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Universal quadratic forms over number fields

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I wouldn't have written this thesis without the support of my mathematical advisors and mentors throughout the last 15 years, mainly Valentin Blomer, Aleš Drápal, Tomáš Kepka, Leonard Lipshitz, and Freydoon Shahidi – thank you.

I'm grateful to my colleagues and students for all the fun we've had with math together, and in particular, to the co-authors (Valentin Blomer, Alexander Dahl, Tomáš Hejda, and Josef Svoboda) of the papers contained in this thesis.

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Contents

I. Introduction

The arithmetics of quadratic forms has a long and rich history. In this thesis we focus on the situation over rings of integers of number fields, but before moving to them, let us start by introducing some relevant results and history over the integers. Then we will turn to a review of the number field case before describing the important results from each of the seven articles included in this thesis:

[1] V. Blomer, V. Kala, *Number fields without universal n-ary quadratic forms*, Math. Proc. Cambridge Philos. Soc. **159** (2015), 239–252.

[2] V. Kala, *Universal quadratic forms and elements of small norm in real quadratic fields*, Bull. Aust. Math. Soc. **94** (2016), 7–14.

[3] V. Kala, *Norms of indecomposable integers in real quadratic fields*, J. Number Theory **166** (2016), 193–207.

[4] V. Blomer, V. Kala, *On the rank of universal quadratic forms over real quadratic fields*, Doc. Math. **23** (2018), 15–34.

[5] A. Dahl, V. Kala, *Distribution of class numbers in continued fraction families of real quadratic fields*, Proc. Edinb. Math. Soc. **61** (2018), 1193–1212.

[6] V. Kala, J. Svoboda, *Universal quadratic forms over multiquadratic fields*, Ramanujan J. **48** (2019), 151–157.

[7] T. Hejda, V. Kala, *Additive structure of totally positive quadratic integers*, 16 pp., Manuscr. Math. (2020), to appear.

We will then conclude by discussing some follow-up works and some of my further research directions.

Parts of this Introduction are taken from my papers [1]–[7], [HK], [KY] and from my grant proposals without mentioning this later in the text.

I.1 Quadratic forms over $\mathbb{Z}$

Let $r \in \mathbb{N}$ be a positive integer and let $Q(X_1, \dots, X_r)$ be an r-ary quadratic form over $\mathbb{Z}$, i.e.,

$$Q(X_1, \dots, X_r) = \sum_{1 \le i \le j \le r} a_{ij} X_i X_j \text{ with } a_{ij} \in \mathbb{Z}.$$

We say that the quadratic form Q over $\mathbb{Z}$

- is positive (definite) if $Q(x_1, \dots, x_r) > 0$ for all $(x_1, \dots, x_r) \in \mathbb{Z}^r \setminus \{(0, \dots, 0)\}$,
- is indefinite if $Q(x_1, \dots, x_r) > 0$ and $Q(y_1, \dots, y_r) < 0$ for some $(x_1, \dots, x_r)$, $(y_1, \dots, y_r) \in \mathbb{Z}^r$,
- is diagonal if $a_{ij} = 0$ for all $i \neq j$,

- is classical if $2 \mid a_{ij}$ for all $i \neq j$,
- represents an integer $a \in \mathbb{Z}$ if there are $(x_1, \dots, x_r) \in \mathbb{Z}^r$ such that $Q(x_1, \dots, x_r) = a$,
- is universal if it represents all positive integers.

The study of representations of integers by quadratic forms has a long history; let us briefly mention here only a few highlights:

One can perhaps argue that they were first considered as Pythagorean triples, i.e., solutions of the Diophantine equation $X^2 + Y^2 = Z^2$, or, equivalently, representations of 0 by the indefinite ternary form $X^2 + Y^2 - Z^2$. A list of 15 such triples occurs already on the Babylonian clay tablet Plimpton 322 [Rob] from around 1800 BC!

The Pell equation, i.e., representation of 1 and other small integers by the binary form $X^2 - dY^2$ (for some $d \in \mathbb{Z}$ that is not a square), was considered as early as 400 BC by Greek mathematicians in connection with approximating $\sqrt{2}, \sqrt{3}$ by rational numbers. Later it was studied, e.g., by Archimedes (3rd century BC) and Diophantus (3rd century AD) and in India by Brahmagupta (7th century AD) and Bhaskara (12th century AD) [wi].

