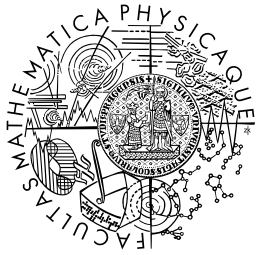

FACULTY
OF MATHEMATICS
AND PHYSICS
Charles University

MASTER THESIS

Bc. Anna Gajdová

Structural properties of random networks with dynamics

Computer Science Institute of Charles University

Supervisor of the master thesis: Ing. David Hartman, Ph.D.
Study programme: Mathematics
Study branch: Mathematical Structures

Prague 2021

I declare that I carried out this master thesis independently, and only with the cited sources, literature and other professional sources. It has not been used to obtain another or the same degree.

I understand that my work relates to the rights and obligations under the Act No. 121/2000 Sb., the Copyright Act, as amended, in particular the fact that the Charles University has the right to conclude a license agreement on the use of this work as a school work pursuant to Section 60 subsection 1 of the Copyright Act.

In ..... date ....

Author's signature

I am grateful to my thesis supervisor Ing. David Hartman, Ph.D for his time, advice and encouragment. I would like to thank my family for their support during my studies. Lastly,  .

Title: Structural properties of random networks with dynamics

Author: Bc. Anna Gajdová

Institute: Computer Science Institute of Charles University

Supervisor: Ing. David Hartman, Ph.D., Computer Science Institute of Charles University

Abstract: Real systems are often represented by so-called complex networks. These networks have a specific connectivity structure given by the specifics of the studied systems. Since often insufficient or inaccurate data are available, a common approach is to model these systems at the level of this connectivity using random networks replicating specific properties such as ease of connectivity, modularity or specific sparsity. The representation of these properties in basic complex network models is a widely explored area. However, if the presence of edges is controlled by a specific distributions or if an element of the dynamics of the overall graph is added to the model, the analysis of such models becomes more complex. This thesis aims to investigate the properties of such dynamically dependent random models.

Keywords: random graphs, dynamical properties, probabilistic method, global properties of networks

Contents

Introduction

Many real-world complex systems, such as the brain or the Internet, consists of small components that are connected, they have a network structure. Among other examples of networks, that describe the world around us, is the social network mapping friendships, the transportation network showing which cities are connected, or the food web network depicting who-eats-whom in a given ecosystem. [1].

Network science, a subfield of complexity science, is an interdisciplinary field that combines knowledge and ideas from mathematics, physics, and computer science but also from biology, ecology, neuroscience and many other areas. It most heavily draws from graph theory, but also from probability and statistics. [2]

Network science applications ranges from fighting epidemics to designing robust power networks that won't be affected by blackouts. Studying network helps us develop new drugs or understand the climate better.

One of the building blocks of network science are *network models*, which are used for studying networks of similar properties and can be represented as a sequence of graphs. On the other side, we have the rigorous mathematical field of graph theory. This thesis attempts to both these fields closer together.

In 2008 Nešetřil and de Mendez introduced graph classes with *bounded expansion* [3]. In later works, they build a whole new theory of structurally sparse graphs and showed that structural sparsity is in some sense *the right* notion of sparsity and has many algorithmic applications. [4]

In the last few years, a lot of work in the intersection of structural sparsity and network science has been done, such as [5], [6], [7] or [8]. To our knowledge, this thesis is the most comprehensive review of what is known about the intersection of these to topics, and we attempted to bridge the gap between them even further.

Furthermore, as most of real-world networks are not static, but evolve in time, we also focus on the question of structural sparsity in networks that grow and change in time, which is our largest contribution.

In the first chapter, we introduce and formalize complex networks concepts. We classify network models and distinguish between two types of dynamic networks: *evolving* networks that involve growth, and time-varying, which do not.

In chapter 2, we present the theory of structural sparsity by defining structurally sparse classes bounded expansion and nowhere dense.

In chapter 3, we dive deeper into properties of bounded expansion classes and mention some of the algorithmic application they offer.

Chapter 4 connects structural sparsity and network models. We present and discuss previous results about sparsity of networks are static, they don't change in time.

In chapter 5, we examine sparsity of evolving networks, which grow in time. In addition to previous results, we suggest possible future directions.

Chapter 6 is dedicated to time-varying networks. We discuss how can be sparsity affected by random changes and generalize a previous result. We also suggest two possible definitions of sparsity in dynamic networks that might be used in future works.

Lastly, in chapter 7, we discuss how can be structural sparsity exploited in network science applications.

1. Complex networks

In this chapter, we will introduce basic network notions, give examples and present several applications. Some examples are included because they are relevant in the later chapters, some because they are important in the context of the field or its intersection with graph theory.

Remark. Unless stated otherwise, we use *network* to denote a simple undirected graph. We use the notions *graph* and *network* interchangeably, depending on the context. Sometimes we also denote edges by *links* and vertices by *nodes,* as is the custom in the network science literature.

The set of edges and vertices of a graph G is denoted by $E(G)$ and $V(G)$, respectively. By $|G|$ we denote the size of the graph, that is $V(G)$, and by $||G||$ the size of $E(G)$.

1.1 Network models

Network models are used for studying networks with similar properties or of the same generating method. A natural way of modeling complex networks is to use a sequence of random graphs [9].

Definition 1 (random graph model)**.** *A* random graph model *is a sequence* $(G_n)_{n\in\mathbb{N}}$ *of random variables over* n*-vertex graphs.*

Graphs in the sequence can be dependent (i.e. the sequence represents growth of a graph) or generated independently. The second case can be formalized via the following definition.

Definition 2 (parametrization [5])**.** *A* parametrization *of a random graph model is a function* $\rho : \mathbb{N} \to \mathbb{R}^t$ *that creates a tuple of* t *parameters depending on* n*, the number of vertices, which parameterize the probability distribution of each variable* G_n*. The random variable* G_n *that is drawn from the probability distribution prescribes by* $\rho(n)$ *is denoted by* $G(n, \rho(n))$*.*

With $\rho(n)$ possibly different for each n, we allow different parameterization with graph size, although often $\rho(n) = \text{const}$..

One of the most studied examples of a random graph model is the Erdős–Rényi model, where $\rho(n) \in [0, 1]$ and $G(n, \rho(n))$ is a graph with n edges in which each edge is present in the graph with probability $\rho(n)$, independently of every other edge in the graph. Often, $\rho(n) = p \in [0, 1]$, that is, the probability is constant.

Observation. Erdős–Rényi graph $G(n, p)$ has in expectation $\frac{n(n-1)}{2}p$ edges and the expected degree of each vertex is $(n - 1)p$.

Degree distributions

An important property of networks and network models is the *degree distribution*, that is, the probability distribution of degrees of vertices. We will follow notation from [5].

Name	p_k	**Parameters**
Exponential	$p_k = e^{-\lambda k}$	$\lambda > 0$
Power law	$p_k = k^{-\alpha}$	$\alpha > 2$
Power law w/ cut-off	$p_k = k^{-\alpha} e^{-\lambda k}$	$\alpha > 2, \lambda > 0$

Table 1.1: Common degree distributions of network models

Definition 3 (degree distribution)**.** *An n-vertex degree distribution is a random variable D with probability mass function p such that*

- $p(k) = 0$ *for* $k \leq 0$ *and* $k \geq n-1$*, and*
- $nf(k) \in \mathbb{N}_0$ *for all* $k \in \mathbb{N}$.

Let G be an n-vertex graph and D its associated degree distribution. Then the probability mass function is given by

$$p(k) = \frac{1}{n} \sum_{i=1}^{n} 1_{\{d_i = k\}}.$$

Thus $p(k)$ describes the probability, that a vertex has degree k.

For example, the Erdős–Rényi graph $G(n,p)$ has a Poisson degree distribution, that is

$$p(k) = \frac{(np)^k e^{-np}}{k!}.$$

To describe the degree distributions of random graph models, we use degree distribution sequences.

Definition 4 (degree distribution sequence)**.** *A* degree distribution sequence *is an infinite sequence $(D_n)_{n \in \mathbb{N}}$ of n-vertex degree distributions. If $(D_n) \to D$, we say that the limiting probability distribution D is the limit of the sequence.*

Degree distribution sequence of a random graph model is a sequence $(D_n)_{n \in \mathbb{N}}$, where D_n is the degree distribution of G_n. If the sequence has a limit D, we call it the degree distribution of the model and denote by p_k the probability that a vertex has degree k.

It has been shown that real-world networks are often *scale-free* [10], which means that their degree distributions follow a power law, i.e. $p_k = k^{-\alpha}$ for $\alpha > 2$. An important example of a scale free network is the Barabási-Albert model, which will be formally defined in section 5.1.1. Recently, it has been discussed that other distributions such as exponential or power law distribution with a cut-off might be better suitable for modelling real-world networks [11]. That is also one of the indications that the Erdős–Rényi model, in which individual graphs follow a Poisson degree distribution, is not a good choice for modelling real-world networks. See table 1.1 for a comparison of common degree distributions.

In later chapters, we will need the concept of a tail-bound of a distribution.

Definition 5 (tail-bound as stated in [5]). *A degree distribution sequence* $(D_n)_{n\in\mathbb{N}}$ *with limit* D *has the function* h *as its* tail-bound *if there is a constant* $\tau \geq 0$ *such that for all* $d \geq \tau$ *and large enough* n *it holds that*

$$\Pr[D_n \geq d] = O\left(\frac{1}{h(d)}\right).$$

All degree distributions in the table 1.1 have a tail-bound that is at least quadratic.

Classification of network models

Barabasi [12] divides network models into three main classes: *static*, *generative*, and *evolving network* models.

- **Static models** have a given number of nodes and place links between them using some stochastic algorithm. The topology of the network is not time dependent. An example is the Erdős–Rényi model or the Watts-Strogatz small world model (see section 1.2.3).
- **Generative models** generate a network with a specific predefined degree distribution. That enables studying properties of networks with certain degree distributions. Similarly to static networks, they are not time dependent, but contrary to them, they can't tell us why a certain degree distribution develops. Both configuration and Chung-Lu models described in section 4.3.1 are examples of generative models.
- **Evolving network models** are used to describe how the network changes in time, in particular, how the network growths. They help us understand how different ways of network creation and aging influence the network topological properties. The entire chapter 5 is dedicated to evolving networks and provides many examples.

1.2 Network measures and metrics

1.2.1 Network centralities

One of the questions we can ask about a network is which nodes or edges are the most important or central. There are many ways how could the question be answered. We will define a couple measures of centrality that are commonly studied [1].

Degree centrality

The simplest centrality measure of a node is the degree centrality, which is simple just its degree.

Betweenness centrality

Another example is the betweenness centrality, which measures how often a node lies on paths between other nodes [1]. Let n_{uv}^i be 1 the if the node i lies on some shortest path between nodes u and v and 0 otherwise. Then we define the betweenness centrality as

$$c_b(i) = \sum_{u,v \in V_G} n_{uv}^i.$$

For example, in the case of the transportation network, the nodes with the highest betweenness centrality are exactly the transportation hubs.

In section 1.1 we will see that localized versions of some other centrality measures can be effectively computed on classes with bounded expansion.

1.2.2 Clustering coefficient

Clustering coefficient is an example of a simple measure that is very useful for studying network structure.

