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Homoclinic Chaos in Black-hole Fields

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Title: Homoclinic Chaos in Black-hole Fields

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Abstract: The existence of homoclinic orbits is an intrinsic feature of static black hole's space-time. The regularity of the geodesic motion around these black holes can therefore be quickly disrupted due to the small changes in the original space-time. The nature of the dynamics around a perturbed orbit depends on the manner of intersection of the surrounding stable and unstable manifolds. If they intersect transversally, the homoclinic orbit splits into chaotic layers.
In this thesis, the mathematical formulation of chaotic dynamical systems and main properties of the geodesic motion in circular space-times are discussed. Thereupon, the space-time around a static black hole is reproduced by classical approximations by using Paczyński-Wiita and logarithmic pseudo-Newtonian potentials. By means of the effective potential method, the homoclinic orbits are found for these potentials. In addition, the analysis of the general circular space-time is done and the equations of geodesic motion in axially symmetric space-times are examined. Finally, the motion in a Schwarzschild space-time with a static axially symmetric external source is inspected.

Keywords: black holes, homoclinic orbit, geodesic motion, chaotic behaviour

Contents

Introduction

The word $\chi\acute{\alpha}o\zeta$ meaning a void, an emptiness in ancient Greek, was described in a Greek mythology [1] as the beginning of everything - time and space. It was personified as a female who was the power behind the energy, mass, and dark matter. The difference between contemporary chaos is noticeable - chaos is now referred to as a system with unpredictable behaviour mathematically and to a state of confusion or a state with no order elsewhere [2]. From the mathematical point of view, linear equations appear to be just an approximation of nonlinear depiction of the world around, which might lead to the bewilderment of the system. From the deterministic point of view, the state of chaos as an erratic state eventually leads back to the very beginnings of everything. Hence, the definitions unite.

Whichever description - linear or nonlinear - is nonetheless a mathematical description. So, what is mathematics? Plato's world of form is the reality of our shadow world. The forms can be understood only by thinking. However, the formal world is abstract, we cannot sense it, so it is easy to refuse. Aristotle viewed numbers as properties of the world we see. Due to the objectivity of numbers, this idea was refuted. Nevertheless, if something is true mathematically, it is necessarily true. So how can we know something which is necessarily true?
There does not exist a general agreement on what exactly mathematics means. It is, indeed, used to describe the physical world around us. Although if we do not know what mathematics is, is it possible to say that it describes everything perfectly? Even though physicists try to generalize things they observe using mathematical language, there will always be a dubiety, whether the term represents what we want it to be.

The nonlinear dynamical systems that eventually lead to chaotic behaviour are described in many books, and more are being written. It probably is because every book concentrates on a specific branch and stress diverse parts of the classical chaos theory. There are various layers from which one can look at the theory and later apply the knowledge to concrete fields. There is usually one thing in common. The books commence with linear systems and their dynamical behaviour and slowly and non-violently evolve into nonlinear dynamics. However, if a person tries to find a definition of chaos itself, it is usually not easy to do so - even by using mathematics, the concrete terms can differ. The chaos in general relativity is fascinating because space-time itself must be understood as a dynamical system. Unlike in other nonlinear dynamics, in general relativity, one of the variables is time, and the question of how to grasp its evolution is crucial. Even though there is a possibility of a mathematical description, the apprehension is often beyond one's imagination.

This thesis should summarize important mathematical concepts which are essential for understanding the background of chaotic dynamical systems. It concentrates on characterizing homoclinic orbits, their occurrence in static space-times and their importance in revealing chaotic behaviour. The dynamics of the geodesic motion is introduced, followed by a specific form of the equations in symmetrical space-times. Firstly, the static space-time around a black hole is simulated by pseudo-Newtonian potentials, to which one can find the appropriate homoclinic orbits by using the method of the effective potential. Consequently, similar calculations are done for a general circular space-time using the generalized method of the effective potential. At last, the equations for geodesic motion are simplified by applying various conditions on them in an axially symmetric space-time around a black-hole surrounded by a thin Bach-Weyl ring and a Schwarzschild field with a generally unknown external source.

Even though the chaos theory mainly uses linearisation (or other simplifications) to determine the system's all-time behaviour, it is just an approximation. It admits the possibility of not knowing, recognizes that reality cannot be easily summarized in a few equations. Moreover, even if it might be, we will not necessarily know what will happen next.

1. Mathematical Preface

This chapter aims to summarise the basic mathematical definitions and practical concepts, leading to a better understanding of the later calculations. Firstly, we will introduce dynamical systems and their stability analysis, followed by a brief overview of essential terms from Hamiltonian mechanics and perturbation theory. Lastly, we will define the chaotic behaviour of a dynamical system and illustrate the Smale horseshoe mapping and Poincaré's surface of section - important terms in chaos theory.
More useful and complex information about the dynamical systems can be found in the following books [3], [4], [5], [6], [7], [8], [9] which also served as the references for this chapter.

1.1 Dynamical Systems

The physical dynamical systems are equations describing the evolution of a time dependence point(s) in space. These systems are therefore described by differential equations (or iterated maps for discrete systems) [4]. In this thesis, we will work strictly with continuous systems.

Definition 1.1.1 (Dynamical System)**.**
A pair (Ω, ϕ), where Ω is a topological space and $\phi = \phi(t, \boldsymbol{x}) : \mathbb{R} \times \Omega \to \Omega$ is a map with following properties:

1. $\phi(0, \boldsymbol{x}) = \boldsymbol{x} \quad \forall \boldsymbol{x} \in \Omega,$
2. $\phi(s, \phi(t, \boldsymbol{x})) = \phi(s + t, \boldsymbol{x}) \quad \forall s, t \in \mathbb{R}, \boldsymbol{x} \in \Omega,$
3. $(t, \boldsymbol{x}) \to \phi(t, \boldsymbol{x})$ is continuous,

is called a *dynamical system.*

The general form of a linear system of ordinary differential equation is denoted by

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \mathbf{A}(t)\boldsymbol{x} + \boldsymbol{f}(t) \tag{1.1}$$

where $\boldsymbol{x}$ stands for a system of functions of variable t, $\boldsymbol{f}(t)$ denotes the vector function of the time variable and $\mathbf{A}(t)$ is a matrix of time-dependent functions. The autonomous system of differential equations has the form:

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \mathbf{A}\boldsymbol{x} \tag{1.2}$$

where $\mathbf{A}$ denotes a matrix with constant coefficients.
There are several methods for solving linear differential equations e. g. the integration by quadrature, the method of separation of variables and so froth. However, for general nonlinear equations, it is not always possible to solve the

integrals analytically. A standard method for inspecting their behaviour is linearisation which eventually allows using the usual methods. For a general system of equations in the form

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{F}(\boldsymbol{x}) \tag{1.3}$$

where $\boldsymbol{F}(\boldsymbol{x})$ can be any smooth function of $\boldsymbol{x}$ (e.g. polynomial, exponential) a fixed point $\boldsymbol{x}_0$ is a point satisfying $\boldsymbol{F}(\boldsymbol{x}_0) = 0$.

