CHARLES UNIVERSITY
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On multifractality and predictability of financial time series

Master's thesis

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Declaration of Authorship

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Prague, July 24, 2021

Michael Heller

Abstract

The aim of this thesis is to examine an empirical relationship between multifractality of financial time series and its returns. We approach the multifractality of a given time series as a measure of its complexity. Multifractal financial time series exhibit repeating self-similar patterns. Multifractality could be a good predictor of stock returns or a factor which can be used in asset pricing. We expected that capturing the complexity of a given time series by a model, a positive or a negative risk premia for investing into "more multifractal assets" could be found. Daily prices of 31 stock indices and daily returns of 10-years US government bonds were downloaded. All the data were recorded between 2012 and 2021. After estimation the multifractal spectra, applying MF-DFA method, of all stock indices, we ordered all stock indices from the lowest to the most multifractal. Then, we constructed a "multifractal portfolio" holding a long position in the 7 most multifractal and holding a short position in the 7 least multifractal stock indices. Fama-MacBeth regression with market risk premia and multifractal variable as independent variables was applied. Multifractality in all examined financial time series was found. We also found a very low negative risk premia for holding "a multifractal portfolio". These results suggest that "more multifractal assets" are connected with lower returns and with lower risks in average. According to our results, an investor seeking a lower risk and lower return in average should hold a long position in "more multifractal assets" and hold a short position in "less multifractal assets" with respect to the market.

JEL Classification	G0, G10, G17
Keywords	fractality, multifractality, financial time series, Hurst exponent, CAPM, Fama-French three factor model, Fama-MacBeth regression
Title	On multifractality and predictability of financial time series

Abstrakt

Cílem této práce je ověřit empirický vztah mezi multifraktalitou finančních časových řad a jejich výnosy. K multifraktalitě přistupujeme jako k míře komplexity dané finanční časové řady. Multifraktální časové řady vykazují sebeopakující se vzorce. Multifraktalita by mohla být dobrým prediktorem akciových výnosů nebo faktorem, který by mohl být využit k oceňování aktiv. V naší práci popisujeme komplexitu dané finanční časové řady pomocí modelů a poté zkoumáme, zda najdeme kladnou nebo zápornou rizikovou prémii za investování do „více multifraktálních aktiv". Získali jsme denní ceny 31 akciových indexů a denní výnosy 10-letých amerických státních dluhopisů. Všechna data pochází z období let 2012 až 2021. Pomocí metody MF-DFA jsme odhadli multifraktální spektra na všech 31 akciových indexech. Všechny akciové indexy jsme seřadili od nejméně po nejvíce multifraktální. Zkonstruovali jsme „multifraktální portfolio" držením dlouhé pozice v 7 nejvíce multifraktálních akciových indexech a držením krátké pozice v 7 nejméně multifraktálních indexech. Poté jsme použili regresi Fama-MacBeth s nezávislými proměnnými, očekávaným tržním výnosem a rizikovou prémií. Ve všech zkoumaných finančních časových řadách jsme našli multifraktalitu. Rovněž jsme nalezli velmi nízkou zápornou rizikovou prémii za držení „multifraktálního portfolia". Náš výsledek naznačuje, že více multifraktální aktiva jsou spojena s nižšími výnosy a s nižším rizikem v průměru. Podle našich výsledků, investor, který vyhledává nízké riziko spojené s nízkým výnosem v průměru, by měl držet dlouhou pozici ve „více multifraktálních aktivech" a krátkou pozici v „méně multifraktálních aktivech" ve srovnání s trhem.

Klasifikace JEL	G0, G10, G17
Klíčová slova	fraktalita, multifraktalita, finanční časové řady, Hurstův exponent, CAPM, Fama-French tří faktorový model, Fama-MacBeth regrese
Název práce	Multifraktalita a prediktabilita finančních časových řad

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Acronyms

MF-DFA Multifractal detrended fluctuation analysis

MF-DMA Multifractal detrended moving average analysis

DFA detrended fluctuation analysis

DMA detrended moving average analysis

GARCH Generalized Autoregressive Conditional heteroskedasticity

E-GARCH Exponential Generalized Autoregressive Conditional heteroskedasticity

I-GARCH Integrated Generalized Autoregressive Conditional heteroskedasticity

T-GARCH Threshold Generalized Autoregressive Conditional heteroskedasticity

Q-GARCH Quadratic Generalized Autoregressive Conditional heteroskedasticity

GARCH-M Generalized Autoregressive Conditional heteroskedasticity - in mean

MSM Markov Switching Multifractal

BSE Bombay Stock Exchange

NSE National Stock Exchange of India

EMH Efficiency Market hypothesis

FMH Fractal Market hypothesis

SMB Small Minus Big

HML High Minus Low

CAPM Capital Asset Pricing Model

.AEX AEX index

.AORD All Ordinaries

.BFX Bell 20 Index

.BSESN S&P BSE Sensex

.BVLG PSI All-Share Index

.BVSP BVSP BOVESPA Index

.DJI Dow Jones Industrial Average

.FCHI CAC 40

.FTMIB FTSE MIB

.FTSE FTSE 100

.GDAXI DAX

.GSPTSE S&P/TSX Composite index

.HSI HANG SENG Index

.IBEX IBEX 35 Index

.IXIC Nasdaq 100

.KS11 Korea Composite Stock Price Index (KOSPI)

.KSE Karachi SE 100 Index

.MXX IPC Mexico

.N225 Nikkei 225

.NSEI NIFTY 50

.OMXC20 OMX Copenhagen 20 Index

.OMXHPI OMX Helsinki All Share Index

.OMXSPI OMX Stockholm All Share Index

.OSEAX Oslo Exchange All-share Index

.RUT Russel 2000

.SMSI Madrid General Index

.SPX S&P 500 Index

.SSEC Shanghai Composite Index

.SSMI Swiss Stock Market Index

.STI Straits Times Index

.STOXX50E EURO STOXX 50

Master's Thesis Proposal

Author	Bc. Michael Heller
Supervisor	prof. PhDr. Ladislav Krištoufek, Ph.D.
Proposed topic	On multifractality and predictability of financial time series

Motivation According to Kukacka, J., & Kristoufek, L. (2020), the dynamics of financial markets is highly complex. It is probably caused by huge number of traders who are heterogeneous, and they are trading the stocks and commodities according to a lot of different trading strategies. Many people, both theoretically and or practically, try to understand the behavior of a financial market. They try to find some pattern of financial time series on the financial market, and they hope that their model or some pattern could improve their trading and increase their returns.

It is difficult to model and predict financial time series by econometric tools, which assume linearity. Due to the fact, that financial time series are often (multi) fractal, it is reasonable to examine the (multi) fractality, in order to model, and to predict financial time series. In a nutshell, the fractality of time series shows us the autocorrelation of the process generating that time series, and the multifractality of time series shows us the complexity of the process generating that time series. The complexity of a process can be caused by the autocorrelation of the process and by the distribution of the generated time series with heavier tails than for the normal distribution.

One way to estimate the (multi) fractality of a time series is the Hurst exponent. The Hurst exponent also evaluates the dependency and autocorrelation of a time series. It measures the long-term memory of a process generating a time series. The Hurst exponent ranges between 0 and 1. For a white noise, the Hurst exponent is 0.5. If the Hurst exponent is higher than 0.5, the process generating the time series is more persistent (trending). If the Hurst exponent is lower than 0.5, the process generating the time series is more anti-persistent (mean-reverting). Using a moving window, I will estimate (multi) fractality of a time series. I would like to find out whether there is some significant dependency between (multi) fractality and the

predictability of that financial time series. Then, I will construct a trading strategy that will utilize the information about (multi) fractality and I will test whether the information about (multi) fractality improves my trading strategy or not.

The idea is that I would like to discover the switching between trending and mean reverting time periods in a (multi) fractal financial time series. Then I expect that I could trade in the trending periods with lower losses, than without using the information about (multi) fractality, and stop the trade in the mean reverting periods, where the future development of time series is more difficult to predict.

Hypotheses

Hypothesis #1: Multifractal financial series are more predictable than non-multifractal financial series.

Hypothesis #2: : A trading strategy is more successful when the strategy is utilizing the fractal dimension of traded series.

Hypothesis #3: : Multifractality as a predictability indicator is better utilized by more complex econometric models/ML models.

Methodology In the first part of my thesis, I will examine the (multi) fractality of several financial time series. I will estimate the Hurst exponent for all time series. At the same time, I will estimate ARIMA-GARCH models to capture the behavior of the time series. Then, I will estimate dynamic forecasts using the estimated ARIMA-GARCH models, and I will calculate the prediction errors using Mean Squared Error (MSE). Then, I will regress the prediction errors on the values of Hurst exponent. In this part, I would like to show how much the financial time series are (multi) fractal.

In the second part, I will construct some trading strategy based on technical trading indicators such as Simple Moving Average (SMA), Parabolic Stop and Reverse (PSAR), Commodity Channel Index (CCI), Rate of Change (ROC), Stochastic Momentum Index (SMI) and Williams Percent Range (W%R). SMA and PSAR indicators show us the trends of the market and potentially can show us the points of reversions. CCI, SMI and W%R are indicators which show us whether the price of an asset is overestimated or underestimated. ROC indicates us the speed of change of prices and it can be utilized to predict reversions in the market.

I will apply the strategy, based on the aforementioned trading indicators, to all selected time series without utilizing the values of the Hurst exponent. Then I will apply the trading strategy to all selected time series but now I will utilize the values of Hurst exponent. In the end, I will compare the fictious profits of these two trading strategies. Using this way of comparison, I would like to evaluate the strength of the Hurst exponent as an information for profitable trading.

During the process, I will retrieve some time series of stock prices. I will utilize three estimators, in order to estimate the generalized Hurst exponent. These are MF-DFA (Multifractal Detrended Fluctuation Analysis), MF-DMA (membrane filtration-differential mobility analyzer) and GHE (Generalized Hurst Exponent) method. Then I will apply some trading strategy to the time series with the Hurst exponent and without Hurst exponent. Finally, I will compare these to trading strategies in order to find out, which of them give us the better results in terms of profit from the simulated trading.

Expected Contribution I would like to find out whether the multifractal time series are more predictable. I am also interested in the hypothesis that the information about the multifractality of a time series could improve the results of some trading strategy. According to my best knowledge, the most of studies focus mainly on Hurst exponent but not on the generalized Hurst exponent and its utilization for predictions. I believe that the understanding of multifractality of financial time series could improve potential investment returns. At least, I hope that my thesis could perhaps broaden the awareness about multifractal financial time series.

Outline

1. Introduction: I will introduce the topic of multifractality of financial time series.
2. Literature review: I will summarize the previous research regarding (multi) fractality in finance.
3. Data: I will explain how I will collect the data. I will provide the descriptive statistics of the data.
4. Methodology: I will estimate the Hurst exponent and I will apply some trading strategy to the data.
5. Results: I will discuss the results and I will compare the results of the model with Hurst exponent and without the Hurst exponent.
6. Concluding remarks: I will summarize my findings and their implications for dependency between (multi) fractality and the predictability of time series.

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Chapter 1

Introduction

Predicting and forecasting financial time series is very popular phenomenon, as old as the financial markets are, since people would like to beat the market and to get rich without too much effort or without spending a long time period. There are many different ways and approaches how to analyze the market. Two main approaches are fundamental and technical analysis. Fundamental analysis focus on the "fundamental" or the intrinsic value of an asset. In the case of a stock, it means that traders consider financial statements, the revenues, expenditures and profit of a given company, the economic development of a relevant industry, etc. On the contrary, the technical analysis tries to predict the price by (seasonal) patterns and trends from the past. The technical traders calculate various technical indicators such MA (Moving Average), RSI (Relative Strength Index), MACD (Moving Average Convergence Divergence) and many others. They try to utilize aforementioned technical indicators in order to predict a trend or a breaking point on a financial market. Another approach in technical analysis is to estimate the behavior of a market by a statistical or an econometric model and to estimate the predictions, or to utilize some statistical property of the data.

It is also worth mentioning that there are two opposite groups in economics, who argue whether it is possible to predict financial markets at all. EMH (Efficiency Market hypothesis) proposed by Malkiel & Fama (1970) states that a market is efficient, if all available information relevant to a given asset is already fully incorporated in the current price of that asset. Malkiel & Fama (1970) distinguish three types of EMH. The "weak-form" EMH studies the information contained in historical prices. If the "weak-form" EMH holds, then the technical analysis is out of use and only the fundamental analysis can be used.

The "semi-strong form" EMH studies publicly available information. If the "semi-strong form" EMH holds, then both technical analysis and fundamental analysis are out of use and only a private information can be theoretically utilized in order to beat the market. The "strong-form" EMH studies only private information. If the "strong form" holds, then the markets are fully efficient and there is no way how to systematically beat the market.

On the one hand there are economists who support the EMH and they claim that it is not possible to predict an asset price via technical or fundamental analysis. There are many scientific works which support the claim that financial markets follow a random walk, i.e. the financial markets are efficient and thus unpredictable (Tong *et al.* (2014), Konak & Şeker (2014), and Erdem & Ulucak (2016)).

On the other hand there are economists who dispute the EMH and some of them believe that it is possible to predict future prices and therefore beat the market (Nicholson (1968), Soros (2015), Rosenberg *et al.* (1985), Darrat & Zhong (2000), and Harrison *et al.* (2012)). Moreover, there are millions of traders and investors, with various trading strategies and models, who try to beat the market every day, which implies that they at least believe that they are able to beat the market.

In our thesis, we would like to examine the multifractality of stock index returns and find out whether there is an association between the risk premia in the returns and the estimated multifractality of the data. According to (Kukacka & Kristoufek 2020, p. 2): "Multifractality is a statistical characteristic of a sequence or time series connected to its complex scaling properties. Extending the fractality characteristic by describing scaling properties of the series with a single exponent, multifractality in a series explains complex behavior by a spectrum of exponents." Firstly, the concept of multifractality and multifractal measures was proposed by Mandelbrot (1972) and it was utilized mainly in physics.

We can distinguish between two types of fractal processes. One type is called "linear" or "monofractal", because we can describe the long-term correlations of such a process by just a single exponent. Another type is called "non-linear" or "multifractal". In the second case, we need to use entire spectrum of q order scaling exponents in order to describe more complex dynamics of such a process with long-term correlations. It is important to distinguish between these two types of processes in order to understand the dynamics and modeling of these types of processes (Kalisky *et al.* (2005)).

According to Kantelhardt (2008), multifractality of time series can be caused by two sources. The first source of multifractality can be a broad distribution function with heavier tails than for normally distributed data. This can be observed very often in stock returns. The second possible source of multifractality can be different long-term correlations of small and large fluctuations in a given time series. If someone would be able to utilize these long-term correlations in a prediction model or in a portfolio construction, he or she could maybe systematically increase their average returns and therefore beat the market.

In our thesis, we would like to approach the multifractality of financial time series as a measure of complexity and a measure of self-similar repeating patterns in given time series. Such a complex financial time series could be theoretically more predictable than a random walk.

We would like to broaden the awareness about the association between multifractality of financial time series and the pricing of financial assets. We approach the multifractality as a measure of complexity of a given financial time series. If there would be a model which could be able to describe the complexity of a given financial time series well enough, then we could theoretically expect higher expected returns than the average returns. We stated our alternative hypothesis in the following way:

H_1 : There is a significant risk premia for holding a "multifractal portfolio".

Financial times series with broader multifractal spectra, estimated by the Generalized Hurst exponent ΔH and/or the Singularity strength $\Delta\alpha$, theoretically could be more predictable, and therefore the trader could possible pay higher or lower risk premia for holding the assets performing "a multifractal behaviour" in its returns.

In our thesis, we would like to test the hypothesis by the following procedure. We obtain stock indices returns and we will obtain the risk free rate of return. We will estimate the multifractality spectra on the all retrieved stock index log-returns. Then, we will employ the Fama-MacBeth regression and observe whether the beta for our multifractal factor will be significantly different from zero or not.

In the section 2, we provide the review of literature associated with multifractality and its role in finance. We also summarize various predicting approaches and asset pricing approaches. In chapter 3, we provide the definitions of (multi)-fractal processes, procedures for multifractality estimation and de-

scriptions of models. In chapter 4, we obtain the data, calculate the descriptive statistics, test the normality and the stationarity of the data. In chapter 5, we describe how we estimate the multifractal spectra, and how we employ the Fama-MacBeth procedure. In chapter 6, we present our results and in chapter 7, we provide the discussion between our results and previous findings, and we interpret and comment on our results. In chapter 8, we conclude the thesis and we suggest a several ideas for a further research. We also comment on the limitations of our methodology.

Chapter 2

Literature review

According to Kukacka & Kristoufek (2020), the dynamics of financial markets is highly complex. It is probably caused by huge number of traders who are heterogeneous, and they are trading the stocks and commodities according to a wide range of different trading strategies. Many people, both theoretically and or practically, try to understand the behavior of a financial market. They try to find some pattern of financial time series on the financial market, and they hope that their model or some pattern could improve their trading strategies and increase their returns. According to (Kukacka & Kristoufek 2020, p. 2): "Multifractality is a statistical characteristic of a sequence or time series connected to its complex scaling properties. Extending the fractality characteristic by describing scaling properties of the series with a single exponent, multifractality in a series explains complex behavior by a spectrum of exponents." In chapter 3, we provide more formal definitions of (multi) fractality and scaling. Multifractality of time series has been examined in various scientific fields such as: hydrometeorology (Veneziano *et al.* (2006)), medicine (Lopes & Betrouni (2009)), image processing (Arneodo *et al.* (2000)), geochemistry (Liu *et al.* (2016)), geology (Dellino & Liotino (2002)), climate change (Ashkenazy *et al.* (2003)) and many other fields. For instance, Matia *et al.* (2003) found that the price fluctuation for commodities have a substantially broader multifractal spectrum than the price fluctuation for stocks. They also showed that the multifractal properties of stocks and commodities are mainly caused by the broad probability distribution, and secondly due to the power law distribution of price fluctuations. They examined daily returns for 29 commodities and for 2449 stocks. They showed that commodity returns are more clustered into patches of small and large fluctuations than stock returns. In order to

remove any temporal correlations, they randomly shuffled the returns many times. They found that the shuffled commodity returns lost its clustering. On the other hand, the shuffled stock returns resembled similar patterns as the original stock returns.

They showed that a significant part of the multifractal spectrum of stocks and commodities comes from the power law distribution of the returns. They also found that the multifractal spectrum for unshuffled commodity returns is broader than for shuffled commodity and for stock returns. They claim that the demand fluctuations drive price fluctuations, and thus it is plausible that stocks respond more quickly than commodities to changes in demand.

Sedlaříková (2016) wrote that there are two major known differences between the traditional simple stock market model and empirical financial data. The first difference is that empirical financial time series exhibit volatility clustering. This stylized fact was firstly described by Mandelbrot (1997), who observed that large price fluctuations are usually followed by other large fluctuations and vice versa. The second stylized fact is that empirical financial time series tend to have much fatter tails than the Gaussian distribution (Mandelbrot *et al.* (1997)).

In order to model volatility, which is changing in time, Engle (1982) proposed the autoregressive conditional heteroskedasticity (ARCH) model. This model was extended by Bollerslev (1986) allowing much more flexible structure. Nowadays, we have well-known general autoregressive conditional heteroskedasticity model, GARCH(p,q), where the conditional heteroskedasticity is expressed as a linear combination of the last p lagged squared returns and the last q lagged variances. There are various GARCH type models, for instance E-GARCH which allows also negative values of volatility, I-GARCH which imports a unit root into the GARCH process, T-GARCH which distinguish between positive and negative values, Q-GARCH which is able to model asymmetric effects of positive and negative shocks or GARCH-M which adds a heteroskedasticity term into the mean equation. These models are powerful tool for econometric modeling and they are also good benchmark for modeling volatility. However, GARCH type models have also certain drawbacks. According to Sedlaříková (2016), GARCH does not assume the two additionally recognized stylized facts of financial time series, which are long memory and multi-scaling. According to (Sedlaříková 2016, p. 2): "long memory can be described by an idea that the longest cycle in the sample is proportional to the length of the underlying time series". In order to capture the long memory of

a process, there is an alternative to GARCH model, the fractionally integrated GARCH and ARFIMA model.

Another disadvantage of the ARCH-type models is the fact that they assume scale consistency of the data. Assuming that data reveals equivalent statistical representations regardless its time scale, the researcher adds an additional restriction to the model when he or she chooses the frequency of collecting the data (Mandelbrot *et al.* (1997)). One way to solve this issue, is the scaling analysis, which looks at the financial data in different time intervals. The scaling analysis originally comes from econophysics but it has been also utilized in another fields, including research of financial markets (Di Matteo *et al.* (2005)). For instance, Di Matteo *et al.* (2005) conducted an empirical study of scaling properties of daily Foreign Exchange rates, Stock Market indices and fixed income instruments. They applied the generalized Hurst approach and found that the scaling exponents are associated with characteristics of the specific markets and moreover, they can utilize these characteristics in order to differentiate markets in their stage of development. According to Di Matteo *et al.* (2005), there are two types of scaling behavior that are studied in financial literature. The first one is the behavior of some forms of volatility measure (e.g.,: variance of returns, absolute value of returns) as a function of the time interval on which the returns are measured. These studies lead to the estimation of a scaling exponent related to the Hurst exponent. The second one is the behavior of the tails of the distribution of returns as a function of the size of the movement but keeping the time interval of the returns constant. These studies lead to the estimation of the tail index of the examined distribution. In a summary, Di Matteo *et al.* (2005) showed that the generalized Hurst exponent $H(q)$ is a powerful tool to characterize and to differentiate the structure of scaling properties of various financial time series. They also confirmed that $qH(q)$ exhibits a non-linear dependence on q. Therefore, they claim that it is a clear sign of the fact that the examined financial time series does not follow pure Brownian motion, and/or other additive or uni-scaling models. They also showed that there is a relationship between $H(2)$ and the level of development of financial markets. Developed markets such as: the NASDAQ 100 (US), the S&P500 (US), the Nikkei 225 (Japan), the Dow Jones Industrial Average (US), the CAC 40 (France), all exhibit $H(2) < 0.5$, i.e. more mean-reverting (anti-correlated) process than for the pure Brownian motion. On the other hand, for instance, the Russian AK&M, the Indonesian JSXC, and the Peruvian LSEG, all exhibit $H(2) > 0.5$, i.e. more trending process (serially correlated) than for

the pure Brownian motion. These results suggest that developing markets still exhibit some inefficiency, and there might be some information, which is not incorporated in the current price, and which could be possibly exploited.

Calvet *et al.* (1997), in their paper, discuss the local behavior of multifractal processes. They analyse the Multifractal Model of Asset Returns (MMAR) and show that the heterogeneity of local scales along the price process are caused by trading time. The MMAR model incorporates 4 stylized facts: volatility clustering, heavy tails, long-term dependence, and multiscaling.

Another stylized fact is multiscaling of moments. According to Kantelhardt (2008) in the field of time series, scaling in time of a quantity $F(s)$ (e.g., a moment of a sample statistics) means that with time re-scaling the behaviour of the quantity has asymptotically a following form: $F(s)^{\alpha}$ where α is a scaling exponent and s is a time interval on which the function F is computed.

Therefore, we can find some association between computed moments and varying sampling intervals. In case of fractal nature of time series, this number is non-integer (Kantelhardt (2008)).

If the behavior of different moments cannot be described by a single exponent, we speak about multi-scaling of moments and about multifractality of time series. Multifractality of time series can be caused by two sources. The first source of multifractality can be a broad distribution function. The second source of multifractality can be different long-term correlations of small and large fluctuations (Kantelhardt (2008)). Multifractal systems are typically studied in natural sciences where high number of micro components interact among each other and cause observable regularities on macro level (Čechová (2013)). We believe that the situation at the financial markets can be very similar. There are a lot of different traders with various trading strategies and different investment time periods.

According to Kantelhardt (2008), it is difficult to characterize and to understand of complex systems. It is not possible to split complex systems into simpler ones without tampering the dynamical properties.

One approach in studying complex systems is to obtain long time series of several selected variables of our interest which capture the behavior of the system in a dimensionally reduced representation. Some of these complex systems can be characterized by periodic or nearly periodic behaviour, which might be caused by oscillatory components or closed-loop regulation chains contained in a given system. Kantelhardt (2008) argues that in truly complex systems, there are wide spectrum of frequency bands, fluctuations on many time scales and

broad distributions of the values. Such complex systems can be characterized by scaling laws which are valid over a wide range of time scales or frequencies. Such a complex system, we denote fractal if only one scaling exponent is present. If there is a multitude of scaling exponents, we denote it as multifractal system. Kantelhardt (2008) also claims that fractal or multifractal scaling behavior or a complex system cannot be assumed a priori, because some fake scaling behavior can be caused by e.g., non-stationarities in the data. He also mentions two major issues of application of conventional statistical methods of complex time series.

The first issue is that the number and the length of available time series are limited. Therefore it can be difficult to gain significant information about given time series in a reliable way. The second issue, which Kantelhardt (2008) mentioned, is that the length of the data can be extended using computer-based recording techniques or by historical (proxy) data, but then non-stationarities in that time series tend to be superimposed upon the intrinsic fluctuation properties and measurement noise. These non-stationarities can be a major issue for the characterization of the complex dynamics and given time series, and in particular for finding the scaling properties of given data.

According to Kantelhardt (2008), there are two general types of multifractality in time series that can be distinguished. The first type of multifractality is due to a broad probability distribution (density function) for the values of the time series, e.g., a stable Levy distribution. In this case, we cannot remove the multifractality by random shuffling of the time series. The second type of multifractality is due to different long-term correlations of the small and large fluctuations. In this case, the probability density function of the values generated by a given process, can have a regular distribution with finite moments, e.g., a Gaussian distribution. In this case, we can randomly shuffle given time series and the corresponding shuffled series will no longer exhibit multifractal scaling, since all long-range correlations are destroyed by the shuffling procedure. Consequently, we can also observe a time series with both kinds of multifractality at the same time. In that case, the randomly shuffled time series will exhibit weaker multifractality than the original one.

Kantelhardt (2008) in his work wrote that a multifractal analysis of time series will also reveal higher order correlations. Moreover, multifractal scaling is also observed if the scaling behaviour of small and large fluctuations is different. For instance, extreme events in a given time series could be more or less correlated than typical events in a given time series.

Similarly, Kumar & Deo (2009), investigated multifractal properties of logarithmic returns of Indian financial indices Bombay Stock Exchange (BSE) and National Stock Exchange of India (NSE). By applying Multifractal Detrended Fluctuation Analysis (MF-DFA) procedure they found that the logarithmic returns exhibit multifractality caused by long-term correlations as well as by broad probability distribution. All the stock returns exhibit multifractality.

In another paper by Kwapień *et al.* (2005), multifractal properties of 5 minutes stock returns of 100 largest US companies according to their market capitalization were analyzed. They showed that the multifractality is caused by the long term temporal correlation for the vast majority of the stock returns. Moreover, they calculated the multifractality caused by long-term correlations $H(q)_{corr} = H(q) - H(q)_{shuffled}$. When they calculated the ratio $R = H(q)_{corr}/H(q)_{shuffled}$, the majority of stocks had the ratio R between 1 and 5. These findings that the multifractality comes mainly from temporal correlation are in contradiction with findings of Matia *et al.* (2003). Kwapień *et al.* (2005) attributes this discrepancy to the fact that they studied 5 minutes data while Matia *et al.* (2003) studied daily data. They claim that the daily data lack the level of richness of temporal correlation such as the high-frequency data do.

According to Jiang *et al.* (2019), there are two main approaches in multifractal analysis. The first one is the structure function approach and the second one is the partition function approach. In the first wave of multifractal analysis in econophysics, the structure function approach dominated. Since 2002, when Kantelhardt *et al.* (2002) published paper called Multifractal detrended fluctuation analysis of nonstationary time series, the partition function approach started to dominate in the field of financial time series. They developed the method Multifractal detrended fluctuation analyses (MF-DFA) which is based on the generalization of the Detrended fluctuation analysis (DFA). Testing of their new method, showed that it is reliable method for detecting the scaling behavior of multifractal time series. Moreover, by random shuffling the time series, they are able to distinguish between two main sources of multifractality, i.e. the long-term memory and the broad distribution density function. In the case that we are interested only in the multifractality caused by long-term correlations, we have to estimate the generalized Hurst exponent, then randomly shuffle given time series and then estimate the generalized Hurst exponent again. Consequently, we have to substract the generalized Hurst exponent of shuffled series from the generalized Hurst exponent of the unshuffled

series and then we get the value of the generalized Hurst exponent caused by the long-term correlations in a given time series.

In 2008, the third wave of simultaneous multifractal analysis of two time series started. It was triggered by the Podobnik & Stanley (2008), who introduced the detrended cross-correlation analysis for nonstationary time series. This work was consequently extended by Zhou *et al.* (2008) for analysis of multifractal time series. In many studies, multifractality of time series has been studied on daily data as well as on high frequency data, i.e. more frequent than daily data. In 2009, Su *et al.* (2009) showed that high frequency data exhibit stronger multifractality than the daily data. He applied the MF-DFA on the Taiwan Stock Exchange Capitalization Weighted Stock Index and also on 150 individual companies from Taiwan.

There is a tempting question, how to explain multifractality in time series from the economic perspective. In 2008, Dacorogna *et al.* (2001), proposed heterogeneous market agents hypothesis. He claims that financial markets are composed from many heterogeneous agents who have different investing frequencies. It seems to us as a very realistic view that the agents who are trading on the markets are heterogeneous and they have various investment strategies with different investment horizons. In accordance with this hypothesis, developed markets have all types of different agents. On the contrary, developing markets have only some types of agents while lacking another types of agents. The Hurst exponent should reflect the characteristics of financial markets described above. In 2009, Zunino *et al.* (2009) conducted a study which supports this heterogeneous market agents hypothesis. They studied the multifractal nature of daily price and volatility returns of Latin-American stock markets. They showed that emerging markets exhibit higher degree of multifractality than developed markets such as US market. They also analyzed the main sources of multifractality and concluded that for Latin-American markets, fat-tailed probability distributions are more significant than the long-term correlations. Another interesting studies track the development of the scaling exponents in time using sliding window for the computation. For instance, Bayraktar *et al.* (2004) conducted a study on how the $H(2)$, (i.e. the Hurst exponent with the moment q equal to 2), of returns of the S&P 500 index reacted to the change in frequency of trading in time around 1997. They showed that with the increase in the usage of internet trading, i.e. the number of traders and the availability of information, the Hurst exponent decreased, much closer to 0.5.

Barunik & Kristoufek (2010) also examined an interesting development of

the Hurst exponent in time. They applied Generalized Hurst exponent procedure (GHE) on high frequency data and computed the Hurst exponent for each day from 2.1.1983 to 30.11.2009. They found that there has been a significant shift from persistent time series to independent (or short-term correlated) time series. They proposed that an increase in trading volumes can be responsible for this result.

Another interesting study was conducted by Morales *et al.* (2012) who examined multifractal behavior of individual companies. They used a moving window in order to analyze changes in the scaling properties during the financial crisis. They suggested that the multifractality of returns increased in the financial crisis, and that the multifractal properties could be possibly used as an indicator of stability of a given company.

Grech & Mazur (2004) tried to utilize $H(2)$ as an indicator of financial crises. They found that the $H(2)$ of Dow Jones Industrial Average index (DJIA) exhibits decreasing trends before financial crisis in years 1929, 1987, and in 2008.

Kristoufek (2012) examined three most liquid stock US indices, DJIA, NASDAQ and S&P 500, in order to test Fractal market hypothesis (FMH). In its core, the Fractal market hypothesis is based on a variable completely omitted in Efficiency market hypothesis (EMH), which is liquidity. In the view of FMH, there are many different investors with different investment horizons in the market. If these investors are uniformly or almost uniformly distributed, the supply and demand sides can relatively easily meet their desires. If they are not, there is excess supply or excess demand for some group of investors at the market, and therefore the market can get unstable. In accordance with the FMH, liquidity provides smooth pricing process in the market, and consequently making market stable. If there is not enough liquidity on a market, the market becomes unstable and consequently extreme movements occur. Kristoufek (2012) analyzed whether it would be possible to predict financial crisis utilizing the Hurst exponent during the Global Financial crisis 2007/2008. He utilized a sliding window estimation of generalized Hurst exponent $H(q)$ with $q = 2$, which is usually called local or time-dependent Hurst exponent. The three examined indices DJIA, NASDAQ and S&P 500 reached their maximum values around the end of 2007 and then significantly declined for more than 50% of their values. Kristoufek (2012) found the local Hurst exponent $H(2)$ followed strong decreasing trend from the break of 2005 and 2006 until the beginning of 2007. After that, $H(2)$ remained below the critical value of 0.5,

which is observed for time series generated by random processes, for a rather long period. In a summary, Kristoufek (2012) found that the behavior of investors with different investment horizons can be quite well described by FMH in times before and during the Global Financial Crisis 2007/2008.

Another model for volatility forecasting and pricing assets is the Markov Switching Multifractal (MSM) model proposed by Calvet & Fisher (2004). According to Calvet & Fisher (2008), this model provides reliable volatility forecasts, especially for long time periods. According to Barunik *et al.* (2012) and Morales *et al.* (2013), this model outperforms GARCH(1,1) and Fractionally Integrated Generalized Autoregressive Conditional heteroskedasticity (FIGARCH), in sample as well as out of sample. MSM is used by academics as well as by practitioners in the financial industry in order to forecast volatility, compute value-at-risk, and in order to price derivatives.