For a vertex $v_i \in G$ with degree k_i, the *local clustering coefficient* is defined as

$$C_i = \frac{2L_i}{k_i(k_i - 1)},$$

where L_i denotes the number of edges between the k_i neighbors of v_i.

The clustering coefficient indicates how much the neighbors of a given degree link to each other, and ranges from 0 (no edges among the neighbors of i) to 1 (the neighborhood of i is a complete graph).

The global clustering coefficient of a graph of size n is defined as the average of local clustering coefficients of all vertices.

$$C_C = \sum_{i=1}^{n} \frac{C_i}{n}$$

The clustering coefficient evaluates the density of triangles in a graph. The fact that the clustering coefficient of Erdős–Rényi graphs is different to real-world networks is another indicator that for many applications, this model is not the most suitable [1].

1.2.3 Small world networks

A *small world phenomenon* first describes the property of real-life social networks, where any two individuals can be connected by a short chain of acquaintances, also known as the concept called "the six degrees of separations". [12]

Mathematically, we say that a class of graphs or a network model have the *small world* property if the average path length is at most logarithmic in the size of the network (up to multiplicative constants).

An example of a model with the small world property is the Watts-Strogatz model introduced in [13]. Informally, the model starts with a square grid of vertices, each of them connected to its neighbors. With some probability p, each edge is rewired to a randomly chosen vertex, thus creating short distances in the network. We will discuss a generalization of this model in section 4.2.3.

1.3 Sparsity in graphs and complex networks

In both graph theory and network science, sparsity doesn't have a single formal meaning. Often it is just used as an informal label stating that the number of edges in the network or a graph is quite small, e.g. linear with the number of vertices.

Claiming that a graph G is sparse also sometimes mean that it's density is low, where density is defined as

$$\rho = \frac{||G||}{\binom{|G|}{2}},$$

that is, as the ratio of the number of edges to the maximal number of possible edges. In this context, dense graphs are those with density bounded away from 0 by a constant.

This definition has deep limitations: consider what happens, when we subdivide each edge of a graph: the structure of the graph remains in many important respects very similar, but the density dropped significantly.

In the rest of the thesis, we will almost always use "sparsity" in the sense of structural sparsity defined by Nešetřil and de Mendez [4]. Structural sparsity is not defined for individual graphs, but for whole graph classes, which removes the limitations of unstable density mentioned in this section.

1.4 Dynamic networks

Most real-world networks such as social networks, the Internet and others are not static but change over time. The terminology is not fixed but in this thesis, we will continue using the term *evolving network* for time evolution that involves growth (i.e. the addition of nodes) and *temporal* or *time-varying* network for a network with fixed number of nodes (i.e. changes of link activation only). A comprehensive survey of time-varying networks and their properties can be found in [14].
Chapter 5 is dedicated to structural sparsity in evolving networks, while chapter 6 discusses structural sparsity in time-varying networks.

1.5 Dynamical processes on networks

Dynamical network processes are operations or procedures that occur on top of the networks. Real-world examples of dynamical processes on networks are disease spreading, or a blackout affecting the power network. Studying dynamical processes on various network models can help us understand how network topology influence the behavior of a given process. [15]

Percolation and resilience

Percolation theory has roots in fluid physics and provides a framework to understand the topological properties of a network when nodes or links are removed.

For example, when $np > 1$, an Erdős–Rényi graph $G(n, p)$ has a *giant component*, i.e. it has a connected component that contains a finite fraction of all nodes. [12]

Resilience refers to the study of attacks that remove nodes, where attacks can be either targeted or affecting all nodes uniformly.

Spreading phenomena

Spreading phenomena refers to variations of a process, that starts with a subset of "infected" nodes, which can spread the infection to their neighbors. A prominent example is the SIR model used for modeling epidemics, in which each node can either be susceptible, infected or recovered and the status of each node changes with time depending on the status of its neighbors.

1.6 Algorithmic problems

1.6.1 Counting network motifs

It was observed that in some networks, certain small patterns appear repeatedly. These small subgraphs are called *motifs*. Evaluation of motif density and comparing it to a random graph can help us understand the network.

Then makes the problem of efficient motif (i.e. subgraph) counting an important algorithmic question. In chapter 7, we will discuss a fast algorithm for motif counting in structurally sparse networks.

1.6.2 Community structure

Another algorithmic problem with real-world applications is determining if a given network has a community structure. Informally, a network has a community structure, if the nodes can be divided into groups, such that nodes inside the same group are densely connected, but there are only a few links between groups.

2. Structural sparsity in graphs

In [3], Nešetřil et al. introduced structural sparsity as a property of graph classes.

In this section, we will define the building block of structural sparsity, shallow minors, and the structurally sparse nowhere dense classes and structurally dense somewhere dense classes, and we will describe why somewhere and nowhere dense classes create a natural dichotomy of sparsity. We will define the structurally sparse classes with bounded expansion, which, as we will show in later chapters, have many desirable features and offer many algorithmic applications, and provide examples of classes with this property.

We will follow the terminology of [4] unless stated otherwise.

2.1 Shallow minors

We'll continue using standard graph terminology. The set of edges and vertices of a graph G is denoted by $E(G)$ and $V(G)$ respectively and by $|G|$ we denote the size of the graph, that is $V(G)$, and by $||G||$ the size of $E(G)$. Given a subset of vertices $V' \subset V$, $G[V']$ is the subgraph of G induced by V'. The *distance* of vertices x, y in a graph G is the length of the minimum path connecting these two vertices, or ∞ if they don't belong to the same component. The *diameter* of a graph is the greatest distance between any two vertices of the graph. A *subdivision* of a graph G is a graph created by subdividing edges of G, i.e. . replacing any of its edges by paths of any length.

The building block of the theory of structural sparsity are *shallow minors*, which are, informally, created by contracting subgraphs. In contrast with regular graph minors, which are formed by contracting edges and deleting edges and vertices, the subgraphs contracted to form a shallow minor need to have small diameter.

Definition 6 (shallow minor)**.** *For any graphs H and G and any integer d, the graph H is said to be a* shallow minor *of G at depth d if there exists a collection $\mathcal{P}$ of disjoint subsets $V_1, ..., V_p$ of $V(G)$ such that:*

- *Each graph $G[V_i]$ has radius at most d: there exists in each set V_i a vertex x_i (a center) such that every vertex in V_i is at distance at most d from x_i in $G[V_i]$,*
- *H is a subgraph of the graph $G/\mathcal{P}$: each vertex v of H corresponds (in an injective way) to a set $V_{i(v)} \in \mathcal{P}$ and two adjacent vertices u and v of H correspond to two sets $V_{i(u)}$ and $V_{i(v)}$ linked by at least one edge.*

See Figure 2.1 for an illustration.

We denote the set of all shallow minors of G at depth d by $G\nabla d$. In particular, $G\nabla 0$ is the set of all subgraphs of G and $G\nabla\infty$ is the set of (non-shallow) minors of G.

Observation. For every graph G, we have the following sequence of sets

$$G \in G\nabla 0 \subseteq G\nabla 1 \subseteq \cdots \subseteq G\nabla d \subseteq \cdots \subseteq G\nabla\infty.$$

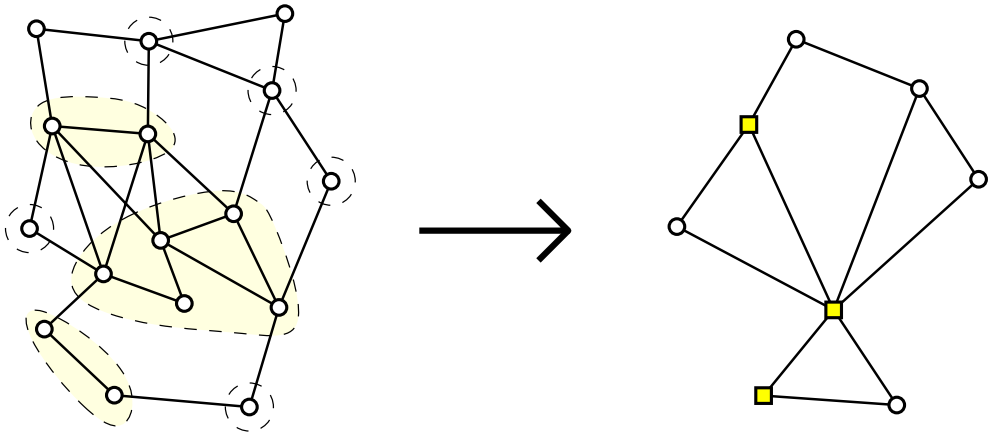

Figure 2.1: Example of a shallow minor operation at depth 2. The dashed lines indicate the contracted subgraphs $G[V_i]$, two vertices are left out and removed. Non-trivial subgraphs contract to the highlighted square vertices.

A stricter notion than minor is one of *topological minor*. A graph H is the topological minor of a graph G if a subdivision of H is isomorphic to a subgraph of G. Similarly, we can define a *shallow topological minor.*

Definition 7 (shallow topological minor)**.** *A* shallow topological minor *of a graph G of depth a (where a is a half-integer) is a graph H obtained from G by taking a subgraph and then replacing an internally vertex disjoint family of paths of length at most $2a+1$ by single edges.*

Alternatively, H is a shallow topological minor at depth a if a $\leq 2a$-subdivision of H is a subgraph of G, where k-subdivision may replace each edge of G by a path of length at most k (only the original edges of G may get replaced this way). See Figure 2.2 for an illustration.

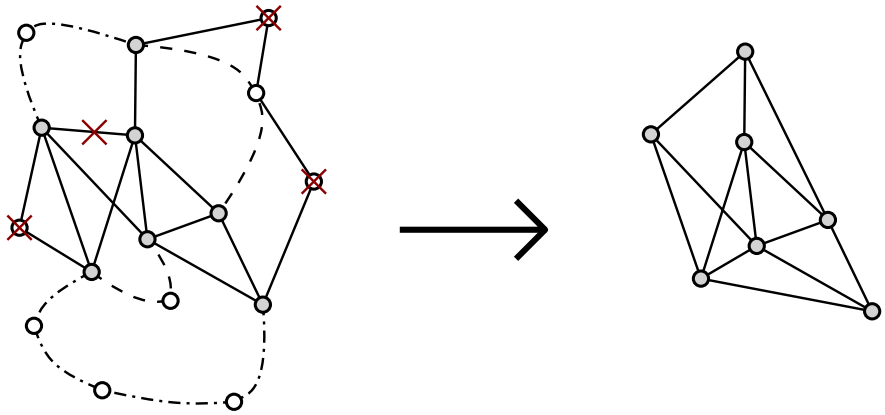

Figure 2.2: Example of a shallow topological minor at depth 2. Some vertices and edges are deleted, the internal vertices of the 4 dashed and dash-dotted paths are replaced with an edge. Note that the contracted paths may share endpoints.

We denote the set of all shallow topological minors of G at depth d by $G\widetilde{\nabla}d$. As in the case of shallow minors, $G\widetilde{\nabla}0$ is the set of all subgraphs of G and $G\widetilde{\nabla}\infty$ corresponds to all (non-shallow) topological minors of G.

