Univerzita Karlova v Praze
Matematicko-fyzikální fakulta

BAKALÁŘSKÁ PRÁCE

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Algebraické vlastnosti barevnosti grafů

Katedra aplikované matematiky
Vedoucí bakalářské práce: RNDr. Daniel Král', Ph.D.,
Institut teoretické informatiky

Studijní program: Informatika, Obor: Programování

2008

Rád bych poděkoval Danielu Kráľovi za mnoho hodin strávených jazykovými a stylistickými korekcemi mé práce, poskynutí podnětných zdrojů a diskuze o nich a především ohromnou trpělivost.

Prohlašuji, že jsem svou bakalářskou práci napsal samostatně a výhradně s použitím citovaných pramenů. Souhlasím se zapůjčováním práce.

V Praze dne

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Abstrakt: V práci se zabýváme algebraickými metodami, pomocí kterých lze rozhodnout, zda existuje obarvení daného grafu. Zaměříme se především na Alon-Tarsiho větu, která bude dokázána, předvedeme její známé aplikace a ukážeme nové použití při barvení druhých mocnin cyklů.

Klíčová slova: Alon-Tarsi, barvení druhých mocnin cyklů

Title:	Algebraic aspects of graph colorings
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Abstract: We study algebraic tools for proving the existence of graph colorings and focus a classical method of Alon-Tarsi from this area. In particular, we prove Alon-Tarsi Theorem, provide examples of some known applications and give a new application to colorings of squares of cycles.

Keywords: Alon-Tarsi, coloring of squares of cycles

1 Graph coloring

The central topic of my bachelor thesis is graph coloring and its variant called list coloring. In this thesis, we focus on vertex colorings. A vertex coloring of a graph $G(V, E)$ is a mapping $c : V \to S$, such that $c(v_i) \neq c(v_j)$ for every $(u, v) \in E$. The set S is referred as the set of colors. A variant of vertex coloring called list coloring was introduced by Erdős, Rubin and Taylor [3] in 1979 and a similar notion was also independently analyzed by Vizing [7]. Instead of having a common set of colors for all vertices, every vertex has its own set of available colors and it is then required that a vertex is asigned a color from its set.

In this chapter, we introduce basic notions on graph coloring and list coloring. In the next chapter, we present a theorem of Alon and Tarsi [2] which relates the existence of a coloring to orientations of graphs through algebra. In Chapters 3 and 4, we present two applications of the theorem of Alon and Tarsi. The first one has been found by the author of the thesis and the second one comes from a paper of Fleischner and Stiebitz [4]. Further applications of the method can be found in one of the surveys [1, 6] in this area.

The formal definition of a list coloring of a graph is the following:

Definition 1. *Let $G(V, E)$ be a graph and assume that each vertex $v \in V(G)$ is assigned a set S_v of available colors. A vertex coloring c of G from the lists S_v is a mapping such that $c(v) \in S_v$ for all $v \in V$ and $c(v_i) \neq c(v_j)$ for every edge v_iv_j of G. A graph G is k-list-colorable (k-choosable) if for every choice of k-element lists S_v there exists a vertex coloring from the lists. The least such integer k is the list-chromatic number (the choice number) of G and is denoted by $ch(G)$ of G.*

First we look at the relation between the chromatic number of G and its list-chromatic number and observe that:

$$ch(G) \geq \chi(G).$$

This inequality holds because if we choose $S_v = \{1, \ldots, k\}$ for every $v \in V$, we can obtain coloring of G with k colors. The inequality can be strict in general. A graph in Figure 1 has chromatic number equal to 2 but its list chromatic number is 3.

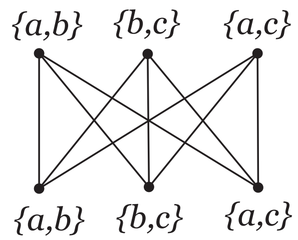

Figure 1: A bipartite graph with $ch(G) = 3$. The list of sizes two witnessing that the choice number is greater than two are given in the figure.

The question is, whether there is a relation between $ch(G)$ and $\chi(G)$. The corollary of the next proposition is that there is not, since we can obtain graph G with arbitraly large $ch(G)$ and with $\chi(G) = 2$.

Proposition 1. *The list chromatic number of the complete bipartite graphs with parts of sizes $\binom{2k-1}{k}$ is at least k.*

Proof. We exhibit sets S_v assigning each vertex $k-1$ colors such that there is no coloring of the complete graph $G = K_{\binom{2k-1}{k}}$ from these lists. Let S be a set of $2k-1$ colors and assign the vertices lists in such a way that every subset of k different colors from S is assigned to exactly one vertex in every part of G. For contradiction assume that there exists a coloring c of the considered complete bipartite graph from these lists. Let P_1 be the set of vertices of one part of G and S_1 be the set of colors of all vertices belonging to P_1 assigned by the coloring c. From the definition of c it holds, that $S_1 \cap S_v \neq \emptyset$ for every $v \in P_1$. The size of the smallest set S_1 intersecting all lists of the vertices of P_1 is k. Otherwise there would exists $S_v \subseteq S \setminus S_1$ for $v \in P_1$, since $|S \setminus S_1| \geq k$. In the same way, we obtain that the vertices of the other part P_2 of G must be assigned at least k colors. Since there are $2k-1$ colors in total, there must exist a vertex of P_1 and a vertex of P_2 with the same color. Since G is a complete bipartite graph, the coloring c cannot then be proper.

□

As an example of techniques used in the area of list colory, let us state and prove a list variant of the Five Color Theorem.

Theorem 1 (Thomassen [5] 1994)**.** *Every planar graph is 5-list-colorable.*

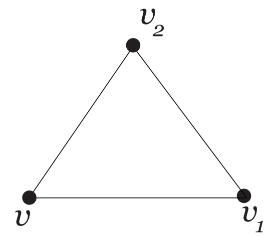

Figure 2: The base case of the induction in the proof of Proposition 2.

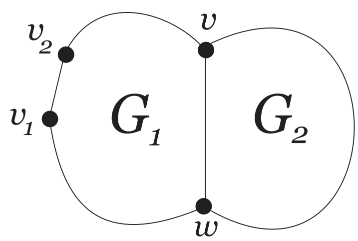

Figure 3: The case where the outer face of graph G constains a chord vw.

To prove Theorem 1, we established the next proposition. Recall, that G is a plane triangulation if every face of G (including the outer face) is bounded by a triangle.

Proposition 2. *Let G be a plane triangulation and $|V(G)| \geq 3$. Let C denote the cycle obtained from the vertices $v_1, v_2, \ldots, v_k$ of the outer face of G. Assume an assigment L satisfying the conditions (*) stated below is given*

i) There are v_1, v_2 that are neighbours on C and have already be colored.

ii) For every $w \neq v_1, v_2$ on the outer face of G, it holds that $|L(w)| = 3$.

iii) For every w in the inner face of G, the size of its list is equal to 5.