The modern European history starts with giants such as Fermat, Euler, and Gauss, who seriously considered representations of primes by binary definite forms $X^2 + dY^2$ (for $d \in \mathbb{Z}_{>0}$) [Cox] and obtained results such as *a prime number p is of the form $X^2 + Y^2$ if and only if $p = 2$ or $p \equiv 1 \pmod 4$.*

In 1770 Lagrange proved the four square theorem stating that *every positive integer n is of the form $X^2 + Y^2 + Z^2 + W^2$*; Jacobi then in 1834 gave a formula for the number of representations of n in this form. In a similar vein, Legendre in 1790 proved the three square theorem that characterizes the integers of the form $X^2 + Y^2 + Z^2$ [wi]. These results eventually led, e.g., to the still active Waring problem, and to using modular forms for studying the representations of integers by quadratic forms.

It is typically easier for an indefinite quadratic form to represent a given integer: for example, every odd integer n can be represented by the binary form $X^2 - Y^2$, as can be seen by using the factorization $1 \cdot n = (x - y) \cdot (x + y)$. From this it then follows that, e.g., every ternary form $X^2 - Y^2 + dZ^2$ with $4 \nmid d$ is universal. The situation is more complicated for anisotropic forms (i.e., forms that do not represent 0), but still it seems that positive forms have a richer and harder theory.

The early 20th century saw the characterization of all universal quaternary diagonal positive forms $aX^2 + bY^2 + cZ^2 + dW^2$ by Ramanujan [Ra], and an extension of this work to non-diagonal forms by Dickson [Di], who also introduced the term "universal quadratic form".

Finally in the 90's, Conway and his students Miller, Schneeberger, and Simons [Bh, BH] came up with the following fascinating criteria for universality:

Theorem 1. *Let Q be a positive quadratic form over $\mathbb{Z}$. Then:*

(a) *If Q is classical and represents the nine integers*

$$1, 2, 3, 5, 6, 7, 10, 14, \textit{ and } 15,$$

then it is universal.

(b) *If Q represents the twenty nine integers*

$$1, 2, 3, 5, 6, 7, 10, 13, 14, 15, 17, 19, 21, 22, 23, 26,$$
$$29, 30, 31, 34, 35, 37, 42, 58, 93, 110, 145, 203, \text{ and } 290,$$

then it is universal.

(c) *Both of these sets of integers are minimal in the sense that for each integer n in the set, there exists a corresponding quadratic form that represents all of $\mathbb{N} \setminus \{n\}$, but does not represent n.*

While the 15-theorem in part (a) is not too hard to prove, the 290-theorem in part (b), that was proved by Bhargava and Hanke [BH], is very challenging, not only because of the large amount of computations needed.

There have been a number of further exciting developments related to universal quadratic forms over $\mathbb{Z}$, such as the conjectural 451-theorem by Rouse [Ro], but let us now focus on the main topic of this thesis, that is, to universal forms over number fields.

I.2 Totally real number fields

Let us start by introducing some basic terminology concerning number fields (that we will use throughout this Introduction, unless stated otherwise).

Let K be a number field of degree d, i.e., a finite field extension of the field of rational numbers $\mathbb{Q}$ of degree $d = [K : \mathbb{Q}]$. We will denote the ring of algebraic integers of K by $\mathcal{O}_K$.

Let us further assume that K is totally real, i.e., that K has d distinct real embeddings $\sigma_1 = \mathrm{id}, \dots, \sigma_d : K \hookrightarrow \mathbb{R}$. The norm and trace of an element $\alpha \in K$ are then $N(\alpha) = \sigma_1(\alpha) \cdots \sigma_d(\alpha)$ and $\mathrm{Tr}(\alpha) = \sigma_1(\alpha) + \cdots + \sigma_d(\alpha)$.

For two elements $\alpha, \beta \in K$ we define that $\alpha \succ \beta$ if $\sigma_i(\alpha) > \sigma_i(\beta)$ for all i; further $\alpha \succeq \beta$ if $\alpha \succ \beta$ or $\alpha = \beta$. We say that α is totally positive if $\alpha \succ 0$; the semiring of totally positive integers in K is denoted $\mathcal{O}_K^+$.

Finally, of critical importance for our study of universal forms is the notion of indecomposability: A totally positive integer $\alpha \in \mathcal{O}_K^+$ is indecomposable, if it cannot be decomposed as a sum $\alpha = \beta + \gamma$ of two totally positive integers $\beta, \gamma \in \mathcal{O}_K^+$.

Brunotte [Br1, Br2] gave a general upper bound on the norm of an indecomposable integer, and so in each K, there are finitely many indecomposables up to multiplication by totally positive units. Unfortunately, the bound is exponential in the regulator of the number field, and so is not very useful. Hence for most applications, it is important to obtain more information about indecomposables, ideally in the form of an explicit construction. In real quadratic fields, this is possible using continued fractions, as we shall see in Section I.4.