Observation. It is easy to see from definitions that for all integers d, a shallow topological minor at depth d is also a shallow minor at depth d, that is

$$G\widetilde{\nabla}d \subseteq G\nabla d.$$

The inclusion does not hold in the opposite direction. An example of a shallow minor that is not a topological minor can be seen at figure 2.3.

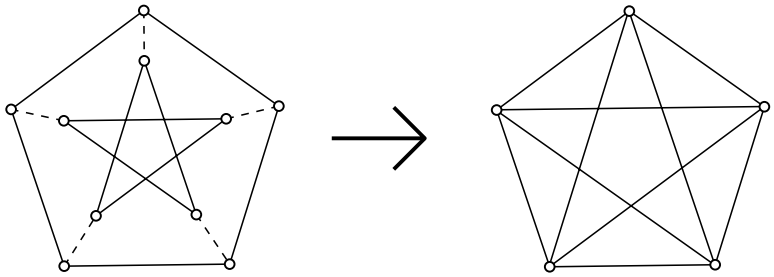

Figure 2.3: K_5 *(right)* is a 1-shallow minor of the Petersen graph *(left)*, since we can obtain K_5 by contracting the dashed edges. K_5 is, however, not a topological minor of the Petersen graph of any depth, for example, since topological minors do not increase the maximum degree of the graph.

Remark. The definition of shallow minors can be extended to half integers, i.e. we can define $G\nabla(d-\frac{1}{2})$ for $d > 1$. We will not describe the details here, as it has little importance in the later chapters, but we will consider shallow minors also for half integers in the rest of this chapter. For all integers $d > 1$, it holds

$$G\widetilde{\nabla}\left(d-\frac{1}{2}\right) \subseteq G\nabla\left(d-\frac{1}{2}\right).$$

2.2 Dichotomy of sparsity

The theory of shallow minors enables us to divide all infinite graph classes into two types: sparse *nowhere dense* classes and dense *somewhere dense* classes.

The *clique number* $\omega(G)$ is the maximum order of a clique of G, that is, the maximum order of a complete subgraph of G. We define the clique number of a graph class $\mathcal{C}$ as $\omega(\mathcal{C}) = \sup\{\omega(G) : G \in \mathcal{C}\}$.

Definition 8 (somewhere dense and nowhere dense)**.** *An infinite class $\mathcal{C}$ of graphs is* somewhere dense *if there exists a finite half-integer t such that $\omega(\mathcal{C}\nabla t) = \infty$. Otherwise, if $\omega(\mathcal{C}\nabla i) < \infty$ for all half-integers i, the class $\mathcal{C}$ is* nowhere dense*.*

Note that if it holds that $\omega(\mathcal{C}\nabla t) = \infty$ for some graph class $\mathcal{C}$ and for some half integer t , then the class $\mathcal{C}\nabla t$ contains complete subgraphs of all sizes and their subgraphs, hence it is equal to the class containing all graphs $\mathcal{GRAPH}$. Thus, an equivalent definition of a somewhere dense class is that $\mathcal{C}\nabla t = \mathcal{GRAPH}$ for some half integer t.

The following theorem shows that somewhere and nowhere dense classes can be analogously defined using shallow topological minors.

Theorem 1. *Let G be a graph and let a be a half-integer. Then*

$$\omega(G\widetilde{\nabla}a) \leq \omega(G\nabla a) \leq 2\omega(G\widetilde{\nabla}a)^{|a|+1}$$

.

This result is very practical, as shallow topological minors are usually easier to work with.

2.2.1 Logarithmic density

In an informal sense, dense classes are those with quadratically many edges. Thus, it makes sense to consider the following measure of logarithmic density.

Definition 9 (logarithmic density)**.** *Let G be a graph, we define the* logarithmic density $l\text{dens}(G)$ *of G as*

$$l\text{dens}(G) = \begin{cases} -\infty & \text{if } ||G|| = 0 \\ \frac{\log ||G||}{\log |G|} & \text{otherwise} \end{cases}$$

The following trichotomy theorem shows that the distinction between nowhere and somewhere dense classes is not arbitrary, but that it actually corresponds to a form of a natural dichotomy of structural sparseness.

Theorem 2 (Trichotomy theorem)**.** *Let $\mathcal{C}$ be an infinite class of graphs (asymptotically not all edgeless). Then the limit*

$$l\text{dens}(\mathcal{C}^{\nabla}) = \lim_{i\to\infty} \limsup_{G\in\mathcal{C}\nabla i} \frac{\log ||G||}{|G|}$$

may take only three values, namely 0, 1, and 2. Moreover, we have:

- $l\text{dens}(\mathcal{C}^{\nabla}) = 0$ *or* 1 *if and only if $\mathcal{C}$ is nowhere dense,*
- $l\text{dens}(\mathcal{C}^{\nabla}) = 2$ *if and only if $\mathcal{C}$ is somewhere dense.*

We will introduce other characterizations in the next chapter.

2.3 Bounded expansion

In this section, we will define the sparse class bounded expansion. As we will see in the next chapters, this subclass of the nowhere dense class is particularly useful and important because it is useful for algorithmic applications while also covering a wide range of graph classes.

2.3.1 Grads and top-grads

To define bounded expansion, we will introduce the notion of the *greatest reduced average density*, which describes the density of shallow minors of a graph.

Definition 10 (greatest reduced average density)**.** *The* greatest reduced average density *(shortly* grad*) with rank r of a graph G is defined by a formula*

$$\nabla_r(G) = \max\left\{\frac{||H||}{|H|} : H \in G\nabla r\right\}$$

Analogously, we define topological greatest reduced average density.

Definition 11 (topological greatest reduced average density)**.** *The* topological greatest reduced average density *(shortly* top-grad*) with rank r of a graph G is defined by a formula*

$$\widetilde{\nabla}_r(G) = \max\left\{\frac{||H||}{|H|} : H \in G\widetilde{\nabla} r\right\}$$

Polynomial equivalence

From our previous observation about the inclusion of the set of shallow topological minors and the set of shallow minors of the same depth, it follows that $\widetilde{\nabla}_r(G) \leq \nabla_r(G)$. The opposite does not hold, but as in the case of the clique number, there is polynomial equivalence between grad and top-grad.

Theorem 3. *For every graph H and every integer $r \geq 1$ holds*

$$\widetilde{\nabla}_r(G) \leq \nabla_r(G) \leq 4(\widetilde{\nabla}_r(G))^{(r+1)^2}.$$

Now we have everything we need to define the class of bounded expansion.

2.3.2 Defining bounded expansion

We can say that graphs in classes with bounded expansion have a type of community structure: they consist of small, dense parts that are only sparsely connected. Formally, they have bounded grad.

Definition 12 (bounded expansion)**.** *A class $\mathcal{C}$ has* bounded expansion *if for every t there exists $c(t)$ such that*

$$\frac{||G||}{|G|} \leq c(t)$$

for every graph $G \in \mathcal{C}\nabla t$. In other words, for every t, we have

$$\nabla_t(\mathcal{C}) \leq c(t).$$

We call the function $c(t)$ the expansion function. As per Theorem 3, the class of bounded expansion can be alternatively defined as a one with bounded top-grad, that is for every t, we have

$$\widetilde{\nabla}_t(\mathcal{C}) \leq c(t).$$

Again, this allows us to choose the type of shallow minor that is better suitable for our application. Bounded expansion was originally defined in [3] using grad, but top-grad is often easier to work with.

It is easy to see that bounded expansion is a subclass of the nowhere dense class

Remark. In nowhere dense classes, the edge density of every graph G in $G\nabla t$ is bounded by $|G|^{o(1)}$

2.3.3 Examples of classes with bounded expansion

Here, we will give examples of some notable classes with bounded expansion. More classes and inclusion between them are shown at figure 2.4.

Classes with bounded tree-width

Informally, tree-width of a graph tells how dissimilar is the given graph to a tree. To define tree-width, we will first define a *tree decomposition* of a graph.

Definition 13 (tree decomposition)**.** *A* tree decomposition *of a graph G is a pair (X, T), where $X = X_1, ..., X_n$ is a family of subsets of $V(G)$, and T is a tree whose nodes are the subsets such that*

- $V(G) = \bigcup_{i=1}^{n} X_i$,
- $E(G) \subset \bigcup_{i=1}^{n} \binom{X_i}{2}$, *and*
- $\forall v \in V(G), \quad T[\{X_i : v \in X_i\}]$ *is connected.*

Definition 14 (tree-width)**.** *The* tree-width $\mathrm{tw}(G)$ *of a graph G is the minimum over all tree decompositions (X, T) of G of* $\max_i |X_i| - 1$.

Classes with bounded tree-width have bounded expansion.

Classes with bounded tree-depth

Informally, tree-width of a graph tells how dissimilar is the given graph to a star, that is from a tree with only one internal node.

Definition 15 (tree-depth)**.** *Let* closure $\mathrm{clos}(F)$ *of a rooted forest be the graph with vertex set $V(F)$ and edge set $\{\{x, y\} : x$ is an ancestor of y in $F\}$. Then the* tree-depth $\mathrm{td}(G)$ *of a graph G is the minimum height of a rooted forest F such that $G \subset \mathrm{clos}(F)$.*

Classes with bounded tree-depth have bounded expansion.

Minor closed and topologically closed classes

Definition 16 (minor closed class)**.** *A class $\mathcal{C}$ is* minor closed *if for every graph $G \in \mathcal{C}$ and every minor G' of G holds, that $G' \in \mathcal{C}$.*

Definition 17 (topologically minor closed class)**.** *A class $\mathcal{C}$ is* topologically closed *if for every graph $G \in \mathcal{C}$ and every topological minor G' of G holds, that $G' \in \mathcal{C}$.*

Minor closed and topologically closed classes have bounded expansion.

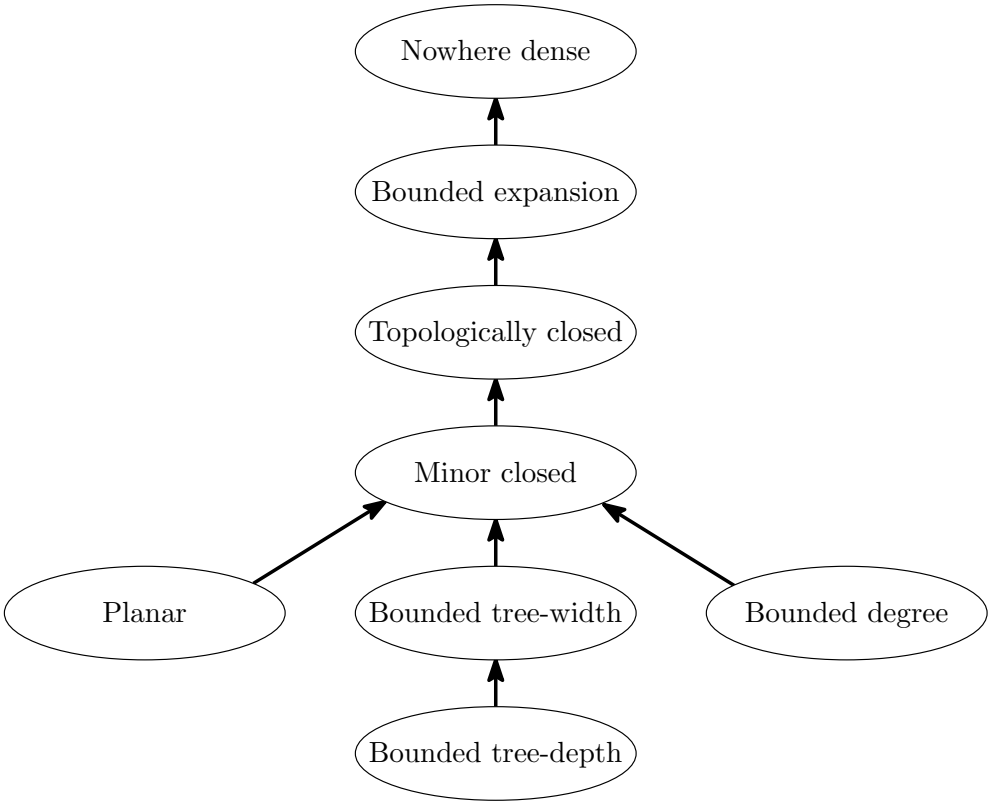

Figure 2.4: Inclusion between structurally sparse graph classes [16].

