. prostoru, právě když tzv. Calderónův operátor je omezený na odpovídajícím reprezentačním prostoru. Určitým okrajovým chováním se v případě Calderónovy věty rozumí, že operátory jsou omezené z* $L^{p_1,1}$ *do* $L^{q_1,\infty}$ *a zároveň z* $L^{p_2,1}$ *do* $L^{q_2,\infty}$*. Ukážeme, jakým způsobem lze Calderónovu větu zobecnit pro nestandardní okrajové chování a jak v takovém případě bude vypadat Calderónův operátor, přičemž jako okrajové prostory interpolačního segmentu zde vystupují Lorentzovy a Marcinkiewiczovy prostory. Pojednáme o dvojím druhu nestandardního chování, jednak prozkoumáme operátory omezené z* Λ_{X_1} *do* M_{Y_1} *a zároveň z* M_{X_2} *do* M_{Y_2}*, jednak operátory omezené z* Λ_{X_1} *do* Λ_{Y_1} *a zároveň z* Λ_{X_2} *do* M_{Y_2}*. K tomu účelu napřed spočteme Peetreho K-funkcionál pro různé dvojice Lorentzových a Marcinkiewiczových prostorů.*

Klíčová slova: *Calderónova věta, Calderónův operátor, interpolace kvazilineárních operátorů, Lorentzovy prostory, Marcinkiewiczovy prostory*

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Abstract: *Calderón Theorem is a fundamental theorem of modern theory of operator interpolation. It states that every quasilinear operator of certain endpoint behavior is bounded on some r.i. space if and only if so-called Calderón operator is bounded on corresponding representation space. In the case of Calderón Theorem, the certain endpoint behavior means boundedness of the operator both from* $L^{p_1,1}$ *to* $L^{q_1,\infty}$*, and from* $L^{p_2,1}$ *to* $L^{q_2,\infty}$*. We will show a way Calderón Theorem can be generalized for nonstandard endpoint behavior, and we will try and find Calderón operator in such a case where Lorentz and Marcinkiewicz spaces will be the endpoints of interpolation segment. Two distinctive types of nonstandard behavior are to be discussed; first, we'll explore the operators bounded both from* Λ_{X_1} *to* M_{Y_1}*, and from* M_{X_2} *to* M_{Y_2}*, next, operators bounded both from* Λ_{X_1} *to* Λ_{Y_1}*, and from* Λ_{X_2} *to* M_{Y_2}*. For that purpose, we evaluate the Peetre's K-functional for varied pairs of Lorentz and Marcinkiewicz spaces.*

Keywords: *Calderón Theorem, Calderón operator, Interpolation of quasilinear operators, Lorentz spaces, Marcinkiewicz spaces*

Introduction

Even though Lebesgue spaces play a main role in many areas of mathematical analysis, there are other important classes of Banach spaces of measurable functions, which are studied for their qualitative properties. The classical theory of interpolation makes use of weak Lebesgue spaces, which are a special case of Lorentz spaces. The modern theory goes even further and works with the so-called Banach function spaces, and particularly with rearrangement-invariant spaces.

The aim of the thesis is to study interpolation properties of quasilinear operators of nonstandard endpoint behavior. Calderón theorem is an example of a traditional result for operators of standard endpoint behavior. It states that every quasilinear operator of weak types (p_0, q_0) and (p_1, q_1) is bounded from an r.i. space X to an r.i. space Y if and only if a single operator, i.e., the Calderón operator corresponding to the interpolation segment $[(1/p_0, 1/q_0), (1/p_1, 1/q_1)]$, is bounded from X to Y.

We will establish an analogue of the Calderón theorem for operators bounded on generic Lorentz Λ_X and Marcinkiewicz M_X spaces. By a nonstandard endpoint behavior we mean two distinct cases. First, the operators bounded both from Λ_{X_1} to M_{Y_1}, and from M_{X_2} to M_{Y_2} will be studied. Under certain conditions we find a quasilinear operator, which plays the same role in the theory as the Calderón operator for the traditional Calderón Theorem does. The operator consists of two parts, one of which is linear, however the other one is merely quasilinear. Under more restrictive conditions, we show that the quasilinear part is insignificant, which leads to a stronger version of the Calderón-type theorem.

Then we will focus on the operators which are bounded both from Λ_{X_1} to Λ_{Y_1}, and from Λ_{X_2} to M_{Y_2}. This case is not covered in such a generality as the previous one, nevertheless a linear "Calderón" operator is found and correspondent Calderón-type theorem proven.

Before we can start the search for the appropriate form of the single operator, it is of utmost importance to evaluate the Peetre's K-functional for various couples of Lorentz and Marcinkiewicz spaces. Therefore, the second chapter is devoted to this calculation.

The first chapter then gives a brief overview of basic properties of Banach function spaces, rearrangement-invariant spaces, and finally Lorentz and Marcinkiewicz spaces.

Chapter 1

Preliminaries

Before we can start a pursuit of any new ideas in the modern theory of interpolation of operators, we're obliged to settle the essential terms and define the objects we will later work with. This chapter is supposed to give a brief overview of function spaces emerging in the later parts of the thesis. Therefore, the undermentioned propositions and theorems are asserted without proofs. Details can be found in chapters 1 and 2 of Bennett, Sharpley [1], as well as in chapters 1 and 2 of Kreĭn, Petunin, Semenov [4].

First, we need to agree on the notation and terminology used throughout the thesis. It is reasonable to strictly distinguish *positive* from *nonnegative* values (and *negative* from *nonpositive*). Often X^+ will be used to denote the positive members of the set X, whereas X_0^+ would mean the nonnegative members. Analogously, we would use X^- and X_0^-.

In order to discuss monotonicity of a function f, we will use the term *increasing* for the case when $t_1 < t_2$ implies $f(t_1) \leq f(t_2)$, whereas *strictly increasing* would mean the latter inequality to be strict. Analogously, the terms (*strictly*) *decreasing* will be used.

We are going to estimate function norms frequently by a multiple of other function norms to prove the boundedness of an operator, however, the actual multiplier is insignificant, therefore, $L \lesssim R$ will denote that there is a constant $c > 0$ such that $L \leq cR$. Furthermore, $L \approx R$ will be used to denote the *equivalence* of the expressions L and R, i.e., $L \lesssim R$ and $R \lesssim L$, simultaneously.

Since we will discuss estimates of integral expressions frequently, it will come in handy to define the *integral average* as

$$\fint_E f \, d\mu = \frac{1}{\mu(E)} \int_E f \, d\mu$$

for any μ-measurable set E of finite measure and any μ-measurable function f.

1.1 Banach Function Spaces

Since we are going to study interpolation of (quasi)linear operators on certain spaces of measurable functions, we need to specify in which sense the phrase "function space" is used so that we could work with such an object in both functional-analytic, and measure-theoretic ways.

Let $(\mathcal{R}, \mu)$ be a σ-finite measure space. Let $\mathcal{M}$ be the space of μ-measurable function on $\mathcal{R}$, and let $\mathcal{M}_0^+$ be the cone of nonnegative functions in $\mathcal{M}$.

Definition 1.1. A mapping $\rho : \mathcal{M}_0^+ \to [0, \infty]$ is called a *Banach function norm* if, for all $f, g, f_n, (n = 1, 2, 3, \dots)$, in $\mathcal{M}_0^+$, for all constants $a \geq 0$, and for all μ-measurable subsets E of $\mathcal{R}$, the following properties hold:

(B1) $\rho(f) = 0$ if and only if $f = 0$ μ-a.e.,
$\rho(af) = a\rho(f)$,
$\rho(f + g) \leq \rho(f) + \rho(g)$;

(B2) if $g \leq f$ μ-a.e., then $\rho(g) \leq \rho(f)$;

(B3) if $f_n \uparrow f$ μ-a.e., then $\rho(f_n) \uparrow \rho(f)$;

(B4) if $\mu(E) < \infty$, then $\rho(\chi_E) < \infty$;

(B5) if $\mu(E) < \infty$, then there is a constant $C_E \in (0, \infty)$ depending on E and ρ but independent of f such that $\int_E f \, d\mu \leq C_E \rho(f)$.

Definition 1.2. Let ρ be a Banach function norm. The set $X_\rho = \{f \in \mathcal{M} : \rho(|f|) < \infty\}$ is called a *Banach function space*. For each $f \in X_\rho$, define $\|f\|_{X_\rho} = \rho(|f|)$.

Definition of a function space by these axioms turns out to be a good choice for the space has several important properties then.

Theorem 1.3. *Let ρ be a Banach function norm, and $X = X_\rho$ be the corresponding Banach function space. Then $(X, \|.\|_X)$ is a Banach space. Moreover, the following properties hold for all $f, g, f_n \in \mathcal{M}, (n = 1, 2, \dots)$, and all measurable sets $E \subset \mathcal{R}$:*

(i) (the lattice property) *If $|g| \leq |f|$ μ-a.e. and $f \in X$, then $g \in X$ and $\|g\|_X \leq \|f\|_X$.*

(ii) (the Fatou property) *Assume $f_n \in X, f_n \geq 0, (n = 1, 2, \dots)$, and $f_n \uparrow f$ μ-a.e. Then $\|f_n\|_X \uparrow \|f\|_X$ if $f \in X$, and $\|f_n\|_X \uparrow \infty$ otherwise.*

(iii) (Fatou's lemma) *If $f_n \in X, (n = 1, 2, \dots), f_n \to f$ μ-a.e., $\liminf_{n\to\infty} \|f_n\|_X < \infty$, then $f \in X$, and $\|f\|_X \leq \liminf_{n\to\infty} \|f_n\|_X$.*

(iv) *Every simple function belongs to X.*

(v) *Provided that E is of finite measure, there is a constant* $C_E \in (0, \infty)$ *such that* $\int_E |f|\,d\mu \le C_E \|f\|_X$ *for all* $f \in X$.

(vi) *If* $f_n \to f$ *in* X, *then* $f_n \to f$ *in measure on every set of finite measure. In particular, there is a subsequence of* $\{f_n\}$ *converging to* f *pointwise* μ*-a.e.*

Proof of these properties based on the axioms of Banach function space can be found in Chapter 1, Section 1 of Bennett, Sharpley [1].

The Hölder inequality plays an important role in the theory of Lebesgue spaces. It links $L^{p'}$ space to L^p space and since $L^{p'}$ represents a dual of L^p, the Hölder inequality is sharp in the sense that $\|f\|_{p'} = \sup\{\int fg : \|g\|_p \le 1\}$. We can construct a function norm based on an analogue of Hölder inequality for general Banach function space, and it can be proven that such a norm is an actual Banach function norm.

Definition 1.4. Let ρ be a function norm. Its *associate norm* ρ' is defined on $\mathcal{M}_0^+$ by

$$\rho'(g) = \sup\left\{\int_{\mathcal{R}} fg\,d\mu : f \in \mathcal{M}_0^+, \rho(f) \le 1\right\}.$$

The space $X_{\rho'}$ determined by ρ' is then called the *associate space* of X and is denoted by X'.

Theorem 1.5. *Let* ρ *be a Banach function norm, then the associate norm* ρ' *is itself a Banach function norm. In particular, the associate space* X' *is a Banach function space.*

Corollary 1.6 (Hölder's inequality). *Let* X *be a Banach function space and* X' *its associate space. If* $f \in X$ *and* $g \in X'$, *then* fg *is integrable and*

$$\int_{\mathcal{R}} |fg|\,d\mu \le \|f\|_X \|g\|_{X'}.$$

The following result shows that an analogous term for reflexivity in the terms of associate spaces has declined in importance.

Theorem 1.7 (Lorentz, Luxemburg). *Every Banach function space* X *coincides with its second associate space* X''. *Furthermore,* $\|f\|_X = \|f\|_{X''}$ *for every* $f \in X$.

Corollary 1.8. *Let* X *be a Banach function space and* X' *its associate space. Then,*

$$\|f\|_X = \sup\left\{\int_{\mathcal{R}} |fg|\,d\mu : g \in X', \|g\|_{X'} \le 1\right\},$$
$$\|g\|_{X'} = \sup\left\{\int_{\mathcal{R}} |fg|\,d\mu : f \in X, \|f\|_X \le 1\right\}.$$

These results on the associated spaces are compiled in Chapter 1, Section 2 of Bennett, Sharpley [1].

1.2 Rearrangement-Invariant Function Spaces

The essential class of Banach function spaces consists of function spaces whose norm is resistant to a special measure-preserving transformation, namely decreasing rearrangement.

Definition 1.9. Let $f \in \mathcal{M}(\mathcal{R}, \mu)$ be finite μ-a.e. The *distribution function* of f is given for any $\lambda \geq 0$ by

$$\mu_f(\lambda) = \mu\{x \in \mathcal{R} : |f(x)| > \lambda\}.$$

Two functions $f \in \mathcal{M}(\mathcal{R}, \mu)$ and $g \in \mathcal{M}(\mathcal{S}, \nu)$ are called *equimeasurable* if their distribution functions coincide on $[0, \infty)$.

Definition 1.10. Let $f \in \mathcal{M}(\mathcal{R}, \mu)$ be finite μ-a.e. The *decreasing rearrangement* of f is the function f^* defined on $[0, \infty)$ by

$$f^*(t) = \inf\{\lambda \geq 0 : \mu_f(\lambda) \leq t\}$$

where the convention that $\inf \varnothing = \infty$ is used.

The operator of decreasing rearrangement possesses some useful qualities which are summed up in the next proposition. Alas, the decreasing rearrangement is not subadditive, i.e., the inequality $(f + g)^*(t) \leq f^*(t) + g^*(t)$ does not necessarily hold. This fact causes quite a few difficulties in the theory. The detailed treatise on decreasing rearrangement is given in Chapter 2, Sections 1 & 2 of Bennett, Sharpley [1], as well as in Chapter 2, Section 2 of Kreĭn, Petunin, Semenov [4].

Proposition 1.11. *Let f, g, and f_n, $(n = 1, 2, \dots)$ belong to $\mathcal{M}$ and be finite μ-a.e. The decreasing rearrangement f^* is a nonnegative, decreasing, right-continuous function on $[0, \infty)$. Furthermore,*

(i) *if $|g| \leq |f|$ μ-a.e., then $g^* \leq f^*$;*

(ii) *$(af)^* = |a| f^*$, whenever $a \in \mathbb{R}$;*

(iii) *$(f + g)^*(t_1 + t_2) \leq f^*(t_1) + g^*(t_2)$, whenever $t_1, t_2 \geq 0$;*

(iv) *if $|f| \leq \liminf_{n\to\infty} |f_n|$ μ-a.e., then $f^* \leq \liminf_{n\to\infty} f_n^*$;*
in particular, if $|f_n| \uparrow |f|$ μ-a.e., then $f_n^* \uparrow f^*$;

(v) *f and f^* are equimeasurable.*

Theorem 1.12 (Hardy, Littlewood). *If $f, g \in \mathcal{M}(\mathcal{R}, \mu)$ are finite μ-a.e., then*

$$\int_{\mathcal{R}} |fg| \, d\mu \leq \int_0^{\mu(\mathcal{R})} f^*(s) g^*(s) \, ds.$$

Definition 1.13. A σ-finite measure space $(\mathcal{R},\mu)$, is said to be *resonant* if, for each $f,g \in \mathcal{M}(\mathcal{R},\mu)$, the identity

$$\int_0^{\mu(\mathcal{R})} f^*(s)g^*(s)\,ds = \sup\left\{\int_{\mathcal{R}} |f\tilde{g}|\,d\mu : \tilde{g} \in \mathcal{M}(\mathcal{R},\mu) \text{ is equimeasurable with } g\right\}$$

holds.

Proposition 1.14. *A σ-finite nonatomic measure space is resonant.*

The notion of resonant spaces will prove to be worthwhile later when we represent a rearrangement-invariant space by a function space consisting of functions on positive real line.

Definition 1.15. Let $f \in \mathcal{M}(\mathcal{R},\mu)$ be finite μ-a.e., then the *maximal function* of f^* is defined by

$$f^{**}(t) = \fint_0^t f^*(s)\,ds \quad (t > 0).$$

The maximal function of f^* has similar qualities as f^*, moreover, it is subadditive, i.e., $(f+g)^{**}(t) \le f^{**}(t) + g^{**}(t)$ holds. Therefore, f^{**} often replaces f^* if subadditivity is essential, e.g., in the definition of Marcinkiewicz spaces.