Proof. We prove this proposition by induction on the number of vertices of G. For $|V(G)| = 3$, the situation is shown on figure 2. By (*) it holds that $|L(v)| = 3$. Hence, we can color v because it has just two neighbours.

Now let $|V(G)| > 3$. First suppose that C contains a chord vw. The chord vw divides the graph G into two subgraphs, G_1 and G_2 (Figure 3). Since v_1v_2 is edge of G and $v_1v_2 \neq vw$ it belongs either to G_1 or to G_2.

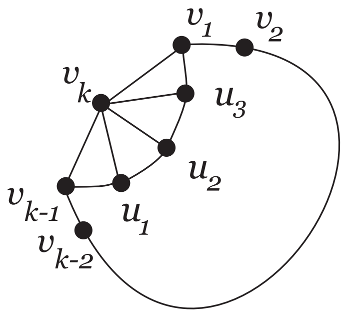

Figure 4: Graph G without a chord.

Assume that $v_1v_2 \in E(G_1)$. Using the induction , we can properly color G_1. Such a coloring assign any colors to vertices v and w so then we can use induction also to the graph G_2. After we simply connect G_1 and G_2 back together and we obtain a coloring of G.

If the outerface C of G contains no chord, let $u_1, u_2, \ldots, u_m$ be the neighbours of v_k in inner face of G (figure 4). Obviously v_1, v_2, …, v_{k-1}, u_1, u_2, …, u_m is the outer face of the graph obtained from G by removing v_k. Let l, k denote the colors from the $L(v_k)$ such that $c(v_1) \neq l, k$. If we remove these colors from the lists of the vertices $u_1, u_2, \ldots, u_m$ there still remains three colors in their lists and thus for every vertex v belonging to the outer face of $G - v_k$, it holds that $|L(v)| = 3$. The lists of other vertices are unchanged thus we can apply the induction on the graph $G - v_k$. Now we have to color v_k using the coloring obtained by induction. It is easy since the vertices $u_1, \ldots u_m$ and v_1 have surely colors different from l, k. The vertex v_{k-1} used at most one of the color l and k to be assigned to v_k and thus one color of these two can be assigned to v_k. □

Finally we have to show that Proposition 2 implies Theorem 1. For every planar graph G, we can obtain a triangulation G_T by adding egdes to G and obiously any coloring of G_T is alse a proper coloring of G. Let $v_1, v_2, \ldots, v_k$ be the outer face of G. Precolor v_1 and v_2, remove arbitraly two colors from the lists of $v3, \ldots, v_k$ and apply the Proposition 2. The statement of Theorem 1 now follows.

2 Alon-Tarsi Theorem

In this chapter, we explain how the bounds on the list chromatic number of graphs can be obtained using an algebraic approach. Before we start the exposition, some notation needs to be introduced. An Eulerian subgraph H of a directed graph G is a subdigraph where $deg^+(v) = deg^-(v)$ for each vertex in H. We do not require H to be connected. Also notice that H can be understood as a union of directed cycles since every vertex of H has the same in-degree and out-degree. If such a subgraph of H has an even number of edges, we call it even, otherwise it is odd. Finally $EE(D)$ denotes the number of even Eulerian subgraphs of G and $EO(D)$ denotes the number of odd Eulerian subgraphs of G.

Using this notation, we can formulate the theorem of Alon and Tarsi [2]:

Theorem 2 (Alon and Tarsi)**.** *Let $G = (V, E)$ be a graph and D one of its orientations. For each $v \in V$, let $S(v)$ be a set of $d_D^+(v) + 1$ distinct colors where $d_D^+(v)$ is the outdegree of v in D. If $EE(D) \neq EO(D)$ then there is a proper vertex-coloring c such that $c(v) \in S(v)$ for all $v \in V$.*

Before proving Theorem 2 we establish a simple algebraic lemma.

Lemma 1. *Let $P = P(x_1, x_2, \dots, x_n)$ be a polynomial in n variables over the ring of integers $\mathbb{Z}$. Suppose that for $1 \leq i \leq n-1$ the degree of P as a polynomial in x_i is at most d_i and let $S_i \in \mathbb{Z}$ be a set of d_i+1 distinct integers. If $P(x_1, x_2, \dots x_n) = 0$ for all n-tuples $(x_1, \dots, x_n) \in S_1 \times S_2 \times \dots \times S_n$, then $P \equiv 0$, i.e., $P(x_1, \dots, x_n) = 0$ for all $(x_1, \dots, x_n) \in \mathbb{Z}^n$.*

Proof. We apply induction on n. For $n = 1$ P is a polynomial in x_1 which has degree at most d_1. Since any polynomial P of degree d_1 has at most d_1 different roots, if $P(x_1) = 0$ for all $x_1 \in S$, then P is identicaly equal to zero, i.e., $P \equiv 0$.

Assume now that the lemma holds for $n-1$. For $n > 1$ we can write the polynomial as the following sum

$$\sum_{i=0}^{d_n} P_i(x_1, \dots, x_{n-1}) x_n^i.$$

Fix on an arbitraty $n-$tuple $(x_1, \dots, x_{n-1}) \in S_1 \times \dots \times S_{n-1}$ and define

$$P'(x_n) = \sum_{i=0}^{d_n} P_i(x_1, \dots, x_{n-1}) x_n^i$$

Since $P'(x_n)$ equals to zero for at least $d_n + 1$ choices of x_n, $P' \equiv 0$. Hence $P_i(x_1, \ldots, x_{n-1}) = 0$ for this particular choice of $x_1, \ldots, x_{n-1}$ for all $i = 0, \ldots, d_n$. Since the choice of the tuple was arbitrary, $P_i(x_1, \ldots, x_{n-1}) = 0$ for all $(x_1, \ldots, x_{n-1}) \in S_1 \times \ldots \times S_n$. By induction hypothesis is $P_i \equiv 0$ for $i = 1, \ldots, n$, which implies that $P \equiv 0$. □

We next define graph polynomials. The graph polynomial $f_G(x_1, \ldots, x_n)$ of an undirected graph $G = (V, E)$ with vertices $V = v_1, \ldots, v_n$ is

$$f_G(x_1, x_2, \ldots, x_n) = \prod (x_i - x_j)$$

where the product ranges over all $i < j, (v_i, v_j) \in E$. Let us describe a connection between graph polynomials and graph orientations. In order to present this connection we assign every arc (x_i, x_j) weight $w(e)$, where $w(e) = x_i$ if $i < j$ and $w(e) = -x_i$ otherwise. The weight of an orientation D of G is then defined as

$$w(D) = \prod_{e \in E(D)} w(e).$$

Finally the relation between graph polynomials and graph orientations is captured by an equality

$$f_G(x_1, \ldots, x_n) = \sum_D w(D)$$

where D ranges over all orientations of G. Hence every monomial of the polynomial $f_g(x_1, \ldots, x_n)$ corresponds to exactly one orientation of G. To prove the equality above proceed by induction on the number of edges of the graph. The equality is obvious for a one-edge graph. If $f_{G-e}(x_1, \ldots, x_n) = \sum_D w(D)$ where the sum ranges over orientations of $G - e$ and $e = (x_i, x_j)$, then

$$f_G(x_1, \ldots, x_n) = x_i f_{G-e}(x_1, \ldots, x_n) - x_j f_{G-e}(x_1, \ldots, x_n) = \sum_D w(D)$$

where D ranges over all orientations of G.