3. Properties of bounded expansion classes

In this chapter, we will seek deeper understanding of bounded expansion classes by stating different equivalent definitions.

Then we will focus on coloring and decomposition of graph in bounded expansion classes and other applications that these classes offer.

3.1 Characterizations

The classes with bounded expansion admit various different characterization. Often, controlling some parameter allows us to characterize both bounded expansion and nowhere dense classes, as was the case with shallow minors.

Intuitively, these characterizations have in common that all of them capture in some sense a concept of depth.

3.1.1 Controlling dense parts

Theorem 4 ([4]). *Let the class $\mathcal{C}$ be a class of graphs. Then $\mathcal{C}$ has bounded expansion if and only if there exists functions $F_{\text{ord}}, F_{\text{deg}}, F_{\triangledown}, F_{\text{prop}} : \mathbb{R}^+ \to \mathbb{R}$ such that the following two conditions hold:*

- $\forall \epsilon > 0, \forall G \in \mathcal{C},\ |G| > F_{\text{prop}} \Rightarrow \frac{|\{v \in G : d(v) \geq F_{\text{deg}}(\epsilon)\}|}{|G|} \leq \epsilon$
- $\forall r \in \mathbb{N}, \forall H \subset G \in \mathcal{C},\ \widetilde{\nabla}_r(H) > F_{\triangledown}(r) \Rightarrow |H| > F_{\text{prop}}(r)|G|$

The first condition can be informally paraphrased, that large enough graphs in bounded expansion classes have only a small fraction of vertices of a large degree. The second condition tells us, that subgraphs with large density necessarily cover a large portion of the graph.

This definition is particularly useful for showing which random graph models have bounded expansion.

3.1.2 Degeneracy

Degeneracy is another property that is sometimes used to informally measure sparsity.

Definition 18 (degeneracy). *A graph G is* k-degenerate *if each nonempty subgraph of G contains a vertex of degree at most k.*

Degeneracy has the following connection with bounded expansion.

Theorem 5 ([4]). *The class $\mathcal{C}$ has bounded expansion if and only if for each t, the class $\mathcal{C}\triangledown t$ is a class of degenerate graphs.*

3.1.3 Neighborhood complexity

Reidel et al. [17] introduced a measure of *neighborhood complexity*. Let us denote by $N(v)$ the neighborhood of a vertex v and by $N^r[v]$ the *r-th closed neighborhood* around a vertex v, i.e. the set of all vertices w such that $d(v, w) \leq r$.

Definition 19 (neighborhood complexity)**.** *For a graph G, the neighborhood complexity is a function v_r defined as*

$$v_r(G) = \max_{H \subseteq G, \emptyset \neq X \subseteq V(H)} \frac{|\{N^r[v] \cap X\}_{v \in V[H]}|}{|X|}$$

We extend the definition to a graph class $\mathcal{C}$ as $v_r(\mathcal{C}) = \sup_{G \in \mathcal{C}} v_r(G)$.

This value tells us in how many ways can vertices be connected to the vertex set X via paths of length at most r.

Theorem 6. *A class $\mathcal{C}$ has bounded expansion if and only if it has bounded neighborhood complexity, i.e. there exists a function f such that for all r it holds that $v_r(\mathcal{C}) \leq f(r)$.*

3.1.4 Low tree-depth and tree-width coloring

One way of studying the structure of graphs is looking at the number of colors in vertex coloring, in which every few-color classes induce some simple subgraph. In the case of structural sparsity, we look at colorings where these induced subgraphs have low tree-depth or tree-width.

Definition 20 (low tree-depth coloring)**.** *A p-*tree-depth coloring *of a graph G is a vertex coloring of G where each $p' \leq p$ colors induce a subgraph with tree-depth at most p'.*

A class $\mathcal{C}$ has a low tree-depth coloring *if for every $p \geq 1$ there is $N(p)$ such that every $G \in \mathcal{C}$ has p-tree-depth-coloring using at most $N(p)$ colors.*

Definition 21 (low tree-width coloring)**.** *A p-*tree-width coloring *of a graph G is a vertex coloring of G where each $p' \leq p$ colors induce a subgraph with tree-width at most $p - 1$.*

A class $\mathcal{C}$ has a low tree-width coloring *if for every $p \geq 1$ there is $N(p)$ such that every $G \in \mathcal{C}$ has p-tree-width-coloring using at most $N(p)$ colors.*

The following theorem then states that having low tree-depth and low tree-width colorings is equivalent with bounded expansion.

Theorem 7 ([4], [3] Thm 7.1)**.** *Let $\mathcal{C}$ be a graph class, then the following conditions are equivalent:*

- *$\mathcal{C}$ has bounded expansion,*
- *each $G \in \mathcal{C}$ has a low tree-depth colorings,*
- *each $G \in \mathcal{C}$ has a low tree-width colorings.*

3.2 Robustness

It is easy to see that if we add a fixed number of edges to each graph in a bounded expansion class, the resulting class still has bounded expansion (but the grad of the class might have increased). In such cases, we say that a bounded expansion class is *BE-robust* under taking such a (class) operation.

Observation. All bounded expansion classes are robust under operations which add a fixed number of edges, or under operations which add a vertex with a fixed number of edges.

The following two operations are examples of nontrivial class operations under which are all bounded expansion classes BE-robust.

Adding an apex

Let $G + v$ denote the operation of adding an apex v to a graph G, that is, the addition of a new universal vertex. Let $\mathcal{C}$ be a bounded expansion class, then the class $\mathcal{C} + v = \{G + v : G \in \mathcal{C}\}$ also has bounded expansion. [6]

Lexicographical product with a small graph

Definition 22 (lexicographical product)**.** *Let G and H be graphs. The* lexicographical product *$G \bullet H$ is defined by*

- $V(G \bullet H) = V(G) \times V(H)$
- $E(G \bullet H) = \{\{(x, y), (x', y')\} : \{x, x'\} \in E(G)$ *or* $x = x'$ *and* $\{y, y'\} \in E(H)\}$.

Let $p \geq 2$ be an integer and let $\mathcal{C}$ be a bounded expansion class. Then the class $\mathcal{C} \bullet K_p = \{G \bullet K_p : G \in \mathcal{C}\}$ has bounded expansion.[4]

3.3 Coloring and decomposition

In this section, we will again follow [4].

In addition to low tree-depth and tree-with colorings, we will define a notion of a p-centered coloring.

Definition 23 (*p-centered coloring*)**.** *A* p-centered coloring *of a graph G is a vertex coloring such that, for any (induced) connected subgraph H, either some color $c(H)$ appears exactly once in H, or H gets at least p colors.*

The following lemma tells us that having a p-centered colorings, we can obtain low tree-depth colorings.

Lemma 8. *Let p be an integer, let G be a graph, and let c be a p-centered coloring of G. Then any $i < p$ colors induce a subgraph of tree-depth at most i.*

In [4], a procedure using so-called *transitive fraternal augmentation*, that can be used to find p-centered colorings, is described. We won't give details of the procedure, but present the following result.

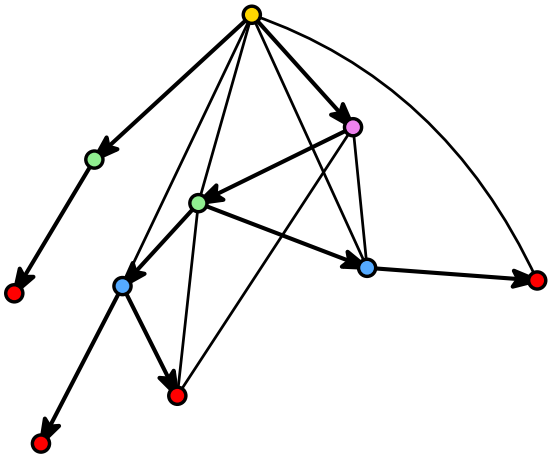

Figure 3.1: An example of a graph of tree-depth $\mathrm{td}(G) = 5$. The directed edges of G constitute a rooted forest T such that $G \subseteq \mathrm{closure}(T)$, showing $\mathrm{td}(G) \leq 5$. The coloring is an example of a 5-centered coloring of G, and also of a 5-tree-dept coloring.

Theorem 9. *There exists a procedure such that for every integer p, there exists a polynomial P_p (of degree about 2^{2^p}) such that for every graph G a $(p+1)$-centered coloring of G with $N_p(G) \leq P_p(\widetilde{\nabla}_{2^{p-2}+\frac{1}{2}}(G))$ colors can be find in time $O(N_p(G)n)$.*

In particular, theorem 9 with combination with lemma 8 tells us, that for graphs in classes with bounded expansion, a p-tree-depth coloring can be computed in linear time.

4. Network models and structural sparsity

In this chapter, we will first give the definitions of structural sparsity for random graph models. Then, we will use the classification of network models from section 1.1, and we will focus on sparsity of static and generative network models including the Erdős–Rényi and the configuration model.

Evolving network models will be discussed separately in the following chapter.

4.1 Structural sparsity in random graph models

So far, we have defined the notions of bounded expansion and nowhere dense only for classes. To be able to describe the sparsity of random graph models, we need to adapt it for sequences of graphs. We will follow terminology from [5].

For a random graph model $G(n, \rho(n))_{n\in\mathbb{N}}$ and an integer r, the notation $G(n, \rho(n))\nabla r$ denotes a random variable over graphs with n vertices whose probability distribution is given by

$$\Pr[G(n, \rho(n))\nabla r = A] = \sum_{H:H=G\nabla r} \Pr[G(n, \rho(n)) = G].$$

This way, a grad of $G(n, \rho(n))$ is correctly defined.

We will start with the definition of bounded expansion.

Definition 24 (bounded expansion a.a.s., w.h.p.)**.** *A graph model $G(n, \rho(n))_{n\in\mathbb{N}}$ has* asymptotically almost surely (a.a.s.) bounded expansion *if there exists a function f such that for all $r \geq 0$*

$$\lim_{n\to\infty} \Pr[\nabla_r(G(n, \rho(n))) < f(r)] = 1.$$

It has bounded expansion with high probability (w.h.p.) *if for every $c \geq 1$ there exists a function f such that, for all $r \geq 0$,*

$$\Pr[\nabla_r(G(n, \rho(n))) < f(r)] \geq 1 - O(n^{-c}).$$

The definition of somewhere and nowhere dense is analogous.