Regarding forecasting and predicting the financial markets, there are plenty of econometric and machine learning models. As a basic econometric model for modeling the financial returns, there is the ARIMA model, proposed by Box & Jenkins (1976). There are a several studies, where the ARIMA model was applied for forecasting stock markets returns Al-Shiab (2006), Ojo & Olatayo (2009), Mondal *et al.* (2014) and Adebiyi *et al.* (2014). Financial markets are very complex with various structural breaks, jumps and non-linearities. In order to capture some non-linearities, for instance Hamilton (1989) and Tong *et al.* (2014) applied Markov regime-switching models and threshold autoregressive (TAR).

In the last decades, Artificial Neural Networks (ANN) and Recurrent Neural Networks (RNN) became very popular, not only in forecasting of financial markets but also in another fields, e.g.: image recognition Yang *et al.* (2000), face recognition Magesh Kumar *et al.* (2011), handwriting recognition Graves *et al.* (2008), or speech recognition Sak *et al.* (2014).

According to Mallikarjuna & Rao (2019), the ANN models are very popular in forecasting, because they are data-driven, nonlinear, and self-adaptive. There are many studies which have shown that prediction models based on ANN and RNN exhibit more accurate forecasts compared to more naive and/or linear models (e.g., Aras & Kocakoç (2016), Qiu *et al.* (2016) and Mostafa (2010)).

A possible way how to connect assets pricing and explain the variation of asset's and market's returns is Fama-MacBeth regression. We describe the Fama-MacBeth regression in more detail in section 5.2. This procedure was

firstly described by Fama & MacBeth (1973) and it is a useful method to estimate the parameters of Capital Asset Pricing Model (CAPM) (Sharpe (1964) and Lintner (1965)) and its extensions. The CAPM models the relationship between the expected return of an asset and the market risk. Fama & French (2021) extended the CAPM model to three-factor model. They added two new factors. The first factor is "small minus big" factor (SMB factor) according to firm's market capitalization. The second factor is "book-to-market"/"High minus Low" factor (HML factor), which compares the ratio of book value of a firm and market value of that firm. They showed that the three-factor model explains the expected returns better than the CAPM. Carhart (1997) estimated and compared three different models: CAPM, three-factor and four-factor model. The CAPM model explains the variation in returns by the average market return and by a risk free rate of return. The three factor model is extended by SMB factor and HML factor. The four-factor model is the three-factor model extended by "one year momentum in stock returns" (MOM). He showed that the three-factor model outperforms the CAPM and the four-factor model outperforms the three-factor model. Fama & French (2015) extended the three-factor model to five-factor model by adding "Robust minus Weak " factor (RMW factor) and "Conservative minus Aggressive" factor (CMA factor). RMW factor distinguishes firms between firms with high ("Robust") and low ("Weak") operating profit. CMA factor distinguishes firms between firms that invest conservatively and firms that invest aggressively.

In our thesis, we would like to move the research of multifractality to the next level. We would like to examine and to test, whether there is or there is not a risk premia on financial markets for holding a "multifractal portfolio". We expect that there could be a different risk and different price for holding a more "multifractal assets" whose price should contain more patterns than for less or even not at all "multifractal assets" such as a random walk. We employ Multifractal Detrended Fluctuation Analysis (MF-DFA) in order to estimate the generalized Hurts exponent and the singularity strength in order to evaluate the multifractality of our examined financial time series. Then, we employ the Fama-MacBeth procedure in order to find whether there is a risk premia for holding "a multifractal portfolio" or not. According to our best knowledge, there is no scientific work which employs the Fama-MacBeth regression with factor based on estimated multifractality.

Chapter 3

Methodology

3.1 Definitions

3.1.1 Definition of multifractal process

Process X_t is considered multifractal if it has stationary increments which scale as:

$$\langle \mid X_{t+\tau} - X_t \mid^q \rangle \quad \propto \quad \tau^{qH(q)}$$

for integer $\tau > 0$ and for all q.

$H(q)$ is called generalized Hurst exponent and its dependence on q separates the processes into two categories monofractal (or unifractal) for constant $H(q)$ and multifractal when $H(q)$ is a function of q. For $q = 2$, we consider long-range dependence of the increments of the process X_t (Calvet & Fisher (2008), Kristoufek (2012)).

3.1.2 Definition of scaling law

Scaling law is a power law with a scaling exponent (e.g. α) describing the behavior of a quantity F as a function of a scale parameter s at least asymptotically: $F(s) \sim s^{\alpha}$. The power law should be valid for a large range of s values, e.g., at least for one order of magnitude.

3.1.3 Definition of multifractality of time series

According to Calvet & Fisher (2002) a stochastic process X_t is called multifractal if it has stationary increments and it holds that:

$$(\mid X_t \mid^q) = c(q)t^{\tau(q)+1}.$$

Here, $\tau(q)$ is the scaling function of the multifractal process, which is linear for unifractal (monofractal) processes and concave for multifractal processes, and q is the moment order. The scaling function $\tau(q)$ can be translated into the language of the generalized Hurst exponents $H(q)$ as:

$$H(q) = \frac{1 + \tau(q)}{q}.$$

3.1.4 Definition of Hurst exponent

The Hurst exponent, H, is defined in terms of the asymptotic behaviour of the rescaled range as a function of the time span of a time series as follows:

$$\left[\frac{R(n)}{S(n)}\right] = Cn^H \quad as \quad n \to \infty$$

where

- $R(n)$ is the range of the first n cumulative deviations from the mean
- $S(n)$ is the series (sum) of the first n standard deviations
- E is the expected value
- n is the time span of the observation (number of data points in a time series)
- C is a constant.

The values of Hurst exponent range between 0 and 1. For an asymptotically uncorrelated process, we get $H = 0.5$. For more trending (persistent) processes, we get $H > 0.5$, and for more mean-reverting processes, we get $H < 0.5$. It is also important to mention that even uncorrelated financial time series can have $H \neq 0.5$ mainly due to heteroskedasticity, short-term memory and fat tails (Kristoufek & Vosvrda (2013)).

3.1.5 Singularity strength and singularity spectrum

Singularity strength α and singularity spectrum $f(\alpha)$ are defined as:

$$\alpha = \frac{\partial \tau(q)}{\partial q}$$

$$f(\alpha) = q\alpha - \tau(q)$$

The relationships among the Hurst exponent, singularity strength and singularity spectrum can be captured by following equations:

$$\alpha(q) = H(q) + qH'(q))$$

$$f(\alpha) = q(\alpha - H(q)) + 1$$

3.2 Procedures

3.2.1 Multifractal detrended fluctuation analysis (MF-DFA)

Following the paper of Matia *et al.* (2003):

1. Calculate the profile

$$Y_i \equiv \sum_{k=1}^{i} (x_k - \bar{x}), i = 1, \ldots, N,$$

   where N is the length of the time series, and $\bar{x}$ is the mean of x_k.

2. Divide Y_i into $N_s \equiv int(N/s)$ segments, and s is the length of one segment.

3. Calculate the local trend $y_\nu(i)$ for segments $\nu = 1, \cdots, N_s$ by least square polynomial fit.

4. Determine the mean square fluctuation in each segment

$$F_2(s, \nu) \equiv \frac{1}{s} \sum_{i=1}^{s} (Y_{(\nu-1)s+i} - Y_\nu(i))^2$$

5. Evaluate

$$F_q(s) \equiv \frac{1}{N_s} \sum_{\nu=1}^{N_s} F_2(s, \nu)^{(q/2)}$$

The scaling function of moment $q, F_q(s)$ follows scaling law:

$$F_q(s) \sim s^{\tau(q)}$$

Negative q values give more weight to small fluctuations, while positive values of q give more weight to large fluctuations. When the contribution of the small fluctuations is comparable to the large fluctuations, the series is $monofractal$, and $\tau(q) = hq$, where $h = constant$ and it is the Hurst exponent. If $\tau(q)$ non-linearly depends on q, the series is $multifractal$.

3.2.2 Multifractal detrended moving average method (MF-DMA)

Following the master thesis of Sedlaříková (2016):
The multifractal detrending moving average (MF-DMA) method consists of six steps:
Step 1: Consider a time series $x_t, \quad t = 1, \ldots, N$. Then, construct the integrated time series, the "profile", as:

$$Y(i) = \sum_{k=1}^{i} (x_k - \langle x_i \rangle), \quad i = 1, \ldots, N.$$

Note that, as in the MF-DFA procedure, the mean subtraction is not mandatory, since it would be eliminated by local detrending later (Wang *et al.* (2011)).
Step 2: Calculate the moving average function $\tilde{Y}(i)$ in a moving window

$$\tilde{Y}(i) = \frac{1}{s} \sum_{k=-\lfloor (s-1)\theta \rfloor}^{\lceil (s-1)(1-\theta) \rceil} Y(i-j)$$

where s denotes the windows size, $\lfloor x \rfloor$ is the largest integer not greater than x, $\lceil x \rceil$ is the smallest integer not smaller than x, and θ is the position parameter varying in the range $[0, 1]$. Hence, the moving average function considers the past $\lceil (s-1)(1-\theta) \rceil$ data points and the future $\lfloor (s-1)\theta \rfloor$ data points. Gu *et al.* (2010) consider the following three special cases. The first case $\theta = 0$ refers to the backward moving average, in which the moving average function $\tilde{Y}(i)$ is calculated over all the past $s-1$ data points. The second case $\theta = 0.5$ corresponds to the centered moving average, where $\tilde{Y}(i)$ contains half past and half future data in each window. The third case $\theta = 1$ refers to the forward moving average, where $\tilde{Y}(i)$ considers the trend of s future data points (Arianos

& Carbone (2007), Gu *et al.* (2010)).

Step 3: Detrend the signal series by removing the moving average function $\tilde{Y}(k)$ from $Y(k)$, and obtain the residual sequence $\epsilon(k)$ through:

$$\epsilon(k) = Y(k) - \tilde{Y}(k)$$

where $s - \lfloor (s-1)\theta \rfloor \leq k \leq N - \lfloor (s-1)\theta \rfloor$.

Step 4: Divide the residual sequence into N_s non-overlapped segments with the equal length s, where $N_s = \lfloor N/s - 1 \rfloor$. Each segment can be denoted by $_\nu$ such that $\epsilon_\nu(k) = \epsilon(l+k)$ for $1 \leq k \leq s$, where $l = (\nu - 1)s$. Then we can define the fluctuation variance as:

$$F_\nu^2(s) = \frac{1}{s} \sum_{k=1}^{s} \epsilon_\nu^2(k).$$

Step 5: Calculate the $q-th$ order fluctuation function using the equation:

$$F_q(s) = \left\{ \frac{1}{N_s} \sum_{\nu=1}^{N_s} F_\nu^q(s) \right\}^{1/q},$$

where q can take any real value except for $q = 0$. When $q = 0$, according to L'Hôspital's rule, we have:

$$ln[F_0(s)] = \frac{1}{N_s} \sum_{\nu=1}^{N_s} ln[F_\nu(s)]$$

Step 6: Varying the value of segment size s, we can determine the power law relation between the function $F_q(s)$ and the size scale n through the following equation:

$$F_q(s) \sim s^{h(q)}$$

3.3 Models

3.3.1 CAPM model

Capital Asset Pricing Model (CAPM) was independently developed by several authors, namely Treynor (1961), Sharpe (1964), Lintner (1965), and Mossin (1966), who extended the earlier work of Harry Markowitz on diversification and modern portfolio theory. The CAPM model can be described by the following

equation:

$$\mathbb{E}(R_i) = R_f + \beta_i(\mathbb{E}(R_m) - R_f)$$

where:

- $\mathbb{E}(R_i)\ldots$ is the expected return on the capital asset
- $R_f\ldots$ is the risk-free rate of interest such as interest arising from government bonds
- $\beta_i\ldots$ is the sensitivity of the expected excess asset returns to the expected excess market returns
- $\mathbb{E}(R_m)\ldots$ is the expected return of the market
- $\mathbb{E}(R_m) - R_f\ldots$ is sometimes known as the market premium (the difference between the expected market rate of return and the risk-free rate of return)
- $\mathbb{E}(R_i) - R_f\ldots$ is also known as the risk premium

This CAPM can be also described in the terms of risk premium:

$$\mathbb{E}(R_i) - R_f = \beta_i(\mathbb{E}(R_m) - R_f)$$

which states that the individual risk premium equals the market premium times β.

Assumptions of CAPM

Following Arnold (2008), all investors:

1. Aim to maximize economic utilities (Asset quantities are given and fixed).
2. Are rational and risk-averse.
3. Are broadly diversified across a range of investments.
4. Are price takers, i.e., they cannot influence prices.
5. Can lend and borrow unlimited amounts under the risk free rate of interest.
6. Trade without transaction or taxation costs.
7. Deal with securities that are all highly divisible into small parcels (All assets are perfectly divisible and liquid).

8. Have homogeneous expectations.

9. Assume all information is available at the same time to all investors.

According to the CAPM, we can describe the expected return $\mathbb{E}(R_i)$ of an asset i by the risk-free rate of return R_f plus beta (β_i) times the market premium $\mathbb{E}(R_m) - R_f$. The market premium is the expected return of overall market $\mathbb{E}(R_m)$ minus the risk-free rate of return R_f. If the beta is greater than 1, than such asset‘s returns jumps up and down more than the market return. If the beta is lower than 1, such asset's return jumps up and down less than the market return. If the beta is equal to 1, such asset‘s return follows the market return.

3.3.2 Fama-French three factor model

Fama-French three factor model was developed by Fama & French (2021). It is an extension of CAPM model, where there are two other factors added, namely: SMB (Small Minus Big) and HML (High Minus Low). We can distinguish between “big” and “small” companies according to their market capitalization. According to the theory, “big” companies are more stable and therefore the volatility of their stocks is lower, thus expected risk is lower and expected returns are lower than for “small” companies. Regarding the HML factor, we can distinguish between companies with “high” book-to-market ratio and companies with “low” book-to-market ratio. When the market price of all stocks is lower than the book price of a given firm, then these stocks are called “value stocks”. When the market price of all stocks is higher than the book price of a given firm, then these stocks are called “growth stocks”.

The Fama-French three factor model can be described by the following equation:

$$\mathbb{E}(R_i) = R_f + \beta(\mathbb{E}(R_m) - R_f) + \beta_1 SMB + \beta_2 HML$$

The β_1 measures the sensitivity of given stock return on the overall markets returns. The β_2 is the sensitivity of expected price of a given stock on SMB factor and β_3 is the sensitivity of a given stock on HML factor.

3.3.3 Fama-MacBeth regression

The Fama-MacBeth regression or procedure was introduced by Fama & MacBeth (1973). This method is used to estimate parameters for asset pricing models such as the CAPM. The method works with a panel data, i.e. with one cross-sectional dimension (usually for multiple different assets) and with one time dimension (e.g.: daily returns of given assets). We can apply this method in order to estimate the betas and risk premia for any risk factors that are expected to determine asset prices. The parameters are estimated in two steps.

1. Firstly, we regress each of n assets returns against m proposed risk factors to determine each asset's beta exposures:

$$R_{1,t} = \alpha_1 + \beta_{1,F_1} F_{1,t} + \beta_{1,F_2} F_{2,t} + \cdots + \beta_{1,F_m} F_{m,t} + \epsilon_{1,t}$$

$$R_{2,t} = \alpha_2 + \beta_{2,F_1} F_{1,t} + \beta_{2,F_2} F_{2,t} + \cdots + \beta_{2,F_m} F_{m,t} + \epsilon_{2,t}$$

$$\vdots$$

$$R_{n,t} = \alpha_n + \beta_{n,F_1} F_{1,t} + \beta_{n,F_2} F_{2,t} + \cdots + \beta_{n,F_m} F_{m,t} + \epsilon_{n,t}$$

where

- $R_{1,t}, R_{2,t}, \ldots, R_{n,t}$ are assets returns through time t
- $F_{1,t}, F_{2,t}, \ldots, F_{m,t}$ are proposed factors
- $\alpha_1, \alpha_2, \ldots, \alpha_n$ are intercepts
- $\beta_{i,F_1}, \beta_{i,F_2}, \ldots, \beta_{i,F_m}, i = 1, \ldots, n$ are the sensitivities of asset's returns on factors
- $\epsilon_{1,t}, \epsilon_{2,t}, \ldots, \epsilon_{n,t}$ are the error terms

From the first step, we obtain $(n \times m)$ beta's, which tell us how the returns of individual assets react to changes in given factors $F_{1,t}, F_{2,t}, \ldots, F_{m,t}$.

2. In the second step, we calculate the sample mean of individual asset's returns with respect to time. Then, we estimate the following equation:

$$\overline{R}_i = \gamma_0 + \gamma_1 \hat{\beta}_{i,F_1} + \gamma_2 \hat{\beta}_{i,F_2} + \ldots \gamma_m \hat{\beta}_{i,F_m} + \nu_i$$

where

$$\overline{R}_i = \frac{\sum_{t=1}^{T} R_{i,t}}{T}, \quad i = 1, \ldots, n.$$

Chapter 4

Data

4.1 Data analysis

4.1.1 Obtaining data

We downloaded the dataset from the Oxford-Man Institute of Quantitative Finance, which provides free datasets on returns and multiple versions of realized volatility stock indices. The "Oxford-Man Institute's realised library" is a project organised by the Oxford-Man Institute at the University of Oxford established in 2007. The data samples cover the period from January 3, 2000 to June 1, 2021. The data, its documentation and its pre-processing is available on: https://realized.oxford-man.ox.ac.uk/. The dataset contains 20 different variables of 31 stock indices from whole world, e.g.: Dow Jones Industrial Average, S&P500, FTSE 100, NASDAQ 100, EURO STOXX 50, AEX index, DAX, NIFTY, Oslo Exchange All-share index, Shanghai Composite index or Swiss Stock Market Index. The second part of our dataset is the risk-free rate of return proxied by the 10-year Treasury Constant Maturity Rate government bonds downloaded from: https://fred.stlouisfed.org/series/DGS10. These data are rate of returns per annum. Therefore, we divided the data by 365 in order to get daily returns.

In order to have equally long time series, we restricted all time series to 2192 observation, what is the length of the shortest one, .BVLG stock index. In the table 4.2, we provide the sample mean and the sample standard deviation of closing prices and log-returns for all 31 indices, and the risk-free rate of return through the time period from October 29, 2012 until June 1, 2021, i.e. 2192 days.

Table 4.1: Lengths of time series

Lengths of time series		
ticker	name	number of observations
.AEX	AEX index	5457
.AORD	All Ordinaries	5407
.BFX	Bell 20 Index	5455
.BSESN	S&P BSE Sensex	5312
.BVLG	PSI All-Share Index	2192
.BVSP	BVSP BOVESPA Index	5267
.DJI	Dow Jones Industrial Average	5364
.FCHI	CAC 40	5459
.FTMIB	FTSE MIB	3040
.FTSE	FTSE 100	5398
.GDAXI	DAX	5425
.GSPTSE	S&P/TSX Composite index	4770
.HSI	HANG SENG Index	5243
.IBEX	IBEX 35 Index	5424
.IXIC	Nasdaq 100	5368
.KS11	Korea Composite Stock Price Index (KOSPI)	5267
.KSE	Karachi SE 100 Index	5214
.MXX	IPC Mexico	5368
.N225	Nikkei 225	5203
.NSEI	NIFTY 50	5308
.OMXC20	OMX Copenhagen 20 Index	3886
.OMXHPI	OMX Helsinki All Share Index	3927
.OMXSPI	OMX Stockholm All Share Index	3927
.OSEAX	Oslo Exchange All-share Index	4915
.RUT	Russel 2000	5365
.SMSI	Madrid General Index	4053
.SPX	S&P 500 Index	5367
.SSEC	Shanghai Composite Index	5169
.SSMI	Swiss Stock Market Index	5361
.STI	Straits Times Index	3424
.STOXX50E	EURO STOXX 50	5456
risk-free rate	10-year Treasury Constant Maturity Rate	2192

4.1.2 Differences between stock indices

The different characteristics and behavior of prices and volatilities of examined stock indices can be influenced by many factors. For instance, what country the index comes from, how many of individual stocks is the index composed, what are the industries or business sectors of traded companies, how old is the index etc. AEX is Amsterdam Exchange index established in 1983. It is based on 25 most frequently traded stock on the Euronext Amsterdam, formerly known as the Amsterdam stock exchange. AORD stands for "All ordinaries" and the index was established in 1980. It is the oldest share index in Australia and it is composed by the 500 shares of largest companies traded on the Australian Securities Exchange (ASX). BFX stands for Bell 20 index and it is the benchmark stock market index of Euronext Brussels. This index is composed by a minimum of 10 and a maximum of 20 companies traded at the Brussels Stock Exchange. Bell 20 index was founded in 1990. S&P BSE Sensex is a free-float market-weighted stock market index of 30 well-established and financially sound companies traded on Bombay Stock Exchange. This index was founded in 1986 and it is composed by stocks of companies from different industries in India. BVLG stands for PSI All-Share Index and the index was initiated in 1992. It is a Portuguese benchmark stock market index of companies that trade on Euronext Lisbon. It is composed by the 20 largest stocks according to the market capitalization of a given underlying company. BVSP stands for BVSP BOVESPA Index, Brazilian stock index, established in 1968. It is composed by 20 stock the most frequently traded or with the largest market capitalization on B3 (Brasil Bolsa Balcão). DJI stands for Dow Jones Industrial Average and the index was established in 1896. It is a stock market index that measures the stock performance of 30 large companies listed on stock exchanges in the United States. FCHI stands for CAC 40 is a benchmark French stock market index, which represents a capitalization-weighted measure of the 40 most significant stocks among the 100 largest market caps on the Euronext Paris. It was founded in 1987. FTMIB stands for FTSE MIB what is Italian stock index composed of the 40 most-traded stock classes on the exchange. FTSE stands for FTSE 100 (Financial Times Stock Exchange 100). It is a stock index consisting of 100 companies with the largest market capitalization, located UK, and traded on the London Stock Exchange. The index was founded in 1984. GDAXI stands for DAX and it is German Stock index consisting of the 30 major German companies trading on the Frankfurt

Stock Exchange. DAX was founded in 1988. GPTSE stands for S&P/TSX Composite index is the Canadian benchmark index composed by around 250 stocks traded on Toronto Stock Exchange. HSI stands for HANG SENG Index and it is a free float-adjusted market-capitalization-weighted stock-market index in Hong Kong. This index is composed by 50 stocks traded on Hong Kong stock exchange and it was founded in 1969. IBEX stands for IBEX 35 Index and it is Spanish stock index composed by 35 most liquid stock traded in the Madrid Stock Exchange General Index and is reviewed twice annually. IXIC stands for NASDAQ 100, which a stock index composed mainly by big technological companies such as Alphabet Inc. (Google), Amazon.com or Apple Inc. In is composed by 102 equity securities issued by 100 of the largest non-financial companies traded on Nasdaq stock market. KS11 stands for Korea Composite Stock Price Index (KOSPI). As of February 2021, KOSPI has over 880 components and it represents the stock market of South Korea. KSE stands for Karachi SE 100 Index and it is composed by stocks from Pakistan Stock Exchange (PSE). These stocks are the top largest companies in terms of market capitalization selected from 34 different sectors on the PSX. MXX stands for IPC Mexico index. This index is based on the 35 largest and the most liquid stocks listed on the Bolsa Mexicana de Valores. N225 stands for Nikkei 225, the index composed by the 225 most important stocks traded on the Tokyo Stock Exchange. NSEI stands for NIFTY 50 and it is a benchmark Indian stock market index that is composed by the weighted average of 50 of the largest Indian companies listed on the National Stock Exchange. OMXC20 stands for OMX Copenhagen 20 Index and this index is composed by the 20 most actively traded Copenhagen Stock Exchange. OMXHPI stands for OMX Helsinki All Share Index. This stock index is composed by all stocks listed on the Helsinki Stock Exchange. OMXSPI stands for OMX Stockholm All-share Index and it is composed by all the stocks traded on OMX Nordic Exchange Stockholm. OSEAX stands for Oslo Exchange All-share Index which is composed by all stocks traded on the Oslo Stock Exchange. RUT stands for Russel 2000 and this index is composed by 2000 stock with the lowest market capitalization listed on the Russel 3000 index. This index was founded in 1984. SMSI stands for Madrid General Index which is market capitalization-weighted index that measures the performance of Madrid Stock Exchange. SPX stands for S&P 500 which is a free-float, weighted measurement stock market index of the 500 large companies traded on stock exchanges in the United States. S&P 500 is composed by well-known companies such as: Apple Inc. Microsoft, Ama-

zon.com, Facebook, Alphabet Inc. or Tesla Inc. SSEC stands for Shanghai Composite Index composed by all stocks traded on the Shanghai Stock Exchange. SSMI stands for Swiss Stock Market Index is Switzerland's blue-chip stock market index. This index is composed by 20 the largest and the most liquid stocks of Swiss Performance Index. STI stands for Straits Times Index. It is a market capitalisation weighted index that tracks the performance of the top 30 companies traded on the Singapore Stock Index. STOXX50E stands for EURO STOXX 50, a stock index which is composed by the largest and most liquid European stocks, e.g.: Adidas, Allianz, BMW, ING, SAP SE or Volkswagen Group. In a summary, all indices are very different in the numbers of stocks, history or the business sector of traded companies.

4.1.3 Descriptive statistics of the data

In this subsection we will provide descriptive statistics of our data. In the table 4.1, we provide the list of names, abbreviations (tickers) and lengths of all 31 examined times series. We can observe that these indices come from a lot of different countries with economies developed at different levels. For instance, the indices comes from USA, Canada, Mexico, Germany, Switzerland, Norway, Spain, China, South Korea, Japan, Brazil, India, Pakistan and many others. We also include these information about the risk-free rate variable. We can observe that the lengths ranges between 2192 observations for the .BVLG stock index, and 5459 observations for the .FCHI stock index. Firstly, we shortened all time series to the length of the shortest one, with 2192 observations, what corresponds approximately to 8 years and 7 months. We calculated sample mean and sample standard deviation for close-prices and for log-returns. The results are provided in table 4.2. From the table, we can observe that the sample mean of close price ranges between 419 for the .OMXSPI stock index, and 128267 for .BVSP stock index. Sample standard deviation of close prices ranges between 86 for .AEX stock index, and 22604 for the .BVSP stock index. Thus, we can observe that our stock indices differ a lot also in terms of sample mean and standard deviation. We provide also table 4.3, where are captured 6 standard statistic measures, minimum, first quartile, median, sample mean, third quartile, and the maximum value. All the statistics for prices are positive numbers as we would expect. 14 out of 31 stock index prices have mean higher than median which tells us that the distribution of prices are skewed right. 2 stock index prices have equal mean and median, which suggest that their

distributions are not skewed. The distribution of the rest 15 stock index prices are skewed left.

Table 4.2: Descriptive statistics

Descriptive statistics				
	close price		log-returns	
index	sample mean	std	sample mean	std
.AEX	489	86	0.00036	0.01082
.AORD	5786	650	0.00024	0.00889
.BFX	3444	433	0.00025	0.01113
.BSESN	30392	8095	0.00050	0.01083
.BVLG	2834	396	0.00025	0.01133
.BVSP	71244	22604	0.00040	0.01638
.DJI	21137	5275	0.00044	0.01085
.FCHI	4847	636	0.00029	0.01196
.FTMIB	20194	2469	0.00021	0.01480
.FTSE	6783	539	0.00009	0.01003
.GDAXI	11064	1851	0.00034	0.01220
.GSPTSE	14996	1618	0.00023	0.00904
.HSI	24854	3055	0.00019	0.01122
.IBEX	9318	1137	0.00007	0.01314
.IXIC	6429	2655	0.00068	0.01204
.KS11	2146	284	0.00026	0.00931
.KSE	34874	8962	0.00055	0.01057
.MXX	44099	3568	0.00011	0.00972
.N225	18770	4643	0.00055	0.01335
.NSEI	9156	2377	0.00050	0.01086
.OMXC20	925	248	0.00054	0.01093
.OMXHPI	8631	1448	0.00037	0.01069
.OMXSPI	541	120	0.00047	0.01046
.OSEAX	777	185	0.00044	0.01070
.RUT	1349	307	0.00047	0.01343
.SMSI	940	120	0.00005	0.01283
.SPX	2417	639	0.00050	0.01063
.SSEC	2897	566	0.00020	0.01365
.SSMI	8889	1004	0.00026	0.00977
.STI	2938	421	0.00020	0.00982
.STOXX50E	3260	325	0.00023	0.01203
risk-free rate			5.76e-05	1.72e-05

We calculated log-returns for all 31 close prices and their descriptive statistics. From the table 4.2, we can observe that the sample mean of log-returns ranges between 0.00005 for the .SMSI index and 0.00068 for the .IXIC index.

Table 4.3: Descriptive statistics of prices

Distribution of close prices						
index	Min.	1st Qu.	Median	Mean	3rd Qu.	Max.
.AEX	320	417	495	489	552	718
.AORD	4360	5317	5724	5786	6183	7427
.BFX	2302	3168	3523	3444	3774	4198
.BSESN	16657	25266	28650	30392	36236	52216
.BVLG	2054	2504	2829	2834	3118	4114
.BVSP	37516	53034	61817	71244	87940	128267
.DJI	12541	16844	19872	21137	25602	34773
.FCHI	3348	4362	4889	4847	5349	6490
.FTMIB	14856	18392	20494	20194	22106	25478
.FTSE	4966	6391	6783	6783	7257	7897
.GDAXI	6958	9658	11319	11064	12485	15569
.GSPTSE	11722	13831	15127	14996	16108	19978
.HSI	18322	22627	24350	24854	27405	33112
.IBEX	6058	8591	9352	9318	10223	11869
.IXIC	2837	4549	5550	6429	7824	14139
.KS11	1457	1970	2044	2146	2251	3252
.KSE	14339	29940	35198	34874	41494	52920
.MXX	32964	41904	44045	44099	46660	51707
.N225	8296	15587	19400	18770	21954	30468
.NSEI	5047	7629	8747	9156	10859	15576
.OMXC20	474	744	961	925	1018	1590
.OMXHPI	5285	7645	8821	8631	9653	12226
.OMXSPI	319	454	541	541	593	922
.OSEAX	462	628	750	777	951	1211
.RUT	770	1141	1284	1349	1543	2360
.SMSI	606	863	943	940	1035	1205
.SPX	1353	1965	2269	2417	2834	4232
.SSEC	1951	2331	2976	2897	3269	5166
.SSMI	6363	8224	8830	8889	9440	11435
.STI	2038	2576	3012	2938	3242	3884
.STOXX50E	2394	3037	3279	3260	3501	4072

The standard deviations ranges between 0.00889 for .AORD and 0.01638 for .BVSP. We calculated also the 6 standard measures, minimum, first quartile, median, sample mean, third quartile and the maximum value, for the realized variance. These results are displayed in the table 4.4. The distributions of log-returns are approximately centered around zero, as we would expect. However, all the means and medians are small positive numbers.