Let $DE(d_1, \ldots, d_n)$ denote the set of orientations of G with even number of edges oriented v_i to v_j with $i > j$ such that $deg_D^+ v_i = d_i$ for each $v_i \in V(G)$. Similarly $DO(d_1, \ldots, d_n)$ denotes the set of such orientations of G with odd number of edges oriented from v_i to v_j with $i > j$.

The arguments of the last paragraph yield:

Lemma 2. *For every graph G the following equation holds*

$$f_G(x_1, \dots, x_n) = \sum_{d_1, \dots, d_n \geq 0} (|DE(d_1, \dots, d_n)| - |DO(d_1, \dots, d_n)|) \prod_{i=1}^{n} x_i^{d_i}.$$

The next lemma describes a relation between the number of Eulerian subgraphs of G and the number of orientations of G with prescribed out-degrees.

Lemma 3. *Let G be an undirected graph and $d_1, \dots, d_n$ integers. For every orientation $D \in DE(d_1, \dots, d_n) \cup DO(d_1, \dots, d_n)$, it holds that*

$$||DE(d_1, \dots, d_n) - |DO(d_1, \dots, d_n)|| = |EE(D) - EO(D)|.$$

Proof. We assume $D \in DO(d_1, \dots, d_n)$. The arguments readily translate to the case $D \in DE(d_1, \dots, d_n)$. For any orientation D', such that $D' \in DE(d_1, \dots, d_n) \cup DO(d_1, \dots, d_n)$, let $D \oplus D'$ denote the set of all oriented edges with different orientation in D and D'. Since the outdegree of any vertex v in D is equal to its outdegree in D', the number of outgoing edges from v in D with different orientation in D' has to be equal to the number of incoming edges to v in D that have different orientation in D'. This implies that $D \oplus D'$ is an Eulerian subgraph of D. Moreover, if D' is odd, then $D \oplus D'$ is an even Eulerian subgraph and if D' is even, then $D \oplus D'$ is odd. Thus the mapping $D' \to D \oplus D'$ is bijection between $D' \in DE(d_1, \dots, d_n) \cup DO(d_1, \dots, d_n)$ and the set of all Eulerian subgraphs of D. All even orientations are mapped to odd Eulerian subgraphs and all odd orientations are mapped to even Eulerian subgraphs. Hence

$$||DE(d_1, \dots, d_n)| - |DO(d_1, \dots, d_n)|| = |EE(D) - EO(D)|$$

□

If we combine Lemmas 2 and 3, we obtain by replacing the coefficients of the monomials the following equation:

$$f_G(d_1, \dots, d_n) = \sum_{d_1, \dots, d_n \geq 0} (|EE(D)| - |EO(D)|) \prod_{i=1}^{n} x_i^{d_i}.$$

This equation is formally stated in the next lemma.

Lemma 4. *Let D be an orientation of an undirected graph $G = (V, E)$ on a set $V = \{v_1, \ldots, v_n\}$ of n vertices. For $1 \le i \le n$, let $d_i = d_D^+(v_i)$ be the outdegree of v_i in D. The absolute value of the coefficient of the monomial $\prod_{i=1}^n x_i^{d_i}$ in the standard representation of $f_G = f_G(x_1, \ldots, x_n)$ as a linear combination of monomials is $|EE(D) - EO(D)|$. In particular, if $EE(D) \ne EO(D)$ then this coefficient is not zero.*

We are now ready to prove Theorem 2.

Proof of Theorem 2. Let $V = \{v_1, \ldots, v_n\}$ and $d_i = d_D^+(v_i)$. It is possible to assume that the lists S_i of colors are subsets of integers of cardinality $d_i + 1$. We want to show that there exists a proper vertex coloring $c : V \to \mathbb{Z}$. If $f_G(x_1, \ldots, x_n)$ is the graph polynomial of G, the non-existence of a proper coloring is equivalent to the following statement

$$f_G(x_1, \ldots, x_n) = 0 \text{ for every } n\text{-tuple } (x_1, \ldots, x_n) \in S_1 \times S_2 \times \ldots \times S_n$$

Let $Q_i(x_i)$, $1 \le i \le n$, be the polynomial

$$Q_i(x_i) = \prod_{s \in S_i} (x_i - s) = x_i^{d_i+1} - \sum_{j=0}^{d_i} q_{ij} x_i^j.$$

Clearly, if $x_i \in S_i$, then $Q_i(x_i) = 0$, i.e., the following holds for all $x_i \in S_i$

$$x_i^{d_i+1} = \sum_{j=0}^{d_i} q_{ij} x_i^j. \tag{2.1}$$

Our aim is to get rid of the powers of x_i with order higher than d_i in the expansion of f_G in order to apply Lemma 1. It can be easily achieved by substituting (2.1) for $x_i^{d_i+1}$ in each occurence of x_i^f where $f > d_i$ with $\sum_{j=0}^{d_i} q_{ij} x_i^j$. By repeated applications until such powers of x_i exist, we obtain a polynomial f'_G. Note that the degree of x_i in f'_G is at most d_i for all $1 \le i \le n$. In addition, for every tuple $(x_1, \ldots, x_n) \in S_1 \times \ldots \times S_n$ the value of $f_G(x_1, \ldots, x_n)$ is equal to $f'_G(x_1, \ldots, x_n)$ since f'_G was obtained from f_G by substituting (2.1), and equation that holds for all $x_i \in S_i$. In particular, it holds that $f'_G(x_1, \ldots, x_n) = 0$ for every $(x_1, \ldots, x_n) \in S_1 \times \ldots \times S_n$. Lemma 1 now yields that $f'_G \equiv 0$.

By Lemma 4, the coefficient of $\prod_{x=1}^{n} x_i^{d_i}$ in f_G is nonzero, because we assume that $EE(D) \neq EO(D)$. Finally we should prove, that the coefficient of this monomial cannot be affected by substitutions we have performed.

The degree of every monomial of f_G is equal in the number of edges of G. If we substitute (2.1) for $x_i^{d_i+1}$ to any monomial of f_G we obtain d_i monomials with strictly smaller degree. But such new monomials cannot affect coefficient of $\prod_{x=1}^{n} x_i^{d_i}$ because they have different degree. Thus the coefficient of $\prod_{x=1}^{n} x_i^{d_i}$ in f'_G is the same as in f_G. In particular, it is non-zero. Since we have obtained contradiction there exists a proper coloring $c : V \to \mathbb{Z}$ such that$c(v_i) \in S_i$.

