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Special classes of P-matrices in the interval setting

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To all my family, loves, friends and mentors. You all have shaped me into who I am today. Thank you.

Název práce: Speciální třídy P-matic v intervalovém prostředí

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Abstrakt: Tato práce se soustředí na zobecnění některých jednoduše rozpoznatelných podtříd P-matic do intervalového prostředí spolu s některými výsledky ohledně těchto tříd. Těmito třídami jsou B-matice, doubly B-matice a B_{π}^{R}-matice. V této práci pro ně pak odvozujeme charakterizace, některé nutné, či postačující podmínky a navíc uvedeme i některé jejich vlastnosti, ať už uzávěrové, nebo některé podmínky na jednotlivé prvky matice, které jsou pro danou třídu splněny. Nakonec předvedeme postup, jak generovat instance některých z těchto tříd intervalových matic.

Klíčová slova: B-matice, Doubly B-matice, B_{π}^{R}-matice, Intervalová analýza, Intervalová matice, P-matice

Title: Special classes of P-matrices in the interval setting

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Abstract: This work focuses on generalizing some easily recognizable subclasses of P-matrices into interval settings, including some results regarding these classes. Those classes are those of B-matrices, doubly B-matrices and B_{π}^{R}-matrices. We derive characterizations, some necessary conditions and sufficient ones, plus we introduce some of their properties, such as are the closure ones and a few conditions the entries of such matrices satisfy. Then we proceed to state a way to generate instances of some of these interval matrix classes.

Keywords: B-matrix, Doubly B-matrix, B_{π}^{R}-matrix, Interval analysis, Interval matrix, P-matrix

Contents

Introduction

In 1968 mathematicians Cottle and Dantzig proposed the linear complementarity problem, denoted $LCP(M, q)$, where M is a matrix and q a vector. LCP is used e.g. to characterize Nash equilibria of normal-form games. Another example of where the LCP occurs is the optimal solution of quadartic programming that can be expressed as a solution of some $LCP(M, q)$. Later, in 1992, in [1] Cottle et al. showed that $LCP(M, q)$ has a unique solution for every vector q if and only if M is a P-matrix, i.e. all its principal minors are positive. However, this class of matrices is computationally complex to recognize. As shown by Coxson in [2], such task to verify given matrix on being a P-matrix is co-NP-complete.

This leads us to try and define some subclasses of P-matrices, that are easily recognizable. Such classes are e.g. B-matrices, doubly B-matrices or B_π^R-matrices, which we will work with in this thesis. To add some subclasses that are not mentioned throughout this work, there are nonsingular M-matrices or positive definite matrices for an example. What more, the B-matrices and doubly B-matrices found their use in localization of eigenvalues, as shown by Peña in [3] and [4].

Since the beginning of rigorous measurements, mathematicians had to deal with inaccuracy in data or any form of uncetainty they may encounter. And an answer emerged in a form of interval analysis, sometimes called interval computing or interval mathematics as well. This answer is about enveloping the data into intervals and then working with the resulting intervals instead of with the data itself. With that we are not only able to overcome problems with measuring equipement, which might give us inaccurate data because of its physical limits, but even to tackle the problems of machines that manipulate our data, e.g. modern computers and their way of storing data, which has only finite precision and often succumbs to rounding errors. By using an interval instead of the discrete data points, we ensure that after the computation is done, we have our desired result in the resulting interval.

It sure is interesting, that there exists a connection between interval analysis and P-matrices, more precisely between regularity of interval matrices and P-matrices, which is shown in [5] by Hladík.

In this thesis we will generalize our special subclasses of P-matrices, namely B-matrices, doubly B-matrices or B_π^R-matrices into interval settings, thus interconnecting these two topics. We shall dive into real cases of our matrix classes and try to deepen our understanding of them up to the point, where we will feel comfortable enough with them to understand even their interval analogies, which we will introduce and explore. We will lay grounds to recognizing the interval variants through characterization, necessary conditions and sufficient ones. Also we shall take a closer look at closure properties of theirs. And in the end, we shall try to formulate methods to generate instances of some of these classes, so it is easier for any following research to test and / or refute hypotheses.

1. Preliminaries

Let us start by stating notation used throughout the thesis (Section 1.1) and introducing interval computation (Section 1.2). In the end we will introduce some matrix classes that are referred to throughout this work (Section 1.3).

1.1 Notation

Let us start by stating our notation of special sets. Let $\mathbb{N}$ be a set of natural numbers, then for any $n \in \mathbb{N}$ by $[n]$ we denote a subset of $\mathbb{N}$, which is of a form $\{1, 2, \ldots, n\}$. By $\mathbb{R}$ we denote the set of all real numbers, whereas by $\mathbb{IR}$ we denote the set of all real intervals. By $\mathbb{F}^{m \times n}$ we denote a class of all matrices of dimension $m \times n$ with its entries from $\mathbb{F}$. Consequently $\mathbb{F}^n$ denotes a class of vectors of length n over $\mathbb{F}$.

For any two matrices $A, B \in \mathbb{F}^{m \times n}$ and vectors $u, v \in \mathbb{F}^{n'}$, where on $\mathbb{F}$ there is defined relation $\geq$ and a zero element 0, $A \geq 0$ means that for every entry a of A: $a \geq 0$ (analogously for relation $>$ and for $v \geq 0$, $v > 0$), and $A \geq B$ is defined as $A - B \geq 0$ (analogously for relation $>$ and for $v \geq u$, $v > u$).

Let $\mathbb{F}$ be a set with defined relation $\geq$ and a zero element 0. Then $\mathbb{F}^+$ indicates subset of $\mathbb{F}$ such that for its every element α: $\alpha > 0$ and $\mathbb{F}_0^+$ designates subset of $\mathbb{F}$ such that for its every element α: $\alpha \geq 0$.

Let $\mathbb{F}$ be a set with defined zero element 0. Then by $o \in \mathbb{F}^n$ we denote zero vector, i.e. $o = (0, \ldots, 0)^T$.

Let $A \in \mathbb{F}^{n \times n}$ for $\mathbb{F}$ with defined relation $\geq$. Then $\forall i \in [n] : r_i^+$is defined as $\max\{0, a_{ij} | j \neq i\}$.

1.2 Interval computation

In this section we will introduce some basics of interval analysis. For further information, see [6], where the authors give reader a wide understanding of the topic, or [7], which contains some more information, or [8] that is a handbook of results in the field.

Let us start by defining of arrangement on intervals as follows. For any two real intervals $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{IR}$, the notation of $\boldsymbol{\alpha} \geq \boldsymbol{\beta}$ means that $\forall \alpha \in \boldsymbol{\alpha}, \forall \beta \in \boldsymbol{\beta} : \alpha \geq \beta$ (analogously for $>$).

Definition 1.1 (interval matrix)**.** *An interval matrix $\boldsymbol{A}$, denoted by $\boldsymbol{A} \in \mathbb{IR}^{m \times n}$ is defined as*

$$\boldsymbol{A} = \left[\underline{A}, \overline{A}\right] = \left\{ A \in \mathbb{R}^{m \times n} \middle| \underline{A} \leq A \leq \overline{A} \right\},$$

where $\underline{A}, \overline{A}$ are called lower, respectively upper bound matrices of $\boldsymbol{A}$.

We can as well look at $\boldsymbol{A}$ as matrix, which has entries from $\mathbb{IR}$, hence $\forall i \in [m], \forall j \in [n] : \boldsymbol{a}_{ij} = \left[\underline{a_{ij}}, \overline{a_{ij}}\right]$.

If we define matrices $A^C = \frac{1}{2}\left(\overline{A} + \underline{A}\right)$ and $A^\Delta = \frac{1}{2}\left(\overline{A} - \underline{A}\right)$, then we can define $\boldsymbol{A}$ alternatively as

$$\boldsymbol{A} = \left[A^C \pm A^\Delta\right] = \left[A^C - A^\Delta, A^C + A^\Delta\right].$$

Definition 1.2 (interval vector)**.** *An interval vector* $\boldsymbol{v} \in \mathbb{IR}^{n'}$ *can be defined as a special case of interval matrix for* $m = n'$ *and* $n = 1$*, just as ordinary vector.*

Now let us take a look at how the interval arithmetic works. For any binary operation $*$ that is defined on the real numbers, we can define the interval extension of the given operation as follows:

$$\boldsymbol{a} * \boldsymbol{b} = \{a * b \mid a \in \boldsymbol{a}, b \in \boldsymbol{b}\}$$

For our common operations we can rewrite this definition into an explicit formula:

- $\boldsymbol{a} + \boldsymbol{b} = \left[\underline{a} + \underline{b}, \overline{a} + \overline{b}\right]$
- $\boldsymbol{a} - \boldsymbol{b} = \left[\underline{a} - \overline{b}, \overline{a} - \underline{b}\right]$
- $\boldsymbol{a} \cdot \boldsymbol{b} = [\min(S), \max(S)]$, where $S = \left\{\underline{a} \cdot \underline{b},\ \underline{a} \cdot \overline{b},\ \overline{a} \cdot \underline{b},\ \overline{a} \cdot \overline{b}\right\}$
- $\boldsymbol{a}/\boldsymbol{b} = [\min(S), \max(S)]$, where $S = \left\{\frac{\underline{a}}{\underline{b}},\ \frac{\underline{a}}{\overline{b}},\ \frac{\overline{a}}{\underline{b}},\ \frac{\overline{a}}{\overline{b}}\right\}$ if $0 \notin \boldsymbol{b}$

What is important to notice, is that if we have an arithmetic expression with interval variables a we evaluate it using interval arithmetic, then if every variable is present at most once, then what we get is the exact smallest interval containing all the possible solutions. (Thus if the soultion is interval, then we get exactly that one and nothing more.) This fact will see use e.g. in proofs of Propositions 3.5 or 5.5.

Often in interval computation, the problem of testing the interval for certain property is reduced to testing just finite number of instances of the interval for the property, as can be seen for example in [9] or [10]. This method we will use in subsections 3.1.1, 4.1.1 or 5.1.1.

1.3 Useful matrix classes

Here we shall introduce some later usefull matrix classes.

Definition 1.3 (P-matrix)**.** *Let* $A \in \mathbb{R}^{n \times n}$*. We say that* A *is a P-matrix, if all its principal minors are positive.*

Definition 1.4 (interval P-matrix)**.** *Let* $\boldsymbol{A} \in \mathbb{IR}^{n \times n}$*. We say that* $\boldsymbol{A}$ *is an interval P-matrix, if* $\forall A \in \boldsymbol{A} : A$ *is a P-matrix.*

Definition 1.5 (Z-matrix)**.** *Let* $A \in \mathbb{R}^{n \times n}$*. We say that* A *is a Z-matrix, if all its off-diagonal elements are non-positive.*

Definition 1.6 (interval Z-matrix)**.** *Let* $\boldsymbol{A} \in \mathbb{IR}^{n \times n}$*. We say that* $\boldsymbol{A}$ *is an interval Z-matrix, if* $\forall A \in \boldsymbol{A} : A$ *is a Z-matrix.*

Definition 1.7 (circulant matrix)**.** *Let* $A \in \mathbb{R}^{n \times n}$*. We say that* A *is a circulant matrix, if all its rows are each cyclic permutations of the first row with offset equal to the row index minus one, hence if it takes the following form:*

$$\begin{pmatrix} c_0 & c_1 & \cdots & c_{n-2} & c_{n-1} \\ c_{n-1} & c_0 & \ddots & & c_{n-2} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ c_2 & & \ddots & c_0 & c_1 \\ c_1 & c_2 & \cdots & c_{n-1} & c_0 \end{pmatrix}$$

2. Real B-matrices and their generalizations

2.1 B-matrices

In this section we will introduce real B-matrices. Subsection 2.1.1 shows us possible characterizations derived by Peña in [3], while in subsection 2.1.2 we will mention some fundamental properties of B-matrices and facts about them. In the last subsection, 2.1.3 we will focus on closure properties of B-matrices.

Definition 2.1 (B-matrix)**.** *Let $A \in \mathbb{R}^{n \times n}$. Then we say that A is a B-matrix, if $\forall i \in [n]$ the following holds:*

$$a) \quad \sum_{j=1}^{n} a_{ij} > 0$$

$$b) \quad \forall k \in [n] \setminus \{i\} : \quad \frac{1}{n} \sum_{j=1}^{n} a_{ij} > a_{ik}$$

Remark 2.2. From Definition 2.1 it can be deduced that every B-matrix A must fulfill following condition for all $i \in [n]$:

$$a_{ii} > r_i^+$$

2.1.1 Characterizations

Following results are taken from Peña [3], therefore we will not state the proofs, those are shown in the original work.

Proposition 2.3. *Let $A \in \mathbb{R}^{n \times n}$. Then A is a B-matrix if and only if $\forall i \in [n]$ the following holds:*

$$\sum_{j=1}^{n} a_{ij} > n \cdot r_i^+$$

Proposition 2.4. *Let $A \in \mathbb{R}^{n \times n}$. Then A is a B-matrix if and only if $\forall i \in [n]$ the following holds:*

$$a_{ii} - r_i^+ > \sum_{j \neq i} \left(r_i^+ - a_{ij} \right)$$

2.1.2 Fundamental properties

Following result is shown in Peña [3] as Corollary 2.6. However it is quite fundamental for the meaning of this thesis, so we will state it here as well:

Proposition 2.5. *B-matrices are P-matrices as well.*

Corollary 2.6. *B-matrices are non-singular.*

Proof. This is a corollary of Proposition 2.5. It comes from the fact that every singular matrix has zero determinant, therefore it is not a P-matrix, therefore it cannot be a B-matrix, because Proposition 2.5 tells us that every B-matrix is P-matrix. ☒

Corollary 2.7. *Let us have two B-matrices $A, B \in \mathbb{R}^{n\times n}$ and let $C \in \mathbb{R}^{n\times n}$ matrix which satisfies the following:*

$$\forall i \in [n]: \quad C_{i*} = A_{i*} \quad \vee \quad C_{i*} = B_{i*}$$

Then C is a B-matrix.

Proof. In Definition 2.1 we can see that there are no conditions intertwining the rows, so each row is inspected on its own. The only thing needed to know, is which of its element is diagonal, thus what row it is. Therefore every row is independent on all the other rows, so if we combine some rows, which satisfy the conditions (where rows of A and B clearly satisfy the conditions, because the matrices are B-matrices), while keeping the order of the rows, hence ensuring that elements that were diagonal still are, then we surely get a B-matrix. ⊠

Next we will introduce a result shown and proved in Peña [3] as Proposition 2.8:

Proposition 2.8. *Let $A \in \mathbb{R}^{n\times n}$ be a Z-matrix. Then the following is equivalent:*

1) A is a B-matrix,

2) The row sums are positive.

3) A is strictly diagonally dominant by rows with positive diagonal entries.

2.1.3 Closure properties

Proposition 2.9. *Let $A, B \in \mathbb{R}^{n\times n}$ be B-matrices, $\alpha \in \mathbb{R}^+$, $D \in \mathbb{R}^{+n\times n}$ be a positive diagonal matrix. Then following holds:*

a) $A + B$ is a B-matrix,
b) αA is a B-matrix,
c) $D \cdot A$ is a B-matrix and
d) principal submatrices of A are B-matrices

Proof. Points *a*) and *b*) can be rather straightforwardly checked from Definition 2.1. Part *c*) holds from combination of part *b*) and Proposition 2.7. For proof of *d*) see Peña [3], Proposition 2.5.

⊠

Remark 2.10. Of course, the B-matrices are not closed under the multiplication by negative scalar, or zero, because then the diagonal would be non-positive, which cannot be, as stated in 2.2.

Proposition 2.11. *A matrix product of two B-matrices is not necessarily a B-matrix.*

Proof. Let

$$A = \begin{pmatrix} 1 & \frac{1}{4} \\ -\frac{1}{4} & 1 \end{pmatrix} \quad \textit{and} \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}.$$

It can be easily seen that both A and B are B-matrices. Nevertheless their product, which is

$$AB = \begin{pmatrix} 1 & 1 \\ -\frac{1}{4} & 4 \end{pmatrix},$$

is not a B-matrix, where the problem is in the first row, which violates the b) condition from the Definition 2.1.

⊠

Proposition 2.12. *An inverse of a B-matrix is not necessarily a B-matrix.*

Proof. Let

$$A = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

It can be easily seen that A indeed is a B-matrix. However its inverse, which is

$$A^{-1} = \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

is not a B-matrix, because of the first row, which violates the a) condition from the Definition 2.1.

⊠

Proposition 2.13. *A power of a B-matrix is not necessarily a B-matrix.*

Proof. Let

$$A = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix}.$$

It can be easily seen that A indeed is a B-matrix. But its second power, which is

$$A^2 = \begin{pmatrix} 1 & 1 & \frac{5}{4} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

is not a B-matrix, because of the first row, which violates the b) condition from the Definition 2.1.

⊠

2.2 Doubly B-matrices

This section will introduce real doubly B-matrices as defined in [4], in subsection 2.2.1 we will inspect some for us interesting properties and in subsection 2.2.2 we will take a closer look at what closure properties they posses.

Definition 2.14 (doubly B-matrix)**.** *Let $A \in \mathbb{R}^{n\times n}$. Then we say that A is a doubly B-matrix, if $\forall i \in [n]$ the following holds:*

a) $a_{ii} > r_i^+$

b) $\forall j \in [n] \setminus \{i\} : \left(a_{ii} - r_i^+\right)\left(a_{jj} - r_j^+\right) > \left(\sum_{k\neq i}\left(r_i^+ - a_{ik}\right)\right)\left(\sum_{k\neq j}\left(r_j^+ - a_{jk}\right)\right)$

2.2.1 Fundamental properties

Proposition 2.15. *Let $A \in \mathbb{R}^{n\times n}$. If A is a B-matrix, then A is a doubly B-matrix.*

Proof. If A is a B-matrix, then, from Proposition 2.4, the following holds for every $i \in [n]$:

$$a_{ii} - r_i^+ > \sum_{j\neq i} \left(r_i^+ - a_{ij}\right) \geq 0$$

From that follows that $\forall i, j \in [n], j \neq i$:

$$\left(a_{ii} - r_i^+\right)\left(a_{jj} - r_j^+\right) > \left(\sum_{k\neq i} \left(r_i^+ - a_{ik}\right)\right)\left(\sum_{k\neq j} \left(r_j^+ - a_{jk}\right)\right),$$

which is exactly the b) part of the Definition 2.14. The a) part of the definition is obtained from Remark 2.2. Therefore A is a doubly B-matrix.

⊠

Remark 2.16. We can now show that the opposite implication does not hold. As a counterexample we can take e.g. matrix

$$A = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}.$$

The matrix A is a doubly B-matrix, but it is not a B-matrix.

The following proposition is stated in Peña [4] as Theorem 2.5 where it is also proved. Nevertheless it holds importance to the essence of this thesis, so we will introduce it here too:

Proposition 2.17. *Doubly B-matrices are P-matrices as well.*

Corollary 2.18. *Doubly B-matrices are non-singular.*

Proof. This is a corollary of Proposition 2.17. It comes from the fact that every singular matrix has zero determinant, therefore it is not a P-matrix, therefore it cannot be a doubly B-matrix, because Proposition 2.17 tells us that every doubly B-matrix is P-matrix. ⊠

Proposition 2.19. *Let $A \in \mathbb{R}^{n\times n}$ be a doubly B-matrix. Then exactly one of the following applies:*

a) Either A is a B-matrix, or

b) there exists a unique $j \in [n]$ that

$$a_{jj} - r_j^+ \leq \sum_{m\neq j} \left(r_j^+ - a_{jm}\right)$$

and for every other $i \in [n] \setminus \{j\}$:

$$a_{ii} - r_i^+ > \sum_{m\neq i} \left(r_i^+ - a_{im}\right).$$

(I.e. there is only one row that does not satisfy the condition stated in Corollary 2.4.)

Proof. Let $a)$ hold, so A is a B-matrix and thus from Corollarty 2.4 $\forall i \in [n]$:

$$a_{ii} - r_i^+ > \sum_{m \neq i} \left(r_i^+ - a_{im} \right),$$

thus $b)$ does not hold.

Now let $a)$ not apply, so A is not a B-matrix. Then it contains a row, which does not fulfills the condition stated in Corollary 2.4. (Else it would fulfill the characterization stated ibidem, thus it would be a -matrix, hence we obtain a contradiction.) We will show that there cannot exist two such rows.

For contradiction, let there be two such rows j and j' that

$$a_{jj} - r_j^+ \leq \sum_{m \neq j} \left(r_j^+ - a_{jm} \right)$$

and

$$a_{j'j'} - r_{j'}^+ \leq \sum_{m \neq j'} \left(r_{j'}^+ - a_{j'm} \right).$$

(It should be noted that because A is a doubly B-matrix, then from Definition 2.14 we get that $0 < a_{jj} - r_j^+$ and $0 < a_{j'j'} - r_{j'}^+$.) Then

$$\left(a_{jj} - r_j^+ \right) \left(a_{j'j'} - r_{j'}^+ \right) \leq \left(\sum_{m \neq j} \left(r_j^+ - a_{jm} \right) \right) \left(\sum_{m \neq j'} \left(r_{j'}^+ - a_{j'm} \right) \right),$$

but that leads us to a contradiction with the Definition 2.14, because A should have been a doubly B-matrix. Therefore such a row which breaks the condition from Corollary 2.4 is exactly one, all the others have to satisfy this condition.

⊠

Definition 2.20 (proper doubly B-matrix)**.** *Let $A \in \mathbb{R}^{n \times n}$ be a doubly B-matrix. Then we say that A is a proper doubly B-matrix, if it is not a B-matrix.*

The next proposition shows us easily testable class of both B- and doubly B-matrices.

Theorem 2.21. *Let $A \in \mathbb{R}^{n \times n}$ be a circulant matrix. Then the following are equivalent:*

1) A is a B-matrix.

2) A is a doubly B-matrix.

3) $a_{11} - r_1^+ > \sum_{j \neq 1} \left(r_1^+ - a_{1j} \right)$

Proof. "1) ⇒ 2)": See Proposition 2.15.

"2) ⇒ 3)": A is a doubly B-matrix, hence for arbitrary $j \neq 1$:

$$\left(a_{11} - r_1^+ \right) \left(a_{jj} - r_j^+ \right) > \left(\sum_{k \neq 1} \left(r_1^+ - a_{1k} \right) \right) \left(\sum_{k \neq j} \left(r_j^+ - a_{jk} \right) \right) \quad \Leftrightarrow$$

$$\Leftrightarrow \quad \left(a_{11} - r_1^+ \right)^2 > \left(\sum_{k \neq 1} \left(r_1^+ - a_{1k} \right) \right)^2 \quad \Leftrightarrow$$

$$\Leftrightarrow \quad \left(a_{11} - r_1^+ \right) > \left(\sum_{k \neq 1} \left(r_1^+ - a_{1k} \right) \right)$$

The first equivalence holds, because the A is circulant, whereas the second one comes from the fact that both sides of the resulting inequality are non-negative, which is based on following:

For left side: A is doubly B-matrix $\Rightarrow \forall i \in [n] : a_{ii} > r_i^+$ (From condition a) of Definition 2.14.)

For right side: From definition of r_i^+: $\forall i \in [n]\ \forall j \neq i : r_i^+ \geq a_{ij}$.

Therefore the implication holds.

"3) $\Rightarrow$ 1)": Because A is circulant, the following implication holds:

$$a_{11} - r_1^+ > \sum_{k\neq 1} \left(r_1^+ - a_{1k}\right) \quad \Rightarrow \quad a_{ii} - r_i^+ > \sum_{k\neq i} \left(r_i^+ - a_{ik}\right)$$

Thus from Proposition 2.4 A is a B-matrix.

⊠

2.2.2 Closure properties

Unlike B-matrices, doubly B-matrices aren't in general closed under the operation of addition, however it can still be shown that they are closed under multiplication by a positive scalar and that their principal submatrices are doubly B-matrices too.

Proposition 2.22. *Let $A \in \mathbb{R}^{n\times n}$ be a doubly B-matrix, $\alpha \in \mathbb{R}^+$. Then αA is a B-matrix.*

Proof. Let $A \in \mathbb{R}^{n\times n}$ be a doubly B-matrix, $\alpha \in \mathbb{R}^+$. First we should mention that $(\alpha A)_{ij} = \alpha \cdot a_{ij}$ and $\max\{0, \alpha \cdot a_{ij} | j \neq i\} = \alpha \cdot \max\{0, a_{ij} | j \neq i\} = \alpha \cdot r_i^+$, so the property a) from Definition 2.14 holds. (Both sides of the inequality are multiplied by the same positive number.) Then $\forall i, j \in [n], j \neq i$:

$$\begin{aligned} & \left(\alpha \cdot a_{ii} - \alpha \cdot r_i^+\right)\left(\alpha \cdot a_{jj} - \alpha \cdot r_j^+\right) & = \\ = \ & \alpha^2 \left(a_{ii} - r_i^+\right)\left(a_{jj} - r_j^+\right) & > \\ > \ & \alpha^2 \left(\sum_{k\neq i}\left(r_i^+ - a_{ik}\right)\right)\left(\sum_{k\neq j}\left(r_i^+ - a_{jk}\right)\right) & = \\ = \ & \left(\sum_{k\neq i}\left(\alpha \cdot r_i^+ - \alpha \cdot a_{ik}\right)\right)\left(\sum_{k\neq j}\left(\alpha \cdot r_i^+ - \alpha \cdot a_{jk}\right)\right). & \end{aligned}$$

The inequality is obtained from the fact that A is a doubly B-matrix. Therefore the b) condition of the definition of doubly B-matrices is satisfied as well. Ergo αA is a doubly B-matrix.

⊠

Proposition 2.23. *Let $A \in \mathbb{R}^{n\times n}$ be a doubly B-matrix, then principal submatrices of A are doubly B-matrices too.*

Proof. Let $A \in \mathbb{R}^{n\times n}$ be a doubly B-matrix and let $\widetilde{A} \in \mathbb{R}^{\widetilde{n}\times\widetilde{n}}$ be its principal submatrix. So $\exists S' \subset [n] : \widetilde{A}$ was formed from A by deletion of rows and columns indexed by the elements of S'. Let $S = [n] \setminus S'$. Then $\forall i \in [\widetilde{n}]\ \exists$ a unique $k_i \in S$ such that $\widetilde{A}_i$ corresponds to the row A_{k_i} (without the entries from the columns from S'). Then, because be removing elements from the set, we cannot increase its maximum, $\forall i \in [\widetilde{n}] : \widetilde{r}_i^+ \leq r_{k_i}^+$. Thus $\forall i, j \in [\widetilde{n}], j \neq i$:

$$\begin{aligned}
\left(\widetilde{a}_{ii} - \widetilde{r}_i^+\right)\left(\widetilde{a}_{jj} - \widetilde{r}_j^+\right) &= \left(a_{k_ik_i} - \widetilde{r}_i^+\right)\left(a_{k_jk_j} - \widetilde{r}_j^+\right) \geq \\
\geq \left(a_{k_ik_i} - r_{k_i}^+\right)\left(a_{k_jk_j} - r_{k_j}^+\right) &> \\
> \left(\sum_{\substack{m=1\\ m\neq k_i}}^{n} \left(r_{k_i}^+ - a_{k_im}\right)\right)\left(\sum_{\substack{m=1\\ m\neq k_j}}^{n} \left(r_{k_j}^+ - a_{k_jm}\right)\right) &\geq \\
\geq \left(\sum_{\substack{m\in S\\ m\neq k_i}} \left(r_{k_i}^+ - a_{k_im}\right)\right)\left(\sum_{\substack{m\in S\\ m\neq k_j}} \left(r_{k_j}^+ - a_{k_jm}\right)\right) &\geq \\
\geq \left(\sum_{\substack{m=1\\ m\neq i}}^{\widetilde{n}} \left(\widetilde{r}_i^+ - \widetilde{a}_{im}\right)\right)\left(\sum_{\substack{m=1\\ m\neq j}}^{\widetilde{n}} \left(\widetilde{r}_j^+ - \widetilde{a}_{jm}\right)\right)
\end{aligned}$$

The third, strict inequality comes from A being a doubly B-matrix.

⊠

Proposition 2.24. *A sum of two doubly B-matrices is not necessarily a doubly B-matrix.*

Proof. Let

$$A = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \quad \textit{and} \quad B = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}.$$

It is easy to check that these two matrices are doubly B- matrices. But their sum, which is

$$A + B = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix},$$

is not a doubly B-matrix, because it is singular and so, as stated in Remark 2.18, cannot be one.

⊠

Proposition 2.25. *A matrix product of two doubly B-matrices is not necessarily a doubly B-matrix.*

Proof. Let

$$A = \begin{pmatrix} 1 & \frac{1}{4} \\ -\frac{1}{4} & 1 \end{pmatrix} \quad \textit{and} \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}.$$

It can be easily seen that both A and B are B-matrices, therefore also a doubly B-matrices, as shown in Proposition 2.15. Nevertheless their product, which is

$$AB = \begin{pmatrix} 1 & 1 \\ -\frac{1}{4} & 4 \end{pmatrix},$$

is not a doubly B-matrix, because of the first row, which violates condition a) from the Definition 2.14.

⊠

Proposition 2.26. *An inverse of a doubly B-matrix is not necessarily a doubly B-matrix.*

Proof. Let

$$A = \begin{pmatrix} 1 & -2 \\ -\frac{1}{3} & 1 \end{pmatrix}.$$

It can be easily seen that A indeed is a doubly B-matrix. However its inverse, which is

$$A^{-1} = \begin{pmatrix} 3 & 6 \\ 1 & 3 \end{pmatrix},$$

is not a doubly B-matrix, once again because the first row violates condition $a)$ from the Definition 2.14.

⊠

Proposition 2.27. *A power of a doubly B-matrix is not necessarily a doubly B-matrix.*

Proof. Let

$$A = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix}.$$

It can be easily seen that A is a B-matrix, therefore also a doubly B-matrix, as shown in Proposition 2.15. But its second power, which is

$$A^2 = \begin{pmatrix} 1 & 1 & \frac{5}{4} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

is not a doubly B-matrix, because of the first row, which violates the a) condition from the Definition 2.14.

⊠

2.3 B_π^R-matrices

In this section we shall introduce real B_π^R-matrices as defined in [11]. Subsection 2.3.1 will show us a characterization of B_π^R-matrices, in subsection 2.3.2 we shall inspect some for us interesting properties and in subsection 2.3.3 we will focus on the closure properties of B_π^R-matrices.