Definition 1.1.2 (Stable and Asymptotically Stable Points)**.**
A fixed point $\boldsymbol{x}_0$ is called stable if

$$\forall \varepsilon > 0 : \quad \exists \delta > 0 \quad |\boldsymbol{x}(0) - \boldsymbol{x}_0| < \delta \implies |\boldsymbol{x}(t) - \boldsymbol{x}_0| < \varepsilon \ \forall t \geq 0 \, ,$$

where $\boldsymbol{x}(t)$ is a solution of (1.1) with initial condition $\boldsymbol{x}(0) = \boldsymbol{x}_0$ at $t = 0$.

The fixed point is called asymptotically stable if it is stable and

$$\exists \Delta > 0 : \quad |\boldsymbol{x}(0) - \boldsymbol{x}_0| < \Delta \implies \lim_{t \to \infty} \boldsymbol{x(t)} = \boldsymbol{x}_0 \, .$$

The point $\boldsymbol{x}_0$ is called unstable if it is not stable.

The second condition alone states that $\boldsymbol{x}_0$ is a *local attractor* which generally does not imply stability. The counterexample can be found in [10].
The nature of a fixed point can be examined by linearisation. Let us consider a system of differential equation of the first order

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t)) \, , \tag{1.4}$$

where $\boldsymbol{x}_0$ is its fixed point and the overdot denotes the total derivative with respect to t, i.e. $\dot{\boldsymbol{x}} = \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}$.
Let $\boldsymbol{\delta}_{\boldsymbol{x}}$ denote the small displacement from the fixed point

$$\boldsymbol{x}(t) = \boldsymbol{x}_0 + \boldsymbol{\delta}_{\boldsymbol{x}}(t) \, . \tag{1.5}$$

Consequently the differential equations (1.4) are rewritten in terms of the new variables

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \frac{\mathrm{d}(\boldsymbol{x}_0 + \boldsymbol{\delta}_{\boldsymbol{x}})}{\mathrm{d}t} = \frac{\mathrm{d}\boldsymbol{\delta}_{\boldsymbol{x}}}{\mathrm{d}t} = \boldsymbol{f}(\boldsymbol{x}) \, , \tag{1.6}$$

where the term $\frac{\mathrm{d}\boldsymbol{x}_0}{\mathrm{d}t} = 0$ since $\boldsymbol{x}_0$ is a constant. The Taylor expansions of $\boldsymbol{f}$ at $\boldsymbol{x_0} + \boldsymbol{\delta}_{\boldsymbol{x}}$ is of the form

$$\boldsymbol{f}(\boldsymbol{x}_0 + \boldsymbol{\delta}_{\boldsymbol{x}}) = \boldsymbol{f}(\boldsymbol{x}_0) + \sum_{i=1}^{N} \frac{\partial \boldsymbol{f}}{\partial x_i}\Big|_{(\boldsymbol{x}_0)} \delta_{x_i} + \mathcal{O}(\boldsymbol{\delta}^2) \, , \tag{1.7}$$

where the first term trivially equals zero. The $\mathcal{O}$ denotes the higher order terms which neglection leads to the linearisation of the system. If the total derivative at the fixed point equals zero, the linearisation does not work. An example can be found in [4].
The displacement evolves as

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\delta}_{\boldsymbol{x}} = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}(\boldsymbol{x}_0) \cdot \boldsymbol{\delta}_{\boldsymbol{x}} \, , \tag{1.8}$$

which rewritten in the vector components reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\delta_{x_1}\\ \delta_{x_2}\\ \vdots\\ \delta_{x_N}\end{pmatrix}=\begin{pmatrix}\frac{\partial f_1}{\partial x_1}(\boldsymbol{x}_0) & \cdots & \frac{\partial f_1}{\partial x_N}(\boldsymbol{x}_0)\\ \vdots & \ddots & \vdots\\ \frac{\partial f_N}{\partial x_1}(\boldsymbol{x}_0) & \cdots & \frac{\partial f_N}{\partial x_N}(\boldsymbol{x}_0)\end{pmatrix}\cdot\begin{pmatrix}\delta_{x_1}\\ \delta_{x_2}\\ \vdots\\ \delta_{x_N}\end{pmatrix}\,. \tag{1.9}$$

The $N \times N$ matrix in (1.9) from now to be denoted by $\mathbf{M}$ is often called a *stability matrix* and corresponds to the Jacobian matrix at the point $\boldsymbol{x}_0$.

For the (1.9) $\mathbf{M}$ matrices, the stability of the fixed points is determined by calculating and comparing the non-degenerate eigenvalues $\lambda_i\,, i = 1 \ldots N$ of the matrix $\mathbf{M}$. If the eigenvalues are degenerate, the stability is determined by comparing the eigenvectors.

To demonstrate the physical meaning of fixed points, let us presume a two-dimensional space with the eigenvalues $\lambda_{1,2}$. The behaviour of the fixed point depends on the relationship between the eigenvalues λ_1 and λ_2 and their nature. If both eigenvalues are real numbers we distinguish between three types of fixed points.

- If the eigenvalues are less than zero $\lambda_1 < \lambda_2 < 0$, the fixed point is called a *stable node.*
- If both eigenvalues are strictly greater than zero $\lambda_1 > \lambda_2 > 0$, the fixed point is called an *unstable node.*
- If $\lambda_1 < 0$ and $0 < \lambda_2$ holds, the fixed point is called a *hyperbolic point.*

For the generally complex eigenvalues $\lambda_{1,2} \in \mathbb{C}$ and $\alpha, \beta \in \mathbb{R}\,, \alpha, \beta > 0$ there are two possible outcomes.