Proposition 1.16. *Let f, g, f_n comply with the assumptions of Proposition 1.11. Then f^{**} is nonnegative, decreasing, and continuous on $(0,\infty)$. Moreover, it has the following properties:*

(i) $f^{**} \equiv 0$ *if and only if* $f = 0$ μ*-a.e.;*

(ii) $f^* \le f^{**}$*;*

(iii) *if* $|g| \le |f|$ μ*-a.e., then* $g^{**} \le f^{**}$*;*

(iv) $(af)^{**} = |a|f^{**}$*, whenever* $a \in \mathbb{R}$*;*

(v) *if* $|f_n| \uparrow |f|$ μ*-a.e., then* $f_n^{**} \uparrow f^{**}$*;*

(vi) $(f+g)^{**}(t) \le f^{**}(t) + g^{**}(t)$*, whenever* $t > 0$*.*

Proposition 1.17 (Hardy's lemma). *Let f_1 and f_2 be nonnegative measurable functions on $(0,\infty)$, which conform to*

$$\int_0^t f_1(s)\,ds \le \int_0^t f_2(s)\,ds$$

for every $t > 0$. Suppose g is a nonnegative decreasing function on $(0,\infty)$. Then,

$$\int_0^\infty f_1(s)g(s)\,ds \le \int_0^\infty f_2(s)g(s)\,ds.$$

Definition 1.18. A Banach function norm ρ over a σ-finite measure space $(\mathcal{R}, \mu)$ is said to be *rearrangement-invariant* if $\rho(f) = \rho(g)$ for every pair of nonnegative equimeasurable functions f and g. The Banach function space X_ρ determined by ρ is then called *rearrangement-invariant space,* or *r.i. space*.

Proposition 1.19. *Let X be a Banach function space over a resonant measure space. Then X is rearrangement-invariant if and only if the associate space X' is. Then, the norms are given by*

$$\|f\|_X = \sup\left\{\int_0^\infty f^*(s)g^*(s)\,ds : \|g\|_{X'} \le 1\right\},$$
$$\|g\|_{X'} = \sup\left\{\int_0^\infty f^*(s)g^*(s)\,ds : \|f\|_X \le 1\right\}.$$

Corollary 1.20 (Hölder's inequality). *Let X be a rearrangement-invariant space over a resonant measure space $(\mathcal{R}, \mu)$. If $f \in X$ and $g \in X'$, then fg is integrable and*

$$\int_{\mathcal{R}} |fg|\,d\mu \le \int_0^\infty f^*(s)g^*(s)\,ds \le \|f\|_X\|g\|_{X'}.$$

Corollary 1.21 (Hardy-Littlewood-Pólya Principle). *Assume X is an r.i. space over a resonant measure space $(\mathcal{R}, \mu)$. Let $f \in \mathcal{M}(\mathcal{R}, \mu)$, and $g \in X$. If $f^{**}(t) \le g^{**}(t)$ for each $t > 0$, then f belongs to X and $\|f\|_X \le \|g\|_X$.*

The Hardy's lemma together with the Hardy-Littlewood-Pólya principle could be referred to as key lemmas of the theory of rearrangement-invariant spaces since they introduce an elegant way how to show estimates of norms of functions.

No less important is the following theorem asserting that every rearrangement-invariant space over a resonant measure space can be represented by a function space over positive real line with Lebesgue measure. Unfortunately, this representation is not unique neither for spaces of finite measure, nor for atomic spaces. No less interesting is the fact that the associate space can be represented by the associate space of the representation space. For details cf. Chapter 2, Section 4 of Bennett, Sharpley [1].

Theorem 1.22 (Luxemburg Representation Theorem). *Let ρ be a rearrangement-invariant function norm over a resonant measure space $(\mathcal{R}, \mu)$. Then there is a rearrangement-invariant function norm $\bar{\rho}$ over $(\mathbb{R}_0^+, \lambda)$ such that $\rho(f) = \bar{\rho}(f^*)$ for every $f \in \mathcal{M}$. Moreover, the associate norm ρ' of ρ can be represented in the same way by the associate norm $\bar{\rho}'$ of $\bar{\rho}$.*

Next, we are about to define the fundamental function of an r.i. space. This function characterizes the space in many ways. Moreover, the definitions of Lorentz and Marcinkiewicz spaces are directly based on the fundamental functions.

Definition 1.23. Let X be an r.i. space over a resonant measure space $(\mathcal{R}, \mu)$. For every $t > 0$ which belongs to the range of μ, define $\Phi_X(t) = \|\chi_E\|_X$, where E is any subset of $\mathcal{R}$, whose measure equals t. The function Φ_X is called the *fundamental function* of X.

Example 1.24. Let's find the fundamental function of the Lorentz spaces $L^{p,q}$ for any $p, q \in [1, \infty]$. These spaces are defined by the Lorentz (quasi)norm, i.e.,

$$\begin{aligned} \|f\|_{L^{p,q}} &= \|f^*(t)t^{\frac{1}{p}-\frac{1}{q}}\|_{L^q}, && \text{if } p < \infty \text{ and } q < \infty, \\ \|f\|_{L^{p,\infty}} &= \|f^*(t)t^{\frac{1}{p}}\|_{L^\infty}, && \text{if } p < \infty, \\ \|f\|_{L^{\infty,\infty}} &= \|f^*(t)\|_{L^\infty} = \|f\|_{L^\infty}. \end{aligned}$$

Analogously, $L^{\infty,q}$ "norm" could be defined for $q < \infty$, however, a space determined by such a norm would be trivial. Now, considering $f^*(t) = \chi_{(0,x)}(t)$, for $x > 0$, we can easily calculate that

$$\begin{aligned} \Phi_{L^{p,q}}(t) &= (q/p)^{1/q} t^{1/p}, && \text{if } p < \infty \text{ and } q < \infty, \\ \Phi_{L^{p,\infty}}(t) &= t^{1/p}, && \text{if } p < \infty, \\ \Phi_{L^{\infty,\infty}}(t) &= \chi_{(0,\infty)}(t). \end{aligned}$$

Theorem 1.25. *Let X be an r.i. space over a resonant measure space (R, μ) and let X' be its associate space. Then the fundamental function has the following properties:*

(i) Φ_X *is increasing;* $\Phi_X(t) = 0$ *if and only if* $t = 0$*;*

(ii) $\Phi_X(t)\Phi_{X'}(t) = t$*, whenever t is a finite value in the range of μ;*

(iii) $\Phi_X(t)/t$ *is decreasing;*

(iv) Φ_X *is continuous, except perhaps at the origin.*

Definition 1.26. Let Φ be a nonnegative increasing function defined on $\mathbb{R}_0^+$. Given $\Phi(t) = 0$ if and only if $t = 0$, and $\Phi(t)/t$ is decreasing on $\mathbb{R}^+$, then Φ is said to be *quasiconcave.*

Lemma 1.27. *Let Φ be a quasiconcave function. Then*

$$\frac{\Phi(u+h) - \Phi(u)}{h} \leq \frac{\Phi(u)}{u}$$

holds for all positive values h and u. In particular, if Φ is (unilaterally) differentiable at $u > 0$, and φ denotes its (unilateral) derivative, then

$$\varphi(u) \leq \frac{\Phi(u)}{u}. \tag{1.1}$$

Proposition 1.28. *The least concave majorant* $\tilde{\Phi}$ *of a quasiconcave function* Φ *satisfies* $\tilde{\Phi}/2 \leq \Phi \leq \tilde{\Phi}$.

Properties of the fundamental functions are thoroughly covered in Chapter 2, Section 5 of Bennett, Sharpley [1] and in Chapter 2, Section 4 of Kreĭn, Petunin, Semenov [4].

Proposition 1.29. *Let* X *be an r.i. space over a resonant measure space* $(\mathcal{R}, \mu)$*. Then* X *can be equivalently renormed with an r.i. norm in such a way that the resulting fundamental function is concave.*

Throughout the thesis, all rearrangement-invariant spaces are supposed to be renormed in such a fashion that their fundamental functions are concave.

1.3 The Peetre K-functional

The K-functional is an object which takes four parameters – a function, a positive real number, and a couple of Banach function spaces. Therefore, it can be regarded either as a functional equivalent to the norm of the sum of the two spaces involved, or as a function of a real argument in which case it gives us a powerful instrument in the interpolation theory.

Definition 1.30. Let X and Y be Banach function spaces. The pair (X, Y) is called a *compatible couple* if there exists a Hausdorff topological vector space in which each of X and Y is continuously embedded.

Definition 1.31. Let (X, Y) be a compatible couple where $\mathcal{X}$ is the corresponding Hausdorff space. Set $X + Y = \{f \in \mathcal{X} : f = g + h, g \in X, h \in Y\}$ whose norm is defined by

$$\|f\|_{X+Y} = \inf\{\|g\|_X + \|h\|_Y : f = g + h\}.$$

Definition 1.32. Let (X, Y) be a compatible couple. For each $f \in X + Y$ and $t > 0$ define the *K-functional* by

$$K(f, t; X, Y) = \inf\{\|g\|_X + t\|h\|_Y : f = g + h, g \in X, h \in Y\}.$$

Lemma 1.33. *For each* $f \in X + Y$*, the K-functional* $K(f, t; X, Y)$ *is a nonnegative quasiconcave function of* $t > 0$*. Moreover,*

$$t^{-1}K(f, t; X, Y) = K(f, t^{-1}; Y, X).$$

For any fixed $t > 0$*, the functional* $f \mapsto K(f, t; X, Y)$ *defines a norm equivalent to* $\|.\|_{X+Y}$ *since*

$$\min\{1, t\}\|f\|_{X+Y} \leq K(f, t; X, Y) \leq \max\{1, t\}\|f\|_{X+Y}.$$

Observation 1.34. *Let* $f \in X$ *and* $g \in Y$*. Then, for any* $t > 0$*,* $K(f, t; X, Y) \leq \|f\|_X$ *and* $K(g, t; X, Y) \leq t\|g\|_Y$.

1.4 Lorentz and Marcinkiewicz Spaces

Lorentz and Marcinkiewicz spaces play a key role in the area of rearrangement-invariant spaces since they are the smallest, and the largest r.i. space with a given fundamental function, respectively. Properties of these types of r.i. spaces in detail are presented in Chapter 2, Section 5 of Kreĭn, Petunin, Semenov [4]. Bennett, Sharpley [1] introduce Lorentz and Marcinkiewicz spaces in Chapter 2, Section 5.

Definition 1.35. Let X be an r.i. space over a resonant nonatomic measure space $(\mathcal{R}, \mu)$. Suppose X has been renormed so that its fundamental function Φ_X is concave. The *Lorentz space* Λ_X consists of all $f \in \mathcal{M}(\mathcal{R}, \mu)$ for which

$$\|f\|_{\Lambda_X} = \int_0^\infty f^*(s)d\Phi_X(s)$$

is finite.

The *Marcinkiewicz space* M_X consists of all $f \in \mathcal{M}(\mathcal{R}, \mu)$ for which

$$\|f\|_{M_X} = \sup_{0<t<\infty} \{f^{**}(t)\Phi_X(t)\}$$

is finite.

Observation 1.36. *Since Φ_X is nonnegative, continuous (except perhaps at the origin), increasing and concave, hence it is absolutely continuous (except perhaps at the origin). Thus, it may be represented as the integral of a nonnegative, decreasing function φ_X on $\mathbb{R}^+$. Therefore, the Lebesgue-Stieltjes integral in the definition of Lorentz norm may be rewritten in the form*

$$\|f\|_{\Lambda_X} = \|f\|_{L^\infty}\Phi_X(0+) + \int_0^\infty f^*(s)\varphi_X(s)\, ds.$$

From now on, let the derivative of a fundamental function (in the sense of the previous observation) be denoted by the same letter as the function itself, however lowercase.

Theorem 1.37. *Let X be a rearrangement-invariant Banach function space over a resonant nonatomic measure space $(\mathcal{R}, \mu)$. Suppose X has been renormed to have concave fundamental function Φ_X. Then the Lorentz and Marcinkiewicz spaces, Λ_X and M_X, are rearrangement-invariant Banach function spaces and both have fundamental function equal to Φ_X. Moreover,*

$$\Lambda_X \hookrightarrow X \hookrightarrow M_X$$

and each of the embeddings has norm 1*.*

Definition 1.38. Let X be an r.i. space over a resonant nonatomic measure space $(\mathcal{R}, \mu)$. Suppose X has been renormed so that its fundamental function Φ_X is concave. Then, the *strong Lorentz space* Λ_X^* consists of all $f \in \mathcal{M}(\mathcal{R}, \mu)$ for which

$$\|f\|_{\Lambda_X^*} = \int_0^\infty f^*(s)\Phi_X(s)\frac{ds}{s}$$

is finite.

The *weak Marcinkiewicz space* M_X^* consists of all $f \in \mathcal{M}(\mathcal{R}, \mu)$ for which

$$\|f\|_{M_X^*} = \sup_{0<t<\infty} \{f^*(t)\Phi_X(t)\}$$

is finite.

Remark 1.39. We can see from the definition of the Λ_X^* space that the fundamental function needs to obey $\Phi_X(0+) = 0$ so that the Λ_X^* space can be nontrivial.

Proposition 1.40. *Let X be a rearrangement-invariant Banach function space over a resonant nonatomic measure space $(\mathcal{R}, \mu)$. Suppose X has been renormed to have concave fundamental function Φ_X. Then the strong Lorentz space Λ_X^* is a rearrangement-invariant Banach function space whose fundamental function equals*

$$\Phi_{\Lambda_X^*}(t) = \int_0^t \Phi_X(s)\frac{ds}{s}.$$

The weak Marcinkiewicz space M_X^* is a quasinormed linear space compliant with all conditions of rearrangement-invariant Banach function spaces except (perhaps) the condition (B5) *of Definition 1.1 and (perhaps) being a normed space. Its fundamental function equals Φ_X. In particular, the condition* (B5) *is satisfied if and only if* $1/\Phi_X$ *is integrable at the origin.*

Moreover,

$$\Lambda_X^* \hookrightarrow \Lambda_X \hookrightarrow X \hookrightarrow M_X \hookrightarrow M_X^*,$$

and, for any μ-measurable function f,

$$\|f\|_{M_X^*} \leq \|f\|_{M_X} \leq \|f\|_X \leq \|f\|_{\Lambda_X} \leq \|f\|_{\Lambda_X^*}. \tag{1.2}$$

Chapter 2

Evaluation of the K-Functional

In order to find an analogue of the Calderón Theorem for Lorentz and Marcinkiewicz spaces, it is of utmost importance to evaluate the Peetre's K-functional for various couples of these spaces. Results in this chapter are based on the work of Mario Milman [5] who computed the K-functionals for the pairs (M^*_X, Y) and (Λ^*_X, Y). The assumptions we are about to impose on the involved spaces are rather strong, nevertheless, there are numerous spaces compliant with them.

2.1 The (M_X, Y) Pair

Theorem 2.1. *Let X and Y be r.i. spaces over a σ-finite nonatomic measure space $(\mathcal{R}, \mu)$ having fundamental functions Φ and Ψ, respectively. Let $s = \Phi/\Psi$; suppose that there exists an $r > 0$ such that $s(u)/u^r$ is increasing on $\mathbb{R}^+$. Then there exist $c_1 > 0$, and $c_2 > 0$ such that, for any $t > 0$ and $f \in M_X + Y$, the estimate*

$$\begin{aligned} c_1 \left(\|f^* \chi_A\|_{\overline{M_X}} + t\|f^*(2u)\chi_B(u)\|_{\overline{Y}} \right) &\leq K(f, t; M_X, Y) \\ &\leq c_2 \left(\|f^* \chi_A\|_{\overline{M_X}} + t\|f^* \chi_B\|_{\overline{Y}} \right) \end{aligned}$$

holds, where $A = \{u \in \mathbb{R}^+ : s(u) < t\}$, $B = \mathbb{R}^+ \setminus A$.

Remark 2.2. In the preceding theorem we supposed there was an $r > 0$ such that $s(u)/u^r$ was an increasing function, where $s = \Phi/\Psi$ was a ratio of fundamental functions. Therefore, $s(u)$ is a strictly increasing function and $s(0+) = 0$. Consequently, $\Phi(0+) = 0$ holds whereas $\Psi(0+)$ may be of any nonnegative value. Pondering the fact that the derivative of $s(u)/u^r$ is nonnegative, we will come up with the following equivalences,

$$s'(u) \approx \frac{s(u)}{u}, \qquad \varphi(u) \approx \frac{\Phi(u)}{u}.$$

A weaker form of the latter equivalence is going to play a key role in the Calderón-type theorem for the case of operators of $(\Lambda_{X_1}, \Lambda_{Y_1}; \Lambda_{X_2}, M_{Y_2})$ type. Instances of spaces compliant with this condition can be found in Example 3.15.

