DOCTORAL THESIS REVIEW – SUPERVISOR'S REPORT

NONLINEAR STABILITY OF STEADY STATES IN THERMOMECHANICS OF VISCOELASTIC FLUIDS

BY MARK DOSTALÍK

Before I proceed with the comments on Mark's thesis, let me briefly recall the history of our mutual cooperation. I know Mark since 2013, when I started to supervise his master thesis focused on stability of Rayleigh–Bénard convection in an extended Boussinesq approximation with depth dependent material parameters. At that time Mark did an excellent job and the results obtained in his thesis later constituted the core part of our joint paper Dostalík et al. (2021). After finishing his master thesis Mark got a job in a software company, but after one year he decided to return back to academia.

When looking for a topic of his PhD thesis, we decided to work on nonlinear stability of nonequilibrium steady states, and as time went we focused the subject matter on stability of flows of viscoelastic fluids. This is a challenging subject, because the mathematical models for flows of viscoelastic fluids are quite complex, and the governing equations contain—compared to the standard Navier–Stokes–Fourier fluids—additional nonlinear terms. We wanted to attack the stability problems via a thermodynamic approach proposed in Bulíček et al. (2019), hence we first investigated thermodynamic background of viscoelastic rate-type models. The results achieved at this step are summarised in Dostalík et al. (2019), where we discussed thermodynamic underpinnings of various phenomenological viscoelastic rate-type models, especially models for incompressible viscoelastic fluids with the so-called stress diffusion, while thermodynamics background of compressible micro-macro models is discussed in detail in Dostalík et al. (2020).

The core of our joint work was however stability, while the ambition was to study stability of non-equilibrium steady states in externally driven systems such as flow between rotating cylinders and so forth. This has been achieved in Dostalík et al. (2019), where we have derived bounds on Reynolds and Weissenberg number that guarantee the stability of the steady flow. This result is to my best knowledge the first result on nonlinear (finite amplitude) stability of non-trivial steady flows of Giesekus viscoelastic rate-type fluids, and it is based on several novel ideas (thermodynamically motivated construction of a Lyapunov type functional for the non-equilibrium steady state, choice of metric measuring the distance between the perturbation and the steady state). Naturally, the result leads to new questions, since our condition is just a sufficient condition of stability, and what happens when it is violated remains unclear.

In other works we dealt with complete thermomechanical systems, that is we considered also the corresponding temperature evolution equation. We were able to prove unconditional stability of steady temperature distribution in vessels with non-uniformly heated walls, both for the classical incompressible Navier–Stokes fluid and for a class of incompressible viscoelastic rate-type models, see Dostalík et al. (2021) and Dostalík et al. (2021). Again, even for the standard Navier–Stokes fluid the result is new, and it is based on several novel ideas (thermodynamically motivated construction of a Lyapunov type functional for the non-equilibrium steady state, temperature rescaling argument).

At the end of his PhD study Mark spent some time in Paris (Ecole des Ponts ParisTech / Laboratoire d'Hydraulique Saint-Venant) working with Sébastien Boyaval. Although the stay in Paris was hampered by the COVID pandemic, Mark managed to do some nice work there, see Boyaval and Dostalík (2021), this time focused on hyperbolic systems describing viscoelastic rate-type fluids including the temperature field.

In my opinion, Mark Dostalík has clearly proved the ability to work independently and creatively and to work on long term projects. The thesis that is essentially based on Mark's published papers Dostalík et al. (2019), Dostalík et al. (2019), Dostalík et al. (2020), Dostalík et al. (2021) and Dostalík et al. (2021), and the thesis documents that Mark is able to handle equally well *mathematics* (stability theory) as well as *physics* (thermodynamics of continuous medium), and that he is able to present his results in a concise and well-organised form. For me it was a great pleasure to work with him, and I strongly recommend Mark Dostalík to be awarded by PhD degree.

Praha, 14th September 2021

Vít Průša

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