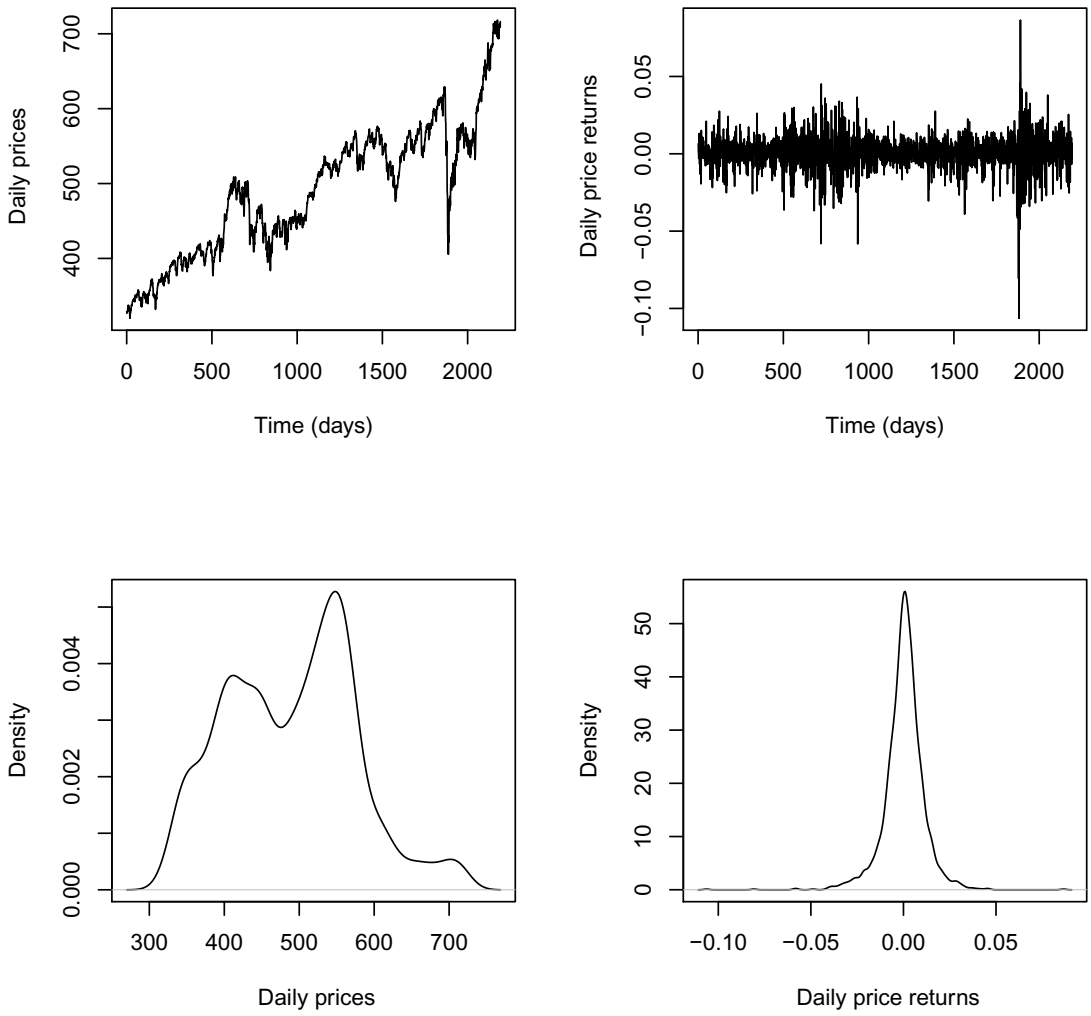

Figure 4.1: Close price and log-returns of AEX index

In the figure A.1, we present the plots for AEX index. In the first row from the left, we present daily close price and daily log-returns. In the second row, we present probability density function of daily close prices and probability density function of daily log-returns. We can observe that the daily close

Table 4.4: Distribution of log-returns

Distribution of log-returns						
index	Min.	1st Qu.	Median	Mean	3rd Qu.	Max.
.AEX	-0.10639	-0.00458	0.00078	0.00036	0.00569	0.08646
.AORD	-0.07059	-0.00429	0.00082	0.00024	0.00501	0.04443
.BFX	-0.14221	-0.00477	0.00063	0.00025	0.00584	0.06897
.BSESN	-0.13781	-0.00457	0.00068	0.00050	0.00592	0.08245
.BVLG	-0.10935	-0.00528	0.00065	0.00025	0.00627	0.06532
.BVSP	-0.15994	-0.00823	0.00030	0.00040	0.00930	0.13023
.DJI	-0.13807	-0.00322	0.00068	0.00044	0.00509	0.10754
.FCHI	-0.11997	-0.00499	0.00053	0.00029	0.00605	0.07789
.FTMIB	-0.18541	-0.00668	0.00070	0.00021	0.00802	0.08549
.FTSE	-0.10137	-0.00454	0.00043	0.00009	0.00516	0.07782
.GDAXI	-0.11863	-0.00519	0.00088	0.00034	0.00642	0.09763
.GSPTSE	-0.13176	-0.00323	0.00076	0.00023	0.00433	0.09120
.HSI	-0.05980	-0.00560	0.00044	0.00019	0.00646	0.04374
.IBEX	-0.12718	-0.00629	0.00040	0.00007	0.00709	0.08111
.IXIC	-0.13141	-0.00382	0.00114	0.00068	0.00671	0.08926
.KS11	-0.10179	-0.00382	0.00046	0.00026	0.00503	0.07111
.KSE	-0.07258	-0.00446	0.00072	0.00055	0.00621	0.04684
.MXX	-0.06937	-0.00495	0.00013	0.00011	0.00553	0.05088
.N225	-0.08253	-0.00566	0.00081	0.00055	0.00744	0.07731
.NSEI	-0.13633	-0.00468	0.00078	0.00050	0.00614	0.08047
.OMXC20	-0.07821	-0.00530	0.00092	0.00054	0.00693	0.05143
.OMXHPI	-0.10788	-0.00494	0.00069	0.00037	0.00616	0.06192
.OMXSPI	-0.11805	-0.00437	0.00087	0.00047	0.00573	0.07014
.OSEAX	-0.09832	-0.00474	0.00067	0.00044	0.00620	0.05842
.RUT	-0.15229	-0.00532	0.00110	0.00047	0.00731	0.08906
.SMSI	-0.14061	-0.00599	0.00050	0.00005	0.00678	0.08165
.SPX	-0.12670	-0.00322	0.00065	0.00050	0.00522	0.08961
.SSEC	-0.08876	-0.00537	0.00060	0.00020	0.00620	0.05641
.SSMI	-0.10134	-0.00425	0.00056	0.00026	0.00506	0.06780
.STI	-0.14717	-0.00399	0.00041	0.00020	0.00493	0.06000
.STOXX50E	-0.12005	-0.00514	0.00046	0.00023	0.00600	0.08655
risk-free rate	1.43e-05	4.80e-05	6.03e-05	5.76e-05	7.04e-05	8.88e-05

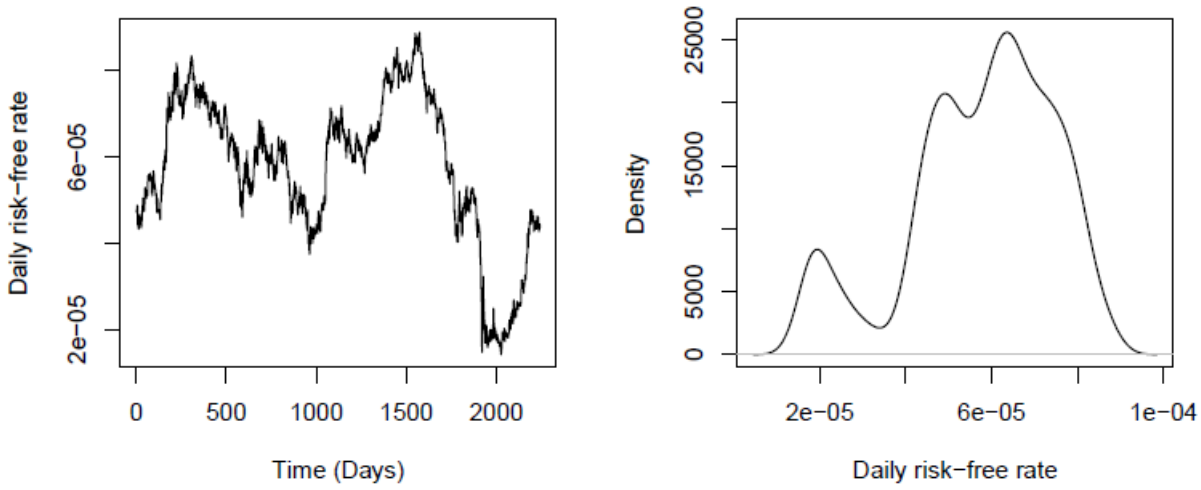

Figure 4.2: Risk-free rate

prices are fluctuating, and very probably are not stationary. Therefore, we transformed the daily close prices to daily log-returns, which look as weakly stationary time series. However, we will test the stationarity by ADF (Augmented Dickey Fuller) test and by KPSS (Kwiatkowski-Phillips-Schmidt-Shin) test. The density plot of AEX index close prices, shows that the data are probably not normally distributed. There are more than one pronounced peak and the data are positively skewed. The density plot of daily log-returns is much less skewed due to logarithmic transformation. However, we will test the normality of log-returns by Jarque-Bera test and KS (Kolmogorov-Smirnov) test. The time series plots and density plots for all other 30 stock indices are included in the appendix.

Regarding the risk-free rate, we present its summary statistics and its distribution in tables 4.2 and 4.4. We also present the time series plot and the density plot of the risk-free rate in the figure 4.2.

4.1.4 Log-returns transformation

Due to the non-stationarity of our close prices time series, we applied traditional log-difference transformation, which transforms the price (levels) time series to logarithmic returns (log returns), i.e. $r_t = ln(P_t) - ln(P_{t-1}), \quad t = 1, \ldots, T,$ and P_t is the daily price. Due to this transformation, we lost one datapoint of each examined time series.

4.2 Testing normality and stationarity of the data

4.2.1 Testing normality of log-returns

In order to test the normality of log-returns, we applied one-sided Jarque-Bera test with the null hypothesis that the data are normally distributed. We also employed one-sample two-sided KS (Kolmogorov-Smirnov) test with the null hypothesis that the data are normally distributed. Then, we calculated skewness and kurtosis of log-returns. The results are presented in the table 4.5. We reject the null hypothesis of Jarque-Bera test at the 1% significance level for all 31 examined log-returns. We also reject the null hypothesis of Kolmogorov-Smirnov test at the 1% significance level for all 31 examined log-returns. The positive values of kurtosis for all 31 log-returns suggest heavier tails than for normally distributed data, i.e. log-returns have leptokurtic distribution. The negative values of skewness suggest that the log-returns are negatively skewed. According to the Jarque-Bera test and the KS test, we reject the null hypothesis that the risk-free rate is normally distributed at the 5% significance level for both tests.

4.2.2 Testing the stationarity of log-returns

In this part, we applied the Augmented Dickey-Fuller test (ADF test) and the Kwiatkowski-Phillips-Schmidt-Shin test (KPSS test) in order to test the stationarity of all 31 time series of log returns. According to the results of the ADF tests, we can reject the null hypothesis of non-stationarity at the 5% significance level for all 31 log returns. According to the results of the KPSS tests we cannot reject the null hypothesis of stationarity at the 5% significance level, for all 31 log returns. In fact for the ADF test, $p.value = 0.01$ means $p.value \leq 0.01$ and for the KPSS test, $p.value = 0.1$ means $p.value \geq 0.1$. Therefore, we conclude that the log-returns are at least weakly stationary. The results for AEX index are in table A.1. In table 4.7, we present the results of ADF and KPSS test for the risk-free rate. On the one hand according to the p-values, we cannot reject the non-stationarity as the null hypothesis of the ADF test at the 5% significance level. On the other hand, we cannot reject the null hypothesis of stationarity as the null hypothesis of the KPSS test. Therefore, these results are inconclusive. All the results of ADF and KPSS tests are included in the appendix.

Table 4.5: Normality of log-returns

Normality of log-returns						
			Jarque-Bera test		KS test	
index	Skewness	Kurtosis	Statistics	p-value	Statistics	p-value
.AEX	-0.7269	12.5573	8532	<0.001	0.497	<0.001
.AORD	-1.0002	10.7934	5910	<0.001	0.4938	<0.001
.BFX	-1.397	20.1995	27719	<0.001	0.5002	<0.001
.BSESN	-1.3218	22.545	35512	<0.001	0.492	<0.001
.BVLG	-0.9664	10.9094	6052	<0.001	0.4893	<0.001
.BVSP	-0.8973	16.084	15922	<0.001	0.4943	<0.001
.DJI	-1.1508	30.9574	71839	<0.001	0.487	<0.001
.FCHI	-0.842	13.1001	9572	<0.001	0.4847	<0.001
.FTMIB	-1.5104	20.2726	28069	<0.001	0.4833	<0.001
.FTSE	-0.6961	13.0561	9409	<0.001	0.4884	<0.001
.GDAXI	-0.6075	11.6603	6982	<0.001	0.4938	<0.001
.GSPTSE	-2.4297	45.036	163470	<0.001	0.5011	<0.001
.HSI	-0.3578	5.291	526	<0.001	0.4843	<0.001
.IBEX	-1.1714	16.3611	16798	<0.001	0.4797	<0.001
.IXIC	-0.9568	16.582	17175	<0.001	0.4979	<0.001
.KS11	-0.5098	14.5195	12209	<0.001	0.4838	<0.001
.KSE	-0.6489	7.3493	1881	<0.001	0.4852	<0.001
.MXX	-0.5376	8.2101	2584	<0.001	0.4879	<0.001
.N225	-0.2791	7.4413	1829	<0.001	0.4975	<0.001
.NSEI	-1.3121	21.2673	31092	<0.001	0.4847	<0.001
.OMXC20	-0.4848	6.7954	1401	<0.001	0.4902	<0.001
.OMXHPI	-0.9343	12.3069	8226	<0.001	0.4916	<0.001
.OMXSPI	-1.1429	15.1077	13860	<0.001	0.4888	<0.001
.OSEAX	-0.8093	11.5884	6973	<0.001	0.5002	<0.001
.RUT	-1.3074	20.1423	27451	<0.001	0.4897	<0.001
.SMSI	-1.2288	17.6522	20151	<0.001	0.4911	<0.001
.SPX	-1.0426	24.9791	44498	<0.001	0.4856	<0.001
.SSEC	-1.0484	10.2205	5161	<0.001	0.4879	<0.001
.SSMI	-1.1378	15.3139	14315	<0.001	0.5039	<0.001
.STI	-1.9691	32.31	79843	<0.001	0.4884	<0.001
.STOXX50E	-0.8056	12.8716	9133	<0.001	0.4797	<0.001
risk-free rate	-0.6967	-0.0377	181	<0.001	0.8130	<0.001

Table 4.6: Stationarity tests: AEX log returns

Stationarity tests: AEX log-returns						
	no drift, no trend		with drift, no trend		with drift and trend	
lag	ADF	p.value	ADF	p.value	ADF	p.value
0	-46.4702	0.01	-46.5094	0.01	-46.4988	0.01
1	-31.8241	0.01	-31.8666	0.01	-31.8594	0.01
2	-25.2151	0.01	-25.2603	0.01	-25.2546	0.01
3	-22.4757	0.01	-22.5271	0.01	-22.522	0.01
4	-20.6935	0.01	-20.7493	0.01	-20.7446	0.01
5	-19.1792	0.01	-19.2409	0.01	-19.2366	0.01
6	-16.5551	0.01	-16.6132	0.01	-16.6096	0.01
7	-16.9561	0.01	-17.0234	0.01	-17.0198	0.01
lag	KPSS	p.value	KPSS	p.value	KPSS	p.value
10	0.6303	0.1	0.0302	0.1	0.0306	0.1

For ADF tests $p.value = 0.01$ means $p.value \leq 0.01$.
For KPSS test $p.value = 0.1$ means $p.value \geq 0.1$.

Table 4.7: Stationarity tests: risk-free rate

Stationarity tests: risk-free rate						
	no drift, no trend		with drift, no trend		with drift and trend	
lag	ADF	p.value	ADF	p.value	ADF	p.value
0	-0.5106	0.4963	-0.5106	0.4963	-0.5106	0.4963
1	-0.4977	0.5007	-0.4977	0.5007	-0.4977	0.5007
2	-0.5024	0.4992	-0.5024	0.4992	-0.5024	0.4992
3	-0.5035	0.4988	-0.5035	0.4988	-0.5035	0.4988
4	-0.4874	0.5037	-0.4874	0.5037	-0.4874	0.5037
5	-0.5167	0.4941	-0.5167	0.4941	-0.5167	0.4941
6	-0.4889	0.5033	-0.4889	0.5033	-0.4889	0.5033
7	-0.441	0.517	-0.441	0.517	-0.441	0.517
lag	KPSS	p.value	KPSS	p.value	KPSS	p.value
10	0.3009	0.1	0.2965	0.1	0.102	0.1

For ADF tests $p.value = 0.01$ means $p.value \leq 0.01$.
For KPSS test $p.value = 0.1$ means $p.value \geq 0.1$.

Chapter 5

Applying the methodology on data

5.1 Estimating multifractal spectra

In this part, we applied the Multifractal Detrended Fluctuation Analysis (MF-DFA) on the 31 time series of log returns. In order to use MF-DFA method, we have to choose a several parameters. Namely a range of moments q, and a range of segment lengths s. In general, the moment q can go from $-\infty$ to ∞. According to Sedlaříková (2016), higher moments of q can significantly bias the estimated multifractality. Therefore, we followed Kwapień *et al.* (2005), and set the moment q going from -3 to 3 by the step of 0.1. For the segment length, we determined the minimal value of s, s_{min}, the maximal value of s, s_{max}, and the step of s, s_{step} according to the guidelines by Thompson & Wilson (2016). These guidelines determine the segment length s dynamically with respect to the length of the examined time series, and follow these rules:

$$s_{min} = max(20, \frac{N}{100})$$

$$s_{max} = min(20 * s_{min}, \frac{N}{10}))$$

$$s_{step} = \frac{s_{max} - s_{min}}{100}$$

where N is the length of a time series.

We applied the MF-DFA procedure to all 31 stock index log returns. We obtained the generalized Hurst exponent $H(q)$ and the singularity strength $\alpha(q)$ for whole spectrum of moments q. Then, we calculated the differences between the Hurst exponent evaluated at the lowest moment $q = -3$ and at the highest moment $q = 3$, i.e.: $\Delta H = H(q_{max}) - H(q_{min})$. We followed analogous

procedure for the singularity strength, i.e.: $\Delta\alpha = \alpha(q_{max}) - \alpha(q_{min})$. These measures capture the overall multifractality strength caused by heavy-tailed distributions of the data as well as by the temporal correlation of a given time series. In order to estimate the multifractality strength caused by the heavy-tailed distributions, we shuffled the time series 30 times, estimated the $H(q)$ and the $\alpha(q)$ again, calculated $\Delta H = H(q_{max}) - H(q_{min})$ and $\Delta H = \alpha(q_{max}) - \alpha(q_{min})$, and calculated the mean value of the ΔH and the $\Delta\alpha$ and also the 0.05 quantile and the 0.95 quantile for the mean of ΔH and $\Delta\alpha$ estimated on shuffled time series. The results of the generalized Hurst exponent H estimated on log returns are presented in table 5.1. The results of the singularity strength α estimated on log returns are presented in table 5.2. In further parts of our thesis, we will denote the multifractality cause by temporal correlations only as following: $\Delta\Delta H = \Delta H - \Delta H_{shuff-mean}$ and $\Delta\Delta\alpha = \Delta\alpha - \Delta\alpha_{shuff-mean}$. We can observe that the ranges between $q_{0.05-shuff}$ and $q_{0.95-shuff}$ are relatively wide, for $\Delta H_{shuff-mean}$ as well as for $\Delta\alpha_{shuff-mean}$.

We also calculated the correlation between ΔH and $\Delta\alpha$. We found very strong positive correlation, $Corr(\Delta H, \Delta\alpha) = 0.997$, as we would expected, because these two measures are trying to capture the same characteristics of a time series but from a different perspective. The correlation is also displayed by scatterplot in figure 5.1.

5.2 Models estimation

In this part, we sorted the 31 stock indices according to the wide of multifractal spectra, from the highest multifractality to the lowest. Then, we constructed two portfolios. The first portfolio consists of 7 the most multifractal stock indices and it is calculated as the average value of returns with respect to time, i.e.: $r_{MF-low,t} = \frac{1}{7}(r_{1,t} + r_{2,t} + ... + r_{7,t}), \quad t = 1, \ldots, T$. The second portfolio consist of 7 the less multifractal stock indices and it is calculated as the average value of returns with respect to time, i.e.: $r_{MF-high,t} = \frac{1}{7}(r_{25,t} + r_{26,t} + ... + r_{31,t}), \quad t = 1, \ldots, T$. We chose 7 stock indices, because it is approximately 25% of all 31 stock indices. Finally, we constructed the resulting portfolio as the difference between the $MF - high$ and $MF - low$ portfolio, i.e.: $r_{MF,t} = r_{MF-high,t} - r_{MF-low,t}, \quad t = 1, \ldots, T$. This portfolio should mirror the long position in "more multifractal" stock indices and the short position in "less multifractal" stock indices.

The construction of the portfolio described above is based on the following

Table 5.1: Generalized Hurst exponent of log-returns

Generalized Hurst exponent of log-returns					
index	ΔH	$\Delta H_{shuffle-mean}$	$q_{0.05-shuffle}$	$q_{0.95-shuffle}$	$\Delta\Delta H$
.AEX	0.1250	0.0716	0.0170	0.1339	0.0534
.AORD	0.0696	0.0766	0.0196	0.1183	-0.0070
.BFX	0.0963	0.0819	0.0136	0.1342	0.0144
.BSESN	0.0389	0.0957	0.0461	0.1642	-0.0568
.BVLG	0.1217	0.0622	0.0092	0.1147	0.0595
.BVSP	0.0798	0.0769	0.0150	0.1293	0.0029
.DJI	0.1489	0.1364	0.0659	0.2092	0.0125
.FCHI	0.1298	0.0801	0.0186	0.1449	0.0496
.FTMIB	0.0858	0.0738	0.0142	0.1539	0.0120
.FTSE	0.0581	0.0732	0.0142	0.1391	-0.0151
.GDAXI	0.0963	0.0730	0.0215	0.1388	0.0233
.GSPTSE	0.1523	0.1358	0.0534	0.2140	0.0166
.HSI	0.0370	0.0493	0.0121	0.1097	-0.0123
.IBEX	0.0813	0.0990	0.0204	0.1625	-0.0177
.IXIC	0.0478	0.0948	0.0342	0.1748	-0.0470
.KS11	0.0875	0.0839	0.0201	0.1607	0.0037
.KSE	0.0398	0.0633	0.0135	0.1170	-0.0234
.MXX	0.1111	0.0580	0.0133	0.1084	0.0531
.N225	0.2033	0.0546	0.0127	0.1002	0.1486
.NSEI	0.0237	0.0791	0.0246	0.1216	-0.0554
.OMXC20	0.1671	0.0524	0.0187	0.1019	0.1147
.OMXHPI	0.0689	0.0572	0.0145	0.1304	0.0117
.OMXSPI	0.0782	0.0737	0.0169	0.1396	0.0045
.OSEAX	0.1294	0.0587	0.0183	0.1143	0.0707
.RUT	0.0395	0.0885	0.0248	0.1717	-0.0489
.SMSI	0.0822	0.0727	0.0171	0.1249	0.0095
.SPX	0.1030	0.1218	0.0583	0.2007	-0.0188
.SSEC	0.0688	0.0907	0.0252	0.1432	-0.0219
.SSMI	0.1597	0.0804	0.0249	0.1420	0.0793
.STI	0.1698	0.0986	0.0333	0.1625	0.0711
.STOXX50E	0.1316	0.0744	0.0230	0.1370	0.0573

Table 5.2: Singularity strength α of log-returns

Singularity strength α of log-returns					
index	$\Delta\alpha$	$\Delta\alpha_{shuffle-mean}$	$q_{0.05-shuffle}$	$q_{0.95-shuffle}$	$\Delta\Delta\alpha$
.AEX	0.1475	0.0782	0.0198	0.1450	0.0693
.AORD	0.0886	0.0831	0.0215	0.1291	0.0055
.BFX	0.1144	0.0900	0.0151	0.1481	0.0244
.BSESN	0.0503	0.1048	0.0508	0.1782	-0.0545
.BVLG	0.1413	0.0676	0.0094	0.1243	0.0737
.BVSP	0.0980	0.0838	0.0183	0.1397	0.0142
.DJI	0.1703	0.1477	0.0723	0.2249	0.0226
.FCHI	0.1486	0.0873	0.0216	0.1570	0.0612
.FTMIB	0.1020	0.0813	0.0154	0.1667	0.0207
.FTSE	0.0730	0.0798	0.0167	0.1498	-0.0068
.GDAXI	0.1157	0.0795	0.0250	0.1501	0.0362
.GSPTSE	0.1799	0.1499	0.0628	0.2325	0.0300
.HSI	0.0416	0.0535	0.0132	0.1187	-0.0119
.IBEX	0.0944	0.1080	0.0228	0.1767	-0.0136
.IXIC	0.0601	0.1027	0.0352	0.1879	-0.0426
.KS11	0.1090	0.0915	0.0219	0.1745	0.0175
.KSE	0.0496	0.0685	0.0145	0.1260	-0.0189
.MXX	0.1200	0.0632	0.0150	0.1174	0.0568
.N225	0.2210	0.0592	0.0139	0.1087	0.1618
.NSEI	0.0337	0.0864	0.0263	0.1329	-0.0527
.OMXC20	0.1872	0.0568	0.0209	0.1097	0.1304
.OMXHPI	0.0884	0.0626	0.0167	0.1417	0.0258
.OMXSPI	0.0970	0.0806	0.0187	0.1562	0.0164
.OSEAX	0.1506	0.0645	0.0206	0.1234	0.0861
.RUT	0.0546	0.0969	0.0295	0.1848	-0.0424
.SMSI	0.0959	0.0799	0.0184	0.1360	0.0160
.SPX	0.1204	0.1321	0.0656	0.2168	-0.0117
.SSEC	0.0777	0.0979	0.0288	0.1540	-0.0202
.SSMI	0.1793	0.0879	0.0283	0.1547	0.0914
.STI	0.1888	0.1088	0.0361	0.1813	0.0800
.STOXX50E	0.1513	0.0814	0.0250	0.1482	0.0699

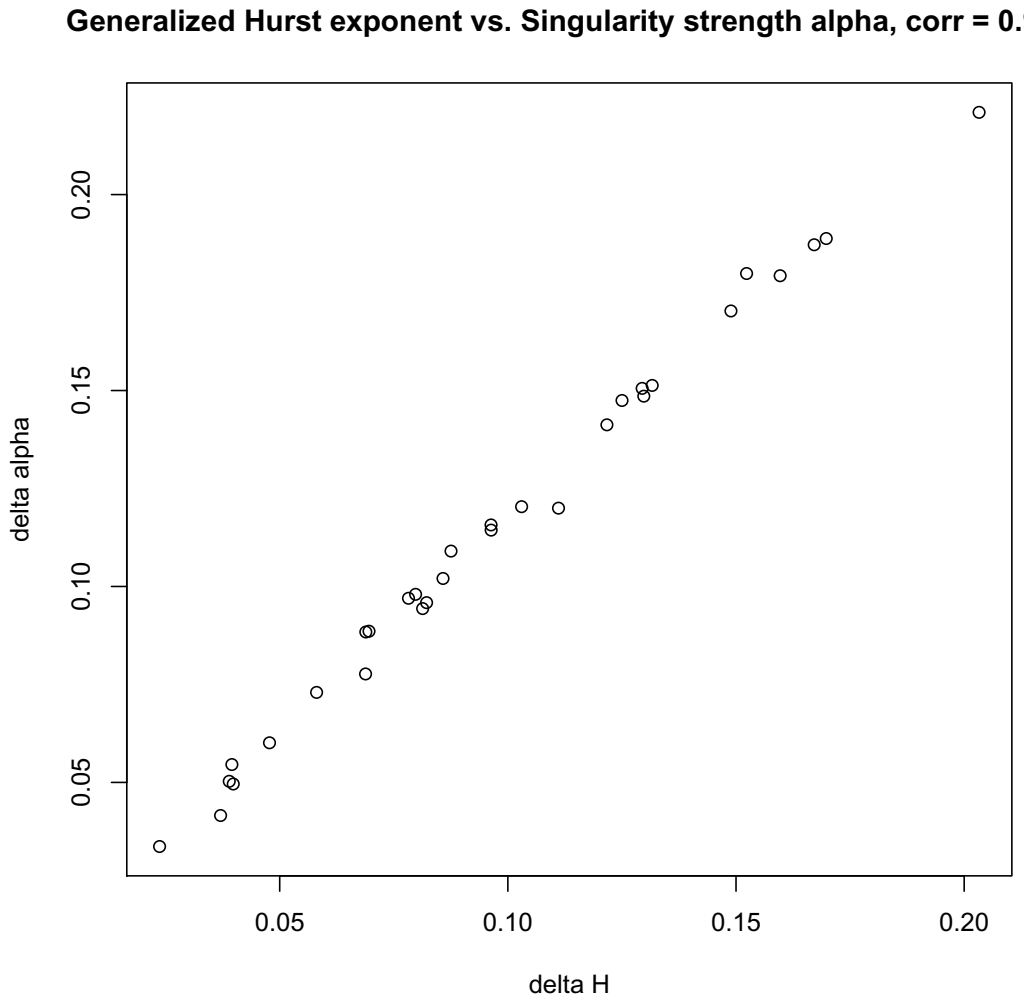

Figure 5.1: Change of generalized Hurst exponent (ΔH) vs. Change of Singularity strength alpha ($\Delta\alpha$)

idea. The higher is the multifractality of a given time series, the higher are the patterns and self-similarities of the time series, and the higher divergence from the random Gaussian returns, one could find the lower risk and the lower risk premia for holding such a portfolio.

The resulting orders of stock indices sorted by the generalized Hurst exponent $\Delta\Delta H$ and by the singularity strength $\Delta\Delta\alpha$ are presented in the table 5.3. We can observe that the orders are different when sorted by $\Delta\Delta H$ and by $\Delta\Delta\alpha$. We constructed the "multifractal portfolio" based the 7 most multifractal and the 7 less multifractal stock indices. In our case, we hold a long position in $r_{MF-high,t}$ which consist of: .N225, .OMXC20, .SSMI, .STI, .OSEAX, .BVLG, and .STOXX50E stock index. We hold a short position in $r_{MF-low,t}$ which consist of: .BSESN, .NSEI, .RUT, .IXIC, .KSE, .SSEC, and .SPX.

5.3 Fama-McBeth regression

We have estimated 31 linear regressions by OLS, described by the following equation:

$$r_{i,t}-r_{risk-free,t} = \alpha_i+\beta_{i,1}(r_{market}-r_{risk-free,t})+\beta_{i,2}r_{MF,t}+\epsilon_{i,t}, \quad t = 1,\ldots,2191.$$

where $r_{i,t}$ stands for log-return of individual stock index at time t, $r_{MF,t}$ stands for log-return of "multifractal portfolio" at time t, $r_{free-rate}$ is the risk-free rate of return proxied by the returns of 10-year US government bonds, r_{market} is the market returns calculated as the average return of all 31 stock indices, α_i and β_i are the parameters, and ϵ_t is the error term. We also had to divide the risk-free rate by 365 in order to get approximately daily returns, because originally we obtained the risk-free rate of return per annum.

From the 31 equations, we got 31 different betas for individual stock indices. We present the results in the tables 6.1 and 6.2.

After obtaining 31 betas measuring the sensitivity of individual stock indices, we calculated 31 average stock index returns with respect to time. This step can be described by the following equation:

$$\overline{R}_i = \frac{\sum_{t=1}^{T}(r_{i,t} - r_{risk-free,t})}{T}, \quad i = 1,\ldots,n.$$

where $\overline{R}_i$ are the average stock index returns through the whole examined time

Table 5.3: Stock indices sorted by $\Delta\Delta H$ and $\Delta\Delta\alpha$ from the lowest to the highest

Stock indices sorted by $\Delta\Delta H$ and $\Delta\Delta\alpha$ from the lowest to the highest				
#	index	sorted by $\Delta\Delta H$	index	sorted by $\Delta\Delta\alpha$
1	.BSESN	-0.0568	.BSESN	-0.0545
2	.NSEI	-0.0554	.NSEI	-0.0527
3	.RUT	-0.0489	.IXIC	-0.0426
4	.IXIC	-0.0470	.RUT	-0.0424
5	.KSE	-0.0234	.SSEC	-0.0202
6	.SSEC	-0.0219	.KSE	-0.0189
7	.SPX	-0.0188	.IBEX	-0.0136
8	.IBEX	-0.0177	.HSI	-0.0119
9	.FTSE	-0.0151	.SPX	-0.0117
10	.HSI	-0.0123	.FTSE	-0.0068
11	.AORD	-0.0070	.AORD	0.0055
12	.BVSP	0.0029	.BVSP	0.0142
13	.KS11	0.0037	.SMSI	0.0160
14	.OMXSPI	0.0045	.OMXSPI	0.0164
15	.SMSI	0.0095	.KS11	0.0175
16	.OMXHPI	0.0117	.FTMIB	0.0207
17	.FTMIB	0.0120	.DJI	0.0226
18	.DJI	0.0125	.BFX	0.0244
19	.BFX	0.0144	.OMXHPI	0.0258
20	.GSPTSE	0.0166	.GSPTSE	0.0300
21	.GDAXI	0.0233	.GDAXI	0.0362
22	.FCHI	0.0496	.MXX	0.0568
23	.MXX	0.0531	.FCHI	0.0612
24	.AEX	0.0534	.AEX	0.0693
25	.STOXX50E	0.0573	.STOXX50E	0.0699
26	.BVLG	0.0595	.BVLG	0.0737
27	.OSEAX	0.0707	.STI	0.0800
28	.STI	0.0711	.OSEAX	0.0861
29	.SSMI	0.0793	.SSMI	0.0914
30	.OMXC20	0.1147	.OMXC20	0.1304
31	.N225	0.1486	.N225	0.1618

period. Then, we regress the 31 average stock index returns on 31 betas, i.e.:

$$\overline{R}_i = \gamma_0 + \gamma_1 \hat{\beta}_i + \nu_i.$$

We present the results of this regression in the table 6.3.

Chapter 6

Results

6.1 Multifractality in stock indices

We found some multifractality all of 31 examined stock indices. We present the estimates of the generalized Hurst exponent in the table 5.1 and the estimates of singularity strength alpha in the table 5.2. Then, we present the stock indices by order by multifractality in table 5.3. According to both measures, the generalized Hurst exponent $\Delta\Delta H$, and the singularity strength $\Delta\Delta\alpha$, the lowest multifractality $\Delta\Delta H = -0.0568$, $\Delta\Delta\alpha = -0.0545$ has the .BSESN stock index and the highest multifractality $\Delta\Delta H = 0.1486$, $\Delta\Delta\alpha = 0.1618$ has the .N225 stock index. However, the overall order of stock indices is slightly different.