Proof of Theorem 2.1. Let $f \in M_X + Y$ be decomposed as $f = f_0 + f_1$ where $f_0 \in M_X$ and $f_1 \in Y$. Then,

$$2(\|f_0\|_{M_X} + t\|f_1\|_Y) \geq \sup_{u>0}\{f_0^{**}(u)\chi_A(u)\Phi(u)\} + t\|f_1^*\chi_A\|_{\overline{Y}} \qquad (I_1)$$
$$+ \sup_{u>0}\{f_0^{**}(u)\chi_B(u)\Phi(u)\} + t\|f_1^*\chi_B\|_{\overline{Y}}. \qquad (I_2)$$

We estimate the right hand side of the inequality for the terms on the first line and for the terms on the second line separately.

To obtain the estimate of I_1 we use inequality (1.2) and the definition of the set A,

$$\begin{aligned} t\|f_1^*\chi_A\|_{\overline{Y}} &\geq t\sup_{u>0}\{f_1^{**}(u)\chi_A(u)\Psi(u)\} \\ &\geq \sup_{u>0}\{f_1^{**}(u)\chi_A(u)\Phi(u)\}. \end{aligned}$$

Thus,

$$\begin{aligned} I_1 &\geq \sup_{u>0}\{f_0^{**}(u)\chi_A(u)\Phi(u)\} + \sup_{u>0}\{f_1^{**}(u)\chi_A(u)\Phi(u)\} \\ &\geq \sup_{u>0}\{f^{**}(u)\chi_A(u)\Phi(u)\} = \|f^*\chi_A\|_{\overline{M_X}}. \end{aligned} \qquad (2.1)$$

The estimate of I_2 will be reached by working backwards, i.e., we're going to search estimates of the term we want to get until we reach I_2. We proceed from (1.2),

$$\begin{aligned} \|f_0^*\chi_B\|_{\overline{Y}} &\leq \|f_0^*\chi_B\|_{\overline{\Lambda_Y}} \\ &= (f_0^*\chi_B)^*(0+)\Psi(0+) + \int_0^\infty (f_0^*\chi_B)^*(u)\psi(u)du \\ &= f_0^*(s^{-1}(t))\Psi(0+) + \int_0^\infty f_0^*(u + s^{-1}(t))\psi(u)du \\ &\leq f_0^*(s^{-1}(t))\Psi(0+) + \int_0^{s^{-1}(t)} f_0^*(u + s^{-1}(t))\psi(u)du \\ &\quad + \int_{s^{-1}(t)}^\infty f_0^*(u)\psi(u)du \\ &= f_0^*(s^{-1}(t))\Psi(0+) + J_1 + J_2. \end{aligned} \qquad (2.2)$$

Using the monotonicity of f_0^* and the definition of the function s, we have,

$$\begin{aligned} f_0^*(s^{-1}(t))\Psi(0+) + J_1 &\leq f_0^*(s^{-1}(t))\Psi(0+) + f_0^*(s^{-1}(t))\int_0^{s^{-1}(t)} \psi(u)du \\ &= f_0^*(s^{-1}(t))\Psi(0+) + f_0^*(s^{-1}(t))[\Psi(s^{-1}(t)) - \Psi(0+)] \\ &= f_0^*(s^{-1}(t))\Psi(s^{-1}(t)) \\ &= t^{-1}f_0^*(s^{-1}(t))\Phi(s^{-1}(t)) \\ &\leq t^{-1}\sup_{u>0}\{f_0^{**}(u)\chi_B(u)\Phi(u)\}. \end{aligned} \qquad (2.3)$$

Using (1.1) and the assumption on the growth of $s(u)/u^r$ as indicated in Remark 2.2, we get

$$\begin{aligned} J_2 &\le \int_{s^{-1}(t)}^{\infty} f_0^*(u)\Psi(u)\frac{du}{u} = \int_{s^{-1}(t)}^{\infty} f_0^*(u)\Psi(u)\frac{s(u)}{u}\frac{du}{s(u)} \\ &\approx \int_t^{\infty} f_0^*(s^{-1}(z))\Psi(s^{-1}(z))\frac{dz}{z} \le \int_t^{\infty} f_0^*(s^{-1}(z))\Phi(s^{-1}(z))\frac{dz}{z^2} \\ &\le \frac{1}{t}\sup_{u>0}\{f_0^*(u)\chi_B(u)\Phi(u)\}. \end{aligned} \tag{2.4}$$

From (2.2), (2.3) and (2.4) it follows that

$$t\|f_0^*\chi_B\|_{\overline{Y}} \le c\sup_{u>0}\{f_0^{**}(u)\chi_B(u)\Phi(u)\}.$$

Therefore,

$$I_2 \ge ct(\|f_0^*\chi_B\|_{\overline{Y}} + \|f_1^*\chi_B\|_{\overline{Y}}) \ge ct\|f^*(2u)\chi_B(u)\|_{\overline{Y}}. \tag{2.5}$$

Combining (2.1) and (2.5) we get the desired estimate of the K-functional from below.

To get the upper bound, let's consider the following decomposition of $f = f_0 + f_1$:

$$f_0 = \begin{cases} f - f^*(s^{-1}(t)) & \text{if } f > f^*(s^{-1}(t)), \\ f + f^*(s^{-1}(t)) & \text{if } f < -f^*(s^{-1}(t)), \\ 0 & \text{otherwise,} \end{cases}$$

$$f_1 = f - f_0.$$

It follows that $f^* = f_0^* + f_1^*$, $f_0^*(u) = 0$ if $u \ge s^{-1}(t)$, and $f_1^*(u) = f^*(s^{-1}(t))$ for $u < s^{-1}(t)$. Then, using the definition of the set A,

$$\begin{aligned} t\|f_1^*\|_{\overline{Y}} &\le t\|f_1^*\chi_A\|_{\overline{Y}} + t\|f_1^*\chi_B\|_{\overline{Y}} \\ &= tf^*(s^{-1}(t))\Psi(s^{-1}(t)) + t\|f^*\chi_B\|_{\overline{Y}} \\ &\le \sup_{u>0}\{f^*(u)\chi_A(u)\Phi(u)\} + t\|f^*\chi_B\|_{\overline{Y}} \\ &\le \|f^*\chi_A\|_{\overline{M_X}} + t\|f^*\chi_B\|_{\overline{Y}}. \end{aligned}$$

Moreover,

$$\|f_0\|_{M_X} = \|f_0^*\|_{\overline{M_X}} = \|f_0^*\chi_A\|_{\overline{M_X}} \le \|f^*\chi_A\|_{\overline{M_X}},$$

which finishes the proof of the upper bound. □

Observation 2.3. *Let X be an r.i. space over a σ-finite nonatomic measure space $(\mathcal{R}, \mu)$. Then, whenever $a > 0$,*

$$\|f^*(2u)\chi_{(0,a)}(u)\|_{\overline{X}} \approx \|f^*(u)\chi_{(0,a)}(u)\|_{\overline{X}},$$
$$\|f^*(2u)\chi_{[a,\infty)}(u)\|_{\overline{X}} \approx \|f^*(u)\chi_{[a,\infty)}(u)\|_{\overline{X}}.$$

Proof. Whenever $u \in \mathbb{R}^+$, $f^*(2u) \le f^*(u)$ due to the monotonicity of f^*. Therefore, $\|f^*(2u)\chi_I(u)\|_{\overline{X}} \le \|f^*(u)\chi_I(u)\|_{\overline{X}}$, where I denotes the interval of either $(0,a)$, or $[a,\infty)$ type.

On account of the Hardy-Littlewood-Pólya principle, it suffices to show that the estimate

$$\int_0^t (f^*(2\cdot)\chi_I(\cdot))^*(u)du \gtrsim \int_0^t (f^*\chi_I)^*(u)du$$

holds for every $t \in \mathbb{R}^+$ in order to show the other inequality. Let's consider the case of the interval $(0,a)$ first. Then, the outer rearrangement operator can be omitted. Thus,

$$\int_0^t f^*(2u)\chi_I(u)du = \frac{1}{2}\int_0^{2t} f^*(u)\chi_I(u/2)du \ge \frac{1}{2}\int_0^t f^*(u)\chi_I(u)du.$$

Now, let $I = [a,\infty)$. Then,

$$\begin{aligned}\int_0^t (f^*(2\cdot)\chi_I(\cdot))^*(u)du &= \int_0^t f^*(2u+a)du = \frac{1}{2}\int_0^{2t} f^*(u+a)du \\ &\ge \frac{1}{2}\int_0^t f^*(u+a)du = \frac{1}{2}\int_0^t (f^*\chi_I)^*(u)du.\end{aligned}$$

□

Corollary 2.4. *Under the assumptions of Theorem 2.1, the following estimate holds,*

$$K(f,t;M_X,Y) \approx \|f^*\chi_A\|_{\overline{M_X}} + t\|f^*\chi_B\|_{\overline{Y}}.$$

Proof. The result immediately follows from Theorem 2.1 and Observation 2.3. □

2.2 The (Λ_X, Y) Pair

Theorem 2.5. *Let X and Y be r.i. spaces over a σ-finite nonatomic measure space $(\mathcal{R},\mu)$ having fundamental functions Φ and Ψ, respectively. Let $s = \Phi/\Psi$; suppose that there exists $r > 0$ such that $s(u)/u^r$ is increasing on $\mathbb{R}^+$. Then there exist $c_1 > 0$, and $c_2 > 0$ such that for any $t > 0$ and $f \in \Lambda_X + Y$ the estimate*

$$\begin{aligned} c_1\left(\|f^*(2u)\chi_A(u)\|_{\overline{\Lambda_X}} + t\|f^*(2u)\chi_B(u)\|_{\overline{Y}}\right) &\le K(f,t;\Lambda_X,Y) \\ &\le c_2\left(\|f^*\chi_A\|_{\overline{\Lambda_X}} + t\|f^*\chi_B\|_{\overline{Y}}\right), \end{aligned}$$

holds, where $A = \{u \in \mathbb{R}^+ : s(u) < t\}$, $B = \mathbb{R}^+ \setminus A$.

The proof of this theorem resembles the proof of Theorem 2.1 in many ways, nevertheless, we consider it appropriate not to omit any important details.

Proof. Let $f \in \Lambda_X + Y$ be decomposed to a sum $f = f_0 + f_1$ where $f_0 \in \Lambda_X$ and $f_1 \in Y$. Then,

$$2(\|f_0\|_{\Lambda_X} + t\|f_1\|_Y) \geq \|f_0^* \chi_A\|_{\overline{\Lambda_X}} + t\|f_1^* \chi_A\|_{\overline{Y}} \qquad (I_1)$$
$$+ \|f_0^* \chi_B\|_{\overline{\Lambda_X}} + t\|f_1^* \chi_B\|_{\overline{Y}}. \qquad (I_2)$$

We estimate the right hand side of the inequality for the terms on the first line and for the terms on the second line separately.

To get the estimate of I_1, we use (1.1), the growth of $s(u)/u^r$ as indicated in Remark 2.2, and (1.2);

$$\begin{aligned}\|f_1^* \chi_A\|_{\Lambda_X} &= \int_0^{s^{-1}(t)} f_1^*(u)\varphi(u)du \leq \int_0^{s^{-1}(t)} f_1^*(u)\Phi(u)\frac{du}{u} \\ &= \int_0^{s^{-1}(t)} f_1^*(u)\Psi(u)\frac{s(u)}{u}\,du \approx \int_0^t f_1^*(s^{-1}(u))\Psi(s^{-1}(u))du \\ &\leq t \sup_{u\in A}\{f_1^*(u)\Psi(u)\} = t\|f_1^*\chi_A\|_{\overline{M_Y^*}} \leq t\|f_1^*\chi_A\|_{\overline{Y}}.\end{aligned}$$

Thus,

$$I_1 \geq c\left(\|f_0^*\chi_A\|_{\overline{\Lambda_X}} + \|f_1^*\chi_A\|_{\overline{\Lambda_X}}\right) \geq c\|f^*(2u)\chi_A(u)\|_{\overline{\Lambda_X}}. \qquad (2.6)$$

To estimate I_2, we proceed from (1.2),

$$\begin{aligned}t\|f_0^*\chi_B\|_{\overline{Y}} &\leq t\|f_0^*\chi_B\|_{\overline{\Lambda_Y}} \\ &= t(f_0^*\chi_B)^*(0+)\Psi(0+) + t\int_0^\infty (f_0^*\chi_B)^*(u)\psi(u)du \\ &= tf_0^*(s^{-1}(t))\Psi(0+) + t\int_0^\infty f_0^*(u+s^{-1}(t))\psi(u)du \\ &\leq tf_0^*(s^{-1}(t))\Psi(0+) + t\int_0^{s^{-1}(t)} f_0^*(u+s^{-1}(t))\psi(u)du \\ &\qquad + t\int_{s^{-1}(t)}^\infty f_0^*(u)\psi(u)du \\ &= tf_0^*(s^{-1}(t))\Psi(0+) + tJ_1 + tJ_2.\end{aligned} \qquad (2.7)$$

Using the monotonicity of f_0^* and the definition of the function s, we have

$$\begin{aligned}tJ_1 &\leq tf_0^*(s^{-1}(t))\int_0^{s^{-1}(t)}\psi(u)du = tf_0^*(s^{-1}(t))[\Psi(s^{-1}(t)) - \Psi(0+)] \\ &= f_0^*(s^{-1}(t))\Phi(s^{-1}(t)) - tf_0^*(s^{-1}(t))\Psi(0+).\end{aligned}$$

Therefore,

$$tf_0^*(s^{-1}(t))\Psi(0+) + tJ_1 \leq \sup_{u>0}\{f_0^*(u)\chi_A(u)\Phi(u)\} = \|f_0^*\chi_A\|_{\overline{M_X^*}} \leq \|f_0^*\chi_A\|_{\overline{\Lambda_X}}. \qquad (2.8)$$

By (1.1), monotonicity of s, and Remark 2.2 we obtain

$$\begin{aligned} tJ_2 &\le t\int_{s^{-1}(t)}^{\infty} f_0^*(u)\Psi(u)\frac{du}{u} = t\int_{s^{-1}(t)}^{\infty} f_0^*(u)\frac{\Phi(u)}{s(u)}\frac{du}{u} \\ &\le \frac{t}{s(s^{-1}(t))}\int_{s^{-1}(t)}^{\infty} f_0^*(u)\frac{\Phi(u)}{u}du \approx \int_{s^{-1}(t)}^{\infty} f_0^*(u)\varphi(u)du \\ &\le \int_{s^{-1}(t)}^{\infty} f_0^*(u)\varphi(u-s^{-1}(t))du = \|f_0^*\chi_B\|_{\overline{\Lambda_X}}. \end{aligned} \tag{2.9}$$

From (2.7), (2.8), and (2.9) it follows that

$$t\|f_0^*\chi_B\|_{\overline{Y}} \lesssim \|f_0^*\chi_A\|_{\overline{\Lambda_X}} + \|f_0^*\chi_B\|_{\overline{\Lambda_X}}.$$

Therefore,

$$\begin{aligned} I_1 + I_2 &\gtrsim \|f^*(2u)\chi_A(u)\|_{\overline{\Lambda_X}} + t(\|f_0^*\chi_B\|_Y + \|f_1^*\chi_B\|_{\overline{Y}}) \\ &\ge \|f^*(2u)\chi_A(u)\|_{\overline{\Lambda_X}} + t\|f^*(2u)\chi_B(u)\|_{\overline{Y}}, \end{aligned}$$

which is the desired estimate of the K-functional from below.

To get the estimate from above we should consider the following decomposition of $f = f_0 + f_1$,

$$\begin{aligned} f_0 &= \begin{cases} f - f^*(s^{-1}(t)) & \text{if } f > f^*(s^{-1}(t)), \\ f + f^*(s^{-1}(t)) & \text{if } f < -f^*(s^{-1}(t)), \\ 0 & \text{otherwise,} \end{cases} \\ f_1 &= f - f_0. \end{aligned}$$

It follows that $f^* = f_0^* + f_1^*$, $f_0^*(u) = 0$ if $u > s^{-1}(t)$, and $f_1^*(u) = f^*(s^{-1}(t))$ for $u < s^{-1}(t)$. Then, using the definition of the set A,

$$\begin{aligned} t\|f_1^*\|_{\overline{Y}} &\le t\|f_1^*\chi_A\|_{\overline{Y}} + t\|f_1^*\chi_B\|_{\overline{Y}} \\ &= tf^*(s^{-1}(t))\Psi(s^{-1}(t)) + t\|f^*\chi_B\|_{\overline{Y}} \\ &\le \sup_{u>0}\{f^*(u)\chi_A(u)\Phi(u)\} + t\|f^*\chi_B\|_{\overline{Y}} \\ &= \|f^*\chi_A\|_{\overline{M_X^*}} + t\|f^*\chi_B\|_{\overline{Y}} \\ &\le \|f^*\chi_A\|_{\overline{\Lambda_X}} + t\|f^*\chi_B\|_{\overline{Y}}. \end{aligned}$$

Moreover,

$$\|f_0\|_{\Lambda_X} \le \|f_0^*\|_{\overline{\Lambda_X}} = \|f_0^*\chi_A\|_{\overline{\Lambda_X}} \le \|f^*\chi_A\|_{\overline{\Lambda_X}},$$

which finishes the proof of the estimate from above. □

Corollary 2.6. *Under the assumptions of Theorem 2.5 the following estimate holds,*

$$K(f,t;\Lambda_X,Y) \approx \|f^*\chi_A\|_{\overline{\Lambda_X}} + t\|f^*\chi_B\|_{\overline{Y}}.$$

Proof. The result immediately follows from Theorem 2.5 and Observation 2.3. □

2.3 The Case of Reversed Monotonicity of Φ/Ψ

Since we are going to try and find an analogue of Calderón theorem for various cases, it is prudent to consider the case of the other monotonicity of s. Nevertheless, more general cases will not be covered in this thesis.