Definition 25 (nowhere dense a.a.s., w.h.p.)**.** *A graph model $G(n, \rho(n))_{n\in\mathbb{N}}$ is* a.a.s. nowhere dense *if there exists a function f such that for all $r \geq 0$*

$$\lim_{n\to\infty} \Pr[\omega(G(n, \rho(n))\nabla r) < f(r)] = 1.$$

It is nowhere dense w.h.p. *if for every $c \geq 1$ there exists a function f such that, for all $r \geq 0$,*

$$\Pr[\omega(G(n, \rho(n))\nabla r) < f(r)] \geq 1 - O(n^{-c}).$$

Definition 26 (somewhere dense a.a.s.)**.** *A graph model $G(n, \rho(n))_{n\in\mathbb{N}}$ is* a.a.s. somewhere dense *if there exists a function f such that for all $r \geq 0$*

$$\lim_{n\to\infty} \Pr[\omega(G(n, \rho(n))\triangledown r) > f(r)] = 1.$$

In contrast to classes, the definitions of a.a.s. nowhere and somewhere dense are not a complementary, i.e. there exists random graph models that are neither a.a.s. nowhere nor a.a.s. somewhere dense.

For that reason, we will also give the definition of a random graph model that is *not nowhere dense a.a.s.*.

Definition 27 (not nowhere dense a.a.s.)**.** *A graph model $G(n, \rho(n))_{n\in\mathbb{N}}$ is* not nowhere dense a.a.s. *if there exists $r \in \mathbb{N}$ such that for all functions f it holds that,*

$$\lim_{n\to\infty} \Pr[\omega(G(n, \rho(n))\triangledown r) > f(r)] > 0.$$

All definition from this section could be also equivalently stated via shallow *topological* minors.

4.2 Static network models

Let us recall that static models are those with a fixed number of vertices and with edges generated according to some stochastic process.

4.2.1 Erdős–Rényi random graphs

We described the well studied random model in section 1.1. In [4], it was shown that the Erdős–Rényi model with bounded average degree belongs a.a.s. to a bounded expansion class. That is, the random graph model $G(n, d/n)$ has a.a.s. bounded expansion.

In [7], another proof was given by Dreier et al. that additionally provided bounds on the size of r-shallow topological minors.

Theorem 10. *For $d \geq 16$ the probability that the Erdős–Rényi random graph $G(n, d/n)$ contains some r-shallow topological minor of size k and at least $8kd^{2r+1}$ edges is at most $\{n^{-2k}, 2^{-n^{2/3}}\}$. For $d < 16$ the same result holds for at least $8k16^{2r+1}$ edges.*

In the same paper, the theorem following was proved

Theorem 11 ([7])**.** *Let $d(n) = n^{o(1)}$, then the Erdős–Rényi random graph $G(n, d(n)/n)$ has a.a.s. locally bounded treewidth.*

From that follows that $G(n, d(n)/n)$ for $d(n) = n^{o(1)}$ is a.a.s. nowhere dense.

4.2.2 Random intersection graphs

Random intersection graphs were introduced by Singer in 1996 [18]. Their creation was motivated by the idea that nodes with the same attributes or properties are often connected, that is, a link between two nodes might represent the fact that they have something in common. Recently, it has been shown, that random intersection graph can have many properties that are desirable for modelling real-world networks [19].

Formally, the random intersection graph $G(n, m, p)$ consists of n vertices and every vertex can have some of the m attributes, each of them with probability p. Vertices that share the same attribute are connected.

In [20], Farrell et al. investigated structural properties of random intersection graphs, including sparsity. They provided the following modified parametrized definition.

Definition 28 (random intersection graph)**.** *Fix positive constants α, β and γ. Let B be a random bipartite graph on parts of size n and $m = \beta n^{\alpha}$ with each edge present independently with probability $p = \gamma n^{-(1+\alpha)/2}$. Let V (vertices) denote the part of size n and A (attributes) the part of size m. The associated* random intersection graph *$G = G(n, m, p)$ is defined on the vertices V: two vertices are connected in G if they share at least one attribute in A, or in other words, if they are both adjacent to the same vertex in B.*

Next, they proved that the structural sparsity of the random intersection graphs can be completely characterized using the parameter α.

Theorem 12 ([20])**.** *Fix constants α, β and γ. Let $m = \beta n^{\alpha}$ and $p = \gamma n^{-(1+\alpha)/2}$. Let $G\sigma G(n, m, p)$. Then the following hold with high probability.*

- *If $\alpha < 1$, $G(n, m, p)$ is somewhere dense.*
- *If $\alpha = 1$, $G(n, m, p)$ is somewhere dense.*
- *If $\alpha > 1$, $G(n, m, p)$ has bounded expansion.*

Additionally, they checked their results experimentally and concluded that not only is the grad for $\alpha > 1$ bounded, but also small enough to be used in algorithmic applications such as those described in section 7.

4.2.3 Kleinberg model

In section 1.2.3 we introduced the small world property and the Watts-Strogatz small world model. Though this model generates networks with the small world property, i.e. the distances between nodes are short, these paths cannot be effectively found. We call the problem of finding the paths the *small world routing problem.*

In [21], Kleinberg provided a generalization of the Watts-Strogatz model, in which the routing problem can be solved using decentralized search.

The Kleinberg model graph $G(n, p, q, r)$ for $n = m \times m$ is constructed from nodes on a $m \times m$ grid. Each node is connected to all other nodes within a grid distance p. Each node also has q random long distance neighbors, which are

determinated by a clustering exponent r: Let $d(u, v)$ denote the grid distance between nodes u and v. The random long distance link from a node u has the probability proportional to $d(u, w)^{-r}$ of ending in a node w.

The Watts-Strogatz model is a special case of the Kleinberg model for $q = 0$ and $p = 1$.

It can be shown that the routing problem can be solved the most efficiently for $r = 2$, that is, when the random links follow an inverse square root distribution.

In [5], Demaine et al. proved that the Kleinberg model which enables efficient routing not only doesn't have bounded expansion, but is somewhere dense with high probability.

Theorem 13 ([5]). *The Kleinberg model $G(n, p, q, r)$ with parameters $p = q = 1$ and $r = 2$ is somewhere dense w.h.p.*

This result might suggest that networks which were created with the aim that all nodes can efficiently communicate, might be inherently dense.

4.3 Generative network models

Generative network models are those that allow generating networks with a certain prescribed degree distribution.

4.3.1 Chung-Lu and configuration models

Given a degree distribution sequence $(D_n)_{n\in\mathbb{N}}$, we want to be able to sample graphs that match this distribution. Among other things, this would allow as to generate and study scale-free graphs.

Configuration and Chung-Lu models offer two ways of doing so. Next, we will describe formalizations of these models from [5].

Configuration model

Configuration model is a classic method for generating graphs with a given degree sequence.

To sample a graph G of size n with vertices $v_1, ..., v_n$, we first construct a degree sequence $(d_i)_{1\leq i\leq n}$ that matches (D_n). Then, we build a vertex set $V_C = \{v_i^1, ..., v_i^{d_i}\}$, which consists of d_i copies of each vertex v_i. Next, we generate an auxiliary graph H on the vertex set V_C and random matching as its edge set. We assemble the multigraph G' with vertex set $v_1, ..., v_n$ and the edge set corresponding to the auxiliary graph H, that is, we connect vertices v_i and v_j with as many edges as there are between all their copies in H. Finally, we get the graph G by removing multiedges and loops from G'.

Graphs generated this way are denoted by $G^{CF}(D_n)$.

Chung-Lu model

Chung-Lu model was first introduced in [22] and its generating process is simpler than in the case of the configuration model.

To sample a Chung-Lu graph of size n, we again construct a degree sequence $(d_i)_{1\leq i\leq n}$ that matches (D_n). We call d_i the weight of vertex v_i. Next, we create a graph on n vertices $v_1, ..., v_n$ and connect each pair of vertices v_i, v_j, $i < j$ with probability $\frac{d_i d_j}{m}$, where $m = \Sigma_{k=0}^{n} d_k$ is the total sum of weights.

Graphs generated this way are denoted by $G_{CL}(D_n)$.

Structural sparsity of configuration and Chung-Lu model

To describe structural sparsity of these model, we need to define one more property of distribution sequences.

Definition 29 (sparse degree distribution sequence)**.** *We say that a degree distribution sequence $(D_n)_{n\in\mathbb{N}}$ is sparse, if the following holds*

- $\mathrm{E}[D] < \infty$*, and*
- $(\mathrm{E}[D_n])_{n\in\mathbb{N}} \to \mathrm{E}[D]$.

We are now ready to fully describe structural sparsity of Chung-Lu and configuration models with degree distributions that are often found in real-world networks.

Theorem 14 ([5])**.** *Let (D_n) be a sparse degree distribution sequence with tail $O(1/d^\gamma)$. Then both the configuration model $G^{CF}(D_n)n \in \mathbb{N}$ and the Chung-Lu model $G^{CL}(D_n)_{n\in\mathbb{N}}$, with high probability*

- *have bounded expansion for $\gamma > 3$,*
- *are nowhere dense (with unbounded expansion) for $\gamma = 3$*
- *and are somewhere dense for $\gamma < 3$.*

5. Sparsity in evolving networks

Many real-world networks were created by gradual growth, and evolving models are trying to imitate this process. Studying them can help us understand how can be structural sparsity formed, or what type of growth processes lead to structurally sparse or dense graphs.

We will use the following formalization of evolving networks.

Definition 30 (evolving graph model)**.** Evolving graph model *is a random graph model* $(G_n)_{n\in\mathbb{N}}$*, in which each* G_t *is build from* G_{t-1} *by a specific* generating process *that adds a single vertex and adds or removes arbitrary many edges.*

In this chapter, we will first focus on evolving models generated only by the attachment of a new vertex, describe two important representative models and discuss if these models or their variations are structurally sparse. Then we will consider how edge changes influences sparsity.

5.1 Attachment models

A generating process is called an *attachment process*, if it consists only of adding a new vertex and connecting it to older ones, that is, it doesn't add or remove edges between older vertices. Models generated this way are called *attachment models*. Often, the number of added edges that connect the new vertex to older ones is fixed, and we call it the *attachment parameter*.

Some real-life networks such us the citation network don't lose or gain links between older nodes and are well suited for modelling with attachment models. But many other networks (e.g. the social network) change as they continue to grow. Nevertheless, attachment models might still be useful as they are easier to analyze.

The following theorem shows us that all attachment processes that have attachment parameter at least two and can attach new edges to all vertices can't be nowhere dense a.a.s.

Theorem 15 ([6])**.** *Let* (G_i) *be an evolving random graph model generated by an attachment process that*

- *attaches a new vertex by at least two edges, and*
- *has a non-zero attachment probability for all vertices*

Then for every $t \geq 1$*,* G_n *contains a one-subdivision of* K_t *with probability at least* $f(t) > 0$*, for some function* f *that depends on the model.*

As was discussed in section 4.1, this does not imply that these model are somewhere dense.