3 The list chormatic number of the square of cycles

In this chapter, we will prove that the list chromatic number of the square of a cycle is equal to its chromatic number. Let us start with some definitions. Recall that $C_{\ell_0}^2$ denotes the square of the cycle of length ℓ_0.

We state an easy observation on Eulerian digraphs.

Proposition 3. *Let G be and Eulerian digraph and H one of its subgraph. If H is Eulerian, then $G - H$ i.e. the digraph obtained by removing the edges of H, is also Eulerian .*

Proof. Since G is Eulerian, the in-degree and out-degree of each vertex of G is the same. This also holds for H. Hence, there is the same number of ingoing and outgoing arcs removed at each vertex of G and thus the in-degree and the out-degree of each vertex are the same in $G - H$. Therefore, $G - H$ is Eulerian.

□

A direct corollary of Proposition 3 is the following.

Proposition 4. *Every subgraph of $C_{\ell_0}^2$ obtained by substracting an Eulerian subgraph from $C_{\ell_0}^2$ is Eulerian.*

Proposition 3 yields a correspondence between Eulerian subgraphs of $C_{\ell_0}^2$ with few and with a lot of edges. Let $\mathcal{S}$ denotes the set of Eulerian subgraphs of $C_{\ell_0}^2$ with less than ℓ_0 edges, $\mathcal{M}$ are subgraphs with exactly ℓ_0 edges and finally $\mathcal{L}$ is the set of subgraphs with more than ℓ_0 edges. It is easy to prove

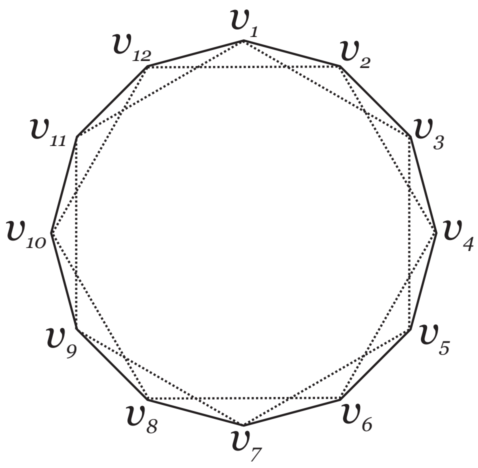

Figure 5: solid : jump by one, dashed : jump by two

that the mapping $\varphi : \mathcal{L} \longrightarrow \mathcal{S}$ given by $\varphi(G) = C^2_{\ell_0} - G$ is bijection. Since $C^2_{\ell_0}$ has an even number of edges, the bijection φ brings an even subgraph to an even subgraph and odd subgraph to odd subgraph. In particulary, the numbers of odd subgraphs in $\mathcal{S}$ and in $\mathcal{L}$ are equal. The same holds for even subgraphs.

Proposition 5. *Every non-empty Eulerian digraph contains at least one cycle.*

Proof. Consider a longest oriented path in G. Let $v_1, v_2, \ldots, v_k$ be this path (clearly $k > 2$). Since G is Eulerian, v_k has another neighbour, say w. By the choice of the path, w is one of the vertices v_i ($i \in 1, 2, \ldots, k-1$). Then $v_i, v_{i+1}, \ldots, v_k$ is a cycle. □

Proposition 6. *Every subgraph H of $C^2_{\ell_0}$ contained in $\mathcal{S}$ is a single oriented cycle.*

Proof. If the vertices of $C^2_{\ell_0}$ are numbered from 1 to ℓ_0 then the arcs of $C^2_{\ell_0}$ "jump" by one or two (figure 5). Hence the length of the shortest cycle in $C^2_{\ell_0}$ is $\lceil \ell_0/2 \rceil$. Assume the opposite, that H contains two cycles. If we remove one of these cycles (which is also Eulerian subgraph of $C^2_{\ell_0}$), we obtain again Eulerian subgraph of $C^2_{\ell_0}$ (Proposition 4). But such graph would have less than $\lceil \ell_0/2 \rceil$ edges and it is impossible while every Eulerian graph contains at least one cycle and the shortest length of cycle is $\lceil \ell_0/2 \rceil$. □

Claim 1. *There are exactly two Eulerian subgraphs H of $C_{\ell_0}^2$ with ℓ_0 edges. In particular $|\mathcal{M}| = 2$.*

Proof. Using the notation from figure 5 one such graph is obviously cycle $v_1, v_2, v_3, \ldots, v_{\ell_0}$. The other subgraph is the cycle $v_1, v_3, \ldots, v_{\ell_0}, v_2, \ldots, v_{\ell_0-1}$ if ℓ_0 is odd and the union of two cycles $v_1, v_3, \ldots, v_{\ell_0-1}$ and $v_2, v_4, \ldots, v_{\ell_0}$ if ℓ_0 is even. In order to show that there are no other subgraphs we distinguish two cases based on the parity of ℓ_0. If ℓ_0 is odd there is exactly one cycle. We prove this analogously to Proposition 6. Assume that H contains two cycles. While the smallest cycle's length is $\lceil \ell_0/2 \rceil$, if we remove this cycle forms H, we obtain a graph with less than $\lceil \ell_0/2 \rceil$ arcs which is contradiction. This implies that any vertex has no more than one incoming and one outgoing arc and because H has ℓ_0 arcs it contains all vertices. We now prove that every subgraph H of $C_{\ell_0}^2$ constains either all arcs of C_{ℓ_0} or no arcs of C_{ℓ_0}. Otherwise H contains a path v_i, v_{i+1}, v_{i+3} or a path v_i, v_{i+2}, v_{i+3}. In the former case, $\deg^+_{v_{i+2}}$ has to be equal to 0 which contradicts that all vertices have non-zero degrees in H. In the latter case, there cannot be any arc leaving from v_{i+1} and we again obtain a contradiction.

Assume now that ℓ_0 is even. If H is connected, i.e., H is a single cycle, we argue as in the case where ℓ_0 is odd that H is the cycle $v_1, v_2, \ldots, v_{\ell_0}$. Otherwise, H is comprised of two cycles of length $\ell_0/2$ each. Such cycles must be formed by "jump-by-two"-edges. If they had a common vertex, they would be identical. Hence they are disjoint and consequently H must be the union of the cycles $v_1, v_3, \ldots$ and $v_2, v_4, \ldots$. □

By the bijection φ between $\mathcal{S}$ and $\mathcal{L}$ it is enough to analyze the number of of even and odd subgraphs of $C_{\ell_0}^2$ in $\mathcal{S}$. By Proposition 6, any subgraph contained in $\mathcal{S}$ is an oriented cycle. Such a cycle can be identified with a cyclic sequence of length ℓ_0 of zeroes and ones with no two consequentive zeroes as follows: the $i-th$ position is equal to 1 if the vertex v_i is not an isolated vertex of the considered subgraph of $C_{\ell_0}^2$.

This allows us to reformulate the problem to counting sequences with odd and even number of ones.

In the final part of proof, we show how to count the number of such sequences. Defina odd sequences as the sequences with odd number ones and even sequences are the sequences with even nubmer of ones. We also find a formula for transforming number of sequences to number of graphs.