Theorem 2.7. *Let X and Y be r.i. spaces over a σ-finite nonatomic measure space $(\mathcal{R},\mu)$ having fundamental functions Φ and Ψ, respectively. Let $s = \Phi/\Psi$, suppose that there exists $r > 0$ such that $s(u)u^r$ is decreasing on $\mathbb{R}^+$. Then, for any $t > 0$ and $f \in X + \Lambda_Y$, $g \in X + M_Y$,*

$$K(f,t;X,\Lambda_Y) \approx \|f^*\chi_A\|_{\overline{X}} + t\|f^*\chi_B\|_{\overline{\Lambda_Y}},$$
$$K(g,t;X,M_Y) \approx \|g^*\chi_A\|_{\overline{X}} + t\|g^*\chi_B\|_{\overline{M_Y}},$$

where $A = \{u \in \mathbb{R}^+ : s(u) < t\}$, $B = \mathbb{R}^+ \setminus A$.

Proof. Since $K(h,t;W,Z) = tK(h,1/t;Z,W)$ holds, whenever W, Z are arbitrary compatible r.i. spaces, $h \in W + Z$, and $t > 0$, hence the assertion is a simple consequence of Corollaries 2.4 and 2.6. □

Remark 2.8. Similarly as in Remark 2.2, the assumption on $s(u)u^r$ being a decreasing function implies that $s(u)$ is strictly decreasing, $s(0+) = \infty$, $\Psi(0+) = 0$, as well as the following equivalences,

$$s'(u) \approx \frac{-s(u)}{u}, \qquad \psi(u) \approx \frac{\Psi(u)}{u}.$$

Chapter 3

Calderón-type Theorems

This chapter contains the actual substance of the thesis. Main ideas of the following discoveries are based on work of Gogatishvili, Pick [3]. First, we set up a fundamental proposition for our further work, that is an equivalence of certain boundedness of a quasilinear operator and an inequality between the K-functional applied at the operator image of a function and the K-functional applied at the function itself.

Later, we shall require the fundamental functions of some of the involved rearrangement-invariant spaces to satisfy the following condition,

$$\sup_{u>0} \Phi(u) \fint_0^u \frac{ds}{\Phi(s)} < \infty. \tag{M}$$

Lemma 3.1. *Let Φ be the fundamental function of an r.i. space X over a σ-finite nonatomic measure space $(\mathcal{R}, \mu)$. Then, the condition* (M) *is equivalent to the coincidence of M_X and M_X^*.*

Proof. Suppose that the condition (M) holds for Φ. Let $f \in M_X^*$. Then,

$$\begin{aligned}
\|f\|_{M_X} &= \sup_{u>0} \{f^{**}(u)\Phi(u)\} = \sup_{u>0} \left\{\Phi(u) \fint_0^u f^*(t)dt\right\} \\
&= \sup_{u>0} \left\{\Phi(u) \fint_0^u f^*(t)\Phi(t)\frac{dt}{\Phi(t)}\right\} \\
&\le \sup_{u>0} \left\{\Phi(u) \fint_0^u \frac{dt}{\Phi(t)}\right\} \sup_{u>0} \{f^*(u)\Phi(u)\} \\
&= \|f\|_{M_X^*} \sup_{u>0} \left\{\Phi(u) \fint_0^u \frac{dt}{\Phi(t)}\right\}.
\end{aligned}$$

On the other hand, if Φ does not satisfy (M), then it suffices to consider a function f such that $f^*(u) \approx 1/\Phi(u)$ on $\mathbb{R}^+$. Then, $f \in M_X^* \setminus M_X$ □

Example 3.2. Let's explore which of the common r.i. spaces are compliant with such a condition,

(i) $L^{1,q}$, $q \in [1, \infty]$ spaces, for which $\Phi(u) = c(q)u$, do not obey (M).

(ii) $L^{p,q}$, $p \in (1, \infty)$, $q \in [1, \infty]$ spaces, for which $\Phi(u) = c(p, q)u^{1/p}$, obey (M),

(iii) L^∞, whose $\Phi(u) = \chi_{(0,\infty)}(u)$, obeys (M),

(iv) Any r.i. space which has the fundamental function $\Phi(u) = u^\alpha(\log(1 + u))^\beta$, where $\alpha \in [0, 1)$, $\beta \in [-\alpha, 1 - \alpha)$, obeys (M), whereas it does not in the case of $\alpha = 1$ or $\beta = 1 - \alpha$,

(v) Any r.i. space whose fundamental function is $\Phi(u) = u^\alpha(\log(1+u^{-1}))^\beta$, where $\alpha \in [0, 1)$, $\beta \in (\alpha - 1, \alpha]$, obeys (M), whereas it does not in the case of $\alpha = 1$ or $\beta = \alpha - 1$.

Proof. The first example follows from the divergence of the integral in condition (M) for any $u > 0$.

In order to see why examples (ii) and (iii) are compliant with (M), it suffices to realize the expression, which we're taking supremum of, is constant.

The argument supporting the assertion in the other two examples can be led in an analogous way, therefore, we're going to show the validity of example (iv), only.

Firstly, if $\alpha + \beta = 1$ then the integral diverges, since the integrand in (M) is comparable to s^{-1} near zero. Let $\alpha = 1, \beta \in [-1, 0)$. Suppose that $u > e - 1$. Then,

$$\begin{aligned}
(\log(1 + u))^\beta \int_0^u \frac{1}{s(\log(1 + s))^\beta}\, ds &\geq (\log(1 + u))^\beta \int_{e-1}^u \frac{1}{s(\log(1 + s))^\beta}\, ds \\
&\geq (\log(1 + u))^\beta \int_{e-1}^u \frac{1}{s}\, ds \\
&= \frac{\log u - C}{(\log(1 + u))^{|\beta|}},
\end{aligned}$$

which is an unbounded function of u. Finally, let's assume $\alpha \in [0, 1)$, and $\beta \in [0, 1-\alpha)$. Then the expression to be taken supremum of can be estimated by

$$\begin{aligned}
&u^{\alpha-1}(\log(1 + u))^\beta \int_0^u s^{-\alpha}(\log(1 + s))^{-\beta} ds \\
&= u^{\alpha-1}(\log(1 + u))^\beta \int_0^u s^{-\alpha-\beta} \frac{s^\beta}{(\log(1 + s))^\beta} ds \leq u^{\alpha+\beta-1} \int_0^u s^{-\alpha-\beta} ds = \frac{1}{1 - \alpha - \beta}.
\end{aligned}$$

Now, let $\alpha \in [0, 1)$, and $\beta \in [-\alpha, 0)$. Then,

$$\begin{aligned}
&u^{\alpha-1}(\log(1 + u))^\beta \int_0^u s^{-\alpha}(\log(1 + s))^{-\beta} ds \\
&\leq u^{\alpha-1}(\log(1 + u))^\beta (\log(1 + u))^{-\beta} \int_0^u s^{-\alpha} ds = \frac{1}{1 - \alpha}.
\end{aligned}$$

□

Proposition 3.3. *Let X_1, X_2, and Y_1, Y_2 be rearrangement-invariant Banach function spaces over σ-finite nonatomic measure spaces $(\mathcal{R}, \mu)$, and $(\mathcal{S}, \nu)$, respectively. Let Φ_j, Ψ_j denote the fundamental functions of X_j, Y_j, respectively ($j = 1, 2$). Assume that Ψ_1 complies with* (M). *Suppose that the functions $s_0(u) = \Phi_1(u)/\Phi_2(u)$ and $s_1(u) = \Psi_1(u)/\Psi_2(u)$ are strictly increasing and that there exist $r_0 > 0$ and $r_1 > 0$ such that $s_j(u)/u^{r_j}$ are increasing ($j = 0, 1$). Then, whenever T is a quasilinear operator, the following conditions are equivalent,*

(i) *T is a bounded operator from Λ_{X_1} to M_{Y_1} and from X_2 to Y_2,*

(ii) $K(Tf, t; M_{Y_1}, Y_2) \lesssim K(f, t; \Lambda_{X_1}, X_2)$.

Proof. The implication (i) ⇒ (ii) is just a simple consequence of the definition of Peetre's K-functional. The proof of the other one is led in a more interesting way.

Let $A_0 = \{s_0 < t\}$, and $A_1 = \{s_1 < t\}$, B_0 and B_1 be their respective complements on positive real line. Then, according to Corollary 2.4, and inequality (1.2), we have

$$K(Tf, t; M_{Y_1}, Y_2) \gtrsim \|(Tf)^*\chi_{A_1}\|_{\overline{M_{Y_1}}} + t\|(Tf)^*\chi_{B_1}\|_{\overline{Y_2}},$$

$$\begin{aligned}\|(Tf)^*\chi_{A_1}\|_{\overline{M_{Y_1}}} &\geq \|(Tf)^*\chi_{A_1}\|_{\overline{M^*_{Y_1}}} \\ &= \sup_{0<u<s_1^{-1}(t)} \{(Tf)^*(u)\Psi_1(u)\} \\ &\geq (Tf)^*(s_1^{-1}(t))\Psi_1(s_1^{-1}(t)) \\ &\geq (Tf)^*(2s_1^{-1}(t))\Psi_1(s_1^{-1}(t)),\end{aligned}$$

$$\begin{aligned}t\|(Tf)^*\chi_{B_1}\|_{\overline{Y_2}} &\geq t\|(Tf)^*\chi_{B_1}\|_{\overline{M^*_{Y_2}}} \\ &= t\sup_{u>0}\left\{(Tf)^*(u + s_1^{-1}(t))\Psi_2(u)\right\} \\ &\geq t(Tf)^*(2s_1^{-1}(t))\Psi_2(s_1^{-1}(t)) \\ &= (Tf)^*(2s_1^{-1}(t))\Psi_1(s_1^{-1}(t)).\end{aligned}$$

Therefore, using the concavity of Ψ_1,

$$K(Tf, t; M_{Y_1}, Y_2) \gtrsim (Tf)^*(2s_1^{-1}(t))\Psi_1(s_1^{-1}(t)) \gtrsim (Tf)^*(2s_1^{-1}(t))\Psi_1(2s_1^{-1}(t)),$$

which leads to the result that, for every $t \in \mathbb{R}^+$,

$$(Tf)^*(2s_1^{-1}(t))\Psi_1(2s_1^{-1}(t)) \lesssim K(f, t; \Lambda_{X_1}, X_2). \tag{3.1}$$

Now, provided $f \in \Lambda_{X_1}$, we have got an estimate independent on t,

$$(Tf)^*(2s_1^{-1}(t))\Psi_1(2s_1^{-1}(t)) \lesssim \|f\|_{\Lambda_{X_1}}.$$

Using condition (M), and bijectivity of s_1, we have proved that T is a bounded operator from Λ_{X_1} to M_{Y_1}.

Now assume that $f \in X_2$. Using Corollary 2.4 and the trivial decomposition, we have

$$\|(Tf)^* \chi_{B_1}\|_{\overline{Y_2}} \lesssim K(Tf, t; M_{Y_1}, Y_2)/t \lesssim K(f, t; \Lambda_{X_1}, X_2)/t \leq \|f\|_{X_2}.$$

Since $B_1 = [s_1^{-1}(t), \infty)$, $(Tf)^* \chi_{B_1}$ converges monotonously up to $(Tf)^*$ as t approaches 0, we have shown the desired boundedness of T from X_2 to Y_2. □

Straightaway, we are going to formulate a little modified version of the previous proposition. Albeit, it is more general in the sense that it covers all the combinations of monotonicity of s_0 and s_1, it is more restrictive about the spaces X_2 and Y_2, since it requires them to be either Lorentz, or Marcinkiewicz spaces.

Proposition 3.4. *Let X_1, X_2, and Y_1, Y_2 be rearrangement-invariant Banach function spaces over σ-finite nonatomic measure spaces $(\mathcal{R}, \mu)$, and $(\mathcal{S}, \nu)$, respectively. Let Φ_j, Ψ_j denote the fundamental functions of X_j, Y_j, respectively ($j = 1, 2$). Assume that Ψ_1 complies with* (M)*. Let X_2 and Y_2 be Lorentz or Marcinkiewicz spaces (independently of each other). Suppose that the functions $s_0(u) = \Phi_1(u)/\Phi_2(u)$ and $s_1(u) = \Psi_1(u)/\Psi_2(u)$ are strictly monotone and that there exist $r_0 > 0$ and $r_1 > 0$ such that either $s_j(u)/u^{r_j}$ is increasing if s_j is increasing, or $s_j u^{r_j}$ is decreasing if s_j is decreasing ($j = 0, 1$). Then, whenever T is a quasilinear operator, the following conditions are equivalent,*

(i) *T is a bounded operator from Λ_{X_1} to M_{Y_1} and from X_2 to Y_2,*

(ii) $K(Tf, t; M_{Y_1}, Y_2) \lesssim K(f, t; \Lambda_{X_1}, X_2)$.

(iii) $K(Tf, t; Y_2, M_{Y_1}) \lesssim K(f, t; X_2, \Lambda_{X_1})$.

Proof. The implication (i) ⇒ (ii) is just a simple consequence of the definition of Peetre's K-functional. The equivalence of (ii) and (iii) follows from the basic arithmetics of the K-functional. It remains to prove the implication (ii) ⇒ (i).

Let $A_0 = \{s_0 < t\}$, and $A_1 = \{s_1 < t\}$, B_0 and B_1 be their respective complements on positive real line. Then, according to Corollary 2.4 or Theorem 2.7, and inequality (1.2), we have

$$K(Tf, t; M_{Y_1}, Y_2) \gtrsim \|(Tf)^* \chi_{A_1}\|_{\overline{M_{Y_1}}} + t\|(Tf)^* \chi_{B_1}\|_{\overline{Y_2}}.$$

By methods analogous to those used in the proof of the previous theorem (if s_1 is increasing, the procedure is exactly the same, otherwise the procedures used to estimate the two terms are switched), we obtain

$$K(Tf, t; M_{Y_1}, Y_2) \gtrsim (Tf)^*(2s_1^{-1}(t))\Psi_1(2s_1^{-1}(t)),$$

from where we can prove that T is a bounded operator from Λ_{X_1} to M_{Y_1} by taking precisely the same steps as in the previous proof.

