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Dear Professor Kratochvil,

This is my report on Dr. Vítězslav Kala's habilitation thesis "*Universal quadratic forms over number fields*". I met Dr. Kala in a couple of professional meetings in the past. He and I share interests in the arithmetic theory of quadratic forms and related problems, which is a robust area of research in number theory. I am glad that I was offered the opportunity to read his habilitation thesis and to learn much more about his recent work.

Let K be a totally real number field, $\mathcal{O}_K$ be its ring of integers, and $\mathcal{O}_K^+$ be the set of totally positive elements in $\mathcal{O}_K$. A positive definite quadratic form $Q(x_1, \ldots, x_n)$ is called *universal* if it represents every element in $\mathcal{O}_K^+$ i.e. for every α in $\mathcal{O}_K^+$, there exist $a_1, \ldots, a_n$ in $\mathcal{O}_K$ such that $Q(a_1, \ldots, a_n) = \alpha$. It is a consequence of the fundamental theorem of representation of quadratic forms by Hsia-Kitaoka-Kneser that there are universal quadratic forms over any given totally real number field K. However, it is unclear from the HKK-Theorem that how many variables a universal quadratic form must have.

It is obvious that once there is one universal quadratic form, there will be infinitely many of them; simply adding an extra variable to a universal quadratic form will result in a new universal quadratic form. Therefore, it is important to know the least number of variables needed for the existence of a universal quadratic form over a totally real number field K. Following the notation introduced by Andrew Earnest in 1999, we let

$$m(K) := \min\{n : \text{ there is an } n\text{-ary universal quadratic form over } K\ \}.$$

The main question is, of course, to determine the exact value of $m(K)$ when K is explicitly given. The classical case when $K = \mathbb{Q}$ is already nontrivial. It requires the existence of a quaternary universal quadratic form over $\mathbb{Q}$. Thanks to Lagrange's Four-Square Theorem we know that the sum of four squares is universal over $\mathbb{Q}$. This, combining with the simple fact that there are no ternary universal quadratic forms over $\mathbb{Q}$, shows that $m(\mathbb{Q}) = 4$.

Maass in 1941 showed that the sum of three squares is universal over $\mathbb{Q}(\sqrt{5})$ and from this we can deduce that $m(\mathbb{Q}(\sqrt{5})) = 3$. One may hope that the sum of squares would be universal over other totally real number fields, but Siegel in 1944 showed that $\mathbb{Q}$ and $\mathbb{Q}(\sqrt{5})$ are the only totally real number fields over which a sum of squares is universal. Not much progress had been made since the work of Maass and Siegel until in 1996 Chan-Kim-Raghavan proved that $m(\mathbb{Q}(\sqrt{D})) = 3$ ($D > 1$ square-free) exactly when $D = 2, 3$, and 5. Their work has revived the interest of universal quadratic forms over totally real number fields and many interesting work have subsequently appeared. However, these more recent results often only address quadratic forms with a small number of variables or over very specific real quadratic fields. Then entered Dr. Kala. In two papers (one joint with V. Blomer), appeared in 2015 and 2016 respectively, he opened a new chapter of investigation on universal quadratic forms and obtained the striking result that for any given positive integer M, there are infinitely many real quadratic fields $\mathbb{Q}(\sqrt{D})$ such that $m(\mathbb{Q}(\sqrt{D})) \geq M$. The main ingredient of his argument is the continued fractions of the irrational number $\sqrt{D}$. Continued fractions are a classical subject and it is a surprise that Dr. Kala can still find new applications of them, especially to the study of quadratic forms. For a given positive integer M, he showed that there are infinitely many D for which he managed to construct explicitly a sequence of totally positive integers $\alpha_1, \ldots, \alpha_M$ in $\mathbb{Q}(\sqrt{D})$ from the continued fraction of $\sqrt{D}$, and these α_i's are "hard" to be represented by a universal quadratic form. What it really means is that the vectors that represent $\alpha_1, \ldots, \alpha_M$ must be mutually orthogonal (with respect to the bilinear form induced by the universal quadratic form) and this ensures that the universal quadratic form must have at least M variables. The construction of the sequence $\alpha_1, \ldots, \alpha_M$ is quite simple, but showing that these α_i's are hard to be represented is tricky. Of course, proving that there are infinitely many those D is nontrivial. All in all, Dr. Kala's results is a breakthrough in the investigation of $m(K)$ when K is real quadratic.

The technique Dr Kala and his collaborator developed in the aforementioned papers have been generalized and extended in two later papers. The first one, appeared in 2017 and coauthored with V. Blomer, is an investigation on $m_{\text{diag}}(K)$ which is the minimal number of variables of a diagonal universal quadratic form over K. In this paper, Dr. Kala obtained explicit lower and upper bounds on $m_{\text{diag}}(\mathbb{Q}(\sqrt{D}))$ in terms of the sum of the partial quotients of $\sqrt{D}$. This is the first result of this type for *all* real quadratic fields.

In a 2019 paper, coauthor with J. Svoboda, Dr. Kala extended his techniques to multiquadratic fields. The main result in that paper is very general: for all positive integers k and M, there are infinitely many totally real multiquadratic fields K of degree 2^k with $m(K) \geq M$. This is a quite interesting work; as a matter of fact it is the first nontrivial result about $m(K)$ when the degree of K is > 2.

As is indicated earlier, lower bounds for $m(K)$ are often obtained by constructing totally positive integers in K which are hard to be represented by a universal quadratic form. These totally positive integers must be indecomposable elements in the semigroup $\mathcal{O}_K^+$. As a semigroup, $\mathcal{O}_K^+$ is generated by the indecomposables. Two natural questions arise. First, what are the relations between the indecomposables? Second, does $\mathcal{O}_K^+$ determine K? In a 2019 paper with T. Hejda, Dr. Kala settled these two questions when K is real quadratic. To the first question, they found all the relations satisfied by the indecomposables. These relations are surprisingly simple and easy to describe. Their answer to the second question is even more surprising: the additive semigroups $\mathcal{O}_K^+$, for real quadratic fields K, are pairwise non-isomorphic. As a group, $\mathcal{O}_K$ is simply $\mathbb{Z} \times \mathbb{Z}$ which is completely useless in the classification of the isomorphism class of K. But $\mathcal{O}_K^+$, a subset of $\mathcal{O}_K$ which happens to be a semigroup, completely determines K. This is striking!

Overall, in his thesis Dr. Kala has accumulated many interesting and important results concerning the arithmetic of universal quadratic forms. With these results he has shown both depth and breadth in his knowledge in number theory. I have no doubt that he will continue to thrive in the research community. I strongly recommend the thesis to be defended without major changes.

For the sake of being cautious, I have gone through the check of originality of the thesis done by the system Turnitin and it is absolutely clear that the thesis represents an original work with minimum overlap with the existing literature.

Sincerely yours,

[redacted]

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