I.3 Universal forms over number fields: the basics and history

Let $r \in \mathbb{N}$ be a positive integer and let $Q(X_1, \ldots, X_r)$ be an r-ary quadratic form over $\mathcal{O}_K$, i.e.,

$$Q(X_1, \ldots, X_r) = \sum_{1 \leq i \leq j \leq r} a_{ij} X_i X_j \text{ with } a_{ij} \in \mathcal{O}_K.$$

We say that the quadratic form Q over $\mathcal{O}_K$

- is totally positive (definite) if

$$Q(x_1, \ldots, x_r) \succ 0 \text{ for all } (x_1, \ldots, x_r) \in \mathcal{O}_K^r \setminus \{(0, \ldots, 0)\},$$

- is diagonal if $a_{ij} = 0$ for all $i \neq j$,
- is classical if $2 \mid a_{ij}$ (in $\mathcal{O}_K$) for all $i \neq j$,
- represents an algebraic integer $\alpha \in \mathcal{O}_K$ if there are $(x_1, \ldots, x_r) \in \mathcal{O}_K^r$ such that $Q(x_1, \ldots, x_r) = \alpha$,
- is universal over K if it represents all totally positive algebraic integers $\alpha \in \mathcal{O}_K^+$.

Note that, as stated, the first and last of these definitions make sense only over a totally real number field K. While one can naturally extend them also to fields with complex embeddings, the resulting universal forms behave similarly as indefinite forms over $\mathbb{Z}$ and do not have as rich and hard theory as totally positive forms. Before fully restricting ourselves to this case, let us thus only briefly remark that, for example, Siegel [Si] and Estes-Hsia [EH] characterized general number fields with universal sums of five and three squares. Another interesting topic is the study of universal Hermitian quadratic forms over imaginary quadratic fields, e.g., [EK2, KP1].

From now on, let us assume that K is totally real and consider only totally positive universal quadratic forms over K.

Their study started in 1941, when Maaß [Ma] used theta series to prove that the sum of three squares is universal over the ring of integers of $\mathbb{Q}(\sqrt{5})$. Conversely, in 1945 Siegel [Si] showed that the sum of any number of squares is universal only over the number fields $\mathbb{Q}, \mathbb{Q}(\sqrt{5})$. For our further discussion it will be interesting to note that indecomposable integers figured prominently in his proof (under the name "extremal elements").

In order to study universal forms, it is thus necessary to consider forms with more general coefficients a_{ij}.

Hsia, Kitaoka, and Kneser [HKK] in 1978 established a version of local-global principle for universal forms over number fields. In particular, it follows from it that a totally positive universal form exists over every totally real number field K.

This was then followed by numerous results attempting to understand the structure of universal forms over K, and in particular, to study the existence of universal forms of small rank r. It is easy to see that there is never a universal form of rank $r = 1$ or 2.

Moreover, when the degree d of K is odd, it quickly follows from Hilbert reciprocity law that there is no ternary universal form [EK1].

As we have seen with the universality of the sum of three squares over $\mathbb{Q}(\sqrt{5})$, ternary universal forms may exist in even degrees. Nevertheless, Kitaoka formulated the influential conjecture that *there are only finitely many totally real fields K admitting a ternary universal form.*

Thus if we denote by $m(K)$ the minimal rank of a universal form over K (following Earnest [Ea]), then $m(K) \geq 3$ for every totally real number field K and, conjecturally, $m(K) = 3$ for only finitely many fields K. Further let $m_{\text{diag}}(K)$ be the minimal rank of a diagonal universal form over K.

Motivated by Kitaoka's conjecture, Chan, M.-H. Kim, and Raghavan [CKR] found all classical universal forms over real quadratic number fields $\mathbb{Q}(\sqrt{D})$ – they exist only when $D = 2, 3, 5$. Several other authors investigated the universality of forms of other small ranks over specific real quadratic fields, in particular, Deutsch [De1, De2, De3], Lee [Le], and Sasaki [Sa2]. See also the nice survey by M.-H. Kim [Km].

Considering infinite families of real quadratic fields $K = \mathbb{Q}(\sqrt{D})$, B. M. Kim [Ki1, Ki2] proved that there are only finitely many K over which there is a diagonal 7-ary universal quadratic form (specifically, $m_{\text{diag}}(\mathbb{Q}(\sqrt{D})) \geq 8$ if $D \geq 153\,721$ is squarefree), and constructed explicit 8-ary diagonal universal forms for each squarefree $D = n^2 - 1$, establishing that $m_{\text{diag}}(\mathbb{Q}(\sqrt{D})) = 8$ for infinitely many values of D.