Now, assume that $f \in X_2$. Using Corollary 2.4 or Theorem 2.7 and the trivial decomposition, we have

$$\|(Tf)^* \chi_{B_1}\|_{\overline{Y_2}} \lesssim K(Tf, t; M_{Y_1}, Y_2)/t \lesssim K(f, t; \Lambda_{X_1}, X_2)/t \le \|f\|_{X_2}.$$

If s_1 is increasing, B_1 has the form of an interval $[s_1^{-1}(t), \infty)$, otherwise it is an interval $(0, s_1^{-1}(t)]$. In any case, $(Tf)^* \chi_{B_1}$ converges monotonously up toward $(Tf)^*$ as t approaches 0, which leads to the desired boundedness of T from X_2 to Y_2. □

3.1 Operators of $(\Lambda_{X_1}, M_{Y_1}; M_{X_2}, M_{Y_2})$ Type

In the proof of the previous two propositions we came to inequality (3.1), which happens to be crucial when looking for a suitable Calderón operator for the case of a quasilinear operator of $(\Lambda_{X_1}, M_{Y_1}; M_{X_2}, M_{Y_2})$ type. Then, using Corollary 2.6 or Theorem 2.7, we have,

$$(Tf)^*(2s_1^{-1}(t))\Psi_1(2s_1^{-1}(t)) \lesssim K(f, t; \Lambda_{X_1}, M_{X_2}) \approx \|f^* \chi_{A_0}\|_{\overline{\Lambda_{X_1}}} + t\|f^* \chi_{B_0}\|_{\overline{M_{X_2}}}.$$

Choose $t = s_1(v/2)$. Then, according to the type of monotonicity of s_0, we get

$$(Tf)^*(v)\Psi_1(v) \lesssim \int_0^{\Gamma(v/2)} f^* \varphi_1 + s_1(v/2) \sup_{u>0} \{f^*(u + \Gamma(v/2))\Phi_2(u)\}$$

if s_0 is increasing, and

$$(Tf)^*(v)\Psi_1(v) \lesssim f^*(\Gamma(v/2))\Phi_1(0+) + \int_0^{\infty} f^*(u + \Gamma(v/2))\varphi_1(u)du$$
$$+ s_1(v/2) \sup_{0<u<\Gamma(v/2)} \{f^*(u)\Phi_2(u)\}$$

otherwise, where $\Gamma = s_0^{-1} \circ s_1$. Therefore, it seems reasonable to try and define the Calderón operator with respect to the monotonicity of s_0. If s_0 is increasing, then

$$Rh(v) = \frac{1}{\Psi_1(v)} \int_0^{\Gamma(v/2)} h\varphi_1 + \frac{1}{\Psi_2(v)} \sup_{u>0} \{h^*(u + \Gamma(v/2))\Phi_2(u)\}; \tag{3.2a}$$

otherwise

$$Rh(v) = \frac{1}{\Psi_1(v)} \left(h(\Gamma(v/2))\Phi_1(0+) + \int_0^{\infty} h^*(u + \Gamma(v/2))\varphi_1(u)du \right)$$
$$+ \frac{1}{\Psi_2(v)} \sup_{0<u<\Gamma(v/2)} \{h^*(u)\Phi_2(u)\}. \tag{3.2b}$$

Theorem 3.5. *Under the assumptions of Proposition 3.3, the following statements are equivalent,*

(i) *T is bounded both from Λ_{X_1} to M_{Y_1}, and from M_{X_2} to M_{Y_2},*

(ii) *$(Tf)^* \lesssim Rf^*$ on $\mathbb{R}^+$ where R is defined either by* (3.2a)*, or by* (3.2b)*, depending on the monotonicity of s_0.*

Proof. In both (3.2a), and (3.2b) the operator R was defined in such a fashion that condition (i) will immediately imply the condition (ii).

The other implication will be proved using Proposition 3.3. Using condition (ii), and Corollary 2.4 or Theorem 2.7, we get,

$$\begin{aligned} K(Tf,t;M_{Y_1},M_{Y_2}) &\lesssim K(Rf^*,t;M_{Y_1},M_{Y_2}) \\ &\approx \|(Rf^*)^*\chi_{\{s_1<t\}}\|_{M_{Y_1}} + t\|(Rf^*)^*\chi_{\{s_1\geq t\}}\|_{M_{Y_2}}. \end{aligned} \tag{3.3}$$

Now, we can also see that no matter which of the two definitions were used to define R, we have

$$Rf^*(v) \approx \frac{1}{\Psi_1(v)} K(f, s_1(v/2); \Lambda_{X_1}, M_{X_2}) \tag{3.4}$$

and

$$\Psi_2(v)Rf^*(v) \approx \frac{\Psi_2(v)}{\Psi_1(v)} K(f, s_1(v/2); \Lambda_{X_1}, M_{X_2}) \approx \frac{1}{s_1(v)} K(f, s_1(v); \Lambda_{X_1}, M_{X_2}). \tag{3.5}$$

Now, it is necessary to split the proof into two parts according to the type of monotonicity of s_1.

First, let s_1 be increasing. Then, $(Rf^*)^* \approx Rf^*$ owing to (3.5). Moreover, $\Psi_2 Rf^*$ is equivalent to a decreasing function, whereas $\Psi_1 Rf^*$ to an increasing one due to (3.4). Proceeding with (3.3),

$$\begin{aligned} K(Tf,t;M_{Y_1},M_{Y_2}) &\lesssim \sup_{0<u<s_1^{-1}(t)} \{\Psi_1(u)(Rf^*)^*(u)\} + t\sup_{u>0}\{\Psi_2(u)(Rf^*)^*(u+s_1^{-1}(t))\} \\ &\lesssim \Psi_1(s_1^{-1}(t))Rf^*(s_1^{-1}(t)) + t\Psi_2(s_1^{-1}(t))Rf^*(s_1^{-1}(t)) \\ &\approx \Psi_1(s_1^{-1}(t))Rf^*(s_1^{-1}(t)) \approx K(f, s_1(s_1^{-1}(t)/2); \Lambda_{X_1}, M_{X_2}) \\ &\leq K(f,t;\Lambda_{X_1},M_{X_2}). \end{aligned}$$

Proposition 3.3 implies the desired boundedness of operator T.

Finally, let s_1 be decreasing. Then, $(Rf^*)^* \approx Rf^*$ owing to (3.4). Moreover, $\Psi_1 Rf^*$ is equivalent to a decreasing function, whereas $\Psi_2 Rf^*$ to an increasing one due to (3.5). Proceeding with (3.3),

$$\begin{aligned} K(Tf,t;M_{Y_1},M_{Y_2}) &\lesssim \sup_{u>0}\{\Psi_1(u)(Rf^*)^*(u+s_1^{-1}(t))\} + t\sup_{0<u<s_1^{-1}(t)} \{\Psi_2(u)(Rf^*)^*(u)\} \\ &\lesssim \Psi_1(s_1^{-1}(t))Rf^*(s_1^{-1}(t)) + t\Psi_2(s_1^{-1}(t))Rf^*(s_1^{-1}(t)) \\ &\approx \Psi_1(s_1^{-1}(t))Rf^*(s_1^{-1}(t)) \approx K(f, s_1(s_1^{-1}(t)/2); \Lambda_{X_1}, M_{X_2}) \\ &\approx K(f,t;\Lambda_{X_1},M_{X_2}). \end{aligned}$$

Again, Proposition 3.3 implies the desired boundedness of operator T. □

Now we're ready to formulate the desired Calderón-type theorem.

Theorem 3.6. *Let X, X_1, X_2, and Y, Y_1, Y_2 be rearrangement-invariant Banach function spaces over σ-finite nonatomic measure spaces $(\mathcal{R}, \mu)$, and $(\mathcal{S}, \nu)$, respectively. Φ_j, Ψ_j denoting the fundamental functions of X_j, Y_j, respectively ($j = 1, 2$), suppose that $s_0 := \Phi_1/\Phi_2$ and $s_1 := \Psi_1/\Psi_2$ are strictly monotone. Suppose there is a constant $r_0 > 0$ such that $s_0(u)/u^{r_0}$ is increasing. In addition, assume there is a constant $r_1 > 0$ such that if s_1 is increasing, the ratio $s_1(u)/u^{r_1}$ is increasing as well, and if s_1 is decreasing, the product $s_1(u)u^{r_1}$ is decreasing as well. Suppose that the fundamental function Ψ_1 complies with condition* (M). *Let $\Gamma = s_0^{-1} \circ s_1$. Then, the following conditions are equivalent:*

(i) *every quasilinear operator of (Λ_{X_1}, M_{Y_1}) and (M_{X_2}, M_{Y_2}) types is bounded from X to Y,*

(ii) *the operator R is bounded from $\overline{X}$ to $\overline{Y}$, where R is defined on measurable functions on $\mathbb{R}^+$ by* (3.2a), *i.e.,*

$$Rh(v) = \frac{1}{\Psi_1(v)} \int_0^{\Gamma(v/2)} h\varphi_1 + \frac{1}{\Psi_2(v)} \sup_{u>0} \{h^*(u + \Gamma(v/2))\Phi_2(u)\}.$$

Proof. First, let's show (i) implies (ii). Operator R is quasilinear, hence, it suffices to show $(Rf)^* \lesssim Rf^*$, since Theorem 3.5 would ensure that R is of $(\Lambda_{X_1}, M_{Y_1}; M_{X_2}, M_{Y_2})$ type, then. Condition (i) then implies the desired boundedness of R. Observe that the Hardy-Littlewood inequality gives $|Rf| \le Rf^*$, thus, $(Rf)^* \le (Rf^*)^* \approx Rf^*$.

Now, let's show that condition (ii) implies condition (i). Let T be a quasilinear operator of $(\Lambda_{X_1}, M_{Y_1}; M_{X_2}, M_{Y_2})$ type. Then Tf is defined whenever $Rf^*(1) < \infty$, thus, whenever $f \in X$. Furthermore, $(Tf)^*(t) \lesssim Rf^*(t)$ by Theorem 3.5. Therefore, by the definition of the representation space and condition (ii), we get

$$\|Tf\|_Y = \|(Tf)^*\|_{\overline{Y}} \lesssim \|Rf^*\|_{\overline{Y}} \lesssim \|f^*\|_{\overline{X}} = \|f\|_X,$$

wherefore T is bounded from X to Y, as required. □

Theorem 3.7. *Let X, X_1, X_2, and Y, Y_1, Y_2 be rearrangement-invariant Banach function spaces over σ-finite nonatomic measure spaces $(\mathcal{R}, \mu)$, and $(\mathcal{S}, \nu)$, respectively. Φ_j, Ψ_j denoting the fundamental functions of X_j, Y_j, respectively ($j = 1, 2$), suppose that $s_0 := \Phi_1/\Phi_2$ and $s_1 := \Psi_1/\Psi_2$ are strictly monotone. Suppose there is a constant $r_0 > 0$ such that $s_0(u)u^{r_0}$ is decreasing. In addition, assume there is a constant $r_1 > 0$ such that if s_1 is increasing, the ratio $s_1(u)/u^{r_1}$ is increasing as well, and if s_1 is decreasing, the product $s_1(u)u^{r_1}$ is decreasing as well. Suppose that the fundamental function Ψ_1 complies with condition* (M). *Let $\Gamma = s_0^{-1} \circ s_1$. Then, the following conditions are equivalent:*

(i) *every quasilinear operator of* (Λ_{X_1}, M_{Y_1}) *and* (M_{X_2}, M_{Y_2}) *types is bounded from* X *to* Y,

(ii) *the operator* $\tilde{R}$ *is bounded from* $\overline{X}$ *to* $\overline{Y}$, *where* $\tilde{R}$ *is defined on measurable functions on* $\mathbb{R}^+$ *by*

$$\tilde{R}h(v) = \frac{1}{\Psi_1(v)} \int_0^\infty h^*(u+\Gamma(v/2))\varphi_1(u)\,du + \frac{1}{\Psi_2(v)} \sup_{0<u<\Gamma(v/2)} \{h^*(u)\Phi_2(u)\}.$$

Proof. First, the forward implication can be shown similarly as in the proof of the preceding theorem. Operator $\tilde{R}$ is quasilinear due to subadditivity of the maximal operator $(.)^{**}$ and the Hardy lemma, hence, it suffices to show $(\tilde{R}f)^* \lesssim Rf^*$, where the operator R is defined by (3.2b), since Theorem 3.5 then ensures that $\tilde{R}$ is of $(\Lambda_{X_1}, M_{Y_1}; M_{X_2}, M_{Y_2})$ type. Condition (i) then implies the desired boundedness of $\tilde{R}$. Observe that by triangle inequality we have $|\tilde{R}f| \le \tilde{R}f^*$, and $\tilde{R}f^* \le Rf^*$, trivially. Therefore, we can estimate $(\tilde{R}f)^* \le (\tilde{R}f^*)^* \le (Rf^*)^* \approx Rf^*$.

Now, let's show that condition (ii) implies condition (i). Let T be a quasilinear operator of $(\Lambda_{X_1}, M_{Y_1}; M_{X_2}, M_{Y_2})$ type. Then Tf is defined whenever $Rf^*(1) < \infty$, thus, whenever $\tilde{R}f^*(1) < \infty$, therefore, whenever $f \in X$. The operator $\tilde{R}$ is actually the operator R after omitting the term $f(\Gamma(v/2))\Phi_1(0+)/\Psi_1(v)$. The Hardy-Littlewood-Pólya principle shows that this term is negligible, for

$$\int_0^t \frac{f^*(\Gamma(v/2))\Phi_1(0+)}{\Psi_1(v)}\,dv \le \int_0^t \frac{f^*(\Gamma(v/2))\Phi_1(\Gamma(v/2))}{\Psi_1(v)}\,dv$$
$$\approx \int_0^t \frac{f^*(\Gamma(v/2))\Phi_2(\Gamma(v/2))}{\Psi_2(v)}\,dv \le \int_0^t \sup_{0<u<\Gamma(v/2)} \left\{\frac{f^*(u)\Phi_2(u)}{\Psi_2(v)}\right\} dv.$$

Consequently, $\|Rf^*\|_{\overline{Y}} \le \|\tilde{R}f^*\|_{\overline{Y}}$ for any $f \in X$. Furthermore, $(Tf)^*(t) \lesssim Rf^*(t)$ by Theorem 3.5. Therefore, by the definition of the representation space and condition (ii), we get

$$\|Tf\|_Y = \|(Tf)^*\|_{\overline{Y}} \lesssim \|Rf^*\|_{\overline{Y}} \lesssim \|\tilde{R}f^*\|_{\overline{Y}} \lesssim \|f^*\|_{\overline{X}} = \|f\|_X,$$

wherefore T is bounded from X to Y, as required. □

The preceding theorem establishes the Calderón-type theorem for relatively general r.i. spaces, which may obscure its sense for common spaces. Therefore, let's formulate a couple of theorems, each being a special case of Theorem 3.6.

Theorem 3.8. *Let* X *and* Y *be rearrangement-invariant function spaces over* σ*-finite nonatomic measure spaces* $(\mathcal{R}, \mu)$ *and* $(\mathcal{S}, \nu)$, *respectively. Let* $1 \le p_1 < p_2 \le \infty$, *and* $1 < q_1, q_2 \le \infty$ *where* $q_1 \ne q_2$. *Then, the following conditions are equivalent:*

(i) *every quasilinear operator bounded both from* $L^{p_1,1}(\mathcal{R})$ *to* $L^{q_1,\infty}(\mathcal{S})$, *and from* $L^{p_2,\infty}(\mathcal{R})$ *to* $L^{q_2,\infty}(\mathcal{S})$, *is bounded from* X *to* Y, *as well,*

(ii) *operator R is bounded from $\overline{X}$ to $\overline{Y}$, where R is defined on measurable functions on $\mathbb{R}^+$ by*

$$Rh(v) = v^{-\frac{1}{q_1}} \int_0^{\Gamma(v/2)} h(u)u^{\frac{1}{p_1}-1}\, du + v^{-\frac{1}{q_2}} \sup_{u>0} \left\{ h^*(u + \Gamma(v/2))u^{\frac{1}{p_2}} \right\}$$

where

$$\Gamma(z) = z^{\frac{\frac{1}{q_1}-\frac{1}{q_2}}{\frac{1}{p_1}-\frac{1}{p_2}}},$$

and analogously for the case when some of the coefficients p_2, q_1, q_2 are infinity.

Sketch of the proof. The theorem is a consequence of the general one. Nevertheless, we need to be aware that Λ_{L^p} coincides with $L^{p,1}$ for any $p \in [1, \infty)$, whereas Λ_{L^∞} coincides with L^∞. On the other hand, $M^*_{L^p}$ coincides with $L^{p,\infty}$ for any $p \in [1, \infty]$, however, only for $p \in (1, \infty]$ the spaces $M^*_{L^p}$ and M_{L^p} coincide. Therefore, we disallowed the case of $q_i = 1$, $i = 1, 2$. □

Theorem 3.9. *Let X, and Y be rearrangement-invariant Banach function spaces over σ-finite nonatomic measure spaces $(\mathcal{R}, \mu)$, and $(\mathcal{S}, \nu)$, respectively. Suppose $1 \le p_2 < p_1 \le \infty$, and $1 < q_1, q_2 \le \infty$ where $q_1 \ne q_2$. Then, the following conditions are equivalent:*

(i) *every quasilinear operator bounded both from $L^{p_1,1}$ to $L^{q_1,\infty}$, and from $L^{p_2,\infty}$ to $L^{q_2,\infty}$, is bounded from X to Y, as well,*

(ii) *operator R is bounded from $\overline{X}$ to $\overline{Y}$, where R is defined on measurable functions on $\mathbb{R}^+$ by*

$$Rh(v) = v^{-\frac{1}{q_1}} \int_0^{\infty} h^*(u + \Gamma(v/2))u^{\frac{1}{p_1}-1}\, du + v^{-\frac{1}{q_2}} \sup_{0<u<\Gamma(v/2)} \left\{ h^*(u)u^{\frac{1}{p_2}} \right\}$$

where

$$\Gamma(z) = z^{\frac{\frac{1}{q_1}-\frac{1}{q_2}}{\frac{1}{p_1}-\frac{1}{p_2}}},$$

with natural analogue for the case when some of the coefficients p_1, q_1, q_2 have the value of infinity.