We start with a few definitions.

Definition 2. *Let o_n^1 denote the number of sequences of length n containing an odd number of ones that end with one. Similarly, let o_n^0 be the number of sequences with an odd number of ones that end with* 0*. Analogously, we define e_n^1 and e_n^0 for sequences with even number of ones.*

It is easy to infer the following recurent formulas:

$$\begin{aligned} o_n^1 &= e_{n-1}^1 + e_{n-1}^0 \\ o_n^0 &= o_{n-1}^1 \\ e_n^1 &= o_{n-1}^1 + o_{n-1}^0 \\ e_n^0 &= e_{n-1}^1 \end{aligned}$$

The following equation exhibit the correspondence between the number of the sequences and the number N_e of Eulerian subgraphs of G with even and odd number of arcs. The number of even Eulerian subgraphs of G where $|V(G)| = n$ is

$$N_e = e_n^1 + e_n^0 - o_{n-2}^0 \tag{3.1}$$

Analogously for the odd subgraphs of G

$$N_o = o_n^1 + o_n^0 - e_{n-2}^0 \tag{3.2}$$

First, we prove the equation 3.2. It is not possible sum o_n^1 and o_n^0, because these sequences are not cyclic and we can obtain illegal subgraphs of G (every cyclic sequence corresponds to exactly one Eulerian subgraph of G), since it could contain two consecutive zeros. Note that this problem is caused by sequences starting with 01 and ending with 0. But such sequences of length n can be written as $01\sigma0$, where σ stands for an even sequence of length $n-2$ ending with zero. Hence, the number of such sequences is e_{n-2}^0. Analogously, we derive that the number of even subgraphs of G is equal to $e_n^1+e_n^0-o_{n-2}^0+1$, where the added one stands for empty graph.

In the following lemma we show that the numbers of odd and even Eulerian subgraphs of G differ and so we can use Alon-Tarsi theorem.

Lemma 5. *The number of odd sequences of length n and even sequences of length n differs for $n = 3 * k$.*

Proof. First we prove that the following equations hold:

$$o_n^1 = e_n^1 - 1$$

$$o_n^0 = e_n^0$$

for n divisible by three.

Let us have a brief look at th values of o_n^1, o_n^0, e_n^1 and e_n^0 for $n = 1, \ldots, 6$.

$$o_n^1 : 1, 1, 1, 3, 4, 6$$

$$o_n^0 : 0, 1, 1, 1, 3, 4$$

$$e_n^1 : 0, 1, 2, 2, 4, 7$$

$$e_n^0 : 1, 0, 1, 2, 2, 4$$

If n=3 or n=6, the claimed equations are true. Assume the equations in question hold for n let us prove them for n+3. We infer the following from the induction:

$$o_{n+1}^1 = e_n^1 + e_n^0 = o_n^1 + 1 + o_n^0 \wedge e_{n+1}^1 = o_n^1 + o_n^0 \Rightarrow o_{n+1}^1 = e_{n+1}^1 + 1$$

By applying the recurence formula once more, we obtain

$$o_{n+1}^0 = e_{n+1}^0 - 1$$

and by yet another application of the recurence formula, we get

$$o_{n+2}^1 = e_{n+2}^1$$

$$o_{n+2}^0 = e_{n+2}^0 + 1$$

and then

$$o_{n+3}^1 = e_{n+3}^1 - 1$$

$$o_{n+3}^0 = e_{n+3}^0$$

This finishes the proof of the equations.

Recall, that the number of odd Eulerian subgraphs of G is equal to

$$o_n^1 + o_n^1 - e_{n-2}^0.$$

After substituting for o_n^1 and o_n^1 using equations from previous paragraph we obtain

$$o_n^1 + o_n^0 - e_{n-2}^0 = e_n^1 - 1 + e_n^0 - (o_{n-2}^0 + 1) = e_n^1 + e_n^0 - o_{n-2}^0 + 1 - 3.$$

This can readily be interpreted as

$$N_o = N_e - 3$$

which finishes the proof.

□

The result is that we have found one such orientation D of G, that number of even Eulerian subgraphs is not equal to the number of odd Eulerian subgraph, which is enough to state, that there exists proper coloring. Moreover, because the largest out-degree is 2, we know, that the size of lists is 3.

□

4 Application of Alon-Tarsi Theorem

In this chapter, we present as an application of Alon-Tarsi Theorem the following result of Fleischner and Stiebitz [4] which answers a problem posed by Erdős at the Julius Petersen Graph Theory Conference in 1990.

Theorem 3. *Let n be a positive integer, and let G be a 4-regular graph on $3n$ vertices. Assume that G has a decomposition into a Hamilton cycle and n pairwise vertex disjoint triangles. Then $\chi(G) = 3$.*

Before proving this theorem, we introduce some notation which was not explained yet. Let D be a digraph. If E is a subset of the edge-set $E(D)$ such that the digraph $(V(D), E)$ is Eulerian, then E is called an *Eulerian set of arcs.* Moreover, we use the following notation

$$\varepsilon(D) := \{E \subseteq E(D), E \text{ is an Eulerian set of arcs in } D\},$$

$$\varepsilon_e(D) := \{E \subseteq E(D), |E| \text{ is even}\}, \text{ and}$$
$$\varepsilon_o(D) := \{E \subseteq E(D), |E| \text{ is odd}\}.$$

The cardinalities of these three sets are denoted by $e(D)$, $e_e(D)$ and $e_o(D)$, respectively.

The set of the outgoing arcs from a vertex v in the graph G is denoted by $E_G^+(v)$ and the set of the incoming arcs by $E_G^-(v)$. Finally $E_G(v)$ is the union of the sets $E_G^+(v)$ and $E_G^-(v)$.

We now present notation related to Eulerian subdigraphs of Eulerian digraphs. If D is a digraph, $a_1, \ldots, a_p, b_1, \ldots, b_q$ are some of its arcs and $x_1, \ldots, x_r, y_1, \ldots, y_s$ are some of its vertices, then

$$\varepsilon := \varepsilon(D, a_1, \ldots, a_p, \overline{b_1}, \ldots, \overline{b_q}, x_1, \ldots x_r, \overline{y_1}, \ldots, \overline{y_s})$$

denotes the set of Eulerian sets of arcs E such that $a_1, \dots, a_p \in E$, $b_1, \dots, b_q \notin E$, $E_G(x_i) \subseteq E$, $i = 1, \dots, r$ and $E_G(y_i) \cap E = \emptyset$, $i = 1, \dots, s$. Furthermore, set

$$e(D, a_1, \dots, a_p, \overline{b_1}, \dots, \overline{b_q}, x_1, \dots x_r, \overline{y_1}, \dots, \overline{y_s}) := |\varepsilon|.$$

Finally, the set E^R is defined as the set of the arcs xy such that $yx \in E$, i.e. E^R is the set of the arcs E with their orientations reversed.