6.2 First Fama-MacBeth regression

In the tables 6.1 and 6.2, we present the results for 31 linear regressions of stock index returns regressed on the market-risk premia and on the "multifractal portfolios". We can observe that all 31 beta coefficients for market-risk premia, $r_{market} - r_{free}$, are significant even at the 1% significance level and they are all positive. 15 are greater or equal than 1, which indicates that such stock indices should be more sensitive than the overall market. The beta coefficients for the rest 16 stock indices are less than 1, which indicates that such stock indices should be less sensitive than the overall market. The highest beta equals to 1.6681 is for the .FTMIB index. The lowest beta equals to 0.4689 is for the .KSE index.

Regarding the beta coefficients for the risk premia for holding "multifractal

Table 6.1: Results of 31 linear regressions (part I)

Results of 31 linear regressions (part I)				
index	intercept	$r_{market} - r_{free}$	r_{MF}	R^2
.AEX	1e-05	1.1170	0.1031	0.0835
	(0.0002)	(0.0793)	(0.0340)	
.AORD	-2e-05	0.7716	0.1129	0.0610
	(0.002)	(0.0659)	(0.0283)	
.BFX	-0.0001	1.1260	0.0146	0.0812
	(0.002)	(0.0816)	(0.0350)	
.BSESN	0.0001	1.0950	-0.7024	0.3016
	(0.002)	(0.0692)	(0.0297)	
.BVLG	-5e-05	1.1025	0.4483	0.1221
	(0.002)	(0.0812)	(0.0348)	
.BVSP	-0.0001	1.4080	-0.0606	0.0615
	(0.0003)	(0.1214)	0.0521	
.DJI	0.0001	0.8628	-0.1537	0.0656
	(0.0002)	(0.0803)	(0.0344)	
.FCHI	-3e-05	1.0682	0.2336	0.0705
	(0.0002)	(0.0882)	(0.0378)	
.FTMIB	-0.0003	1.6681	-0.1797	0.1156
	(0.0003)	(0.1065)	(0.0457)	
.FTSE	-0.0001	0.5998	0.0933	0.0292
	(0.0002)	(0.0756)	(0.0324)	
.GDAXI	-0.0001	1.1973	-0.1600	0.0917
	(0.0002)	(0.0890)	(0.0382)	
.GSPTSE	-0.0001	0.9458	0.0269	0.0861
	(0.0002)	(0.0661)	(0.0284)	
.HSI	-0.0001	0.6863	0.0178	0.0295
	(0.0002)	(0.0845)	(0.0363)	
.IBEX	-0.0003	1.3672	0.3502	0.1018
	(0.0003)	(0.0953)	(0.0409)	
.IXIC	0.0003	1.0398	-0.5889	0.1873
	(0.0002)	(0.0830)	(0.0356)	
.KS11	0.0001	0.5671	0.0502	0.0291
	(0.0002)	(0.0702)	(0.0301)	

The values in parenthesis are estimated standard errors.

Table 6.2: Results of 31 linear regressions (part II)

Results of 31 linear regressions (part II)				
index	intercept	$r_{market} - r_{free}$	r_{MF}	R^2
.KSE	0.0003	0.4689	-0.3317	0.0663
	(0.0002)	(-0.0782)	(0.0335)	
.MXX	-0.0002	0.7904	-0.0724	0.0586
	(0.0002)	(0.0722)	(0.031)	
.N225	0.0003	0.9854	0.5904	0.1101
	(0.0003)	(0.0964)	(0.0413)	
.NSEI	5e-05	1.0673	-0.7531	0.3245
	(0.0002)	(0.0683)	(0.0293)	
.OMXC20	0.0003	0.9656	0.3899	0.1000
	(0.0002)	(0.0793)	(0.034)	
.OMXHPI	4e-05	1.0535	0.0990	0.0761
	(0.0002)	(0.0786)	(0.0337)	
.OMXSPI	0.0002	0.9827	0.1395	0.0712
	(0.0002)	(0.0771)	(0.0331)	
.OSEAX	0.0002	0.8446	0.3899	0.0913
	(0.0002)	(0.078)	(0.0335)	
.RUT	-0.0001	1.5379	-0.6611	0.2417
	(0.0003)	(0.0895)	(0.0384)	
.SMSI	-0.0003	1.1465	0.1233	0.0630
	(0.0003)	(0.095)	(0.0408)	
.SPX	0.0001	0.9967	-0.2939	0.1182
	(0.0002)	(0.0763)	(0.0327)	
.SSEC	-0.0001	0.6153	-0.5378	0.0937
	(0.0003)	(0.0994)	(0.0426)	
.SSMI	4e-05	0.7722	0.3960	0.1033
	(0.0002)	(0.0707)	(0.0303)	
.STI	-3e-05	0.7841	0.3120	0.0806
	(0.0002)	(0.072)	(0.0309)	
.STOXX50E	-0.0001	1.3665	0.6046	0.1807
	(0.0002)	(0.0833)	(0.0357)	

The values in parenthesis are estimated standard errors.

portfolio" defined in the section 5.2, we can observe that 26 out of 31 are statistically significant at the 5% significance level. Betas for .BFX, .BVSP, .GSPTSE, .HSI, and for .KS11 are not significant at the 5% significance level.

These 26 significant beta coefficients range between -0.7531 for the .NSEI stock index and 0.6046 for the .STOXX50E index. There are 15 positive and 11 negative. These results suggest that for the 15 stock indices, there is a positive risk premia for holding the "multifractal portfolio" and for the rest 11 stock indices, there is a negative risk premia. The R^2 of the 31 linear regressions range between 0.0291, for .KS11 stock index, and 0.3245 for .NSEI stock index, and the average value of R^2 is 0.1063.

6.3 Second Fama-MacBeth regression

The results of the second Fama-MacBeth regression are presented in the table 6.3. The coefficient $\hat{\gamma}_1 = -0.00016$ captures sensitivity how the average returns react on the beta coefficients from the first Fama-MacBeth regression. This $\hat{\gamma}_1$ is statistically significant only at the 10% significance level. The R^2 of this regression is 0.125. These results suggest that there is some, although not strong and very significant, relationship between the multifractality of stock indices and the risk premia on the financial stock markets. These results are illustrated by the scatterplot in figure 6.1.

After estimation of 31 linear regressions for each examined stock index and 1 linear regression for the average stock index return, we tested the residuals for normality by calculation of skewness, kurtosis, Jarque-Bera test and KS test. We present the results in the table 6.4. The skewness and kurtosis of residuals are quite different for various stock indices. According to the Jarque-Bera test and the KS test, we can reject the null hypothesis of normality of residuals at the 1% significance level for all 31 examined stock indices. For the residuals of linear regression of average returns $\overline{R}_i$, we can reject the null hypothesis of normality of the KS test at the 1% significance level but we cannot reject the null hypothesis of normality of Jarque-Bera tests at the 5% significance level.

Table 6.3: Average stock index returns vs. betas

Results of linear regression: $\overline{R}_i \sim \hat{\beta}_i$				
	intercept	$\hat{\beta}_i$	R^2	# observations
$\overline{R}_i$	0.00027	-0.00016	0.125	31
	(0.00003)	(0.00008)		

The values in parenthesis are estimated standard errors.

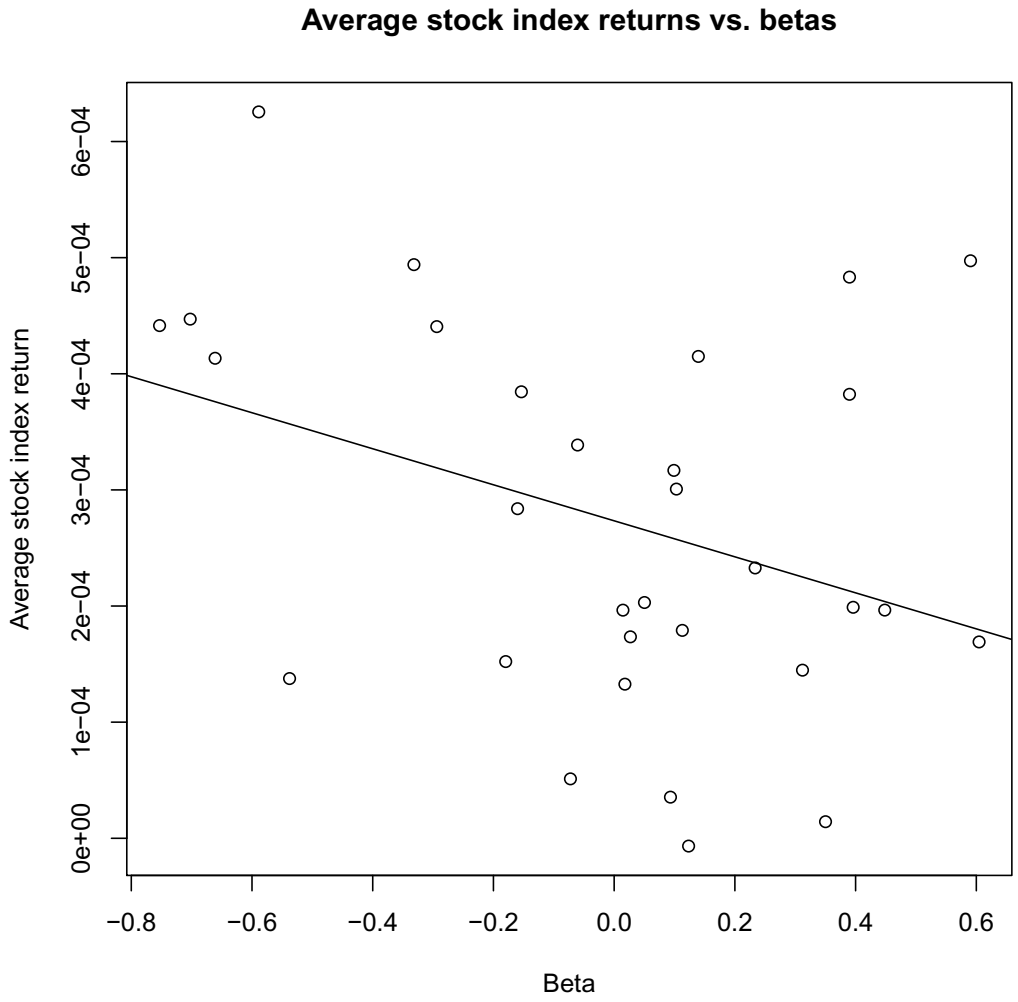

Figure 6.1: Average stock index return vs. beta

Table 6.4: Normality of residuals from linear regressions

Normality of residuals from linear regressions						
			Jarque-Bera test		KS test	
index	Skewness	Kurtosis	Statistics	p.value	Statistics	p.value
.AEX	-0.1605	12.1024	7573	<0.001	0.4925	<0.001
.AORD	-0.4292	8.2010	2537	<0.001	0.4957	<0.001
.BFX	-0.9610	16.9945	18216	<0.001	0.4829	<0.001
.BSESN	-0.6099	11.6993	7045	<0.001	0.4847	<0.001
.BVLG	-0.4519	7.8133	2190	<0.001	0.4884	<0.001
.BVSP	-0.9077	13.7732	10896	<0.001	0.4792	<0.001
.DJI	-0.4920	26.6658	51218	<0.001	0.4906	<0.001
.FCHI	-0.2862	10.6995	5442	<0.001	0.4833	<0.001
.FTMIB	-0.6104	10.7372	5601	<0.001	0.4792	<0.001
.FTSE	-0.4353	12.0112	7482	<0.001	0.4961	<0.001
.GDAXI	-0.1375	7.3313	1720	<0.001	0.5007	<0.001
.GSPTSE	-1.6585	45.6738	167251	<0.001	0.4979	<0.001
.HSI	-0.3866	5.2572	520	<0.001	0.4824	<0.001
.IBEX	-0.5289	13.8885	10926	<0.001	0.4870	<0.001
.IXIC	0.1741	12.0856	7547	<0.001	0.4806	<0.001
.KS11	-0.4843	15.1645	13594	<0.001	0.5030	<0.001
.KSE	-0.7016	8.9936	3459	<0.001	0.4989	<0.001
.MXX	-0.3300	7.7346	2086	<0.001	0.5107	<0.001
.N225	-0.3899	6.7363	1330	<0.001	0.4792	<0.001
.NSEI	-0.5413	11.0865	6077	<0.001	0.4934	<0.001
.OMXC20	-0.4630	6.5966	1259	<0.001	0.4966	<0.001
.OMXHPI	-0.5139	10.5595	5313	<0.001	0.4888	<0.001
.OMXSPI	-0.6880	13.0433	9381	<0.001	0.4856	<0.001
.OSEAX	-0.7267	10.9602	5978	<0.001	0.4893	<0.001
.RUT	-0.0440	9.5105	3870	<0.001	0.4929	<0.001
.SMSI	-0.7499	13.1467	9604	<0.001	0.4893	<0.001
.SPX	-0.4164	29.2774	63100	<0.001	0.4833	<0.001
.SSEC	-0.9587	9.5787	4287	<0.001	0.4961	<0.001
.SSMI	-0.7048	12.3318	8131	<0.001	0.4906	<0.001
.STI	-1.6411	28.5682	60664	<0.001	0.4874	<0.001
.STOXX50E	-0.0589	7.8754	2171	<0.001	0.4874	<0.001
$\overline{R_i}$	0.1628	2.3975	0.6058	0.7386	0.5484	<0.001

Chapter 7

Discussion

We examined the multifractality of financial time series as a possible predictor of stock index returns or as a possible factor influencing the prices of examined assets. According to our results, presented in tables 5.1 and 5.2, we found some multifractality in daily price returns in all 31 examined stock indices. For instance, Di Matteo *et al.* (2005), Kwapień *et al.* (2005) and Kumar & Deo (2009) also found multifractality in financial time series. We can understand the multifractality as a measure of complexity of a given time series. The higher is the the generalized Hurst exponent $\Delta\Delta H$ and the higher is the Singularity strength $\Delta\Delta\alpha$, the more is the time series complex and it contains self-similar and repeating patterns.

The multifractality in our financial time series is caused by the long-term correlations and by the heavy-tailed distributions of our returns. Therefore, we estimated the multifractality on original time series, then we randomly shuffled each time series in order to loose serial correlations. Finally, we subtracted the multifractality caused only by the distribution from the overall multifractality and we got the multifractality caused only by the serial correlations in our time series.

We found that for 11 stock indices out of 31, the multifractality, measured by the generalized Hurst exponent, raised after shuffling the time series. For the change of singularity strength, we observe this phenomenon for 10 stock indices out of 31. This interesting phenomenon was also found by Barunik *et al.* (2012).

In the table 5.3, we present 31 stock indices sorted from the lowest to the highest multifractal spectra. We can observe that orders are partially different when we sort the indices by the generalized Hurst exponent $\Delta\Delta H$ and by the

Singularity strength $\Delta\Delta\alpha$.

According to tables 6.1 and 6.2, we can observe that all 31 beta coefficients for market risk are significant at the 5% significance level. These results are consistent with theory of CAPM. We got 26 out of 31 significant beta coefficients for the "multifractal" factor at the 5% significance level. These results suggest that there is some risk premia connected to the multifractality present at the financial markets.

The beta coefficient estimated by the second equation of Fama-MacBeth procedure is very low, $\hat{\gamma}_1 = -0.00016$ and it is significant only at the 10% significance level. This result also suggests a relationship between multifractality of stock indices and the variation of returns of these stock indices.

We interpret these results as a negative risk premia for holding a "multifractal portoflio". In other words, investors and traders expect less risk connected with holding "multifractal assets", i.e. assets whose price performs higher multifractal behavior than we would expect for a random walk. Therefore, these investors and traders are willing to pay less for holding more "multifractal assets" with less risk than for holding less "multifractal assets" with higher risk.

Therefore, we can reject our null hypothesis at the 10% significance level that H_0 : There is no a significant risk premia for holding a "multifractal portfolio".

This result could be theoretically utilized on the market. For instance, a risk-averse investor who is willing to accept lower returns with lower risk exposition, could estimate multifractality for hundreds of assets and buy the relatively more multifractal assets and sell the relatively less multifractal assets. Applying such a trading strategy, he or she could theoretically achieve lower average risk and returns than the overall market.

On the other way around, an investor who would like to reach higher return at the cost of higher risk, he or she could buy less multifractal assets and sell more multifractal assets. In such a case, he or she could theoretically achieve higher average return and higher average risk than the overall market.

Nonetheless, the historical behavior of stock markets do not guarantee the future behavior. We also estimated very small sensitivity for holding a more "multifractal" portfolio. We have to add that we did not take into account any cost for making analysis of multifractality of assets and any fees for trading. Therefore, there remains a question, whether such a trading model or a trading strategy would be profitable at all or at least if we would be able to overperform Buy Hold strategy.

Chapter 8

Conclusion

In our thesis, we obtained daily stock index prices 31 stock indices. We transformed the daily stock index prices to daily logarithmic returns, in order to get weakly stationary data. We also obtained the daily data of 10-year US government bonds returns in order to proxy a risk free rate of return. We estimated the multifractality of all 31 log-returns by MF-DFA procedure. Then, we sorted the 31 stock-indices by the estimated multifractality. We constructed a "multifractal portfolio" of returns, holding a long position in the 7 most multifractal returns and holding a short position in the 7 lowest multifractal returns. We applied the Fama-MacBeth regression.

We found both, positive and negative beta coefficients connected to the multifractal factor. From the second equation of Fama-MacBeth procedure, we found we very low and negative gamma coefficient capturing the overall sensitivity of stock index returns to the multifractal factor. However, the gamma coefficient is significant only at the 10% significance level.

Therefore, we conclude there is a relationship between multifractality of stock indices returns and the risk premia for holding a "multifractal portfolio". We find our results as an interesting contribution in the field of multifractality and asset pricing of financial time series. According to our best knowledge, there is no such a work applying the Fama-MacBeth regression utilizing a multifractal factor. We are aware of limitations of our experiment. Firstly, the estimated multifractality in the log-returns can be caused by more factors than just long-term serial correlations and by heavier tails than for normal distribution. Secondly, we had a relatively small number of observations, only 31, for calculating the gamma coefficient. According to the Central limit theorem, we should use at least 30 observations in order to have normally distributed data.

Therefore, we provide a several suggestions for further research. It might help to examine higher number of stock indices than 31. It would be interesting to examine another type of financial time series instead of stock indices, e.g.: stock prices, commodity prices or foreign exchange rates. Then we could observe whether a multifractality of another financial time series is a significant factor for pricing respective assets at its financial markets.

It would be also interesting to extend our model by another factors such as SMB or HML which are usually incorporated in Fama-French three factor model but these factors are not available for stock indices.

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Disclaimer:

All investments, especially in the stock markets, involve substantial degree of risk. The primary purpose of this thesis is purely academic and it should not be considered as an investment advice. The author is not liable for any potential losses stemming from the application of the described methods and the whole responsibility lies on an individual making particular investment decisions.