Not much more has been known about universal forms over totally real fields by 2014. Before turning to these most recent developments that constitute the principal topic of this thesis, let us briefly comment on three closely related fields of interest:

Regular quadratic forms are forms that represent all elements that are not ruled out by local obstructions [CI, Ea]. Their theory in many aspects parallels the theory of universal forms; in fact, tools such as Watson's transformations [CE+, Wa] allow one to convert regular forms into universal ones. In connection to this, let us also briefly mention the recent computational results by Kirschmer and Lorch [LK, Kr] that classify 1-class genera of quadratic lattices over number fields.

We have already seen [Si] that typically not all totally positive integers are sums of squares, but we can ask: What is the smallest integer m such that if an element is the sum of squares, then it is the sum of at most m squares? This integer m is called the Pythagoras number of the order $\mathcal{O}_K$ and is known to be always finite, but can be arbitrarily large [Sc2] (cf. also [Po]). In the case of real quadratic number fields $K = \mathbb{Q}(\sqrt{D})$ the Pythagoras number is always ≤ 5, and this bound is sharp [Pe]. In fact, one can show that $\mathrm{P}(\mathcal{O}_K) = 3$ for $D = 2, 3, 5$ [Co, Sc1] and determine all D for which $\mathrm{P}(\mathcal{O}_K) = 4$ (as in [CP]).

Finally, let us note that besides from studying representations of integers by quadratic forms, there have been numerous works considering representations of quadratic forms by quadratic forms and, in particular, by the sum of squares, e.g., [Mo1, Mo2, Ko, BI, Ic, KO1, Sa1, Oh, KO2, KO3, JKO, BC+]. Most of them deal with forms over $\mathbb{Z}$, but it is another exciting direction of future research to consider the situation over number fields in detail.

Let us now turn our attention to the articles comprising this thesis.

I.4 [1] Number fields without universal n-ary quadratic forms

We have already mentioned that it immediately follows from the theorem of Hsia, Kitaoka, and Kneser [HKK] that there exists a universal form over every totally real number field K. Unfortunately, this result is not very explicit and, in particular, does not allow us to estimate the minimal rank $m(K)$ of such a form. It is not even clear whether over every K there is a universal form in, say, 1 000 000 variables.

An answer to this question is the main result of the paper [1], at least in the case of classical forms.

Theorem 2 (Blomer-Kala 2015, [1, Theorem 1])**.** *For every positive integer M there are infinitely many real quadratic fields $K = \mathbb{Q}(\sqrt{D})$ such that every* classical *universal quadratic form over K has rank greater than M.*

As far as I know, this is the first result dealing with universal quadratic forms of large rank. There are three main ideas behind this theorem:

First is the observation that indecomposable elements are hard to be represented by quadratic forms, and so they often have to appear as diagonal coefficients of universal quadratic forms. As the simplest example, consider the representation of an indecomposable α by a *diagonal* universal form, $\alpha = Q(x_1, \dots, x_r) = \sum_{1\le i\le r} a_i x_i^2$. By the indecomposability of α, we see that $\alpha = a_i x_i^2$ for some i. Thus the number of different classes of indecomposables modulo squares gives a lower bound for the rank r.

Second, we can use continued fractions to construct real quadratic fields $K = \mathbb{Q}(\sqrt{D})$ with many indecomposables. Let

$$\sqrt{D} = u_0 + \cfrac{1}{u_1 + \frac{1}{u_2+\dots}} = [u_0, \overline{u_1, \dots, u_s}]$$

be the periodic continued fraction expansion with $u_i \in \mathbb{Z}_{>0}$. Let $\frac{p_i}{q_i} = [u_0, \dots, u_i]$ be the *convergents* and define $\alpha_i = p_i + q_i\sqrt{D}$. Then $\alpha_i + r\alpha_{i+1}$ with odd i and $0 \le r < u_{i+2}$ are precisely all indecomposables > 1. Here we for simplicity assumed that $D \equiv 2, 3 \pmod 4$, although the description is analogous when $D \equiv 1 \pmod 4$ with the exception that one considers the continued fraction for $\frac{1+\sqrt{D}}{2}$.

This characterization is somewhat technical, but the important thing is that we have very explicit control over the indecomposables: they come in certain arithmetic sequences, which are combined with a recurrent structure, i.e., $\alpha_{i+1} = u_i\alpha_i + \alpha_{i-1}$.

It is also important that we can control the norms of convergents and indecomposables using the well-known estimates

$$\frac{2\sqrt{D}}{u_{i+1}+2} < |N(\alpha_i)| < \frac{2\sqrt{D}}{u_{i+1}}. \tag{1}$$

Finally, we can choose the continued fraction coefficients almost arbitrarily in order to construct the real quadratic fields needed in Theorem 2.