As the next step towards the proof of the result of Fleischner and Stiebitz, we establish three lemmas.

Lemma 6. *Let D be an Eulerian digraph with $m \geq 1$ arcs. If φ is a mapping defined as $\varphi(E) := E(D) \setminus E$, then the following statements hold:*

1. *The mapping φ is a bijection from $\varepsilon(D)$ onto itself without a fixed point. In particular, $e(D) \equiv 0$ mod 2.*

2. *For $a_1, \dots, a_p, b_1, \dots, b_q \in E(D)$ and $x_1, \dots, x_r, y_1, \dots, y_s \in V(D)$, the mapping φ is also a bijection from*

$$\varepsilon(D, a_1, \dots, a_p, \overline{b_1}, \dots, \overline{b_q}, x_1, \dots x_r, \overline{y_1}, \dots, \overline{y_s}) \text{ onto}$$

$$\varepsilon(D, \overline{a_1}, \dots, \overline{a_p}, b_1, \dots, b_q, \overline{x_1}, \dots \overline{x_r}, y_1, \dots, y_s).$$

Consequently,

$$e(D, a_1, \dots, a_p, \overline{b_1}, \dots, \overline{b_q}, x_1, \dots x_r, \overline{y_1}, \dots, \overline{y_s})$$

$$= e(D, \overline{a_1}, \dots, \overline{a_p}, b_1, \dots, b_q, \overline{x_1}, \dots \overline{x_r}, y_1, \dots, y_s).$$

3. *If m is odd, then φ is a bijection from $\varepsilon_o(D)$ onto $\varepsilon_e(D)$, and $\varepsilon_o(D) = \varepsilon_e(D)$.*

4. *If m is even, then φ is bijection from $\varepsilon_o(D)$ onto itself and $\varepsilon_e(D)$ onto itself, and $\varepsilon_o(D) \equiv \varepsilon_e(D) \equiv 0$ mod 2.*

5. *If m is even and $e(D) \equiv 2$ mod 4, then $\varepsilon_o(D) \neq \varepsilon_e(D)$.*

Proof. 1. The mapping φ is a bijection from $\varepsilon(D)$ to $\varepsilon(D)$ by Proposition 4. Clearly, the bijection has no fixed point. Since $\varphi^{-1} = \varphi$, φ couples Eulerian sets of arcs of D. Hence $e(D)$ is even.

2. The claim follows from the definition of the set $\varepsilon(D, a_1, \dots, \overline{y_s})$

3. If m is odd, then φ maps odd-size Eulerian sets of arcs to even-size ones by its definition.

4. As in the previous case, φ is a bijection from $\varepsilon_e(D)$ onto itself. Since φ has no fixed point and $\varphi^{-1} = \varphi$, $e_e(D)$ is even. Similarly we can argue in the odd case.

5. Since $e(D) \equiv 2 \bmod 4$, $e_o(D)$ and $e_e(D)$ are both even by the previous claim, the numbers $e_e(D)$ and $e_o(D)$ cannot be congruent modulo 4. In particular, they are different.

□

Lemma 7. *Let D be an Eulerian digraph and C a (directed) cycle of length m in D. Set D_1 to be a digraph $(D-C)\cup(C)^R$. Let the mapping φ_C is defined as $\varphi_C(E) := (E \setminus E(C)) \cup (E(C) \setminus E)^R$ for an eulerian arc set $E \in \varepsilon(D)$. Then the following statements hold:*

1. *D_1 is an Eulerian digraph.*

2. *The mapping φ_C is a bijection from $\varepsilon(D)$ onto $\varepsilon(D_1)$. In particular, $e(D) = e(D_1)$.*

3. *If m is odd, then φ_C is a bijection from $\varepsilon_o(D)$ onto $\varepsilon_e(D_1)$. Symetrically, it is also a bijection from $\varepsilon_e(D)$ onto $\varepsilon_o(D)$. Consequently, $e_o(D) - e_e(D) = e_e(D_1) - e_o(D_1)$.*

4. *If m is even, then φ_C is a bijection from $\varepsilon_o(D)$ onto $\varepsilon_o(D_1)$ and it is also a bijection from $\varepsilon_e(D)$ onto $\varepsilon_e(D_1)$. Hence, $e_o(D) - e_e(D) = e_o(D_1) - e_e(D_1)$.*

Proof. 1. We observe that every vertex v in D_1 has the same in-degree and out-degree as in D. If v is not contained in C, then the in-degree and the out-degree are clearly preserved. If v is contained in C, then one arc incoming to v is changed to an outgoing arc and one arc outgoing from v is changed to an incoming arc. Hence, the in-degree and the out-degree of v are unchanged, and thus D_1 is Eulerian.

2. Let us fix a set $E \in \varepsilon(D)$. D' denotes the Eulerian subgraph of D consisting of the edges E and D_1' denotes the Eulerian subgraph of D with the edge set $\varphi_C(E)$. We first prove that D_1' is also an Eulerian digraph. We distinguish two cases based on whether a vertex x of D

is in C or not. The latter case is easier to analyze, since it is enough to realize that the set $E_{D'}(x)$ is unchanged. Because D' is an Eulerian digraph, it holds that $deg^+_{D'_1}(x) = deg^-_{D'_1}(x)$. If x belongs to C, then we further distinguish three subcases. If $E_{D'}(x) \cap E(C) = \emptyset$, then $E_{D'_1}(x)$ contains in addition the two arcs of C^R incident with x. Hence, $deg^+_{D'_1}(x) = deg^-_{D'_1}(x)$. If $|E_{D'}(x) \cap E(C)| = 2$, then the two arcs of C are removed from $E_{D'}(x)$ to get $E_{D'_1}$ and thus $deg^+_{D'_1}(x) = deg^-_{D'_1}(x)$.

Finally assume that $|E_{D'}(x) \cap E(C)| = 1$. Let $e_1, e_2 \in E(C)$ be the edges incident with x. We can assume that $e_1 \in D'$ and $e_2 \notin D'$ by symetry. By the definition of φ_C, e_1 is removed from D' and e_2^R is added to D'_1. Since one of these edges is incoming to x in D' and the another one is outgoing from x in D', the mapping φ_C does not change the sizes of the sets $E^+_{D'}(x)$ and $E^-_{D'}(x)$. This implies that $deg^+_{D'_1}(x) = deg^-_{D'_1}(x)$.

Observe that the mapping φ_C from $\varepsilon(D_1)$ is the inverse mapping for φ_C. Hence, φ_C is a bijection between $\varepsilon(D)$ and $\varepsilon(D_1)$.

3. Assume that the cardinality of the intersection $E \cap E(C)$ is odd, then the cardinality of $E(C) \setminus E$ is even since the length of C is odd. In particular, the parity of the size of the set $E \setminus E(C)$ is different from the parity of $|E|$. Hence, $|E| \not\equiv |\varphi_C(E)| \bmod 2$. We obtain that $\varphi_C(E)$ is a bijection from $\varepsilon_e(D)$ onto $\varepsilon_o(D_1)$ and also from $\varepsilon_o(D)$ onto $\varepsilon_e(D_1)$. This implies that $e_o(D) - e_e(D) = e_e(D_1) - e_o(D_1)$.