Appendix A

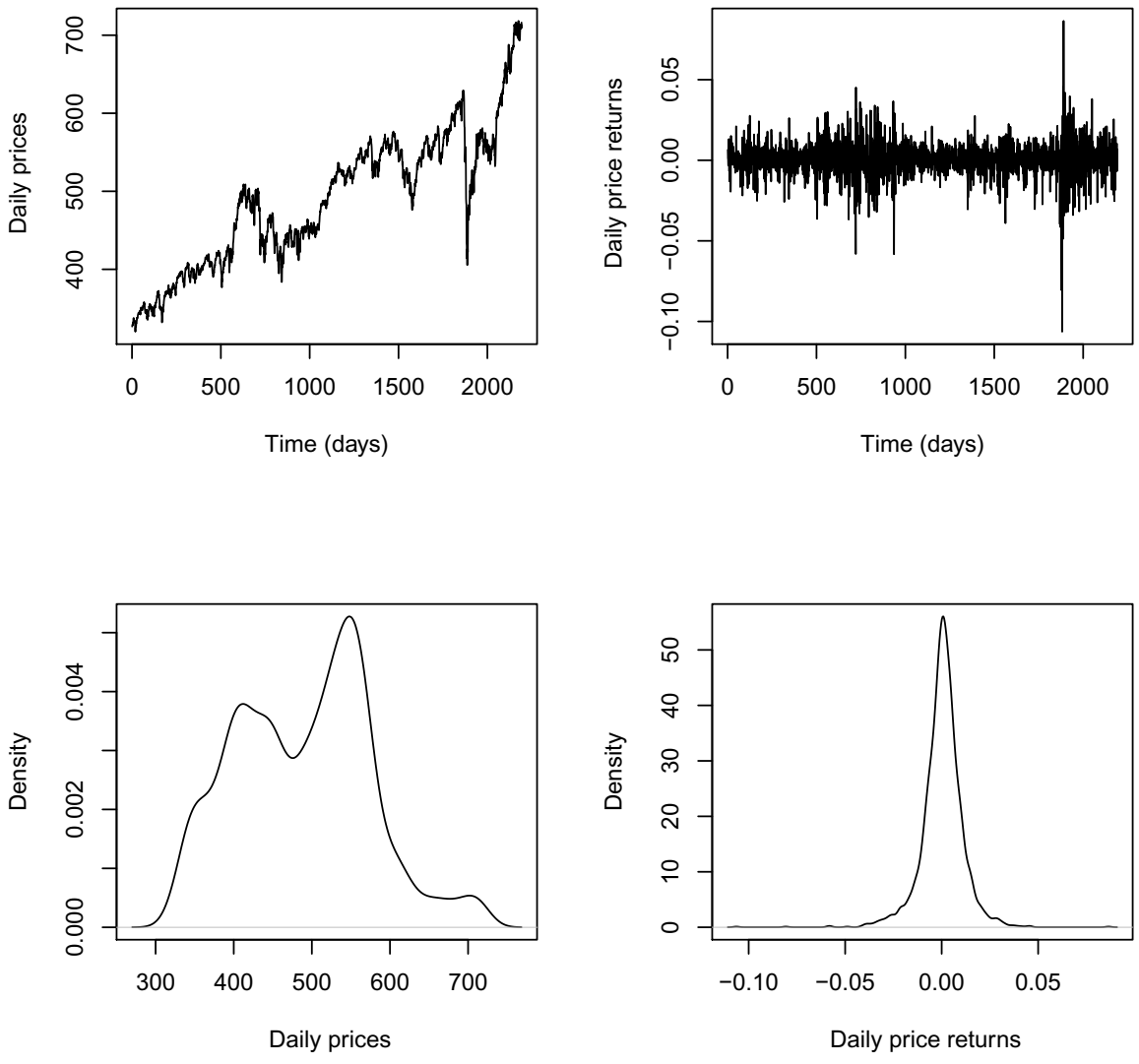

Figure A.1: AEX index

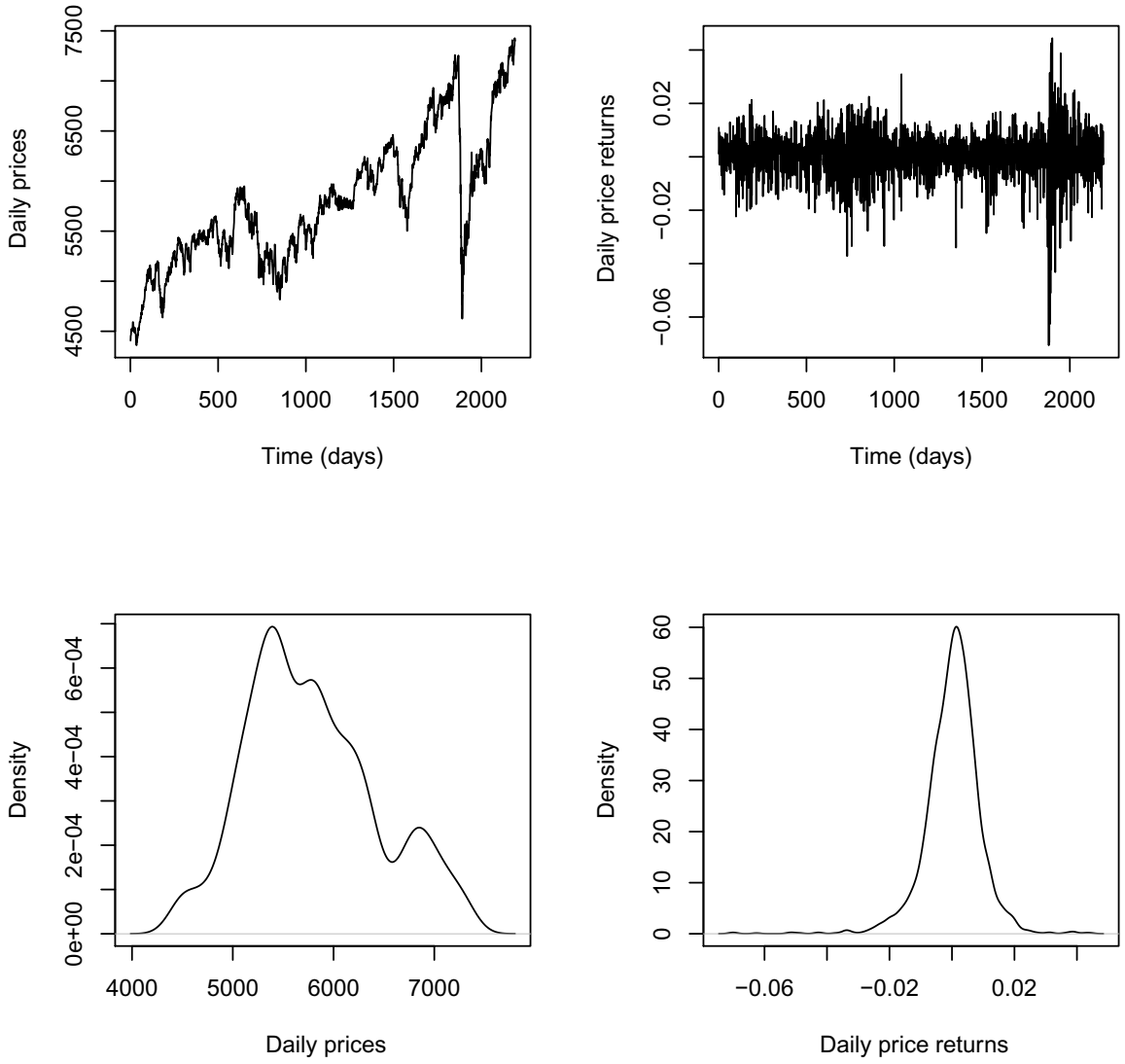

Figure A.2: AORD index

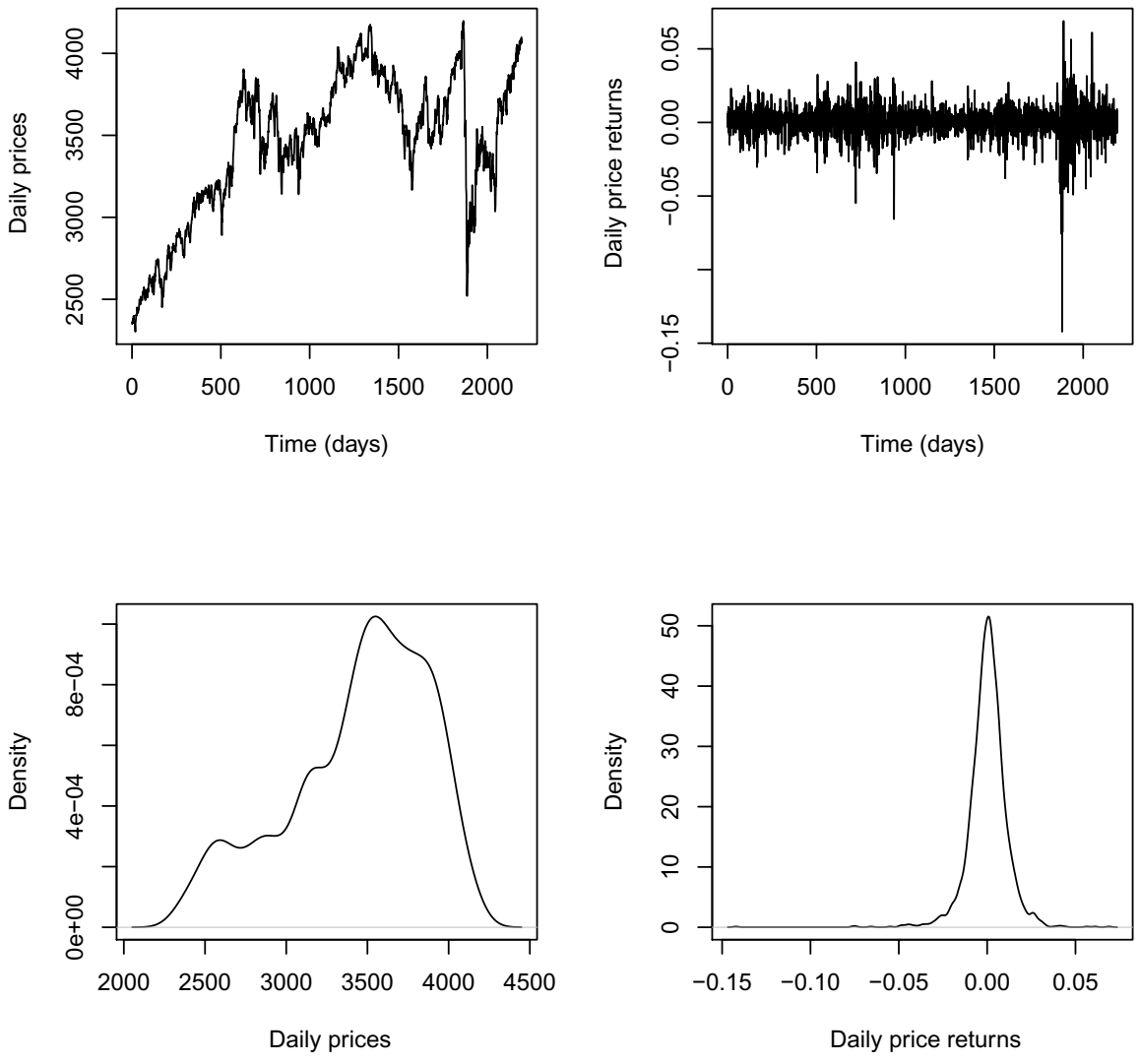

Figure A.3: BFX index

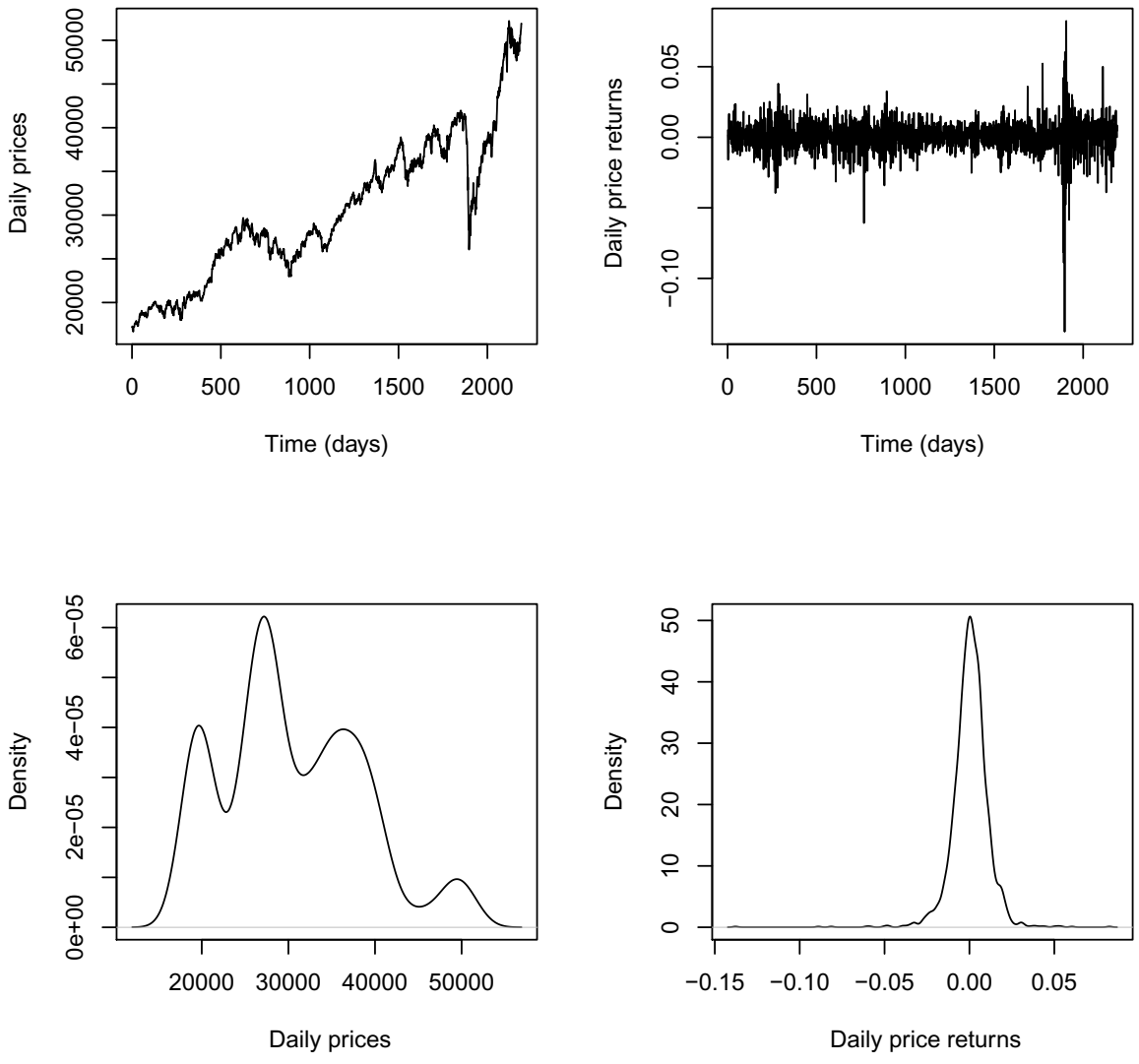

Figure A.4: BSESN index

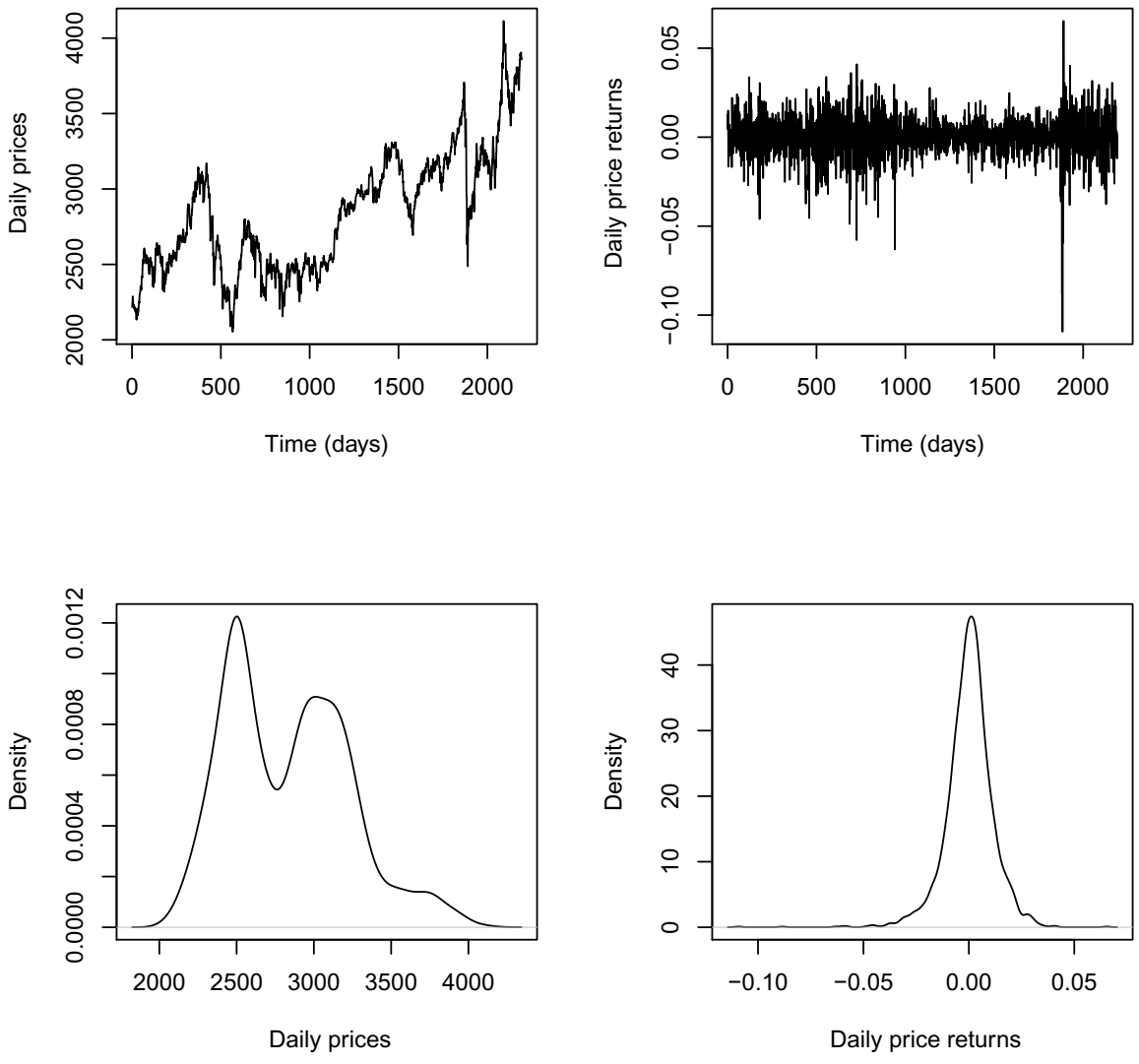

Figure A.5: BVLG index

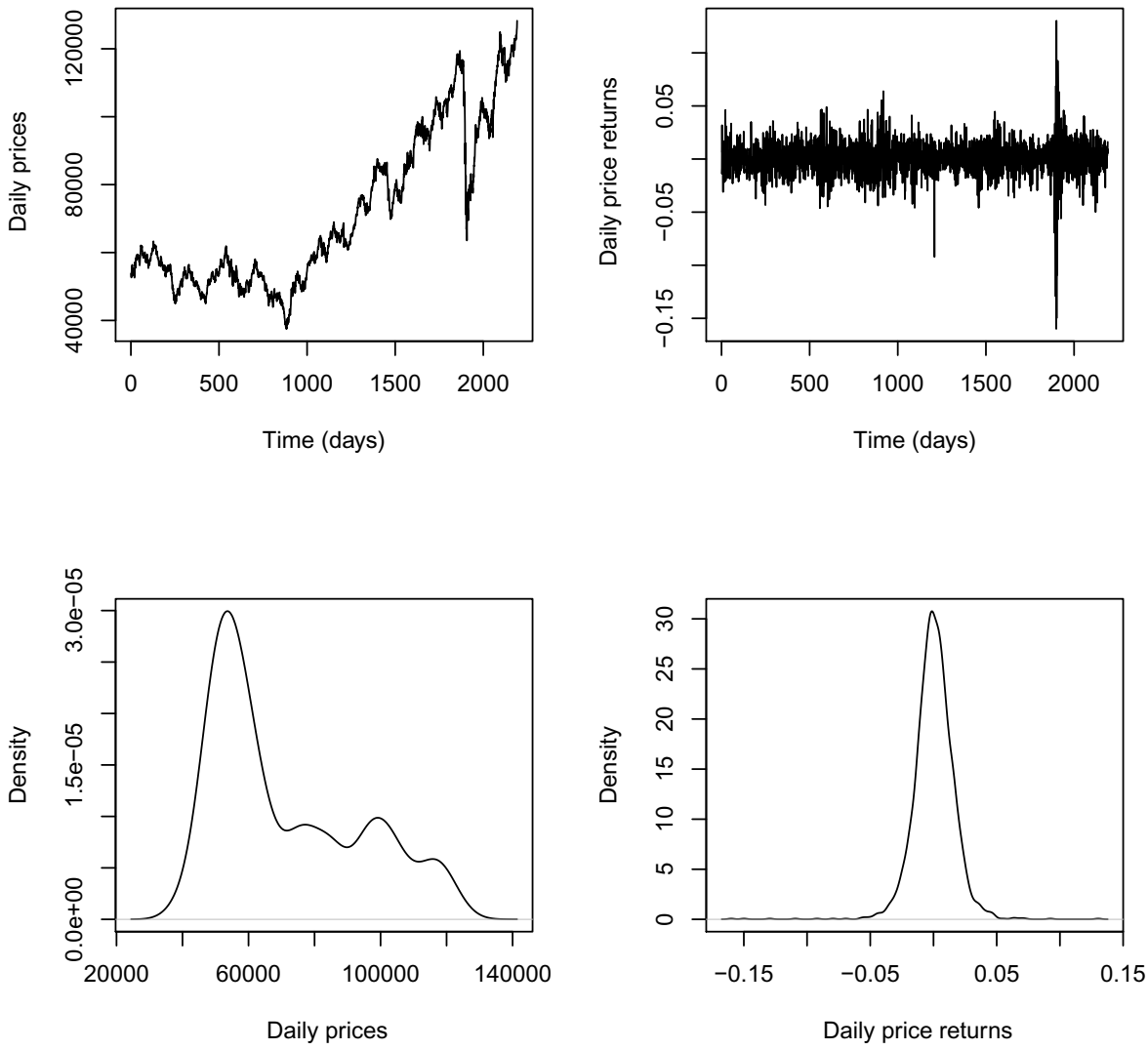

Figure A.6: BVSP index

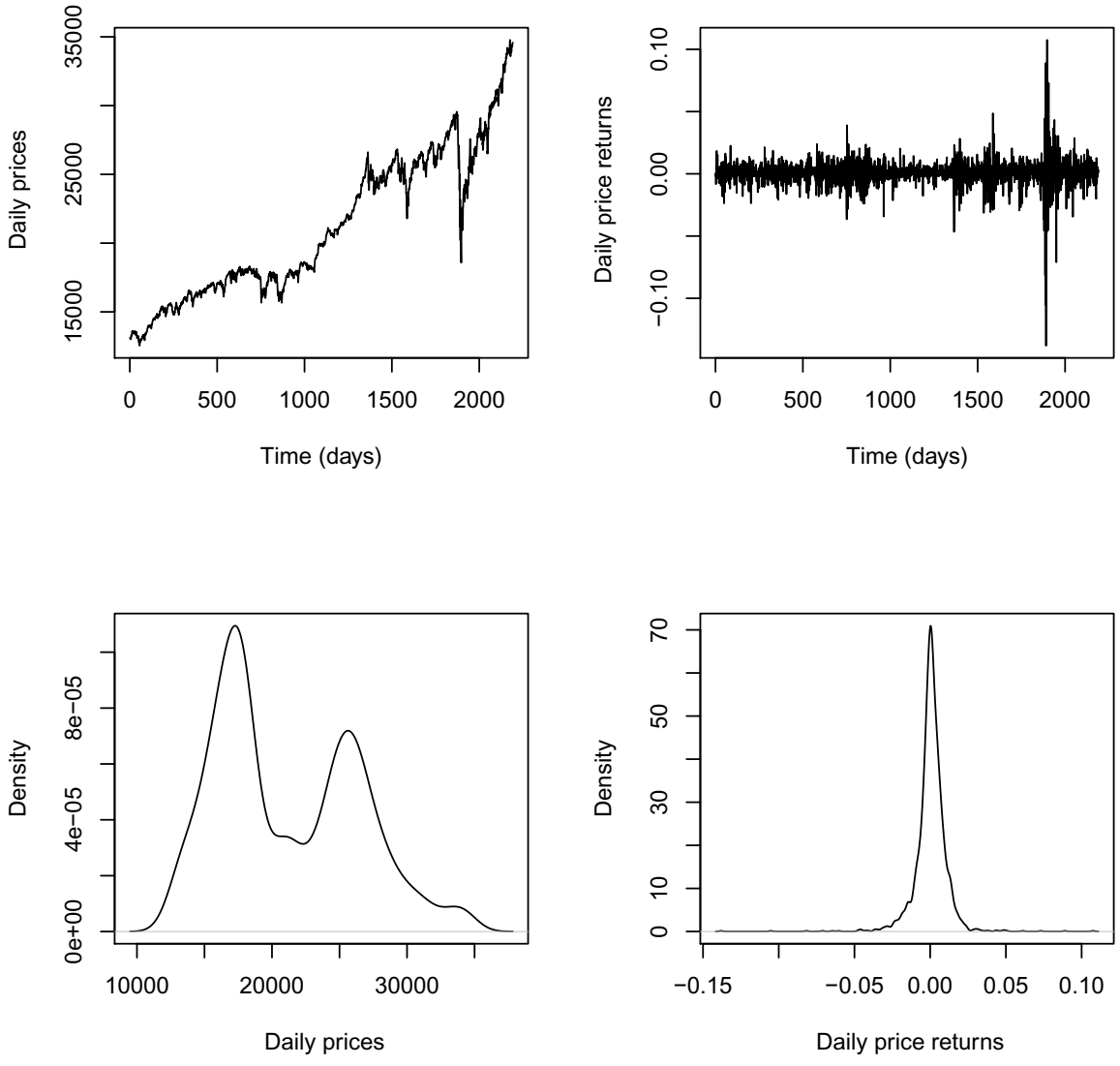

Figure A.7: DJI index

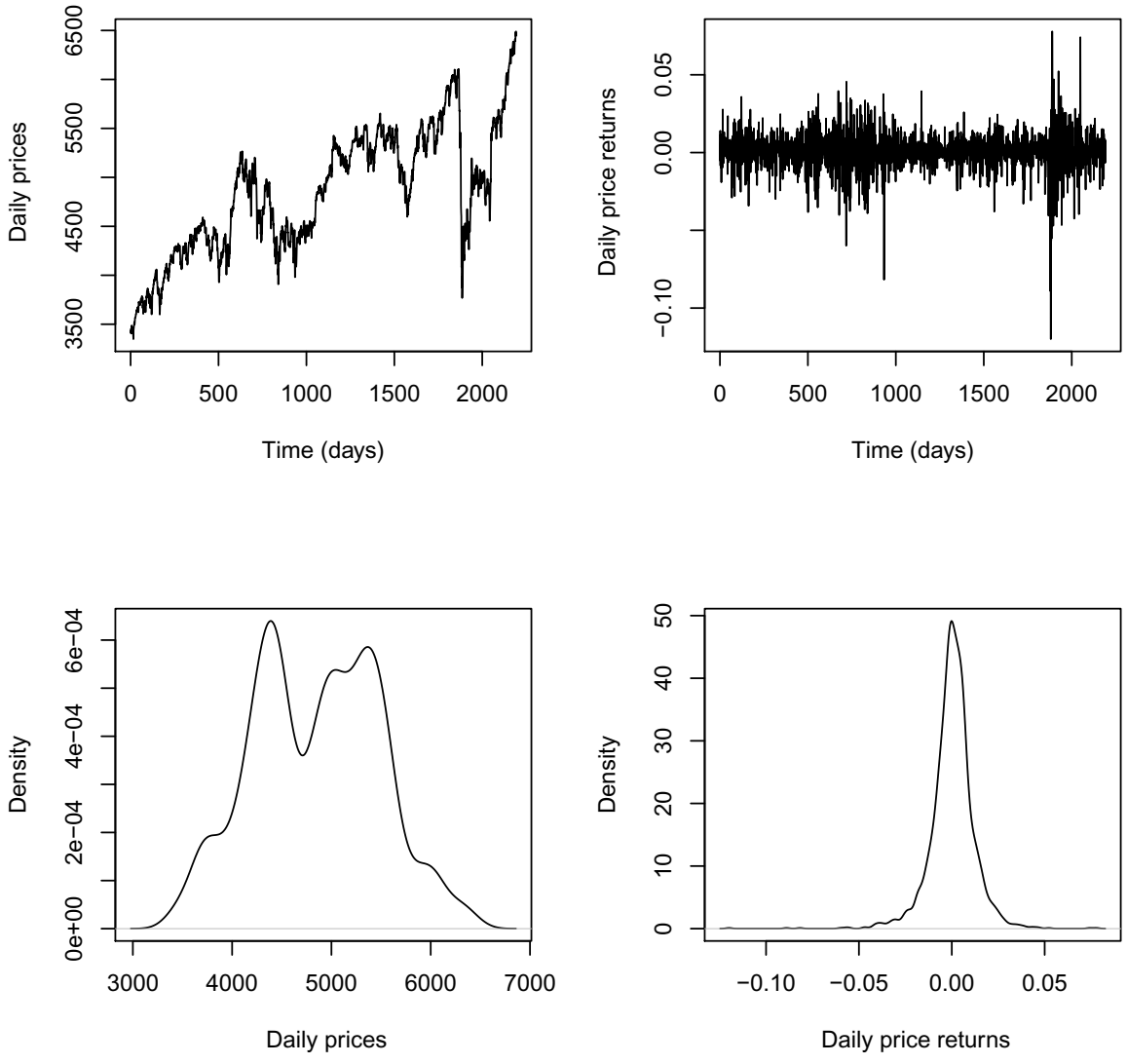

Figure A.8: FCHI index

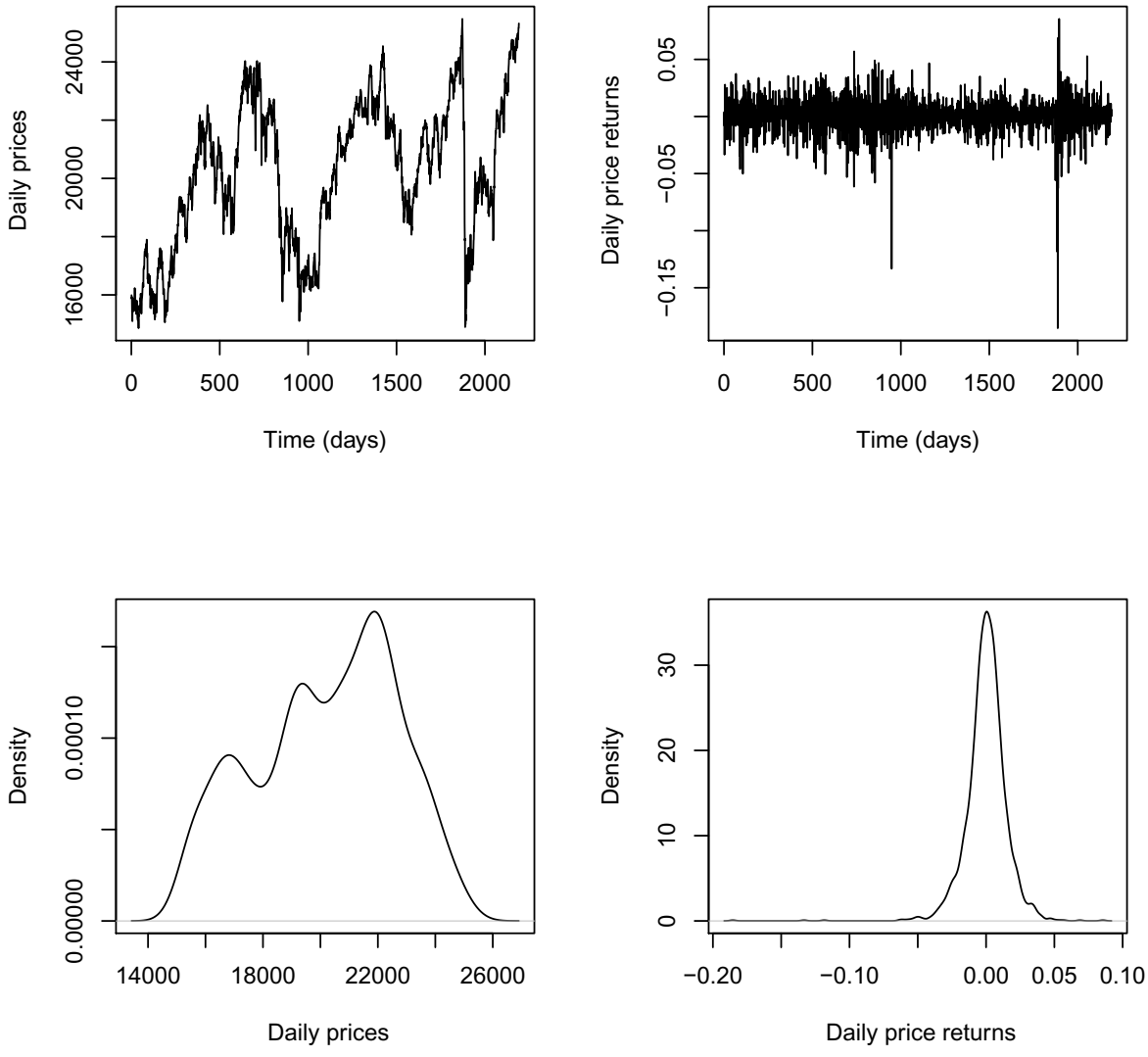

Figure A.9: FTMIB index

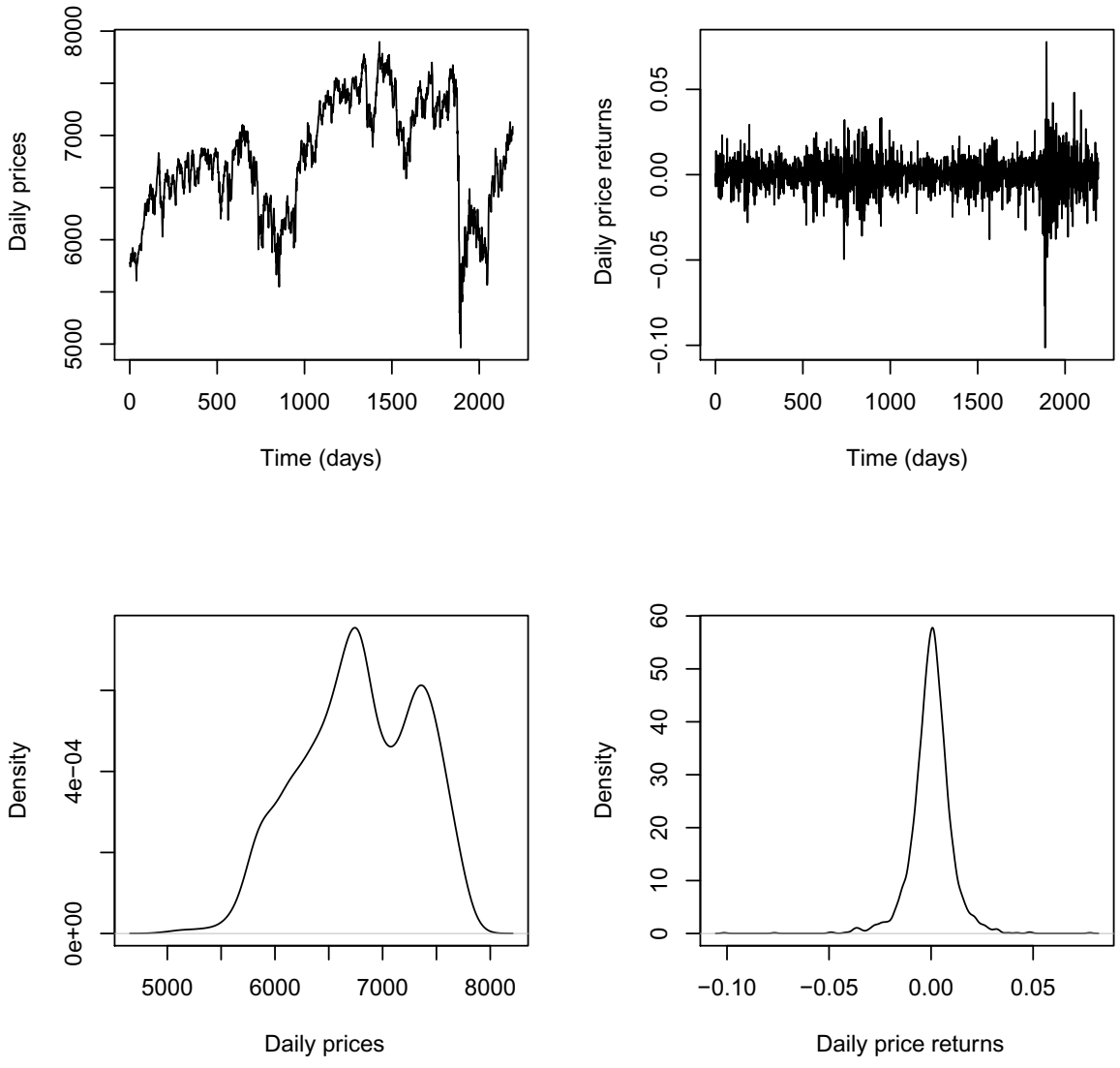

Figure A.10: FTSE index

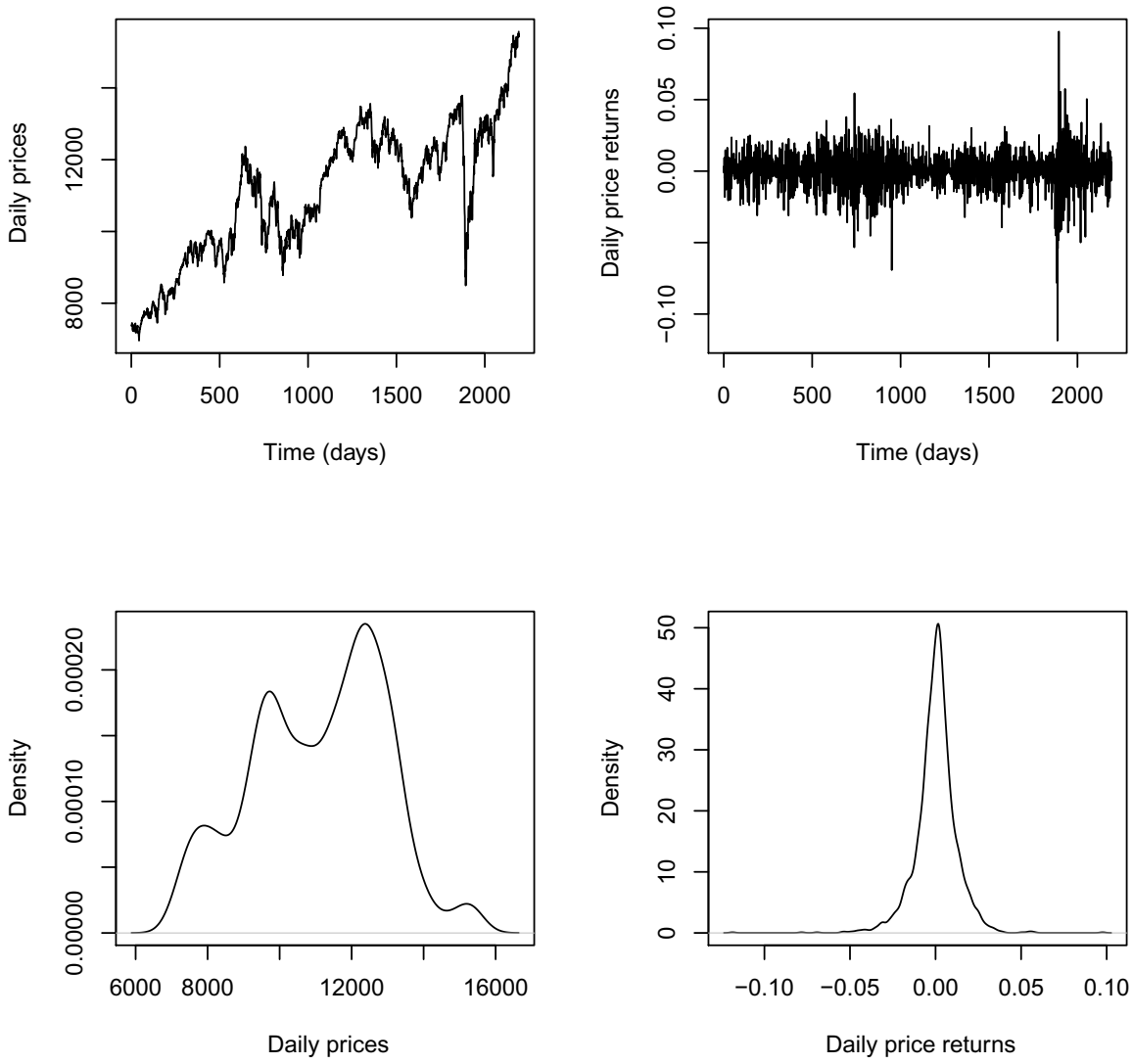

Figure A.11: GDAXI index

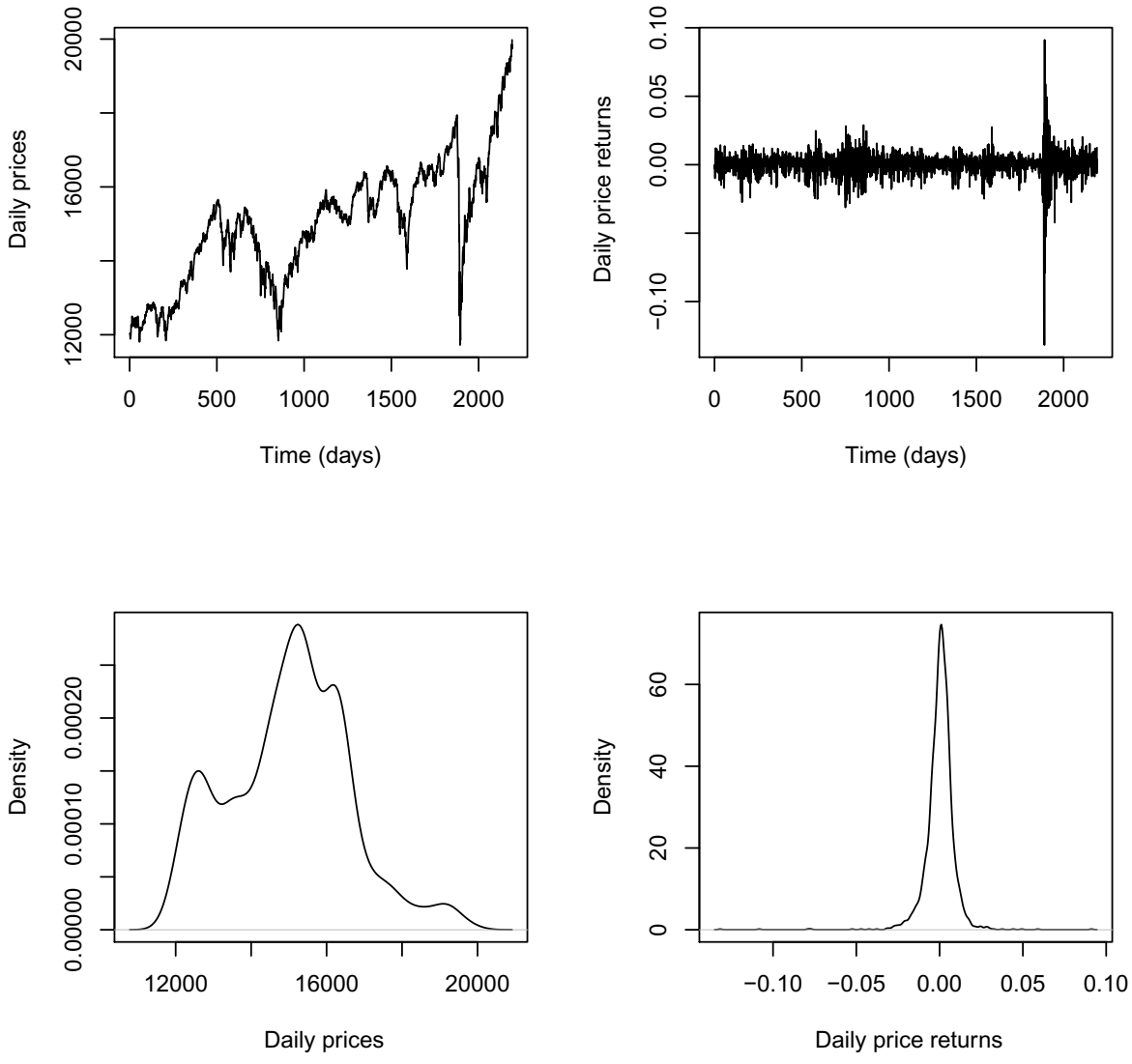

Figure A.12: GPTSE index

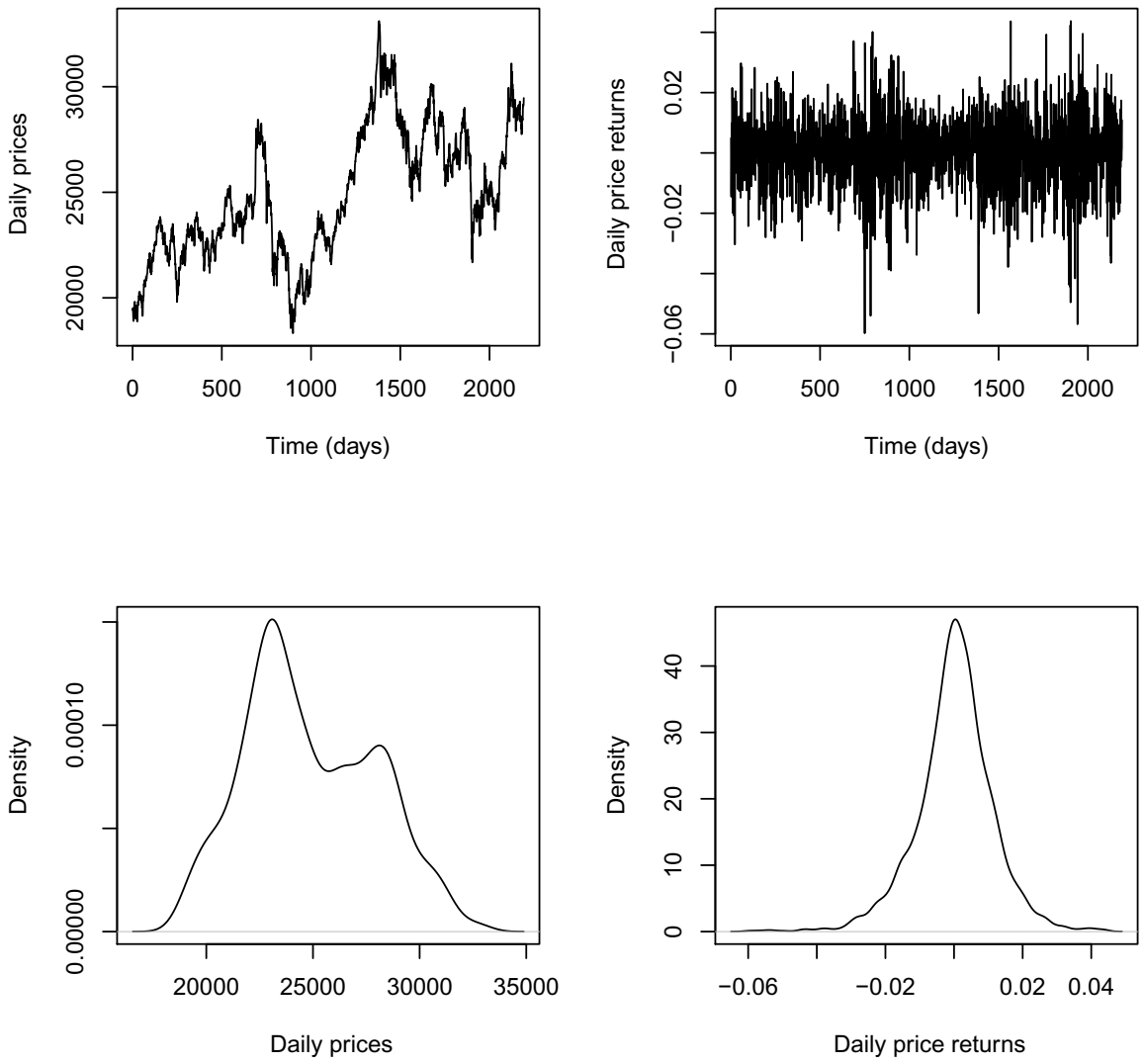

Figure A.13: HSI index

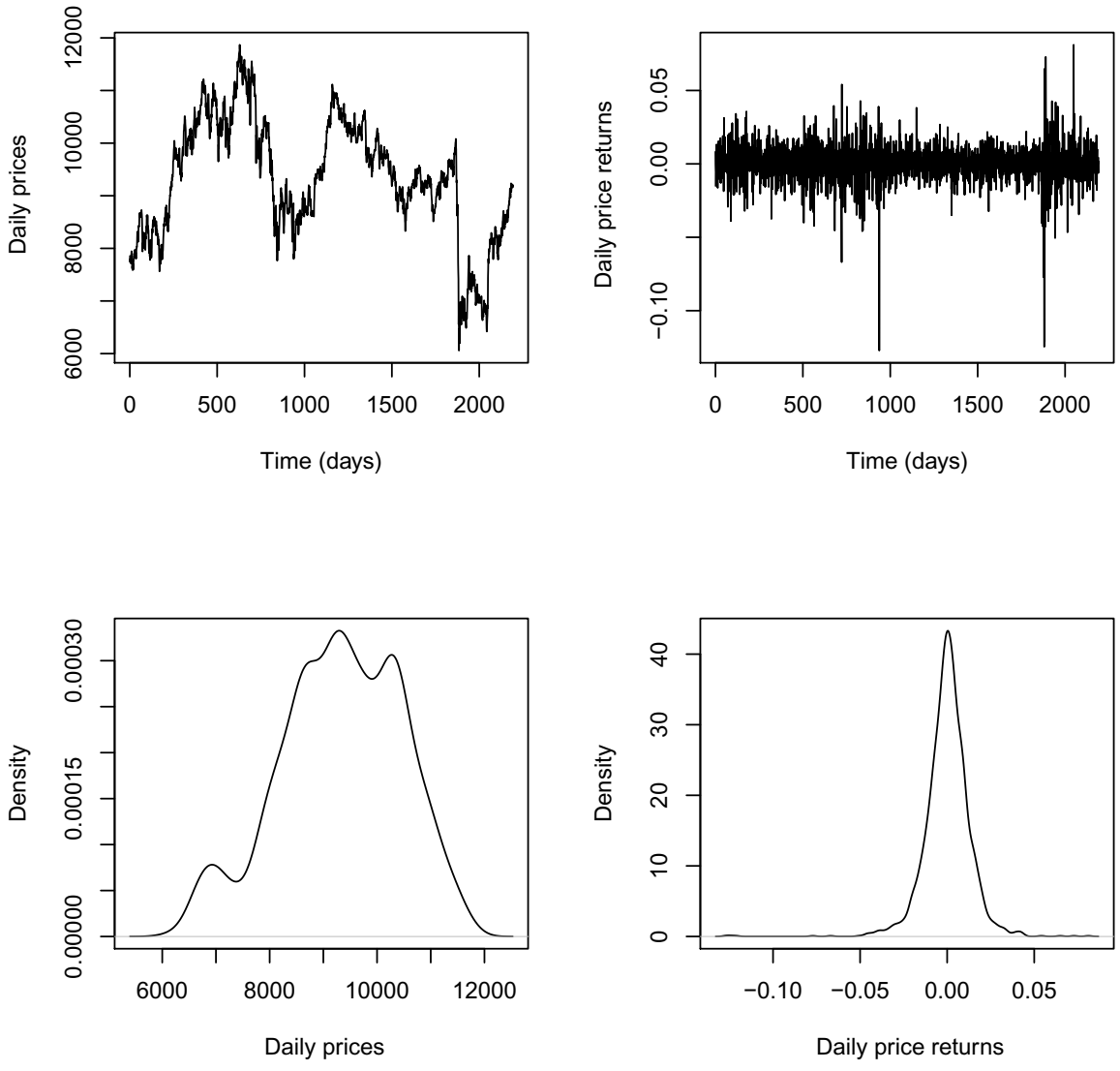

Figure A.14: IBEX index