Definition 2.28 (B_π^R-matrix)**.** *Let $A \in \mathbb{R}^{n\times n}$, let $\pi \in \mathbb{R}^n$ such that $0 < \sum_{j=1}^n \pi_j \leq 1$ and let $R \in \mathbb{R}^n$ be a vector formed by the row sums of A (hence $\forall i \in [n] : R_i = \sum_{j=1}^n a_{ij}$). Then we say that A is a B_π^R-matrix, if $\forall i \in [n]$:*

$$\begin{aligned} &a) \quad R_i > 0 \\ &b) \quad \forall k \in [n] \setminus \{i\} : \quad \pi_k \cdot R_i > a_{ik} \end{aligned}$$

Remark 2.29. We can observe that for $\pi = \left(\frac{1}{n}, \ldots, \frac{1}{n}\right)^T$ the previous definition gives us B-matrices, thus, just as the doubly B-matrices, the B_π^R-matrices are in a certain sense a generalization of the class of B-matrices.

Now that we established a relation between B- and B_π^R-matrices, we might want to do the same for B_π^R- and doubly B-matrices. But for that we might need a characterization stated in Proposition 2.30.

2.3.1 Characterizations

The following proposition is stated and proved in [11] as Observation 3.2, yet it shows us a characterization of which matrices $A \in \mathbb{R}^{n\times n}$ are B_π^R-matrices for some π and its proof shows us a way to find such a $\pi \in \mathbb{R}^n$, thus we will state both.

Proposition 2.30. *Let $A \in \mathbb{R}^{n\times n}$ be a square matrix with positive row sums and let $R \in \mathbb{R}^n$ be a vector formed by the row sums of A (hence $\forall i \in [n] : R_i = \sum_{j=1}^n a_{ij} > 0$). Then there exists a vector $\pi \in \mathbb{R}^n$ satisfying $0 < \sum_{j=1}^n \pi_j \leq 1$ such that A is a B_π^R-matrix if and only if*

$$\sum_{j=1}^{n} \max\left\{ \frac{a_{ij}}{R_i} \middle| i \neq j \right\} < 1.$$

Proof. "$\Rightarrow$": A is a B_π^R-matrix for some π satisfying the property, hence $\forall j \in [n] : \max\left\{ \frac{a_{ij}}{R_i} \middle| i \neq j \right\} < \pi_j$. But then

$$\sum_{j=1}^{n} \max\left\{ \frac{a_{ij}}{R_i} \middle| i \neq j \right\} < \sum_{j=1}^{n} \pi_j \leq 1.$$

"$\Leftarrow$": Let

$$\epsilon = 1 - \sum_{j=1}^{n} \max\left\{ \frac{a_{ij}}{R_i} \middle| i \neq j \right\} > 0$$

and for every $j \in [n]$ set the $\pi_j = \max\left\{ \frac{a_{ij}}{R_i} \middle| i \neq j \right\} + \frac{\epsilon}{n}$. Then A is a B_π^R-matrix. ⊠

Remark 2.31. As shown in the proof, if for any matrix $A \in \mathbb{R}^{n\times n}$ the condition from the Proposition 2.30 is satisfied, then we can construct a vector $\pi \in \mathbb{R}^n$ satisfying $0 < \sum_{j=1}^n \pi_j \leq 1$ such that A is a B_π^R-matrix in the following manner:

1) We define $\epsilon \in \mathbb{R}$ as

$$\epsilon = 1 - \sum_{j=1}^{n} \max\left\{ \frac{a_{ij}}{R_i} \middle| i \neq j \right\}$$

and then

2) for every $j \in [n]$ we define π_j as

$$\pi_j = \max\left\{ \frac{a_{ij}}{R_i} \middle| i \neq j \right\} + \frac{\epsilon}{n}.$$

Of course instead of $\frac{\epsilon}{n}$ in the second step we can use any constant $0 < c \leq \frac{\epsilon}{n}$, or we might use a vector $\xi \in \mathbb{R}^{+n}$ such that $0 < \sum_{j=1}^n \xi_j \leq \epsilon$ and define π_j as

$$\pi_j = \max\left\{ \frac{a_{ij}}{R_i} \middle| i \neq j \right\} + \xi_j.$$

(It is easy to verify that this holds from Definition 2.28, because thus defined π meets condition *b*) for the above mentioned definition and also satisfies that $0 < \sum_{j=1}^n \pi_j \leq 1$.)

Now let us return and show the relation of B_π^R-matrices to the class of doubly B-matrices.

Remark 2.32. B_π^R-matrices and doubly B-matrices are two distinct classes of matrices, even though with nonempty intersection (at least B-matrices are in the intersection). Let us show the following two examples of why this holds:

Example. Doubly B-matrix, that is not a B_π^R-matrix:

$$\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$$

(It does not have positive row sum of the first row, and therefore it cannot be a B_π^R-matrix.)

Example. B_π^R-matrix, that is not a doubly B-matrix:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

(It fulfills the Definition 2.28 for $\pi = (\frac{1}{3}, \frac{2}{3})^T$, hence is a B_π^R-matrix, but it is not a doubly B-matrix, because it does not fulfill the condition a) from Definition 2.14, which is that $\forall i \in [n] : a_{ii} > \max\{0, a_{ij} | j \neq i\}$.)

2.3.2 Fundamental properties

The following proposition is stated in [12] as Theorem 1. Nevertheless it holds importance to the essence of this thesis, so we will introduce it here too:

Proposition 2.33. *B_π^R-matrices with $\pi \geq 0$ are P-matrices as well.*

Remark 2.34. We can show an example of B_π^R-matrix with some $\pi_i < 0$, which is not a B_ψ^R-matrix for any $\psi > 0$. (To verify this fact, reader may use the properties of B_π^R-matrices stated in the next proposition, more precisely part 1).)

Example.

$$A = \begin{pmatrix} \frac{3}{2} & -1 \\ 2 & -\frac{1}{2} \end{pmatrix}$$

It is easy to check that A is a B_π^R-matrix for $\pi = (2, -1)^T$. (And, for interest, is clearly not a P-matrix nor doubly B-matrix.)

Ergo for the purpose of this thesis, we are interested only in such B_π^R-matrices that have $\pi \geq 0$. What does it mean for example for Proposition 2.30? Almost nothing, just that not only do we have to calculate the characterization, but we also have to check, which exact π is the certificate of the matrix being a B_π^R-matrix (as shown in Remark 2.31). Even though we will continue to state and prove the results for general π, from now on in places where it will seem to be needed we will state how is the given result related to the subclass of B_π^R-matrices with $\pi \geq 0$.

The next proposition and its corollary, which are stated and proved in [13] as Proposition 2.1 and Corollary 2.2, show us some properties that relates entries of π with that of a corresponding B_π^R-matrix.

Proposition 2.35. *Let $\pi \in \mathbb{R}^n$ such that $0 < \sum_{j=1}^n \pi_j \leq 1$ and let $A \in \mathbb{R}^{n\times n}$ be a B_π^R-matrix, where $R \in \mathbb{R}^n$ is the vector of row sums of A. Then the following holds:*

1) $\forall i \in [n]: \quad a_{ii} > \pi_i \cdot R_i,$

2) $\forall i, j \in [n], j \neq i: \quad \pi_i \geq \pi_j \;\Rightarrow\; a_{ii} > a_{ij},$

3) let $k = \operatorname{argmax}\{\pi_i \mid i \in [n]\ \}$*, then* $\forall j \neq k: \quad a_{kk} > a_{kj}$ *and*

4) $\forall i, j \in [n], j \neq i: \quad \pi_j \leq 0 \;\Rightarrow\; a_{ij} < 0.$

Corollary 2.36. *Let $\pi \in \mathbb{R}^n$ such that $0 < \sum_{j=1}^n \pi_j \leq 1$ with $\pi_1 = \ldots = \pi_n = r$ and let $A \in \mathbb{R}^{n\times n}$ be a B_π^R-matrix, where $R \in \mathbb{R}^n$ is the vector of row sums of A. Then the following holds:*

1) $\forall i, j \in [n], j \neq i: \quad a_{ii} > a_{ij}$ *and*

2) $\operatorname{tr} A > r \cdot \sum_{i=1}^{n} R_i.$

The following result was shown alongside its proof in [13] as Theorem 2.3.

Proposition 2.37. *Let $A \in \mathbb{R}^{n\times n}$ be a Z-matrix and let $\pi \in \mathbb{R}^{+^n}$ such that $0 < \sum_{j=1}^n \pi_j \leq 1$. Then the following is equivalent:*

1) A is a B_π^R-matrix,

2) the row sums of A are positive,

3) A is strictly diagonally dominant by row with positive diagonal entries and

4) A is a B-matrix.

The next two propositions were again introduced and proved in [13] as Proposition 2.5 and 2.6, respectively.

Proposition 2.38. *Let $\pi \in \mathbb{R}^n$ such that $0 < \sum_{j=1}^n \pi_j \leq 1$ and let $A \in \mathbb{R}^{n\times n}$ be a B_π^R-matrix. If $\alpha \in \mathbb{R}^n$ such that $0 < \sum_{j=1}^n \alpha_j \leq 1$ and $\alpha \geq \pi$, then A is a B_α^R-matrix.*

Proposition 2.39. *Let $P = \operatorname{perm}(i_1, \ldots, i_n)$, where $i_1, \ldots, i_n = [n]$ be the permutation matrix of order n defined by*

$$P = (p_{m_1 m_2}); \quad p_{m_1 m_2} = \begin{cases} 1 & \text{if } m_2 = i_{m_1}, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\pi \in \mathbb{R}^n$ such that $0 < \sum_{j=1}^n \pi_j \leq 1$ and let $A \in \mathbb{R}^{n\times n}$ be a B_π^R-matrix. Let $\alpha = (\pi_{i_1}, \ldots, \pi_{i_n})^T$. Then PAP^T is a $B_\alpha^{R'}$-matrix.

2.3.3 Closure properties

In this section we will inspect what closure properties the B_π^R-matrices posses. And because in this thesis we are interested in P-matrices, we will try to show eventual counterexamples from the subclass of B_π^R-matrices for $\pi \geq 0$.

The following closure properties are shown and proved in [13] as Theorem 2.7, Corollaries 2.8 and 2.9 and Proposition 2.12.

Proposition 2.40. *Let $\alpha, \beta \in \mathbb{R}^n$ such that $0 < \sum_{j=1}^n \alpha_j \leq 1$ and $0 < \sum_{j=1}^n \beta_j \leq 1$ and let $\pi \in \mathbb{R}^n$ defined by $\forall i \in [n] : \ \pi_i = \max\{\alpha_i, \beta_i\}$. Let $A, B \in \mathbb{R}^{n\times n}$ be a B_α^R-matrix and a B_β^Q-matrix, respectively. If $\sum_{j=1}^n \pi_j \leq 1$, then $A + B$ is a B_π^{R+Q}-matrix.*

Remark 2.41. We can see that for $\alpha, \beta \geq 0$, we again get $\pi \geq 0$, therefore even the subclass of B_π^R-matrices for $\pi \geq 0$ is closed in the same manner as above.

Corollary 2.42. *Let $\pi \in \mathbb{R}^n$ such that $0 < \sum_{j=1}^n \pi_j \leq 1$ and let $A, B \in \mathbb{R}^{n\times n}$ be a B_π^R-matrix and a B_π^Q-matrix. Then $A + B$ is a B_π^{R+Q}-matrix.*

Corollary 2.43. *Let $\pi \in \mathbb{R}^n$ such that $0 < \sum_{j=1}^n \pi_j \leq 1$ and let $s, t \in \mathbb{R}_0^+$ with $s + t > 0$. Let $A, B \in \mathbb{R}^{n\times n}$ be a B_π^R-matrix and a B_π^Q-matrix, respectively. Then $s \cdot A + t \cdot B$ is a B_π^{sR+tQ}-matrix.*

Proposition 2.44. *Let $\pi \in \mathbb{R}^n$ such that $0 < \sum_{j=1}^n \pi_j \leq 1$ and let $A \in \mathbb{R}^{n\times n}$ be a B_π^R-matrix. Let $D \in \mathbb{R}^{n\times n}$ be a positive diagonal matrix. Then $D \cdot A$ is a $B_\pi^{R'}$-matrix.*

The next corollary can be derived from multiple previous statements, more precisely from Corollary 2.43 and Proposition 2.44

Corollary 2.45. *Let $\alpha \in \mathbb{R}^+$, $\pi \in \mathbb{R}^n$ such that $0 < \sum_{j=1}^n \pi_j \leq 1$ and let $A \in \mathbb{R}^{n\times n}$ be a B_π^R-matrix. Then $\alpha \cdot A$ is a $B_\pi^{\alpha R}$-matrix.*

Now what about other operations:

Proposition 2.46. *An inverse of a B_π^R-matrix is not necessarily a B_ψ^R-matrix for any ψ.*

Proof. Let

$$A = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

It can be easily seen that A indeed is a B_π^R-matrix (it satisfies Proposition 2.30), its π might be, for example, $(\frac{1}{6}, \frac{5}{12}, \frac{5}{12})^T$. However its inverse, which is

$$A^{-1} = \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

is not a B_ψ^R-matrix for any ψ, because of the first row, which violates the a) condition from Definition 2.28.

⊠

Proposition 2.47. *A power of a B^R_π-matrix is not necessarily a B^R_ψ-matrix for any psi.*

Proof. Let

$$A = \begin{pmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \end{pmatrix}.$$

It can be easily seen that A indeed is a B^R_π-matrix for example for $\pi = (\frac{1}{2}, \frac{1}{2})^T$. But its second power, which is

$$A^2 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix},$$

is not a B^R_ψ-matrix for any ψ, because of the first row, which violates the a) condition from Definition 2.28.

⊠

Corollary 2.48. *A matrix product of two B^R_π-matrices does not have to necessarily be a B^R_ψ-matrix for any ψ.*

Corollary 2.49. *A matrix product of a B^R_π-matrix and a B^R_ψ-matrix is not necessarily a B^R_φ-matrix for any φ.*

3. Interval B-matrices

In this chapter we will generalize B-matrices into interval B-matrices. In section 3.1 we shall introduce some characterizations, whereas in section 3.2 we will try to derive necessary conditions and sufficient ones to help us recognize this matrix class even more efficiently. And in section 3.3 we will take a closer look at which operations are the interval B-matrices closed under.

Definition 3.1 (interval B-matrix)**.** *Let* $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$. *Then we say that* $\boldsymbol{A}$ *is an interval B-matrix, if* $\forall A \in \boldsymbol{A}$: *A is a (real) B-matrix.*

Remark 3.2. Because for every interval B-matrix $\boldsymbol{A}$ holds that $\forall A \in \boldsymbol{A}$: A is a B-matrix, thus even matrix A', defined as

$$A' = \begin{pmatrix} \underline{a}_{11} & \overline{a}_{12} & \cdots & & \overline{a}_{1n} \\ \overline{a}_{21} & \underline{a}_{22} & \cdots & & \overline{a}_{2n} \\ \vdots & \vdots & \ddots & & \vdots \\ \overline{a}_{(n-1)1} & & \cdots & \underline{a}_{(n-1)(n-1)} & \overline{a}_{(n-1)n} \\ \overline{a}_{n1} & & \cdots & \overline{a}_{n(n-1)} & \underline{a}_{nn} \end{pmatrix},$$

is a B-matrix, thus it must hold (from Remark 2.2) that

$$\forall i \in [n]: \quad \underline{a}_{ii} > \max\{0, \overline{a}_{ij} | j \neq i\}.$$

Next we should mention a simple corollary of Proposition 2.5 and our definition:

Corollary 3.3. *Every interval B-matrix is an interval P-matrix.*

Proof. It holds for every instance, hence it holds for whole interval matrix.
(Every instance is P-matrix by Proposition 2.5 and that is exactly the definition of interval P-matrix.)

⊠

Corollary 3.4. *Let us have two interval B-matrices* $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^{n\times n}$ *and let* $\boldsymbol{C} \in \mathbb{R}^{n\times n}$ *matrix which satisfies the following:*

$$\forall i \in [n]: \quad \boldsymbol{C}_{i*} = \boldsymbol{A}_{i*} \quad \vee \quad \boldsymbol{C}_{i*} = \boldsymbol{B}_{i*}$$

Then $\boldsymbol{C}$ *is an interval B-matrix.*

Proof. From Definition 3.1 and from Corollary 2.7 we can see that it holds for every instance, thus it holds for whole interval matrix. ⊠

3.1 Characterizations

Let us start with stating a few characterizations to help us with recognition of interval B-matrices in finite time. Thereafter in subsection 3.1.1 we will try to derive a characterization through reduction of Definition 3.1 to finite number of instances to check for the property of being a B-matrix.

Theorem 3.5. *Let $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$. Then $\boldsymbol{A}$ is an interval B-matrix if and only if $\forall i \in [n]$ the following two properties hold:*

$$a) \quad \sum_{j=1}^{n} \underline{a}_{ij} > 0$$

$$b) \quad \forall k \in [n] \setminus \{i\} : \quad \sum_{j \neq k} \underline{a}_{ij} > (n-1) \cdot \overline{a}_{ik}$$

Proof. The most important thing, we need for the following proof, is to realize that for every interval $[\underline{\alpha}, \overline{\alpha}]$ and every $\alpha \in [\underline{\alpha}, \overline{\alpha}]$ the following applies: $\underline{\alpha} \le \alpha \le \overline{\alpha}$.

As shown in Definition 2.1, square real matrix A is a B-matrix, if for its every row i holds that the row sums are positive (marked as condition a)) and every non-diagonal element of the ith row is bounded above by the corresponding row mean (b) condition).

The a) condition is surely satisfied $\forall A \in \boldsymbol{A}$, because of the a) condition of Proposition, whereas the a) condition of Proposition always holds true for an interval B-matrix $\boldsymbol{A}$ because $\underline{A} \in \boldsymbol{A}$, thus $\underline{A}$ is a B-matrix and fulfills the condition a) of Definition 2.1

Now let's take a look at conditions b). The b) condition of the Definition 2.1 can be for every $k \neq i$ rewritten as follows:

$$\begin{aligned} & \frac{1}{n}\sum_{j=1}^{n} a_{ij} > a_{ik} \quad \Leftrightarrow \\ \Leftrightarrow \quad & \sum_{j=1}^{n} a_{ij} > n a_{ik} \quad \Leftrightarrow \\ \Leftrightarrow \quad & \sum_{j \neq k} a_{ij} > (n-1) \cdot a_{ik} \end{aligned}$$

In the last inequality, we can see there is no element twice. Consequently, if we use intervals in this inequality, by substitution (of specific values from the intervals) we obtain exact values, not a superset. So now we can see that the condition b) of real case applies for every $A \in \boldsymbol{A}$ iff it holds for $\underline{a}_{ij}$ on the left side and $\overline{a}_{ik}$ on the right side, which is exactly the b) condition of Proposition.

⊠

Remark 3.6. This characterization has time complexity $O(n^2)$, which is the same as the time complexity of the characterization for real case from Definition 2.1. (And even though there are characterizations like the one in Proposition 2.3, which might seem to have $O(n)$ complexity, it is vital to realize that for every $i \in [n]$ we have to compute sum of n elements, therefore the complexity is $O(n^2)$ as well.)

Corollary 3.7. *Let $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$. Then $\boldsymbol{A}$ is an interval B-matix iff $\boldsymbol{A}$ with the diagonal fixed on lower bounds ($a_{ii} = \underline{a}_{ii}$) is an interval B-matrix.*

Proof. In the characterization given in Theorem 3.5 we see that every time any a_{ii} occures, it occures in form of $\underline{a}_{ii}$, hence we are not interested in any other value of a_{ii}. (So the reduced matrix has to fulfill exactly the same conditions as the matrix $\boldsymbol{A}$)

⊠

Corollary 3.8. *Let* $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$ *and let*

$$S = \Big\{(i,j) \mid i,j \in [n] : \exists k \in [n] \setminus \{i,j\} : \underline{a}_{ij} \leq \underline{a}_{ik} \ \wedge \ \overline{a}_{ij} \leq \overline{a}_{ik}\Big\}.$$

We have that $\boldsymbol{A}$ *is an interval B-matix iff* $\boldsymbol{A}$ *with every element, whose indices are in* S*, set to its lower bound (*$\forall (i,j) \in S : a_{ij} = \underline{a}_{ij}$*) is an interval B-matrix.*

Proof. The only time, when for every i and $j \neq i$ the $\overline{a}_{ij}$ occures in Theorem 3.5, is the b) condition. Let us show that in the case that $(i,j) \in S$ this condition is not necessary and is substituted by one of the others.

Let $(i,j) \in S$ arbitrary and let $k \in [n] \setminus \{i\} : \underline{a}_{ij} \leq \underline{a}_{ik} \ \wedge \ \overline{a}_{ij} \leq \overline{a}_{ik}$. Because $(i,j) \in S$, then surely such k exists. Then:

$$\sum_{m\neq j} \underline{a}_{im} \quad \geq \quad \sum_{m\neq k} \underline{a}_{im} \quad > \quad (n-1)\cdot \overline{a}_{ik} \quad \geq \quad (n-1)\cdot \overline{a}_{ij}$$

First inequality is obtained from $\underline{a}_{ij} \leq \underline{a}_{ik}$ and the third from $\overline{a}_{ij} \leq \overline{a}_{ik}$. The second one holds, if condition b) holds for (i,k), so we see that if the condition holds for (i,k), then it holds for (i,j) as well. Thus the implication "$\Leftarrow$" holds.

The second implication is trivial, because $\boldsymbol{A}$ is a superset of the reduced matrix.

⊠

Although the next corollary is obtained rather straightforwardly from the previous proposition, we will state it, as it will prove to be a useful step in the derivation of characterization of interval B-matrices that will help us clarify the relation between interval B- and doubly B-matrices.

Corollary 3.9. *Let* $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$*. Then* $\boldsymbol{A}$ *is an interval B-matrix if and only if* $\forall i \in [n]$ *the following holds:*

$$\forall k \in [n] \setminus \{i\} : \sum_{j=1}^{n} \underline{a}_{ij} > \max\{0, (n-1)\cdot \overline{a}_{ik} + \underline{a}_{ik}\}$$

Proof. "$\Rightarrow$"

$\boldsymbol{A}$ is interval B-matrix, so $\boldsymbol{A}$ satisfies both conditions from Theorem 3.5. Thus for arbitrary $k \neq i$:

$$\sum_{j\neq k} \underline{a}_{ij} > (n-1)\cdot \overline{a}_{ik} \quad \Leftrightarrow$$
$$\Leftrightarrow \sum_{j=1}^{n} \underline{a}_{ij} > (n-1)\cdot \overline{a}_{ik} + \underline{a}_{ik}$$

And combined with condition a) from Theorem 3.5 we get that this implication clearly holds.

"$\Leftarrow$"

We will show that if matrix fulfills condition stated in this corollary, then it fulfills the conditions of Theorem 3.5 as well.

The condition $a)$ holds trivially.

$b)$ condition:

$$\forall k \neq i :$$

$$\sum_{j=1}^{n} \underline{a}_{ij} \quad > \quad \max\{0, (n-1) \cdot \overline{a}_{ik} + \underline{a}_{ik}\} \quad \geq \quad (n-1) \cdot \overline{a}_{ik} + \underline{a}_{ik}$$

$$\Rightarrow$$

$$\sum_{j \neq k} \underline{a}_{ij} > (n-1) \cdot \overline{a}_{ik}$$

So the $b)$ condition holds too.

Thus this implication also holds.

⊠

By realignment of the previous corollary, we get subsequent one.

Corollary 3.10. *Let $\boldsymbol{A} \in \mathbb{IR}^{n \times n}$. Then $\boldsymbol{A}$ is an interval B-matrix if and only if $\forall i \in [n]$ the following holds:*

$$a) \quad \sum_{j=1}^{n} \underline{a}_{ij} > 0$$

$$b) \quad \forall k \in [n] \setminus \{i\} : \quad \underline{a}_{ii} - \underline{a}_{ik} > \sum_{j \neq i} \left(\overline{a}_{ik} - \underline{a}_{ij} \right)$$

Proof. Obtained by realignment of inequalities from Corollary 3.9.

⊠

Or we might realign it in a different way and obtain our future connection to interval doubly B-matrices.

Corollary 3.11. *Let $\boldsymbol{A} \in \mathbb{IR}^{n \times n}$. Then $\boldsymbol{A}$ is an interval B-matrix if and only if $\forall i \in [n]$ the following holds:*

$$a) \quad \sum_{j=1}^{n} \underline{a}_{ij} > 0$$

$$b) \quad \forall k \in [n] \setminus \{i\} : \quad \underline{a}_{ii} - \overline{a}_{ik} > \sum_{\substack{j \neq i \\ j \neq k}} \left(\overline{a}_{ik} - \underline{a}_{ij} \right)$$

Proof. Obtained by realignment of inequalities from Corollary 3.9.

⊠

3.1.1 Characterization through reduction

Proposition 3.12. *Let $\boldsymbol{A} \in \mathbb{IR}^{n \times n}$ and let A_i be matrices defined as follows:*

$$A_i = (a_{m_1 m_2}); \quad a_{m_1 m_2} = \begin{cases} \overline{a}_{m_1 m_2} & \text{if } m_1 \neq i, m_2 = i, \\ \underline{a}_{m_1 m_2} & \text{otherwise.} \end{cases}$$

Then $\boldsymbol{A}$ is interval B-matrix if and only if $\forall i \in [n] : A_i$ is a B-matrix.

Proof. "⇒" This holds trivially, because $\forall i \in [n] : A_i \in \boldsymbol{A}$

"⇐"

a)$\forall i \in [n] : \sum_{j=1}^{n} \underline{a}_{ij} > 0$, because A_i is a B-matrix and $(A_i)_{i,*} = (\underline{A})_{i,*}$, thus the row sums of $\underline{A}$ are positive.

b)$\forall i \in [n]\ \forall k \neq i :\quad A_k$ is a B-matrix ⇒ (From Proposition 2.3:)

$$\overline{a}_{ik} + \sum_{j \neq k} \underline{a}_{ij} \quad = \quad \sum_{j=1}^{n} (A_k)_{ij} \quad > \quad n \cdot r_i^{+} \quad \geq \quad n \cdot (A_k)_{ik} \quad = \quad n \cdot \overline{a}_{ik}$$

⇒

$$\sum_{j \neq k} \underline{a}_{ij} \quad > \quad (n-1) \cdot \overline{a}_{ik}$$

⇒ $\boldsymbol{A}$ fulfills the conditions of Theorem 3.5 ⇒ $\boldsymbol{A}$ is an interval B-matrix.

⊠

Proposition 3.13. *The characterization of interval B-matrices through reduction given by Proposition 3.12 is minimal in inclusion.*

Proof. If we ditched any A_i for arbitrary $i \in [n]$, then we can construct a counterexample, e.g. a unit matrix with interval $[0, 1]$ on position (j, i) for arbitrary $j \neq i$. Then $\forall k \neq i : A_k = I_n$, which surely is a B-matrix, but A_i does not fulfill *b)* condition from Definition 2.1 in jth row. (Sum of the jth row is equal to 2, so we get $2/n > 1 = (A_i)_{ji}$, which does not hold for $n \geq 2$.)

⊠

Remark 3.14. This reduction reduces the problem of verifying, whether any given interval matrix is an interval doubly B-matrix, into testing n matrices, whether they are real doubly B-matrices.

3.2 Necessary or sufficient conditions

Here in this section we will derive a few necessary or sufficient conditions that might help us with even quicker recognition of a class of interval B-matrices.

First, we will introduce a simple consequence of Remark 3.2:

Corollary 3.15. *Let $\boldsymbol{A} \in \mathbb{IR}^{n \times n}$. We have that $\boldsymbol{A}$ is an interval B-matrix only if* $\forall i \in [n] :\quad \underline{a}_{ii} > \max\{0, \overline{a}_{ij} | j \neq i\}$.

Theorem 3.16. *Let $\boldsymbol{A} \in \mathbb{IR}^{n \times n}$ be an interval Z-matrix. Then the following is equivalent:*

1) $\boldsymbol{A}$ *is an interval B-matrix,*

2) $\forall i \in [n] :\quad \sum_{j=1}^{n} \underline{a}_{ij} > 0$,

3) $\forall i \in [n] :\quad \underline{a}_{ii} > \sum_{j \neq i} |\underline{a}_{ij}|$.

4) $\underline{A}$ *is a B-matrix.*

Proof. "1) ⇒ 2)": From Theorem 3.5

"2) ⇔ 3) ⇔ 4)": From Proposition 2.8.

"3) ⇒ 1)": $\forall A \in \boldsymbol{A}: \quad \forall i \in [n]:$

$$a_{ii} \geq \underline{a}_{ii} \quad \wedge \quad \forall j \in [n] \setminus \{i\}: \quad |a_{ij}| \leq |\underline{a}_{ij}|$$

⇒

$$a_{ii} \quad \geq \quad \underline{a}_{ii} \quad > \quad \sum_{j \neq i} |\underline{a}_{ij}| \quad \geq \quad \sum_{j \neq i} |a_{ij}| \quad \geq \quad 0$$

⇒

A is strictly diagonally dominant with positive diagonal. Thus, according to Proposition 2.8, A is a B-matrix.

And because that applies $\forall A \in \boldsymbol{A} \quad \Rightarrow \quad \boldsymbol{A}$ is an interval B-matrix.

⊠

The following results hold both, an information about some properties of interval B-matrices, as well as a necessary condition for B-matrices.

Proposition 3.17. *Let $\boldsymbol{A} \in \mathbb{IR}^{n \times n}$ be an interval B-matrix. Then $\forall i \in [n]$ the following two properties hold:*

1) $\underline{a}_{ii} > \sum_{j \in S} |\underline{a}_{ij}|$, where $S = \{j \in [n] \mid \underline{a}_{ij} < 0\}$ and

2) $\forall j \in [n] \setminus \{i\}: \quad \underline{a}_{ii} > \max\{|\overline{a}_{ij}|, |\underline{a}_{ij}|\}$.

Proof. 1) Let us distinguish the following two cases for arbitrary $i \in [n]$:

I. $\forall j \in [n] \setminus \{i\}: \quad \underline{a}_{ij} \leq 0$

Then it follows directly from Theorem 3.5, condition a). (Because it holds that $\forall j \in [n] \setminus \{i\}: -\underline{a}_{ij} = |\underline{a}_{ij}|$.)