4. The proof of this claim proceed alongs the lines of the proof of the previous claim.

□

Recall that if D_1 and D_2 are two digraphs on the same vertex set, then $D_1 - D_2$ is the digraph containing the arcs of D_1 that are not present in D_2 with the same orientation.

Lemma 8. *Let D_1 and D_2 be Eulerian orientations of the Eulerian graph G. Set $D_0 := D_1 - D_2 = (D_2 - D_1)^R$, and $D_3 := D_1 \cap D_2$. D_0 and D_3 are Eulerian digraphs.*

Proof. We first prove that D_3 is an Eulerian digraph. Both orientations are Eulerian, thus $|E^+_{D_1}(v)| = |E^+_{D_2}(v)| = deg(v)/2$ for any vertex v of G. The number of outgoing arcs from v in D_1 with different orientation than in D_2

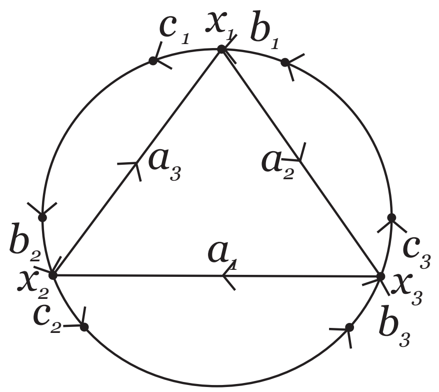

Figure 6: The graph D from the proof of Lemma 9.

is equal to the number of incoming arcs to v in D_1 with different orientation in D_2. Thus the number of incoming arcs to v in D_3 is equal to the number of outgoing arcs from v in D_3. It follows that D_3 is Eulerian.

Obviously $D_1 - D_2 = D_1 - D_3$. Lemma4 implies that D_0 is an Eulerian digraph. Observe that if D_1 and D_2 are orientations of the same graph, then $D_1 - D_2 = (D_2 - D_1)^R$. □

Now we are ready to prove the following lemma, which yields Theorem 3 as argued further.

Lemma 9. *Let D be an Eulerian digraph. Assume that D has a decomposition into a (directed) Hamilton cycle and $n \geq 0$ pairwise vertex disjoint (directed) triangles. Then $e(D) \equiv 2 \mod 4$.*

Proof of Theorem 3 using Lemma 9. Lemma 6.5 implies that $e_o(D) \neq e_e(D)$. Since D is Eulerian and every two triangles are pairwise disjoint, the out-degree of every vertex is at most 2 and thus we infer from Alon-Tarsi Theorem $\chi_l \leq 3$. □

Proof of Lemma 9. We prove this lemma by the induction on n. For $n = 0$ it holds $e(D) = 2$ since there are exactly two Eulerian subgraphs of D, an empty graph and D itself.

Let $n \geq 1$ and T be one of the triangles with vertices x_1, x_2 and x_3 and arcs a_1, a_2 and a_3 (see Figure 6).

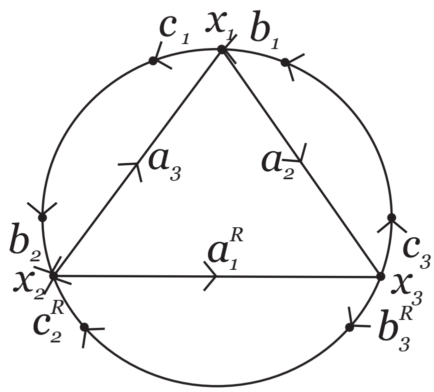

Figure 7: The graph D_i from the proof of Lemma 9.

Set

$$e^*(D) := e(D, \overline{a_1}, a_2, a_3) + e(D, a_1, \overline{a_2}, \overline{a_3}), +e(D, a_1, \overline{a_2}, a_3)$$

$$+e(D, \overline{a_1}, a_2, \overline{a_3}) + e(D, a_1, a_2, \overline{a_3}) + e(D, \overline{a_1}, \overline{a_2}, a_3).$$

Consequently,

$$e(D) = e(D, a_1, a_2, a_3) + e(D, \overline{a_1}, \overline{a_2}, \overline{a_3}) + e^*(D).$$

Obviously, we can apply the induction on $e(D, \overline{a_1}, \overline{a_2}, \overline{a_3})$ since it contains $n-1$ triangles and a Hamilton cycle. In addition, since T is a directed cycle, say $x_1x_3x_2$, if we add it to any Eulerian subgraph D' of the graph $D - T$, then we obtain an Eulerian graph and if we add T to any non-Eulerian graph, the resulting graph will also be non-Eulerian, which implies that $e(D, a_1, a_2, a_3) = e(D, \overline{a_1}, \overline{a_2}, \overline{a_3}) \equiv 2 \mod 4$. This implies

$$e(D) \equiv e^*(D) \mod 4.$$

Therefore it is enough to show that $e^*(D) \mod 4 \equiv 2 \mod 4$. We now construct three digraphs D'_1, D'_2 and D'_3 to which the induction can be used.

For $i = 1, 2, 3$, let x_i^+ be the succesor of x_i on C and x_i^- the predeccesor of x_i on C (see Figure 7). Finally, b_i denotes arc $x_i^- x_i$ and c_i denotes arc $x_i x_i^+$. By Lemma 2, reversing the orientation of C does not change $e(D)$. Hence, we assume that C is oriented from x_1 to x_2, from x_2 to x_3 and from x_3 to x_1 as in Figure 6. The cycle C_i denotes the cycle containing arc a_i and a part

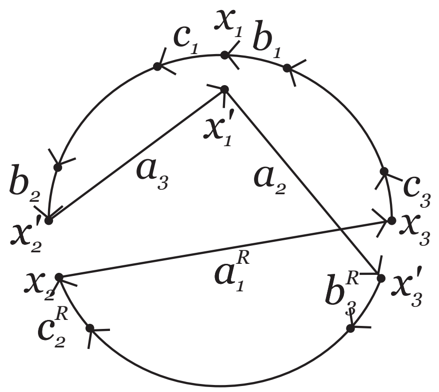

Figure 8: The graph D_i' from the proof of Lemma 9.

of C in such a way that all C_i are arc disjoint. If we reverse the orientation of the cycle C_i in D, we obtain the graph D_i (Figure 7).

In the following, (i, j, k) is one of the triples $(1, 2, 3), (2, 3, 1)$ and $(3, 1, 2)$ and we illustrate the proof for $(i, j, k) = (1, 2, 3)$ in the figures.