- If the eigenvalues are complex conjugate and have the form $\lambda_{1,2} = -\alpha \pm i\beta$, the fixed point is called a *stable spiral point.*
- For the eigenvalues which satisfy $\lambda_1 = \alpha + i\beta$ $\lambda_2 = \alpha - i\beta$ the fixed point is called an *unstable spiral point.*

For pure imaginary values of $\lambda_{1,2}$ there exists only one fixed point called an *elliptic point* or a *centre.* For the eigenvalues then holds $\lambda_1 = +i\omega$ and $\lambda_2 = -i\omega$, where $\omega \in \mathbb{R}$.

If none of these conditions is satisfied, we cannot state which fixed point we work with without additional information. Moreover, for degenerate roots, the stability of fixed points will also depend on the associated eigenvectors [3].

Recognising hyperbolic fixed points in generic $\mathbb{R}^N$ space employing a definition of hyperbolic linear isomorphism is one of two equivalent definitions described in [8].

Definition 1.1.3 (Hyperbolic Isomorphism)**.**
Let $\mathcal{L}$ be a vector space between two normed spaces. An isomorphism $A \in \mathcal{L}(\mathbb{R}^N)$ is called *hyperbolic* if its spectrum has properties

$$|\lambda| \neq 1\,, \lambda \in \sigma(A) \tag{1.10}$$

and there exist points λ in the spectrum satisfying $|\lambda| < 1$ and aslo points satisfying $|\lambda| > 1$.

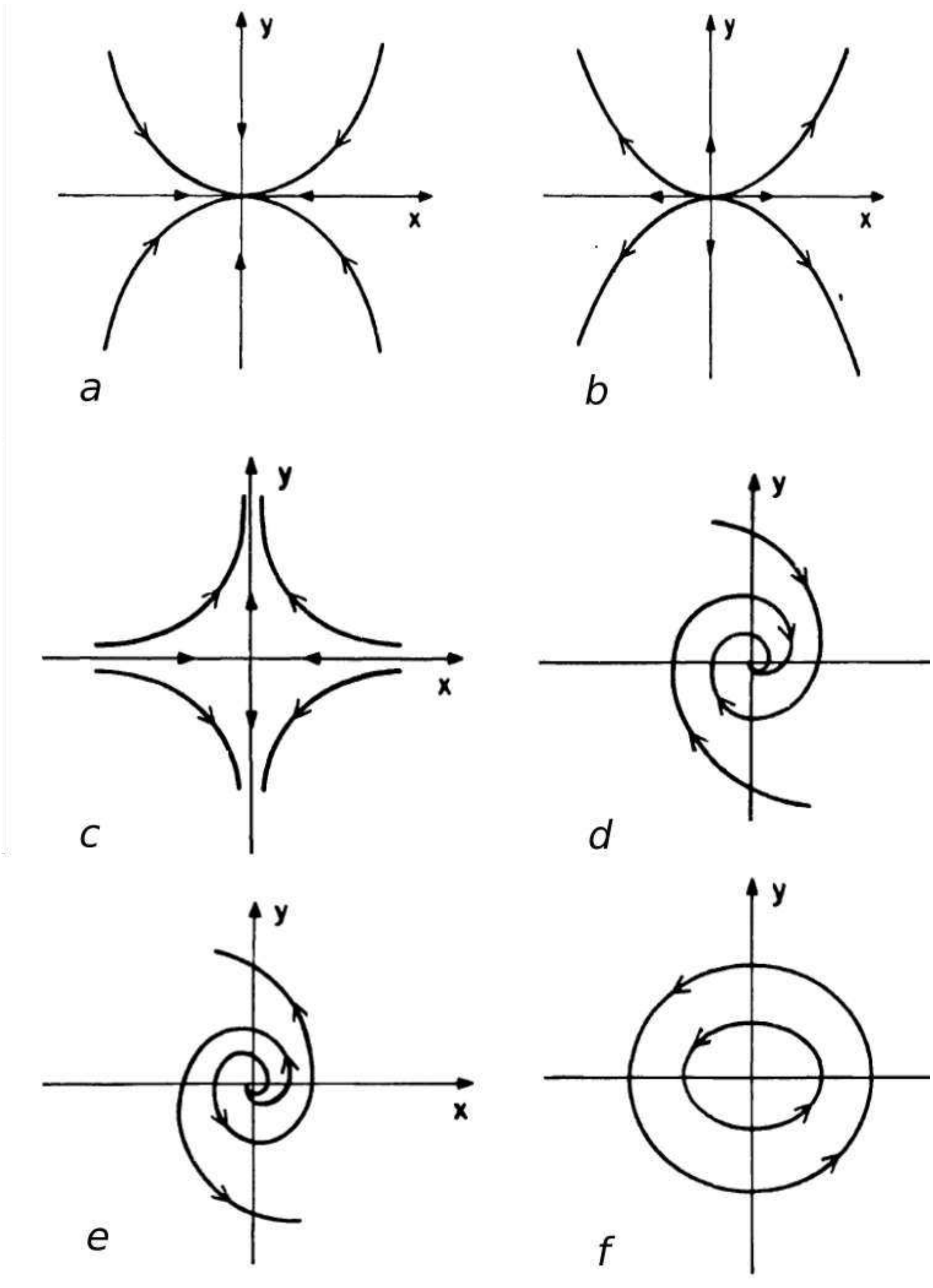

Figure 1.1: Different types of fixed points in two dimensions with displayed flows approaching or receding them. a) a stable node, b) an unstable node, c) a hyperbolic fixed point, d) a stable spiral point, e) an unstable spiral point and f) an elliptic point. Drawings are taken from [3].

Definition 1.1.4 (Bifurcation).
Let

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, r) \tag{1.11}$$

denote a system where $\boldsymbol{x} : \mathbb{R} \to \mathbb{R}^N$ is an unknown function and $r \in \mathbb{R}$ is a parameter.
The equation (1.11) has a *bifurcation* in point $(\boldsymbol{x}_0, r_0)$ if the behaviour of the solution in the neighborhood of x_0 is significantly changed at $r = r_0$.

By the significant change, we usually understand the creation or destruction of a fixed point or change in the nature of the dynamics.

1.2 Conservative Systems

Since the systems we are interested in are primarily conservative, it is convenient to define their meaning properly.