Fix positive integers s and $u_1, \dots, u_{s-1}$ and consider continued fractions of the form $\sqrt{D} = [u_0, \overline{u_1, \dots, 2u_0}]$. If the sequence $u_1, \dots, u_{s-1}$ is symmetric (i.e., $u_i = u_{s-i}$) and satisfies certain mild parity condition, then by a theorem of Friesen and Halter-Koch [Fr, H-K], all the possible Ds are given by the values of a quadratic polynomial $q(t)$ and the values of $u_0 = \lfloor \sqrt{D} \rfloor$ are given by a linear polynomial in t. Further, there are infinitely many such Ds that are squarefree. These families generalize most of the well-known 1-parameter families of real quadratic fields, such as Chowla's $D = 4n^2 + 1$ or Yokoi's $D = n^2 + 4$.

In [1], we considered continued fractions of the form $\sqrt{D} = [u_0, \overline{u, u, \dots, u, 2u_0}]$ with $s-1$ elements u in the period. Using quite delicate arithmetic arguments we then established that, when u is suitably chosen and s is even, every classical universal form over $\mathbb{Q}(\sqrt{D})$ has rank at least $s/2 - 1$, which proved Theorem 2.

I.5 [2] Universal quadratic forms and elements of small norm in real quadratic fields

The results of the paper [1] have at least two big interconnected disadvantages: they apply only in the case of classical forms, and they depend on very delicate arguments and specific calculations.

In the follow-up paper [2] I managed to overcome these issues and to develop a more systematic approach that led to the following theorem.

Theorem 3 (Kala 2016, [2, Theorem 1.1])**.** *For every positive integer M there are infinitely many real quadratic fields $K = \mathbb{Q}(\sqrt{D})$ such that every universal quadratic form over K has rank greater than M, i.e., $m(K) > M$.*

As one tool served the following general proposition.

Proposition 4 ([2, Proposition 2.1])**.** *Assume that there are M elements $\gamma_1, \dots, \gamma_M \in \mathcal{O}_K^+$ such that for all $1 \le i < j \le M$,*

$$4\gamma_i\gamma_j \succeq \gamma^2 \text{ for some } \gamma \in \mathcal{O}_K \text{ implies } \gamma = 0. \qquad (2)$$

Then each universal form over K has rank at least M.

Further, it turned out to be better to choose different coefficients of the continued fraction than in [1], namely, rapidly growing ones such as $u_i = 3^{3^{i-1}}$. This choice allowed me to use the bounds (1) to establish the property (2), which proved Theorem 3.

I.6 [3] Norms of indecomposable integers in real quadratic fields

As we have just seen, the estimates (1) for the norms of the convergents α_i were one of the key steps in the proofs of Theorems 2 and 3. Is it possible to prove analogous results also for the norms of general indecomposable $\alpha_{i,r} := \alpha_i + r\alpha_{i+1}$?

Dress and Scharlau [DS] established the basic bound $N(\alpha_{i,r}) \le D$ which was then improved by Jang and Kim [JK] to $N(\alpha_{i,r}) \le D/k$, where $-k$ is the maximum negative norm of an element of $\mathcal{O}_K$ (both of these results can be further naturally strengthened when $D \equiv 1 \pmod 4$).

Jang and Kim further conjectured that in fact $N(\alpha_{i,r}) \le \frac{D-a^2}{k}$, where $a \in \mathbb{Z}_{\ge 0}$ is smallest such that $k \mid D - a^2$.

In the article [3] I disproved their conjecture. In order to do that, I first established explicit formulas for the norms of the indecomposables in $K = \mathbb{Q}(\sqrt{D})$ in terms of the continued fraction coefficients u_i. These formulas then guided me towards discovering suitable shapes of continued fractions that could yield counterexamples, and finally to the specific example $D = 24\,009\,857\,226\,825\,282\,345\,490$.

I.7 [4] On the rank of universal quadratic forms over real quadratic fields

While Theorems 2 and 3 established that ranks of universal forms can be arbitrarily large, they leave much unanswered, for they apply only to very sparse sets of real quadratic fields and, even for these fields, do not provided explicit bounds on the ranks.

In the article [4] we addressed these issues, at least for *diagonal* forms.