II. $\exists j \in [n] \setminus \{i\}: \quad \underline{a}_{ij} > 0$

Let us take $k \in \operatorname{argmax} \left\{\underline{a}_{ij} | j \neq i\right\}$. Then, according to Corollary 3.10, the following applies:

$$\underline{a}_{ii} - \underline{a}_{ik} > \sum_{j \neq i} \left(\overline{a}_{ik} - \underline{a}_{ij}\right).$$

And because

$$\underline{a}_{ii} > \underline{a}_{ii} - \underline{a}_{ik} \quad \wedge \quad \forall j \neq i: \overline{a}_{ik} - \underline{a}_{ij} \geq 0$$

(because of the presumption of this case and definition of k), then

$$\underline{a}_{ii} > \underline{a}_{ii} - \underline{a}_{ik} > \sum_{j \neq i} \left(\overline{a}_{ik} - \underline{a}_{ij}\right) \geq \sum_{j \in S} \left(\overline{a}_{ik} - \underline{a}_{ij}\right) > \sum_{j \in S} -\underline{a}_{ij} = \sum_{j \in S} |\underline{a}_{ij}|.$$

2) For arbitrary $j \neq i$, let us distinguish two cases:

I. $|\underline{a}_{ij}| \geq |\overline{a}_{ij}|$

⇒ $\underline{a}_{ij} \leq 0$, thus from property 1. of this proposition:

$$\underline{a}_{ii} \quad > \quad \sum_{k \in S} |\underline{a}_{ik}| \quad \geq \quad |\underline{a}_{ij}|,$$

because $j \in S$.

II. $|\overline{a}_{ij}| > |\underline{a}_{ij}|$

⇒ $\overline{a}_{ij} > 0$, thus from Remark 3.2 ⇒ $\underline{a}_{ii} > \overline{a}_{ij} = |\overline{a}_{ij}|$.

⊠

3.3 Closure properties

In this section, we shall expand our understanding of what operations the class of B-matrices is closed under into interval environment, plus we will add how it is with multiplication of B-matrix by positive interval.

So let us start with an extension of facts already known from section 2.1.3:

Corollary 3.18. *Interval B-matrices are closed under following operations:*

1) matrix addition,

2) multiplication by positive scalar and

3) matrix multiplication from the left by a real positive diagonal matrix.

Also it holds that for interval B-matrices, their principal submatrices are interval B-matrices as well.

Proof. All four properties of interval B-matrices come directly from Proposition 2.9 and Definition 3.1. For example we will show the proof of closure under matrix addition, the rest is analogous:

Let $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{IR}^{n \times n}$ be interval B-matrices. Then $\forall A \in \boldsymbol{A}, \forall B \in \boldsymbol{B} : A + B$ is B-matrix, by Proposition 2.9. Therefore $\boldsymbol{A} + \boldsymbol{B}$ is an interval B-matrix.

⊠

Remark 3.19. Because the real B-matrices are not closed under the following operations (see Propositions 2.11, 2.12 and 2.13), then surely even interval B-matrices are in general not closed under them. Such operations are:

- matrix product,
- matrix inverse and
- matrix power.

Now, that we have dealt with the legacy of real B-matrices, comes time for the interesting part, which is a behaviour of a multiplication of B-matrices by intervals.

Remark 3.20. As direct corollary of Remark 2.10, we can see that multiplying by an interval containing non-positive number can not be an operation that interval B-matrices might be closed under.

Proposition 3.21. *Interval B-matrices are in general not closed under multiplication by positive interval.*

Proof. Let

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

and $\boldsymbol{\alpha} = [1, 2]$. It is easy to verify that such A is an interval B-matrix. But matrix

$$\boldsymbol{\alpha} \cdot A = \begin{pmatrix} [2,4] & [1,2] \\ [1,2] & [2,4] \end{pmatrix}$$

contains a (singular) non-B-matrix instance, namely matrix

$$\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}.$$

⊠

Well that was rather anticlimactic. But let us try to derive some conditions, under which B-matrices, or rather interval B-matrices will be closed under a positive interval multiplication.

Lemma 3.22. *Let $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$ be an interval B-matrix and $\boldsymbol{\alpha} \in \mathbb{IR}^{+}$ be an interval, such that:*

1) $\forall i \in [n]:$

$$\underline{\alpha} \cdot \sum_{j:\underline{a}_{ij}>0} \underline{a}_{ij} > \overline{\alpha} \cdot \sum_{j:\underline{a}_{ij}<0} -\underline{a}_{ij}$$

and

2) $\forall i \in [n] \quad \forall k \in [n] \setminus \{i\}:$

$$\underline{\alpha} \cdot \sum_{\substack{j:\underline{a}_{ij}>0 \\ j\neq k}} \underline{a}_{ij} + \overline{\alpha} \cdot \sum_{\substack{j:\underline{a}_{ij}<0 \\ j\neq k}} \underline{a}_{ij} > \begin{cases} \overline{\alpha} \cdot (n-1) \cdot \overline{a}_{ik} & \text{if } \overline{a}_{ik} > 0, \\ \underline{\alpha} \cdot (n-1) \cdot \overline{a}_{ik} & \text{otherwise.} \end{cases}$$

Then matrix $\boldsymbol{\alpha} \cdot \boldsymbol{A}$ is also an interval B-matrix.

Proof. We can observe that the first condition can be rewritten as

$$\sum_{j=1}^{n} \underline{(\boldsymbol{\alpha} \cdot \boldsymbol{A})}_{ij} > 0$$

and the second one as

$$\sum_{j\neq k} \underline{(\boldsymbol{\alpha} \cdot \boldsymbol{A})}_{ij} > (n-1) \cdot \overline{(\boldsymbol{\alpha} \cdot \boldsymbol{A})}_{ik}.$$

Consequently, interval fulfills exactly those conditions, thanks to which $\boldsymbol{\alpha} \cdot \boldsymbol{A}$ still meets the characterization shown in Theorem 3.5.

⊠

Lemma 3.23. *Let $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$ be an interval B-matrix and $\boldsymbol{\alpha} \in \mathbb{IR}^{+}$ be an interval, such that:*

1) $\forall i \in [n]:$

$$\underline{\alpha} \cdot \sum_{j:\underline{a}_{ij}>0} \underline{a}_{ij} > \overline{\alpha} \cdot \sum_{j:\underline{a}_{ij}<0} -\underline{a}_{ij}$$

and

2) $\forall i \in [n] \quad \forall k \in [n] \setminus \{i\}:$

$$\overline{a}_{ik} > 0 \quad \Rightarrow \quad \underline{\alpha} \cdot \sum_{\substack{j:\underline{a}_{ij}>0 \\ j\neq k}} \underline{a}_{ij} + \overline{\alpha} \cdot \sum_{\substack{j:\underline{a}_{ij}<0 \\ j\neq k}} \underline{a}_{ij} > \overline{\alpha} \cdot (n-1) \cdot \overline{a}_{ik}.$$

Then matrix $\boldsymbol{\alpha} \cdot \boldsymbol{A}$ is also an interval B-matrix.

Proof. We will prove this lemma by showing that its conditions are equivalent to those stated in Lemma 3.22. Then if for a given interval B-matrix an interval fulfills the condition of this lemma, we know that it fulfills the condition of the previous lemma as well, so we know that the interval times the matrix is an interval B-matrix.

What we can observe is that the conditions in this lemma are subset of conditions from Lemma 3.22. So let us see, what is different:

The first condition of this lemma is the same as in the previous case, so that one is trivial. Now for the second one. If $\overline{a}_{ik} > 0$, then the condition is again the same. If $\overline{a}_{ik} \leq 0$, then necessarily even $\underline{a}_{ik} \leq 0$. In addition, let us WLOG assume that the first condition holds. (Otherwise it is unnecessary to check the second one...) Then:

$$\begin{aligned} & \underline{\alpha} \cdot \sum_{\substack{j:\underline{a}_{ij}>0 \\ j\neq k}} \underline{a}_{ij} + \overline{\alpha} \cdot \sum_{\substack{j:\underline{a}_{ij}<0 \\ j\neq k}} \underline{a}_{ij} \quad \geq \\ \geq \quad & \underline{\alpha} \cdot \sum_{j:\underline{a}_{ij}>0} \underline{a}_{ij} + \overline{\alpha} \cdot \sum_{j:\underline{a}_{ij}<0} \underline{a}_{ij} \quad > \\ > \quad & 0 \quad \geq \quad \underline{\alpha} \cdot (n-1) \cdot \overline{a}_{ik} \end{aligned}$$

The first inequality holds, because we are adding a non-positive number to the left side. The second inequality holds because of the first condition of this lemma and the last one is a result of $\overline{a}_{ik} \leq 0$ and $\underline{\alpha}$ being positive.

Thus $\boldsymbol{\alpha}$ implicitly meets even the last condition from Lemma 3.22.

Therefore, from the above mentioned lemma, $\boldsymbol{\alpha} \cdot \boldsymbol{A}$ is an interval B-matrix.

⊠

Proposition 3.24. *Let $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$ be an interval B-matrix and $\boldsymbol{\alpha} \in \mathbb{IR}^{+}$ be an interval, such that:*

$$\underline{\alpha}/\overline{\alpha} \quad > \quad \max\left(\left\{ \frac{\sum\limits_{j:a_{ij}<0} -\underline{a}_{ij}}{\sum\limits_{j:a_{ij}>0} \underline{a}_{ij}} \,\middle|\, i \in [n] \right\} \cup \left\{ \frac{\sum\limits_{\substack{j:\underline{a}_{ij}<0 \\ j\neq k}} -\underline{a}_{ij} + (n-1)\cdot \overline{a}_{ik}}{\sum\limits_{\substack{j:\underline{a}_{ij}>0 \\ j\neq k}} \underline{a}_{ij}} \,\middle|\, i \in [n], k \in [n]\setminus\{i\} : \overline{a}_{ik} > 0 \right\}\right).$$

Then matrix $\boldsymbol{\alpha} \cdot \boldsymbol{A}$ is also an interval B-matrix.

Proof. The condition is obtained by rearranging and grouping of the conditions from Lemma 3.23. (Because for interval B-matrix always holds that $\forall i \in [n]$: $\underline{a}_{ii} > 0$, then $\forall i \in [n]$:

$$\sum_{j:a_{ij}>0} \underline{a}_{ij} \quad \geq \quad \sum_{\substack{j:\underline{a}_{ij}>0 \\ j\neq k}} \underline{a}_{ij} \quad \geq \quad \underline{a}_{ii} \quad > \quad 0.$$

Thus the expressions in the definitions of the sets are correct, because we never divide by zero.)

⊠

Now Proposition 3.24 gives us for every interval B-matrix a ratio of boundaries of interval, which, if it is sufficed by any given interval, guarantees that the interval B-matrix multiplied by the interval is still an interval B-matrix. That may come in handy, if we have one of the boundaries given, respectively fixed, and want to found an interval, which we can multiply given interval B-matrix by, while still remaining in space of interval B-matrices.

Next we shall show a second approach to the problem: We have a given (positive) center of the interval and for a given matrix want to find out the possible width of the interval.

Lemma 3.25. *Let* $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$ *be an interval B-matrix and* $\boldsymbol{\alpha} = \left[\alpha^C \pm \alpha^\Delta\right] \in \mathbb{IR}^+$*, such that:*

1) $\forall i \in [n]:$

$$\alpha^C \cdot \sum_{j=1}^{n} \underline{a}_{ij} > \alpha^\Delta \cdot \sum_{j=1}^{n} |\underline{a}_{ij}|$$

and

2) $\forall i \in [n]: \quad \forall k \in [n] \setminus \{i\}:$

$$\alpha^C \cdot \left(\sum_{j\neq k} \underline{a}_{ij} - (n-1)\cdot \overline{a}_{ik}\right) > \alpha^\Delta \cdot \left(\sum_{j\neq k} |\underline{a}_{ij}| + (n-1)\cdot |\overline{a}_{ik}|\right).$$

Then matrix $\boldsymbol{\alpha}\cdot\boldsymbol{A}$ *is also an interval B-matrix.*

Proof. In the proof we just need to realize the following fact: $\forall \beta \in \mathbb{R}: \beta \cdot \left[\gamma^C \pm \gamma^\Delta\right] = [\beta\cdot\gamma^C \pm |\beta|\cdot\gamma^\Delta]$

Now if we count in Theorem 3.5, we get our conditions. Because if we want the *a*) condition of Theorem 3.5 to hold for $\boldsymbol{\alpha}\cdot\boldsymbol{A}$, then our first condition must hold. (Because $\underline{(\boldsymbol{\alpha}\cdot\boldsymbol{A})}_{ij} = \underline{\left(\boldsymbol{\alpha}\cdot\underline{a}_{ij}\right)} = \alpha^C\cdot\underline{a}_{ij} - \alpha^\Delta\cdot|\underline{a}_{ij}|$.)

Analogically, if the *b*) condition should hold, then our second condition have to hold. ($\underline{(\boldsymbol{\alpha}\cdot\boldsymbol{A})}_{ij} = \alpha^C\cdot\underline{a}_{ij} - \alpha^\Delta\cdot|\underline{a}_{ij}| \quad \wedge \quad \overline{(\boldsymbol{\alpha}\cdot\boldsymbol{A})}_{ik} = \alpha^C\cdot\overline{a}_{ik} + \alpha^\Delta\cdot|\overline{a}_{ik}|$.) ⊠

Proposition 3.26. *Let* $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$ *be an interval B-matrix and* $\boldsymbol{\alpha} = \left[\alpha^C \pm \alpha^\Delta\right] \in \mathbb{IR}^+$*, such that:*

$$\alpha^\Delta < \min\left(\left\{ \frac{\alpha^C\cdot\sum_{j=1}^{n}\underline{a}_{ij}}{\sum_{j=1}^{n}|\underline{a}_{ij}|} \,\middle|\, i\in[n] \right\} \cup \left\{ \frac{\alpha^C\cdot\left(\sum_{j\neq k}\underline{a}_{ij} - (n-1)\cdot\overline{a}_{ik}\right)}{\sum_{j\neq k}|\underline{a}_{ij}| + (n-1)\cdot|\overline{a}_{ik}|} \,\middle|\, i\in[n], k\in[n]\setminus\{i\} \right\}\right).$$

Then matrix $\boldsymbol{\alpha}\cdot\boldsymbol{A}$ *is also an interval B-matrix.*

Proof. Obtained by rearranging and grouping of conditions from Lemma 3.25. (Because for interval B-matrix always holds that $\forall i \in [n] : \underline{a}_{ii} > 0$, then $\forall i \in [n]$:

$$\sum_{j=1}^{n} |\underline{a}_{ij}| \quad \geq \quad \sum_{j \neq k} |\underline{a}_{ij}| \quad \geq \quad |\underline{a}_{ii}| \quad > \quad 0.$$

Thus the expressions in the definitions of the sets are correct, because we never divide by zero.)

⊠

Now we have shown how to ensure, that after multiplication of a B-matrix by positive interval, we still have a B-matrix. Next we will show that something similar applies to multiplying just one row of the matrix by an interval. For that we shall use the previous propositions as well as Corollary 3.4.

Proposition 3.27. *Let $\boldsymbol{A} \in \mathbb{IR}^{n \times n}$ be an interval B-matrix, $i \in [n]$ and $\boldsymbol{\alpha} \in \mathbb{IR}^{+}$ be an interval, such that:*

$$\underline{\alpha}/\overline{\alpha} \quad > \quad \max\left(\left\{\frac{\sum\limits_{j:a_{ij}<0} -\underline{a}_{ij}}{\sum\limits_{j:a_{ij}>0} \underline{a}_{ij}}\right\} \cup \left\{\frac{\sum\limits_{\substack{j:\underline{a}_{ij}<0 \\ j \neq k}} -\underline{a}_{ij} + (n-1) \cdot \overline{a}_{ik}}{\sum\limits_{\substack{j:\underline{a}_{ij}>0 \\ j \neq k}} \underline{a}_{ij}} \,\middle|\, k \in [n] \setminus \{i\} : \overline{a}_{ik} > 0\right\}\right).$$

Let $\boldsymbol{A}(\boldsymbol{\alpha}, i)$ be defined as follows:

$$\boldsymbol{A}(\boldsymbol{\alpha}, i)_{i*} = \boldsymbol{\alpha} \cdot \boldsymbol{A}_{i*} \quad \wedge \quad \forall j \in [n] \setminus \{i\} : \quad \boldsymbol{A}(\boldsymbol{\alpha}, i)_{j*} = \boldsymbol{A}_{j*}$$

Then $\boldsymbol{A}(\boldsymbol{\alpha}, i)$ is also an interval B-matrix.

Proof. Let us define $\boldsymbol{I}(\boldsymbol{A}_{i*}, i)$ as follows:

$$\boldsymbol{I}(\boldsymbol{A}_{i*}, i)_{i*} = \boldsymbol{A}_{i*} \quad \wedge \quad \forall j \in [n] \setminus \{i\} : \quad \boldsymbol{I}(\boldsymbol{A}_{i*}, i)_{j*} = (I_n)_{j*}$$

Surely $\boldsymbol{I}(\boldsymbol{A}_{i*}, i)$ is an interval B-matrix. (It is trivial to verify that I_n is a B-matrix and because $\boldsymbol{A}$ is an interval B-matrix from assumtion, then from Corollary 3.4 it follows that $\boldsymbol{I}(\boldsymbol{A}_{i*}, i)$ is an interval B-matrix too.)

Now let us take a look at what the conditions from Proposition 3.24 on interval $\boldsymbol{\alpha}$ and matrix $\boldsymbol{I}(\boldsymbol{A}_{i*}, i)$ are:

$$\underline{\alpha}/\overline{\alpha} \quad > \quad \max\left(\left\{0, \frac{\sum\limits_{j:a_{ij}<0} -\underline{a}_{ij}}{\sum\limits_{j:a_{ij}>0} \underline{a}_{ij}}\right\} \cup \left\{0, \frac{\sum\limits_{\substack{j:\underline{a}_{ij}<0 \\ j \neq k}} -\underline{a}_{ij} + (n-1) \cdot \overline{a}_{ik}}{\sum\limits_{\substack{j:\underline{a}_{ij}>0 \\ j \neq k}} \underline{a}_{ij}} \,\middle|\, k \in [n] \setminus \{i\} : \overline{a}_{ik} > 0\right\}\right),$$

where the zeros in both sets are what we get for rows $j \neq i$. We can observe that the fact that our $\boldsymbol{\alpha}$ fulfills the conditions from assumption is equivalent to $\boldsymbol{\alpha}$ fulfilling these conditions. (Among other things because $\boldsymbol{\alpha} = [\underline{\alpha}, \overline{\alpha}] \in \mathbb{IR}^+$, so necessarily $\overline{\alpha} \geq \underline{\alpha} > 0$) Thus $\boldsymbol{\alpha} \cdot \boldsymbol{I}(\boldsymbol{A}_{i*}, i)$ is an interval B-matrix.

Now what remains is to realize that $\boldsymbol{A}(\boldsymbol{\alpha}, i)_{i*}$ can be defined even as follows:

$$\boldsymbol{A}(\boldsymbol{\alpha}, i)_{i*} = (\boldsymbol{\alpha} \cdot \boldsymbol{I}(\boldsymbol{A}_{i*}, i))_{i*} \quad \wedge \quad \forall j \in [n] \setminus \{i\} : \quad \boldsymbol{A}(\boldsymbol{\alpha}, i)_{j*} = \boldsymbol{A}_{j*},$$

hence it is combination (in the meaning of Corollary 3.4) of rows of the two B-matrices, therefore it itself is an interval B-matrix.

⊠

Proposition 3.28. *Let $\boldsymbol{A} \in \mathbb{IR}^{n \times n}$ be an interval B-matrix, $i \in [n]$ and $\boldsymbol{\alpha} = \left[\alpha^C \pm \alpha^\Delta\right] \in \mathbb{IR}^+$ be an interval, such that:*

$$\alpha^\Delta < \min\left(\left\{\frac{\alpha^C \cdot \sum_{j=1}^{n} \underline{a}_{ij}}{\sum_{j=1}^{n} |\underline{a}_{ij}|}\right\} \cup \left\{\frac{\alpha^C \cdot \left(\sum_{j \neq k} \underline{a}_{ij} - (n-1) \cdot \overline{a}_{ik}\right)}{\sum_{j \neq k} |\underline{a}_{ij}| + (n-1) \cdot |\overline{a}_{ik}|} \,\middle|\, k \in [n] \setminus \{i\}\right\}\right).$$

Let $\boldsymbol{A}(\boldsymbol{\alpha}, i)$ be defined as follows:

$$\boldsymbol{A}(\boldsymbol{\alpha}, i)_{i*} = \boldsymbol{\alpha} \cdot \boldsymbol{A}_{i*} \quad \wedge \quad \forall j \in [n] \setminus \{i\} : \quad \boldsymbol{A}(\boldsymbol{\alpha}, i)_{j*} = \boldsymbol{A}_{j*}$$

Then $\boldsymbol{A}(\boldsymbol{\alpha}, i)$ is also an interval B-matrix.

Proof. Let us define $\boldsymbol{I}(\boldsymbol{A}_{i*}, i)$ as follows:

$$\boldsymbol{I}(\boldsymbol{A}_{i*}, i)_{i*} = \boldsymbol{A}_{i*} \quad \wedge \quad \forall j \in [n] \setminus \{i\} : \quad \boldsymbol{I}(\boldsymbol{A}_{i*}, i)_{j*} = (I_n)_{j*}$$

Surely $\boldsymbol{I}(\boldsymbol{A}_{i*}, i)$ is an interval B-matrix. (It is trivial to verify that I_n is a B-matrix and because $\boldsymbol{A}$ is an interval B-matrix from assumtion, then from Corollary 3.4 it follows that $\boldsymbol{I}(\boldsymbol{A}_{i*}, i)$ is an interval B-matrix too.)

Now let us take a look at what are the conditions from Proposition 3.24 on interval $\boldsymbol{\alpha}$ and matrix $\boldsymbol{I}(\boldsymbol{A}_{i*}, i)$:

$$\alpha^\Delta < \min\left(\left\{\alpha^C, \frac{\alpha^C \cdot \sum_{j=1}^{n} \underline{a}_{ij}}{\sum_{j=1}^{n} |\underline{a}_{ij}|}\right\} \cup \left\{\alpha^C, \frac{\alpha^C \cdot \left(\sum_{j \neq k} \underline{a}_{ij} - (n-1) \cdot \overline{a}_{ik}\right)}{\sum_{j \neq k} |\underline{a}_{ij}| + (n-1) \cdot |\overline{a}_{ik}|} \,\middle|\, k \in [n] \setminus \{i\}\right\}\right),$$

where the α^C in both sets is what we get for rows $j \neq i$. We can observe that the fact that our $\boldsymbol{\alpha}$ fulfills the conditions from assumption is equivalent to $\boldsymbol{\alpha}$

fulfilling these conditions. (Among other things because $\boldsymbol{\alpha} = \left[\alpha^C \pm \alpha^\Delta\right] \in \mathbb{IR}^+$, so necessarily $\alpha^\Delta < \alpha^C$) Thus $\boldsymbol{\alpha} \cdot \boldsymbol{I}(\boldsymbol{A}_{i*}, i)$ is an interval B-matrix.

Now what remains is to realize that $\boldsymbol{A}(\boldsymbol{\alpha}, i)_{i*}$ can be defined even as follows:

$$\boldsymbol{A}(\boldsymbol{\alpha}, i)_{i*} = \left(\boldsymbol{\alpha} \cdot \boldsymbol{I}(\boldsymbol{A}_{i*}, i)\right)_{i*} \quad \wedge \quad \forall j \in [n] \setminus \{i\} : \quad \boldsymbol{A}(\boldsymbol{\alpha}, i)_{j*} = \boldsymbol{A}_{j*},$$

hence it is combination (in the meaning of Corollary 3.4) of rows of the two B-matrices, therefore it itself is an interval B-matrix.

⊠

4. Interval doubly B-matrices

In this chapter we shall generalize real doubly B-matrices into interval doubly B-matrices. In section 4.1 we shall introduce some characterizations, whereas in section 4.2 we will try to derive necessary conditions and sufficient ones to help us recognize this matrix class even more efficiently. That comes quite handy here, because the characterization of interval doubly B-matrices has time complexity $O(n^4)$. And in section 4.3 we will take a closer look at which operations are the interval doubly B-matrices closed under.

Definition 4.1 (interval doubly B-matrix)**.** *Let $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$. Then we say that $\boldsymbol{A}$ is an interval doubly B-matrix, if $\forall A \in \boldsymbol{A}$: A is a (real) doubly B-matrix.*

Corollary 4.2. *Every interval doubly B-matrix is an interval P-matrix.*

Proof. It holds for every instance, thus it holds for whole interval matrix.

(Every instance is P-matrix by Proposition 2.17 and that is exactly the definition of interval P-matrix.)

⊠

4.1 Characterizations

Let us start with introducing a characterization to help us recognize interval doubly B-matrices in finite time. Thereafter in subsection 4.1.1 we will try to derive a characterization through reduction of Definition 4.1 to finite number of instances to check for the property of being a doubly B-matrix.

Theorem 4.3. *Let $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$. Then $\boldsymbol{A}$ is an interval doubly B-matrix if and only if the following two properties holds:*

a) $\forall i \in [n]: \quad \underline{a}_{ii} > \max\{0, \overline{a}_{ij} | j \neq i\}$ *and*

b) $\forall i, j \in [n], j \neq i, \forall k, l \in [n], k \neq i, l \neq j:$

I. $$\Big(\underline{a}_{ii} - \overline{a}_{ik}\Big)\Big(\underline{a}_{jj} - \overline{a}_{jl}\Big) > \left(\max\left\{0, \sum_{\substack{m\neq i\\ m\neq k}} \Big(\overline{a}_{ik} - \underline{a}_{im}\Big)\right\}\right)\left(\max\left\{0, \sum_{\substack{m\neq j\\ m\neq l}} \Big(\overline{a}_{jl} - \underline{a}_{jm}\Big)\right\}\right)$$

II. $$\underline{a}_{ii}\Big(\underline{a}_{jj} - \overline{a}_{jl}\Big) > \left(\max\left\{0, -\sum_{m\neq i} \underline{a}_{im}\right\}\right)\left(\max\left\{0, \sum_{\substack{m\neq j\\ m\neq l}} \Big(\overline{a}_{jl} - \underline{a}_{jm}\Big)\right\}\right)$$

III. $$\underline{a}_{ii} \cdot \underline{a}_{jj} > \left(\max\left\{0, -\sum_{m\neq i} \underline{a}_{im}\right\}\right)\left(\max\left\{0, -\sum_{m\neq j} \underline{a}_{jm}\right\}\right)$$

Proof. Let us recall the Definition 2.14, which we will be using in this proof:

Let $A \in \mathbb{R}^{n \times n}$. Then we know that A is a real doubly B-matrix, if $\forall i \in [n]$ the following holds:

a) $a_{ii} > r_i^+$

b) $\forall j \in [n] \setminus \{i\} : \left(a_{ii} - r_i^+\right)\left(a_{jj} - r_j^+\right) > \left(\sum_{k \neq i} \left(r_i^+ - a_{ik}\right)\right)\left(\sum_{k \neq j} \left(r_j^+ - a_{jk}\right)\right)$

"$\Rightarrow$" $\forall A \in \boldsymbol{A} : A$ is (real) doubly B-matrix, hence:

Our "interval" condition *a)* holds because of "real" condition *a)* from Definition 2.14 for matrix $A' \in \boldsymbol{A}$ with all diagonal elements set on their lower bounds and all the off-diagonal elements set on their upper bounds.

As for "interval" condition *b)* let us fix arbitrary $i, j \in [n], j \neq i$, and arbitrary $k \neq i, l \neq j$. Then:

I. Let $A \in \boldsymbol{A}$, such that

$$a_{m_1 m_2} = \begin{cases} \overline{a}_{ik} & \text{if } (m_1, m_2) = (i, k), \\ \overline{a}_{jl} & \text{if } (m_1, m_2) = (j, l), \\ \underline{a}_{m_1 m_2} & \text{otherwise.} \end{cases}$$

Then for this A :

$$\begin{aligned} \left(\underline{a}_{ii} - \overline{a}_{ik}\right)\left(\underline{a}_{jj} - \overline{a}_{jl}\right) \quad &\geq \quad \left(a_{ii} - r_i^+\right)\left(a_{jj} - r_j^+\right) \quad > \\ > \quad &\left(\sum_{m \neq i} \left(r_i^+ - a_{im}\right)\right)\left(\sum_{m \neq j} \left(r_j^+ - a_{jm}\right)\right) \quad \geq \\ \geq \quad &\left(\max\left\{0, \sum_{\substack{m \neq i \\ m \neq k}} \left(\overline{a}_{ik} - \underline{a}_{im}\right)\right\}\right)\left(\max\left\{0, \sum_{\substack{m \neq j \\ m \neq l}} \left(\overline{a}_{jl} - \underline{a}_{jm}\right)\right\}\right) \end{aligned}$$

The first inequality comes trivially from $\overline{a}_{ik} \leq r_i^+$ and analogically for j and l. The second one is obtained from the fact that A is a doubly B-matrix (because $A \in \boldsymbol{A}$ and $\boldsymbol{A}$ is an interval doubly B-matrix). The third and last inequality is a direct result of the following facts: $\forall m \neq i : a_{im} \leq r_i^+$ (from the definition of r_i^+), so $r_i^+ - a_{im} \geq 0$, thus whole $\sum_{m \neq i} \left(r_i^+ - a_{im}\right)$ is non-negative. Another fact is that what we drop from the sum, i.e. the "k member", is a non-negative element of the sum. And finally $r_i^+ \geq a_{ik} = \overline{a}_{ik} \wedge a_{im} \geq \underline{a}_{ik}$. Again, for j, l it is analogous.

II. Let $A \in \boldsymbol{A}$, such that

$$a_{m_1 m_2} = \begin{cases} \overline{a}_{jl} & \text{if } (m_1, m_2) = (j, l), \\ \underline{a}_{m_1 m_2} & \text{otherwise.} \end{cases}$$

Then for this A :

$$\begin{aligned} \underline{a}_{ii}\left(\underline{a}_{jj} - \overline{a}_{jl}\right) \quad &\geq \quad \left(a_{ii} - r_i^+\right)\left(a_{jj} - r_j^+\right) \quad > \\ > \quad &\left(\sum_{m \neq i} \left(r_i^+ - a_{im}\right)\right)\left(\sum_{m \neq j} \left(r_j^+ - a_{jm}\right)\right) \quad \geq \\ \geq \quad &\left(\max\left\{0, -\sum_{m \neq i} \underline{a}_{im}\right\}\right)\left(\max\left\{0, \sum_{\substack{m \neq j \\ m \neq l}} \left(\overline{a}_{jl} - \underline{a}_{jm}\right)\right\}\right) \end{aligned}$$

The second inequality comes from the same fact as above, i.e. A is a doubly B-matrix. And the last one holds as well because of similar reasons as above plus because, in case of "i part" of the expression, we drop $n \cdot r_i^+$, which is some non-negative quantity.