The operator R as we have defined it by (3.2a) consists of two parts. The first part (integral) is linear, whereas the second part (supremum) is merely quasilinear. Therefore, we couldn't have formulated the Calderón-type theorem containing an assertion for linear operators as an equivalent condition. Nevertheless, on the following pages we will see that in the case when both s_0, and s_1 are increasing we can neglect the quasilinear term of R.

Lemma 3.10. *Let X_j, Y_j be r.i. spaces over σ-finite nonatomic measure spaces $(\mathcal{R}, \mu)$ and $(\mathcal{S}, \nu)$, respectively, having concave fundamental functions Φ_j, Ψ_j; $j = 1, 2$. Denoted $s_0 = \Phi_1/\Phi_2$, $s_1 = \Psi_1/\Psi_2$, and $\Gamma = s_0^{-1} \circ s_1$, suppose there are constants r_0, r_1 such that the fractions $s_i(u)/u^{r_i}$ are increasing on $\mathbb{R}^+$, $i = 0, 1$. Then, there is a constant $c > 0$ such that for every $u > 0$,*

$$\sup_{v \in [u/2, u]} \frac{\Gamma'(u)}{\Gamma'(v)} < c.$$

Moreover, the constant c depends only on the magnitude of r_i's, $i = 0, 1$.

Proof. The growth of $s_i(u)/u^{r_i}$ implies the equivalence $s_i'(u) \approx s_i(u)/u$, $i = 1, 2$, as we've established in Remark 2.2.

Let $u > 0$ and $v \in [u/2, u]$. As s_i is increasing and concave, $i = 1, 2$,

$$\frac{\Gamma'(u)}{\Gamma'(v)} = \frac{s_0'(\Gamma(v))}{s_0'(\Gamma(u))} \frac{s_1'(u)}{s_1'(v)} \approx \frac{s_0(\Gamma(v))}{s_0(\Gamma(u))} \frac{\Gamma(u)}{\Gamma(v)} \cdot \frac{s_1(u)}{s_1(v)} \frac{v}{u} \le \frac{\Gamma(u)}{\Gamma(v)}.$$

For Γ is increasing, we can estimate

$$\sup_{v \in [u/2, u]} \frac{\Gamma'(u)}{\Gamma'(v)} \lesssim \sup_{v \in [u/2, u]} \frac{\Gamma(u)}{\Gamma(v)} \le \frac{\Gamma(u)}{\Gamma(u/2)}.$$

Concavity of the fundamental functions implies $\lambda s_i(u) \le s_i(\lambda u)$ whenever $\lambda \in (0, 1)$; $i = 1, 2$. In addition, the growth of $s_i(u)/u^{r_i}$ implies that $u^{1/r_i}/s_i^{-1}(u)$ is increasing as well. Those facts together with the definition of Γ give the desired boundedness of the supremum,

$$\frac{\Gamma(u)}{\Gamma(u/2)} \le \frac{s_0^{-1}(s_1(u))}{s_0^{-1}(s_1(u)/2)} \le 2^{1/r_0}.$$

□

Lemma 3.11. *Assume the r.i. spaces and their fundamental functions comply with the assumptions of Proposition 3.3. In addition, suppose Ψ_2 satisfies* (M)*. As previously, $\Gamma = s_0^{-1} \circ s_1$. Then, whenever $h \ge 0$ is a decreasing function on $\mathbb{R}^+$ and $t > 0$, the following equivalence holds with constants dependent neither on h, nor on t,*

$$\int_0^t \frac{1}{\Psi_2(s)} \sup_{s/2 \le y \le t} \{\Phi_2(\Gamma(y) - \Gamma(s/2)) h(\Gamma(y))\}\, ds$$
$$\approx \int_0^t \frac{\Phi_2(\Gamma(s) - \Gamma(s/2))}{\Psi_2(s)} h(\Gamma(s)) ds. \tag{3.6}$$

Proof. The inequality "$\gtrsim$" is easily proved by taking a specific y in the supremum, viz. $y = s$. The other inequality can be shown using the Fubini theorem,

$$\begin{aligned} I &= \int_0^t \frac{1}{\Psi_2(s)} \sup_{s/2 \le y \le t} \{\Phi_2(\Gamma(y) - \Gamma(s/2))h(\Gamma(y))\}\, ds \\ &= \int_0^t \frac{1}{\Psi_2(s)} \sup_{s/2 \le y \le t} \left\{ h(\Gamma(y)) \left(\Phi_2(0+) + \int_{s/2}^y \varphi_2(\Gamma(z) - \Gamma(s/2))\Gamma'(z) dz \right) \right\} ds \\ &\le \int_0^t \frac{1}{\Psi_2(s)} \sup_{s/2 \le y \le t} \left\{ \int_{s/2}^y h(\Gamma(z)) \varphi_2(\Gamma(z) - \Gamma(s/2))\Gamma'(z) dz \right\} ds && (I_1) \\ &\quad + \int_0^t \frac{h(\Gamma(s/2))\Phi_2(0+)}{\Psi_2(s)}\, ds. && (I_2) \end{aligned}$$

Let's focus on the (I_1) first where the Fubini theorem is to be used,

$$\begin{aligned} I_1 &= \int_0^t \frac{1}{\Psi_2(s)} \int_{s/2}^t h(\Gamma(z))\varphi_2(\Gamma(z) - \Gamma(s/2))\Gamma'(z) dz\, ds \\ &= \int_0^t h(\Gamma(z))\varphi_2(\Gamma(z) - \Gamma(s/2))\Gamma'(z) \int_0^{2z} \frac{ds}{\Psi_2(s)}\, dz. \end{aligned}$$

Now, let's use (1.1), the Lagrange mean value theorem with $\zeta \in (z/2, z)$, the condition (M), and Lemma 3.10 to see that

$$\begin{aligned} I_1 &\le \int_0^t h(\Gamma(z))\Phi_2(\Gamma(z) - \Gamma(z/2)) \frac{\Gamma'(z)}{\Gamma(z) - \Gamma(z/2)} \int_0^{2z} \frac{ds}{\Psi_2(s)}\, dz \\ &\approx \int_0^t h(\Gamma(z))\Phi_2(\Gamma(z) - \Gamma(z/2)) \frac{\Gamma'(z)}{z\Gamma'(\zeta)} \int_0^{2z} \frac{ds}{\Psi_2(s)}\, dz \\ &\lesssim \int_0^t \frac{\Phi_2(\Gamma(z) - \Gamma(z/2))}{\Psi_2(2z)} h(\Gamma(z)) dz \\ &\le \int_0^t \frac{\Phi_2(\Gamma(z) - \Gamma(z/2))}{\Psi_2(z)} h(\Gamma(z)) dz. \end{aligned}$$

It remains to show that the term I_2 can be estimated by the same term, which we accomplish by a simple substitution,

$$\begin{aligned} I_2 &= 2 \int_0^{t/2} \frac{h(\Gamma(s))\Phi_2(0+)}{\Psi_2(2s)}\, ds \\ &\approx \int_0^t \frac{h(\Gamma(s))\Phi_2(0+)}{\Psi_2(s)}\, ds \\ &\le \int_0^t \frac{\Phi_2(\Gamma(s) - \Gamma(s/2))}{\Psi_2(s)} h(\Gamma(s)) ds. \end{aligned}$$

□

Theorem 3.12. *Let Z be an r.i. space. Suppose X_1, X_2, Y_1, Y_2 are r.i. spaces satisfying the assumptions of the preceding lemma. Suppose g is a μ-measurable function. Then*

$$\left\| \frac{1}{\Psi_2(s)} \sup_{\Gamma(s/2)\le y<\infty} \{\Phi_2(y-\Gamma(s/2))g^*(y)\} \right\|_{\overline{Z}} \lesssim \left\| \frac{1}{\Psi_1(s)} \int_0^{\Gamma(s/2)} g^*(v)\varphi_1(v)dv \right\|_{\overline{Z}}.$$

Proof. Using Lemma 3.11 and concavity of a fundamental function, one can see that for any fixed $t>0$ and any measurable function g,

$$\begin{aligned} I &= \int_0^t \frac{1}{\Psi_2(s)} \sup_{\Gamma(s/2)\le y<\infty} \{g^*(y)\Phi_2(y-\Gamma(s/2))\}ds \qquad (3.7)\\ &= \int_0^t \frac{1}{\Psi_2(s)} \sup_{s/2\le y<\infty} \{g^*(\Gamma(y))\Phi_2(\Gamma(y)-\Gamma(s/2))\}ds \\ &\lesssim \int_0^t \frac{\Phi_2(\Gamma(s)-\Gamma(s/2))}{\Psi_2(s)} g^*(\Gamma(s))ds \\ &\le \int_0^t \frac{\Phi_2(\Gamma(s/2))}{\Psi_2(s)} \frac{\Gamma(s)}{\Gamma(s/2)} g^*(\Gamma(s))ds. \end{aligned}$$

Now, with the aid of Lemma 3.10, we can estimate $\Gamma(s)/\Gamma(s/2)\le c$, where $c\in\mathbb{R}^+$ depends only on r_0 and r_1. Therefore, by definition of Γ,

$$\begin{aligned} I &\lesssim \int_0^t \frac{\Phi_2(\Gamma(s/2))}{\Psi_2(s)} g^*(\Gamma(s))ds \\ &\lesssim \int_0^t \frac{\Phi_1(\Gamma(s/2))}{\Psi_1(s)} g^*(\Gamma(s))ds \\ &\lesssim \int_0^t \frac{1}{\Psi_1(s)} g^*(\Gamma(s)) \int_0^{\Gamma(s/2)} \varphi_1(v)dv\,ds \\ &\le \int_0^t \frac{1}{\Psi_1(s)} \int_0^{s/2} g^*(u)\varphi_1(v)dv\,ds \\ &\le \int_0^t \left(\frac{1}{\Psi_1(s)} \int_0^{\Gamma(s/2)} g^*(v)\varphi_1(v)dv \right)^* (s)ds. \end{aligned}$$

Noticing that the integrand in (3.7) is decreasing in s, the Hardy-Littlewood-Pólya principle implies the assertion. □

Theorem 3.13. *Let X, X_1, X_2, and Y, Y_1, Y_2 be rearrangement-invariant Banach function spaces over σ-finite nonatomic measure spaces $(\mathcal{R},\mu)$, and $(\mathcal{S},\nu)$, respectively. Φ_j, Ψ_j denoting the fundamental functions of X_j, Y_j, respectively ($j=1,2$), suppose that $s_0 := \Phi_1/\Phi_2$ and $s_1 := \Psi_1/\Psi_2$ are strictly increasing and that there are constants $r_i>0$ such that the fractions $s_i(u)/u^{r_i}$ are increasing functions ($i=0,1$). Suppose that the condition* (M) *holds for the fundamental functions Ψ_j, $j=1,2$. Let $\Gamma = s_0^{-1}\circ s_1$. Then, the following conditions are equivalent:*

(i) *every linear operator of* (Λ_{X_1}, M_{Y_1}) *and* (M_{X_2}, M_{Y_2}) *types is bounded from* X *to* Y,

(ii) *every quasilinear operator of* (Λ_{X_1}, M_{Y_1}) *and* (M_{X_2}, M_{Y_2}) *types is bounded from* X *to* Y,

(iii) *the operator* R_1 *is bounded from* $\overline{X}$ *to* $\overline{Y}$, *where* R_1 *is defined on measurable functions on* $\mathbb{R}^+$ *as*

$$R_1 f(v) = \frac{1}{\Psi_1(v)} \int_0^{\Gamma(v/2)} f(u)\varphi_1(u)du.$$

Proof. Clearly, (ii) implies (i), for every linear operator is quasilinear as well. Now, let's show that (iii) implies (ii). The operator R as defined by (3.2a) is bounded from $\overline{X}$ to $\overline{Y}$ due to Theorem 3.12. Let T be a quasilinear operator of $(\Lambda_{X_1}, M_{Y_1}; M_{X_2}, M_{Y_2})$ type. Then Tf is defined whenever $Rf^*(1) < \infty$, thus, whenever $f \in X$. Moreover, $(Tf)^*(t) \lesssim Rf^*(t)$. Therefore, by the definition of the representation space, by Theorem 3.12, and (iii), we get

$$\|Tf\|_Y = \|(Tf)^*\|_{\overline{Y}} \lesssim \|Rf^*\|_{\overline{Y}} \lesssim \|R_1 f^*\|_{\overline{Y}} \lesssim \|f^*\|_{\overline{X}} = \|f\|_X,$$

wherefore T is bounded from X to Y, as required.

It remains to show that (i) implies (iii). It suffices to prove that $(R_1 f)^* \lesssim Rf^*$. Simple inequality $|R_1 f| \le R_1|f|$ and the Hardy-Littlewood inequality will do the work. Hence, R_1 is bounded both from $\overline{\Lambda_{X_1}}$ to $\overline{M_{Y_1}}$, and from $\overline{M_{X_2}}$ to $\overline{M_{Y_2}}$ by Theorem 3.5. Since R_1 is a linear operator, the condition (i) gives the condition (iii). □

Let's formulate a special case of the previous theorem so that we could see its applied form.

Theorem 3.14. *Let* X, *and* Y *be rearrangement-invariant Banach function spaces over* σ*-finite nonatomic measure spaces* $(\mathcal{R}, \mu)$, *and* $(\mathcal{S}, \nu)$, *respectively. Suppose* $1 \le p_1 < p_2 \le \infty$, *and* $1 < q_1 < q_2 \le \infty$. *Then, the following conditions are equivalent:*

(i) *every linear operator which is bounded both from* $L^{p_1,1}(\mathcal{R})$ *to* $L^{q_1,\infty}(\mathcal{S})$, *and from* $L^{p_2,\infty}(\mathcal{R})$ *to* $L^{q_2,\infty}(\mathcal{S})$, *is bounded from* X *to* Y, *as well,*

(ii) *every quasilinear operator which is bounded both from* $L^{p_1,1}(\mathcal{R})$ *to* $L^{q_1,\infty}(\mathcal{S})$, *and from* $L^{p_2,\infty}(\mathcal{R})$ *to* $L^{q_2,\infty}(\mathcal{S})$, *is bounded from* X *to* Y, *as well,*

(iii) *the operator* R_1 *is bounded from* $\overline{X}$ *to* $\overline{Y}$, *where* R_1 *is defined on measurable functions on* $\mathbb{R}^+$ *by*

$$Rh(v) = v^{-\frac{1}{q_1}} \int_0^{\Gamma(v/2)} h(u) u^{\frac{1}{p_1}-1}\, du$$

where

$$\Gamma(z) = z^{\frac{\frac{1}{q_1}-\frac{1}{q_2}}{\frac{1}{p_1}-\frac{1}{p_2}}},$$

and analogously for the case when any of the coefficients p_2, q_2 is infinity.

3.2 Operators of $(\Lambda_{X_1}, \Lambda_{Y_1}; \Lambda_{X_2}, M_{Y_2})$ Type

Here again, the equivalence of boundedness of a quasilinear operator and inequality of K-functionals is going to play a key role. From now on, we shall require the r.i. spaces to comply with the condition

$$\int_0^t \frac{\Phi(u)}{u} du \lesssim \int_0^t \varphi(u) du, \tag{L}$$

which, inter alia, ensures the coincidence of Λ_X and Λ_X^* spaces. Recall Remarks 2.2 and 2.8, for at least one of the X spaces and one of the Y spaces are compliant with a stronger condition than (L) given the setting we are using.

Example 3.15. Let's investigate which of the common r.i. spaces are compliant with such a condition,

(i) $L^{p,q}, p \in [1,\infty), q \in [1,\infty]$ spaces, for which $\Phi(u) = c(p,q)u^{1/p}$, comply with (L),

(ii) L^∞, whose $\Phi(u) = \chi_{(0,\infty)}(u)$, does not comply with (L),

(iii) Any r.i. space whose fundamental function does not have a limit of zero at the origin does not comply with (L).

(iv) Any r.i. space whose fundamental function is $\Phi(u) = u^\alpha(\log(1+u))^\beta$, where $\alpha \in [0,1], \beta \in (-\alpha, 1-\alpha]$, complies with (L), whereas it does not in the case of $\beta = -\alpha$,

(v) Any r.i. space whose fundamental function is $\Phi(u) = u^\alpha(\log(1+u^{-1}))^\beta$, where $\alpha \in (0,1], \beta \in [\alpha-1, \alpha)$, complies with (L), whereas it does not if $\alpha = 0$ or $\alpha = \beta$.

Proof. Since $(u^{1/p})' = (u^{-1+1/p})/p \approx u^{1/p}/u$, hence the $L^{p,q}, p \in [1,\infty), q \in [1,\infty]$, spaces comply with even stronger condition than (L), i.e. $\Phi(u)/u \lesssim \varphi(u)$.

On the other hand, the integral on the left hand side diverges for any $t > 0$ whenever the fundamental function satisfies $\Phi(0+) > 0$ which is the case of L^∞, as well.