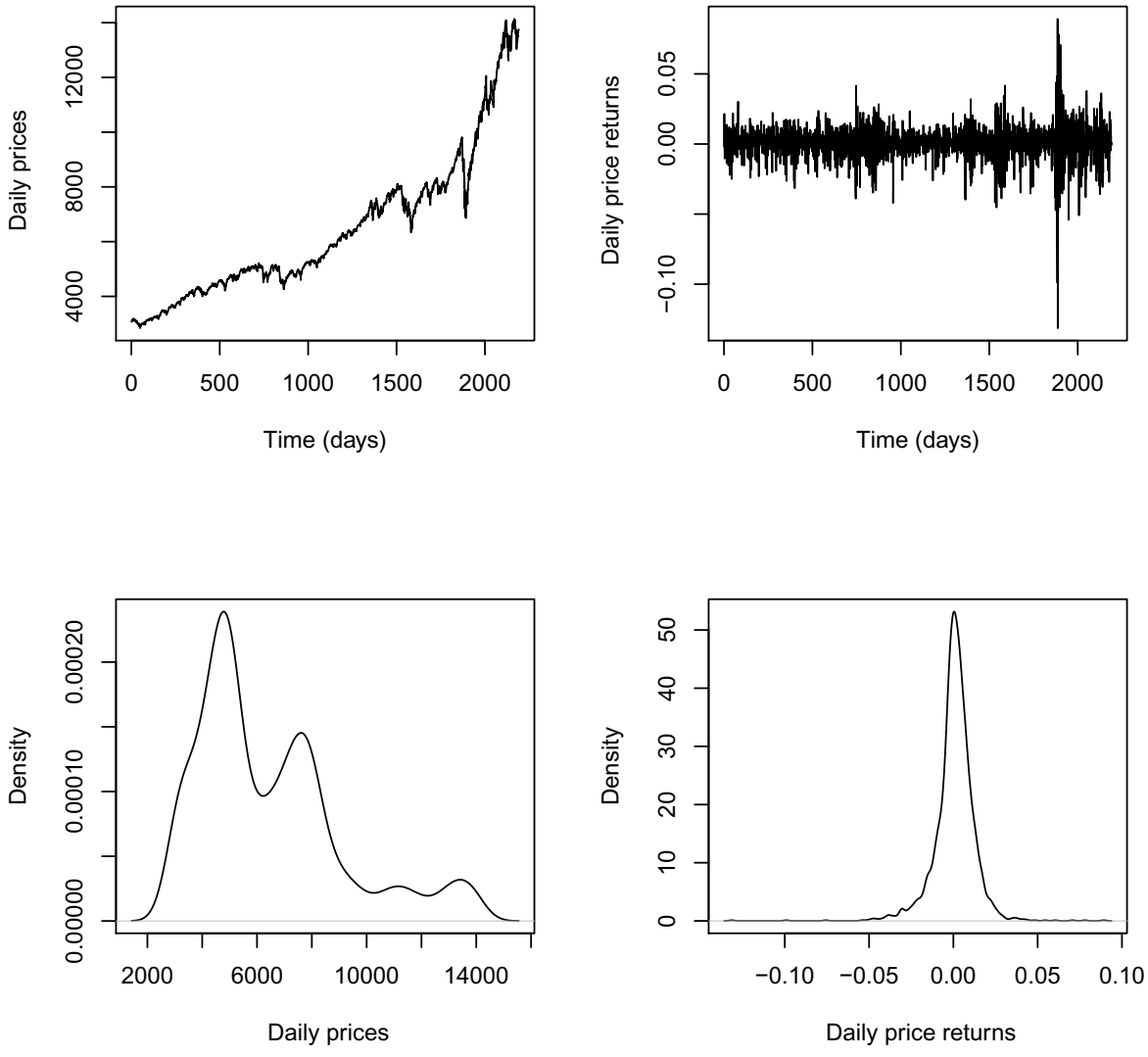

Figure A.15: IXIC index

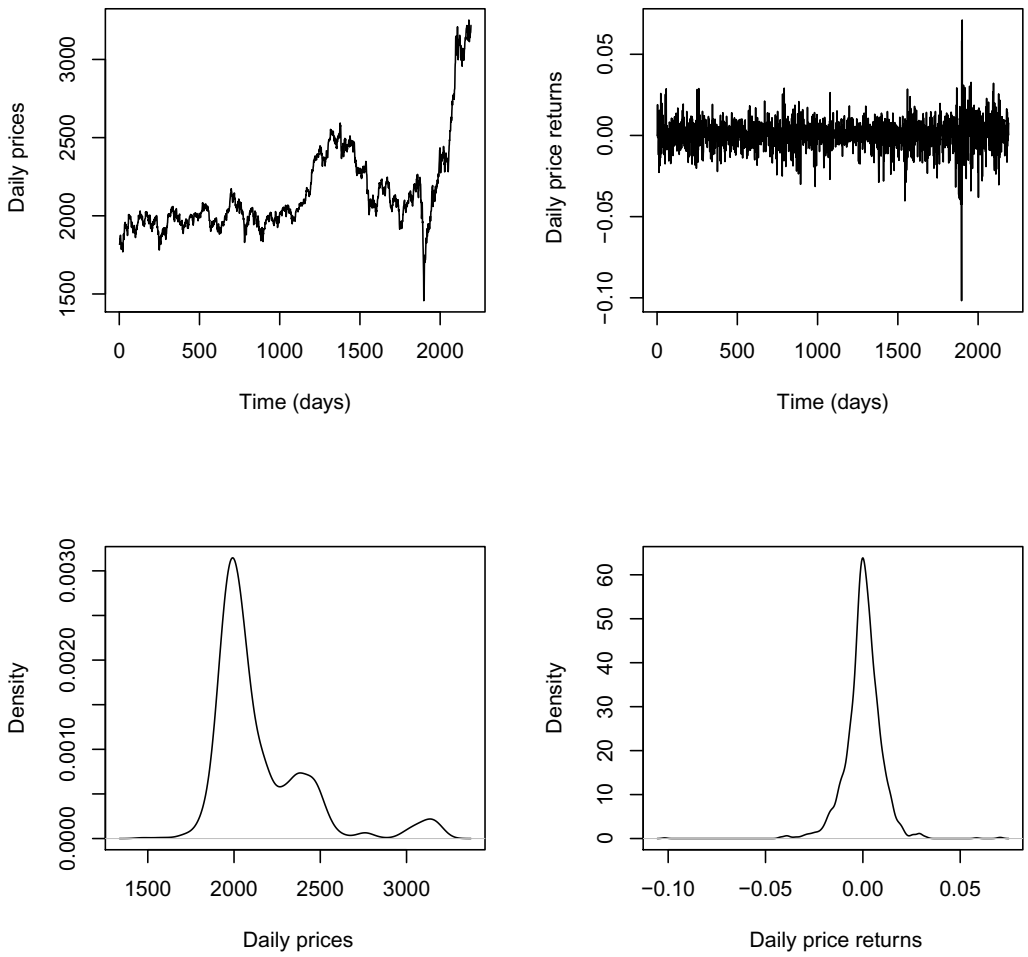

Figure A.16: KS11 index

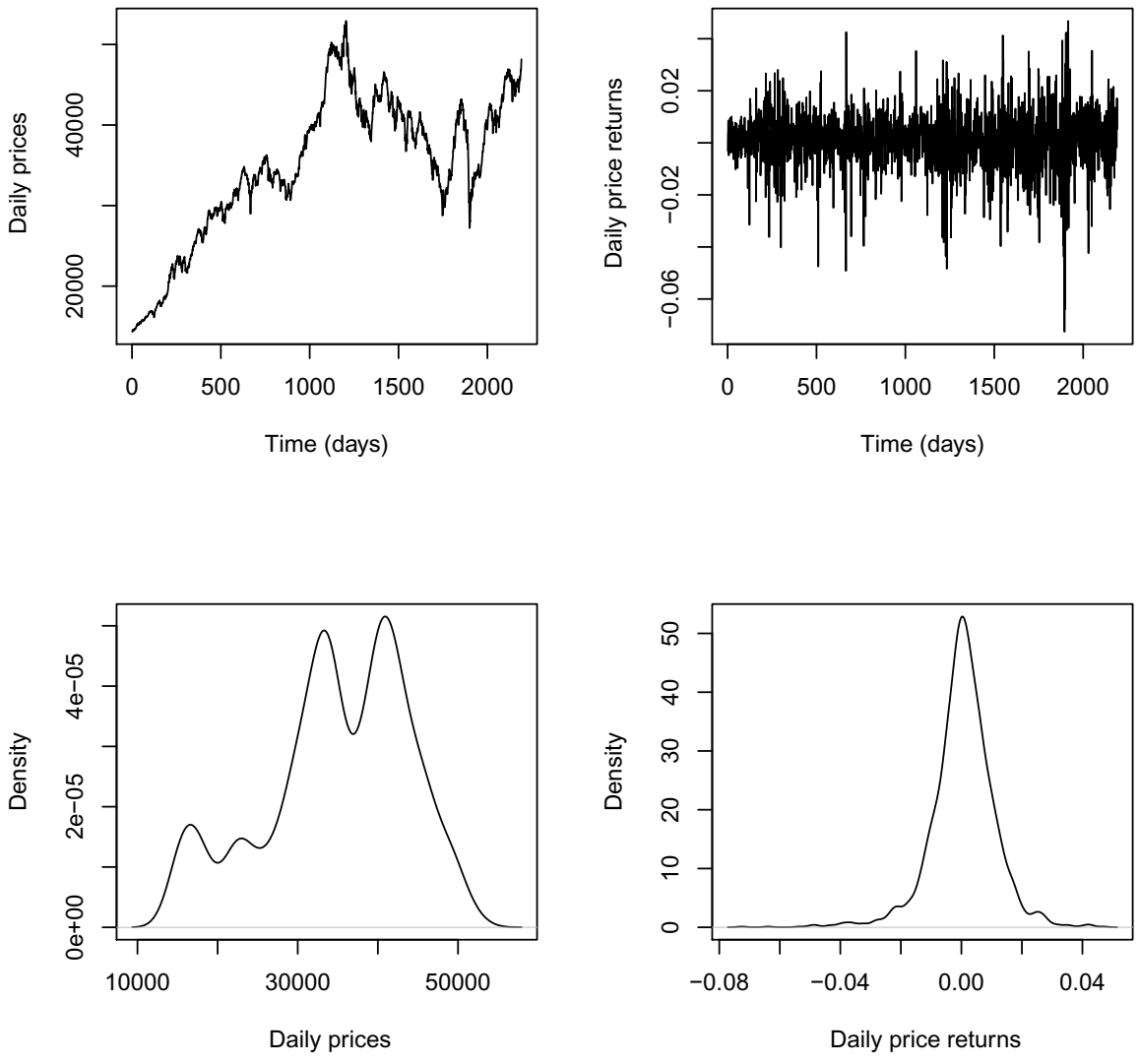

Figure A.17: KSE index

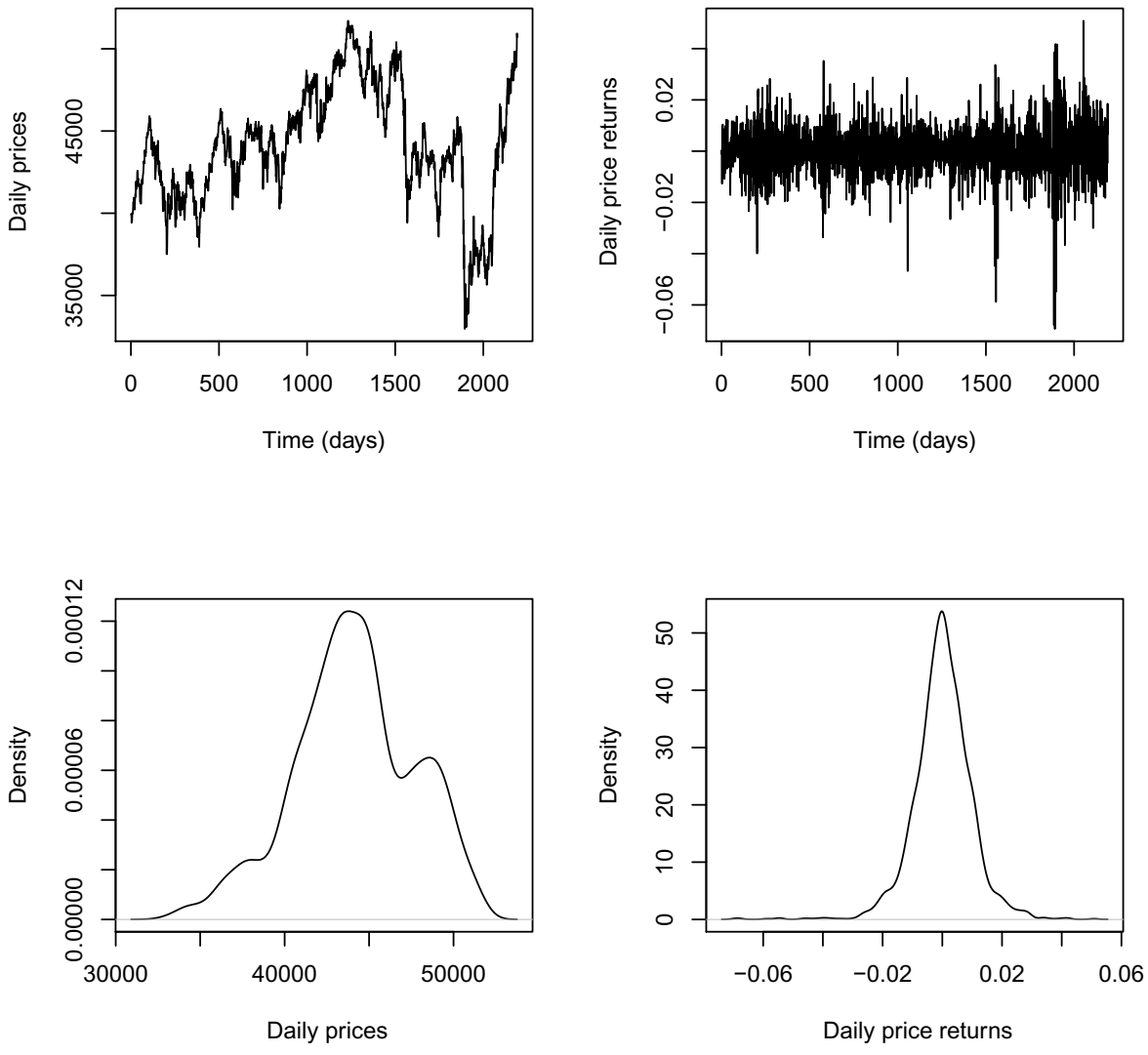

Figure A.18: MXX index

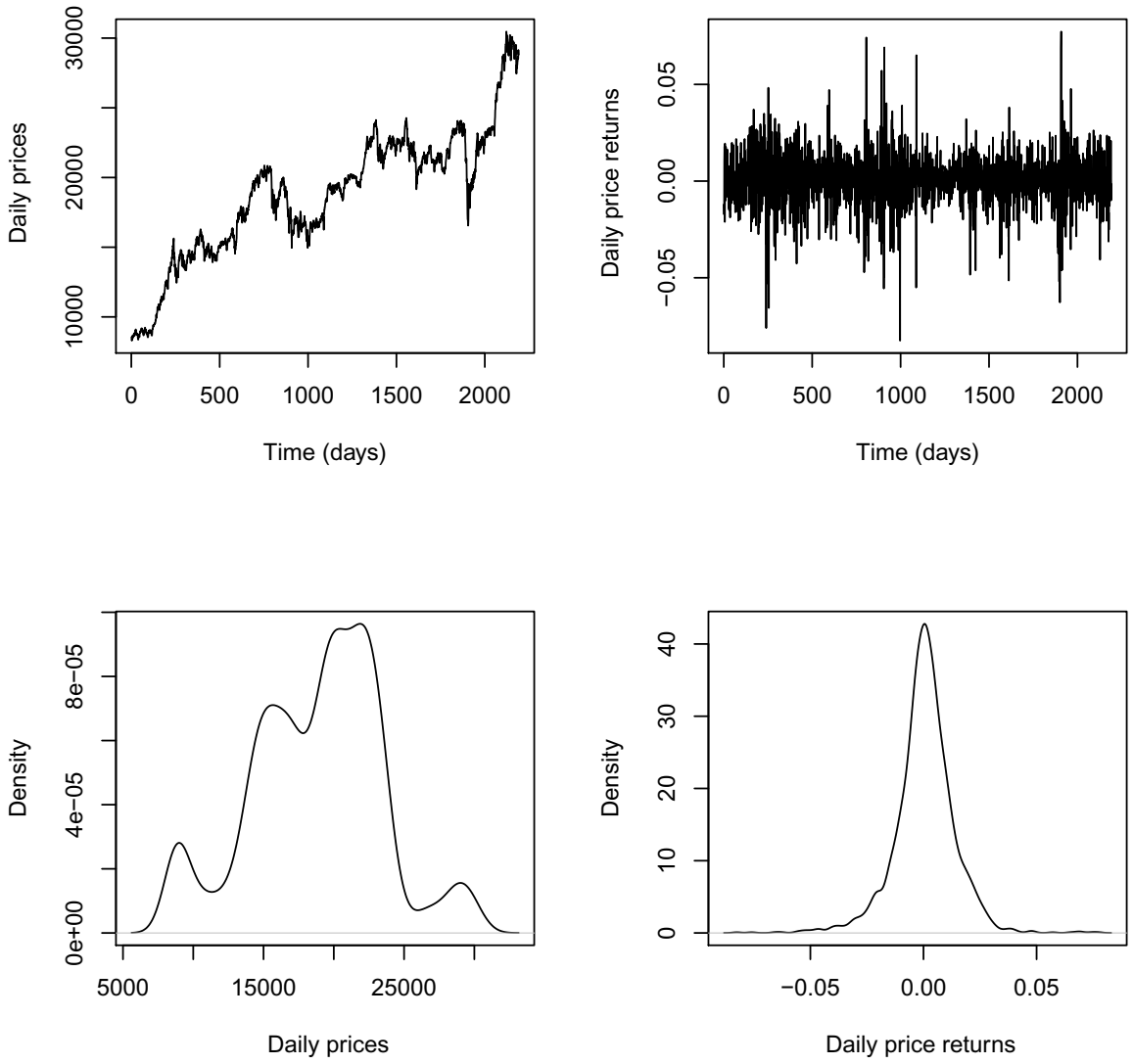

Figure A.19: N225 index

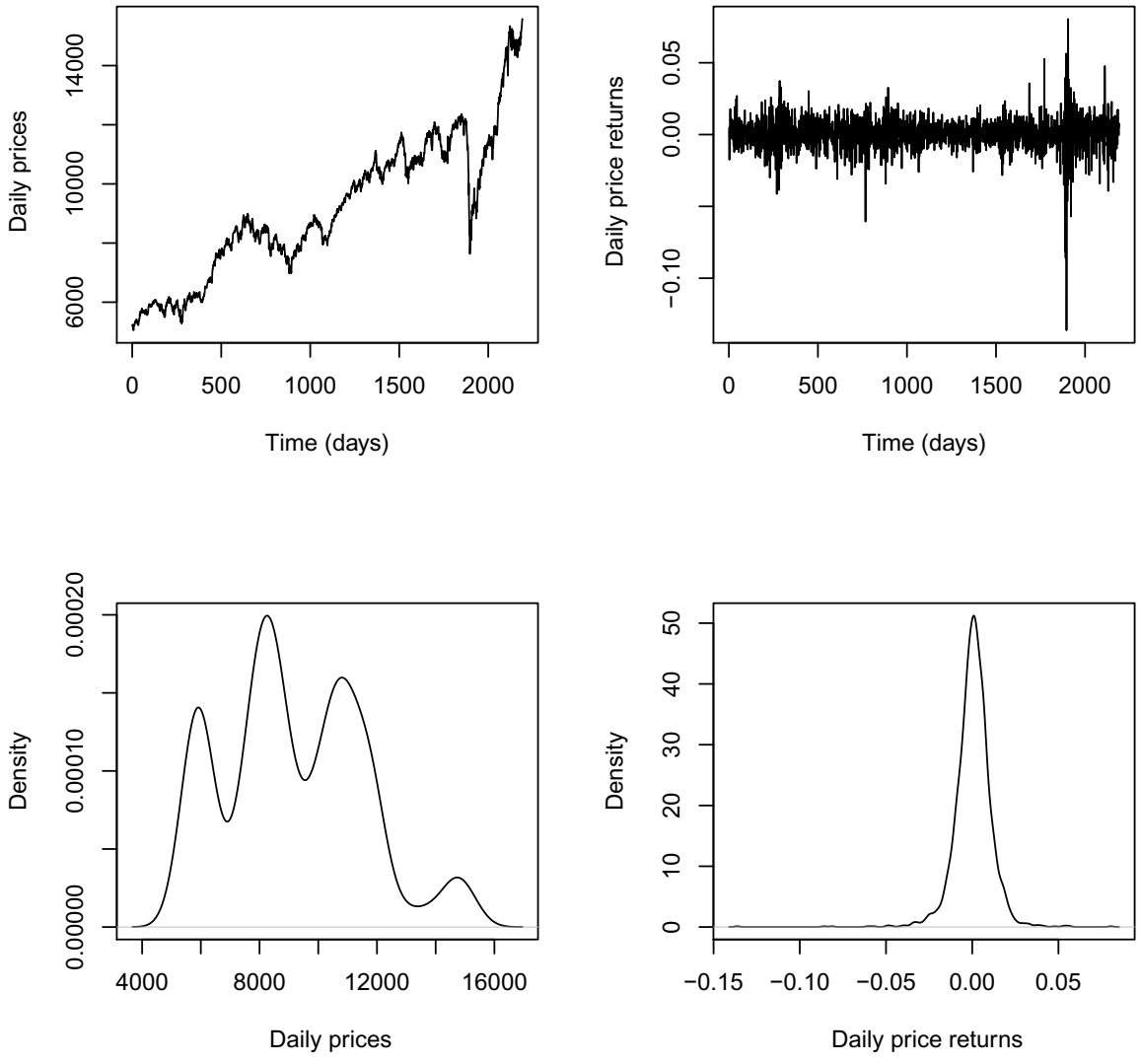

Figure A.20: NSEI index

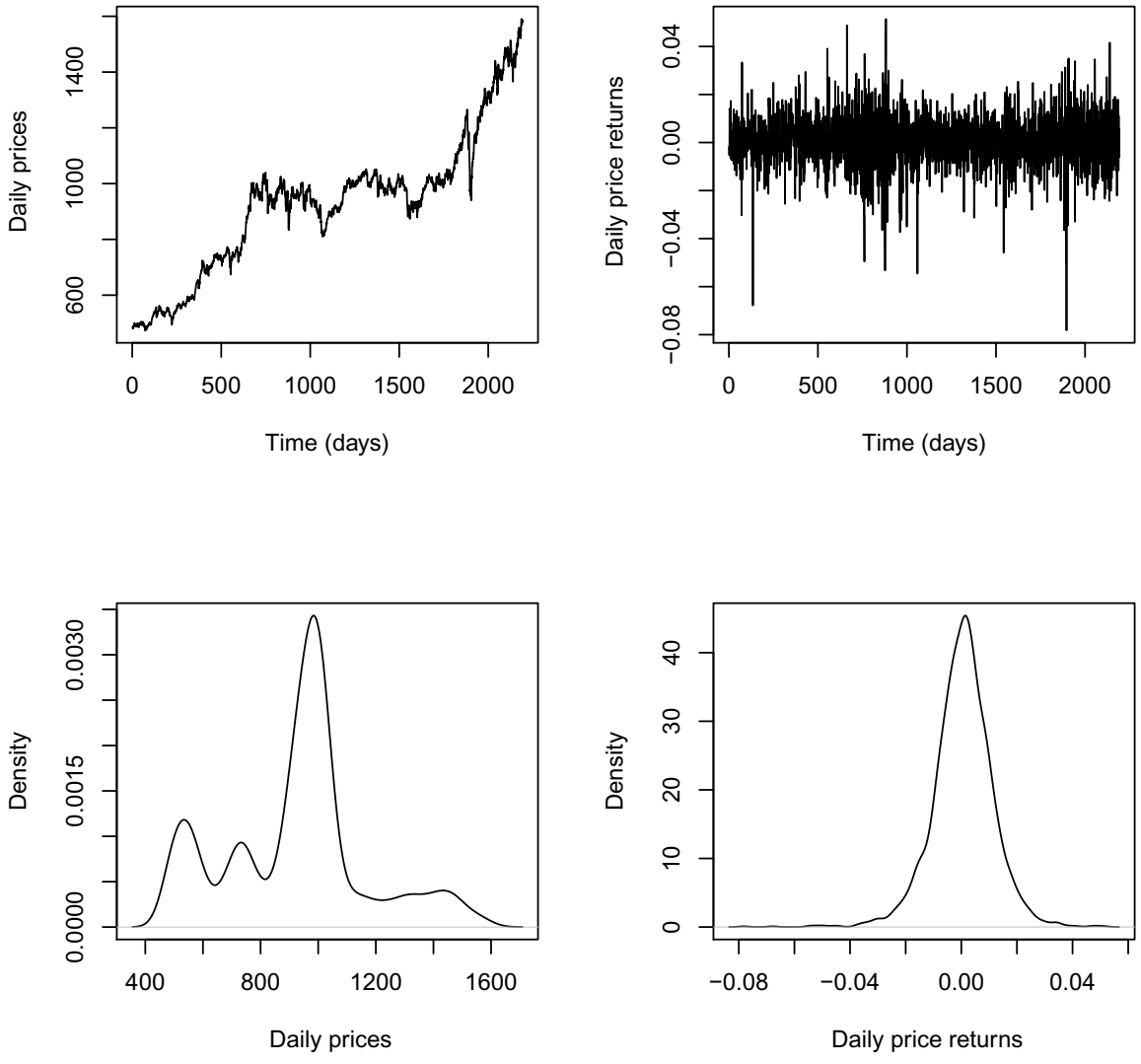

Figure A.21: OMX20 index

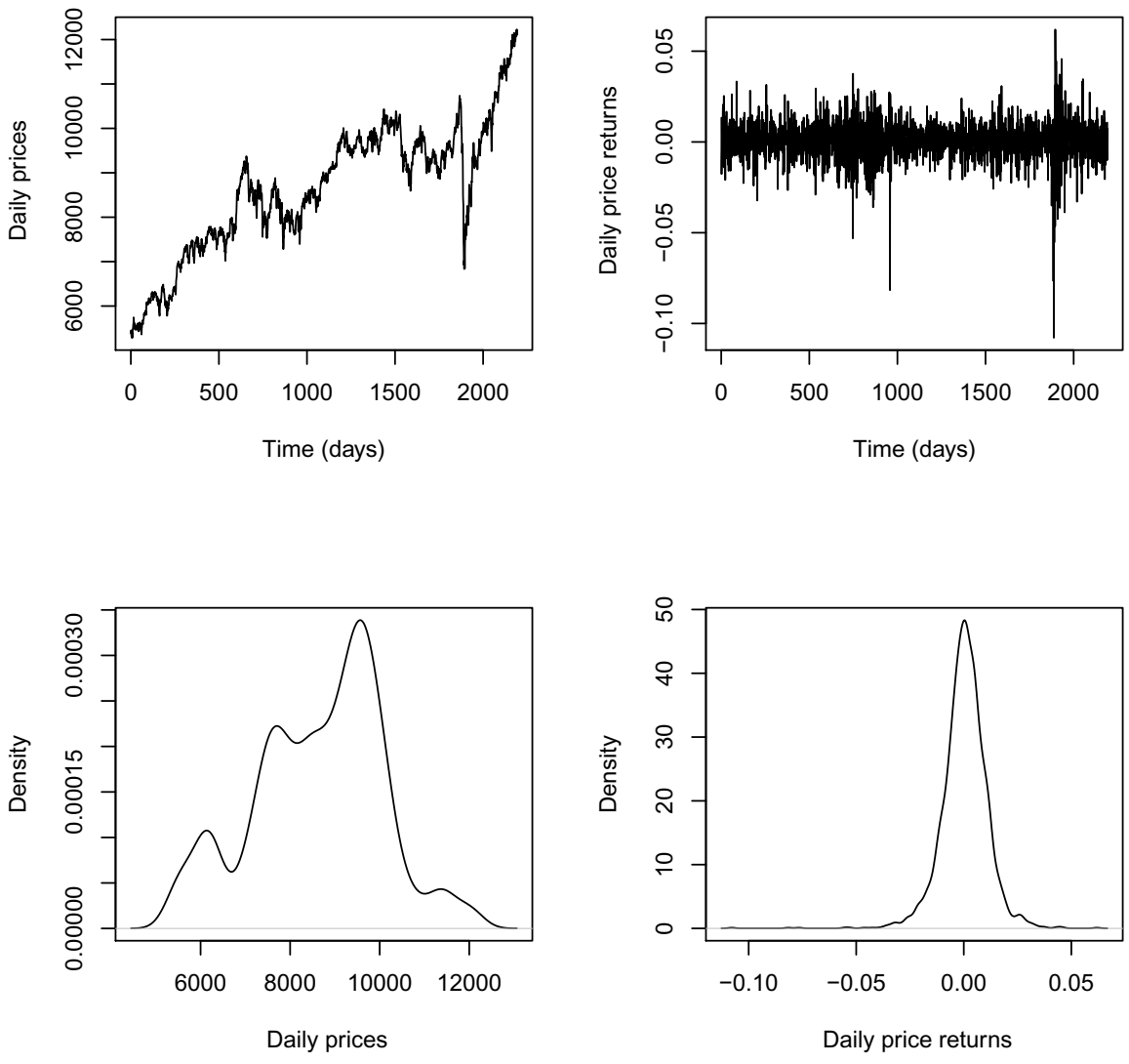

Figure A.22: OMXHPI index

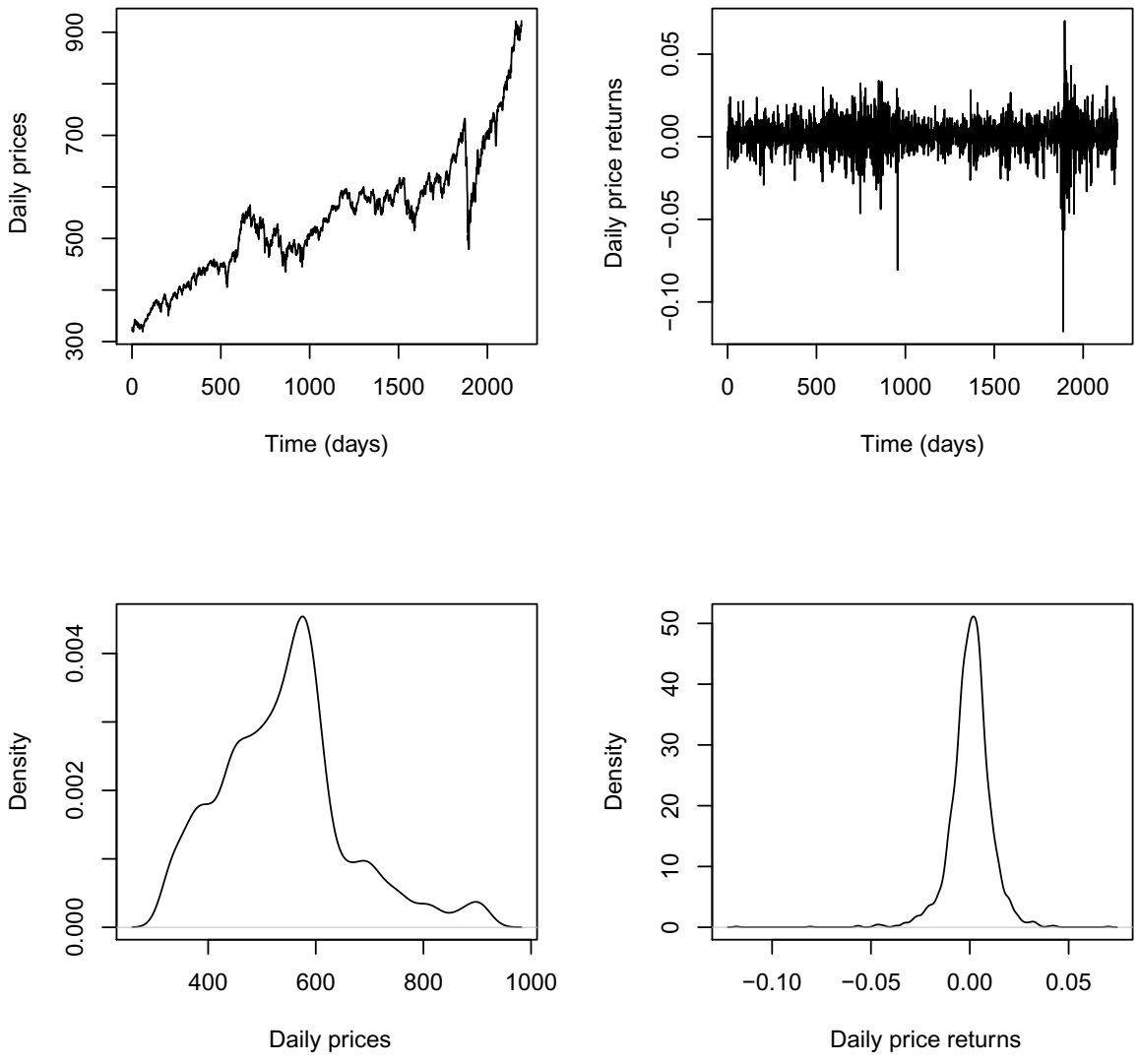

Figure A.23: OMXSPI index

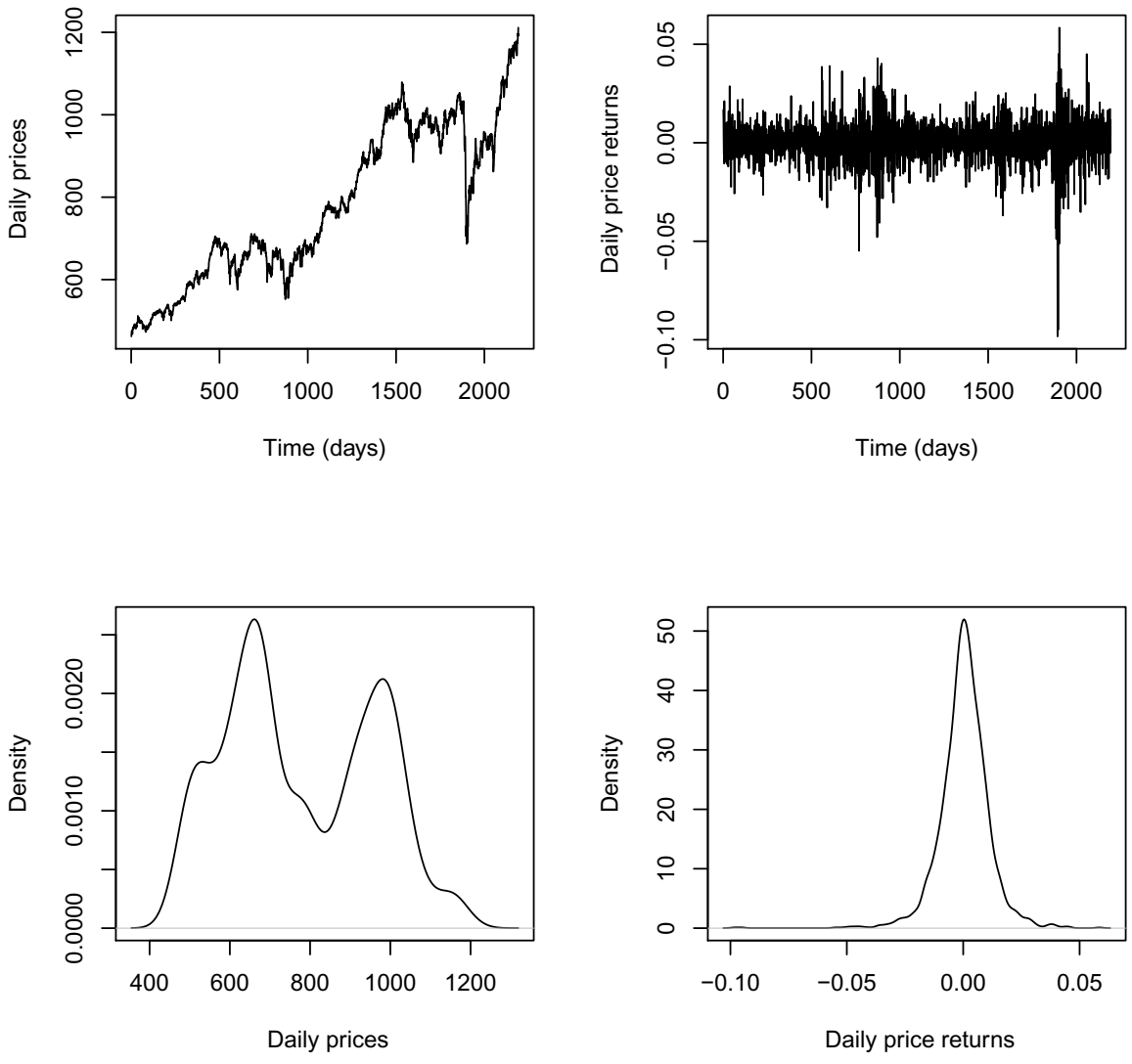

Figure A.24: OSEAX index

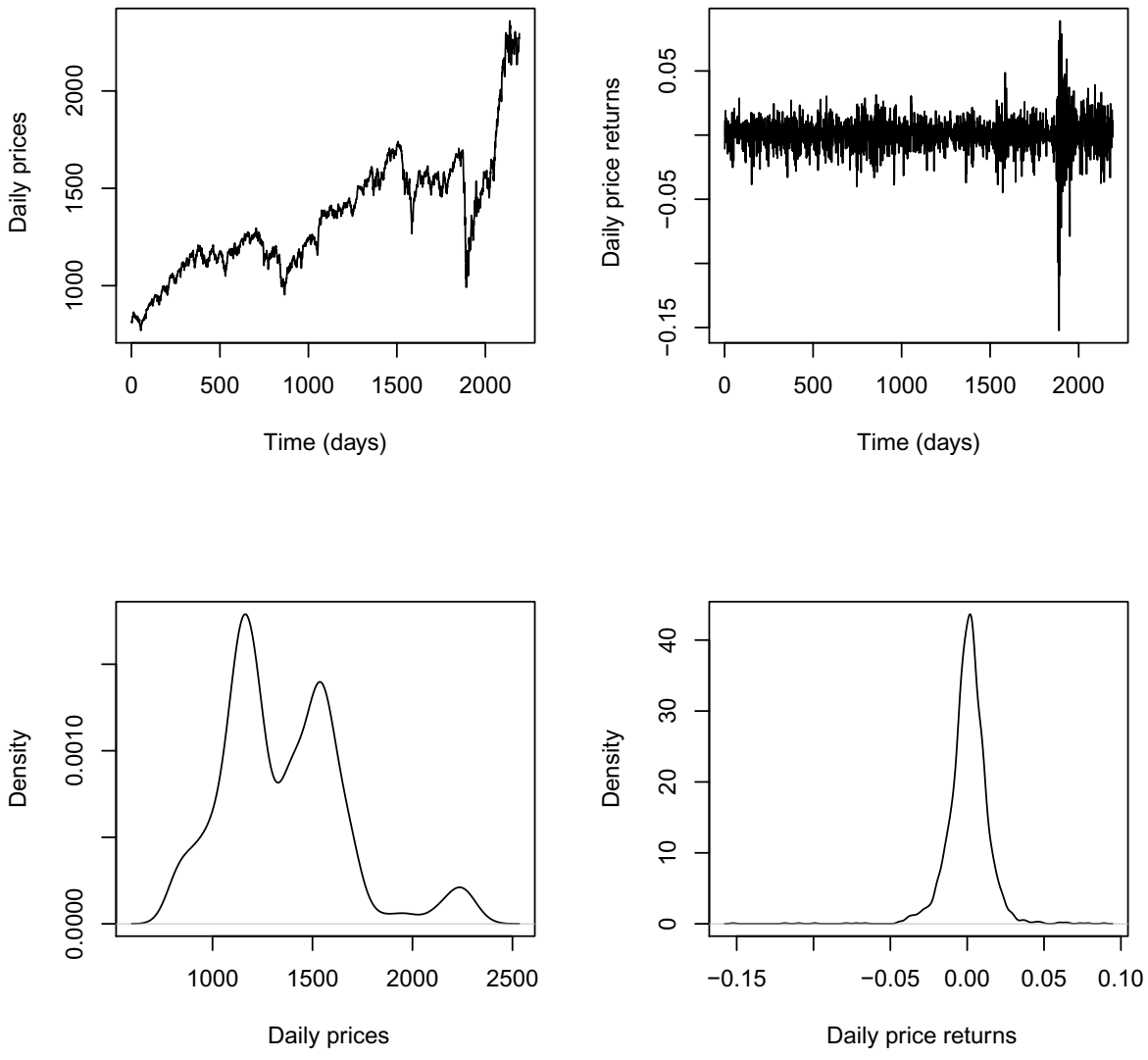

Figure A.25: RUT index

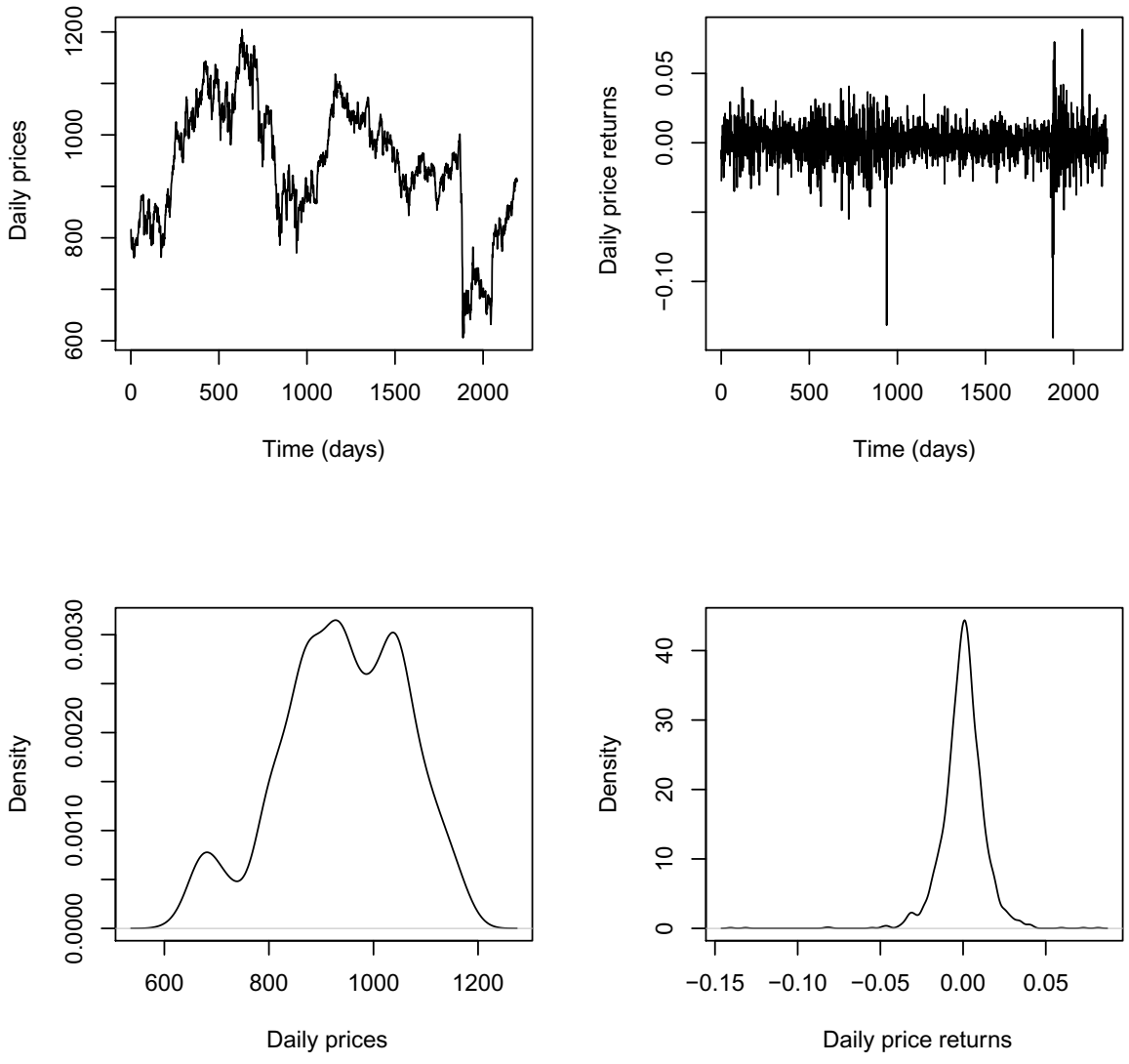

Figure A.26: SMSI index

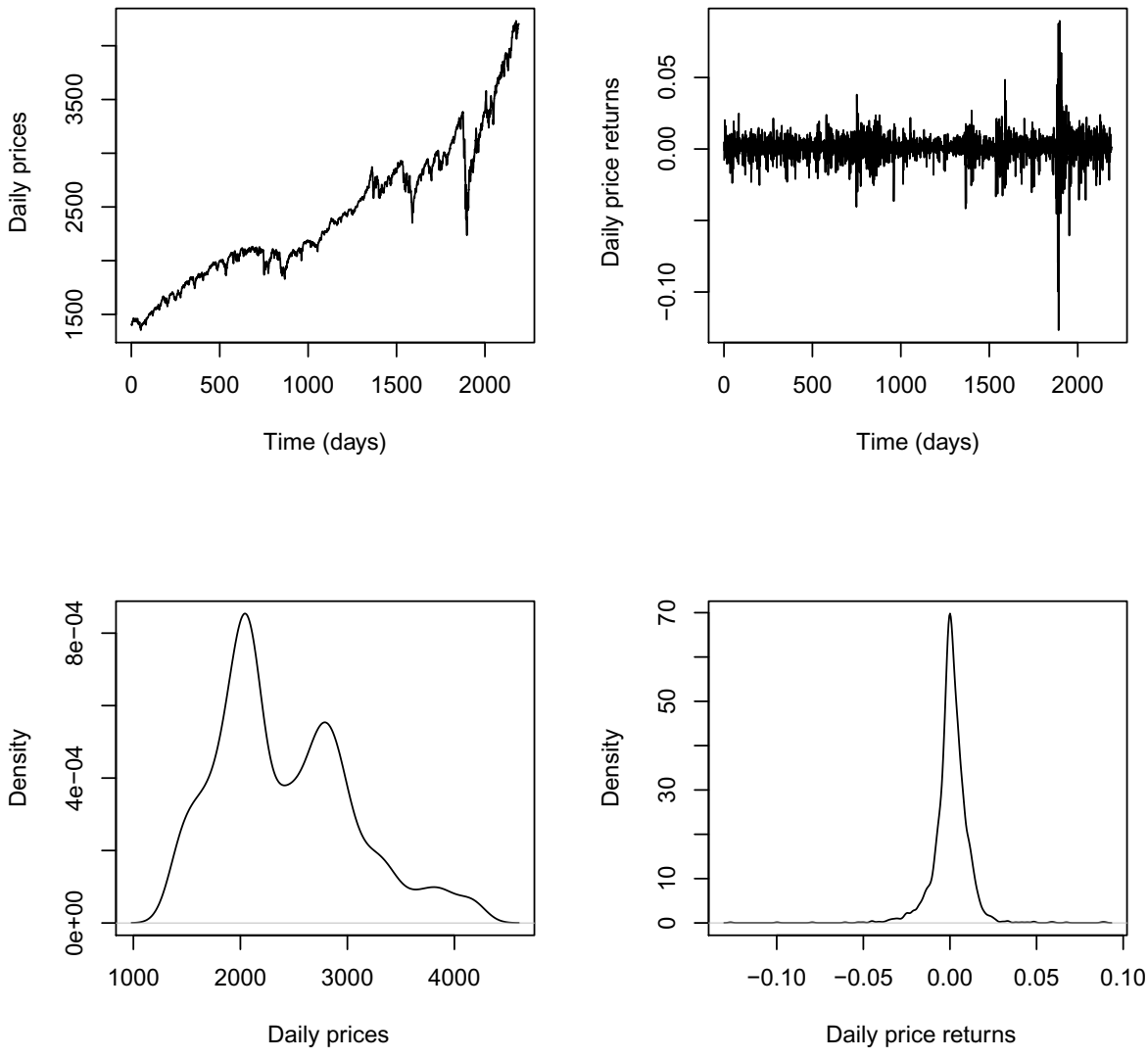

Figure A.27: SPX index

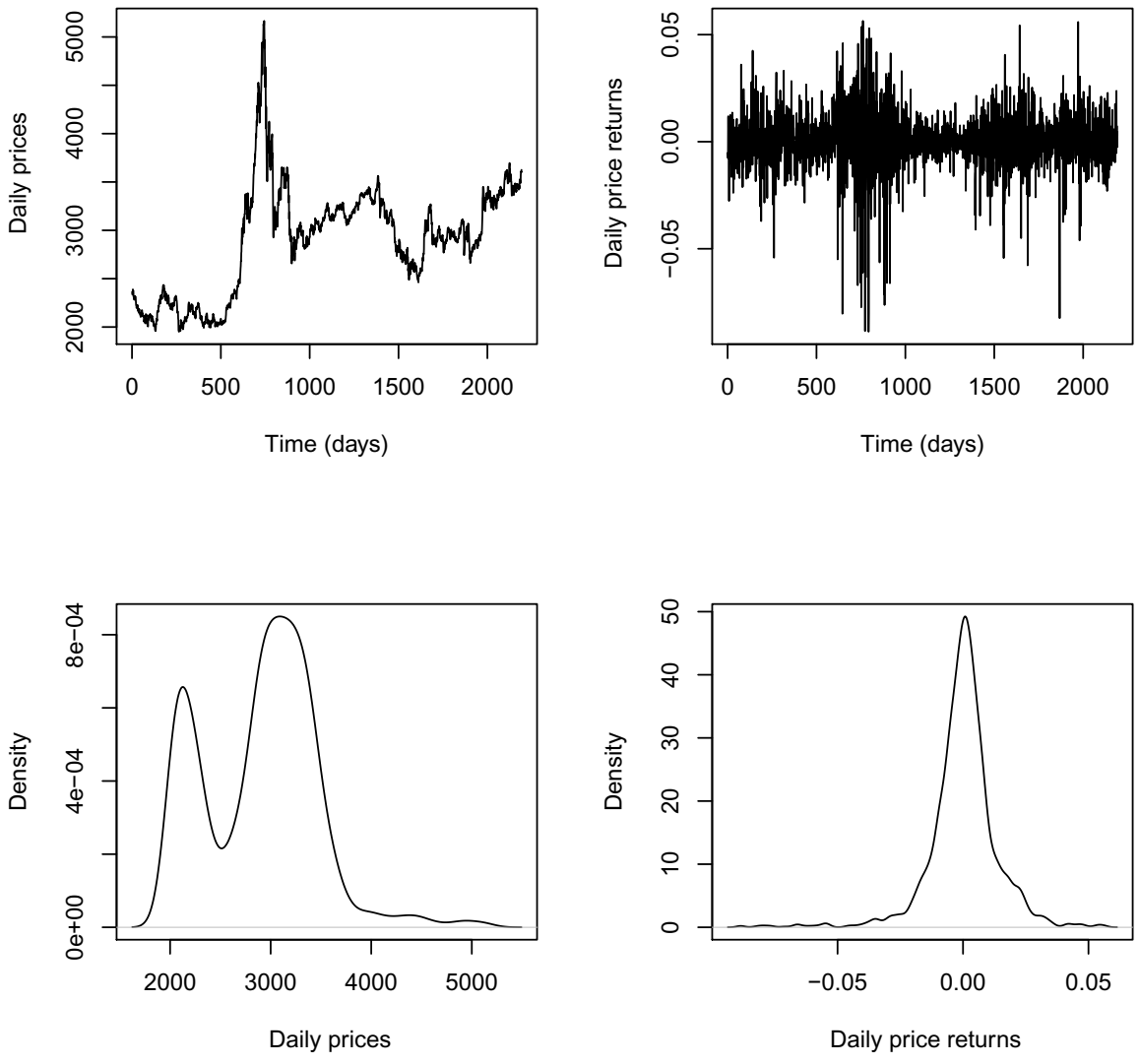

Figure A.28: SSEC index