First we define a digraph D_i' to be a digraph obtained from D_i by splitting off the arcs b_j, a_k, a_j and b_k^R (Figure 8). The new vertices produced by splitting the arcs are x_i', x_j' and x_k' as given in the figure. The obtained graph has $n - 1$ triangles and contains the Hamilton cycle. By the induction we get that

$$e(D_1') \equiv e(D_2') \equiv e(D_3') \equiv 2 \mod 4, \tag{4.1}$$

which in turn implies

$$e(D_1') + e(D_2') + e(D_3') \equiv 2 \mod 4.$$

Obviously, every Eulerian arc set $E \in \varepsilon(D_i')$ contains either both arcs a_k or a_j or none of them, as the degree of x_i' is two. For every triple $(i, j, k) \in \{(1, 2, 3), (2, 3, 1), (3, 1, 2)\}$, it holds that

$$\begin{aligned} e(D_i') \quad = \quad & e(D_i', a_i^R, a_j, a_k) + e(D_i', \overline{a_i^R}, a_j, a_k) + \\ & e(D_i', a_i^R, \overline{a_j}, \overline{a_k}) + e(D_i', \overline{a_i^R}, \overline{a_j}, \overline{a_k}). \end{aligned} \tag{4.2}$$

To find the relations between $e(D_i')$ and $e(D)$ we start with proving the following equation

$$e(D_i', a_i^R, a_j, a_k) = e(D, \overline{a_i}, a_j, a_k). \tag{4.3}$$

Set $\varepsilon' := \varepsilon(D'_i, a_i^R, a_j, a_k)$ and $\varepsilon = \varepsilon(D, \overline{a_i}, a_j, a_k)$. Now we prove that there exists a bijection between ε and ε' which yields the equation 4.3. First we observe that

$$\varepsilon' = \varepsilon(D'_i, a_i^R, a_j, a_k) = \varepsilon(D'_i, c_j^R, a_i^R, b_k^R, c_k, b_j, a_j, a_k).$$

It is easy to see that this equation holds since we just add necessary arcs to keep the in-degrees and out-degrees of x_j and x_k equal. By the same reason the following equation is true

$$\varepsilon = \varepsilon(D, \overline{a_i}, a_j, a_k) = \varepsilon(D, \overline{c_j}, \overline{a_i}, \overline{b_k}, c_k, b_j, a_j, a_k).$$

We next show that the following holds

$$\varepsilon' = \varepsilon(D'_i, c_j^R, a_i^R, b_k^R, c_k, b_j, a_j, a_k) = \varepsilon(D_i, c_j^R, a_i^R, b_k^R, c_k, b_j, a_j, a_k).$$

The question is whether every set of arcs which is Eulerian for D'_i is Eulerian for D_i. For splitted vertices it holds simply from definition of ε. For other vertices the in-degree and out-degree is obviously same in D_i and D'_i.

Let us have a mapping φ_{C_i} defined as $\varphi_{C_i}(E) := (E \setminus E(C_i)) \cup (E(C_i) \setminus E)^R$ for every $E \in \varepsilon(D)$. By Lemma 7 the mapping φ_{C_i} is a bijection from $\varepsilon(D)$ onto $\varepsilon(D_i)$. From the definition of φ_{C_i} Eulerian sets E in ε that avoid an arc $a \in E(C_i)$ are mapped to sets containing a^R and vice versa. The presence of arcs $a \notin E(C_i)$ in E is not affected by the mapping φ_{C_i}. Thus φ_{C_i} is mapping from ε to ε'. Again, mapping φ_{C_i} from ε' is the inverse mapping for φ_{C_i}. Thus φ_{C_i} is a bijection and the equation (4.3) holds.

It directly follows by Lemma 6.2 and (4.3) that

$$e(D'_i, \overline{a_i^R}, \overline{a_j}, \overline{a_k}) = e(D, a_i, \overline{a_j}, \overline{a_k}). \tag{4.4}$$

Before we state the next equation, let us assign $D' := D - T$ where T is the triangle $x_1x_2x_3$. Then it holds that

$$e(D'_i, a_i^R, \overline{a_j}, \overline{a_k}) = e(D', \overline{x_j}, x_k) = e(D', x_i, \overline{x_j}, x_k) + e(D', \overline{x_i}, \overline{x_j}, x_k). \tag{4.5}$$

Set $\varepsilon' := \varepsilon(D'_i, a_i^R, \overline{a_j}, \overline{a_k})$ and $\varepsilon := e(D', \overline{x_j}, x_k)$. Similarly as in (4.3) we obtain

$$\varepsilon' = \varepsilon(D'_i, c_j^R, a_i, c_k, \overline{b_j}, \overline{a_j}, \overline{a_k}, \overline{b_k^R}) = \varepsilon(D_i, c_j^R, a_i, c_k, \overline{b_j}, \overline{a_j}, \overline{a_k}, \overline{b_k^R}),$$

and consequently

$$\varepsilon = \varepsilon(D', \overline{b_j}, \overline{c_j}, b_k, c_k) = \varepsilon(D, \overline{a_j}, \overline{a_k}, \overline{b_j}, \overline{c_j}, b_k, c_k).$$

The last equation follows from the definition of D'. In the same way as in the proof of the equation (4.3), we get $e(D'_i, a_i^R, \overline{a_j}, \overline{a_k}) = e(D', \overline{x_j}, x_k)$, since there exists a bijection from ε onto ε'. Because the degree of x_i is two, the second equality in (4.5) holds an thus the equation (4.5) is proved.

We infer from (4.3) using Lemma 6.2 that

$$e(D'_i, \overline{a_i^R}, a_j, a_k) = e(D', x_j, \overline{x_k}) = e(D', x_i, x_j, \overline{x_k}) + e(D', \overline{x_i}, x_j, \overline{x_k}). \quad (4.6)$$

Set

$$m := e(D_1) + e(D_2) + e(D_3),$$

and

$$\begin{aligned} m' \quad := \textstyle\sum_{(i,j,k)\in\{(1,2,3),(2,3,1),(3,1,2)\}} \quad & e(D', x_i, \overline{x_j}, x_k) + e(D', \overline{x_i}, \overline{x_j}, x_k) + \\ & e(D', x_i, x_j, \overline{x_k}) + e(D', \overline{x_i}, x_j, \overline{x_k}). \end{aligned}$$

if we combine (4.2)-(4.6), we obtain $m = e^*(D) + m'$. Notice that each triple (i, j, k) in m' stands for one digraph D_i. Using Lemma 6.2 twice and rearranging the terms we conclude

$$m' := 2 \cdot \sum_{(i,j,k)\in\{(1,2,3),(2,3,1),(3,1,2)\}} e(D', x_i, \overline{x_j}, x_k) + e(D', \overline{x_i}, \overline{x_j}, x_k)$$

and finally

$$m' := 4 \cdot (e(D', x_1, x_2, \overline{x_3}) + e(D', x_1, \overline{x_2}, x_3) + e(D', \overline{x_1}, x_2, x_3)).$$

Hence, $m' \equiv 0 \mod 4$ which is equivalent to $m \equiv e^*(D) \mod 4$. We now infer from (4.1) that $e(D) \equiv 2 \mod 4$ which implies the assertion $e(D) \equiv 2 \mod 4$ of the lemma. □

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