Considering the case of $\Phi(u) = u^\alpha(\log(1+u))^\beta$, we can see that $\beta = -\alpha$ implies divergence of the integral on the left hand side for any $t > 0$. Now, let $\beta \in (-\alpha, 1-\alpha]$.

Then,

$$\begin{aligned}\varphi(u) &= \alpha u^{\alpha-1}(\log(1+u))^{\beta} + \beta\frac{u^{\alpha}}{1+u}(\log(1+u))^{\beta-1}\\ &= u^{\alpha-1}(\log(1+u))^{\beta}\left(\alpha+\beta\frac{u}{(1+u)\log(1+u)}\right).\end{aligned}$$

Thus, $\Phi(u)/u \leq \alpha^{-1}\varphi(u)$ for $\beta \geq 0$, and $\Phi(u)/u \leq (\alpha+\beta)^{-1}\varphi(u)$ for $\beta < 0$. Therefore, the fundamental function mentioned in the fourth example complies with the stronger version of (L).

Let's examine the last example, where $\Phi(u) = u^{\alpha}(\log(1+u^{-1}))^{\beta}$. If $\alpha = 0$, then the integral on the left hand side diverges. Let $\alpha \in (0,1]$. Then,

$$\begin{aligned}\varphi(u) &= \alpha u^{\alpha-1}(\log(1+u^{-1}))^{\beta} - \beta\frac{u^{\alpha-1}}{1+u}(\log(1+u^{-1}))^{\beta-1}\\ &= u^{\alpha-1}(\log(1+u^{-1}))^{\beta}\left(\alpha-\beta\frac{1}{(1+u)\log(1+u^{-1})}\right).\end{aligned}$$

Thus, $\Phi(u)/u \leq \alpha^{-1}\varphi(u)$ for $\beta \in [\alpha-1,0]$, and $\Phi(u)/u \leq (\alpha-\beta)^{-1}\varphi(u)$ for $\beta \in (0,\alpha)$. Therefore, the fundamental function mentioned in the fifth example complies with the stronger version of (L) whenever $\alpha \in (0,1]$, and $\beta \in [\alpha-1,\alpha)$. It remains to show that this fundamental function does not comply with (L) for $\beta = \alpha \in (0,1]$. In such a situation, $\Phi(u)$ is bounded from above, and $\Phi(u)/u \approx u^{-1}$ for $u > 1$. Thus,

$$\int_0^t \Phi(u)/u\,du \approx 1+\log(t),$$

for any $t > 1$. It is not bounded from above (in variable t), and that's why it cannot be estimated from above by a bounded function. □

Theorem 3.16. *Let X_1, X_2, Y_1 and Y_2 be r.i. spaces over σ-finite nonatomic measure spaces $(\mathcal{R},\mu)$, and $(\mathcal{S},\nu)$, respectively, having fundamental functions Φ_1, Φ_2, Ψ_1 and Ψ_2, respectively. Let Ψ_1 satisfy* (M) *whereas both Φ_1, and Φ_2 satisfy* (L). *Suppose that the functions $s_0(u) = \Phi_1(u)/\Phi_2(u)$ and $s_1(u) = \Psi_1(u)/\Psi_2(u)$ are strictly increasing and that there exist $r_0 > 0$ and $r_1 > 0$ such that $s_j(u)/u^{r_j}$ are increasing ($j = 0,1$). Let $\Gamma = s_0^{-1}\circ s_1$. Then, the following statements are equivalent,*

(i) *T is bounded both from Λ_{X_1} to Λ_{Y_1}, and from Λ_{X_2} to M_{Y_2},*

(ii) *T satisfies the condition*

$$\int_0^t (Tf)^*(u)\psi_1(u)du \lesssim \int_0^t Uf^*(u)\psi_1(u)du, \tag{3.8}$$

for every $t > 0$, where the U operator is defined by

$$(Ug)(u) = \frac{1}{\Psi_2(u)}\int_{\Gamma(u)}^{\infty} g(v)\frac{\Phi_2(v)}{v}dv, \tag{3.9}$$

whenever g is a measurable function on $\mathbb{R}^+$.

Proof. To show the implication (i)⇒(ii), we start from the inequality of K-functionals (having used Proposition 3.4), which are to be written down using Corollary 2.6,

$$\|(Tf)^*\chi_{A_1}\|_{\overline{\Lambda_{Y_1}}} + t\|(Tf)^*\chi_{B_1}\|_{\overline{M_{Y_2}}} \lesssim \|f^*\chi_{A_0}\|_{\overline{\Lambda_{X_1}}} + t\|f^*\chi_{B_0}\|_{\overline{\Lambda_{X_2}}}.$$

Recall that both $\Phi_1(0+) = 0$, and $\Phi_2(0+) = 0$ due to the condition (L). In order to obtain the left hand side of inequality (3.8), just a simple neglecting of the term $t\|(Tf)^*\chi_{B_1}\|_{\overline{M_{Y_2}}}$ will suffice. Nevertheless, quite some diligent work will be needed to get the right hand side of the estimate.

Let's focus on the second term first,

$$\begin{aligned} t\|f^*\chi_{B_0}\|_{\overline{\Lambda_{X_2}}} &= t\int_0^\infty f^*(u+s_0^{-1}(t))\varphi_2(u)du \\ &= t\int_0^{s_0^{-1}(t)} f^*(u+s_0^{-1}(t))\varphi_2(u)du + t\int_{s_0^{-1}(t)}^\infty f^*(u+s_0^{-1}(t))\varphi_2(u)du \\ &= I_1 + tI_2. \end{aligned}$$

Now,

$$\begin{aligned} I_1 &\le tf^*(s_0^{-1}(t))\Phi_2(s_0^{-1}(t)) = f^*(s_0^{-1}(t))\Phi_1(s_0^{-1}(t)) \\ &\le \|f^*\chi_{A_0}\|_{\overline{M^*_{X_1}}} \le \|f^*\chi_{A_0}\|_{\overline{\Lambda_{X_1}}}, \end{aligned}$$

and

$$I_2 \le \int_{s_0^{-1}(t)}^\infty f^*(u)\varphi_2(u)du.$$

Thus,

$$\begin{aligned} \|(Tf)^*\chi_{A_1}\|_{\overline{\Lambda_{Y_1}}} &\lesssim \|f^*\chi_{A_0}\|_{\overline{\Lambda_{X_1}}} + tI_2, \\ \int_0^{s_1^{-1}(t)} (Tf)^*\psi_1 &\lesssim \int_0^{s_0^{-1}(t)} f^*\varphi_1 + t\int_{s_0^{-1}(t)}^\infty f^*\varphi_2. \end{aligned}$$

Then, substituting t by $s_1(t)$, using property (1.1) and Fubini theorem,

$$\begin{aligned} \int_0^t (Tf)^*\psi_1 &\lesssim \int_0^{s_0^{-1}(s_1(t))} f^*\varphi_1 + s_1(t)\int_{s_0^{-1}(s_1(t))}^\infty f^*\varphi_2 \\ &\le \int_0^{s_0^{-1}(s_1(t))} f^*(u)\frac{\Phi_1(u)}{u}du + s_1(t)\int_{s_0^{-1}(s_1(t))}^\infty f^*(u)\frac{\Phi_2(u)}{u}du \\ &= \int_0^{s_0^{-1}(s_1(t))} f^*(u)\frac{\Phi_2(u)}{u}\left(\int_0^u s_0'(v)dv\right)du \\ &\quad + \int_0^{s_0^{-1}(s_1(t))} s_0'(v)dv \int_{s_0^{-1}(s_1(t))}^\infty f^*(u)\frac{\Phi_2(u)}{u}du \\ &= \int_0^{s_0^{-1}(s_1(t))} s_0'(v)\int_v^\infty f^*(u)\frac{\Phi_2(u)}{u}du\,dv. \end{aligned} \tag{3.10}$$

Continuing by change of variables, such that $s_0(v) = s_1(w)$, we get,

$$\int_0^{s_0^{-1}(s_1(t))} s_0'(v) \int_v^\infty f^*(u)\frac{\Phi_2(u)}{u}du\,dv = \int_0^t s_1'(w) \int_{s_0^{-1}(s_1(w))}^\infty f^*(u)\frac{\Phi_2(u)}{u}du\,dw.$$

After considering the fact that

$$s_1'(w) = \frac{\psi_1(w)\Psi_2(w) - \psi_2(w)\Psi_1(w)}{\Psi_2^2(w)} \le \frac{\psi_1(w)}{\Psi_2(w)},$$

we've come to the desired inequality, finalizing the proof of the forward implication.

In agreement with Proposition 3.4, it will suffice to show that (3.8) implies the inequality of K-functionals in order to show the backward implication.

According to Corollary 2.6,

$$K(Tf, t; \Lambda_{Y_1}, M_{Y_2}) \approx \|(Tf)^*\chi_{A_1}\|_{\Lambda_{Y_1}} + t\|(Tf)^*\chi_{B_1}\|_{M_{Y_2}}. \tag{3.11}$$

First, let's estimate the first term of the sum.

Inequalities (3.8) and (1.1), Remark 2.2, change of variables $s_1(v) = s_0(w)$, as well as the Fubini theorem (for details cf. (3.10) – follow the equalities in bottom-up direction) lead to the estimate

$$\begin{aligned}
\int_0^t (Tf)^*\psi_1 &\lesssim \int_0^t \frac{\psi_1(v)}{\Psi_2(v)} \int_{s_0^{-1}(s_1(v))}^\infty f^*(u)\frac{\Phi_2(u)}{u}du\,dv \\
&\le \int_0^t \frac{s_1(v)}{v} \int_{s_0^{-1}(s_1(v))}^\infty f^*(u)\frac{\Phi_2(u)}{u}du\,dv \\
&\approx \int_0^t s_1'(v) \int_{s_0^{-1}(s_1(v))}^\infty f^*(u)\frac{\Phi_2(u)}{u}du\,dv \\
&= \int_0^{s_0^{-1}(s_1(t))} s_0'(w) \int_w^\infty f^*(u)\frac{\Phi_2(u)}{u}du\,dw \\
&= \int_0^{s_0^{-1}(s_1(t))} f^*(u)\frac{\Phi_1(u)}{u}du + s_1(t)\int_{s_0^{-1}(s_1(t))}^\infty f^*(u)\frac{\Phi_2(u)}{u}du.
\end{aligned} \tag{3.12}$$

Now, substituting t with $s_1^{-1}(t)$, using (1.1), condition (L) with the Hardy lemma and considering the concavity of Φ_2,

$$\|(Tf)^*\chi_{A_1}\|_{\overline{\Lambda_{Y_1}}} = \int_0^{s_1^{-1}(t)} (Tf)^*\psi_1 \lesssim \|f^*\chi_{A_0}\|_{\overline{\Lambda_{X_1}}} + t\|f^*\chi_{B_0}\|_{\overline{\Lambda_{X_2}}}. \tag{3.13}$$

To estimate the latter term in (3.11) start by dividing the inequality (3.8) by $s_1(t)$ and taking the supremum over the interval $(s_1^{-1}(t), \infty)$. Let's continue by taking the same steps as in estimating the first term in (3.11), until we come to an analogue of (3.12). Then,

$$\begin{aligned}
&\sup_{s_1^{-1}(t)<\tau<\infty} \frac{1}{s_1(\tau)} \int_0^\tau (Tf)^*\psi_1 \\
&\qquad \lesssim t \sup_{s_1^{-1}(t)<\tau<\infty} \frac{1}{s_1(\tau)} \int_0^\tau s_1'(v) \int_{s_0^{-1}(s_1(v))}^\infty f^*(u)\frac{\Phi_2(u)}{u}du\,dv.
\end{aligned}$$

Realizing that on the right hand side there's a weighted average of a decreasing function, i.e., a decreasing function itself, in the argument of the supremum, we can remove the supremum by evaluating the argument at the lower endpoint of the interval over which the supremum is taken. Then, resume in taking steps of the previous technique stopping at the analogue of (3.13), i.e.

$$\sup_{s_1^{-1}(t)<\tau<\infty} \frac{1}{s_1(\tau)} \int_0^\tau (Tf)^* \psi_1 \lesssim \|f^* \chi_{A_0}\|_{\overline{\Lambda_{X_1}}} + t\|f^* \chi_{B_0}\|_{\overline{\Lambda_{X_2}}}. \tag{3.14}$$

Several steps involving evaluation of the left hand side of (3.14) need to be taken to finish the proof,

$$\begin{aligned}\sup_{s_1^{-1}(t)<\tau<\infty} \frac{1}{s_1(\tau)} \int_0^\tau (Tf)^*(v)\psi_1(v)dv &\geq \sup_{s_1^{-1}(t)<\tau<\infty} (Tf)^*(\tau)\frac{\Psi_1(\tau)}{s_1(\tau)} \\ &= \sup_{s_1^{-1}(t)<\tau<\infty} (Tf)^*(\tau)\Psi_2(\tau) \\ &\geq \sup_{s_1^{-1}(t)<\tau<\infty} (Tf)^*(\tau)\Psi_2(\tau - s_1^{-1}(\tau)) \\ &= \|(Tf)^* \chi_{B_1}\|_{\overline{M_{Y_2}}}. \end{aligned}\tag{3.15}$$

Finally, combining (3.11), (3.13), (3.14), and (3.15), we have proven that,

$$K(Tf, t; \Lambda_{Y_1}, M_{Y_2}) \lesssim K(f, t; \Lambda_{X_1}, \Lambda_{X_2}),$$

which is equivalent to the statement (i) according to Proposition 3.4. □

Lemma 3.17. *Under the assumptions of Theorem 3.16, whenever $f \geq 0$ is a decreasing function on $\mathbb{R}^+$ and $v > 0$, the following equivalence holds with constants dependent neither on f, nor on v,*

$$\int_0^{\Gamma^{-1}(v)} \frac{\psi_1(t)}{\Psi_2(t)} \sup_{t\leq s\leq\Gamma^{-1}(v)} \left\{\frac{f(s)}{\psi_1(s)}\right\} dt \approx \int_0^{\Gamma^{-1}(v)} \frac{f(t)}{\Psi_2(t)} dt.$$

Proof. The method used to prove this lemma is very similar to the one used in the proof of Lemma 3.11. The inequality "$\gtrsim$" is easily proved by taking a specific s in the supremum, viz. $s = t$. The other inequality can be shown using Fubini theorem. First,

$$\begin{aligned}&\int_0^{\Gamma^{-1}(v)} \frac{\psi_1(t)}{\Psi_2(t)} \sup_{t\leq s\leq\Gamma^{-1}(v)} \left\{\frac{f(s)}{\psi_1(s)}\right\} dt \\ &\leq \int_0^{\Gamma^{-1}(v)} \frac{\psi_1(t)}{\Psi_2(t)} \sup_{t\leq s\leq\Gamma^{-1}(v)} \left\{f(s)\left(\frac{1}{\psi_1(s)} - \frac{1}{\psi_1(t)}\right)\right\} dt + \int_0^{\Gamma^{-1}(v)} \frac{\psi_1(t)}{\Psi_2(t)} \cdot \frac{f(t)}{\psi_1(t)} dt \\ &= I_1 + I_2.\end{aligned}$$

The I_2 is exactly what we expect, so we need to focus on I_1 only. By Remark 2.2, we can approximate $s_1'(u) \approx s_1(u)/u$, and $\psi_1(u) \approx \Psi_1(u)/u$. Letting $\Theta(u) = u/\Psi_1(u)$, we see that Θ is differentiable on $\mathbb{R}^+$ (with countably many exceptions). Then,

$$\begin{aligned}
I_1 &\lesssim \int_0^{\Gamma^{-1}(v)} \frac{\Psi_1(t)}{t\Psi_2(t)} \sup_{t\le s\le \Gamma^{-1}(v)} \{f(s)(\Theta(s)-\Theta(t))\}\,dt \\
&\le \int_0^{\Gamma^{-1}(v)} \frac{s_1(t)}{t} \sup_{t\le s\le \Gamma^{-1}(v)} \left\{\int_t^s f(z)\vartheta(z)dz\right\} dt \\
&\lesssim \int_0^{\Gamma^{-1}(v)} s_1'(t) \int_t^{\Gamma^{-1}(v)} f(z)\vartheta(z)dz\,dt \\
&= \int_0^{\Gamma^{-1}(v)} f(z)\vartheta(z) \int_0^z s_1'(t)dt\,dz \\
&= \int_0^{\Gamma^{-1}(v)} f(z)\left(\frac{1}{\Psi_1(z)} - \frac{z}{\Psi_1(z)}\cdot\frac{\psi_1(z)}{\Psi_1(z)}\right)\frac{\Psi_1(z)}{\Psi_2(z)}dz \\
&\le \int_0^{\Gamma^{-1}(v)} \frac{f(z)}{\Psi_2(z)}dz.
\end{aligned}$$