III. Let $A = \underline{A} \in \boldsymbol{A}$. Then for this A :

$$\begin{aligned} \underline{a}_{ii} \cdot \underline{a}_{jj} &\geq \left(a_{ii} - r_i^+\right)\left(a_{jj} - r_j^+\right) > \\ &> \left(\sum_{m \neq i} \left(r_i^+ - a_{im}\right)\right)\left(\sum_{m \neq j} \left(r_j^+ - a_{jm}\right)\right) \geq \\ &\geq \left(\max\left\{0, -\sum_{m \neq i} \underline{a}_{im}\right\}\right)\left(\max\left\{0, -\sum_{m \neq j} \underline{a}_{jm}\right\}\right) \end{aligned}$$

Again these inequalities hold from the reasons stated above.

Ergo this implication holds.

"$\Leftarrow$" Let $A \in \boldsymbol{A}$.

Condition a) from Definition 2.14 follows trivially from our "interval" condition a).

Let us pick arbitrary $i, j \in [n], j \neq i$. Now let us distinguish the following cases:

1) $r_i^+, r_j^+ > 0$

   Then $\exists k \neq i, \exists l \neq j : r_i^+ = a_{ik} \wedge r_j^+ = a_{jl}$. So

$$\begin{aligned} \left(a_{ii} - r_i^+\right)\left(a_{jj} - r_j^+\right) &\geq \left(\underline{a}_{ii} - \overline{a}_{ik}\right)\left(\underline{a}_{jj} - \overline{a}_{jl}\right) > \\ &> \left(\max\left\{0, \sum_{\substack{m \neq i \\ m \neq k}} \left(\overline{a}_{ik} - \underline{a}_{im}\right)\right\}\right)\left(\max\left\{0, \sum_{\substack{m \neq j \\ m \neq l}} \left(\overline{a}_{jl} - \underline{a}_{jm}\right)\right\}\right) = \\ &= \left(\sum_{\substack{m \neq i \\ m \neq k}} \left(\overline{a}_{ik} - \underline{a}_{im}\right)\right)\left(\sum_{\substack{m \neq j \\ m \neq l}} \left(\overline{a}_{jl} - \underline{a}_{jm}\right)\right) \geq \\ &\geq \left(\sum_{\substack{m \neq i \\ m \neq k}} \left(r_i^+ - a_{im}\right)\right)\left(\sum_{\substack{m \neq j \\ m \neq l}} \left(r_j^+ - a_{jm}\right)\right) = \\ &= \left(\sum_{m \neq i} \left(r_i^+ - a_{im}\right)\right)\left(\sum_{m \neq j} \left(r_j^+ - a_{jm}\right)\right) \end{aligned}$$

   The second inequality holds because $A \in \boldsymbol{A}$ (thus because of the assumptions of the implication, more specificaly because of point I. of condition b)). The next equality follows from $\overline{a}_{ik} \geq r_i^+ \geq a_{im} \geq \underline{a}_{im}$, for $m \neq i$, because that implies that the sums are non-negative. (Analogically for j and l.) The same chain of inequalities can be used to verify the fourth inequality. And the last equality arises from the fact that $r_i^+ = a_{ik} \wedge r_j^+ = a_{jl}$. Thus from Definition 2.14, A is a doubly B-matrix.

2) $r_i^+ = 0 \wedge r_j^+ > 0$

Then $\exists l \neq j : r_j^+ = a_{jl}$. So

$$\begin{aligned}
& \left(a_{ii} - r_i^+\right)\left(a_{jj} - r_j^+\right) && = \\
= \quad & a_{ii}\left(a_{jj} - r_j^+\right) \quad \geq \quad \underline{a}_{ii}\left(\underline{a}_{jj} - \overline{a}_{jl}\right) && > \\
> \quad & \left(\max\left\{0, -\sum_{m\neq i} \underline{a}_{im}\right\}\right)\left(\max\left\{0, \sum_{\substack{m\neq j\\ m\neq l}} \left(\overline{a}_{jl} - \underline{a}_{jm}\right)\right\}\right) && = \\
= \quad & \left(-\sum_{m\neq i} \underline{a}_{im}\right)\left(\sum_{\substack{m\neq j\\ m\neq l}} \left(\overline{a}_{jl} - \underline{a}_{jm}\right)\right) && \geq \\
\geq \quad & \left(\sum_{m\neq i} \left(r_i^+ - a_{im}\right)\right)\left(\sum_{\substack{m\neq j\\ m\neq l}} \left(r_j^+ - a_{jm}\right)\right) && = \\
= \quad & \left(\sum_{m\neq i} \left(r_i^+ - a_{im}\right)\right)\left(\sum_{m\neq j} \left(r_j^+ - a_{jm}\right)\right)
\end{aligned}$$

The reasoning for the "j part" of the expressions in the inequalities is the same as in the previous case, so let us focus on the "i part": The third inequality holds, because $A \in \boldsymbol{A}$, so the point $II.$ of condition $b)$ applies. The fourth equality comes from the following: $\forall m \neq i : \underline{a}_{im} \leq a_{im} \leq r_i^+ = 0 \Rightarrow -\sum_{m\neq i} \underline{a}_{im} \geq 0$. Therefore again from Definition 2.14 A is a doubly B-matrix.

3) $r_i^+ > 0 \wedge r_j^+ = 0$

   By swapping i for j we get the previous case.

4) $r_i^+, r_j^+ = 0$

$$\begin{aligned}
& \left(a_{ii} - r_i^+\right)\left(a_{jj} - r_j^+\right) \quad \geq \quad \underline{a}_{ii} \cdot \underline{a}_{jj} && > \\
> \quad & \left(\max\left\{0, -\sum_{m\neq i} \underline{a}_{im}\right\}\right)\left(\max\left\{0, -\sum_{m\neq j} \underline{a}_{jm}\right\}\right) && = \\
= \quad & \left(-\sum_{m\neq i} \underline{a}_{im}\right)\left(-\sum_{m\neq j} \underline{a}_{jm}\right) && \geq \\
\geq \quad & \left(\sum_{m\neq i} \left(r_i^+ - a_{im}\right)\right)\left(\sum_{m\neq j} \left(r_j^+ - a_{jm}\right)\right)
\end{aligned}$$

Now the logic behind this chain of inequalities is the same as in the previous cases. Thus once again from Definition 2.14 A is a doubly B-matrix.

Ergo this implication holds as well.

$\boxtimes$

Remark 4.4. This characterization has time complexity $O(n^4)$, which is two orders of magnitude higher then for the real case, given the $O(n^2)$ complexity of the characterization from Definition 2.14.

Corollary 4.5. *Let $\mathbf{A} \in \mathbb{IR}^{n\times n}$. We have that $\mathbf{A}$ is an interval doubly B-matix iff $\mathbf{A}$ with diagonal fixed on lower bounds ($a_{ii} = \underline{a}_{ii}$) is an interval doubly B-matrix.*

Proof. In the characterization given in Theorem 4.3 we see that every time any a_{ii} occures, it occures in a form of $\underline{a}_{ii}$, hence we are not interested in any other value of a_{ii}. (So the reduced matrix has to fulfill exactly the same conditions as the matrix $\mathbf{A}$)

⊠

Corollary 4.6. *Let $\mathbf{A} \in \mathbb{IR}^{n\times n}$ and let*

$$S = \left\{(i,j) | i,j \in [n] : \exists k \in [n] \setminus \{i\} : \overline{a}_{ij} \leq \underline{a}_{ik}\right\}.$$

We have that $\mathbf{A}$ is an interval doubly B-matix iff $\mathbf{A}$ with every element, whose indices are in S, set to its lower bound ($\forall (i,j) \in S : a_{ij} = \underline{a}_{ij}$) is an interval doubly B-matrix.

Proof. The only time, when for every i and $k \neq i$ the $\overline{a}_{ik}$ occures in Theorem 4.3, are some of the inequalities "$I.$" in the $b)$ condition. (And symmetrically as (j,l) in "$II.$" and "$III.$" in the $b)$ condition, but that is analogous, so we will prove just the first case, where it pops up as (i,k).) Let us show that in the case that $(i,k) \in S$ the inequalities "$I.$" are not necessary to check because they are substituted by some of the others.

Let $(i,k) \in S$ arbitrary and let $k' = \operatorname{argmax}\{\underline{a}_{im} | m \in [n] \setminus \{i\}\}$. Because $(i,k) \in S$, then surely $\overline{a}_{ik} \leq \underline{a}_{ik'}$. (And thus even $\underline{a}_{ik} \leq \underline{a}_{ik'}$ and $\overline{a}_{ik} \leq \overline{a}_{ik'}$.) Let us take any arbitrary $j, l \in [n], l \neq j$. Then:

$$\begin{aligned}
\left(\underline{a}_{ii} - \overline{a}_{ik}\right)\left(\underline{a}_{jj} - \overline{a}_{jl}\right) \quad &\geq \quad \left(\underline{a}_{ii} - \overline{a}_{ik'}\right)\left(\underline{a}_{jj} - \overline{a}_{jl}\right) \quad > \\
> \quad &\left(\max\left\{0, \sum_{\substack{m\neq i\\ m\neq k'}} \left(\overline{a}_{ik'} - \underline{a}_{im}\right)\right\}\right)\left(\max\left\{0, \sum_{\substack{m\neq j\\ m\neq l}} \left(\overline{a}_{jl} - \underline{a}_{jm}\right)\right\}\right) \quad \geq \\
\geq \quad &\left(\max\left\{0, \sum_{\substack{m\neq i\\ m\neq k}} \left(\overline{a}_{ik} - \underline{a}_{im}\right)\right\}\right)\left(\max\left\{0, \sum_{\substack{m\neq j\\ m\neq l}} \left(\overline{a}_{jl} - \underline{a}_{jm}\right)\right\}\right)
\end{aligned}$$

So the inequality "$I.$" for (i,k) holds only if the inequality "$I.$" for (i,k') holds. (The last inequality in the previous chain of inequalities holds, because $\underline{a}_{ik} \leq \underline{a}_{ik'}$ and $\overline{a}_{ik'} \geq \overline{a}_{ik}$, so we subtract more and add less.)

The second implication is trivial, because $\mathbf{A}$ is a superset of the reduced matrix.

⊠

Proposition 4.7. *Let $\mathbf{A} \in \mathbb{IR}^{n\times n}$. If $\mathbf{A}$ is an interval B-matrix, then it is an interval doubly B-matrix as well.*

Proof. It holds for every instance, therefore it holds for whole interval matrix.

(Every instance is doubly B-matrix by Proposition 2.15 and that is exactly the definition of an interval doubly B-matrix.)

⊠

Proposition 4.8. *Let $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$ be an interval doubly B-matrix. Then exactly one of the following applies:*

a) Either $\boldsymbol{A}$ is an interval B-matrix, or

b) there exists a unique $j \in [n]$ that j-th row breaks the condition stated in Corollary 3.11 while for all others $i \in [n] \setminus \{k\}$ this condition holds.

In other words there exists a unique j, for which holds either

$$\sum_{m=1}^{n} \underline{a}_{jm} \leq 0$$

or

$$\exists k \in [n] \setminus \{j\} : \quad \underline{a}_{jj} - \overline{a}_{jk} \leq \sum_{\substack{m\neq j\\ m\neq k}} \left(\overline{a}_{jk} - \underline{a}_{jm}\right),$$

and for all the others $i \in [n] \setminus \{j\}$ hold both

$$\sum_{m=1}^{n} \underline{a}_{im} > 0$$

and

$$\forall k \in [n] \setminus \{i\} : \quad \underline{a}_{ii} - \overline{a}_{ik} > \sum_{\substack{m\neq i\\ m\neq k}} \left(\overline{a}_{ik} - \underline{a}_{im}\right),$$

Proof. Let a) hold, so $\boldsymbol{A}$ is an interval B-matrix and thus from Corollarty 3.11 $\forall i \in [n]$:

$$\sum_{m=1}^{n} \underline{a}_{im} > 0$$

and

$$\forall k \in [n] \setminus \{i\} : \quad \underline{a}_{ii} - \overline{a}_{ik} > \sum_{\substack{m\neq i\\ m\neq k}} \left(\overline{a}_{ik} - \underline{a}_{im}\right),$$

thus b) does not hold.

Now let a) not apply, so $\boldsymbol{A}$ is not an interval B-matrix. Then it contains a row, which does not fulfills the condition stated in Corollary 3.11. (Otherwise it would fulfill the characterization stated ibidem, thus it would be an interval B-matrix → contradiction.) We will show that there cannot exist two such rows. For contradiction, let there be two such rows j and j' that breaks the condition. Let us distinguish the following cases:

1) Let it hold that

$$\sum_{m=1}^{n} \underline{a}_{jm} \leq 0$$

and

$$\sum_{m=1}^{n} \underline{a}_{j'm} \leq 0.$$

Then

$$\underline{a}_{jj} \leq -\sum_{m\neq j} \underline{a}_{jm} \quad \wedge \quad \underline{a}_{j'j'} \leq -\sum_{m\neq j'} \underline{a}_{j'm}$$

and because $\boldsymbol{A}$ is an interval doubly B-matrix, then

$$\forall i \in [n]: \quad \underline{a}_{ii} > \max\{0, \overline{a}_{im} | m \neq i\} \geq 0$$

$$\left(\Rightarrow 0 < \underline{a}_{jj} \leq -\sum_{m\neq j} \underline{a}_{jm} \quad \wedge \quad 0 < \underline{a}_{j'j'} \leq -\sum_{m\neq j'} \underline{a}_{j'm} \right)$$

(see Theorem 4.3, part a)) and so the following is true:

$$\underline{a}_{jj} \cdot \underline{a}_{j'j'} \quad \leq \quad \left(-\sum_{m\neq j} \underline{a}_{jm} \right) \left(-\sum_{m\neq j'} \underline{a}_{j'm} \right) \quad =$$

$$= \quad \left(\max\left\{ 0, -\sum_{m\neq j} \underline{a}_{jm} \right\} \right) \left(\max\left\{ 0, -\sum_{m\neq j'} \underline{a}_{j'm} \right\} \right)$$

But that is a contradiction with the assumption that $\boldsymbol{A}$ is an interval doubly B-matrix, because it violates the b) condition, part III. of characterization of interval doubly B-matrices stated in Theorem 4.3.

2) Let it hold that

$$\sum_{m=1}^{n} \underline{a}_{jm} \leq 0$$

and

$$\exists k \in [n] \setminus \{j'\}: \quad \underline{a}_{j'j'} - \overline{a}_{j'k} \leq \sum_{\substack{m\neq j' \\ m\neq k}} \left(\overline{a}_{j'k} - \underline{a}_{j'm} \right).$$

Then

$$\underline{a}_{jj} \leq -\sum_{m\neq j} \underline{a}_{jm}$$

and because $\boldsymbol{A}$ is an interval doubly B-matrix, then

$$\forall i \in [n]: \quad \underline{a}_{ii} > \max\{0, \overline{a}_{im} | m \neq i\}$$

$$\left(\Rightarrow 0 < \underline{a}_{jj} \leq -\sum_{m\neq j} \underline{a}_{jm} \quad \wedge \quad 0 < \underline{a}_{j'j'} - \overline{a}_{ik} \leq \sum_{\substack{m\neq j' \\ m\neq k}} \left(\overline{a}_{j'k} - \underline{a}_{j'm} \right) \right)$$

(see Theorem 4.3, part a)) and so the following is true:

$$\underline{a}_{jj} \left(\underline{a}_{j'j'} - \overline{a}_{j'k} \right) \quad \leq \quad \left(-\sum_{m\neq j} \underline{a}_{jm} \right) \left(\sum_{\substack{m\neq j' \\ m\neq k}} \left(\overline{a}_{j'k} - \underline{a}_{j'm} \right) \right) \quad =$$

$$= \quad \left(\max\left\{ 0, -\sum_{m\neq j} \underline{a}_{jm} \right\} \right) \left(\max\left\{ 0, \sum_{\substack{m\neq j' \\ m\neq k}} \left(\overline{a}_{j'k} - \underline{a}_{j'm} \right) \right\} \right)$$

But that is a contradiction with the assumption that $\boldsymbol{A}$ is an interval doubly B-matrix, because it violates the $b)$ condition, part $II.$ of characterization of interval doubly B-matrices stated in Theorem 4.3.

3) Let it hold that

$$\exists k \in [n] \setminus \{j\} : \quad \underline{a}_{jj} - \overline{a}_{jk} \leq \sum_{\substack{m\neq j \\ m\neq k}} \left(\overline{a}_{jk} - \underline{a}_{jm}\right)$$

and

$$\sum_{m=1}^{n} \underline{a}_{j'm} \leq 0.$$

Then by swapping j and j' we convert it to the previous case.

4) Let it hold that

$$\exists k \in [n] \setminus \{j\} : \quad \underline{a}_{jj} - \overline{a}_{jk} \leq \sum_{\substack{m\neq j \\ m\neq k}} \left(\overline{a}_{jk} - \underline{a}_{jm}\right)$$

and

$$\exists k' \in [n] \setminus \{j'\} : \quad \underline{a}_{j'j'} - \overline{a}_{j'k'} \leq \sum_{\substack{m\neq j' \\ m\neq k'}} \left(\overline{a}_{j'k'} - \underline{a}_{j'm}\right).$$

Then because $\boldsymbol{A}$ is an interval doubly B-matrix, then

$$\forall i \in [n] : \quad \underline{a}_{ii} > \max\{0, \overline{a}_{im} | m \neq i\}$$

$$\left(\Rightarrow \quad 0 < \underline{a}_{jj} - \overline{a}_{jk} \leq \sum_{\substack{m\neq j \\ m\neq k}} \left(\overline{a}_{jk} - \underline{a}_{jm}\right) \underline{a}_{jm} \right.$$
$$\left. \wedge \quad 0 < \underline{a}_{j'j'} - \overline{a}_{ik} \leq \sum_{\substack{m\neq j' \\ m\neq k}} \left(\overline{a}_{j'k} - \underline{a}_{j'm}\right) \right)$$

(see Theorem 4.3, part $a)$) and so the following is true:

$$\begin{aligned}
&\left(\underline{a}_{jj} - \overline{a}_{jk}\right)\left(\underline{a}_{j'j'} - \overline{a}_{j'k'}\right) \quad \leq \\
\leq \quad & \left(\sum_{\substack{m\neq j \\ m\neq k}} \left(\overline{a}_{jk} - \underline{a}_{jm}\right)\right) \left(\sum_{\substack{m\neq j' \\ m\neq k'}} \left(\overline{a}_{j'k'} - \underline{a}_{j'm}\right)\right) \quad = \\
= \quad & \left(\max\left\{0, \sum_{\substack{m\neq j \\ m\neq k}} \left(\overline{a}_{jk} - \underline{a}_{jm}\right)\right\}\right) \left(\max\left\{0, \sum_{\substack{m\neq j' \\ m\neq k'}} \left(\overline{a}_{j'k'} - \underline{a}_{j'm}\right)\right\}\right)
\end{aligned}$$

But that is a contradiction with the assumption that $\boldsymbol{A}$ is an interval doubly B-matrix, because it violates the $b)$ condition, part $I.$ of characterization of interval doubly B-matrices stated in Theorem 4.3.

Thus, we have shown that there exists exactly one row, which breaks the condition from Corollary 3.11, which finishes the proof.

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Definition 4.9 (proper interval doubly B-matrix)**.** *Let $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$ be an interval doubly B-matrix. Then we say that $\boldsymbol{A}$ is a proper interval doubly B-matrix, if it is not an interval B-matrix.*

4.1.1 Characterizations through reduction

Proposition 4.10. *Let $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$ for $n \geq 4$ and let us define $A_{(i,k),(j,l)} \in \mathbb{R}^{n\times n}$ as follows:*

$$A_{(i,k),(j,l)} = (a_{m_1 m_2})\,; \quad a_{m_1 m_2} = \begin{cases} \overline{a}_{ik} & \text{if } (m_1, m_2) = (i,k), \\ \overline{a}_{jl} & \text{if } (m_1, m_2) = (j,l), \\ \underline{a}_{m_1 m_2} & \text{otherwise.} \end{cases}$$

Then $\boldsymbol{A}$ is an interval doubly B-matrix if and only if $\forall i, j \in [n], j > i, \forall k, l \in [n], k \neq i, l \neq j : A_{(i,k),(j,l)}$ is a doubly B-matrix.

Proof. "$\Rightarrow$" Trivial, for all such matrices: $A_{(i,k),(j,l)} \in \boldsymbol{A}$.

"$\Leftarrow$" We will prove that the conditions of Theorem 4.3 hold:

a) $\forall i \in [n], \forall k \neq i : \quad \underline{a}_{ii} > \max\{0, \overline{a}_{ik}\}$, because for all arbitrary j, l the matrix $A_{(i,k),(j,l)}$ is a doubly B-matrix. Thus $\forall i \in [n] : \quad \underline{a}_{ii} > \max\{0, \overline{a}_{ik} | k \neq i\}$.

b) Let us fix arbitrary $i, j \in [n], j \neq i$ and arbitrary $k, l \in [n], k \neq i, l \neq j$. WLOG $j > i$. (If $j < i$, we swap their values and we swap the values of k and l too.) Let $A = A_{(i,k),(j,l)}$. Then:

I.

$$\begin{aligned} \left(\underline{a}_{ii} - \overline{a}_{ik}\right)\left(\underline{a}_{jj} - \overline{a}_{jl}\right) &\geq \left(a_{ii} - r_i^+\right)\left(a_{jj} - r_j^+\right) > \\ > \left(\sum_{m\neq i} \left(r_i^+ - a_{im}\right)\right)\left(\sum_{m\neq j} \left(r_j^+ - a_{jm}\right)\right) &\geq \\ \geq \left(\max\left\{0, \sum_{\substack{m\neq i\\ m\neq k}} \left(\overline{a}_{ik} - \underline{a}_{im}\right)\right\}\right)&\left(\max\left\{0, \sum_{\substack{m\neq j\\ m\neq l}} \left(\overline{a}_{jl} - \underline{a}_{jm}\right)\right\}\right) \end{aligned}$$

The second inequality holds, because A is a doubly B-matrix.

II.

$$\begin{aligned} \underline{a}_{ii}\left(\underline{a}_{jj} - \overline{a}_{jl}\right) &\geq \left(a_{ii} - r_i^+\right)\left(a_{jj} - r_j^+\right) > \\ > \left(\sum_{m\neq i} \left(r_i^+ - a_{im}\right)\right)\left(\sum_{m\neq j} \left(r_j^+ - a_{jm}\right)\right) &\geq \\ \geq \left(\max\left\{0, -\sum_{m\neq i} \underline{a}_{im}\right\}\right)&\left(\max\left\{0, \sum_{\substack{m\neq j\\ m\neq l}} \left(\overline{a}_{jl} - \underline{a}_{jm}\right)\right\}\right) \end{aligned}$$

The second inequality holds because of the fact that even $A_{(x,y),(j,l)}$ for any $x \neq i$ and $y \neq x$ is a doubly B-matrix.

III.

$$\begin{aligned}\underline{a}_{ii}\cdot\underline{a}_{jj} \quad &\geq \quad \left(a_{ii}-r_i^+\right)\left(a_{jj}-r_j^+\right) \quad > \\ > \quad &\left(\sum_{m\neq i}\left(r_i^+-a_{im}\right)\right)\left(\sum_{m\neq j}\left(r_j^+-a_{jm}\right)\right) \quad \geq \\ \geq \quad &\left(\max\left\{0,-\sum_{m\neq i}\underline{a}_{im}\right\}\right)\left(\max\left\{0,-\sum_{m\neq j}\underline{a}_{jm}\right\}\right)\end{aligned}$$

The second inequality holds because of the fact that even $A_{(x,y),(u,v)}$ for any x, y, u, v, such that $x \neq i, x \neq j, y \neq x, u \neq i, u \neq j, u \neq x$ and $v \neq u$ is a doubly B-matrix.

So, as we have shown, the $\boldsymbol{A}$ fulfills both the conditions of Theorem 4.3, therefore it is an interval doubly B-matrix.

⊠

Remark 4.11. Proposition 4.10 could work even for $n \geq 3$, but we would have to add requirement that $\underline{A}$ is a doubly B-matrix too. Or it could work even for $n \geq 2$, but again we would have to add requirements that $\underline{A}$ is a doubly B-matrix and $\forall j \in [n], l \neq j : A_{(j,l)}$ is a doubly B-matrix, where

$$A_{(j,l)} = (a_{m_1m_2}); \quad \begin{cases} \overline{a}_{jl} & \text{if } (m_1, m_2) = (j, l), \\ \underline{a}_{m_1m_2} & \text{otherwise.} \end{cases} \quad .$$

These requirements are needed for proof of "*II.*" and "*III.*" of *b)* condition of the second (right-to-left) implication. But we can show an example that they are not just formal requirements:

Example. Let $\boldsymbol{A} \in \mathbb{IR}^{3\times 3}$, such that $\boldsymbol{A}_{ij} = \begin{cases} [1,1] = 1 & \text{if } i = j, \\ [-\frac{1}{2}, 0] & \text{otherwise.} \end{cases}$

Then $\forall A_{(i,k),(j,l)}: \quad \forall z, z' \in [3], z' \neq z : r_z^+ = r_{z'}^+ = 0$, so:

$$\left(a_{zz}-r_z^+\right)\left(a_{z'z'}-r_{z'}^+\right) = 1\cdot 1 = 1$$

and

$$\left(\sum_{m\neq z}\left(r_z^+-a_{zm}\right)\right)\left(\sum_{m\neq z'}\left(r_{z'}^+-a_{z'm}\right)\right) \leq \frac{1}{2}\cdot 1 = \frac{1}{2}.$$

Thus every $A_{(i,k),(j,l)}$ is a doubly B-matrix.

But for $\underline{A}: \quad \forall z, z' \in [3], z' \neq z :$

$$\left(a_{zz}-r_z^+\right)\left(a_{z'z'}-r_{z'}^+\right) = 1\cdot 1 = 1$$

and

$$\left(\sum_{m\neq z}\left(r_z^+-a_{zm}\right)\right)\left(\sum_{m\neq z'}\left(r_{z'}^+-a_{z'm}\right)\right) = \left(\frac{1}{2}+\frac{1}{2}\right)^2 = 1^2 = 1.$$

So $\underline{A}$ is not a doubly B-matrix, thus $\boldsymbol{A}$ cannot be an interval doubly B-matrix.

Proposition 4.12. *The characterization of interval doubly B-matrices through reduction given by Proposition 4.10 is minimal in inclusion.*

Proof. If we ditched $A_{(i,k),(j,l)}$ for any arbitrary $i, j, k, l \in [n], j \neq i, k \neq i, l \neq j$, then we can construct a counterexample, e.g. a unit matrix with interval $[0, \frac{1}{2}]$ on positions (i, k) and (j, l). Then $\forall x, y, u, v \in [n], u \neq x, y \neq x, v \neq u$, such that $(x, y, u, v) \neq (i, k, j, l) : A_{(x,y),(u,v)}$ is a doubly B-matrix. That holds because $\forall z, z' \in [n], z' \neq z :$

$$\left(a_{zz} - r_z^+\right)\left(a_{z'z'} - r_{z'}^+\right) \geq \frac{1}{2}$$

and

$$\left(\sum_{m \neq z} \left(r_z^+ - a_{zm}\right)\right)\left(\sum_{m \neq z'} \left(r_{z'}^+ - a_{z'm}\right)\right) = 0.$$

But $A_{(i,k),(j,l)}$ is not a doubly B-matrix, because

$$\left(a_{ii} - r_i^+\right)\left(a_{jj} - r_j^+\right) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

and

$$\left(\sum_{m \neq i} \left(r_i^+ - a_{im}\right)\right)\left(\sum_{m \neq j} \left(r_j^+ - a_{jm}\right)\right) =$$

$$\left(\left(\frac{1}{2} - \frac{1}{2}\right) + (n-2) \cdot \left(\frac{1}{2} - 0\right)\right)^2 = \left(\frac{n-2}{2}\right)^2$$

and for $n \geq 3$ it does not hold that $\frac{1}{4} > \left(\frac{n-2}{2}\right)^2$. (And in Proposition we assume $n \geq 4$.)

Hence the whole interval matrix cannot be an interval doubly B-matrix.

$\boxtimes$

Whereas the previous proposition, Proposition 4.10, reduces the Definition 4.1 to $O(n^4)$ matrices (more precisely for its basic version for $n \geq 4$ it reduces the problem to $\binom{n}{2} \cdot (n-1)^2$ real instances), the following uses a bit different approach and achieves to reduce the definition to $O(n^3)$ (respectively to $O(n^3) + O(n^2)$, and even more precisely to $n^2 \cdot (n-1) + n^2 = n^3$) matrices.

Proposition 4.13. *Let $\boldsymbol{A} \in \mathbb{IR}^{n \times n}$ and let us define $A_{(i,k),(*,l)}$ and ${}_iA_{(*,l)} \in \mathbb{R}^{n \times n}$ as follows:*

$$A_{(i,k),(*,l)} = (a_{m_1 m_2})\,; \quad a_{m_1 m_2} = \begin{cases} \overline{a}_{ik} & \text{if } (m_1, m_2) = (i, k), \\ \overline{a}_{m_1 l} & \text{if } m_2 = l \wedge m_1 \neq i \wedge m_1 \neq l, \\ \underline{a}_{m_1 m_2} & \text{otherwise.} \end{cases}$$

and

$${}_iA_{(*,l)} = \left(a'_{m_1 m_2}\right); \quad a'_{m_1 m_2} = \begin{cases} \overline{a}_{m_1 l} & \text{if } m_2 = l \wedge m_1 \neq i \wedge m_1 \neq l, \\ \underline{a}_{m_1 m_2} & \text{otherwise.} \end{cases}$$

Then $\boldsymbol{A}$ is an interval doubly B-matrix if and only if $\forall i, l \in [n] : ({}_iA_{(*,l)}$ is a doubly B-matrix $\quad\wedge\quad \forall k \in [n] \setminus \{i\} : A_{(i,k),(*,l)}$ is a doubly B-matrix).

Proof. "⇒" Trivial, for all such matrices are in $\boldsymbol{A}$.