Let again $K = \mathbb{Q}(\sqrt{D})$ with $\sqrt{D} = [u_0, \overline{u_1, \dots, u_s}]$ and assume for simplicity that $D \equiv 2, 3 \pmod 4$. Define the following sum of continued fraction coefficients:

$$N_D = \begin{cases} u_1 + u_3 + \cdots + u_{s-1} & \text{if } s \text{ is even,} \\ 2u_0 + u_1 + u_2 + \cdots + u_{s-1} & \text{if } s \text{ is odd.} \end{cases} \tag{3}$$

For $\varepsilon > 0$ define $N^*_{D,\varepsilon}$ as the sum in (3), but ranging only over coefficients $u_i \ge D^{1/8+\varepsilon}$.

Finally recall that $m_{\text{diag}}(K)$ denotes the minimal rank of a diagonal universal quadratic form over K. Then we proved the following estimates of this rank:

Theorem 5 (Blomer-Kala 2018, [4, Theorems 1 and 2])**.** *We have*

$$\max\left(\frac{N_D}{\kappa s}, C_\varepsilon N^*_{D,\varepsilon}\right) \le m_{\text{diag}}(K) \le 8N_D$$

for any $\varepsilon > 0$, where $C_\varepsilon > 0$ is a constant (depending only on ε) and $\kappa = 2$ if s is odd and $\kappa = 1$ otherwise.

Further, $N_D \le c\sqrt{D}\,(\log D)^2$ for an absolute constant $c > 0$. If s is odd, equivalently if $\mathcal{O}_K$ has a unit of negative norm, we have $N^*_{D,\varepsilon} \ge 2\sqrt{D}$ for every $\varepsilon < 1/8$.

The estimates of $m_{\text{diag}}(K)$ are based on studying the representability of indecomposables of $\mathbb{Q}(\sqrt{D})$. To show the upper bound, we generalized Kim's result [Ki2] and constructed an explicit diagonal universal form, whose coefficients are certain indecomposables. The lower bound hinges on showing that sufficiently many indecomposables (essentially) have to appear as the coefficients of any diagonal universal form; the obstacle to obtaining sharper results was the difficulty of dealing with properties of indecomposables such as (2) and, in particular, with estimating the number of squarefree indecomposables. In fact, we suspect that $c_1 N_D \le M_{K,\mathrm{diag}}$ for a constant c_1 that is not very small (e.g., $c_1 = 0.01$).

Moreover, we have established an asymptotic formula for the sum of coefficients, which can be viewed as a variation of Kronecker's limit formula for real quadratic fields and highlights the fascinating connection between special L-values and continued fractions.

Theorem 6 (Blomer-Kala 2018, [4, Theorem 3])**.** *As $D \to \infty$, we have*

$$\sum_{i=1}^{s} u_i \sim \frac{\sqrt{\Delta}}{\zeta^{(\Delta)}(2)} \Big(L(D) + \frac{1}{h} L(1, \chi_\Delta) \log \sqrt{D} \Big),$$

where $\Delta = 4D$, $h = h_D$ is the class number, $\zeta^{(\Delta)}(s)$ is the Riemann zeta function with Euler factors at primes dividing Δ removed, χ_Δ is the usual quadratic character associated with the fundamental discriminant Δ, and $L(D)$ is the constant Taylor coefficient of the ζ-function associated to the class of principal ideals.

Kronecker's limit formula is concerned with finding a closed expression for $L(D)$ (and more general functions). Zagier and Hirzebruch observed that for real quadratic fields there is a connection between $L(D)$ and the coefficients of the continued fraction of $\sqrt{D}$. The exact formula in [Za1, Corollary 2] (which is derived by a completely different method than Theorem 6), however, seems to be hard to use to obtain any sort of asymptotic statement. The beautiful formula [Za2, Satz 2, § 14], on the other hand, is of different nature, since it treats the alternating sum $\sum_{i=1}^{s} (-1)^i u_i$, cf. [Za2, p. 131] (and gives in particular no information if s is odd). Yet another variation of the connection between special values of class group L-functions for real quadratic fields and continued fractions can be found in a nice paper of Biró and Granville [BG, Theorem 1].

I.8 [5] Distribution of class numbers in continued fraction families of real quadratic fields

The *class number* of a number field K measures the extent of the failure of the unique factorization, and so is one of the most important invariants of K. Its behavior is largely unknown, despite very precise predictions in the form of Cohen-Lenstra heuristics [CL] (and numerous improvements [Bh2, CM, Mal]). In particular, the Class Number One problem asking whether there are infinitely many (real quadratic) number fields with class number one is widely open; ditto for similar questions for other small class numbers, even when "small" is allowed to grow with the discriminant of K.

The situation is quite different in *families*, such as $\mathbb{Q}(\sqrt{4n^2+1})$, where class numbers behave similarly as in imaginary quadratic fields: there are only finitely many members of the family with class number one [KT] and the class numbers grow with the discriminant in a fairly clean way [DL].