5.1.1 Barabasi-Albert model

The most notable example of an attachment model is the Barabasi-Albert model, first introduced in [10]. The model is motivated by the observation that in some networks, new nodes tend to link to nodes with higher degree, also called the "rich get richer phenomenon" or preferential attachment. In Barabasi-Albert model, the new vertex links to m older ones and the probability of linking to each vertex is directly proportional to its degree.

The properties of the Barabasi-Albert model are well studied. It has a scale-free degree distribution with parameter $\alpha = 3$, that is $p_k = k^{-3}$, thus it contains a small number of nodes with high degree, so-called hubs. It also has very short path lengths, the average path length increases logarithmically with the size of the graph, which is less than in the case of the Erdős–Rényi graphs.

Generating process

There are few possibly ways how to define the random process generating the Barabasi-Albert model rigorously, as we somehow have to deal with arising multiedges and whether to add new edges simultaneously or not. The following definition of the process is adapted from [8].

For a fixed m, the process starts with a single vertex and iteratively adds vertices with m edges, thus constructing a graph sequence $\bar{G}_m^1, \bar{G}_m^2, ..., \bar{G}_m^t$, where $\bar{G}_m^t$ has t vertices and mt edges, some of which might be self loops. We denote the degree of a vertex v in $\bar{G}_m^t$ by $d_m^t(v)$.

For $m = 1$, the process works in the following way. A random graph $\bar{G}_1^1$ is started that has only one vertex v_1 with exactly one self loop. We define the rest of the process inductively: Given $\bar{G}_1^{t-1}$ with vertex set $\{v_1, ..., v_{t-1}\}$, we construct $\bar{G}_1^t$ by adding a new single vertex v_t together with one edge connecting v_t with v_i, where i is chosen at random from $\{1, ..., t\}$ with

$$\Pr[i = s] = \begin{cases} \frac{d_1^{t-1}(v_s)}{2t-1} & \text{if } 1 \leq s \leq t \\ \frac{1}{2t-1} & \text{if } s = t. \end{cases}$$

For $m > 1$, we create $\bar{G}_m^t$ by merging every m consecutive vertices from $\bar{G}_1^{mt}$ to a single vertex. The graph $\bar{G}_m^t$ is a multigraph with self-loops. To obtain a simple random graph G_m^t, that is, the B-A-graph at time t with attachment parameter m, we take $\bar{G}_m^t$, remove self loops and replace multiple edges by a single edge.

Structural sparsity of the model

Thanks to theorem 15, we know that the B-A model is not a.a.s. nowhere dense. There remained a question if it is actually a.a.s. somewhere dense, or if it doesn't fall into either category.

In [7], Dreier et al. experimentally evaluated structural properties of B-A graphs and formed a conjecture, that the model is a.a.s. somewhere dense. In [8], the same authors studied structural sparsity of the Barabasi-Albert model and confirmed the conjecture by proving the following theorem.

Theorem 16 ([8]). *There exists a constant c such that for $m, n \in \mathbb{N}$, $m \geq 2$, $n \geq c$, the B-A graph G_m^n contains a one-subdivided clique of size at least $\log(n)^{1/4}$ with a probability of at least $1 - e^{-\log(n)^{1/4}/c}$.*

Resilience of the model

It is known that in scale-free networks, removing only a tiny fraction of the hubs, i.e. the highest degree nodes, results in a fragmented network without the giant component [12]. This means that scale-free networks, such us B-A graphs, are vulnerable to targeted attacks.

In [8], similar property was described from structural sparsity point of view. The following theorem was proved.

Theorem 17 ([8]). *There exists a constant c such that for $l, n, m \in \mathbb{N}^+$ and $b - \log\lceil(n)^{cl^2} m^{cl}\rceil$ a.a.s. every r-neighborhood of $G_m^n[v_{b+1}, ..., v_n]$ is either a tree or a tree with one additional edge.*

This means, that removing only polylogarithmically many vertices results in a graph that has a tree-like structure, it is locally extremely sparse.

Variations of the model

If the probability of attaching to a vertex with degree k is proportional to k^α for $\alpha > 1$, we call it *superlinear attachment*. It is easy to see that for large α, the graph resembles a star. Though superlinear attachment still generates models that are not nowhere dense, we hypothesize they are not somewhere dense.

If $\alpha \in (0, 1)$, we call the generating process *sublinear attachment*. When $\alpha = 0$, i.e. all vertices are equally likely to be chosen, we call it *uniform attachment*.

5.1.2 Copying network model

The copying network model, also known as the node-copying model, offers an alternative explanation of how the scale-free property of networks can arise.

The creation of the copying network model was driven by the observation that in some cases, networks share a large fraction of their neighbors. For example, in a citation network, papers from the same field often cite similar papers. Thus, in the copying network model, neighbors of a new vertex are copied from some other one that is picked at random. The copying is not perfect, but has some error rate.

Generating process

Here, we formalize the description of the model given in [1].

Let m be the attachment parameter and let γ be the error probability. While generating the model, we assume that the graphs are oriented, all their vertices have the same outdegree m and multiedges are allowed. To achieve that, the graphs $G_1, ...G_m$ are not defined, and we start with $\bar{G}_{m+1}$ which consist of $m + 1$ vertices, each pointing to m other ones, making $\bar{G}_{m+1}$ is a multigraph.

We define the rest inductively. Given $\bar{G}_{t-1}$, we define by adding a new vertex v_t and making it point to some m older vertices, which we chose the following

way: We choose a random vertex $u \in \{v_1, ...v_{t-1}\}$ at random. For each of the m vertices that u points to, we either

- with probability γ point v_t to this vertex, or
- with probability $1 - \gamma$ point v_t to any other from the graph.

We obtain G_t from $\bar{G}_t$ by forgetting edge orientation and replacing multiedges with single edges.

Model properties and application

The degree distribution of the model follows a power law with exponent $\alpha = 1 + \frac{1}{\gamma}$, that is $p_k = k^{-(1+\frac{1}{\gamma})}$. This shows that there are other ways to generate a scale-free network other than the preferential attachment in the Barabasi-Albert model. However, these models differ in other metrics and thus are not equivalent.

The fact that copying networks are generated by imperfect duplication suggest that they might serve as a good model for some biological networks, such as the protein interaction networks [1].

Sparsity of the model and its variations

First, we will discuss the degenerate cases for $\gamma \in \{0, 1\}$. For $\gamma = 0$, all new vertices connect only to a subset of the first $m + 1$ vertices, and it's easy to see that such a model has bounded expansion. For $\gamma = 1$, the new vertices attach to others purely at random, making the process equivalent to uniform attachment described in section 5.1.1.

For $\gamma > 0$, it follows from theorem 15 that the model is not a.a.s. nowhere dense. The question if it is also a.a.s. somewhere dense remains or if it depends on the parameter γ remains open.

In [23], a different version of a copying model was studied. In this version, the edges weren't oriented, parameter m unbounded and γ set to 0. The authors proved the following.

Theorem 18 ([23])**.** *For all $d \in \mathbb{N}$,*

$$\lim_{t\to\infty} \Pr[G_t \textit{ contains a complete bipartite graph } K_{d,d}].$$

Thus showing that the model is a.a.s. somewhere dense.

5.2 Inner edges changes

We will call the edges between older edges *inner edges.* In some real-world networks, the deletion of older edges corresponds to the aging process.

In this section, we will briefly discuss how can the deletion of inner edges as a step in the generating process of evolving model influence sparsity.

Erdős–Rényi as an evolving model

In [9], it is discussed that some static models can also be seen as evolving models. In particular, we can generate the Erdős–Rényi graph $G(n, d/n)$: We build the graph G_i inductively by adding a new vertex to G_{i-1}, connecting it with each older vertex with probability d/n and deleting (in expectation) $d/2$ older edges chosen at random.

Notice that if we omitted the step of deleting $d/2$ edges, we would get the generating process of uniform attachment described earlier in this chapter.

This tells us, that a deletion of only constantly many edges at each generating step makes the difference between a model with bounded expansion and a model, that is not even nowhere dense.

This observation might be used in future work dedicated to structural sparsity of evolving network models.

6. Sparsity in time-varying networks

In this section, we will discuss structural sparsity in time-varying networks, that is in dynamic networks, that don't change the number of vertices.

First, we will extend the notion of robustness to random graph models and discuss which sparse model stay sparse after addition of a few random edges.

Then, we will state two definitions of structural sparsity for time-varying networks. The first definition describes when time-varying networks gains structural sparsity, the second describes when a time-varying networks keeps structural sparsity within the same bounds.

Throughout this chapter, T will always denote a graph operation that changes only edges, i.e. doesn't add or delete any vertices. For a graph G, we inductively define $T^t(G) = T(T^{t-1}(G))$.

We will call a random graph model $(G_n)_{n\in\mathbb{N}}$ together with a graph operation T a *time-varying random model*. $T^t((G_n))_{n\in\mathbb{N}}$ denotes the time-varying random model at time t.

6.1 Robustness

In section 3.2 we defined what does it mean for a bounded expansion class to be robust under taking some operation. Here, we will define robustness of a random graph model.

Definition 31 (BE-robustness of a random graph model)**.** *Let $(G_n)_{n\in\mathbb{N}}$ be a random graph model that has a.a.s. bounded expansion (or has bounded expansion w.h.p.) and let T be a graph operation on edges (i.e. it doesn't change the number of vertices). If the random graph model $(T(G_n))_{nin\mathbb{N}}$ has a.a.s. bounded expansion (or has bounded expansion w.h.p. respectively), then we say that the random graph model is* BE-robust *under taking the operation T.*

Analogously, we could also define a (nowhere dense)-robustness.

6.1.1 Random perturbations

A question we can ask is whether graph models are BE-robust under adding a few random edges, or which models stay structurally sparse when we do so.

We will use the formalization of this process from [5].

Definition 32 (random perturbations of a graph)**.** *Let G^* be some base graph with n vertices. Then we will denote by $G = G^* + G(n, \mu/n)$ the graph obtained from G^* by adding every possible edge not already contained in G^* independently with probability μ/n. This process is called* random perturbations.

We can view this as a generalization of the sparse Erdős–Rényi model that is in fact more flexible than many other generalizations, such as the generalized random graph [9].

Perturbations of bounded degree graphs

For a fixed graph G, we will denote by $D_{r,G}$ the distribution of $|N^r(x)|$, that is the size of r-th neighborhood of a randomly chosen vertex $x \in G$:

$$\Pr[D_{r,G} = d] = \frac{|\{x \in G : |N_r(x)| = d\}|}{|G|}.$$

In [5], the following lemma describing random perturbation of bounded expansion classes with tail-bounded r-th neighborhood distribution was proved.

Lemma 19 ([5])**.** *Let $\mathcal{G}$ be a class of graphs with the following properties:*

- *$\mathcal{G}$ has bounded expansion, and*
- *for $G \in \mathcal{G}$ and every $r \in \mathbb{N}$ the distribution of $\mathcal{N}^r$ has a tail-bound h with $h(d) = \Omega(d^{3+\epsilon})$ for some $\epsilon > 0$.*

Let $G^{\mathcal{G}}$ be a random graph model which draws graphs from $\mathcal{G}$ with an arbitrary probability distribution. Then $G^{\mathcal{G}}(n)+G(n, \mu/n)$ has bounded expansion with high probability.

The proof relies on estimating grads using the results about structural sparsity of Chung-Lu graphs with the same tail-bound described in section 4.3.1.

