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Report on the thesis of Mark Dostalík

This dissertation, proposed for the defense of a doctorate in Mathematics at the Mathematical Institute of Charles University (Prague), is entitled

"**Nonlinear stability of steady states in thermomechanics of viscoelastic fluids**".

In this PhD thesis, Mark Dostalík is interested in the nonlinear stability of steady states in some complex fluids. This comprehensive work mixes the physical as thermodynamics approach, and mathematics with the stability of PDE. The manuscrit is well written and pleasant to read. There are enough details and explanations to realize the difficulties and the interest of the results. It is particularly appreciated to find "simple" counter-examples that sometimes indicate that results, which could be intuitive, are not right. Similarly, the choices that are made are generally well explained so that there is no ambiguity. This shows a mathematical rigor which is appreciable.

In terms of results, Mark Dostalík proves several stability theorems for various fluid flow models. The physical notions used are all the concepts of thermodynamics leading to the constitutive relations in fluid mechanics. The mathematical tools that Mark Dostalík used are mainly those that allow to obtain Theorems 1 and 2 (part 1) : proof that a Lyapunov functional allows to obtain a stability result. Mathematically speaking, the rest of the manuscript "summarizes" to implement these theorems in different contexts (which is not always easy...).

The honesty of the writing is also highlighted in the Preface where Mark Dostalík immediately indicates that he assume the existence of classical strong solution, while for many models studied here, this existence is unclear. This does not detract from the mathematical value and the difficulties which will be encountered thereafter. On the contrary, the works presented in this thesis could be seen as a new understanding of these complex models and thus give some additional ideas for proofs of existence (or non-existence) of strong or weak solutions.

Concretely, the document is composed of 3 main chapters, and completed by 3 appendices. In the rest of the report, I will comment chapter by chapter, before providing a conclusion.

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The first chapter can be read as an introduction where Mark Dostalík recall some notion of thermodynamics and stability. This chapter allows you to set up many notations and to get a feel for the context.

Section 1.3 on the stability of thermodynamic systems is particularly interesting since it describes precisely the notions of asymptotic stability or Lyapunov stability. The end of this chapter describes, in an elegant way, a method to construct Lyapunov functionals, both in the framework of closed systems (without source term, and thus in the neighborhood of a zero-velocity and spatially homogeneous steady state) and in the more difficult case of open systems via an inhomogeneous boundary conditions.

In the studies that are performed afterwards, Mark Dostalík makes the assumption that no external volumetric force operates. Thus the imbalances can only be active through the boundary conditions. It might be relevant to know if there are technical difficulties in not taking external forces into account, or if it is only a choice of the author.

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In the two next chapters, Mark Dostalík applies the Lyapunov's method previously introduced in order to analyse the stability of thermodynamially closed or open system. More precisely, in chapter 2, he is interested in the case where the system are closed that corresponds to system without source term and where the steady state solutions are spatially homogeneous.

The method is first developed on the standard compressible heat-conducting viscous fluids described by the Navier-Stokes-Fourier equations. In this case, the results are already known but allow to introduce more naturally the following cases : the compressible heat-conducting viscoelastic fluids.

Mark Dostalík distinguishes two types of viscoelastic models : multi-scale models based on the Fokker-Planck equation, and macroscopic models derived mainly from mechanical considerations. In both cases, the equations are of a formidable complexity, and generally, there are no results for the existence of a solution ! The summary of equations at the end of section 2.2.3 is impressive enough to imagine the difficulty that this type of model can involve.

In Fokker-Planck case, despite efforts to obtain a suitable Lyapunov functional, technical difficulties do not allow to reach a stability condition. A natural question would be to know if there is not another choice of variables for which one could conclude favorably. Indeed, Mark Dostalík proved in the newtonian case that this choice was fundamental and that by taking a bad variables, one could not conclude.

The case of macroscopic models is more favorable since a theorem, similar to the Newtonian case, is obtained. Here again it should be noted that the choice of variables is fundamental. The last paragraphs of this chapter illustrate this well.

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In the last chapter, Mark Dostalík always studies the nonlinear stability of viscoelastic (macroscopic) fluids via the Lyapunov's method. The originality lies in the taking into account of non-homogeneous stationary state, that corresponds to system which are thermodynamically open or non-isolated.

After recalling the context, Mark Dostalík considers two cases of non-equilibrium. The first case is the one where the system is thermally open, i.e. where the inhomogeneous boundary condition concerns the temperature. The second case corresponds to a mechanical disequilibrium :

the stationary state considered is not at zero velocity.

The result of this last part is particularly remarkable. It announces the stability in the Lyapunov sense around a inhomogeneous steady state for incompressible Giesekus viscoelastic fluid. This result requires an assumption of smallness of the Reynolds and Weissenberg numbers with respect to the inhomogeneity of the steady state. The hypothesis makes appear λ_{min} which must correspond to the smallest eigenvalue (?). Although this hypothesis seems natural since at large Reynolds or Weissenberg numbers, one can imagine that a non-equilibrium system is unstable, could one imagine a theoretical result the following sense : if Re or We is large, then there is an instability ?

Note that in the latter study, temperature is not taken into account. It might then be natural to ask whether such a stability result is still true in the non-isothermal case.

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Also note that the manuscrit contains 3 appendices that complement the document very well. The Appendix A consists of a review of popular viscoelastic models. There are introduced using the Helmoltz free energy that allows to make a link with the first chapters. This appendix is very useful to have a synthetic vision of the models as well as to have uniform notations on the whole of the manuscript.

The Appendix B simply recalls some classical mathematical inequalities (Young inequality, Lebesgue embeddings, Korn equality, Poincaré inequality and Gronwall lemma). Although these results are frequently used in the thesis, it was perhaps not useful to make an appendix because they are sufficiently well known.

In the Appendix C, Mark Dostalík offers a synthetic view of the results obtained. This is really useful for quickly finding the Lyapunov functions associated with each problem.

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In conclusion, the work proposed in this thesis is well written and very complete. The problems introduced are very well described and denote a mastery of both the physics of continuous media and mathematics applied to these problems.

Mark Dostalík tackled difficult questions that are currently being studied extensively. He was able to answer relevant questions using tools that he has mastered. He also knows and dares to explain why a particular method did not work, which shows a real culture of research in mathematics. All these works also highlight his rigour and pedagogical clarity.

His list of publications also highlights his ability to discuss and work with established researchers.

For all these reasons, I recommend without any hesitation the defense of this doctoral thesis.

14 September 2021,
Laurent Chupin,