"⇐" We will prove that the conditions of Theorem 4.3 hold:

a) $\forall i \in [n], \forall k \neq i: \quad \underline{a}_{ii} > \max\{0, \overline{a}_{ik}\}$, because for any arbitrary l the matrix $A_{(i,k),(*,l)}$ is a doubly B-matrix. Thus $\forall i \in [n]: \quad \underline{a}_{ii} > \max\{0, \overline{a}_{ik} | k \neq i\}$.

b) Let us fix arbitrary $i, j \in [n], j \neq i$ and arbitrary $k, l \in [n], k \neq i, l \neq j$:

I. Let us take $A = A_{(i,k),(*,l)}$. Then:

$$\begin{aligned} \Big(\underline{a}_{ii} - \overline{a}_{ik}\Big)\Big(\underline{a}_{jj} - \overline{a}_{jl}\Big) \quad &\geq \quad \left(a_{ii} - r_i^+\right)\left(a_{jj} - r_j^+\right) \quad > \\ > \quad &\left(\sum_{m \neq i}\left(r_i^+ - a_{im}\right)\right)\left(\sum_{m \neq j}\left(r_j^+ - a_{jm}\right)\right) \quad \geq \\ \geq \quad &\left(\max\left\{0, \sum_{\substack{m \neq i \\ m \neq k}}\left(\overline{a}_{ik} - \underline{a}_{im}\right)\right\}\right)\left(\max\left\{0, \sum_{\substack{m \neq j \\ m \neq l}}\left(\overline{a}_{jl} - \underline{a}_{jm}\right)\right\}\right) \end{aligned}$$

II. Let us take $A = {}_iA_{(*,l)}$. Then:

$$\begin{aligned} \underline{a}_{ii}\Big(\underline{a}_{jj} - \overline{a}_{jl}\Big) \quad &\geq \quad \left(a_{ii} - r_i^+\right)\left(a_{jj} - r_j^+\right) \quad > \\ > \quad &\left(\sum_{m \neq i}\left(r_i^+ - a_{im}\right)\right)\left(\sum_{m \neq j}\left(r_j^+ - a_{jm}\right)\right) \quad \geq \\ \geq \quad &\left(\max\left\{0, -\sum_{m \neq i}\underline{a}_{im}\right\}\right)\left(\max\left\{0, \sum_{\substack{m \neq j \\ m \neq l}}\left(\overline{a}_{jl} - \underline{a}_{jm}\right)\right\}\right) \end{aligned}$$

III. Let us take $A = {}_iA_{(*,j)}$. Then:

$$\begin{aligned} \underline{a}_{ii} \cdot \underline{a}_{jj} \quad &\geq \quad \left(a_{ii} - r_i^+\right)\left(a_{jj} - r_j^+\right) \quad > \\ > \quad &\left(\sum_{m \neq i}\left(r_i^+ - a_{im}\right)\right)\left(\sum_{m \neq j}\left(r_j^+ - a_{jm}\right)\right) \quad \geq \\ \geq \quad &\left(\max\left\{0, -\sum_{m \neq i}\underline{a}_{im}\right\}\right)\left(\max\left\{0, -\sum_{m \neq j}\underline{a}_{jm}\right\}\right) \end{aligned}$$

So, as we have proved, the $\boldsymbol{A}$ fulfills both the conditions of characterization stated in Theorem 4.3, thus it is an interval doubly B-matrix.

⊠

4.2 Necessary or sufficient conditions

Here, in this section, we will derive a few necessary or sufficient conditions that might help us with even quicker recognition of a class of interval doubly B-matrices.

Let us start with two necessary conditions, one through reduction to some instances, the other through properties the matrix must have.

Proposition 4.14. *Let* $\boldsymbol{A} \in \mathbb{IR}^{n\times n}, \forall i \in [n] : k_i \in \operatorname{argmax}\{\overline{a}_{ij} | j \neq i\}$ *and let us define* ${}_iA_{\max} \in \mathbb{R}^{n\times n}$ *as follows:*

$$ {}_iA_{\max} = (a_{m_1m_2})\,; \quad a_{m_1m_2} = \begin{cases} \overline{a}_{m_1k_{m_1}} & \text{if } m_1 \neq i \wedge m_2 = k_{m_1}, \\ \underline{a}_{m_1m_2} & \text{otherwise.} \end{cases} $$

Then $\boldsymbol{A}$ *is an interval doubly B-matrix only if* $\underline{A}$ *and* $\forall i \in [n] : {}_iA_{\max}$ *are doubly B-matrices.*

Proof. It holds that $\underline{A} \in \boldsymbol{A} \wedge \forall i \in [n] : {}_iA_{\max} \in \boldsymbol{A}$. ⊠

Proposition 4.14 gives us quite nice necessary condition through reduction, but to compute it, we have to verify $n+1$ matrices whether they are doubly B-matrices, which takes us verifying $O(n^2)$ inequalities for each. Hence together the time complexity would be $O(n^3)$. So let us state an equivalent condition with better time complexity, more precisely with $O(n^2)$ complexity.

Proposition 4.15. *Let* $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$. *Then* $\boldsymbol{A}$ *is a doubly B-matrix only if the following hold:*

a) $\forall i \in [n] : \quad \underline{a}_{ii} > \max\{0, \overline{a}_{ij} | j \neq i\}$ *and*

b) $\forall i, j \in [n], j \neq i, k \in \operatorname{argmax}\{\overline{a}_{im} | m \neq i\}, l \in \operatorname{argmax}\{\overline{a}_{jm} | m \neq j\}$:

I. $(\overline{a}_{ik} > 0 \quad \wedge \quad \overline{a}_{jl} > 0) \quad \Rightarrow$

$$ \Big(\underline{a}_{ii} - \overline{a}_{ik}\Big)\Big(\underline{a}_{jj} - \overline{a}_{jl}\Big) > \left(\sum_{\substack{m\neq i \\ m\neq k}} \Big(\overline{a}_{ik} - \underline{a}_{im}\Big) \right) \left(\sum_{\substack{m\neq j \\ m\neq l}} \Big(\overline{a}_{jl} - \underline{a}_{jm}\Big) \right) $$

II. $(\overline{a}_{ik} \leq 0 \quad \wedge \quad \overline{a}_{jl} > 0) \quad \Rightarrow$

$$ \underline{a}_{ii}\Big(\underline{a}_{jj} - \overline{a}_{jl}\Big) > \left(-\sum_{m\neq i} \underline{a}_{im} \right) \left(\sum_{\substack{m\neq j \\ m\neq l}} \Big(\overline{a}_{jl} - \underline{a}_{jm}\Big) \right) $$

III. $(\overline{a}_{ik} \leq 0 \quad \wedge \quad \overline{a}_{jl} \leq 0) \quad \Rightarrow$

$$ \underline{a}_{ii} \cdot \underline{a}_{jj} > \left(-\sum_{m\neq i} \underline{a}_{im} \right) \left(-\sum_{m\neq j} \underline{a}_{jm} \right) $$

Proof. We asssume that $\forall A \in \boldsymbol{A} : A$ is a doubly B-matrix. Therefore our condition $a)$ follows from condition $a)$ from the Definition 2.14 for $A \in \boldsymbol{A}$:

$$ A = (a_{m_1m_2}); \quad a_{m_1m_2} = \begin{cases} \overline{a}_{m_1m_1} & \text{if } m_1 = m_2, \\ \underline{a}_{m_1m_2} & \text{otherwise.} \end{cases} $$

To prove condition $b)$, let us take arbitrary $i, j \in [n], j \neq i$ and let $k \in \operatorname{argmax}\{\overline{a}_{im} | m \neq i\}, l \in \operatorname{argmax}\{\overline{a}_{jm} | m \neq j\}$. Then:

I. Let $\overline{a}_{ik} > 0, \overline{a}_{jl} > 0$. Let us take such $A \in \boldsymbol{A}$, that

$$ A = (a_{m_1m_2}); \quad a_{m_1m_2} = \begin{cases} \overline{a}_{ik} & \text{if } (m_1, m_2) = (i, k), \\ \overline{a}_{jl} & \text{if } (m_1, m_2) = (j, l), \\ \underline{a}_{m_1m_2} & \text{otherwise.} \end{cases} $$

Then $r_i^+ = \overline{a}_{ik}, r_j^+ = \overline{a}_{jl}$, so the following holds:

$$\begin{aligned} \Big(\underline{a}_{ii} - \overline{a}_{ik}\Big)\Big(\underline{a}_{jj} - \overline{a}_{jl}\Big) &= \left(a_{ii} - r_i^+\right)\left(a_{jj} - r_j^+\right) > \\ > \left(\sum_{m\neq i}\left(r_i^+ - a_{im}\right)\right)\left(\sum_{m\neq j}\left(r_j^+ - a_{jm}\right)\right) &= \\ = \left(\sum_{\substack{m\neq i\\ m\neq k}}\left(\overline{a}_{ik} - \underline{a}_{im}\right)\right)\left(\sum_{\substack{m\neq j\\ m\neq l}}\left(\overline{a}_{jl} - \underline{a}_{jm}\right)\right) \end{aligned}$$

The last equality arises from the fact that $a_{ik} = r_i^+$, so $r_i^+ - a_{ik} = 0$, and that $\forall m \neq i : r_i^+ \geq a_{im}$. (And of course analogies of that hold for j as well.)

II. Let $\overline{a}_{ik} \leq 0, \overline{a}_{jl} > 0$. Let us take such $A \in \boldsymbol{A}$, that

$$A = (a_{m_1m_2}); \quad a_{m_1m_2} = \begin{cases} \overline{a}_{jl} & \text{if } (m_1, m_2) = (j, l), \\ \underline{a}_{m_1m_2} & \text{otherwise.} \end{cases}$$

Then $r_i^+ = 0, r_j^+ = \overline{a}_{jl}$, so the following holds:

$$\begin{aligned} \underline{a}_{ii}\Big(\underline{a}_{jj} - \overline{a}_{jl}\Big) &= \left(a_{ii} - r_i^+\right)\left(a_{jj} - r_j^+\right) > \\ > \left(\sum_{m\neq i}\left(r_i^+ - a_{im}\right)\right)\left(\sum_{m\neq j}\left(r_j^+ - a_{jm}\right)\right) &= \\ = \left(-\sum_{m\neq i}\underline{a}_{im}\right)\left(\sum_{\substack{m\neq j\\ m\neq l}}\left(\overline{a}_{jl} - \underline{a}_{jm}\right)\right) \end{aligned}$$

III. Let $\overline{a}_{ik} \leq 0, \overline{a}_{jl} \leq 0$. Let us take $A = \underline{A} \in \boldsymbol{A}$. Then $r_i^+ = 0, r_j^+ = 0$, so the following holds:

$$\begin{aligned} \underline{a}_{ii} \cdot \underline{a}_{jj} &= \left(a_{ii} - r_i^+\right)\left(a_{jj} - r_j^+\right) > \\ > \left(\sum_{m\neq i}\left(r_i^+ - a_{im}\right)\right)\left(\sum_{m\neq j}\left(r_j^+ - a_{jm}\right)\right) &= \\ = \left(-\sum_{m\neq i}\underline{a}_{im}\right)\left(-\sum_{m\neq j}\underline{a}_{jm}\right) \end{aligned}$$

Thus if $\boldsymbol{A}$ is an interval doubly B-matrix, then our conditions hold.

⊠

Remark 4.16. The above mentioned semantic equivalence between Propositions 4.14 and 4.15 can be seen from the fact that in the proof of the second one we can use the matrices defined in the first:

In proof of the point *I*. of the *b*) condition: For given i, j, if we had taken ${}_xA_{\max}$ for some $x \neq i, x \neq j$ instead of the matrix that we used, it would have worked even so. (If we restrict our view on the two rows i and j, which we are interested in, the two matrices are the same.)

The same reasoning applies for the case that in the point *II*. of the *b*) condition for given i, j we would have used ${}_iA_{\max}$.

And as for the last part, the point *III.* of the *b*) condition, there we are already using one of the matrices from Proposition 4.14 and that is $\underline{A}$.

Ergo it can be seen that the conditions of Proposition 4.15 are together exactly just the rewritten condition of the real case (from Definition 2.14) for the matrices from Proposition 4.14.

Now let us take a closer look on varied sufficient conditions for being an interval doubly B-matrix.

First, let us demonstrate, for which matrices are the previous necessary conditions sufficient ones too. Then we shall look at a link between interval B- and doubly B-matrices, which will be analogous to Proposition 2.15. And after that we will show that for interval Z-matrices, it is quite easy to recognize, whether they are or are not interval doubly B-matrices.

Proposition 4.17. *Let* $\boldsymbol{A} \in \mathbb{IR}^{n\times n}, n \geq 3$, *such that it fulfills the following condition:*

$$\forall i \in [n]\ \exists k_i \in [n] \setminus \{i\}\ \forall j \in [n] \setminus \{i, k_i\} : \overline{a}_{ij} \leq \underline{a}_{ik_i}$$

Then $\boldsymbol{A}$ *is an interval doubly B-matrix if and only if it fulfills the necessary condition stated in Proposition 4.14.*

Proof. "$\Rightarrow$" Trivially from Proposition 4.14.

"$\Leftarrow$" $\forall i \in [n]$ be k_i from the assumption. Then $\forall A \in \boldsymbol{A} \forall i \in [n] : (r_i^+ > 0 \Rightarrow r_i^+ \leq \overline{a}_{ik_i} \wedge r_i^+ = a_{ik_i})$.

Let $A \in \boldsymbol{A}$ arbitrary.

a) From assumption it holds that $\forall i \in [n] : \quad a_{ii} \geq \underline{a}_{ii} > \max\{0, \overline{a}_{ij} | j \neq i\} \geq \max\{0, a_{ij} | j \neq i\}$.

b) Let us take arbitrary $i, j \in [n], j \neq i$. Let us distinguish the following cases:

1) $r_i^+ > 0 \wedge r_j^+ > 0$

Thus from assumption, $0 < \overline{a}_{ik_i}, 0 < \overline{a}_{jk_j}, r_i^+ = a_{ik_i}, r_j^+ = a_{jk_j}$ and so, because ${}_xA_{\max}$ for some $x \neq i, x \neq j$ is a doubly B-matrix from the assumption of this implication, the following applies.

$$\begin{aligned}
\left(a_{ii} - r_i^+\right)\left(a_{jj} - r_j^+\right) &\geq \left(\underline{a}_{ii} - \overline{a}_{ik_i}\right)\left(\underline{a}_{jj} - \overline{a}_{jk_j}\right) > \\
> &\left(\sum_{\substack{m\neq i\\ m\neq k_i}} \left(\overline{a}_{ik_i} - \underline{a}_{im}\right)\right)\left(\sum_{\substack{m\neq j\\ m\neq k_j}} \left(\overline{a}_{jk_j} - \underline{a}_{jm}\right)\right) \geq \\
\geq &\left(\sum_{\substack{m\neq i\\ m\neq k_i}} \left(r_i^+ - a_{im}\right)\right)\left(\sum_{\substack{m\neq j\\ m\neq k_j}} \left(r_j^+ - a_{jm}\right)\right) = \\
= &\left(\sum_{m\neq i} \left(r_i^+ - a_{im}\right)\right)\left(\sum_{m\neq j} \left(r_j^+ - a_{jm}\right)\right)
\end{aligned}$$

2) $r_i^+ = 0 \wedge r_j^+ > 0$

Thus from assumption, $\underline{a}_{ik_i} \leq 0, 0 < \overline{a}_{jk_j}, r_j^+ = a_{jk_j}$ and so, because ${}_iA_{\max}$ is a doubly B-matrix from the assumption of this implication, the following applies.

$$
\begin{aligned}
\left(a_{ii}-r_i^+\right)\left(a_{jj}-r_j^+\right) &\geq \underline{a}_{ii}\left(\underline{a}_{jj}-\overline{a}_{jk_j}\right) > \\
> \left(-\sum_{m\neq i}\underline{a}_{im}\right)\left(\sum_{\substack{m\neq j\\ m\neq k_j}}\left(\overline{a}_{jk_j}-\underline{a}_{jm}\right)\right) &\geq \\
\geq \left(\sum_{m\neq i}\left(r_i^+-a_{im}\right)\right)\left(\sum_{\substack{m\neq j\\ m\neq k_j}}\left(r_j^+-a_{jm}\right)\right) &= \\
= \left(\sum_{m\neq i}\left(r_i^+-a_{im}\right)\right)\left(\sum_{m\neq j}\left(r_j^+-a_{jm}\right)\right)
\end{aligned}
$$

3) $r_i^+ > 0 \wedge r_j^+ = 0$

By swapping i for j we get the previous case.

4) $r_i^+ = 0 \wedge r_j^+ = 0$

Thus from assumption, $\underline{a}_{ik_i} \leq 0, \underline{a}_{jk_j} \leq 0$ and so, because $\underline{A}$ is a doubly B-matrix from the assumption of this implication, the following applies.

$$
\begin{aligned}
\left(a_{ii}-r_i^+\right)\left(a_{jj}-r_j^+\right) &\geq \underline{a}_{ii}\cdot\underline{a}_{jj} > \\
> \left(-\sum_{m\neq i}\underline{a}_{im}\right)\left(-\sum_{m\neq j}\underline{a}_{jm}\right) &\geq \\
\geq \left(\sum_{m\neq i}\left(r_i^+-a_{im}\right)\right)\left(\sum_{m\neq j}\left(r_j^+-a_{jm}\right)\right)
\end{aligned}
$$

Therefore we have shown that in each case the matrix A is a doubly B-matrix, thus $\boldsymbol{A}$ is an interval doubly B-matrix.

$\boxtimes$

Theorem 4.18. *Let $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$ interval Z-matrix. Then $\boldsymbol{A}$ is an interval doubly B-matrix if and only if $\underline{A}$ is a doubly B-matrix.*

Proof. "⇒" Trivially, because $\underline{A} \in \boldsymbol{A}$.

"⇐" Let $A \in \boldsymbol{A}$. Then

a) From assumption it holds that $\forall i \in [n]: \quad a_{ii} \geq \underline{a}_{ii} > \max\{0, \underline{a}_{ij} | j \neq i\} = 0 = \max\{0, \overline{a}_{ij} | j \neq i\} \geq \max\{0, a_{ij} | j \neq i\}$, because $\underline{A}$ is a doubly B-matrix and also a Z-matrix.

b) Let $i, j \in [n], j \neq i$ arbitrary.

$$
\begin{aligned}
\left(a_{ii}-r_i^+\right)\left(a_{jj}-r_j^+\right) &\geq \left(\underline{a}_{ii}-0\right)\left(\underline{a}_{jj}-0\right) > \\
> \left(\sum_{m\neq i}\left(0-\underline{a}_{im}\right)\right)\left(\sum_{m\neq j}\left(0-\underline{a}_{jm}\right)\right) &\geq \\
\geq \left(\sum_{m\neq i}\left(r_i^+-a_{im}\right)\right)\left(\sum_{m\neq j}\left(r_j^+-a_{jm}\right)\right)
\end{aligned}
$$

The first inequality uses just the fact that $r_i^+ = 0, r_j^+ = 0$ and that $\forall i, j \in [n] : a_{ij} \geq \underline{a}_{ij}$, the second one holds, because $\underline{A}$ is a doubly B-matrix and because it is a Z-matrix too. And the last inequality is again completely trivial using the same facts, as the first one.

Hence we have proved that every $A \in \boldsymbol{A}$ is a doubly B-matrix, therefore $\boldsymbol{A}$ is an interval doubly B-matrix.

⊠

Next we will take a look, what sufficient conditions are there for non-negative interval doubly B-matrices, and then we shall try to at least partially generalize them to general interval matrices.

Proposition 4.19. *Let $\boldsymbol{A} \in \mathbb{IR}_0^{+n\times n}, \forall i \in [n] : k_i \in \operatorname{argmax}\{\overline{a}_{ij} | j \neq i\}$ and let us define $\widetilde{A} \in \mathbb{IR}^{n\times n}$ as follows:*

$$\widetilde{A} = (\widetilde{a}_{m_1m_2})\,; \quad \widetilde{a}_{m_1m_2} = \begin{cases} \overline{a}_{m_1k_{m_1}} & \text{if } m_2 = k_{m_1}, \\ \underline{a}_{m_1m_1} & \text{if } m_2 = m_1, \\ 0 & \text{otherwise.} \end{cases}$$

If $\widetilde{A}$ is a doubly B-matrix, then $\boldsymbol{A}$ is an interval doubly B-matrix.

Proof. Let $A \in \boldsymbol{A}, i, j \in [n], j \neq i$ arbitrary. Then the $a)$ condition of Definition 2.14 is satisfied trivially ($\underline{a}_{ii} > \max\{0, \overline{a}_{ik_i}\} = \overline{a}_{ik_i}$) and as for the $b)$ condition:

$$\begin{aligned}
\left(a_{ii} - r_i^+\right)\left(a_{jj} - r_j^+\right) &\geq \left(\underline{a}_{ii} - \overline{a}_{ik_i}\right)\left(\underline{a}_{jj} - \overline{a}_{jk_j}\right) = \\
= \left(\widetilde{a}_{ii} - \widetilde{r}_i^+\right)\left(\widetilde{a}_{jj} - \widetilde{r}_j^+\right) &> \left(\sum_{m\neq i}\left(\widetilde{r}_i^+ - \widetilde{a}_{im}\right)\right)\left(\sum_{m\neq j}\left(\widetilde{r}_j^+ - \widetilde{a}_{jm}\right)\right) = \\
= \left(\sum_{\substack{m\neq i\\ m\neq k_i}} \overline{a}_{ik_i}\right)\left(\sum_{\substack{m\neq j\\ m\neq k_j}} \overline{a}_{jk_j}\right) &= \left((n-2)\cdot\overline{a}_{ik_i}\right)\left((n-2)\cdot\overline{a}_{jk_j}\right) \geq \\
\geq \left((n-2)\cdot r_i^+\right)\left((n-2)\cdot r_j^+\right) &\geq \left(\sum_{m\neq i}\left(r_i^+ - a_{im}\right)\right)\left(\sum_{m\neq j}\left(r_j^+ - a_{jm}\right)\right)
\end{aligned}$$

(Where $\widetilde{r}_i^+$ is r_i^+due to the matrix $\widetilde{A}$.)

The third inequality holds, because of the assumption that the $\widetilde{A}$ is a doubly B-matix. The last inequality arises from the assumption that $\boldsymbol{A} \in \mathbb{IR}_0^{+n\times n}$ and from the consequential fact that $\forall i \in [n] \exists l_i \neq i : a_{il_i} = r_i^+$. Plus the $\overline{a}_{ik_i}$ is the largest off-diagonal element in its row, so $r_i^+ \leq \overline{a}_{ik_i}$.

Hence we have shown that every $A \in \boldsymbol{A}$ is a doubly B-matrix, thus $\boldsymbol{A}$ is an interval doubly B-matrix.

⊠

Proposition 4.20. *Let $\boldsymbol{A} \in \mathbb{IR}_0^{+n\times n}, \forall i \in [n] : k_i \in \operatorname{argmax}\{\overline{a}_{ij} | j \neq i\}$ and $\forall i \in [n] : k_i' \in \operatorname{argmin}\{\underline{a}_{ij} | j \neq i\}$. Let us define $\widetilde{A} \in \mathbb{IR}^{n\times n}$ as follows:*

$$\widetilde{A} = (\widetilde{a}_{m_1m_2})\,; \quad \widetilde{a}_{m_1m_2} = \begin{cases} \overline{a}_{m_1k_{m_1}} & \text{if } m_2 = k_{m_1}, \\ \underline{a}_{m_1m_1} & \text{if } m_2 = m_1, \\ \underline{a}_{m_1k'_{m_1}} & \text{otherwise.} \end{cases}$$

If $\widetilde{A}$ is a doubly B-matrix, then $\boldsymbol{A}$ is an interval doubly B-matrix.

Proof. Let $A \in \boldsymbol{A}, i, j \in [n], j \neq i$ arbitrary. Then the a) condition of Definition 2.14 is satisfied trivially ($\underline{a}_{ii} > \max\{0, \overline{a}_{ik_i}\} = \overline{a}_{ik_i}$) and as for the b) condition:

$$\begin{aligned}
\left(a_{ii} - r_i^+\right)\left(a_{jj} - r_j^+\right) \quad &\geq \quad \left(\underline{a}_{ii} - \overline{a}_{ik_i}\right)\left(\underline{a}_{jj} - \overline{a}_{jk_j}\right) \quad = \\
= \quad \left(\tilde{a}_{ii} - \tilde{r}_i^+\right)\left(\tilde{a}_{jj} - \tilde{r}_j^+\right) \quad &> \quad \left(\sum_{m\neq i}\left(\tilde{r}_i^+ - \tilde{a}_{im}\right)\right)\left(\sum_{m\neq j}\left(\tilde{r}_j^+ - \tilde{a}_{jm}\right)\right) \quad = \\
= \quad \left(\sum_{\substack{m\neq i\\ m\neq k_i}}\left(\overline{a}_{ik_i} - \underline{a}_{ik'_i}\right)\right)&\left(\sum_{\substack{m\neq j\\ m\neq k_j}}\left(\overline{a}_{jk_j} - \underline{a}_{jk'_j}\right)\right) \quad = \\
= \quad \left((n-2)\cdot\left(\overline{a}_{ik_i} - \underline{a}_{ik'_i}\right)\right)&\left((n-2)\cdot\left(\overline{a}_{jk_j} - \underline{a}_{jk'_j}\right)\right) \quad \geq \\
\geq \quad \left(\sum_{m\neq i}\left(r_i^+ - a_{im}\right)\right)&\left(\sum_{m\neq j}\left(r_j^+ - a_{jm}\right)\right)
\end{aligned}$$

(Where $\tilde{r}_i^+$ is r_i^+due to the matrix $\widetilde{A}$.)

The third inequality holds, because of the assumption that the $\widetilde{A}$ is a doubly B-matix. The last inequality arises from the assumption that $\boldsymbol{A} \in \mathbb{IR}_0^{+n\times n}$ and from the consequential fact that $\forall i \in [n] \exists l_i \neq i : a_{il_i} = r_i^+$ plus from the definition of k'_i and $\widetilde{A}$. Plus the $\overline{a}_{ik_i}$ is the largest off-diagonal element in its row, so $r_i^+ \leq \overline{a}_{ik_i}$.

Hence we have shown that every $A \in \boldsymbol{A}$ is a doubly B-matrix, thus $\boldsymbol{A}$ is an interval doubly B-matrix.

⊠

Proposition 4.21. *Let* $\boldsymbol{A} \in \mathbb{IR}_0^{+n\times n}, \forall i \in [n] : k_i \in \operatorname{argmax}\{\overline{a}_{ij} | j \neq i\}$ *and let us define* $\widetilde{A} \in \mathbb{IR}^{n\times n}$ *as follows:*

$$\widetilde{A} = \left(\tilde{a}_{m_1 m_2}\right); \quad \tilde{a}_{m_1 m_2} = \begin{cases} \overline{a}_{m_1 k_{m_1}} & \text{if } m_2 = k_{m_1}, \\ \underline{a}_{m_1 m_1} & \text{if } m_2 = m_1, \\ \min\{\underline{a}_{m_1 m_2}, \underline{a}_{m_1 k_{m_1}}\} & \text{otherwise.} \end{cases}$$

If $\widetilde{A}$ *is a doubly B-matrix, then* $\boldsymbol{A}$ *is an interval doubly B-matrix.*

Proof. Let $A \in \boldsymbol{A}, i, j \in [n], j \neq i$ arbitrary. Then the a) condition of Definition 2.14 is satisfied trivially ($\underline{a}_{ii} > \max\{0, \overline{a}_{ik_i}\} = \overline{a}_{ik_i}$) and as for the b) condition, let $\forall i \in [n] : l_i \in \text{argmax}\{a_{im} | jm \neq i\}$, then $a_{il_i} = r_i^+$ and:

$$\begin{aligned}
\left(a_{ii} - r_i^+\right)\left(a_{jj} - r_j^+\right) \quad &\geq \quad \left(\underline{a}_{ii} - \overline{a}_{ik_i}\right)\left(\underline{a}_{jj} - \overline{a}_{jk_j}\right) \quad = \\
= \quad \left(\widetilde{a}_{ii} - \widetilde{r}_i^+\right)\left(\widetilde{a}_{jj} - \widetilde{r}_j^+\right) \quad &> \quad \left(\sum_{m \neq i} \left(\widetilde{r}_i^+ - \widetilde{a}_{im}\right)\right)\left(\sum_{m \neq j} \left(\widetilde{r}_j^+ - \widetilde{a}_{jm}\right)\right) \quad = \\
= \quad &\left(\sum_{\substack{m \neq i \\ m \neq k_i}} \left(\overline{a}_{ik_i} - \min\{\underline{a}_{im}, \underline{a}_{ik_i}\}\right)\right)\left(\sum_{\substack{m \neq j \\ m \neq k_j}} \left(\overline{a}_{jk_j} - \min\{\underline{a}_{jm}, \underline{a}_{jk_j}\}\right)\right) \quad \geq \\
\geq \quad &\left(\sum_{\substack{m \neq i \\ m \neq l_i}} \left(a_{il_i} - a_{im}\right)\right)\left(\sum_{\substack{m \neq j \\ m \neq l_j}} \left(a_{jl_j} - a_{jm}\right)\right) \quad = \\
= \quad &\left(\sum_{m \neq i} \left(r_i^+ - a_{im}\right)\right)\left(\sum_{m \neq j} \left(r_j^+ - a_{jm}\right)\right)
\end{aligned}$$

(Where $\widetilde{r}_i^+$ is r_i^+due to the matrix $\widetilde{A}$.)

The third inequality holds, because of the assumption that the $\widetilde{A}$ is a doubly B-matix. The fifth inequality is quite trivial too, it relies only on the following two facts: $\overline{a}_{ik_i} \geq a_{il_i}$ and for $m = l_i : a_{im} = \min\{\underline{a}_{im}, \underline{a}_{ik_i}\} \leq \underline{a}_{ik_i} \leq a_{ik_i}$. And the last equality holds, because $\forall i : a_{il_i} = r_i^+$.

Hence we have shown that every $A \in \boldsymbol{A}$ is a doubly B-matrix, thus $\boldsymbol{A}$ is an interval doubly B-matrix.

$\boxtimes$

Theorem 4.22. *Let $\boldsymbol{A} \in \mathbb{IR}^{n \times n}, \forall i \in [n] : k_i \in \text{argmax}\{\overline{a}_{ij} | j \neq i\}$ and $\forall i \in [n] : k_i' \in \text{argmax}\{\underline{a}_{ij} | j \neq i\}$. Let us define $\widetilde{A} \in \mathbb{IR}^{n \times n}$ as follows:*

$$\widetilde{A} = \left(\widetilde{a}_{m_1 m_2}\right); \quad \widetilde{a}_{m_1 m_2} = \begin{cases} \overline{a}_{m_1 k_{m_1}} & \text{if } m_2 = k_{m_1}, \\ \underline{a}_{m_1 m_1} & \text{if } m_2 = m_1, \\ \min\{\underline{a}_{m_1 m_2}, \underline{a}_{m_1 k_{m_1}}\} & \text{otherwise.} \end{cases}$$

If $\forall i \in [n] : \underline{a}_{ik_i'} \geq 0$ and $\widetilde{A}$ is a doubly B-matrix, then $\boldsymbol{A}$ is an interval doubly B-matrix.