The second condition from lemma 19 is true specifically for bounded-degree graphs, thus the following theorem holds.

Theorem 20 ([5])**.** *Let $\mathcal{G}$ be a class of bounded-degree graphs and μ a constant. Let $G^{\mathcal{G}}$ be a random graph model which draws graphs from $\mathcal{G}$ with an arbitrary probability distribution. Then the composite model $G^{\mathcal{G}}(n)+G(n, \mu/n)$ has bounded expansion with high probability.*

This applies to $G(n, \mu/n)_{n\in\mathbb{N}}$ itself, providing an additional proof that the Erdős–Rényi model is sparse. The result carries over to the sparse stochastic block model, a model with community structure used for studying algorithms for community detection [24].

Perturbations of bounded degree models

Recall that ΔG denotes the maximal degree of a graph G. We will define what does it mean for a graph model to have bounded degree.

Definition 33 (bounded degree of a random graph model)**.** *We say that a random graph model has* bounded degree *if*

$$\Pr[\lim_{n\in\mathbb{N}} \Delta G_n < \infty] = 1.$$

Lemma 19 can be extended such that it holds not only for graph classes with bounded expansion, but also for random graph models that have bounded expansion w.h.p.

Lemma 21. *Let $(G_n)_{n\in\mathbb{N}}$ be a random graph model with the following properties:*

- *$(G_n)_{n\in\mathbb{N}}$ has bounded expansion with high probability, and*

- *for every* $G_n \in (G_n)_{n\in\mathbb{N}}$ *and every* $r \in \mathbb{N}$ *the distribution of* $\mathcal{N}^r$ *has a tail-bound* h *with* $h(d) = \Omega(d^{3+\epsilon})$ *for some* $\epsilon > 0$.

Then the model $(G_n)_{n\in\mathbb{N}} + G(n, \mu/n)$ *has bounded expansion with high probability.*

Proof. We prove the lemma in appendix A as it requires only a small modification of the original proof. □

The lemma induces the following theorem.

Theorem 22. *Let* $(G_n)_{n\in\mathbb{N}}$ *be a random graph model with the following properties:*

- $(G_n)_{n\in\mathbb{N}}$ *has bounded expansion with high probability, and*
- $(G_n)_{n\in\mathbb{N}}$ *has bounded degree*

Then the model $(G_n)_{n\in\mathbb{N}}$ *is BE-robust under random perturbations* $G(n, \mu/n)$.

Or in other words, random perturbations of a bounded degree random graph model with bounded expansion result in a bounded expansion model.

6.2 Gaining structural sparsity

We would like to understand, which time-varying processes on random graph models lead to structural sparsity. To formalize this question, we offer the following definition.

Definition 34 (gaining bounded expansion)**.** *Let* $(G_n)_{n\in\mathbb{N}}$ *be a random graph model and let* T *be a graph operation on edges (i.e. it doesn't change the number of vertices). We say that* $(G_n)_{n\in\mathbb{N}}$ gains bounded expansion *through the operation* T *if one of the following conditions holds.*

- *There exists a function* f *such that for all* $r \geq 0$

$$\lim_{t\to\infty} \lim_{n\to\infty} \Pr[\nabla_r(T^t(G_n) < f(r)] = 1.$$

- *There exists a function* f *such that for all* $r \geq 0$*, there exists* t_0^r *such that for all* $t \geq t_0^r$

$$\Pr[\nabla_r(T^t(G_n) < f(r)] \geq 1 - O(n^{-c}).$$

6.2.1 Time-varying static models

In [25], Zhang et al. proposed a simple time-varying variants of some static random graph models that is governed by continuous-time Markov processes. We will illustrate this idea on the Erdős–Rényi model.

Erdős–Rényi

The process is governed by parameters λ and μ. Let λ be the rate in continuous time at which an edge appears between two nodes where there previously was none. Similarly, let μ be the rate at which an existing edge disappears.

Let G_n be an arbitrary n-vertex graph and let us fix parameters α and β that will govern the transition process. We start with G_n at time $t = 0$. At every time step, each vertex pair not connected by an edge gains an edge with probability α, or not with probability $1 - \alpha$. Similarly, each existing edge disappears with the probability β, or not with probability $1 - \beta$. We denote this operation T and we denote by $T^t(G_n)$ our graph after the time step t.

If the process goes on indefinitely, the average probability of an edge between two arbitrary vertices converges to

$$\frac{\alpha}{\alpha + \beta}.$$

Thus, if we set $\alpha = c \cdot d/n$ and $\beta = c \cdot (1 - d/n)$ where $c \in (0, 1)$ is a constant, the process converges to the Erdős–Rényi graph $G(n, d/n)$.

Therefore, we can say, that any random graph model gains bounded expansion through T.

6.3 Conserving structural sparsity

Another problem we would like to understand is, which time-varying random models, keep the random graph model structurally sparse, while also keeping grads (or some other measure) bounded. Answering this question is useful, because many algorithms exploiting structural sparsity described in chapter 7 have running time dependent on grad.

We offer the following definition to formalize the problem.

Definition 35. *Let $(G_n)_{nin\mathbb{N}}$ be a random graph model that has a.a.s. bounded expansion or has bounded expansion w.h.p., and let T be a graph operation on edges only. We say that the operation T* conserves bounded expansion *of the graph model $(G_n)_{nin\mathbb{N}}$*

- *if there exists a function f such that for all $r \geq 0$ and for all $t \geq 0$*

$$\lim_{n\to\infty} \Pr[\nabla_r(T^t(G_n) < f(r)] = 1,$$

- *or if there exists a function f such that for all $r \geq 0$ and for all $t \geq 0$*

$$\Pr[\nabla_r(T^t(G_n) < f(r)] \geq 1 - O(n^{-c}),$$

respectively.

7. Applications of sparsity in complex networks

In this section, we will discuss how can be the theory of structural sparsity applied to network science problems.

First, we will discuss if real-world networks are actually sparse. Then we will describe how can be network measures such us motif density and network centralities efficiently computed on structurally sparse graphs. The section ends with discussion of the spreading phenomena on structurally spare networks.

7.1 Structural sparsity in real-world networks

An important question is whether structural sparsity can be found in real-world networks. The obvious constraint in answering it is, that the notions of nowhere dense and bounded expansion are defined for classes and not for individual graphs. In chapters 4 and 5, we tried to overcome this obstacle by studying network models which are assumed to be an approximation of how are real-world networks formed or generated, but the results are inconclusive.

In [5], the authors tried a different way of answering the question by calculating the upper bounds on the p-centered coloring number χ_p. To compute it, they implemented the transitive fraternal augmentation procedure mentioned in section 3.3. They computed the number χ_3 of a wide range of real-world networks of various sizes, including the co-authorship network [26] or the protein interaction network [27], and compared it to the 3-centered coloring number of a configuration model with the same degree distribution. As was concluded in section 4.3.1, such configuration models likely have bounded expansion with high probability.

The experimental result showed that χ_3 of real-world networks in the chosen dataset is always smaller, with only one exception. This result suggests that algorithms exploiting bounded expansion such us subgraph counting from the next section can be applicable to real-world networks.

7.2 Motif counting

In section 1.6.1, we introduced the problem of motif counting.

In [4], an algorithm for counting the number of occurrences of small graphs was described. This algorithm runs in time $O(2^{ht}htn)$, where h is the size of the subgraph, n is the size of the host graph and t is its tree-depth. Thus, the algorithm runs in linear time on classes with bounded expansion.

In [5] a faster, though still only linear, algorithm was presented.

Theorem 23 ([5])**.** *Given a graph H and a graph G belonging to a class of bounded expansion, there exists an algorithm to count the appearances of H as a subgraph of G in time*

$$O\left(\binom{f(h)}{h} 6^h h^{h+2} n\right).$$

where f is a function depending only on the graph class.

The algorithm work in the following way: First, the $(h+1)$-tree-depth coloring of the host graph is computed. As described in section 3.3, on graphs from bounded expansion classes, this can be done in linear time. Next, in all parts with $i < h$ colors, we count the number of occurrences of H. We get the resulting number of occurrences by applying the inclusion-exclusion principle.

The algorithm can be also extended to nowhere dense classes, because for any $\epsilon > 0$, sufficiently large graphs have a p-tree-depth coloring with at most n^ϵ colors.

Theorem 24 ([5])**.** *Let $\mathcal{G}$ be a nowhere dense class and let H be a graph. For every $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}$, such that for any graph $G \in \mathcal{G}$, $|G| > N_\epsilon$, there exists an algorithm to count the appearances of H as a subgraph of G in time*

$$O(6^h h^{h+2} n^{1+\epsilon}).$$

Another, slightly faster variant of the algorithms was also described in [28].

Software that realizes the algorithm for subgraph counting that exploits structural sparsity and can visualize the p-tree-depth coloring process was implemented in [29] and [30].

A biological application of fast subgraph counting on bounded expansion graphs was described in [31], where authors exploited the sparsity of DNA assembly graphs and created a data structure that can efficiently retrieve metagenomic data.

7.3 Localized centralities

We introduced network centralities in section 1.2.1. Some centralities can be also defined locally. We will describe two centralities and their local versions, which can be efficiently computed in bounded expansion classes. Throughout this section, we will follow [5].

Closeness centrality

Closeness centrality of a vertex is the inverse of the sum of all shortest paths to this vertex. In the localized variant, only paths to vertices in a certain neighborhood are considered. The higher the closeness centrality is, the closer it is to other vertices, or vertices in its neighborhood, respectively.

Definition 36 (closeness centrality)**.** Closeness centrality *of a vertex v is defined as*

$$c_C(v) = \left(\sum_{u \in V(G)} d(v,u) \right)^{-1},$$

let $r \in \mathbb{N}$ be an integer, localized closeness centrality *of a vertex v is defined as*

$$c_C(v)^r = \left(\sum_{u \in N^r(v)} d(v,u) \right)^{-1}.$$

Harmonic centrality

Harmonic centrality is similar to closeness centrality, but the operations of sum and taking the inverse are reversed.

Definition 37 (harmonic centrality)**.** Harmonic centrality *of a vertex v is defined as*

$$c_H(v) = \sum_{u \in V(G)} d(v,u)^{-1},$$

let $r \in \mathbb{N}$ be an integer, localized harmonic centrality *of a vertex v is defined as*

$$c_H(v)^r = \sum_{u \in N^r(v)} d(v,u)^{-1}.$$

Experimental evidence suggests, that localized variants are good enough approximations that for certain applications, they can successfully be used in practice.

To compute the closeness or the harmonic centrality of all vertices, we need to compute the shortest distance between any two vertices. In general, this requires at least quadratic time. For bounded expansion classes, it was proved that we can compute localized variants of these centralities in linear time.

Theorem 25 ([5])**.** *Let $\mathcal{C}$ be a class with bounded expansion, $G \in \mathcal{C}$ and $r \in \mathbb{N}$ an integer. Then the r-centric closeness centrality and harmonic centrality can be computed for all vertices of G in total time $O|(V(G)|)$.*

7.4 Spreading phenomena

7.4.1 Bootstrap percolation

Bootstrap percolation is an example of a simple spreading process on graphs. Let $r \in \mathbb{N}$ be an integer. At the start of the process, only some vertices are infected. At every time step, all vertices with at least r infected neighbors become infected. The process ends when there aren't any susceptible vertices left, in particular, when the whole graph becomes infected.