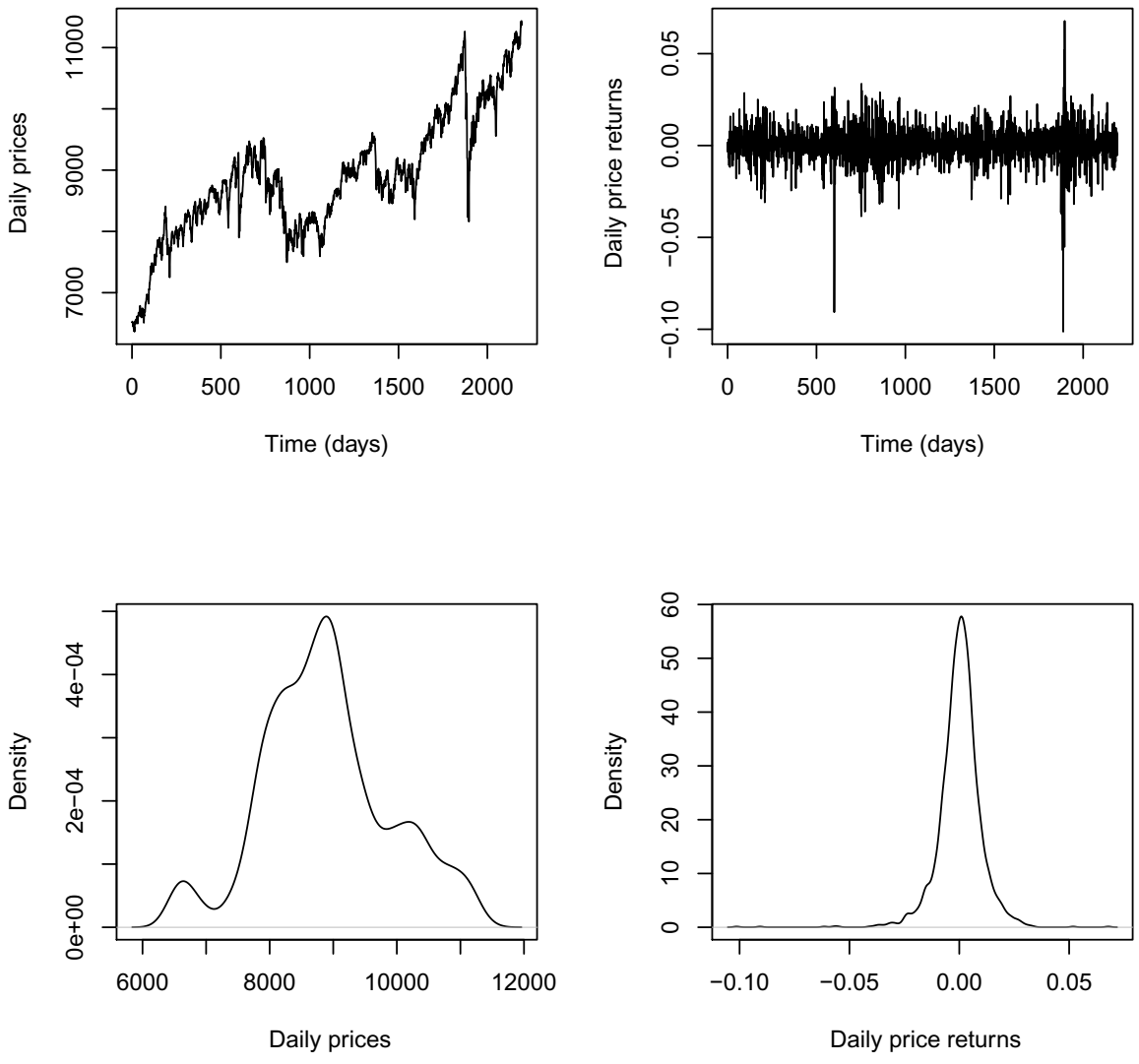

Figure A.29: SSMI index

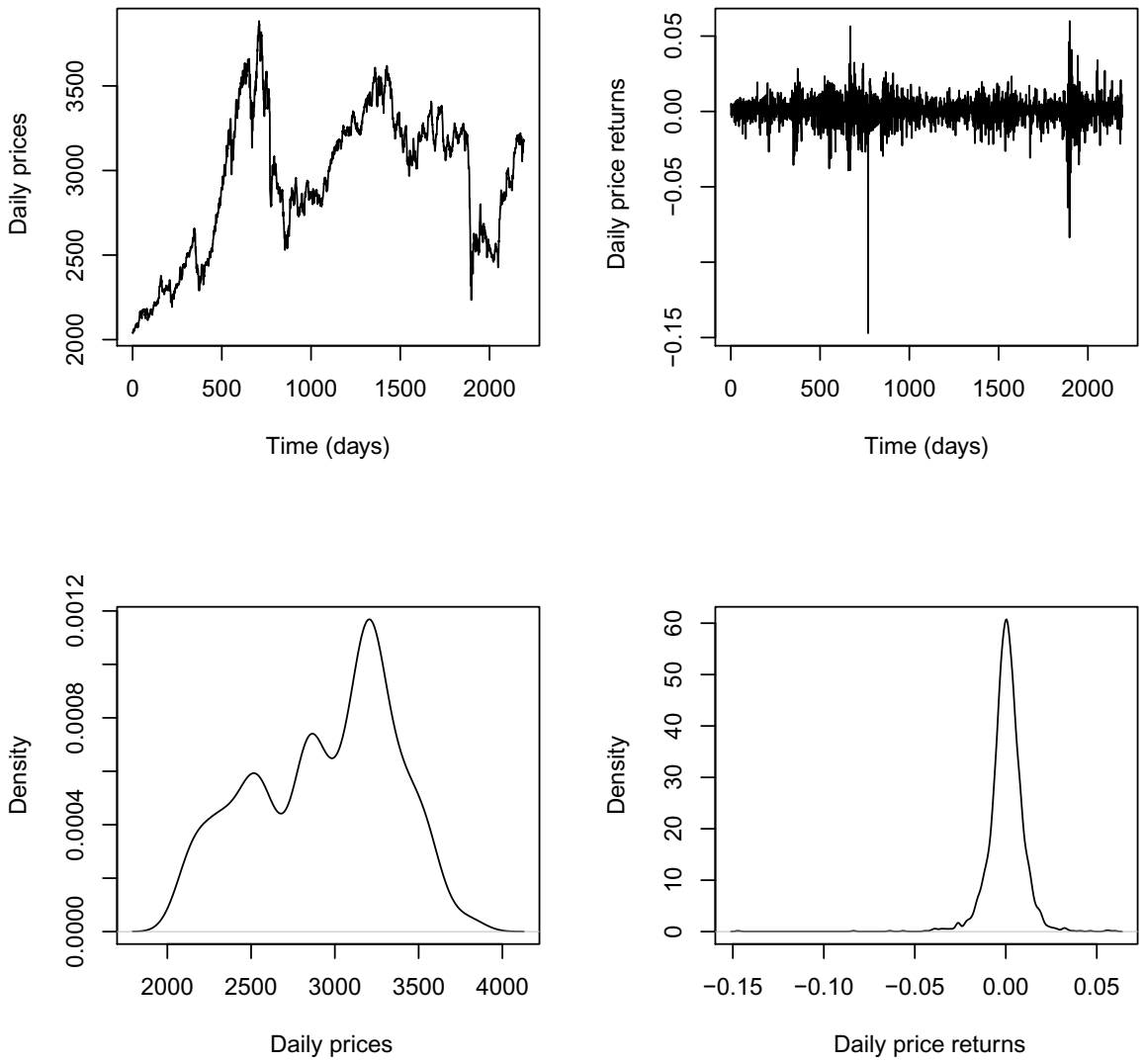

Figure A.30: STI index

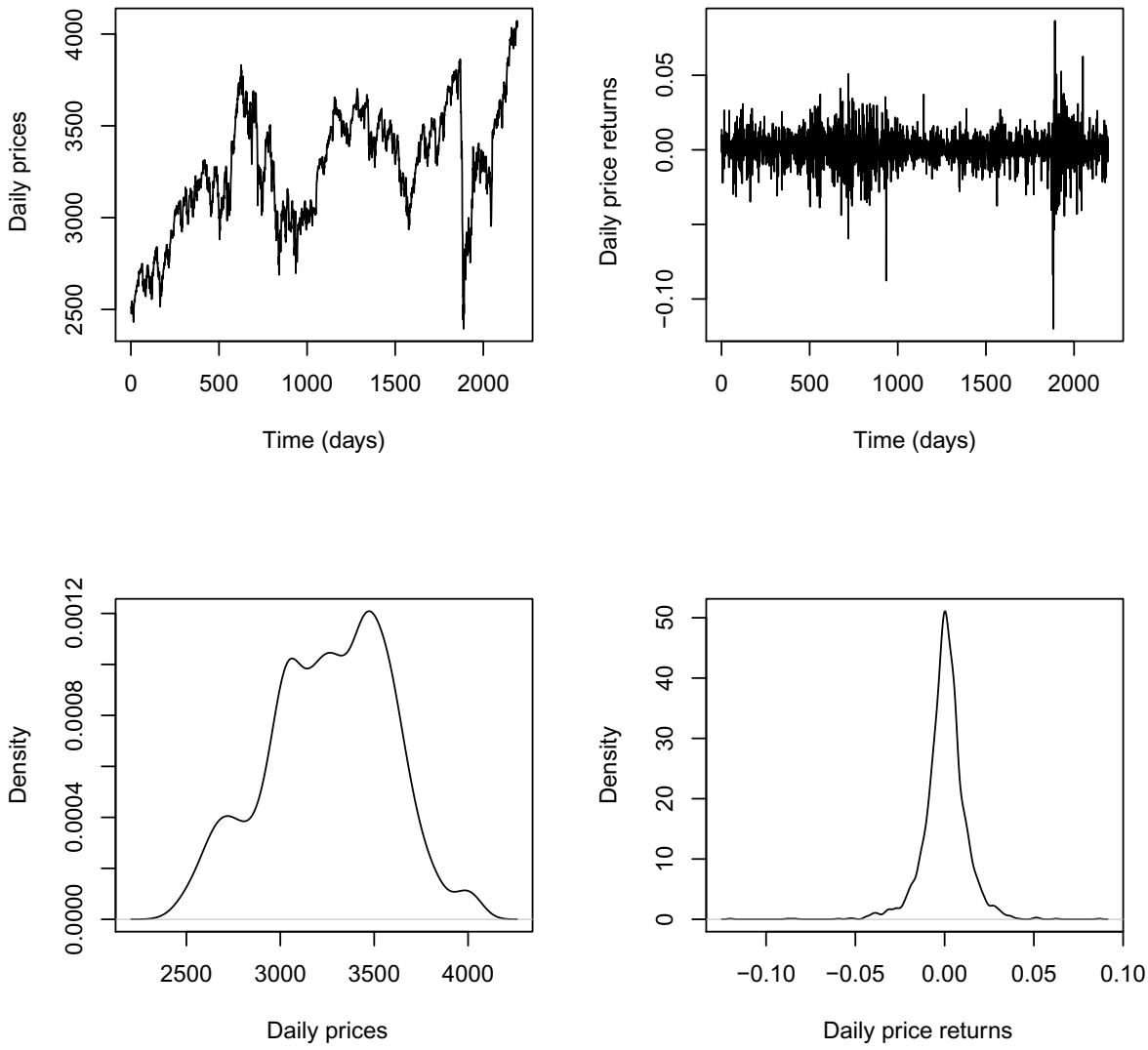

Figure A.31: STOXX50E index

Table A.1: Stationarity tests: AEX log returns

Stationarity tests: AEX log-returns						
	no drift, no trend		with drift, no trend		with drift and trend	
lag	ADF	p.value	ADF	p.value	ADF	p.value
0	-46.4702	0.01	-46.5094	0.01	-46.4988	0.01
1	-31.8241	0.01	-31.8666	0.01	-31.8594	0.01
2	-25.2151	0.01	-25.2603	0.01	-25.2546	0.01
3	-22.4757	0.01	-22.5271	0.01	-22.522	0.01
4	-20.6935	0.01	-20.7493	0.01	-20.7446	0.01
5	-19.1792	0.01	-19.2409	0.01	-19.2366	0.01
6	-16.5551	0.01	-16.6132	0.01	-16.6096	0.01
7	-16.9561	0.01	-17.0234	0.01	-17.0198	0.01
lag	KPSS	p.value	KPSS	p.value	KPSS	p.value
10	0.6303	0.1	0.0302	0.1	0.0306	0.1

For ADF tests $p.value = 0.01$ means $p.value \leq 0.01$.
For KPSS test $p.value = 0.1$ means $p.value \geq 0.1$.