The results of [4] suggested a connection between the minimal rank of universal forms $m(K)$, $m_{\mathrm{diag}}(K)$ and the class number h_K, at least in the real quadratic case $K = \mathbb{Q}(\sqrt{D})$: The class number is roughly inversely proportional to the (logarithm of the) fundamental unit ε by the class number formula. Both the fundamental unit and indecomposables are constructed from the periodic continued fraction for $\sqrt{D}$, and so $\log \varepsilon$ is roughly of the same size as the number of indecomposables (modulo units). Thus the connection between the rank and h_K follows from an explicit relation between the number of indecomposables and ranks of universal forms – which we established for diagonal forms in Theorem 5.

Dahl and Lamzouri [DL] determined very precise asymptotics for the growth of class numbers in the family $\mathbb{Q}(\sqrt{m^2+4})$ (for odd $m = 2k-1$), when the corresponding continued fraction is very short, i.e., $\frac{\sqrt{(2k-1)^2+4}+1}{2} = [k, \overline{2k-1}]$.

With Alexander Dahl [5] in 2018 we then generalized these asymptotics to all *continued fraction families* of real quadratic fields, i.e., to families in which the period length s and all the coefficients $u_1, \ldots, u_{s-1}$ are fixed. The distinguishing property of these families is that one has control over the size of the fundamental unit, and thus over the class number. At the same time, the fields in these families have a very similar structure of their indecomposables.

We constructed a random model for the behavior of class numbers in these families and proved very precise results (that are somewhat too technical for this Introduction).

I.9 [6] Universal quadratic forms over multiquadratic fields

With my Bachelor's student Josef Svoboda [6] in 2019 we extended the results of [2] to all totally real multiquadratic fields, i.e., to fields $K = \mathbb{Q}(\sqrt{p_1}, \ldots, \sqrt{p_k})$ where $p_1, \ldots, p_k$ are positive integers: We proved a precise analogue of Theorem 3 for such fields of given degree 2^k.

The idea was to again apply Proposition 4 which we did by first ensuring the existence of the elements γ_i in the quadratic field $\mathbb{Q}(\sqrt{p_1})$ and then by choosing $p_2, \ldots, p_k$ carefully so that the property (2) was not violated.

I.10 [7] Additive structure of totally positive quadratic integers

In order to progress on the study of properties such as (2), it is crucial to better understand the additive and multiplicative properties of indecomposables. In the article [7] with my postdoc Tomáš Hejda, we have completely described the additive semigroup $\mathcal{O}_K^+(+)$ in the real quadratic case:

Let

$$\mathcal{A} := \{\alpha_{i,r} \mid i \geq -1 \text{ odd and } 0 \leq r \leq u_{i+2}-1\} \backslash \{1\}$$

denote the set of indecomposable elements > 1 and $\mathcal{A}' := \{y' \mid y \in \mathcal{A}\}$ the set of their conjugates, that is, of indecomposables < 1.

Considering the trace, it is easy to see that each element of $\mathcal{O}_K^+$ can be expressed as a (finite) sum of indecomposables; in other words, the indecomposables $\mathcal{A} \cup \mathcal{A}' \cup \{1\}$

generate the semigroup $\mathcal{O}_K^+(+)$. Let us further introduce an alternative notation of the indecomposables for convenience. We define β_j, $j \in \mathbb{Z}$, by the condition that

$$\cdots < \beta_{-3} < \beta_{-2} < \beta_{-1} < \beta_0 = 1 < \beta_1 < \beta_2 < \beta_3 < \cdots$$

is the increasing sequence of the indecomposables. Note that we have $\beta_j' = \beta_{-j}$ for all $j \in \mathbb{Z}$.

Then it is quite easy to see from the definitions that

$$\beta_{j-1} - v_j \beta_j + \beta_{j+1} = 0 \text{ for } j \in \mathbb{Z}, \tag{4}$$

where

$$v_j := \begin{cases} 2 & \text{if } \beta_{|j|} = \alpha_{i,r} \text{ with odd } i \geq -1 \text{ and } 1 \leq r \leq u_{i+2} - 1, \\ u_{i+1} + 2 & \text{if } \beta_{|j|} = \alpha_{i,0} \text{ with odd } i \geq -1. \end{cases}$$

The equalities (4) can be considered as the trivial relations between the indecomposables that always have to hold. A priori it is not at all clear that there can not be any further accidental relations. However, we proved that this is indeed the case.