Definition 1.2.1 (σ-finite space).
A measure space is called *σ-finite* if its sample space is the countable union of finite measure sets.

Definition 1.2.2 (Conservative System).
Let $(X, \mathcal{B}, \mu, T)$ be a measure preserving transformation on an infinite σ-finite, measure space. A set $W \in \mathcal{B}$ is called *wandering*, if $T^{-n}W : n \geq 0$ are pairwise disjoint. A measure preserving transformation is called *conservative* if every wandering set has a measure zero.

The system $(X, \mathcal{B}, \mu, T)$ is a dynamical system with a Borel set $(X, \mathcal{B}$, measure μ and transformation T. Physically the set X is understood to be the phase space of a dynamical system. The transformation $T : X \to X$ is called *non-singular* if $\mu(T^{-1}\sigma) = 0 \quad \forall \sigma \in \mathcal{B} \leftrightarrow \mu(\sigma) = 0$, i.e. we can say a dynamical system is conservative if and only if it is non-singular.
We often read just the fact that conservative systems are dynamical systems that conserve the phase volume. For illustration, in a conservative system, momentum is preserved, and the domain of position and momentum phase point remains. On the contrary, in a dissipative system, the momentum continually decreases (as a function of position because the energy decreases with the motion). [1]

Considering the normal autonomous system of equations

$$\dot{\boldsymbol{x}} = \boldsymbol{F}(\boldsymbol{x}) \tag{1.12}$$

a continuous, real function $u(\boldsymbol{x})$ is called a *constant of motion* or a *first integral* if substituting an arbitrary solution into (1.12) leads to an expression independent of t. This function is constant on all trajectories and solves the relevant equations of motion. However, the following inequality $\mathrm{d}u/\mathrm{d}x^i \neq 0$ must hold [11].

[1] The quantities conserved under canonical transformations are called Poincare invariants. One of those invariants is the phase volume which always has to be preserved in our systems. $\int\int \sum_{i=1}^{n} dp_i dq^i = \int\int \sum_{i=1}^{n} dP_i dQ^i$ is the summation of areas projected onto the set of (p_i, q^i). [3].

1.3 Hamiltonian Mechanics

The Hamiltonian formulation of classical mechanics uses one scalar function, which contains complete information about the system's behaviour. According to [3], the Hamiltonian formulation is more convenient than Lagrangian formulation in terms of integrability[2] of the systems.

The generalized coordinates $q^1, \cdots, q^n$ altogether with generalized momenta $p_1, \cdots, p_n$ form a formal mathematical space - a *phase space*, a space of physical states of $2n$ dimension. A single function called *Hamiltonian* $H(q^i, p_i, t)$ carries all the information about the system's dynamics.
The phase space associated with Hamiltonian equations has some specific properties, e.g. the values of the phase space's coordinates at any time define the system completely at that specific time.
The function of time that represents the solution of the system is referred to as a *phase curve* or a *trajectory*, the motion along such curve is called a *phase flow*. Finally, different phase curves do not cross each other in *integrable systems*. A system is integrable if it has the same amount of constants of motion as it has degrees of freedom. The Hamiltonian equations of n dimensions are integrable if it is possible to find n independent *constants of motion*. Therefore, it is possible to plot solutions of the system with different initial conditions into one graph. The plotted figure is called a *phase portrait*.
Hamilton's canonical equations

$$\frac{\mathrm{d}q^j}{\mathrm{d}t} = \frac{\partial H}{\partial p_j}, \quad \frac{\mathrm{d}p_j}{\mathrm{d}t} = -\frac{\partial H}{\partial q^j}\,, \tag{1.13}$$

are equations which beautifully describe the motion in classical mechanics.
Finally, let us define the mathematical operation called Poisson brackets.

Definition 1.3.1 (Poisson Brackets)**.**
Let (q^i, p_i) be the conjugate coordinates and $f(q^i, p_i, t)$ and $g(q^i, p_i, t)$ functions. The binary operation of the form

$$\{f, g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i} \right)\,, \tag{1.14}$$

is called a *Poisson bracket*.

Poisson brackets are a powerful tool since they are conserved throughout a canonical transformation and thus are helpful in the transformation process.

Now we will take the Hamiltonian as a dynamical system. We will propose the action-angle variables and overview the perturbation theory. These are necessary tools in rewriting non-integrable Hamiltonians. They are later used at the introduction of the Melnikov method in the chapter 2.3.

1.3.1 Action-Angle Variables

A canonical transformation is a technique used to simplify the equations of motion. It is a transformation that preserves the form of Hamiltonian's equations

[2] To be defined in the following text.

(1.13) - the phase volume is conserved, and the whole information about the system is contained in the new Hamiltonian function $H'(Q^j, P_j)$, where (P_i, Q^i) denote the new set of conjugate coordinates. The Hamiltonian equations in the new variables read

$$\dot{Q}^i = \frac{\mathrm{d}H'(Q^j, P_j)}{\mathrm{d}P_i}, \ \dot{P}_i = -\frac{\mathrm{d}H'(Q^j, P_j)}{\mathrm{d}Q^i} \,. \tag{1.15}$$

There are infinitely many canonical transformations with four possible generating functions F_i [12]; for finding the action-angle variables, it is convenient to use the generating function of the $F_2(q^j, P_j)$ type for which the following equations apply

$$H'(Q^j, P_j) = H(q^j, p_j) + \frac{\mathrm{d}F_2}{\mathrm{d}t} \,, \tag{1.16}$$

$$\frac{\mathrm{d}F_2}{\mathrm{d}Q^j} = p_j, \quad \frac{\mathrm{d}F_2}{\mathrm{d}p_k} = Q^k \,. \tag{1.17}$$

For a one-degree-of-freedom system, the intention is to rewrite the Hamiltonian, so it only depends on one variable, i.e. the second variable is conserved. The action-angle variables are such variables where the conjugate coordinate increases by 2π after each period [3]. The equation for the action variable has the form

$$I = \frac{1}{2\pi} \oint \sum_j p_j(q^i, \alpha^j) \mathrm{d}q^j \,, \tag{1.18}$$

where α is the conserved quantity of $H'(I)$ Hamiltonian. The canonical equations for transformed Hamiltonian are given by

$$\dot{I} = -\frac{\partial H'(I)}{\partial \theta} = 0 \,, \tag{1.19}$$

$$\dot{\theta} = \frac{\partial H'(I)}{\partial I} = \omega(I) \,, \tag{1.20}$$

where $\omega(I)$ is the characteristic frequency of motion. The equation (1.19) is trivially zero because I is a constant of motion. From equations (1.19) and (1.20) θ and the generating function can be found in the form of an integral.