Proof. Let $A \in \boldsymbol{A}, i, j \in [n], j \neq i$ arbitrary. Then $\overline{a}_{ik_i} \geq 0$, because $\overline{a}_{ik_i} \geq \underline{a}_{ik_i'} \geq 0$ from the assumption and the definition of k_i (analogically for j). And so the *a*) condition of the Definition 2.14 is satisfied trivially ($\underline{a}_{ii} > \max\{0, \overline{a}_{ik_i}\} = \overline{a}_{ik_i}$) and as for the *b*) condition, let $\forall i \in [n] : l_i \in \text{argmax}\{a_{im} | m \neq i\}$, then, because

$\max\{a_{im}|m \neq i\} \geq \underline{a}_{ik'_i} \geq 0$, it holds that $a_{il_i} = r_i^+$. Hence:

$$
\begin{aligned}
\left(a_{ii} - r_i^+\right)\left(a_{jj} - r_j^+\right) \quad &\geq \quad \left(\underline{a}_{ii} - \overline{a}_{ik_i}\right)\left(\underline{a}_{jj} - \overline{a}_{jk_j}\right) \quad = \\
= \quad \left(\tilde{a}_{ii} - \tilde{r}_i^+\right)\left(\tilde{a}_{jj} - \tilde{r}_j^+\right) \quad &> \quad \left(\sum_{m\neq i}\left(\tilde{r}_i^+ - \tilde{a}_{im}\right)\right)\left(\sum_{m\neq j}\left(\tilde{r}_j^+ - \tilde{a}_{jm}\right)\right) \quad = \\
= \quad \left(\sum_{\substack{m\neq i\\ m\neq k_i}}\left(\overline{a}_{ik_i} - \min\{\underline{a}_{im}, \underline{a}_{ik_i}\}\right)\right)&\left(\sum_{\substack{m\neq j\\ m\neq k_j}}\left(\overline{a}_{jk_j} - \min\{\underline{a}_{jm}, \underline{a}_{jk_j}\}\right)\right) \quad \geq \\
\geq \quad \left(\sum_{\substack{m\neq i\\ m\neq l_i}}\left(a_{il_i} - a_{im}\right)\right)&\left(\sum_{\substack{m\neq j\\ m\neq l_j}}\left(a_{jl_j} - a_{jm}\right)\right) \quad = \\
= \quad \left(\sum_{m\neq i}\left(r_i^+ - a_{im}\right)\right)&\left(\sum_{m\neq j}\left(r_j^+ - a_{jm}\right)\right)
\end{aligned}
$$

(Where $\tilde{r}_i^+$ is r_i^+due to the matrix $\tilde{A}$.)

The third inequality holds, because of the assumption that the $\tilde{A}$ is a doubly B-matix. The fifth inequality is quite trivial too, it relies only on the following two facts: $\overline{a}_{ik_i} \geq a_{il_i} \geq 0$ and for $m = l_i : a_{im} = \min\{\underline{a}_{im}, \underline{a}_{ik_i}\} \leq \underline{a}_{ik_i} \leq a_{ik_i}$. And the last equality holds, because $\forall i : a_{il_i} = r_i^+$.

Hence we have shown that every $A \in \boldsymbol{A}$ is a doubly B-matrix, thus $\boldsymbol{A}$ is an interval doubly B-matrix.

⊠

Remark 4.23. Propositions 4.19, 4.20 and 4.21 are gradually stronger variations of one another. Each deals with non-negative interval matrices, but each uses a better matrix, in the meaning that it is closer to being an instance of the examined interval matrix, then the former one. Proposition 4.19 uses such a matrix, whose examination whether it is a doubly B-matrix is computationally simple, but which can be rather distant from any instance of original interval matrix. Proposition 4.20 uses a bit better matrix, which can be however quite simple to examine too, because in every row almost all the elements are the same even though not necessarily zero, as in the first case. And the third proposition, Proposition 4.21, then uses much better matrix, although it can be a bit harder to examine, because now in every row, every element can be different.

And then there is one more result, Theorem 4.22, which takes the approach of Proposition 4.21 and generalizes it for much wider range of matrices, where instead of complete non-negativity, we just require the existence of a non-diagonal element, which is always non-negative, thus which is a non-negative interval.

Now, what we could be interested in, is for what matrices is the sufficient condition from Theorem 4.22 characterization as well. So let us take a look at that.

Proposition 4.24. *Let* $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$ *such that* $\forall i \in [n]\ \exists k_i \in [n] \setminus \{i\} : \quad \overline{a}_{ik_i} = \max\{\overline{a}_{ij}|j \neq i\} \ \wedge \ \underline{a}_{ik_i} = \max\{0, \underline{a}_{ij}|j \neq i\}$. *Then the sufficient condition stated in Theorem 4.22 is a characterization for* $\boldsymbol{A}$.

Proof. "$\widetilde{A}$ is a doubly B-matrix, then $\boldsymbol{A}$ is an interval doubly B-matrix": Follows from Theorem 4.22.

"$\boldsymbol{A}$ is an interval doubly B-matrix, then $\widetilde{A}$ is a doubly B-matrix": From construction of $\widetilde{A}$ and from assumtions of this proposition it follows that $\widetilde{A} \in \boldsymbol{A}$.

⊠

Next we will show that if lower and upper bound matrices of an interval matrix are circulant, then some nice properties applies.

Theorem 4.25. *Let* $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$ *such that* $\underline{A}$ *and* $\overline{A}$ *are circulant. Then the following is equivalent:*

1) $\boldsymbol{A}$ *is an interval doubly B-matrix*

2) $\boldsymbol{A}$ *is an interval B-matrix*

3) *It holds that*

$$\begin{aligned} &a) \quad \underline{a}_{11} > -\sum_{j\neq 1} \underline{a}_{1j} \\ &b) \quad \forall k \in [n] \setminus \{1\} : \ \underline{a}_{11} - \overline{a}_{1k} > \sum_{\substack{j\neq 1 \\ j\neq k}} \left(\overline{a}_{1k} - \underline{a}_{1j}\right) \end{aligned}$$

Proof. "1) ⇒ 2)" $\boldsymbol{A}$ is a doubly B-matrix, thus it satisfies the characterization given in Theorem 4.3. Let $i \in [n]$ and $k \neq i$ arbitrary. Se let us choose

$$j = \begin{cases} i+1 & \text{if } i < n, \\ 1 & \text{if } i = n. \end{cases} \quad \text{and } l = \begin{cases} k+1 & \text{if } k < n, \\ 1 & \text{if } k = n. \end{cases}$$

(Then $\underline{a}_{ii} = \underline{a}_{jj}$ and $\overline{a}_{ik} = \overline{a}_{jl}$, because both $\underline{A}$ and $\overline{A}$ are circulant.) Hence, because $\boldsymbol{A}$ is an interval doubly B-matrix:

$$\begin{aligned} &\underline{a}_{ii} \cdot \underline{a}_{jj} > \left(\max\left\{0, -\sum_{m\neq i} \underline{a}_{im}\right\}\right)\left(\max\left\{0, -\sum_{m\neq j} \underline{a}_{jm}\right\}\right) \quad \Leftrightarrow \\ \Leftrightarrow \quad &\underline{a}_{ii}^2 > \left(\max\left\{0, -\sum_{m\neq i} \underline{a}_{im}\right\}\right)^2 \quad \Rightarrow \\ \Rightarrow \quad &\underline{a}_{ii} = |\underline{a}_{ii}| > \left|\max\left\{0, -\sum_{m\neq i} \underline{a}_{im}\right\}\right| = \max\left\{0, -\sum_{m\neq i} \underline{a}_{im}\right\} \geq -\sum_{m\neq i} \underline{a}_{im} \quad \Rightarrow \\ \Rightarrow \quad &\sum_{m=1}^{n} \underline{a}_{im} > 0 \end{aligned}$$

Therefore the *a*) condition of Corollary 3.11 is satisfied. Let us take a look at the second one, the $b)$ condition:

$$\begin{aligned}
&\left(\underline{a}_{ii} - \overline{a}_{ik}\right)\left(\underline{a}_{jj} - \overline{a}_{jl}\right) > \\
&\qquad > \left(\max\left\{0, \sum_{\substack{m\neq i\\ m\neq k}} \left(\overline{a}_{ik} - \underline{a}_{im}\right)\right\}\right)\left(\max\left\{0, \sum_{\substack{m\neq j\\ m\neq l}} \left(\overline{a}_{jl} - \underline{a}_{jm}\right)\right\}\right) \quad \Leftrightarrow \\
&\Leftrightarrow \quad \left(\underline{a}_{ii} - \overline{a}_{ik}\right)^2 > \left(\max\left\{0, \sum_{\substack{m\neq i\\ m\neq k}} \left(\overline{a}_{ik} - \underline{a}_{im}\right)\right\}\right)^2 \quad \Rightarrow \\
&\Rightarrow \quad \underline{a}_{ii} - \overline{a}_{ik} = |\underline{a}_{ii} - \overline{a}_{ik}| > \left|\max\left\{0, \sum_{\substack{m\neq i\\ m\neq k}} \left(\overline{a}_{ik} - \underline{a}_{im}\right)\right\}\right| = \\
&\qquad = \max\left\{0, \sum_{\substack{m\neq i\\ m\neq k}} \left(\overline{a}_{ik} - \underline{a}_{im}\right)\right\} \geq \sum_{\substack{m\neq i\\ m\neq k}} \left(\overline{a}_{ik} - \underline{a}_{im}\right) \quad \Rightarrow \\
&\Rightarrow \quad \underline{a}_{ii} - \overline{a}_{ik} > \sum_{\substack{m\neq i\\ m\neq k}} \left(\overline{a}_{ik} - \underline{a}_{im}\right)
\end{aligned}$$

Hence $\boldsymbol{A}$ fulfills the characterization of an interval B-matrix as stated in Corollary 3.11, thus it is an interval B-matrix.

"2) $\Rightarrow$ 3)" $\boldsymbol{A}$ is an interval B-matrix, so it satisfies the characterization given in Corollary 3.11. Hence for $i = 1$ it follows from condition $a)$ of the corollary that our condition $a)$ holds, and the same goes for the $b)$ conditions.

"3) $\Rightarrow$ 2)" From our $a)$ condition we know that row sum of the first row of matrix $\underline{A}$ is positive. And because $\underline{A}$ is circulant, all the row sums of this matrix are the same, thus positive. $\Rightarrow \quad \forall i \in [n] : \sum_{j=1}^{n} \underline{a}_{ij} > 0$

And because both $\underline{A}$ and $\overline{A}$ are circulant and from our condition $b)$, we get

$$\begin{aligned}
&\forall i \in [n]\ \forall k \neq 1\ \exists k_i \neq i : \\
&\qquad \underline{a}_{11} - \overline{a}_{1k} = \underline{a}_{ii} - \overline{a}_{ik_i} \quad \wedge \quad \sum_{\substack{j\neq 1\\ j\neq k}} \left(\overline{a}_{1k} - \underline{a}_{1j}\right) = \sum_{\substack{j\neq i\\ j\neq k_i}} \left(\overline{a}_{ik_i} - \underline{a}_{ij}\right).
\end{aligned}$$

$\Rightarrow \forall i \in [n]\ \forall k \neq i : \quad \underline{a}_{ii} - \overline{a}_{ik} > \sum_{\substack{j\neq i\\ j\neq k}} \left(\overline{a}_{ik} - \underline{a}_{ij}\right)$

Therefore $\boldsymbol{A}$ is an interval B-matrix, because it fulfills the conditions of characterization shown in Corollary 3.11.

"2) $\Rightarrow$ 1)" Trivial (see Proposition 4.7).

⊠

4.3 Closure properties

In this section, we will extend our understanding of what operations the class of B-matrices is closed under into interval environment.

Corollary 4.26. *Interval doubly B-matrices are closed under the multiplication by a positive scalar.*

Proof. It holds for every instance, so it holds for whole interval matrix.

(Every instance multiplied by positive scalar is an interval doubly B-matrix by Proposition 2.22 and that is exactly the definition of interval doubly B-matrix.)

⊠

Corollary 4.27. *Principal submatrices of interval doubly B-matrices are interval doubly B-matrices as well.*

Proof. It holds for every instance, so it holds for the whole interval matrix. (From Proposition 2.23.)

⊠

Remark 4.28. Because the real doubly B-matrices are not closed under the following operations (see Propositions 2.24, 2.25, 2.26 and 2.27), then surely even interval doubly B-matrices are in general not closed under them. Such operations are:

- matrix sum,
- matrix product,
- matrix inverse and
- matrix power.

Remark 4.29. As direct corollary of Definition 2.14, condition a), we can see that multiplying by an interval containing non-positive number can not be an operation that interval doubly B-matrices might be closed under.

Proposition 4.30. *In general, interval doubly B-matrices are not closed under multiplication by positive interval.*

Proof. Let

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

and $\boldsymbol{\alpha} = [1, 2]$. It is easy to verify that such A is an interval doubly B-matrix. But matrix

$$\boldsymbol{\alpha} \cdot A = \begin{pmatrix} [2,4] & [1,2] \\ [1,2] & [2,4] \end{pmatrix}$$

contains a (singular) non-doubly B-matrix instance, namely matrix

$$\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}.$$

⊠

5. Interval $\mathrm{B}_\pi^{\boldsymbol{R}}$-matrices

In this chapter we will generalize B_π^R-matrices into interval $\mathrm{B}_\pi^{\boldsymbol{R}}$-matrices. In sections 5.1 and 5.2 we shall introduce some characterizations, whereas in section 5.3 we will try to come to understanding of some fundamental properties of interval $\mathrm{B}_\pi^{\boldsymbol{R}}$-matrices. Then in section 5.4 we will take a look at closure properties of interval $\mathrm{B}_\pi^{\boldsymbol{R}}$-matrices.

Definition 5.1 (homogeneous interval $\mathrm{B}_\pi^{\boldsymbol{R}}$-matrix)**.** *Let $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$, $\pi \in \mathbb{R}^n$ such that $0 < \sum_{j=1}^n \pi_j \leq 1$ and $\boldsymbol{R} \in \mathbb{IR}^n$. Then we say that $\boldsymbol{A}$ is a homogeneous interval $B_\pi^{\boldsymbol{R}}$-matrix, if $\forall A \in \boldsymbol{A}$: $\exists R \in \boldsymbol{R}$ such that A is a (real) B_π^R-matrix.*

Here, the $\boldsymbol{R}$ in the definition can be perceived as the vector with its entries corresponding to the intervals of row sums of matrices $A \in \boldsymbol{A}$, but in the interval settings it is more of a symbol, than of any greater significance (unlike in the real case). That is because if we have two interval $\mathrm{B}_\pi^{\boldsymbol{R}}$-matrices $\boldsymbol{A}$ and $\boldsymbol{B}$, we cannot say that any two $A \in \boldsymbol{A}$ and $B \in \boldsymbol{B}$ are real B_π^R-matrices for the same R. (Moreover in general it is much more likely that the vectors of the row sums of A and B are different.)

Corollary 5.2. *Every homogeneous interval $B_\pi^{\boldsymbol{R}}$-matrix with $\pi \geq 0$ is an interval P-matrix.*

Proof. It holds for every instance, hence it holds for whole interval matrix.

(Every instance is P-matrix by Proposition 2.33 and that is exactly the definition of interval P-matrix.)

⊠

Definition 5.3 ((heterogeneous) interval $\mathrm{B}_\Pi^{\boldsymbol{R}}$-matrix)**.** *Let $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$ and $\boldsymbol{R} \in \mathbb{IR}^n$. Then we say that $\boldsymbol{A}$ is a (heterogeneous) interval $B_\Pi^{\boldsymbol{R}}$-matrix, if $\forall A \in \boldsymbol{A}$: $\exists R \in \boldsymbol{R}, \exists \pi \in \mathbb{R}^n$ such that $0 < \sum_{j=1}^n \pi_j \leq 1$: A is a (real) B_π^R-matrix.*

Corollary 5.4. *Every homogeneous interval $B_\pi^{\boldsymbol{R}}$-matrix is an interval $B_\Pi^{\boldsymbol{R}}$-matrix.*

Proof. Trivially follows from the definitions.

⊠

5.1 Characterizations of homogeneous interval $\mathrm{B}_\pi^{\boldsymbol{R}}$-matrices

Let us start with stating a characterization that will help us with recognition of the class of homogeneous interval $\mathrm{B}_\pi^{\boldsymbol{R}}$-matrices in finite time. Thereafter, in subsection 5.1.1, we will try to derive a characterization through reduction of Definition 5.1 to finite number of instances to check for the property of being a real B_π^R-matrix.

Theorem 5.5. *Let $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$, let $\pi \in \mathbb{R}^n$ such that $0 < \sum_{j=1}^n \pi_j \leq 1$ and $\boldsymbol{R} \in \mathbb{IR}^n$ be a vector of intervals of the individual row sums in matrix $\boldsymbol{A}$. Then $\boldsymbol{A}$ is* *a homogeneous interval $B_\pi^{\boldsymbol{R}}$-matrix if and only if $\forall i \in [n]$ the following properties hold:*

$$
\begin{aligned}
&a) \quad \underline{R}_i > 0 \\
&b) \quad \forall k \in [n] \setminus \{i\} : \\
&\left(\pi_k > 1 \quad \Rightarrow \quad \sum_{j \neq k} \underline{a}_{ij} \ > \ \left(\frac{1}{\pi_k} - 1 \right) \cdot \underline{a}_{ik} \right) \quad \wedge \\
\wedge \quad &\left(0 < \pi_k \leq 1 \quad \Rightarrow \quad \sum_{j \neq k} \underline{a}_{ij} \ > \ \left(\frac{1}{\pi_k} - 1 \right) \cdot \overline{a}_{ik} \right) \quad \wedge \\
\wedge \quad &\left(\pi_k = 0 \quad \Rightarrow \quad 0 \ > \ \overline{a}_{ik} \right) \quad \wedge \\
\wedge \quad &\left(\pi_k < 0 \quad \Rightarrow \quad \sum_{j \neq k} \overline{a}_{ij} \ < \ \left(\frac{1}{\pi_k} - 1 \right) \cdot \overline{a}_{ik} \right)
\end{aligned}
$$

Proof. For any $A \in \mathbb{R}^{n \times n}$ to be a B_π^R-matrix, it must, according to Definition 2.28, fulfill the following two conditions for every $i \in [n]$:

$$
\begin{aligned}
&a) \quad R_i > 0 \\
&b) \quad \forall k \in [n] \setminus \{i\} : \quad \pi_k \cdot R_i > a_{ik}
\end{aligned}
$$

Thus $\boldsymbol{A} \in \mathbb{IR}^{n \times n}$ is a homogeneous interval $\mathrm{B}_\pi^{\boldsymbol{R}}$-matrix if and only if $\forall A \in \boldsymbol{A}$: A fulfills the conditions above. From here we get that condition a) evaluated for every $A \in \boldsymbol{A}$ is equivalent to $\underline{R}_i > 0$. And as for the b) condition, it can be for every $k \neq i$ modified as follows (while notting that $\pi_k \cdot R_i = \pi_k \cdot \sum_{j=1}^n a_{ij}$):

1) $\pi_k > 1$:

$$
\begin{aligned}
&\pi_k \cdot \sum_{j=1}^n a_{ij} > a_{ik} \quad \Leftrightarrow \\
\Leftrightarrow \quad &\sum_{j=1}^n a_{ij} > \frac{1}{\pi_k} \cdot a_{ik} \quad \Leftrightarrow \\
\Leftrightarrow \quad &\sum_{j \neq k} a_{ij} > \left(\frac{1}{\pi_k} - 1 \right) \cdot a_{ik}
\end{aligned}
$$

We can observe that, because when $\pi_k > 1$, then $\left(\frac{1}{\pi_k} - 1 \right) < 0$, the biggest value of $\left(\frac{1}{\pi_k} - 1 \right) \cdot a_{ik}$ is obtained by the lower bound of the $\boldsymbol{a}_{ik}$. Ergo the condition above holds for every $A \in \boldsymbol{A}$ if and only if the following condition holds:

$$
\sum_{j \neq k} \underline{a}_{ij} > \left(\frac{1}{\pi_k} - 1 \right) \cdot \underline{a}_{ik}
$$

2) $0 < \pi_k \leq 1$:

$$\begin{aligned} \pi_k \cdot \sum_{j=1}^{n} a_{ij} > a_{ik} \quad &\Leftrightarrow \\ \Leftrightarrow \quad \sum_{j=1}^{n} a_{ij} > \frac{1}{\pi_k} \cdot a_{ik} \quad &\Leftrightarrow \\ \Leftrightarrow \quad \sum_{j \neq k} a_{ij} > \left(\frac{1}{\pi_k} - 1\right) \cdot a_{ik} & \end{aligned}$$

We can observe that, because when $0 < \pi_k \leq 1$, then $\left(\frac{1}{\pi_k} - 1\right) \geq 0$, the biggest value of $\left(\frac{1}{\pi_k} - 1\right) \cdot a_{ik}$ is obtained by the upper bound of the $\boldsymbol{a}_{ik}$. Ergo the condition above holds for every $A \in \boldsymbol{A}$ if and only if the following condition holds:

$$\sum_{j \neq k} \underline{a}_{ij} > \left(\frac{1}{\pi_k} - 1\right) \cdot \overline{a}_{ik}$$

3) $\pi_k = 0$:

$$\begin{aligned} \pi_k \cdot \sum_{j=1}^{n} a_{ij} > a_{ik} \quad &\Leftrightarrow \\ \Leftrightarrow \quad 0 > a_{ik} & \end{aligned}$$

The condition above holds for every $A \in \boldsymbol{A}$ if and only if the following condition holds:

$$0 > \overline{a}_{ik}$$

4) $\pi_k < 0$:

$$\begin{aligned} \pi_k \cdot \sum_{j=1}^{n} a_{ij} > a_{ik} \quad &\Leftrightarrow \\ \Leftrightarrow \quad \sum_{j=1}^{n} a_{ij} < \frac{1}{\pi_k} \cdot a_{ik} \quad &\Leftrightarrow \\ \Leftrightarrow \quad \sum_{j \neq k} a_{ij} < \left(\frac{1}{\pi_k} - 1\right) \cdot a_{ik} & \end{aligned}$$

We can observe that, because when $\pi_k < 0$, then $\left(\frac{1}{\pi_k} - 1\right) < 0$, the smallest value of $\left(\frac{1}{\pi_k} - 1\right) \cdot a_{ik}$ is obtained by the upper bound of the $\boldsymbol{a}_{ik}$. Ergo the condition above holds for every $A \in \boldsymbol{A}$ if and only if the following condition holds:

$$\sum_{j \neq k} \overline{a}_{ij} < \left(\frac{1}{\pi_k} - 1\right) \cdot \overline{a}_{ik}$$

⊠

Remark 5.6. This characterization has time complexity $O(n^2)$, which is, surprisingly, the same as a characterization from the definition of the real case, Definition 2.28 (although the interval case has undoubtedly higher implementational complexity).

Let us now introduce an analogy of Proposition 2.30 for homogeneous interval $\mathrm{B}_{\pi}^{\boldsymbol{R}}$-matrices.

Theorem 5.7. *Let $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$ be an interval square matrix with positive row sums intervals (hence $\forall i \in [n] : \sum_{j=1}^{n} \underline{a}_{ij} > 0$). Then there exists a vector $\pi \in \mathbb{R}^n$ satisfying $0 < \sum_{j=1}^{n} \pi_j \leq 1$ such that $\boldsymbol{A}$ is a homogeneous interval $\mathrm{B}_{\pi}^{\boldsymbol{R}}$-matrix if and only if*

$$\sum_{j=1}^{n} \max\left\{ \frac{\overline{a}_{ij}}{\overline{a}_{ij} + \sum\limits_{m\neq j} \underline{a}_{im}}, \frac{\underline{a}_{ij}}{\underline{a}_{ij} + \sum\limits_{m\neq j} \underline{a}_{im}} \middle| i \neq j \right\} < 1.$$

Proof. "$\Rightarrow$": $\boldsymbol{A}$ is a $\mathrm{B}_{\pi}^{\boldsymbol{R}}$-matrix for some π satisfying the property, hence every $A \in \boldsymbol{A}$ is a B_{π}^{R}-matrix, thus even matrix $A' \in \boldsymbol{A}$ defined as follows:

$$A' = (a'_{m_1 m_2}); \quad a'_{m_1 m_2} = \begin{cases} \overline{a}_{m_1 m_2} & \text{if } j = m_2 \ \wedge \ \frac{\overline{a}_{m_1 j}}{\overline{a}_{m_1 j} + \sum\limits_{m\neq j} \underline{a}_{m_1 m}} > \frac{\underline{a}_{m_1 j}}{\underline{a}_{m_1 j} + \sum\limits_{m\neq j} \underline{a}_{m_1 m}}, \\ \underline{a}_{m_1 m_2} & \text{otherwise.} \end{cases}$$

Therefore (if we denote R' the vector of row sums of A') it holds that

$$\forall j \in [n] : \ \max\left\{ \frac{\overline{a}_{ij}}{\overline{a}_{ij} + \sum\limits_{m\neq j} \underline{a}_{im}}, \frac{\underline{a}_{ij}}{\underline{a}_{ij} + \sum\limits_{m\neq j} \underline{a}_{im}} \middle| i \neq j \right\} = \max\left\{ \frac{a'_{ij}}{R'_i} \middle| i \neq j \right\} < \pi_j.$$

But then

$$\sum_{j=1}^{n} \max\left\{ \frac{\overline{a}_{ij}}{\overline{a}_{ij} + \sum\limits_{m\neq j} \underline{a}_{im}}, \frac{\underline{a}_{ij}}{\underline{a}_{ij} + \sum\limits_{m\neq j} \underline{a}_{im}} \middle| i \neq j \right\} < \sum_{j=1}^{n} \pi_j \leq 1.$$

"$\Leftarrow$": Let

$$\epsilon = 1 - \sum_{j=1}^{n} \max\left\{ \frac{\overline{a}_{ij}}{\overline{a}_{ij} + \sum\limits_{m\neq j} \underline{a}_{im}}, \frac{\underline{a}_{ij}}{\underline{a}_{ij} + \sum\limits_{m\neq j} \underline{a}_{im}} \middle| i \neq j \right\} > 0$$

and for every $j \in [n]$ set the $\pi_j = \max\left\{ \frac{\overline{a}_{ij}}{\overline{a}_{ij} + \sum\limits_{m\neq j} \underline{a}_{im}}, \frac{\underline{a}_{ij}}{\underline{a}_{ij} + \sum\limits_{m\neq j} \underline{a}_{im}} \middle| i \neq j \right\} + \frac{\epsilon}{n}$. Then $\boldsymbol{A}$ is a homogeneous interval $\mathrm{B}_{\pi}^{\boldsymbol{R}}$-matrix. That is because for any $A \in \boldsymbol{A}$

$$\begin{aligned} \max\left\{ \frac{\overline{a}_{ij}}{\overline{a}_{ij} + \sum\limits_{m\neq j} \underline{a}_{im}}, \frac{\underline{a}_{ij}}{\underline{a}_{ij} + \sum\limits_{m\neq j} \underline{a}_{im}} \middle| i \neq j \right\} \quad &\geq \\ \geq \quad \max\left\{ \frac{a_{ij}}{a_{ij} + \sum\limits_{m\neq j} \underline{a}_{im}} \middle| i \neq j \right\} \quad &\geq \quad \max\left\{ \frac{a_{ij}}{\sum\limits_{m=1}^{n} a_{im}} \middle| i \neq j \right\}, \end{aligned}$$

thus for every $A \in \boldsymbol{A}$ and for every $k, j \in [n], j \neq k$, it holds that

$$\begin{aligned} \frac{a_{kj}}{R_k} &= \frac{a_{kj}}{\sum\limits_{m=1}^{n} a_{km}} \leq \max\left\{ \frac{a_{ij}}{\sum\limits_{m=1}^{n} a_{im}} \middle| i \neq j \right\} \leq \\ &\leq \max\left\{ \frac{\overline{a}_{ij}}{\overline{a}_{ij} + \sum\limits_{m\neq j} \underline{a}_{im}}, \frac{\underline{a}_{ij}}{\underline{a}_{ij} + \sum\limits_{m\neq j} \underline{a}_{im}} \middle| i \neq j \right\} < \\ &< \max\left\{ \frac{\overline{a}_{ij}}{\overline{a}_{ij} + \sum\limits_{m\neq j} \underline{a}_{im}}, \frac{\underline{a}_{ij}}{\underline{a}_{ij} + \sum\limits_{m\neq j} \underline{a}_{im}} \middle| i \neq j \right\} + \frac{\epsilon}{n} = \pi_j, \end{aligned}$$

ergo $\pi_k \cdot R_k > a_{kj}$. Therefore every $A \in \boldsymbol{A}$ is a B_π^R-matrix

⊠

Remark 5.8. And once again, analogously to what is shown in Remark 2.31 for the real case of B_π^R-matrices (and partly just as shown in the proof of Theorem 5.7), if for any matrix $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$ the condition from the Theorem 5.7 is satisfied, then we can construct a vector $\pi \in \mathbb{R}^n$ satisfying $0 < \sum_{j=1}^n \pi_j \leq 1$ such that $\boldsymbol{A}$ is a homogeneous interval $\mathrm{B}_\pi^{\boldsymbol{R}}$-matrix in the following manner:

1) We define $\epsilon \in \mathbb{R}$ as

$$\epsilon = 1 - \sum_{j=1}^{n} \max\left\{ \frac{\overline{a}_{ij}}{\overline{a}_{ij} + \sum\limits_{m\neq j} \underline{a}_{im}}, \frac{\underline{a}_{ij}}{\underline{a}_{ij} + \sum\limits_{m\neq j} \underline{a}_{im}} \middle| i \neq j \right\}$$

   and then

2) for every $j \in [n]$ we define π_j as

$$\pi_j = \max\left\{ \frac{\overline{a}_{ij}}{\overline{a}_{ij} + \sum\limits_{m\neq j} \underline{a}_{im}}, \frac{\underline{a}_{ij}}{\underline{a}_{ij} + \sum\limits_{m\neq j} \underline{a}_{im}} \middle| i \neq j \right\} + \frac{\epsilon}{n}.$$

Of course instead of $\frac{\epsilon}{n}$ in the second step we can use any constant $0 < c \leq \frac{\epsilon}{n}$, or we might use a vector $\xi \in {\mathbb{R}^+}^n$ such that $0 < \sum_{j=1}^n \xi_j \leq \epsilon$ and define π_j as

$$\pi_j = \max\left\{ \frac{\overline{a}_{ij}}{\overline{a}_{ij} + \sum\limits_{m\neq j} \underline{a}_{im}}, \frac{\underline{a}_{ij}}{\underline{a}_{ij} + \sum\limits_{m\neq j} \underline{a}_{im}} \middle| i \neq j \right\} + \xi_j.$$

(It is easy to verify that this holds from Theorem 5.5, because thus defined π meets condition *b*) for the above mentioned definition and also satisfies that $0 < \sum_{j=1}^n \pi_j \leq 1$.)