Theorem 7 (Hejda-Kala 2020, [7, Theorem 2])**.** *Let $x \in \mathcal{O}_K^+$ be given as a finite sum $x = \sum k_j \beta_j$ with $k_j \in \mathbb{Z}$. Then there exist unique $j_0, e, f \in$ with $e \geq 1$ and $f \geq 0$ such that $x = e\beta_{j_0} + f\beta_{j_0+1}$.*

Every relation of the form $\sum h_j \beta_j = 0$ (with $h_j \in \mathbb{Z}$ and only finitely many non-zeros) is a $\mathbb{Z}$-linear combination of the relations (4); in particular, this is true for $e\beta_{j_0} + f\beta_{j_0+1} - \sum k_j \beta_j = 0$.

This amounts to giving a certain presentation of the semigroup $\mathcal{O}_K^+(+)$; we further used this presentation to show that this semigroup determines the real quadratic field uniquely and to characterize all the *uniquely decomposable elements*, i.e., elements of $\mathcal{O}_K^+(+)$ that can be decomposed as a sum of indecomposables in a unique way. For example, 2 is always uniquely decomposable.

I.11 Follow-up works

As outlined above, the basic approach of using indecomposables has been successful in breaking through the former barriers in the theory of universal forms over real quadratic fields. These results led to a lot of follow-up activity both by other researchers [Co, KP2, Ya] and by my students [CL+, KTZ, TV]. Let me now briefly comment on some of these results:

Collinet [Co] showed that, unlike Theorems 2 and 3, when one works over the ring $\mathcal{O}_K[\frac{1}{2}]$, every element is represented by the sum of five squares.

By working with interlacing polynomials, Yatsyna [Ya] was able to construct some indecomposable integers in certain number fields of degree greater than two and thus to extend Theorem 3 to that case.

B. M. Kim, M.-H. Kim, and D. Park [KP2] very recently proved that there are finitely many real quadratic fields that admit a 7-ary universal quadratic form (without having to assume that such a form is diagonal or classical).

As part of the *Student Number Theory Seminar* that I have been organizing since 2017, my students have worked on the case of biquadratic fields. First, Čech, Lachman, Svoboda, Tinková, and Zemková [CL+] studied their indecomposables and in particular the question of when does an element that is indecomposable in a quadratic field $\mathbb{Q}(\sqrt{D})$ remain indecomposable also in a biquadratic field $\mathbb{Q}(\sqrt{D}, \sqrt{E})$.

Building on these results (and, for example, working with uniquely decomposable elements as in [7]), Krásenský, Tinková, and Zemková [KTZ] then studied universal quadratic forms over biquadratic fields and proved Kitaoka's conjecture for classical forms over them, i.e., they established that there are no ternary classical universal forms over any biquadratic field.

In a joint work [TV] with Voutier, Tinková built on the results of [3]. They significantly improved the estimates on the norms and found the smallest counterexamples to the conjecture of Jang and Kim.

I.12 Future directions

One thing that is conspicuous about all the articles included in this thesis is that they deal only with the situation of real quadratic fields $\mathbb{Q}(\sqrt{D})$ (or, as in the case of [6], with the closely related multiquadratic fields). The principal reason for this is the absence of a good theory of (generalized) continued fractions for higher degree fields and, consequently, the lack of understanding of indecomposables.

Resolving this big problem is the main focus of my current research.

For number fields K of degree d, there are multidimensional generalizations of continued fractions such as the Jacobi-Perron algorithm (JPA) [Be, Sch] that repeatedly applies a simple transformation to $(d-1)$-tuples of elements from K. Notably, if the resulting sequence is periodic, then a suitable "convergent" is a unit in $\mathcal{O}_K$ [BeH].

Surprisingly, it seems that the other convergents to JPA have never been studied. In the case of cubic fields we are working on this with my Ph.D. student Magdaléna Tinková. We are considering the convergents and other elements defined from periodic JPA's over Shanks' simplest cubic fields [Sh] that are indecomposable. The preliminary results look promising and will hopefully soon lead to bounds for the ranks of universal forms over the simplest cubic fields. Of course, extending this to all number fields will be a tall order.

Further, I hope to expand on the connection to the class number. In a general totally real field I would like to ideally establish an explicit correspondence between the minimal rank of a universal quadratic form, the number of indecomposables, and the class number. Such a connection should then enable the transfer of results and insights across these topics.

One can view the articles contained in this thesis as first steps towards establishing this correspondence: The papers [1, 2, 4, 6] primarily consider the universal forms part, [3, 7] develop the theory of indecomposable integers, and [5] deals with class numbers.

Besides from this vision, I also plan to expand the methods to the study of regular forms and of forms over orders in number fields (this is the main topic of my Ph.D. student Jakub Krásenský), and to representations of quadratic forms of higher rank by quadratic forms.

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