1.3.2 Classical Perturbation Theory

Most Hamiltonian systems are not entirely integrable; it is convenient to represent non-integrable systems with an integrable part and a small non-integrable perturbation.

$$H(\boldsymbol{p}, \boldsymbol{q}) = H_0(\boldsymbol{p}, \boldsymbol{q}) + \varepsilon H_1(\boldsymbol{p}, \boldsymbol{q}) \,, \tag{1.21}$$

where the perturbation parameter is assumed to be $\varepsilon \ll 1$. The aim is to represent the original Hamiltonian with an integrable part H_0 and a small corrections part H_1. Consider a one-degree-of-freedom problem. The main idea is to expand the solution $x(t)$ in a power series of ε that is

$$x(t) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \cdots \tag{1.22}$$

where $x_0(t)$ is the exact solution of the integrable part H_0. In canonical perturbation theory the system is described as a function of action-angle variables (I, θ) $H(I, \theta)$, which is to be replaced with new pair of variables (J, θ) so the new Hamiltonian K depends only on J[3]

$$H(I, \theta) = H_0(I) + \varepsilon H_1(I, \theta) + \varepsilon^2 H_2(I, \theta) + \mathcal{O}(\varepsilon^3) \,. \tag{1.23}$$

The original systems then become completely integrable. The generating function $S(J, \theta)$ is expanded in power series of ε following the new Hamiltonian $K(J)$ to be expressed as a power series of ε as well

$$\begin{aligned} K(J) =& K_0(J) + \varepsilon K_1(J) + \varepsilon^2 K_2(J) + \cdots \\ =& H_0(\frac{\mathrm{d}S}{\mathrm{d}\theta}) + \varepsilon H_1(\frac{\mathrm{d}S}{\mathrm{d}\theta}, \theta) + \varepsilon^2 H_2(\frac{\mathrm{d}S}{\mathrm{d}\theta}, \theta) + \cdots \,. \end{aligned} \tag{1.24}$$

The next step is to expand the S function in Taylor series inside the Hamiltonian H. The lastly obtained expression are then compared term-wise. The first terms in both Hamiltonians H and K are equal, the following terms are compared according to the power of ε.

$$H_0(J) = K_0(J) \,, \tag{1.25}$$

$$K_1(J) = \omega_0(J) \frac{\mathrm{d}S_1(\theta, J)}{\mathrm{d}\theta} + H_1(J, \theta) \,. \tag{1.26}$$

The same procedure is done for the higher-order terms $\mathcal{O}$. Expressing and calculating the Hamiltonian in the action-angle variables proves the possibility of rewriting the Hamiltonian into an integrable part and a small perturbation.

1.4 Chaos

Introducing of a *chaotic system* evolves from the study of dynamical systems. However, the words are usually just briefly mentioned, and the literature concentrates more on the methods of solving the system's behaviour. Even though the topic has been thoroughly studied for the last 50 years, it is challenging to find a precise definition. Most frequently, chaos is characterised just by sensitivity to minor changes in the initial conditions. The following section aims to define a chaotic system based on proper mathematical definitions, and the previous approach of [13], [14], [15], [9] and [8].

Definition 1.4.1 (Dense set).
Let (Ω, ρ) be a metric space and $X \subset \Omega$. We say that X is *dense* in Ω if in every neighborhood of every element $m \in \Omega$ there exists an element $x \in X$.

Let X be a set, $x \in X$ and $\phi : X \to X$.

Definition 1.4.2 (Orbit).
Given $x \in X$, the sequence $\{\phi^j(x)\}_{j=0}^{\infty}$ is called the *parametrized orbit* of ϕ through the point x. The set

$$\mathcal{O}^+(x) := \bigcup_{j=0}^{\infty} \{\phi^j(x)\} \subset X \tag{1.27}$$

is called an *orbit* of x under the mapping ϕ.

[3]When such a canonical transformation exists.

Definition 1.4.3 (Periodic Point).
A point $x \in X$ is called a *periodic point* of ϕ, if $\exists k \in \mathbb{N} : \phi^k(x) = x$.

Now we will briefly introduce the concept of topological transitivity and sensitive dependence on initial conditions.
Let (Ω, ϕ) be a topological dynamical system where Ω is a topological space and $\phi : \Omega \to \Omega$ a continuous map. Let $\mathcal{O}(x)$ denote an orbit and $\alpha(\phi(x))$ a set of limit points of the orbit sequence.

Definition 1.4.4 (Topologically Transitive System).
The dynamical system (Ω, ϕ) is called *topologically transitive* if $\alpha(\phi(x)) = \Omega$ for some $x \in \Omega$. Such x is called a *transitive point*.

More profound explanation of what is meant by topological transitivity is provided by [13], [14].

Definition 1.4.5 (Sensitive dependence on initial conditions).
Suppose (Ω, ρ) is a metric space. We say ϕ has *sensitive dependence on initial conditions* if there is a constant $\delta > 0$ with the property:

$$\forall x \in \Omega \, \forall \varepsilon > 0 \, \exists y \in \Omega : \; \rho(x, y) < \varepsilon \wedge \rho\left(\phi(tx), \phi(ty)\right) > \delta \quad \text{for some } t > 0 \,. \tag{1.28}$$

Now that we have formulated all the prerequisite terms, we can introduce the definition of chaos.

Definition 1.4.6 (Chaotic system).
A dynamical system (Ω, ϕ) is considered *chaotic* if

1. ϕ has at least two distinct periodic points which are dense in Ω ,
2. ϕ is topologically transitive ,
3. ϕ has sensitive dependence on initial conditions .

1.4.1 Poincaré's Surface of Section

The most valuable visualising technique of the motion is the surface of section suggested by Poincaré (followed by Birkhoff). The surface of section is a $2n-1$ dimensional subspace of a phase space of $2n$ dimension. For one given point, a "slice" of the phase space is taken, and at every time the corresponding trajectory intersects this area, the values of remaining coordinates and momenta are recorded. Considering a two-degree-of-freedom system that already has four dimension phase space and assuming there exists a constant of motion, the system can be visualised in a two-dimensional surface of section.