Table A.2: Stationarity tests: AORD log returns

Stationarity tests: AORD log-returns						
	no drift, no trend		with drift, no trend		with drift and trend	
lag	ADF	p.value	ADF	p.value	ADF	p.value
0	-48.0954	0.01	-48.1181	0.01	-48.1071	0.01
1	-32.2648	0.01	-32.2905	0.01	-32.2831	0.01
2	-25.465	0.01	-25.4927	0.01	-25.4868	0.01
3	-22.5238	0.01	-22.5539	0.01	-22.5486	0.01
4	-20.0443	0.01	-20.0772	0.01	-20.0726	0.01
5	-18.2223	0.01	-18.2567	0.01	-18.2524	0.01
6	-16.2923	0.01	-16.3278	0.01	-16.324	0.01
7	-14.1988	0.01	-14.2332	0.01	-14.2299	0.01
lag	KPSS	p.value	KPSS	p.value	KPSS	p.value
10	0.3885	0.1	0.0318	0.1	0.0318	0.1

Table A.3: Stationarity tests: BFX log returns

Stationarity tests: BFX log-returns						
	no drift, no trend		with drift, no trend		with drift and trend	
lag	ADF	p.value	ADF	p.value	ADF	p.value
0	-44.3415	0.01	-44.3529	0.01	-44.3501	0.01
1	-30.7723	0.01	-30.787	0.01	-30.7876	0.01
2	-25.1489	0.01	-25.1653	0.01	-25.1669	0.01
3	-22.8362	0.01	-22.8567	0.01	-22.8605	0.01
4	-20.9274	0.01	-20.9498	0.01	-20.9538	0.01
5	-19.3534	0.01	-19.3789	0.01	-19.3847	0.01
6	-16.7347	0.01	-16.7594	0.01	-16.7646	0.01
7	-17.0191	0.01	-17.0477	0.01	-17.054	0.01
lag	KPSS	p.value	KPSS	p.value	KPSS	p.value
10	0.4873	0.1	0.0773	0.1	0.0357	0.1

Table A.4: Stationarity tests: BSESN log returns

Stationarity tests: BSESN log-returns						
	no drift, no trend		with drift, no trend		with drift and trend	
lag	ADF	p.value	ADF	p.value	ADF	p.value
0	-47.4946	0.01	-47.5878	0.01	-47.5774	0.01
1	-32.5097	0.01	-32.6093	0.01	-32.6022	0.01
2	-26.5447	0.01	-26.658	0.01	-26.652	0.01
3	-23.4875	0.01	-23.6142	0.01	-23.6089	0.01
4	-19.1125	0.01	-19.2357	0.01	-19.2313	0.01
5	-19.1869	0.01	-19.3347	0.01	-19.3302	0.01
6	-16.5368	0.01	-16.6786	0.01	-16.6748	0.01
7	-15.7532	0.01	-15.8993	0.01	-15.8957	0.01
lag	KPSS	p.value	KPSS	p.value	KPSS	p.value
10	1.2767	0.0911	0.0448	0.1	0.0446	0.1

Table A.5: Stationarity tests: BVLG log returns

Stationarity tests: BVLG log-returns						
	no drift, no trend		with drift, no trend		with drift and trend	
lag	ADF	p.value	ADF	p.value	ADF	p.value
0	-42.1548	0.01	-42.1635	0.01	-42.1556	0.01
1	-31.2652	0.01	-31.2775	0.01	-31.2724	0.01
2	-24.9023	0.01	-24.9151	0.01	-24.9122	0.01
3	-23.133	0.01	-23.1511	0.01	-23.148	0.01
4	-21.5766	0.01	-21.5976	0.01	-21.5955	0.01
5	-19.6107	0.01	-19.6355	0.01	-19.6331	0.01
6	-17.4463	0.01	-17.4715	0.01	-17.4699	0.01
7	-16.448	0.01	-16.4759	0.01	-16.4743	0.01
lag	KPSS	p.value	KPSS	p.value	KPSS	p.value
10	0.202	0.1	0.0384	0.1	0.0278	0.1

Table A.6: Stationarity tests: BVSP log returns

Stationarity tests: BVSP log-returns						
	no drift, no trend		with drift, no trend		with drift and trend	
lag	ADF	p.value	ADF	p.value	ADF	p.value
0	-52.1503	0.01	-52.1735	0.01	-52.1822	0.01
1	-33.0377	0.01	-33.0644	0.01	-33.0754	0.01
2	-26.8034	0.01	-26.8318	0.01	-26.849	0.01
3	-24.0132	0.01	-24.046	0.01	-24.0679	0.01
4	-20.3144	0.01	-20.3468	0.01	-20.3723	0.01
5	-19.4635	0.01	-19.5001	0.01	-19.5302	0.01
6	-16.8187	0.01	-16.8545	0.01	-16.886	0.01
7	-15.4687	0.01	-15.5075	0.01	-15.5378	0.01
lag	KPSS	p.value	KPSS	p.value	KPSS	p.value
10	0.2809	0.1	0.1149	0.1	0.0263	0.1

Table A.7: Stationarity tests: DJI log returns

Stationarity tests: DJI log-returns						
	no drift, no trend		with drift, no trend		with drift and trend	
lag	ADF	p.value	ADF	p.value	ADF	p.value
0	-54.7485	0.01	-54.8441	0.01	-54.8335	0.01
1	-31.808	0.01	-31.8929	0.01	-31.8872	0.01
2	-25.8465	0.01	-25.9401	0.01	-25.9356	0.01
3	-24.9256	0.01	-25.0418	0.01	-25.0381	0.01
4	-20.1343	0.01	-20.2475	0.01	-20.2446	0.01
5	-21.0041	0.01	-21.1469	0.01	-21.1442	0.01
6	-16.9905	0.01	-17.1178	0.01	-17.1169	0.01
7	-16.3461	0.01	-16.483	0.01	-16.4827	0.01
lag	KPSS	p.value	KPSS	p.value	KPSS	p.value
10	1.1883	0.1	0.0338	0.1	0.0278	0.1

Table A.8: Stationarity tests: FCHI log returns

Stationarity tests: FCHI log-returns						
	no drift, no trend		with drift, no trend		with drift and trend	
lag	ADF	p.value	ADF	p.value	ADF	p.value
0	-46.9669	0.01	-46.9847	0.01	-46.9743	0.01
1	-32.1268	0.01	-32.147	0.01	-32.1398	0.01
2	-25.4659	0.01	-25.4889	0.01	-25.4833	0.01
3	-22.7612	0.01	-22.7867	0.01	-22.7815	0.01
4	-21.045	0.01	-21.0741	0.01	-21.0693	0.01
5	-19.7491	0.01	-19.7833	0.01	-19.779	0.01
6	-16.6921	0.01	-16.7245	0.01	-16.7208	0.01
7	-16.6061	0.01	-16.6452	0.01	-16.642	0.01
lag	KPSS	p.value	KPSS	p.value	KPSS	p.value
10	0.4135	0.1	0.0396	0.1	0.0369	0.1

Table A.9: Stationarity tests: FTMIB log returns

Stationarity tests: FTMIB log-returns						
	no drift, no trend		with drift, no trend		with drift and trend	
lag	ADF	p.value	ADF	p.value	ADF	p.value
0	-50.615	0.01	-50.6148	0.01	-50.6032	0.01
1	-32.488	0.01	-32.4913	0.01	-32.4838	0.01
2	-25.5299	0.01	-25.5366	0.01	-25.5308	0.01
3	-23.0128	0.01	-23.0215	0.01	-23.0163	0.01
4	-20.9437	0.01	-20.9554	0.01	-20.9509	0.01
5	-18.7742	0.01	-18.7855	0.01	-18.7813	0.01
6	-16.7023	0.01	-16.7142	0.01	-16.7104	0.01
7	-16.3243	0.01	-16.3375	0.01	-16.3338	0.01
lag	KPSS	p.value	KPSS	p.value	KPSS	p.value
10	0.1481	0.1	0.0383	0.1	0.0383	0.1

Table A.10: Stationarity tests: FTSE log returns

Stationarity tests: FTSE log-returns						
	no drift, no trend		with drift, no trend		with drift and trend	
lag	ADF	p.value	ADF	p.value	ADF	p.value
0	-47.0306	0.01	-47.0239	0.01	-47.0167	0.01
1	-33.2712	0.01	-33.2682	0.01	-33.2649	0.01
2	-26.6448	0.01	-26.643	0.01	-26.6404	0.01
3	-23.6072	0.01	-23.6067	0.01	-23.6055	0.01
4	-20.7838	0.01	-20.784	0.01	-20.7833	0.01
5	-19.7419	0.01	-19.743	0.01	-19.7432	0.01
6	-16.8677	0.01	-16.8688	0.01	-16.8686	0.01
7	-17.0751	0.01	-17.0772	0.01	-17.0783	0.01
lag	KPSS	p.value	KPSS	p.value	KPSS	p.value
10	0.1432	0.1	0.0476	0.1	0.0283	0.1

Table A.11: Stationarity tests: GDAXI log returns

Stationarity tests: GDAXI log-returns						
	no drift, no trend		with drift, no trend		with drift and trend	
lag	ADF	p.value	ADF	p.value	ADF	p.value
0	-47.1229	0.01	-47.149	0.01	-47.1384	0.01
1	-31.7674	0.01	-31.7966	0.01	-31.7896	0.01
2	-25.1148	0.01	-25.1454	0.01	-25.1397	0.01
3	-22.8888	0.01	-22.9252	0.01	-22.9201	0.01
4	-20.8203	0.01	-20.8608	0.01	-20.8563	0.01
5	-18.9951	0.01	-19.0414	0.01	-19.0377	0.01
6	-16.7655	0.01	-16.8123	0.01	-16.8091	0.01
7	-16.628	0.01	-16.6819	0.01	-16.679	0.01
lag	KPSS	p.value	KPSS	p.value	KPSS	p.value
10	0.5172	0.1	0.0368	0.1	0.0345	0.1

Table A.12: Stationarity tests: GSPTSE log returns

Stationarity tests: GSPTSE log-returns						
	no drift, no trend		with drift, no trend		with drift and trend	
lag	ADF	p.value	ADF	p.value	ADF	p.value
0	-49.4333	0.01	-49.4566	0.01	-49.4475	0.01
1	-34.0189	0.01	-34.0475	0.01	-34.0421	0.01
2	-23.3162	0.01	-23.3437	0.01	-23.3399	0.01
3	-22.5411	0.01	-22.5739	0.01	-22.5709	0.01
4	-20.3529	0.01	-20.3893	0.01	-20.3868	0.01
5	-18.2805	0.01	-18.318	0.01	-18.3165	0.01
6	-16.1159	0.01	-16.1518	0.01	-16.1514	0.01
7	-16.3603	0.01	-16.3992	0.01	-16.3999	0.01
lag	KPSS	p.value	KPSS	p.value	KPSS	p.value
10	0.3083	0.1	0.0455	0.1	0.0403	0.1

Table A.13: Stationarity tests: HSI log returns

Stationarity tests: HSI log-returns						
	no drift, no trend		with drift, no trend		with drift and trend	
lag	ADF	p.value	ADF	p.value	ADF	p.value
0	-47.1213	0.01	-47.1239	0.01	-47.1138	0.01
1	-32.8408	0.01	-32.8481	0.01	-32.8418	0.01
2	-26.0718	0.01	-26.0824	0.01	-26.0782	0.01
3	-23.5628	0.01	-23.5765	0.01	-23.5734	0.01
4	-21.5225	0.01	-21.5381	0.01	-21.5354	0.01
5	-19.3196	0.01	-19.3359	0.01	-19.3332	0.01
6	-17.8005	0.01	-17.8192	0.01	-17.8175	0.01
7	-16.1872	0.01	-16.2048	0.01	-16.202	0.01
lag	KPSS	p.value	KPSS	p.value	KPSS	p.value
10	0.2548	0.1	0.0393	0.1	0.0357	0.1

Table A.14: Stationarity tests: IBEX log returns

Stationarity tests: IBEX log-returns						
	no drift, no trend		with drift, no trend		with drift and trend	
lag	ADF	p.value	ADF	p.value	ADF	p.value
0	-48.3997	0.01	-48.3904	0.01	-48.3855	0.01
1	-31.2434	0.01	-31.2378	0.01	-31.236	0.01
2	-25.7928	0.01	-25.7885	0.01	-25.7883	0.01
3	-23.0688	0.01	-23.0654	0.01	-23.0668	0.01
4	-20.5971	0.01	-20.5946	0.01	-20.5976	0.01
5	-19.3627	0.01	-19.3603	0.01	-19.3632	0.01
6	-17.1907	0.01	-17.1888	0.01	-17.1919	0.01
7	-16.6889	0.01	-16.6871	0.01	-16.6905	0.01
lag	KPSS	p.value	KPSS	p.value	KPSS	p.value
10	0.1221	0.1	0.0799	0.1	0.0482	0.1

Table A.15: Stationarity tests: IXIC log returns

Stationarity tests: IXIC log-returns						
	no drift, no trend		with drift, no trend		with drift and trend	
lag	ADF	p.value	ADF	p.value	ADF	p.value
0	-54.6931	0.01	-54.8901	0.01	-54.8862	0.01
1	-33.0265	0.01	-33.2042	0.01	-33.2061	0.01
2	-26.581	0.01	-26.7719	0.01	-26.7764	0.01
3	-24.0656	0.01	-24.2901	0.01	-24.2953	0.01
4	-20.9434	0.01	-21.1804	0.01	-21.1866	0.01
5	-20.808	0.01	-21.0904	0.01	-21.0991	0.01
6	-17.0996	0.01	-17.3597	0.01	-17.3703	0.01
7	-17.6132	0.01	-17.9168	0.01	-17.9308	0.01
lag	KPSS	p.value	KPSS	p.value	KPSS	p.value
10	2.3084	0.0205	0.074	0.1	0.0391	0.1

Table A.16: Stationarity tests: KS11 log returns

Stationarity tests: KS11 log-returns						
	no drift, no trend		with drift, no trend		with drift and trend	
lag	ADF	p.value	ADF	p.value	ADF	p.value
0	-45.0379	0.01	-45.0617	0.01	-45.0717	0.01
1	-29.2554	0.01	-29.2796	0.01	-29.2917	0.01
2	-24.6175	0.01	-24.6427	0.01	-24.6606	0.01
3	-21.8807	0.01	-21.9093	0.01	-21.9295	0.01
4	-21.025	0.01	-21.0581	0.01	-21.0834	0.01
5	-19.4442	0.01	-19.4805	0.01	-19.5094	0.01
6	-17.753	0.01	-17.7916	0.01	-17.8225	0.01
7	-16.2931	0.01	-16.3348	0.01	-16.3651	0.01
lag	KPSS	p.value	KPSS	p.value	KPSS	p.value
10	0.1733	0.1	0.1438	0.1	0.0781	0.1

Table A.17: Stationarity tests: KSE log returns

Stationarity tests: KSE log-returns						
	no drift, no trend		with drift, no trend		with drift and trend	
lag	ADF	p.value	ADF	p.value	ADF	p.value
0	-45.0379	0.01	-45.0617	0.01	-45.0717	0.01
1	-29.2554	0.01	-29.2796	0.01	-29.2917	0.01
2	-24.6175	0.01	-24.6427	0.01	-24.6606	0.01
3	-21.8807	0.01	-21.9093	0.01	-21.9295	0.01
4	-21.025	0.01	-21.0581	0.01	-21.0834	0.01
5	-19.4442	0.01	-19.4805	0.01	-19.5094	0.01
6	-17.753	0.01	-17.7916	0.01	-17.8225	0.01
7	-16.2931	0.01	-16.3348	0.01	-16.3651	0.01
lag	KPSS	p.value	KPSS	p.value	KPSS	p.value
10	0.1733	0.1	0.1438	0.1	0.0781	0.1

Table A.18: Stationarity tests: MXX log returns

Stationarity tests: MXX log-returns						
	no drift, no trend		with drift, no trend		with drift and trend	
lag	ADF	p.value	ADF	p.value	ADF	p.value
0	-44.0096	0.01	-44.0048	0.01	-43.9953	0.01
1	-32.7012	0.01	-32.6994	0.01	-32.6926	0.01
2	-27.3611	0.01	-27.362	0.01	-27.3561	0.01
3	-25.1064	0.01	-25.1083	0.01	-25.1032	0.01
4	-21.4894	0.01	-21.4929	0.01	-21.4885	0.01
5	-20.9144	0.01	-20.9192	0.01	-20.915	0.01
6	-18.1275	0.01	-18.1324	0.01	-18.129	0.01
7	-17.2288	0.01	-17.2344	0.01	-17.2313	0.01
lag	KPSS	p.value	KPSS	p.value	KPSS	p.value
10	0.0788	0.1	0.0555	0.1	0.0577	0.1

Table A.19: Stationarity tests: N225 log returns

Stationarity tests: N225 log-returns						
	no drift, no trend		with drift, no trend		with drift and trend	
lag	ADF	p.value	ADF	p.value	ADF	p.value
0	-48.3749	0.01	-48.4521	0.01	-48.457	0.01
1	-31.8944	0.01	-31.975	0.01	-31.9852	0.01
2	-27.1755	0.01	-27.2679	0.01	-27.2805	0.01
3	-23.6701	0.01	-23.7695	0.01	-23.7825	0.01
4	-21.0376	0.01	-21.1429	0.01	-21.1564	0.01
5	-19.7289	0.01	-19.8506	0.01	-19.8696	0.01
6	-18.5918	0.01	-18.7216	0.01	-18.7404	0.01
7	-17.1644	0.01	-17.3033	0.01	-17.3252	0.01
lag	KPSS	p.value	KPSS	p.value	KPSS	p.value
10	1.7851	0.043	0.1444	0.1	0.0693	0.1

Table A.20: Stationarity tests: NSEI log returns

Stationarity tests: NSEI log-returns						
	no drift, no trend		with drift, no trend		with drift and trend	
lag	ADF	p.value	ADF	p.value	ADF	p.value
0	-47.3127	0.01	-47.4023	0.01	-47.3919	0.01
1	-32.3938	0.01	-32.4895	0.01	-32.4824	0.01
2	-26.4249	0.01	-26.5338	0.01	-26.5278	0.01
3	-23.4291	0.01	-23.5509	0.01	-23.5456	0.01
4	-19.0628	0.01	-19.1811	0.01	-19.1768	0.01
5	-19.214	0.01	-19.3568	0.01	-19.3523	0.01
6	-16.4045	0.01	-16.5399	0.01	-16.5362	0.01
7	-15.7328	0.01	-15.8726	0.01	-15.8691	0.01
lag	KPSS	p.value	KPSS	p.value	KPSS	p.value
10	1.2214	0.0971	0.0466	0.1	0.0471	0.1

Table A.21: Stationarity tests: OMXC20 log returns

Stationarity tests: OMXC20 log-returns						
	no drift, no trend		with drift, no trend		with drift and trend	
lag	ADF	p.value	ADF	p.value	ADF	p.value
0	-46.3436	0.01	-46.4461	0.01	-46.4357	0.01
1	-32.3866	0.01	-32.4974	0.01	-32.4903	0.01
2	-26.2804	0.01	-26.402	0.01	-26.3964	0.01
3	-23.4067	0.01	-23.5403	0.01	-23.5352	0.01
4	-21.3496	0.01	-21.4973	0.01	-21.4926	0.01
5	-19.3444	0.01	-19.5021	0.01	-19.4978	0.01
6	-17.5575	0.01	-17.7227	0.01	-17.719	0.01
7	-17.2416	0.01	-17.4238	0.01	-17.4199	0.01
lag	KPSS	p.value	KPSS	p.value	KPSS	p.value
10	1.7124	0.0469	0.0834	0.1	0.0799	0.1

Table A.22: Stationarity tests: OMXHPI log returns

Stationarity tests: OMXHPI log-returns						
	no drift, no trend		with drift, no trend		with drift and trend	
lag	ADF	p.value	ADF	p.value	ADF	p.value
0	-44.2903	0.01	-44.33	0.01	-44.3213	0.01
1	-31.8638	0.01	-31.9117	0.01	-31.9068	0.01
2	-25.1205	0.01	-25.1723	0.01	-25.1695	0.01
3	-22.1019	0.01	-22.1595	0.01	-22.1578	0.01
4	-19.9045	0.01	-19.9644	0.01	-19.9627	0.01
5	-18.8455	0.01	-18.9107	0.01	-18.909	0.01
6	-17.0066	0.01	-17.0764	0.01	-17.0762	0.01
7	-16.4846	0.01	-16.563	0.01	-16.5639	0.01
lag	KPSS	p.value	KPSS	p.value	KPSS	p.value
10	0.7661	0.1	0.0521	0.1	0.0422	0.1

Table A.23: Stationarity tests: OMXSPI log returns

Stationarity tests: OMXSPI log-returns						
	no drift, no trend		with drift, no trend		with drift and trend	
lag	ADF	p.value	ADF	p.value	ADF	p.value
0	-47.5912	0.01	-47.6829	0.01	-47.6731	0.01
1	-32.9441	0.01	-33.0426	0.01	-33.0362	0.01
2	-25.517	0.01	-25.6197	0.01	-25.6147	0.01
3	-22.552	0.01	-22.6637	0.01	-22.6597	0.01
4	-20.3345	0.01	-20.4545	0.01	-20.4513	0.01
5	-19.0994	0.01	-19.2349	0.01	-19.2316	0.01
6	-16.451	0.01	-16.5834	0.01	-16.5806	0.01
7	-15.8089	0.01	-15.9451	0.01	-15.9434	0.01
lag	KPSS	p.value	KPSS	p.value	KPSS	p.value
10	1.144	0.1	0.0668	0.1	0.065	0.1

Table A.24: Stationarity tests: OSEAX log returns

Stationarity tests: OSEAX log-returns						
	no drift, no trend		with drift, no trend		with drift and trend	
lag	ADF	p.value	ADF	p.value	ADF	p.value
0	-48.1596	0.01	-48.2297	0.01	-48.2186	0.01
1	-33.9775	0.01	-34.0553	0.01	-34.0474	0.01
2	-25.3938	0.01	-25.4706	0.01	-25.4646	0.01
3	-22.5572	0.01	-22.6441	0.01	-22.6388	0.01
4	-20.008	0.01	-20.1005	0.01	-20.0958	0.01
5	-17.7969	0.01	-17.8944	0.01	-17.8902	0.01
6	-16.5806	0.01	-16.6787	0.01	-16.6745	0.01
7	-15.7068	0.01	-15.8102	0.01	-15.8061	0.01
lag	KPSS	p.value	KPSS	p.value	KPSS	p.value
10	1.0556	0.1	0.0355	0.1	0.0354	0.1

Table A.25: Stationarity tests: RUT log returns

Stationarity tests: RUT log-returns						
	no drift, no trend		with drift, no trend		with drift and trend	
lag	ADF	p.value	ADF	p.value	ADF	p.value
0	-52.7471	0.01	-52.81	0.01	-52.7989	0.01
1	-30.6492	0.01	-30.7044	0.01	-30.6983	0.01
2	-25.3463	0.01	-25.4055	0.01	-25.4009	0.01
3	-23.7591	0.01	-23.8301	0.01	-23.826	0.01
4	-19.8387	0.01	-19.9073	0.01	-19.9046	0.01
5	-20.2516	0.01	-20.3343	0.01	-20.332	0.01
6	-16.8757	0.01	-16.9556	0.01	-16.9539	0.01
7	-16.5219	0.01	-16.6098	0.01	-16.6084	0.01
lag	KPSS	p.value	KPSS	p.value	KPSS	p.value
10	0.7542	0.1	0.0637	0.1	0.062	0.1

Table A.26: Stationarity tests: SMSI log returns

Stationarity tests: SMSI log-returns						
	no drift, no trend		with drift, no trend		with drift and trend	
lag	ADF	p.value	ADF	p.value	ADF	p.value
0	-46.4569	0.01	-46.4474	0.01	-46.4425	0.01
1	-31.3821	0.01	-31.3761	0.01	-31.375	0.01
2	-25.6918	0.01	-25.6874	0.01	-25.6888	0.01
3	-22.7515	0.01	-22.7478	0.01	-22.7502	0.01
4	-20.7313	0.01	-20.7282	0.01	-20.7317	0.01
5	-19.0437	0.01	-19.0412	0.01	-19.0461	0.01
6	-16.7606	0.01	-16.7588	0.01	-16.7648	0.01
7	-16.3941	0.01	-16.3922	0.01	-16.3977	0.01
lag	KPSS	p.value	KPSS	p.value	KPSS	p.value
10	0.1088	0.1	0.0789	0.1	0.0476	0.1

Table A.27: Stationarity tests: SPX log returns

Stationarity tests: SPX log-returns						
	no drift, no trend		with drift, no trend		with drift and trend	
lag	ADF	p.value	ADF	p.value	ADF	p.value
0	-55.8319	0.01	-55.9701	0.01	-55.9595	0.01
1	-32.6577	0.01	-32.7812	0.01	-32.7757	0.01
2	-26.0332	0.01	-26.1643	0.01	-26.1604	0.01
3	-24.6962	0.01	-24.8577	0.01	-24.8544	0.01
4	-21.096	0.01	-21.258	0.01	-21.2568	0.01
5	-21.2121	0.01	-21.4078	0.01	-21.4077	0.01
6	-16.7459	0.01	-16.9234	0.01	-16.9235	0.01
7	-17.0833	0.01	-17.2886	0.01	-17.2893	0.01
lag	KPSS	p.value	KPSS	p.value	KPSS	p.value
10	1.6625	0.0496	0.0496	0.1	0.0426	0.1

Table A.28: Stationarity tests: SSEC log returns

Stationarity tests: SSEC log-returns						
	no drift, no trend		with drift, no trend		with drift and trend	
lag	ADF	p.value	ADF	p.value	ADF	p.value
0	-44.6295	0.01	-44.6283	0.01	-44.6185	0.01
1	-33.1352	0.01	-33.1379	0.01	-33.1307	0.01
2	-26.4935	0.01	-26.4978	0.01	-26.4923	0.01
3	-21.9512	0.01	-21.9561	0.01	-21.9519	0.01
4	-20.0099	0.01	-20.0162	0.01	-20.0124	0.01
5	-19.8004	0.01	-19.8086	0.01	-19.8048	0.01
6	-17.4914	0.01	-17.5003	0.01	-17.4971	0.01
7	-15.9462	0.01	-15.9577	0.01	-15.9542	0.01
lag	KPSS	p.value	KPSS	p.value	KPSS	p.value
10	0.164	0.1	0.0577	0.1	0.0603	0.1

Table A.29: Stationarity tests: SSMI log returns

Stationarity tests: SSMI log-returns						
	no drift, no trend		with drift, no trend		with drift and trend	
lag	ADF	p.value	ADF	p.value	ADF	p.value
0	-45.8616	0.01	-45.882	0.01	-45.8721	0.01
1	-33.9891	0.01	-34.0167	0.01	-34.0097	0.01
2	-27.5675	0.01	-27.5997	0.01	-27.5942	0.01
3	-23.629	0.01	-23.6661	0.01	-23.6618	0.01
4	-21.6276	0.01	-21.6704	0.01	-21.6668	0.01
5	-20.3325	0.01	-20.3805	0.01	-20.3773	0.01
6	-17.7274	0.01	-17.7759	0.01	-17.7731	0.01
7	-17.151	0.01	-17.2066	0.01	-17.2048	0.01
lag	KPSS	p.value	KPSS	p.value	KPSS	p.value
10	0.5295	0.1	0.0461	0.1	0.0447	0.1

Table A.30: Stationarity tests: STI log returns

Stationarity tests: STI log-returns						
	no drift, no trend		with drift, no trend		with drift and trend	
lag	ADF	p.value	ADF	p.value	ADF	p.value
0	-46.8773	0.01	-46.8868	0.01	-46.8975	0.01
1	-32.5192	0.01	-32.5321	0.01	-32.5459	0.01
2	-24.0652	0.01	-24.0783	0.01	-24.0918	0.01
3	-21.2384	0.01	-21.2533	0.01	-21.2681	0.01
4	-18.0673	0.01	-18.0826	0.01	-18.0976	0.01
5	-18.0055	0.01	-18.0241	0.01	-18.0433	0.01
6	-17.0336	0.01	-17.054	0.01	-17.0751	0.01
7	-16.0703	0.01	-16.0926	0.01	-16.1161	0.01
lag	KPSS	p.value	KPSS	p.value	KPSS	p.value
10	0.5765	0.1	0.1491	0.1	0.0516	0.1

Table A.31: Stationarity tests: STOXX50E log returns

Stationarity tests: STOXX50E log-returns						
	no drift, no trend		with drift, no trend		with drift and trend	
lag	ADF	p.value	ADF	p.value	ADF	p.value
0	-47.4025	0.01	-47.4079	0.01	-47.3971	0.01
1	-32.2731	0.01	-32.2821	0.01	-32.2748	0.01
2	-25.5614	0.01	-25.5716	0.01	-25.5658	0.01
3	-22.6061	0.01	-22.6182	0.01	-22.6129	0.01
4	-20.8762	0.01	-20.8914	0.01	-20.8868	0.01
5	-19.5292	0.01	-19.5463	0.01	-19.5419	0.01
6	-16.8029	0.01	-16.822	0.01	-16.8186	0.01
7	-16.3698	0.01	-16.3909	0.01	-16.3877	0.01
lag	KPSS	p.value	KPSS	p.value	KPSS	p.value
10	0.2279	0.1	0.0398	0.1	0.0383	0.1