And just as in section 2.3, the fact that we are interested in such homogeneous interval $\mathrm{B}_\pi^{\boldsymbol{R}}$-matrices that have $\pi \geq 0$ means only that in addition to checking the property from the characterization, we should try to find the corresponding π (using the method stated in Remark 5.8) and verify whether it is nonnegative.

5.1.1 Characterization through reduction

Proposition 5.9. *Let $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$, let $\pi \in \mathbb{R}^n$ such that $0 < \sum_{j=1}^{n} \pi_j \leq 1$ and $\boldsymbol{R} \in \mathbb{IR}^n$ be a vector of intervals of the individual row sums in matrix $\boldsymbol{A}$. Let $\forall i \in [n] : A_i \in \mathbb{R}^{n\times n}$ defined as follows:*

1) if $\pi_i > 1$, then:

$$A_i = \underline{A}$$

2) else if $0 \leq \pi_i \leq 1$, then:

$$A_i = (a_{m_1 m_2}); \quad a_{m_1 m_2} = \begin{cases} \overline{a}_{m_1 m_2} & \text{if } m_1 \neq i, m_2 = i, \\ \underline{a}_{m_1 m_2} & \text{otherwise.} \end{cases}$$

3) else if $\pi_i < 0$, then:

$$A_i = (a_{m_1 m_2}); \quad a_{m_1 m_2} = \begin{cases} \underline{a}_{m_1 m_2} & \text{if } m_1 = i, \\ \overline{a}_{m_1 m_2} & \text{otherwise.} \end{cases}$$

Then $\boldsymbol{A}$ is a homogeneous interval $B_\pi^{\boldsymbol{R}}$-matrix if and only if $\forall i \in [n] :\ A_i$ is a B_π^R-matrix, where $R \in \mathbb{R}^n$ is a vector of values corresponding to the row sums of A_i.

Proof. "$\Rightarrow$" This holds trivially, because $\forall i \in [n] : A_i \in \boldsymbol{A}$ (and the corresponding $R \in \boldsymbol{R}$).

"$\Leftarrow$"

a)$\forall i \in [n] : \underline{R}_i > 0$, because A_i is a B_π^R-matrix and $(A_i)_{i,*} = (\underline{A})_{i,*}$, thus the entries of $\underline{R}$ are positive.

b)$\forall i \in [n]\ \forall k \neq i :\quad A_k$ is a $B_\pi^{\boldsymbol{R}}$-matrix $\Rightarrow$ (From Definition 2.28:)

1) $\pi_k > 1$:

$$\begin{aligned} & \pi_k \cdot \sum_{j=1}^{n} (A_k)_{ij} > (A_k)_{ik} \quad \Leftrightarrow \\ \Leftrightarrow \quad & \pi_k \cdot \sum_{j=1}^{n} \underline{a}_{ij} > \underline{a}_{ik} \quad \Leftrightarrow \\ \Leftrightarrow \quad & \sum_{j\neq k} \underline{a}_{ij} > \left(\frac{1}{\pi_k} - 1\right) \cdot \underline{a}_{ik} \end{aligned}$$

2) $0 < \pi_k \leq 1$:

$$\begin{aligned} & \pi_k \cdot \sum_{j=1}^{n} (A_k)_{ij} > (A_k)_{ik} \quad \Leftrightarrow \\ \Leftrightarrow \quad & \pi_k \cdot \left(\overline{a}_{ik} + \sum_{j\neq k} \underline{a}_{ij}\right) > \overline{a}_{ik} \quad \Leftrightarrow \\ \Leftrightarrow \quad & \sum_{j\neq k} \underline{a}_{ij} > \left(\frac{1}{\pi_k} - 1\right) \cdot \overline{a}_{ik} \end{aligned}$$

3) $\pi_k = 0$:

$$
\begin{aligned}
&\pi_k \cdot \sum_{j=1}^{n} (A_k)_{ij} > (A_k)_{ik} \quad \Leftrightarrow \\
\Leftrightarrow \quad & 0 \cdot \left(\overline{a}_{ik} + \sum_{j \neq k} \underline{a}_{ij} \right) > \overline{a}_{ik} \quad \Leftrightarrow \\
\Leftrightarrow \quad & 0 > \overline{a}_{ik}
\end{aligned}
$$

4) $\pi_k < 0$:

$$
\begin{aligned}
&\pi_k \cdot \sum_{j=1}^{n} (A_k)_{ij} > (A_k)_{ik} \quad \Leftrightarrow \\
\Leftrightarrow \quad & \pi_k \cdot \sum_{j=1}^{n} \overline{a}_{ij} > \overline{a}_{ik} \quad \Leftrightarrow \\
\Leftrightarrow \quad & \sum_{j \neq k} \overline{a}_{ij} < \left(\frac{1}{\pi_k} - 1 \right) \cdot \overline{a}_{ik}
\end{aligned}
$$

$\Rightarrow$ $\boldsymbol{A}$ fulfills the conditions of Theorem 5.5 $\Rightarrow$ $\boldsymbol{A}$ is an interval B-matrix.

⊠

Remark 5.10. This reduction reduces the problem of verifying, whether any given interval matrix is a homogeneous interval $\mathrm{B}_{\pi}^{\boldsymbol{R}}$-matrix, into testing n matrices, if they are real B_{π}^{R}-matrices.

5.2 Characterizations of interval $\mathrm{B}_{\Pi}^{\boldsymbol{R}}$-matrices

In this section we intend to show one major claim and that is the fact, that, in a sense, the classes of the interval $\mathrm{B}_{\Pi}^{\boldsymbol{R}}$-matrices and the homogeneous interval $\mathrm{B}_{\pi}^{\boldsymbol{R}}$-matrices are the same.

Proposition 5.11. *Let $\boldsymbol{A} \in \mathbb{IR}^{n \times n}$. Then $\boldsymbol{A}$ is an interval $B_{\Pi}^{\boldsymbol{R}}$-matrix only if it has positive row sums intervals, ergo $\forall i \in [n]: \sum_{j=1}^{n} \underline{a}_{ij} > 0$.*

Proof. $\boldsymbol{A}$ is a $\mathrm{B}_{\Pi}^{\boldsymbol{R}}$-matrix, hence every $A \in \boldsymbol{A}$ is a B_{π}^{R}-matrix for some π satisfying the property, thus even matrix $\underline{A} \in \boldsymbol{A}$. And from Definition 2.28, part a) we get that $\forall i \in [n]: \sum_{j=1}^{n} \underline{a}_{ij} > 0$.

⊠

Proposition 5.12. *Let $\boldsymbol{A} \in \mathbb{IR}^{n \times n}$. Then $\boldsymbol{A}$ is an interval $B_{\Pi}^{\boldsymbol{R}}$-matrix only if*

$$
\sum_{j=1}^{n} \max \left\{ \frac{\overline{a}_{ij}}{\overline{a}_{ij} + \sum\limits_{m \neq j} \underline{a}_{im}}, \frac{\underline{a}_{ij}}{\underline{a}_{ij} + \sum\limits_{m \neq j} \underline{a}_{im}} \,\middle|\, i \neq j \right\} < 1.
$$

Proof. $\boldsymbol{A}$ is a $\mathrm{B}_{\Pi}^{\boldsymbol{R}}$-matrix, hence every $A \in \boldsymbol{A}$ is a B_{π}^{R}-matrix for some π satisfying the property, thus even matrix $A' \in \boldsymbol{A}$ defined as follows:

$$
A' = (a'_{m_1 m_2}); \quad a'_{m_1 m_2} = \begin{cases} \overline{a}_{m_1 m_2} & \text{if } m_2 = j \ \wedge \ \frac{\overline{a}_{m_1 j}}{\overline{a}_{m_1 j} + \sum\limits_{m \neq j} \underline{a}_{m_1 m}} > \frac{\underline{a}_{m_1 j}}{\underline{a}_{m_1 j} + \sum\limits_{m \neq j} \underline{a}_{m_1 m}}, \\ \underline{a}_{m_1 m_2} & \text{otherwise.} \end{cases}
$$

Therefore (if we denote R' the vector of row sums of A') it holds that

$$\forall j \in [n]:\ \max\left\{ \frac{\overline{a}_{ij}}{\overline{a}_{ij} + \sum\limits_{m \neq j} \underline{a}_{im}}, \frac{\underline{a}_{ij}}{\underline{a}_{ij} + \sum\limits_{m \neq j} \underline{a}_{im}} \,\middle|\, i \neq j \right\} = \max\left\{ \frac{a'_{ij}}{R'_i} \,\middle|\, i \neq j \right\} < \pi_j.$$

But then

$$\sum_{j=1}^{n} \max\left\{ \frac{\overline{a}_{ij}}{\overline{a}_{ij} + \sum\limits_{m \neq j} \underline{a}_{im}}, \frac{\underline{a}_{ij}}{\underline{a}_{ij} + \sum\limits_{m \neq j} \underline{a}_{im}} \,\middle|\, i \neq j \right\} < \sum_{j=1}^{n} \pi_j \leq 1.$$

⊠

Corollary 5.13. *Every interval $B_\Pi^{\boldsymbol{R}}$-matrix is a homogeneous interval $B_\pi^{\boldsymbol{R}}$-matrix for some π fulfilling $0 < \sum_{j=1}^{n} \pi_j \leq 1$.*

Proof. Let $\boldsymbol{A} \in \mathbb{IR}^{n \times n}$ be an interval square matrix with positive row sums intervals. From the previous proposition, Proposition 5.12, we get the following implication:

$\boldsymbol{A}$ is an interval $\mathrm{B}_\Pi^{\boldsymbol{R}}$-matrix $\quad\Rightarrow$

$$\Rightarrow \quad \sum_{j=1}^{n} \max\left\{ \frac{\overline{a}_{ij}}{\overline{a}_{ij} + \sum\limits_{m \neq j} \underline{a}_{im}}, \frac{\underline{a}_{ij}}{\underline{a}_{ij} + \sum\limits_{m \neq j} \underline{a}_{im}} \,\middle|\, i \neq j \right\} < 1.$$

And from the equivalence from Theorem 5.7 we might use the following implication:

$$\sum_{j=1}^{n} \max\left\{ \frac{\overline{a}_{ij}}{\overline{a}_{ij} + \sum\limits_{m \neq j} \underline{a}_{im}}, \frac{\underline{a}_{ij}}{\underline{a}_{ij} + \sum\limits_{m \neq j} \underline{a}_{im}} \,\middle|\, i \neq j \right\} < 1 \quad \Rightarrow$$

$\Rightarrow \quad \exists \pi:\ 0 < \sum_{j=1}^{n} \pi_j \leq 1 \quad \wedge \quad \boldsymbol{A}$ is a homogeneous interval $\mathrm{B}_\pi^{\boldsymbol{R}}$-matrix.

Ergo we can compose these two implications (because from Proposition 5.11 we get that if $\boldsymbol{A}$ is an interval $\mathrm{B}_\Pi^{\boldsymbol{R}}$-matrix, then it has positive row sums intervals as well, therefore fulfilling the assumptions of Theorem 5.7) and thus obtain the following:

$\boldsymbol{A}$ is an interval $\mathrm{B}_\Pi^{\boldsymbol{R}}$-matrix $\quad\Rightarrow\quad \exists \pi:\ 0 < \sum_{j=1}^{n} \pi_j \leq 1 \quad \wedge \quad \boldsymbol{A}$ is a homogeneous interval $\mathrm{B}_\pi^{\boldsymbol{R}}$-matrix.

⊠

What we now obtained is the second inclusion we need to show the equality among our two interval matrix classes, the class of the homogeneous interval $\mathrm{B}_\pi^{\boldsymbol{R}}$-matrices and that of the interval $\mathrm{B}_\Pi^{\boldsymbol{R}}$-matrices.

Theorem 5.14. *Let $\boldsymbol{A} \in \mathbb{IR}^{n \times n}$ be an interval square matrix with positive row sums intervals. Then $\boldsymbol{A}$ is an interval $B_\Pi^{\boldsymbol{R}}$-matrix if and only if $\exists \pi \in \mathbb{R}^n$ such that $0 < \sum_{j=1}^{n} \pi_j \leq 1$ and that $\boldsymbol{A}$ is a homogeneous interval $B_\pi^{\boldsymbol{R}}$-matrix.*

Proof. Follows from combining Corollaries 5.4 and 5.13. ⊠

So we have proved that the two classes we have defined at the beginning of this chapter are the same, hence it does not make any sense to differentiate the two. Thus from now on we will refer to them as "interval $\mathrm{B}_\pi^{\boldsymbol{R}}$-matrices".

Definition 5.15 (interval $\mathrm{B}_\pi^{\boldsymbol{R}}$-matrix)**.** *Let $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$ and $\pi \in \mathbb{R}^n$ such that $0 < \sum_{j=1}^n \pi_j \leq 1$. Then we say that $\boldsymbol{A}$ is an interval $B_\pi^{\boldsymbol{R}}$-matrix if it is a homogeneous interval $B_\pi^{\boldsymbol{R}}$-matrix.*

Corollary 5.16. *Every interval $B_\pi^{\boldsymbol{R}}$-matrix with $\pi \geq 0$ is an interval P-matrix.*

Proof. Direct corollary of Corollary 5.2.

⊠

Remark 5.17. Because of this definition, the characterizations of the interval $\mathrm{B}_\pi^{\boldsymbol{R}}$-matrices (and because of Theorem 5.14 even of the $\mathrm{B}_\Pi^{\boldsymbol{R}}$-matrices) are the same as the characterizations of the homogeneous interval $\mathrm{B}_\pi^{\boldsymbol{R}}$-matrices, thus we may use those stated in section 5.1

5.3 Fundamental properties

In this section let us take a look at some fundamental properties the interval $\mathrm{B}_\pi^{\boldsymbol{R}}$-matrices posses, be it conditions their entries fulfill or some their interesting subclasses, like an interval $\mathrm{B}_\pi^{\boldsymbol{R}}$-matrices that are interval Z-matrices as well.

Proposition 5.18. *Let $\pi \in \mathbb{R}^n$ such that $0 < \sum_{j=1}^n \pi_j \leq 1$ and let $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$ be an interval $B_\pi^{\boldsymbol{R}}$-matrix. Then the following holds:*

1) $\forall i \in [n]: \quad \underline{a}_{ii} > \max\left\{\pi_i \cdot \left(\underline{a}_{ii} + \sum_{j\neq i} \underline{a}_{ij}\right), \pi_i \cdot \left(\underline{a}_{ii} + \sum_{j\neq i} \overline{a}_{ij}\right)\right\}$,

2) $\forall i,j \in [n], j \neq i: \quad \pi_i \geq \pi_j \;\Rightarrow\; \underline{a}_{ii} > \overline{a}_{ij}$,

3) let $k = \operatorname{argmax}\{\pi_i \mid i \in [n]\}$, *then* $\forall j \neq k: \quad \underline{a}_{kk} > \overline{a}_{kj}$ *and*

4) $\forall i,j \in [n], j \neq i: \quad \pi_j \leq 0 \;\Rightarrow\; \overline{a}_{ij} < 0$.

Proof. Let $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$ be an interval $\mathrm{B}_\pi^{\boldsymbol{R}}$-matrix for some $\pi \in \mathbb{R}^n$ such that $0 < \sum_{j=1}^n \pi_j \leq 1$.

1) Let $A_1, A_2 \in \mathbb{R}^{n\times n}$ be defined as follows:

$$A_1 = \underline{A}$$

$$A_2 = (a_{m_1m_2}); \quad a_{m_1m_2} = \begin{cases} \underline{a}_{ii} & \text{if } m_1 = m_2 = i, \\ \overline{a}_{m_1m_2} & \text{otherwise.} \end{cases}$$

Because $A_1, A_2 \in \boldsymbol{A}$, they are both B_π^{R}-matrices, thus from Proposition 2.35, part 1) we get that this point holds.

2) Let $A' \in \mathbb{R}^{n\times n}$ be defined as follows:

$$A' = (a'_{m_1m_2}); \quad a'_{m_1m_2} = \begin{cases} \underline{a}_{ii} & \text{if } m_1 = m_2 = i, \\ \overline{a}_{m_1m_2} & \text{otherwise.} \end{cases}$$

Because $A' \in \boldsymbol{A}$, it is a B_π^{R}-matrix, thus from Proposition 2.35, part 2) we get that this point holds.

3) Direct consequence of the previous point.

4) Because $\overline{A} \in \boldsymbol{A}$, it is a B_π^R-matrix, thus from Proposition 2.35, part 4) we get that this point holds.

⊠

Corollary 5.19. *Let $\pi \in \mathbb{R}^n$ such that $0 < \sum_{j=1}^n \pi_j \leq 1$ with $\pi_1 = \ldots = \pi_n = r$ and let $\boldsymbol{A} \in \mathbb{IR}^{n \times n}$ be an interval $B_\pi^{\boldsymbol{R}}$-matrix. Then the following holds:*

1) $\forall i, j \in [n], j \neq i: \quad \underline{a}_{ii} > \overline{a}_{ij}$ *and*

2) $\operatorname{tr} \underline{A} > r \cdot \sum\limits_{i=1}^{n} \left(\underline{a}_{ii} + \sum\limits_{j \neq i} \overline{a}_{ij} \right)$.

Proof. 1) Direct corollary of Proposition 5.18, part 2).

2) Direct corollary of Proposition 5.18, part 1). What is good to notice (even though not that important) is that for $r = \pi_i > 0$ (which must hold, if $0 < \sum_{j=1}^n r$) it applies that

$$\begin{aligned}
&\max \left\{ \pi_i \cdot \left(\underline{a}_{ii} + \sum_{j \neq i} \underline{a}_{ij} \right), \pi_i \cdot \left(\underline{a}_{ii} + \sum_{j \neq i} \overline{a}_{ij} \right) \right\} \quad = \\
= \quad &\pi_i \cdot \max \left\{ \underline{a}_{ii} + \sum_{j \neq i} \underline{a}_{ij}, \underline{a}_{ii} + \sum_{j \neq i} \overline{a}_{ij} \right\} = \pi_i \cdot \left(\underline{a}_{ii} + \sum_{j \neq i} \overline{a}_{ij} \right).
\end{aligned}$$

⊠

Theorem 5.20. *Let $\boldsymbol{A} \in \mathbb{IR}^{n \times n}$ be an interval Z-matrix and let $\pi \in \mathbb{R}^{+n}$ such that $0 < \sum_{j=1}^n \pi_j \leq 1$. Then the following is equivalent:*

1) $\boldsymbol{A}$ *is an interval $B_\pi^{\boldsymbol{R}}$-matrix,*

2) $\boldsymbol{A}$ *is an interval B-matrix,*

3) $\forall i \in [n]: \quad \sum\limits_{j=1}^{n} \underline{a}_{ij} > 0$,

4) $\forall i \in [n]: \quad \underline{a}_{ii} > \sum\limits_{j \neq i} |\underline{a}_{ij}|$.

5) $\underline{A}$ *is a B-matrix.*

Proof. "1) ⇔ 2)": From Proposition 2.37, we get that $\forall A \in \boldsymbol{A} : A$ is a B_π^R-matrix ⇔ A is a B-matrix. Hence our equivalence holds.

"2) ⇔ 3) ⇔ 4) ⇔ 5)": Follows from Theorem 3.16.

⊠

Proposition 5.21. *Let $\pi \in \mathbb{R}^n$ such that $0 < \sum_{j=1}^n \pi_j \leq 1$ and let $\boldsymbol{A} \in \mathbb{IR}^{n \times n}$ be an interval $B_\pi^{\boldsymbol{R}}$-matrix. If $\alpha \in \mathbb{R}^n$ such that $0 < \sum_{j=1}^n \alpha_j \leq 1$ and $\alpha \geq \pi$, then $\boldsymbol{A}$ is interval $B_\alpha^{\boldsymbol{R}}$-matrix.*

Proof. It holds for every instance of the interval matrix (see Proposition 2.38), thus it holds for the whole interval matrix.

⊠

Proposition 5.22. *Let $P = \mathrm{perm}(i_1, \dots, i_n)$, where $i_1, \dots, i_n = [n]$ be the permutation matrix of order n defined by*

$$P = (p_{m_1 m_2}); \quad p_{m_1 m_2} = \begin{cases} 1 & \text{if } m_2 = i_{m_1}, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\pi \in \mathbb{R}^n$ such that $0 < \sum_{j=1}^{n} \pi_j \leq 1$ and let $\boldsymbol{A} \in \mathbb{IR}^{n \times n}$ be an interval $B_\pi^{\boldsymbol{R}}$-matrix. Let $\alpha = (\pi_{i_1}, \dots, \pi_{i_n})^T$. Then $P\boldsymbol{A}P^T$ is a $B_\alpha^{\boldsymbol{R}'}$-matrix.

Proof. It holds for every instance of the interval matrix (see Proposition 2.39), therefore it holds for the whole interval matrix.

⊠

5.4 Closure properties

In this section, we shall establish our understanding of what closure properties the class of interval $\mathrm{B}_\pi^{\boldsymbol{R}}$-matrices posses.

Proposition 5.23. *Let $\alpha, \beta \in \mathbb{R}^n$ such that $0 < \sum_{j=1}^{n} \alpha_j \leq 1$ and $0 < \sum_{j=1}^{n} \beta_j \leq 1$ and let $\pi \in \mathbb{R}^n$ defined by $\forall i \in [n] : \ \pi_i = \max\{\alpha_i, \beta_i\}$. Let $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{IR}^{n \times n}$ be an interval $B_\alpha^{\boldsymbol{R}}$-matrix and an interval $B_\beta^{\boldsymbol{Q}}$-matrix, respectively. If $\sum_{j=1}^{n} \pi_j \leq 1$, then $\boldsymbol{A} + \boldsymbol{B}$ is an interval $B_\pi^{\boldsymbol{R}+\boldsymbol{Q}}$-matrix.*

Proof. It holds for every pair of instances of $\boldsymbol{A}, \boldsymbol{B}$ (see Proposition 2.40), thus it holds for whole interval matrix $\boldsymbol{A} + \boldsymbol{B}$.

⊠

Remark 5.24. Just as in the real case, we can see that for $\alpha, \beta \geq 0$, we again get $\pi \geq 0$, therefore even the subclass of interval $\mathrm{B}_\pi^{\boldsymbol{R}}$-matrices for $\pi \geq 0$ is closed in the same manner as above.

Corollary 5.25. *Let $\pi \in \mathbb{R}^n$ such that $0 < \sum_{j=1}^{n} \pi_j \leq 1$ and let $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{IR}^{n \times n}$ be an interval $B_\pi^{\boldsymbol{R}}$-matrix and an interval $B_\pi^{\boldsymbol{Q}}$-matrix. Then $\boldsymbol{A} + \boldsymbol{B}$ is an interval $B_\pi^{\boldsymbol{R}+\boldsymbol{Q}}$-matrix.*

Proof. Direct corollary of the previous proposition, Proposition 5.23.

⊠

Proposition 5.26. *Let $\pi \in \mathbb{R}^n$ such that $0 < \sum_{j=1}^{n} \pi_j \leq 1$ and let $s, t \in \mathbb{R}_0^+$ with $s + t > 0$. Let $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{IR}^{n \times n}$ be an interval $B_\pi^{\boldsymbol{R}}$-matrix and an interval $B_\pi^{\boldsymbol{Q}}$-matrix, respectively. Then $s \cdot \boldsymbol{A} + t \cdot \boldsymbol{B}$ is an interval $B_\pi^{s\boldsymbol{R}+t\boldsymbol{Q}}$-matrix.*

Proof. It holds for every pair of instances of $\boldsymbol{A}, \boldsymbol{B}$ (see Proposition 2.43), thus it holds for whole interval matrix $s \cdot \boldsymbol{A} + t \cdot \boldsymbol{B}$.

⊠

Proposition 5.27. *Let $\pi \in \mathbb{R}^n$ such that $0 < \sum_{j=1}^{n} \pi_j \leq 1$ and let $\boldsymbol{A} \in \mathbb{IR}^{n \times n}$ be an interval $B_\pi^{\boldsymbol{R}}$-matrix. Let $\boldsymbol{D} \in \mathbb{IR}^{n \times n}$ be a positive interval diagonal matrix. Then $\boldsymbol{D} \cdot \boldsymbol{A}$ is an interval $B_\pi^{\boldsymbol{R}'}$-matrix.*

Proof. It holds for every pair of instances of $\boldsymbol{D}$ and $\boldsymbol{A}$ (see Proposition 2.44), thus it holds for whole interval matrix $\boldsymbol{D} \cdot \boldsymbol{A}$.

⊠

Corollary 5.28. *Let $\alpha \in \mathbb{R}^+$, $\pi \in \mathbb{R}^n$ such that $0 < \sum_{j=1}^{n} \pi_j \leq 1$ and let $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$ be an interval $B_\pi^{\boldsymbol{R}}$-matrix. Then $\alpha \cdot \boldsymbol{A}$ is an interval $B_\pi^{\alpha\boldsymbol{R}}$-matrix.*

Proof. Direct corollary of Proposition 5.26 or Proposition 5.27 ⊠

Remark 5.29. Because the real B_π^R-matrices are not closed under the following operations (see Propositions 2.46, 2.47, 2.48 and 2.49), then surely even interval $\mathrm{B}_\pi^{\boldsymbol{R}}$-matrices are in general not closed under them. Such operations are:

- matrix inverse,
- matrix power and
- matrix product.

In addition even a matrix product of an interval $\mathrm{B}_\pi^{\boldsymbol{R}}$-matrix and an interval $\mathrm{B}_\psi^{\boldsymbol{R}}$-matrix is not necessarily an interval $\mathrm{B}_\varphi^{\boldsymbol{R}}$-matrix for any φ.

6. Generating interval B-matrices and interval doubly B-matrices

The time may come when we might find ourselves in a need for some arbitrary interval B- or doubly B-matrices, be it for testing our hypothesis or anything else. So here, in this chapter, we shall introduce a mean to generate them.

6.1 Generating interval B-matrices

Let us imagine the following situation. We have a real B-matrix A and we want to inflate it, so it becomes an interval B-matrix. Then the solution is simple. Let $\boldsymbol{A} = [A, A]$. Then we are searching for such $\alpha > 0$, that $\boldsymbol{A}$ and $[1 \pm \alpha]$ fulfill the conditions of Proposition 3.26.

Now let us assume, we don't have any real instance, but want to generate brand new interval B-matrix. First, we will generate a real B-matrix A, next we will choose a positive real number α and decide, which way we want to inflate this instance. Either we choose $\alpha \cdot A$ to be a center of our new interval B-matrix, thus using Proposition 3.26, or we determine $\alpha \cdot A$ to be a lower / upper bound of the interval B-matrix, thus using Proposition 3.24.

How do we generate some real B-matrix for this usage? First we might realize that later we will scale it using α, so we might just focus on matrices with non-diagonal entries in absolute value less than or equal to 1. (Then by choice of α we are able to get values of any arbitrary magnitude.) Hence let us introduce the following lemmata:

Lemma 6.1. *Let $n \in \mathbb{N}$. Let $N \in \mathbb{R}, N \geq 2n-1$ arbitrary and let $\boldsymbol{A} \in \mathbb{IR}^{n \times n}$ defined as follows:*

$$\boldsymbol{A} = (\boldsymbol{a}_{ij}); \quad \boldsymbol{a}_{ij} = \begin{cases} [2n-1, N] & \text{if } i = j, \\ [-1, 1] & \text{if } i \neq j. \end{cases}$$

Then $\boldsymbol{A}$ is an interval B-matrix.

Proof. We will show, that $\boldsymbol{A}$ satisfies the conditions of the characterization stated in Theorem 3.5:

a)

$$\sum_{j=1}^{n} \underline{a}_{ij} = (n-1) \cdot (-1) + (2n-1) = n > 0$$

b) $\forall k \neq i :$

$$\sum_{j \neq k} \underline{a}_{ij} = (n-2) \cdot (-1) + (2n-1) = (n+1) > (n-1) = (n-1) \cdot \overline{a}_{ij}$$

⊠

Lemma 6.2. *Let $n \in \mathbb{N}, n \geq 2$. Let $N \in \mathbb{R}, N \geq n$ arbitrary and let $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$ defined as follows:*

$$\boldsymbol{A} = (\boldsymbol{a}_{ij}); \quad \boldsymbol{a}_{ij} = \begin{cases} [n, N] & \text{if } i = j, \\ [-1, \frac{1}{n-1}] & \text{if } i \neq j. \end{cases}$$

Then $\boldsymbol{A}$ is an interval B-matrix.

Proof. We will show, that $\boldsymbol{A}$ satisfies the conditions of the characterization stated in Theorem 3.5:

a)

$$\sum_{j=1}^{n} \underline{a}_{ij} = (n-1)\cdot(-1) + n = 1 > 0$$

b) $\forall k \neq i:$

$$\sum_{j\neq k} \underline{a}_{ij} = (n-2)\cdot(-1) + n = 2 > 1 = (n-1)\cdot\frac{1}{n-1} = (n-1)\cdot\overline{a}_{ij}$$

⊠

Lemma 6.3. *Let $n \in \mathbb{N}, n \geq 2$. Let $N \in \mathbb{R}, N \geq n$ arbitrary and let $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$ defined as follows:*

$$\boldsymbol{A} = (\boldsymbol{a}_{ij}); \quad \boldsymbol{a}_{ij} = \begin{cases} [n, N] & \text{if } i = j, \\ [\frac{-1}{n-1}, 1] & \text{if } i \neq j. \end{cases}$$

Then $\boldsymbol{A}$ is an interval B-matrix.