1.4.2 Smale Horseshoe Mapping

The trajectories in phase space of integrable systems cannot cross[4]; therefore, if the trajectories fold and repeatedly stretch from some unstable fixed point,

[4]Precisely; they cross exactly at fixed points, assuming they are continuous functions.

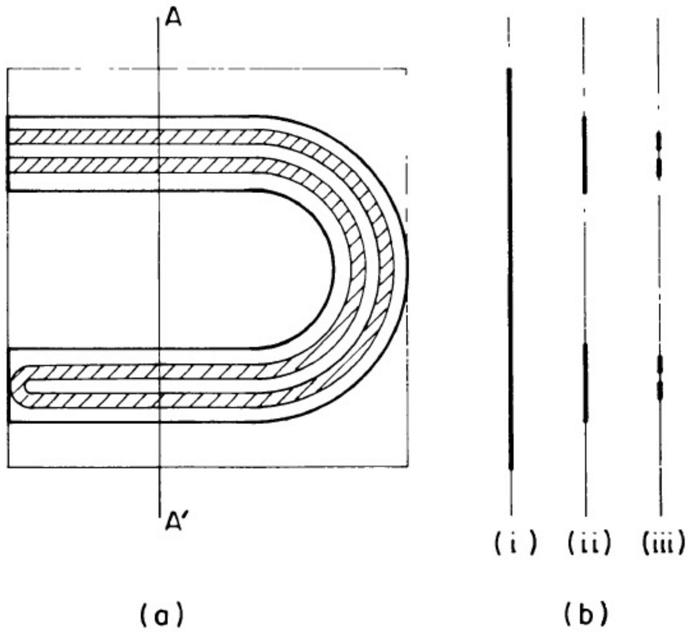

Figure 1.2: a) The characteristic rectangle of Smale Map with a folded object arose after n iterations. b) Successive cross-sections which gradually form a Cantor set structure. Taken from [3].

the resulted object is of vast topological complexity, mathematically speaking, corresponds to a Cantor set.

The simple model of this complexity is known as the Smale Horseshoe mapping. It is a two-dimensional map that consists of a sequence of stretching and folding geometrical operations. The procedure can be characterised on a rectangle that is firstly symmetrically stretched to more than twice the rectangle's width in x direction and consequently contracted in less than half the size of the rectangle in y direction. The object is then folded into the original rectangle without changing its initial area in the form of a horseshoe. The process is then iterated with the result of a complex structure resembling a horseshoe. There arise a set of points previously lying in the rectangle after n iterations located outside of it.

2. Homoclinic Orbits

This chapter will focus on defining homoclinic orbits, their related terms, and relationships with other later used concepts. We shall shortly discuss the Melnikov method of detecting chaotic behaviour and define the Melnikov integral. The references for this chapter are [8], [3], [16], [17], [18], [19], [20], [21], [22]. Now we shall define the last group of terms useful for proper understanding of the following text. We build on the terms from the end of the previous chapter - a map ϕ and set X. The definitions are inspired by [8].

Definition 2.0.1 (Invariant Set)**.** A set $A \subset X$ is called *invariant* under the map $\phi : X \to X$ if

$$\phi^{-1}(A) := \{x \in X | \phi(x) \in A\} = A\,. \tag{2.1}$$

Specifically, for $X = \mathbb{R}^n$ and a map $\phi : \mathbb{R}^n \to \mathbb{R}^n$ being a diffeomorphism, we can define the hyperbolic set.[1]

Definition 2.0.2 (Hyperbolic Set)**.** The subset V is called a *hyperbolic set* of the mapping ϕ if the following conditions hold

- V is compact and invariant under ϕ ,
- for every $x \in V$ there exists a splitting of the tangent space .

Definition 2.0.3 (Local Invariant Manifold)**.** The *local invariant manifolds* of the fixed point x_0 are defined as the sets

$$W^+_{loc} := \{x \in Q | \phi^j(x) \in Q\, \forall j \geq 0\,, \tag{2.2}$$

$$W^-_{loc} := \{x \in Q | \phi^{-j}(x) \in Q\, \forall j \geq 0\,, \tag{2.3}$$

where Q is the neighborhood of the fixed point.

The following part introduces the expression *transversal*, which is an important term for the discovery of chaotic layers. The definition is well known in geometry and differential topology and is adjusted according to [23].

Definition 2.0.4 (Transversality)**.** Let $f : M \to N$ be smooth and $L \subset N$ be a submanifold. We say that f is transversal to L and write $f \pitchfork L \leftrightarrow \forall p \in f^{-1}(L) : T_p f(T_p M) + T_{f(p)} L = T_{f(p)} N$.

The terms $T_p f(T_p M)$ and $T_{f(p)} L$ denote the tangent bundle of the map f and L, respectively.

Definition 2.0.5 (Transversal Intersection)**.** We say U and V are *transverse* (or that U and V *intersect transversally*) and write $U \pitchfork V$ if at every point $x \in U \cap V$

$$T_x U + T_x V = T_x M\ .$$

[1] The definition of a hyperbolic set can be found, for instance, in [8]. We will not build the whole mathematical prerequisites for this definition because if we did, we could write just about the mathematical properties. What will be said in the definition is what has the most importance for our understanding of the term.

2.1 Homoclinic Points

A hyperbolic fixed point H is characterized by two stable manifolds, H^+, and two unstable manifolds, H^-. The points on H^+ approaches the hyperbolic fixed point, whereas the points on H^- recede from a fixed point H. The smooth joining of the stable and unstable manifold forming a single smooth loop is not common. An example of how the manifolds H^+, H^- can intersect is displayed in the figure 2.2. If the intersection points contain manifolds from the same family,[2] the points of intersection are called *homoclinic points*. The points are called *heteroclinic* if they emanate from different families.

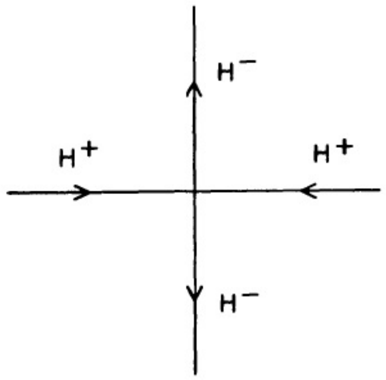

Figure 2.1: Stable manifolds H^+ that access the hyperbolic fixed point H in the middle. And unstable manifolds H^- which recede from the same hyperbolic fixed point H. Taken from [3].