□

Lemma 3.18. *Let X_1, X_2, Y_1 and Y_2 be r.i. spaces over σ-finite nonatomic measure spaces $(\mathcal{R},\mu)$, and $(\mathcal{S},\nu)$, respectively, having fundamental functions Φ_1, Φ_2, Ψ_1 and Ψ_2, respectively. Let both Ψ_1, and Ψ_2 satisfy* (M). *Assume $\Phi_2(0+) = 0$. Suppose that the functions $s_0(u) = \Phi_1(u)/\Phi_2(u)$ and $s_1(u) = \Psi_1(u)/\Psi_2(u)$ are strictly increasing and that there exist $r_0 > 0$ and $r_1 > 0$ such that $s_j(u)/u^{r_j}$ are increasing ($j = 0,1$). As previously, $\Gamma = s_0^{-1} \circ s_1$. Then, whenever $h \ge 0$ is a decreasing function on $\mathbb{R}^+$ and $t > 0$, the following equivalence holds with constants dependent neither on h, nor on t,*

$$\int_0^t \frac{1}{\Psi_2(s)} \sup_{s\le y\le t} \{\Phi_2(\Gamma(y))h(\Gamma(y))\}\,ds \approx \int_0^t \frac{\Phi_2(\Gamma(s))}{\Psi_2(s)} h(\Gamma(s))ds.$$

Proof. Analogous to the proof of Lemma 3.11. □

Theorem 3.19. *Let Z be an r.i. space. Suppose X_1, X_2, Y_1, Y_2 are r.i. spaces satisfying the assumptions of the previous lemma. Let g be a μ-measurable function. Then,*

$$\left\| \frac{1}{\Psi_2(s)} \sup_{\Gamma(s)\le y\le\infty} \{\Phi_2(y)g^*(y)\} \right\|_{\overline{Z}} \lesssim \left\| \frac{1}{\Psi_1(s)} \int_0^{\Gamma(s)} g^*(v)\varphi_1(v)dv \right\|_{\overline{Z}}.$$

Proof. Analogous to the proof of Theorem 3.12. Lemma 3.18 is to be used instead of Lemma 3.11. □

Theorem 3.20. *Let X, X_1, X_2, and Y, Y_1, Y_2 be r.i. spaces over σ-finite nonatomic measure spaces $(\mathcal{R},\mu)$, and $(\mathcal{S},\nu)$, respectively. Φ_j, Ψ_j denoting the fundamental* *functions of X_j, Y_j, respectively ($j = 1, 2$), suppose that $s_0 := \Phi_1/\Phi_2$ and $s_1 := \Psi_1/\Psi_2$ are strictly increasing and that there are constants $r_i > 0$ such that the fractions $s_i(u)/u^{r_i}$ are increasing functions ($i = 0, 1$). Suppose that both fundamental functions Φ_1, and Φ_2 satisfy the condition* (L), *whereas Ψ_1, and Ψ_2 satisfy the condition* (M). *Let $\Gamma = s_0^{-1} \circ s_1$. Then, the following conditions are equivalent:*

(i) *every linear operator of $(\Lambda_{X_1}, \Lambda_{Y_1})$ and (Λ_{X_2}, M_{Y_2}) types is bounded from X to Y,*

(ii) *every quasilinear operator of $(\Lambda_{X_1}, \Lambda_{Y_1})$ and (Λ_{X_2}, M_{Y_2}) types is bounded from X to Y,*

(iii) *the operator U is bounded from $\overline{X}$ to $\overline{Y}$, where U is defined on measurable functions on $\mathbb{R}^+$ as*

$$Uf(v) = \frac{1}{\Psi_2(v)} \int_{\Gamma(v)}^{\infty} f(u) \frac{\Phi_2(u)}{u} du.$$

Proof. Apparently, condition (ii) implies condition (i), since every linear operator is quasilinear as well.

Since U is a linear operator, hence in order to prove that condition (i) implies condition (iii), it suffices to show that the operator U is bounded both from Λ_{X_1} to Λ_{Y_1}, and from Λ_{X_2} to M_{Y_2}. First, let $f \in \Lambda_{X_1}$. We have established the equivalences $\psi_1(v) \approx \Psi_1(v)/v$ and $s_1'(v) \approx s_1(v)/v$ in Remark 2.2. Thus, followed by change of variables $s_1(v) = s_0(w)$, and by the Fubini theorem (details can be found at (3.10), in the proof of Theorem 3.16), using the definition of Γ, the Hardy-Littlewood inequality, and the condition (L), we have,

$$\begin{aligned}
\|Uf\|_{\Lambda_{Y_1}} &= \lim_{t\to\infty} \int_0^t \frac{\psi_1(v)}{\Psi_2(v)} \int_{\Gamma(v)}^{\infty} f(u) \frac{\Phi_2(u)}{u} du\, dv \\
&\approx \lim_{t\to\infty} \int_0^t s_1'(v) \int_{\Gamma(v)}^{\infty} f(u) \frac{\Phi_2(u)}{u} du\, dv \\
&= \lim_{t\to\infty} \int_0^{\Gamma(t)} s_0'(w) \int_{w}^{\infty} f(u) \frac{\Phi_2(u)}{u} du\, dw \\
&= \lim_{t\to\infty} \left(\int_0^{\Gamma(t)} f(u) \frac{\Phi_1(u)}{u} du + s_1(t) \int_{\Gamma(t)}^{\infty} f(u) \frac{\Phi_2(u)}{u} du \right) \\
&\leq \lim_{t\to\infty} \left(\int_0^{\Gamma(t)} f(u) \frac{\Phi_1(u)}{u} du + \int_{\Gamma(t)}^{\infty} f(u) \frac{\Phi_1(u)}{u} du \right) \\
&\leq \int_0^{\infty} f^*(u) \frac{\Phi_1(u)}{u} du \\
&\leq \int_0^{\infty} f^*(u) \varphi_1(u) du \\
&= \|f\|_{\Lambda_{X_1}}.
\end{aligned}$$

Therefore, U is bounded from Λ_{X_1} to Λ_{Y_1}. Now, let $f \in \Lambda_{X_2}$. Then, by the condition (M), monotonicity of Uf, the Hardy-Littlewood inequality, and the condition (L),

$$\begin{aligned}\|Uf\|_{M_{Y_2}} &\approx \sup_{v>0}\left\{\frac{1}{\Psi_2(v)}\int_{\Gamma(v)}^{\infty} f(u)\frac{\Phi_2(u)}{u}du\,\Psi_2(v)\right\}\\ &= \int_0^{\infty} f(u)\frac{\Phi_2(u)}{u}du\\ &\le \int_0^{\infty} f^*(u)\frac{\Phi_2(u)}{u}du\\ &\approx \|f\|_{\Lambda_{X_2}}.\end{aligned}$$

Thus, U is bounded from Λ_{X_2} to M_{Y_2}, as well. Since U is a linear operator, the condition (i) ensures that it is bounded from $\overline{X}$ to $\overline{Y}$.

It remains to prove that condition (iii) implies condition (ii). Let T be a quasilinear operator of the $(\Lambda_{X_1}, \Lambda_{Y_1}; \Lambda_{X_2}, M_{Y_2})$ type. Let $f \in X$, and $g \in Y'$, $\|g\|_{Y'} \le 1$. By the Hardy-Littlewood inequality, we have

$$\begin{aligned}I = \int_{\mathcal{R}} Tf(x)g(x)d\mu(x) &\le \int_0^{\infty}(Tf)^*(t)g^*(t)dt\\ &= \int_0^{\infty}(Tf)^*(t)\psi_1(t)\frac{g^*(t)}{\psi_1(t)}dt\\ &\le \int_0^{\infty}(Tf)^*(t)\psi_1(t)\sup_{t\le s<\infty}\left\{\frac{g^*(s)}{\psi_1(s)}\right\}dt. \qquad (3.16)\end{aligned}$$

Theorem 3.16 and the Hardy lemma, the Fubini theorem, and the Hölder inequality imply

$$\begin{aligned}I &\lesssim \int_0^{\infty} Uf^*(t)\psi_1(t)\sup_{t\le s<\infty}\left\{\frac{g^*(s)}{\psi_1(s)}\right\}dt\\ &= \int_0^{\infty}\psi_1(t)\sup_{t\le s<\infty}\left\{\frac{g^*(s)}{\psi_1(s)}\right\}\frac{1}{\Psi_2(t)}\int_{\Gamma(t)}^{\infty} f^*(v)\frac{\Phi_2(v)}{v}dv\,dt\\ &= \int_0^{\infty} f^*(v)\frac{\Phi_2(v)}{v}\int_0^{\Gamma^{-1}(v)}\frac{\psi_1(t)}{\Psi_2(t)}\sup_{t\le s<\infty}\left\{\frac{g^*(s)}{\psi_1(s)}\right\}dt\,dv\\ &\le \|f^*\|_{\overline{X}}\cdot\left\|\frac{\Phi_2(v)}{v}\int_0^{\Gamma^{-1}(v)}\frac{\psi_1(t)}{\Psi_2(t)}\sup_{t\le s<\infty}\left\{\frac{g^*(s)}{\psi_1(s)}\right\}dt\right\|_{\overline{X}'}. \qquad (3.17)\end{aligned}$$

Now, we need to estimate the latter norm. By splitting the interval $[t,\infty)$ into two parts, viz. $[t,\Gamma^{-1}(v))$, and $[\Gamma^{-1}(v),\infty)$, and by utilizing Lemma 3.17, we have

$$\begin{aligned}&\frac{\Phi_2(v)}{v}\int_0^{\Gamma^{-1}(v)}\frac{\psi_1(t)}{\Psi_2(t)}\sup_{t\le s<\infty}\left\{\frac{g^*(s)}{\psi_1(s)}\right\}dt\\ &\qquad\lesssim \frac{\Phi_2(v)}{v}\int_0^{\Gamma^{-1}(v)}\frac{g^*(t)}{\Psi_2(t)}dt + \frac{\Phi_2(v)}{v}\int_0^{\Gamma^{-1}(v)}\frac{\psi_1(t)}{\Psi_2(t)}\sup_{\Gamma^{-1}(v)\le s<\infty}\left\{\frac{g^*(s)}{\psi_1(s)}\right\}dt.\end{aligned}$$

The growth of $s_j(t)/t^{r_j}$ implies that $s_j'(t) \approx s_j(t)/t$, $j = 0,1$, and $\psi_1(t) \approx \Psi_1(t)/t$. Thus,

$$\begin{aligned}
&\frac{\Phi_2(v)}{v} \sup_{\Gamma^{-1}(v)\le s<\infty} \left\{\frac{g^*(s)}{\psi_1(s)}\right\} \int_0^{\Gamma^{-1}(v)} \frac{\psi_1(t)}{\Psi_2(t)} dt \\
&\qquad\approx \frac{\Phi_2(v)}{v} \sup_{\Gamma^{-1}(v)\le s<\infty} \left\{\frac{sg^*(s)}{\Psi_1(s)}\right\} \int_0^{\Gamma^{-1}(v)} s_1'(t)dt \\
&\qquad= \frac{\Phi_2(v)}{v} \sup_{\Gamma^{-1}(v)\le s<\infty} \left\{\frac{sg^*(s)}{\Psi_1(s)}\right\} \frac{\Phi_1(v)}{\Phi_2(v)} \\
&\qquad= \frac{\Phi_1(v)}{v} \sup_{\Gamma^{-1}(v)\le s<\infty} \left\{\frac{sg^*(s)}{\Psi_1(s)}\right\}.
\end{aligned}$$

Let's utilize Theorem 3.19 for the segment $(\Lambda_{Z_2}, M_{W_2}; M_{Z_1}, M_{W_1})$, where W_j, Z_j are r.i. spaces over $(\mathcal{R},\mu)$, and $(\mathcal{S},\nu)$, respectively, having concave fundamental functions equivalent to $t/\Phi_j(t)$, $t/\Psi_j(t)$, $j = 1,2$, respectively. Since Φ_j's satisfy (L), hence (Id/Φ_j)'s satisfy (M). The growth of $s_1(u)/u^{r_1}$ implies $(Id/\Psi_1)(0+) = 0$. Therefore,

$$\left\|\frac{\Phi_1(v)}{v} \sup_{\Gamma^{-1}(v)\le s<\infty} \left\{\frac{sg^*(s)}{\Psi_1(s)}\right\}\right\|_{\overline{X}'} \lesssim \left\|\frac{\Phi_2(v)}{v} \int_0^{\Gamma^{-1}(v)} \frac{g^*(t)}{\Psi_2(t)} dt\right\|_{\overline{X}'}.$$

Thus,

$$\left\|\frac{\Phi_2(v)}{v} \int_0^{\Gamma^{-1}(v)} \frac{\psi_1(t)}{\Psi_2(t)} \sup_{t\le s<\infty} \left\{\frac{g^*(s)}{\psi_1(s)}\right\} dt\right\|_{\overline{X}'} \lesssim \left\|\frac{\Phi_2(v)}{v} \int_0^{\Gamma^{-1}(v)} \frac{g^*(t)}{\Psi_2(t)} dt\right\|_{\overline{X}'}. \tag{3.18}$$

To evaluate the latter norm, let's consider $h \in \overline{X}$, $\|h\|_{\overline{X}} \le 1$. Then, by the Fubini theorem and the Hölder inequality,

$$\begin{aligned}
\left|\int_0^\infty h(v)\frac{\Phi_2(v)}{v} \int_0^{\Gamma^{-1}(v)} \frac{g^*(t)}{\Psi_2(t)} dt\, dv\right| &= \left|\int_0^\infty g^*(t)\frac{1}{\Psi_2(t)} \int_{\Gamma(t)}^\infty h(v)\frac{\Phi_2(v)}{v} dv\, dt\right| \\
&= \left|\int_0^\infty g^*(t)Uh(t)dt\right| \\
&\le \|g^*\|_{\overline{Y}'} \|Uh\|_{\overline{Y}} \\
&\le \|g^*\|_{\overline{Y}'} \|U\|_{\overline{X}\to\overline{Y}} \|h\|_{\overline{X}}.
\end{aligned}$$

Taking the supremum over the set $\{h \in \overline{X}, \|h\|_{\overline{X}} \le 1\}$, we have

$$\left\|\frac{\Phi_2(v)}{v} \int_0^{\Gamma^{-1}(v)} \frac{g^*(t)}{\Psi_2(t)} dt\right\|_{\overline{X}'} \le \|U\|_{\overline{X}\to\overline{Y}} \|g^*\|_{\overline{Y}'}.$$

By (3.16), (3.17), and (3.18), we obtain

$$\int_{\mathcal{R}} Tf(x)g(x)d\mu(x) \lesssim \|f^*\|_{\overline{X}} \cdot \|g^*\|_{\overline{Y}'} \cdot \|U\|_{\overline{X}\to\overline{Y}}.$$

Taking the supremum over the set $\{g \in Y', \|g\|_{Y'} \le 1\}$, we have $\|Tf\|_Y \lesssim \|f\|_X$, finishing the proof. □

Let's conclude this section by formulating a special case of the preceding theorem in terms of Lorentz $L^{p,1}$ and $L^{p,\infty}$ spaces.

Theorem 3.21. *Let X, and Y be rearrangement-invariant Banach function spaces over σ-finite nonatomic measure spaces $(\mathcal{R},\mu)$, and $(\mathcal{S},\nu)$, respectively. Suppose $1 \le p_1 < p_2 < \infty$, and $1 < q_1 < q_2 \le \infty$. Then, the following conditions are equivalent:*

(i) *every linear operator which is bounded both from $L^{p_1,1}(\mathcal{R})$ to $L^{q_1,1}(\mathcal{S})$, and from $L^{p_2,1}(\mathcal{R})$ to $L^{q_2,\infty}(\mathcal{S})$, is bounded from X to Y, as well,*

(ii) *every quasilinear operator which is bounded both from $L^{p_1,1}(\mathcal{R})$ to $L^{q_1,1}(\mathcal{S})$, and from $L^{p_2,1}(\mathcal{R})$ to $L^{q_2,\infty}(\mathcal{S})$, is bounded from X to Y, as well,*

(iii) *the operator U is bounded from $\overline{X}$ to $\overline{Y}$, where U is defined on measurable functions on $\mathbb{R}^+$ as*

$$Uf(v) = v^{\frac{1}{q_2}} \int_{\Gamma(v)}^{\infty} f(u) u^{\frac{1}{p_2}-1} du$$

where

$$\Gamma(z) = z^{\frac{\frac{1}{q_1}-\frac{1}{q_2}}{\frac{1}{p_1}-\frac{1}{p_2}}},$$

with natural analogy for the case of $q_2 = \infty$.

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