Proof. We will show, that $\boldsymbol{A}$ satisfies the conditions of the characterization stated in Theorem 3.5:

a)

$$\sum_{j=1}^{n} \underline{a}_{ij} = (n-1)\cdot\left(\frac{-1}{n-1}\right) + n = (n-1) > 0$$

b) $\forall k \neq i:$

$$\sum_{j\neq k} \underline{a}_{ij} = (n-2)\cdot\left(\frac{-1}{n-1}\right) + n = \frac{n^2-2n+2}{n-1} >$$
$$> \frac{n^2-2n+1}{n-1} = (n-1) = (n-1)\cdot\overline{a}_{ij}$$

⊠

Therefore, because of Definition 3.1, we proved that in case we want to generate an interval B-matrix we can take any matrix $A \in \mathbb{R}^{n\times n}$, which is of either one of the following forms:

$$A = (a_{ij}); \quad a_{ij} \in \begin{cases} [2n-1, \infty) & \text{if } i = j, \\ [-1, 1] & \text{if } i \neq j. \end{cases}$$

or

$$A = (a_{ij}); \quad a_{ij} \in \begin{cases} [n, \infty) & \text{if } i = j, \\ [-1, \frac{1}{n-1}] & \text{if } i \neq j. \end{cases}$$

or

$$A = (a_{ij}); \quad a_{ij} \in \begin{cases} [n, \infty) & \text{if } i = j, \\ [\frac{-1}{n-1}, 1] & \text{if } i \neq j. \end{cases}$$

or whose rows are taken from such matrices and combined (see Remark 2.7), and use the process mentioned above.

Now let us try to define an even more general approach. Instead of using Propositions 3.24, 3.26 and one interval to multiply the whole matrix by, we will use Propositions 3.27 and 3.28 and a different interval for each row.

So let us start by taking any real B-matrix $A \in \mathbb{R}^{n\times n}$. (We can take any one we already have or use one defined above.) Next for every $i \in [n]$ we randomly choose $\alpha_i \in \mathbb{R}^+$ and decide, whether we want the $\alpha_i \cdot A_{i*}$ to be center of the i-th row of our generated matrix, or its lower / upper bound. Then we use Propositions 3.27 and 3.28 to obtain the respective intervals $\boldsymbol{\alpha}_i$ and generate interval B-matrix by multiplying the rows of A by relevant intervals. (I.e. i-th row by interval $\boldsymbol{\alpha}_i$.)

6.2 Generating interval doubly B-matrices

As stated in Proposition 4.8, if we want a doubly B-matrix, we can either have a B-matrix, or almost a B-matrix, in the meaning that only one row does not fulfill the conditions of B-matrix. Let us start our generation by random choice, whether we will generate an interval B-matrix, or a proper interval doubly B-matrix. In the first case we will use the process described in the previous section. So from now on, let us consider only generation of proper interval doubly B-matrices.

The approach we shall choose in attempt to generate proper interval doubly B-matrices is the following. First we shall generate an interval B-matrix, for example using methods stated in section 6.1. Then we will choose a random $i \in [n]$, which will be the index of row that we are going to tweak in such a way that we might possibly break its conditions on being a B-matrix. So let us assume we already have an interval B-matrix $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$ and $i \in [n]$ and let us show, how to tweak an i-th row.

Let $\underline{x}, \overline{x} \in \mathbb{R}^n$ be two vector variables and let $\boldsymbol{A}(\underline{x}, \overline{x}, i) \in \mathbb{IR}^{n\times n}$ defined as follows:

$$\boldsymbol{A}(\underline{x}, \overline{x}, i)_{i*} = \left[\underline{A}_{i*} + \underline{x}^T, \overline{A}_{i*} + \overline{x}^T\right] \quad \wedge \quad \forall j \in [n] \setminus \{i\}: \quad \boldsymbol{A}(\underline{x}, \overline{x}, i)_{j*} = \boldsymbol{A}_{j*}.$$

We will construct such a linear program which gives us the most interesting values for both $\underline{x}$ and $\overline{x}$ (in the meaning that it has the maximum possible $\sum_{m=1}^{n} \overline{x}_m - \sum_{m=1}^{n} \underline{x}_m$ while both $\underline{x}_i$ and $\overline{x}_i$ are zero, hence it does not change the diagonal entry (otherwise we could increase the diagonal entry up to infinity) but results in the biggest possible overall change) from whose results we can construct such vectors $\underline{x}, \overline{x}$ that $\boldsymbol{A}(\underline{x}, \overline{x}, i)$ is an interval doubly B-matrix:

$$
\begin{aligned}
&\text{maximize} \quad \sum_{m=1}^{n} \overline{x}_m - \sum_{m=1}^{n} \underline{x}_m \\
&\text{subject to} \quad \overline{x}_k \le \underline{a}_{ii} - \overline{a}_{ik} && \text{for } k \ne i \\
&- \sum_{m \ne i} \underline{x}_m \le \frac{\underline{a}_{ii} \cdot \underline{a}_{jj}}{- \sum_{m \ne j} \underline{a}_{jm}} + \sum_{m \ne i} \underline{a}_{im} && \text{for } j \ne i: \\
& && - \sum_{m \ne j} \underline{a}_{jm} > 0 \\
&- \sum_{m \ne i} \underline{x}_m \le \frac{\underline{a}_{ii} \left(\underline{a}_{jj} - \overline{a}_{jl} \right)}{\sum_{\substack{m \ne j \\ m \ne l}} \left(\overline{a}_{jl} - \underline{a}_{jm} \right)} + \sum_{m \ne i} \underline{a}_{im} && \text{for } j \ne i, l \ne j: \\
& && \sum_{\substack{m \ne j \\ m \ne l}} \left(\overline{a}_{jl} - \underline{a}_{jm} \right) > 0 \\
&\left(\frac{\underline{a}_{jj}}{- \sum_{m \ne j} \underline{a}_{jm}} + (n-2) \right) \cdot \overline{x}_k - \sum_{\substack{m \ne i \\ m \ne k}} \underline{x}_m \le \\
&\quad \le \frac{\left(\underline{a}_{ii} - \overline{a}_{ik} \right) \underline{a}_{jj}}{- \sum_{m \ne j} \underline{a}_{jm}} - \sum_{\substack{m \ne i \\ m \ne k}} \left(\overline{a}_{ik} - \underline{a}_{im} \right) && \text{for } k \ne i, j \ne i: \\
& && - \sum_{m \ne j} \underline{a}_{jm} > 0 \\
&\left(\frac{\left(\underline{a}_{jj} - \overline{a}_{jl} \right)}{\sum_{\substack{m \ne j \\ m \ne l}} \left(\overline{a}_{jl} - \underline{a}_{jm} \right)} + (n-2) \right) \cdot \overline{x}_k - \sum_{\substack{m \ne i \\ m \ne k}} \underline{x}_m \le \\
&\quad \le \frac{\left(\underline{a}_{ii} - \overline{a}_{ik} \right) \left(\underline{a}_{jj} - \overline{a}_{jl} \right)}{\sum_{\substack{m \ne j \\ m \ne l}} \left(\overline{a}_{jl} - \underline{a}_{jm} \right)} - \sum_{\substack{m \ne i \\ m \ne k}} \left(\overline{a}_{ik} - \underline{a}_{im} \right) && \text{for } k \ne i, j \ne i, l \ne j: \\
& && \sum_{\substack{m \ne j \\ m \ne l}} \left(\overline{a}_{jl} - \underline{a}_{jm} \right) > 0 \\
&\overline{x}_i = 0 \\
&\underline{x}_i = 0 \\
&\overline{x} \ge 0 \\
&\underline{x} \le 0
\end{aligned}
$$

Theorem 6.4. *Let $\overline{x}^*, \underline{x}^*$ be the output of the previous linear program, thus its optimum. Then for any $0 < \epsilon \le 1$ the matrix $\boldsymbol{A}(\underline{x}^* \cdot (1-\epsilon), \overline{x}^* \cdot (1-\epsilon), i)$ is still an interval doubly B-matrix.*

Proof. We will show that for any pair of $\overline{x}, \underline{x}$ which is a feasible solution of the linear program, the matrix $\boldsymbol{A}(\underline{x} \cdot (1-\epsilon), \overline{x} \cdot (1-\epsilon), i)$ is an interval doubly B-matrix.

First we need to remember that if $\overline{x}, \underline{x}$ is a feasible solution, then it fulfills the conditions of the linear program. And if we take a closer look at what it means for vectors $\underline{x} \cdot (1-\epsilon), \overline{x} \cdot (1-\epsilon)$, we discover that those satisfy strictly sharp versions of the conditions from the linear program (by which we mean the same conditions, but with $<$ instead of $\leq$), except for the last four, which are satisfied in the original form.

Ergo we will show, that we can transform sharp versions of the conditions of the linear program into subset of conditions on $\boldsymbol{A}(\underline{x}, \overline{x}, i)$ from the characterization stated in Theorem 4.3, where $\underline{x}, \overline{x}$ fulfill the sharp versions of our conditions. (Plus there are some conditions which just tell us something about a shape of feasible solutions, but we do not have to care about those here. They just ensure, that $\boldsymbol{A} \subseteq \boldsymbol{A}(\underline{x}, \overline{x}, i)$.) So what we are interested in, are the first five conditions of the linear program. But first, we should mention some observations, which come from the fact, that $\boldsymbol{A}$ is an interval B-matrix, more respectively from Remark 3.2: $\forall j \in [n]: \quad \underline{a}_{jj} > \max\{0, \overline{a}_{jk} | k \neq j\}$, thus $\forall j \in [n]: \quad \underline{a}_{jj} > 0 \wedge \forall k \neq j: \underline{a}_{ii} - \overline{a}_{jk} > 0$.

Now let us take a look at the sharp versions of the conditions of our linear program.

1) $\overline{x}_k < \underline{a}_{ii} - \overline{a}_{ik}$ for $k \neq i$.

   This we can rearrange to

$$\forall k \neq i: \quad \overline{a}_{ik} + \overline{x}_k < \underline{a}_{ii} = \underline{a}_{ii} + \underline{x}_i,$$

   because $\underline{x}_i (= \overline{x}_i) = 0$ for any feasible solution. But that is equal to

$$\forall k \neq i: \quad \overline{\boldsymbol{A}(\underline{x}, \overline{x}, i)}_{ik} < \underline{\boldsymbol{A}(\underline{x}, \overline{x}, i)}_{ii}.$$

   And because $\underline{\boldsymbol{A}(\underline{x}, \overline{x}, i)}_{ii} = \underline{a}_{ii}$ and $\boldsymbol{A}$ is an interval doubly B-matrix, hence $\underline{a}_{ii} > 0$, then we get

$$\underline{\boldsymbol{A}(\underline{x}, \overline{x}, i)}_{ii} > \max\left\{0, \overline{\boldsymbol{A}(\underline{x}, \overline{x}, i)}_{ik} | k \neq i\right\},$$

   which is condition $a)$ of Theorem 4.3 for i-th row of $\boldsymbol{A}(\underline{x}, \overline{x}, i)$. (The resting rows fulfill the $a)$ condition as well, because $\forall j \neq i: \boldsymbol{A}(\underline{x}, \overline{x}, i)_{j*} = \boldsymbol{A}_{j*}$ and $\boldsymbol{A}$ is an interval doubly B-matrix.)

2) $-\sum_{m \neq i} \underline{x}_m < \frac{\underline{a}_{ii} \cdot \underline{a}_{jj}}{-\sum_{m \neq j} \underline{a}_{jm}} + \sum_{m \neq i} \underline{a}_{im}$ for $j \neq i$ such that $-\sum_{m \neq j} \underline{a}_{jm} > 0$.

   This we can rearrange to $\forall j \neq i:$

$$-\sum_{m \neq j} \underline{a}_{jm} > 0 \quad \Rightarrow \quad \underline{a}_{ii} \cdot \underline{a}_{jj} > \left(-\sum_{m \neq i} (\underline{a}_{im} + \underline{x}_m)\right)\left(-\sum_{m \neq j} \underline{a}_{jm}\right).$$

   But that is equivalent to $\forall j \neq i: -\sum_{m \neq j} \underline{\boldsymbol{A}(\underline{x}, \overline{x}, i)}_{jm} > 0 \quad \Rightarrow$

$$\Rightarrow \quad \underline{\boldsymbol{A}(\underline{x}, \overline{x}, i)}_{ii} \cdot \underline{\boldsymbol{A}(\underline{x}, \overline{x}, i)}_{jj} > \left(-\sum_{m \neq i} \underline{\boldsymbol{A}(\underline{x}, \overline{x}, i)}_{im}\right)\left(-\sum_{m \neq j} \underline{\boldsymbol{A}(\underline{x}, \overline{x}, i)}_{jm}\right).$$

And because $\forall j \in [n] : \underline{\boldsymbol{A}(\underline{x}, \overline{x}, i)}_{jj} = \underline{a}_{jj} > 0$ it leads to $\forall j \neq i$:

$$\underline{\boldsymbol{A}(\underline{x}, \overline{x}, i)}_{ii} \cdot \underline{\boldsymbol{A}(\underline{x}, \overline{x}, i)}_{jj} > \\ > \left(\max \left\{ 0, - \sum_{m \neq i} \underline{\boldsymbol{A}(\underline{x}, \overline{x}, i)}_{im} \right\} \right) \left(\max \left\{ 0, - \sum_{m \neq j} \underline{\boldsymbol{A}(\underline{x}, \overline{x}, i)}_{jm} \right\} \right).$$

(If either of the sums is non-positive, then the product of the two maxima is 0, which is surely less then the product of (positive) diagonal elements. The remaining case holds as direct implication of the condition from the linear program.)

This is exactly condition b) of Theorem 4.3, part III. for i-th and any other row j of $\boldsymbol{A}(\underline{x}, \overline{x}, i)$. (The resting rows fulfill this part of the b) condition as well as we will show in the conclusion of the proof.)

3) $- \sum_{m \neq i} \underline{x}_m < \frac{\underline{a}_{ii} \left(\underline{a}_{jj} - \overline{a}_{jl} \right)}{\sum_{\substack{m \neq j \\ m \neq l}} \left(\overline{a}_{jl} - \underline{a}_{jm} \right)} + \sum_{m \neq i} \underline{a}_{im}$

for $j \neq i, l \neq j$ such that $\sum_{\substack{m \neq j \\ m \neq l}} \left(\overline{a}_{jl} - \underline{a}_{jm} \right) > 0$.

This we can rearrange to $\forall j \neq i, \forall l \neq j : \sum_{\substack{m \neq j \\ m \neq l}} \left(\overline{a}_{jl} - \underline{a}_{jm} \right) > 0 \quad \Rightarrow$

$$\Rightarrow \quad \underline{a}_{ii} \left(\underline{a}_{jj} - \overline{a}_{jl} \right) > \left(- \sum_{m \neq i} \left(\underline{a}_{im} + \underline{x}_m \right) \right) \left(\sum_{\substack{m \neq j \\ m \neq l}} \left(\overline{a}_{jl} - \underline{a}_{jm} \right) \right)$$

But that is equivalent to $\forall j \neq i, \forall l \neq j$:

$$\sum_{\substack{m \neq j \\ m \neq l}} \left(\overline{\boldsymbol{A}(\underline{x}, \overline{x}, i)}_{jl} - \underline{\boldsymbol{A}(\underline{x}, \overline{x}, i)}_{jm} \right) > 0 \quad \Rightarrow$$

$$\Rightarrow \quad \underline{\boldsymbol{A}(\underline{x}, \overline{x}, i)}_{ii} \left(\underline{\boldsymbol{A}(\underline{x}, \overline{x}, i)}_{jj} - \overline{\boldsymbol{A}(\underline{x}, \overline{x}, i)}_{jl} \right) > \\ > \left(- \sum_{m \neq i} \underline{\boldsymbol{A}(\underline{x}, \overline{x}, i)}_{im} \right) \left(\sum_{\substack{m \neq j \\ m \neq l}} \left(\overline{\boldsymbol{A}(\underline{x}, \overline{x}, i)}_{jl} - \underline{\boldsymbol{A}(\underline{x}, \overline{x}, i)}_{jm} \right) \right).$$

And because $\forall j \in [n] : \underline{\boldsymbol{A}(\underline{x}, \overline{x}, i)}_{jj} = \underline{a}_{jj} > 0$ and $\forall j \neq i, \forall l \neq j$: $\underline{\boldsymbol{A}(\underline{x}, \overline{x}, i)}_{jj} - \overline{\boldsymbol{A}(\underline{x}, \overline{x}, i)}_{jl} = \underline{a}_{jj} - \overline{a}_{jl} > 0$ it leads to $\forall j \neq i, \forall l \neq j$:

$$\underline{\boldsymbol{A}(\underline{x}, \overline{x}, i)}_{ii} \left(\underline{\boldsymbol{A}(\underline{x}, \overline{x}, i)}_{jj} - \overline{\boldsymbol{A}(\underline{x}, \overline{x}, i)}_{jk} \right) > \\ > \left(\max \left\{ 0, - \sum_{m \neq i} \underline{\boldsymbol{A}(\underline{x}, \overline{x}, i)}_{im} \right\} \right) \cdot \\ \cdot \left(\max \left\{ 0, \sum_{\substack{m \neq j \\ m \neq l}} \left(\overline{\boldsymbol{A}(\underline{x}, \overline{x}, i)}_{jl} - \underline{\boldsymbol{A}(\underline{x}, \overline{x}, i)}_{jm} \right) \right\} \right).$$

(If either of the sums is non-positive, then the product of the two maxima is 0, which is surely less then the product of (positive) diagonal elements. The remaining case holds as direct implication of the condition from the linear program.)

This is exactly one part of the condition b) of Theorem 4.3, part II. for i-th and any other row j of $\boldsymbol{A}(\underline{x}, \overline{x}, i)$. (The resting rows fulfill this part of the b) condition as well as we will show in the conclusion of the proof.)

4)

$$\left(\frac{\underline{a}_{jj}}{-\sum\limits_{m\neq j} \underline{a}_{jm}} + (n-2)\right) \cdot \overline{x}_k - \sum_{\substack{m\neq i\\ m\neq k}} \underline{x}_m <$$

$$< \frac{\left(\underline{a}_{ii} - \overline{a}_{ik}\right)\underline{a}_{jj}}{-\sum\limits_{m\neq j} \underline{a}_{jm}} - \sum_{\substack{m\neq i\\ m\neq k}} \left(\overline{a}_{ik} - \underline{a}_{im}\right)$$

for $k \neq i, j \neq i$: such that $-\sum\limits_{m\neq j} \underline{a}_{jm} > 0$.

This we can rearrange to $\forall k \neq i, \forall j \neq i : -\sum\limits_{m\neq j} \underline{a}_{jm} > 0 \quad \Rightarrow$

$$\Rightarrow \quad \underline{a}_{jj}\Big(\underline{a}_{ii} - (\overline{a}_{ik} + \overline{x}_k)\Big) >$$

$$> \left(-\sum_{m\neq j} \underline{a}_{jm}\right)\left(\sum_{\substack{m\neq i\\ m\neq k}} \Big((\overline{a}_{ik} + \overline{x}_k) - (\underline{a}_{im} + \underline{x}_m)\Big)\right).$$

But that is equivalent to $\forall k \neq i, \forall j \neq i$:

$$-\sum_{m\neq j} \underline{\boldsymbol{A}(\underline{x}, \overline{x}, i)}_{jm} > 0 \quad \Rightarrow$$

$$\Rightarrow \quad \underline{\boldsymbol{A}(\underline{x}, \overline{x}, i)}_{jj}\left(\underline{\boldsymbol{A}(\underline{x}, \overline{x}, i)}_{ii} - \overline{\boldsymbol{A}(\underline{x}, \overline{x}, i)}_{ik}\right) >$$

$$> \left(-\sum_{m\neq j} \underline{\boldsymbol{A}(\underline{x}, \overline{x}, i)}_{jm}\right)\left(\sum_{\substack{m\neq i\\ m\neq k}} \left(\overline{\boldsymbol{A}(\underline{x}, \overline{x}, i)}_{ik} - \underline{\boldsymbol{A}(\underline{x}, \overline{x}, i)}_{im}\right)\right).$$

And because $\forall j \in [n] : \underline{\boldsymbol{A}(\underline{x}, \overline{x}, i)}_{jj} = \underline{a}_{jj} > 0$ and $\forall k \neq i : \underline{\boldsymbol{A}(\underline{x}, \overline{x}, i)}_{ii} > \overline{\boldsymbol{A}(\underline{x}, \overline{x}, i)}_{ik}$ (see the first condition), thus $\forall k \neq i : \underline{\boldsymbol{A}(\underline{x}, \overline{x}, i)}_{ii} - \overline{\boldsymbol{A}(\underline{x}, \overline{x}, i)}_{ik} > 0$, it leads to $\forall k \neq i, \forall j \neq i$:

$$\underline{\boldsymbol{A}(\underline{x}, \overline{x}, i)}_{jj}\left(\underline{\boldsymbol{A}(\underline{x}, \overline{x}, i)}_{ii} - \overline{\boldsymbol{A}(\underline{x}, \overline{x}, i)}_{ik}\right) >$$

$$> \left(\max\left\{0, -\sum_{m\neq j} \underline{\boldsymbol{A}(\underline{x}, \overline{x}, i)}_{jm}\right\}\right) \cdot$$

$$\cdot \left(\max\left\{0, \sum_{\substack{m\neq i\\ m\neq k}} \left(\overline{\boldsymbol{A}(\underline{x}, \overline{x}, i)}_{ik} - \underline{\boldsymbol{A}(\underline{x}, \overline{x}, i)}_{im}\right)\right\}\right).$$

(If either of the sums is non-positive, then the product of the two maxima is 0, which is surely less then the product of (positive) diagonal elements. The remaining case holds as direct implication of the condition from the linear program.)

This is exactly the second yet missing part of the condition b) of Theorem 4.3, part II. for any row $j \neq i$ and the i-th row of $\boldsymbol{A}(\underline{x}, \overline{x}, i)$. (The resting rows fulfill this part of the b) condition as well as we will show in the conclusion of the proof.)

5)

$$\left(\frac{\left(\underline{a}_{jj} - \overline{a}_{jl}\right)}{\sum\limits_{\substack{m \neq j \\ m \neq l}} \left(\overline{a}_{jl} - \underline{a}_{jm}\right)} + (n-2) \right) \cdot \overline{x}_k - \sum_{\substack{m \neq i \\ m \neq k}} \underline{x}_m <$$

$$< - \frac{\left(\underline{a}_{ii} - \overline{a}_{ik}\right)\left(\underline{a}_{jj} - \overline{a}_{jl}\right)}{\sum\limits_{\substack{m \neq j \\ m \neq l}} \left(\overline{a}_{jl} - \underline{a}_{jm}\right)} + \sum_{\substack{m \neq i \\ m \neq k}} \left(\overline{a}_{ik} - \underline{a}_{im}\right)$$

for $k \neq i, j \neq i, l \neq j$ such that $\sum\limits_{\substack{m \neq j \\ m \neq l}} \left(\overline{a}_{jl} - \underline{a}_{jm}\right) > 0$.

This we can rearrange to $\forall k \neq i, \forall j \neq i, \forall l \neq j : \sum\limits_{\substack{m \neq j \\ m \neq l}} \left(\overline{a}_{jl} - \underline{a}_{jm}\right) > 0 \quad \Rightarrow$

$$\Rightarrow \quad \left(\underline{a}_{ii} - (\overline{a}_{ik} + \overline{x}_k)\right)\left(\underline{a}_{jj} - \overline{a}_{jl}\right) >$$

$$> \left(\sum_{\substack{m \neq i \\ m \neq k}} \left((\overline{a}_{ik} + \overline{x}_k) - (\underline{a}_{im} + \underline{x}_m) \right) \right) \left(\sum_{\substack{m \neq j \\ m \neq l}} \left(\overline{a}_{jl} - \underline{a}_{jm}\right) \right)$$

But that is equivalent to $\forall k \neq i, \forall j \neq i, \forall l \neq j :$

$$\sum_{\substack{m \neq j \\ m \neq l}} \left(\overline{\boldsymbol{A}(\underline{x}, \overline{x}, i)}_{jl} - \underline{\boldsymbol{A}(\underline{x}, \overline{x}, i)}_{jm} \right) > 0 \quad \Rightarrow$$

$$\Rightarrow \quad \left(\underline{\boldsymbol{A}(\underline{x}, \overline{x}, i)}_{ii} - \overline{\boldsymbol{A}(\underline{x}, \overline{x}, i)}_{ik} \right) \left(\underline{\boldsymbol{A}(\underline{x}, \overline{x}, i)}_{jj} - \overline{\boldsymbol{A}(\underline{x}, \overline{x}, i)}_{jl} \right) >$$

$$> \left(\sum_{\substack{m \neq i \\ m \neq k}} \left(\overline{\boldsymbol{A}(\underline{x}, \overline{x}, i)}_{ik} - \underline{\boldsymbol{A}(\underline{x}, \overline{x}, i)}_{im} \right) \right) \cdot$$

$$\cdot \left(\sum_{\substack{m \neq j \\ m \neq l}} \left(\overline{\boldsymbol{A}(\underline{x}, \overline{x}, i)}_{jl} - \underline{\boldsymbol{A}(\underline{x}, \overline{x}, i)}_{jm} \right) \right).$$

And because $\forall k \neq i : \underline{\boldsymbol{A}(\underline{x}, \overline{x}, i)}_{ii} > \overline{\boldsymbol{A}(\underline{x}, \overline{x}, i)}_{ik}$ (see the first condition), thus $\forall k \neq i : \underline{\boldsymbol{A}(\underline{x}, \overline{x}, i)}_{ii} - \overline{\boldsymbol{A}(\underline{x}, \overline{x}, i)}_{ik} > 0$, and $\forall j \neq i, \forall l \neq j : \underline{\boldsymbol{A}(\underline{x}, \overline{x}, i)}_{jj} -$

$\overline{\boldsymbol{A}(\underline{x}, \overline{x}, i)}_{jl} = \underline{a}_{jj} - \overline{a}_{jl} > 0$ it leads to $\forall k \neq i, \forall j \neq i, \forall l \neq j$:

$$\left(\underline{\boldsymbol{A}(\underline{x}, \overline{x}, i)}_{ii} - \overline{\boldsymbol{A}(\underline{x}, \overline{x}, i)}_{ik}\right)\left(\underline{\boldsymbol{A}(\underline{x}, \overline{x}, i)}_{jj} - \overline{\boldsymbol{A}(\underline{x}, \overline{x}, i)}_{jl}\right) >$$

$$> \left(\max\left\{0, \sum_{\substack{m \neq i \\ m \neq k}} \left(\overline{\boldsymbol{A}(\underline{x}, \overline{x}, i)}_{ik} - \underline{\boldsymbol{A}(\underline{x}, \overline{x}, i)}_{im}\right)\right\}\right) \cdot$$

$$\cdot \left(\max\left\{0, \sum_{\substack{m \neq j \\ m \neq l}} \left(\overline{\boldsymbol{A}(\underline{x}, \overline{x}, i)}_{jl} - \underline{\boldsymbol{A}(\underline{x}, \overline{x}, i)}_{jm}\right)\right\}\right).$$

(If either of the sums is non-positive, then the product of the two maxima is 0, which is surely less then the product of (positive) diagonal elements. The remaining case holds as direct implication of the condition from the linear program.)

This is exactly condition b) of Theorem 4.3, part I. for i-th and any other row j of $\boldsymbol{A}(\underline{x}, \overline{x}, i)$. (The resting rows fulfill this part of the b) condition as well as we will show in the conclusion of the proof.)

Now if we take any rows $j, j' \neq i$ of matrix $\boldsymbol{A}(\underline{x}, \overline{x}, i)$, then, because they are both the same as in the original matrix $\boldsymbol{A}$ and because $\boldsymbol{A}$ is an interval B-matrix, thus an interval doubly B-matrix, the whole condition b) (all three parts) holds for them.

Hence we have shown that for any feasible $\underline{x}, \overline{x}$, matrix $\boldsymbol{A}(\underline{x}^* \cdot (1-\epsilon), \overline{x}^* \cdot (1-\epsilon), i)$ fulfills all the conditions of characterization stated in Theorem 4.3, therefore it is an interval doubly B-matrix.

⊠

Proposition 6.5. *The above formulated linear program has a non-empty set of feasible solutions.*

Proof. As stated in the proof of the previous theorem, Theorem 6.4, by rearranging the conditions from the linear program, we get a subset of conditions on $\boldsymbol{A}(\underline{x}, \overline{x}, i)$ being an interval doubly B-matrix from Theorem 4.3 (plus four conditions on domain of values of $\underline{x}$ and $\overline{x}$, but which are all satisfied by o). Moreover, we know that $\boldsymbol{A}(o, o, i) = \boldsymbol{A}$ and that $\boldsymbol{A}$ is an interval B-matrix, thus even an interval doubly B-matrix (Proposition 4.7). Hence it must fulfill the conditions from characterization of interval doubly B-matrices stated in Theorem 4.3. Therefore o is a feasible solution of the linear program.

⊠

Remark 6.6. Using this method, we have nothing to guarantee us, that what we get in the end is a proper interval doubly B-matrix, it still might be an interval B-matrix. But still, from the procedure it seems highly plausible, that if it is possible to create a proper interval B-matrix from the specific $\boldsymbol{A}$ by tweaking the specific row i, then this method will achieve it.

Conclusion

The main goal of this thesis was to achieve some understanding of a few easily recognizable subclasses of P-matrices and to build on that to generalize these subclasses into interval settings, i.e. to find their characterizations, necessary conditions and sufficient ones and to inspect their properties.

Results of the thesis

We have achieved to widen our understanding of classes of B-matrices, doubly B-matrices and B_π^R-matrices (Chapter 2), which are polynomially verifiable subclasses of P-matrices. We introduced classes of interval matrices analogous to these, them being interval B-matrices (Chapter 3), interval doubly B-matrices (Chapter 4) and two classes corresponding to real B_π^R-matrices, homogeneous interval $B_\pi^{\boldsymbol{R}}$-matrices and (heterogeneous) interval $B_\Pi^{\boldsymbol{R}}$-matrices, but for them we have proved, that they are, in a certain sense, one and the same, and designated them as one class of interval $B_\pi^{\boldsymbol{R}}$-matrices (Chapter 5). Thus what we attained is a widening of a group of polynomially recognizable subclasses of interval P-matrices.

Not only have we found characterizations, but managed to state necessary conditions as well as the sufficient ones (mosty for the class of interval doubly B-matrices, which has computationally the most complex recognition). On top of that, we inspected what operations these matrix classes are closed under and what general properties they posses.

And as an icing on the cake we have derived methods to generate interval B-matrices and interval doubly B-matrices too, which might, for example, help to test some hypotheses one may have.

Open problems

There are several paths a curious mind might want to explore in order to expand our knowledge in this region. One way is to inspect more properties to interval case, for example try to adapt those introduced in [14], or to look into sub-direct sums of B_π^R-matrices as shown in [13] and to analyze their analogies in the interval settings. Another possibility is to generalize our three classes even further, into parametric matrices, otherwise known as linearly dependent, shown for example in [15]. Or one might want to generalize another subclass of P-matrices, which might be, for example, so called mimes, which stands for "M-matrix and Inverse M-matrix Extension", as they are introduced in [16].

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