2.2 Homoclinic Orbits

A *separatrix* is a boundary dividing two different modes of behaviour of a dynamical system. If the curve connects two different fixed points, it is called a hyperbolic orbit; if the curve tends to the same fixed point in both past and future infinity, it is called a homoclinic orbit. Both orbits separate two parts of the manifolds with different properties, i.e. stable and unstable ones.

The homoclinic orbits are convenient for finding a chaotic behaviour. When a small perturbation is done, the behaviour of the homoclinic orbit can reveal the chaotic tendencies in the system. The Melnikov method is a manner of establishing the nature of a homoclinic orbit after a small perturbation.

[2]Precise description of *family* can be found, e.g. in [3].The family of points means there are new points that form around the original fixed point. Let us state that this term arises from the Kolmogorov-Arnold-Moser theorem, which is one of the most important concepts in classical chaos theory. However, there is no direct use of it in our calculations, and it is not possible to define KAM theorem without the demanding theory behind it. Therefore by this remark, we shall just accent its existence and importance.

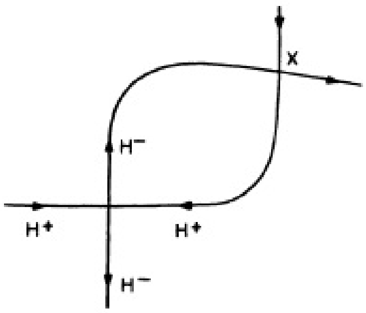

Figure 2.2: An intersection of stable manifold H^+ and unstable manifold H^-, arising from the same hyperbolic fixed point, to give homoclinic point X. Taken from [3].

2.3 Melnikov Method

Melnikov method is an analytical approach for studying the behaviour of near-integrable systems and can be shown in terms of geodesic motion around Schwarzschild black holes. The near-integrable systems mean an integrable Hamiltonian with a periodically dependent perturbation term (where the dependency is on the external source's time). The aim is to identify the invariant sets of the perturbed systems, such that the local dynamics in those sets is chaotic. There are known techniques of identifying the invariant subsets with chaotic behaviour in places where Smale horseshoe exists. Their existence itself also indicates the complexity of the dynamical systems.

Let us consider a one-degree-of-freedom system in which the Hamiltonian can be expressed in form according to the perturbation theory. Let q, p be the conjugate coordinates defining the phase space, t a time and ε a small perturbation parameter.

$$H(q,p,t;\varepsilon) = H_0(q,p) + \varepsilon H_1(q,p,t) + \mathcal{O}(\varepsilon^2) \,. \tag{2.4}$$

The integrable part H_0 has a hyperbolic fixed point $[Q_0, P_0]$ connected to itself by a homoclinic orbit $[q_0(t), p_0(t)]$ and H_1 is a periodic function of t. The exact procedure of calculating the Melnikov integral

$$M(t_0) = \int_{-\infty}^{+\infty} \{H_0, H_1\}(q_0(t), p_0(t), t_0 + t)\mathrm{d}t \,, \tag{2.5}$$

is precisely described in [18],[24],[17]. The term $\{\,\}$ in (2.5) denotes the Poisson brackets. To solve (2.5) the homoclinic orbit needs to be found. Qualitatively, the Melnikov method is a good approach for finding a chaotic behaviour around an already established orbit. It does include only one specific orbit; on the contrary, it is not necessary to know the exact solution of the perturbed system for solving the Melnikov integral.

Smale horseshoe (taken as an indicator of chaos) occurs, where an asymptotic orbit splits under perturbation, i.e. two daughter orbits split transversally.

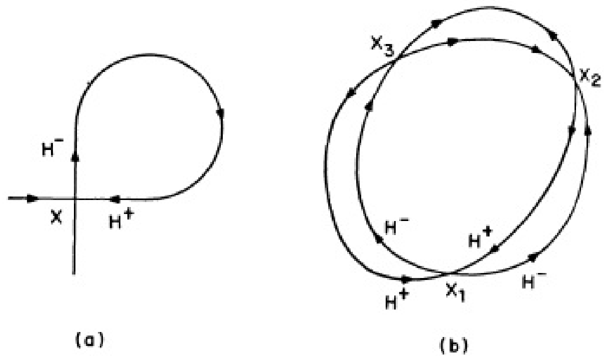

Figure 2.3: (a) A homoclinic orbit receding and returning to one hyperbolic fixed point X. (b) Three hyperbolic fixed points of the same family X_1, X_2, X_3, i.e. homoclinic points, through which a homoclinic orbit moves and eventually returns to its starting point. Taken from [3].

2.4 Unstable Circular Orbits

In stationary and axially symmetric space-times of isolated black holes, the geodesic dynamics is completely integrable, thus regular. However, due to the presence of a separatrix, the motion can quickly become chaotic under a perturbation. Since the geodesic equations are nonlinear and hence can behave chaotically under small displacements, the importance of the homoclinic orbit is fundamental for the evolution of the dynamics.

The homoclinic orbit separates the stable and unstable manifolds of a hyperbolic invariant set, which in general relativity is commonly referred to as *unstable circular orbit*. The derivation of the relationship between those two terms, specifically for the Kerr metric, is to be found in [21]. Hence, the critical conclusion is their presence and connection to each other - the homoclinic orbit whirls an infinite amount of times, and it asymptotes to the unstable circular orbit.

3. Geodesic Motion in Circular Space-times

The geodesic equations represent the equations of motion of free-falling particles in general relativity.[1] In this chapter, we will introduce the main concepts of geodesic motion in symmetrical fields around black holes. Specifically, we will concentrate on the Schwarzschild space-time as the most straightforward metric, and we will try to reproduce it in the following chapters. Furthermore, as an example of a generally circular (but not static!) space-time, we will introduce the Kerr metric. As a preface, we will define the effective potential method and formally compare relativistic and Newtonian cases as simple examples.