Formally, following the notation from [32], we will denote the set of initially infected vertices by A_0 and we will denote by A_t the set of vertices infected at time t, that is $A_t = A_{t-1} \cup \{v \in V : |N(v) \cap A_{t-1} \geq r\}$.

The final set of infected vertices is denoted by A_f. It holds that $A_f = \bigcup_{i \in \mathbb{N}} A_i$. We call the minimum time τ such that $A_f = A_\tau$ the *running time*.

The following holds for a bootstrap percolation process on degenerate graphs.

Theorem 26 ([32])**.** *Let G be a d-degenerate graph with n vertices, and let $r \geq d$ be an integer. Then the running time τ of the r-bootstrap process with a given set A_0 is bounded by*

$$\tau \leq \frac{d}{r-d}|A_0|.$$

Specifically, this holds for graphs from classes with bounded expansion.

7.4.2 Discussion

On the example of bootstrap percolation, it can be seen that notions related to structural sparsity such us degeneracy can be used to describe properties of spreading phenomena.

This observation suggests that structural sparsity could be exploited in this way too, and opens a window for both new theoretic and real-world applications, that can be explored in future work.

Conclusion

This thesis examined the intersection of the theory of structural sparsity and network science. We presented past work that has been done in this area and to our knowledge, this thesis is the most comprehensive review of what is known about the intersection of these to topics, and we attempted to bridge the gap between them even further.

Our main addition lies in extending the theory of structural sparsity to dynamic networks, chapters 5 and 6 are dedicated to this. In chapter 5, we also generalized theorem 22 so that it can be applied to network models.

Dynamic definition of structural sparsity in chapter 6 suggest direction for future research in trying to understand, how can be structural sparsity in networks created or how can be efficient algorithms applied even in time-varying networks.

Bibliography

[1] M. Newman. *Networks: An Introduction*. OUP Oxford, 2010.

[2] R. Hanel, S. Thurner, and P. Klimek. *Introduction to the Theory of Complex Systems*. 2018.

[3] Jaroslav Nešetřil and Patrice Ossona de Mendez. Grad and classes with bounded expansion i. decompositions. *European Journal of Combinatorics*, 29, 09 2005.

[4] J. Nešetřil and P.O. de Mendez. *Sparsity: Graphs, Structures, and Algorithms*. Algorithms and Combinatorics. Springer Berlin Heidelberg, 2012.

[5] Erik D. Demaine, Felix Reidl, Peter Rossmanith, Fernando Sánchez Villaamil, Somnath Sikdar, and Blair D. Sullivan. Structural sparsity of complex networks: Bounded expansion in random models and real-world graphs. *Journal of Computer and System Sciences*, 105:199–241, nov 2019.

[6] Felix Reidl. *Structural sparseness and complex networks*. PhD thesis, Aachen, Techn. Hochsch., 2015.

[7] Jan Dreier, Philipp Kuinke, Ba Le Xuan, and Peter Rossmanith. Local structure theorems for erdős–rényi graphs and their algorithmic applications. In *International Conference on Current Trends in Theory and Practice of Informatics*, pages 125–136. Springer, 2018.

[8] Jan Dreier, Philipp Kuinke, and Peter Rossmanith. Maximum shallow clique minors in preferential attachment graphs have polylogarithmic size. In *Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques (APPROX/RANDOM 2020)*. Schloss Dagstuhl-Leibniz-Zentrum für Informatik, 2020.

[9] R. van der Hofstad. *Random Graphs and Complex Networks*. Number v. 1 in Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, 2017.

[10] Albert-László Barabási and Réka Albert. Emergence of scaling in random networks. *Science*, 286(5439):509–512, 1999.

[11] Anna Broido and Aaron Clauset. Scale-free networks are rare. *Nature Communications*, 10, 03 2019.

[12] A.L. Barabási and M. Pósfai. *Network Science*. Cambridge University Press, 2016.

[13] Duncan J Watts and Steven H Strogatz. Collective dynamics of 'small-world'networks. *nature*, 393(6684):440–442, 1998.

[14] Petter Holme and Jari Saramäki. Temporal networks. *Physics Reports*, 519(3):97–125, 2012. Temporal Networks.

[15] Alain Barrat, Marc Barthélemy, and Alessandro Vespignani. *Dynamical Processes on Complex Networks*. Cambridge University Press, 2008.

[16] Zdeněk Dvořák and Daniel Král'. Algorithms for classes of graphs with bounded expansion. In Christophe Paul and Michel Habib, editors, *Graph-Theoretic Concepts in Computer Science*, pages 17–32, Berlin, Heidelberg, 2010. Springer Berlin Heidelberg.

[17] Felix Reidl, Fernando Sánchez Villaamil, and Konstantinos Stavropoulos. Characterising bounded expansion by neighbourhood complexity. *European Journal of Combinatorics*, 75:152–168, 2019.

[18] Karen B Singer. *Random intersection graphs*. PhD thesis, The Johns Hopkins University, 1996.

[19] Mindaugas Bloznelis, Erhard Godehardt, Jerzy Jaworski, Valentas Kurauskas, and Katarzyna Rybarczyk. Recent progress in complex network analysis: Properties of random intersection graphs. In Berthold Lausen, Sabine Krolak-Schwerdt, and Matthias Böhmer, editors, *Data Science, Learning by Latent Structures, and Knowledge Discovery*, pages 79–88, Berlin, Heidelberg, 2015. Springer Berlin Heidelberg.

[20] Matthew Farrell, Timothy D. Goodrich, Nathan Lemons, Felix Reidl, Fernando Sánchez Villaamil, and Blair D. Sullivan. Hyperbolicity, degeneracy, and expansion of random intersection graphs. In David F. Gleich, Júlia Komjáthy, and Nelly Litvak, editors, *Algorithms and Models for the Web Graph*, pages 29–41, Cham, 2015. Springer International Publishing.

[21] D. Easley and J. Kleinberg. *Networks, Crowds, and Markets: Reasoning about a Highly Connected World*. Cambridge University Press, 2010.

[22] William Aiello, Fan Chung, and Linyuan Lu. A random graph model for power law graphs. *Experimental Mathematics*, 10(1):53 – 66, 2001.

[23] Silvia Gago and Dirk Schlatter. Bounded expansion in web graphs. *Commentationes Mathematicae Universitatis Carolinae*, 50:181–190, 2009.

[24] Brian Karrer and M. E. J. Newman. Stochastic blockmodels and community structure in networks. *Phys. Rev. E*, 83:016107, Jan 2011.

[25] Xiao Zhang, Cristopher Moore, and Mark E. J. Newman. Random graph models for dynamic networks. *The European Physical Journal B*, 90(10), Oct 2017.

[26] Mark EJ Newman. Finding community structure in networks using the eigenvectors of matrices. *Physical review E*, 74(3):036104, 2006.

[27] Dongbo Bu, Yi Zhao, Lun Cai, Hong Xue, Xiaopeng Zhu, Hongchao Lu, Jingfen Zhang, Shiwei Sun, Lunjiang Ling, Nan Zhang, Guo-Jie Li, and Runsheng Chen. Topological structure analysis of the protein-protein interaction network in budding yeast. *Nucleic acids research*, 31:2443–50, 06 2003.

[28] Felix Reidl and Blair D. Sullivan. A color-avoiding approach to subgraph counting in bounded expansion classes, 2020.

[29] Michael P. O'Brien, Clayton G. Hobbs, Kevin Jasnick, Felix Reidl, Brandon Mork, Nishant G. Rodrigues, and Blair D. Sullivan. Concuss: Version 2.0, June 2016.

[30] Yang Ho, Clayton G. Hobbs, Brandon Mork, Felix Reidl, Nishant G. Rodrigues, and Blair D. Sullivan. Beavr: Version 1.0, June 2016.

[31] C. Titus Brown, Dominik Moritz, Michael P. O'Brien, Felix Reidl, Taylor Reiter, and Blair D. Sullivan. Exploring neighborhoods in large metagenome assembly graphs using spacegraphcats reveals hidden sequence diversity. *Genome Biology*, 21(1), 2020.

[32] Marinus Gottschau. Bootstrap percolation on degenerate graphs. In Natalia Kliewer, Jan Fabian Ehmke, and Ralf Borndörfer, editors, *Operations Research Proceedings 2017*, pages 303–308, Cham, 2018. Springer International Publishing.

List of Figures

List of Tables

A. Proof of lemma 21

Lemma . *Let $(G_n)_{n\in\mathbb{N}}$ be a random graph model with the following properties:*

- *$(G_n)_{n\in\mathbb{N}}$ has bounded expansion with high probability, and*
- *for every $G_n \in (G_n)_{n\in\mathbb{N}}$ and every $r \in \mathbb{N}$ the distribution of $\mathcal{N}^r$ has a tail-bound h with $h(d) = \Omega(d^{3+\epsilon})$ for some $\epsilon > 0$.*

Then the model $(G_n)_{n\in\mathbb{N}} + G(n, \mu/n)$ has bounded expansion with high probability.

Proof. Let us fix n and G_n, let $\tilde{G} = G(n, \mu/n)$ and let $G = G_n + \tilde{G}$. Assume H is an r-shallow topological of G and consider an embedding Φ_V, Φ_E of H witnessing the fact. Since $\widetilde{\nabla}_r(G_n)$ is a constant, we want to bound the density of topological minors whose embedding use at least one edge of $\tilde{G}$: each component of P $E(\tilde{G})$ is contained in a subgraph $G_n[N^r(v)]$ for some vertex v. Let $N_1, ..., N_p$ be these subgraphs of the path P: then we can bound the probability that P exists by considering the probability that there exist at least one edge between N_i and N_{i+1} in $\tilde{G}$ for $1 \leq i \leq p-1$.

Since the probability that two r-neighborhoods $N^r(u), N^r(v)$ in G_n are connected by an edge in $\tilde{G}$ is at most

$$\frac{\mu|N^r(u)||N^r(u)|}{n},$$

we can stochastically bound the occurrence of r paths in G by the occurrence of edges in $G^{CL}(D_{r,G})$. Hence, we have that, stochastically,

$$\widetilde{\nabla}_r(G) \leq \widetilde{\nabla}_r(G_n) + \widetilde{\nabla}_r(G^{CL}(D_{r,G})).$$

Let $c \geq 1$ be an integer, from theorem 14, we know that there exist a function f_1 such that

$$P_1 = \Pr[\widetilde{\nabla}_r(G^{CL}(D_{r,G})_n) < f_1(r)] \geq 1 - O(n^{-c}).$$

From assumption, we also know there exist a function f_2 such that

$$P_2 = \Pr[\widetilde{\nabla}_r(G_n) < f_2(r)] \geq 1 - O(n^{-c}).$$

Let $f = f_1 + f_2$, then

$$\Pr[\widetilde{\nabla}_r(G) < f(r)] \geq P_1 \times P_2.$$

From union bound, we get that

$$\Pr[\widetilde{\nabla}_r(G) < f(r)] \geq 1 - O(n^{-c}).$$

Hence, the model $(G_n)_{n\in\mathbb{N}} + G(n, \mu/n)$ has bounded expansion with high probability.

□