From now on, let us express the equations in the geometrized units $G = c = 1$, let M be the mass of the centre of symmetry and $\mathcal{M}$ the mass of the external source. Astrophysically, the mass of the external source is considered to be much less than the mass of the black hole, $\mathcal{M} \ll M$.[2]

3.1 Geodesic Motion as a Dynamical System

In general relativity, the gravitational field itself must be treated as a dynamical system [25]. This dynamical system can be expressed in terms of Hamiltonian dynamics with constraints. Since space-time is four-dimensional, with one variable being time, we can understand it as a history or evolution of space.[3]

Geodesic motion is a motion that occurs in the absence of non-gravitational forces. The motion is regular near isolated black holes [19], the aim is to examine how the behaviour changes when an external source is symmetrically placed into the black hole's field, i.e. perturbing the system so that a possible change of its behaviour can be examined. The potential existence of chaos is due to the presence of an unstable orbit.

A massive particle affected only by a gravitational force is said to be free, and its motion is described by the geodesic equation. The equation for a time-like geodesic can be written in the form

$$\frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} + \Gamma^\mu_{\kappa\lambda} u^\kappa u^\lambda = 0 \,, \tag{3.1}$$

where

$$u^\mu := \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \,, \tag{3.2}$$

is the four-velocity, $\Gamma^\mu_{\kappa\lambda}$ are the appropriate components of affine connection and τ is the proper time. In general relativity, the affine-connection coefficients are

[1]We emphasize that we shall assume the reader's basic knowledge of the general relativistic concepts and will expressly state only terms directly related to the calculations.

[2]Generally, the inequality $\mathcal{M} \ll M$ does not need to be true.

[3]Space-time does not change or evolve throughout the time. Although to verbally express the history and future, i.e. every possible behaviour in space-time, the word evolution seems correct.

represented by the Christoffel symbols of the second kind

$$\Gamma^{\mu}_{\kappa\lambda} := \frac{1}{2} g^{\mu\sigma} \left(g_{\sigma\kappa,\lambda} + g_{\lambda\sigma,\kappa} - g_{\kappa\lambda,\sigma} \right) \, . \tag{3.3}$$

The equation (3.1) completely describes the geodesic dynamics. It is a system of four generally nonlinear ordinary differential equations. There is no conventional method of solving them; however, they can be simplified by constants of motion or restricting the motion itself. The space-time symmetries also play a significant role in the final form of (3.1).

3.2 Schwarzschild Metric

Theorem 3.2.1 (Birkhoff theorem)**.**
In four dimensions, the metric of the Schwarzschild black hole is the unique spherically symmetric solution of the vacuum Einstein field equations.

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right) \mathrm{d}t^2 + \frac{r}{r - 2M} \,\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta \,\mathrm{d}\phi^2) \, . \tag{3.4}$$

This metric is the unique solution expressed in Schwarzschild coordinates(t, r, θ, ϕ) which are coordinates of spherical nature. θ and ϕ are angles on any sphere where $t = \text{const}$, $r = \text{const}$. r is proportional to the proper area of the specific sphere and the time t represents the proper time at radial infinity of an observer staying at rest. [26]

3.3 Circular Space-times

The metric of every circular space-time can be written in isotropic spheroidal coordinates (t, ϕ, r, θ) in the form

$$\mathrm{d}s^2 = -N^2 \,\mathrm{d}t^2 + g_{\phi\phi}(\mathrm{d}\phi - \omega \,\mathrm{d}t)^2 + \frac{e^{2\zeta}}{N^2}(\mathrm{d}r^2 + r^2 \mathrm{d}\theta^2) \, , \tag{3.5}$$

where $N^2 := -g_{tt} - g_{t\phi}\omega$ and $\omega := -g_{t\phi}/g_{\phi\phi}$. [4] Circular space-time is an axially symmetric, stationary and orthogonally transitive space-time which means there must exist a global meridional plane, i.e. a plane everywhere perpendicular to the two existing Killing symmetries [26]. The metric does not explicitly depend on t and ϕ which describes the stationarity and axial symmetry and leads to the conservation of energy $\mathcal{E} = -u_t$ and azimuthal angular momentum $\ell = u_\phi$ in the geodesic motion. In addition there is one more constant of motion that can be expressed from the equation of four-velocity normalization given by

$$g_{\mu\nu} u^\mu u^\nu = -1 \, . \tag{3.7}$$

[4] The metric (3.5) can be eventually rewritten into

$$\mathrm{d}s^2 = -e^{2\nu} \,\mathrm{d}t^2 + B^2 r^2 \sin^2\theta e^{-2\nu}(\mathrm{d}\phi - \omega \,\mathrm{d}t)^2 + e^{2\zeta - 2\nu}(\mathrm{d}r^2 + r^2 \mathrm{d}\theta^2) \, , \tag{3.6}$$

where B, ω, ν, ζ are functions of r, θ.

For time-like geodesics, the four-velocity normalization thus reads

$$g^{\mu\nu}u_\mu u_\nu = g^{tt}\mathcal{E}^2 - 2g^{t\phi}\mathcal{E}\ell + g^{\phi\phi}\ell^2 + g^{rr}(u_r)^2 + g^{\theta\theta}(u_\theta)^2 = -1\,. \tag{3.8}$$

The inverse-metric components are given by

$$g^{tt} = -\frac{1}{N^2}\,,\quad g^{t\phi} = -\frac{\omega}{N^2}\,,\quad g^{\phi\phi} = -\frac{g_{tt}}{g_{\phi\phi}N^2}\,,\quad g^{rr} = \frac{1}{g_{rr}}\,,\quad g^{\theta\theta} = \frac{1}{g_{\theta\theta}}\,, \tag{3.9}$$

hence the normalization yields equation for meridional-plane motion in the form

$$e^{2\zeta}[(u^r)^2 + r^2(u^\theta)^2] = (\mathcal{E} - \omega\ell)^2 - N^2\left(1 + \frac{\ell^2}{g_{\phi\phi}}\right)\,. \tag{3.10}$$

From the metric (3.8) rewritten into more general form

$$\mathrm{d}s^2 = g_{tt}\,\mathrm{d}t^2 + g_{\phi\phi}\,\mathrm{d}\phi^2 + 2g_{t\phi}\,\mathrm{d}t\,\mathrm{d}\phi + g_{rr}\,\mathrm{d}r^2 + g_{\theta\theta}\,\mathrm{d}\theta^2\,, \tag{3.11}$$

and by considering the equation (3.7) one can obtain the generic equation for radial and latitudinal motion

$$g_{rr}(u^r)^2 + g_{\theta\theta}(u^\theta)^2 = -1 - g_{tt}(u^t)^2 - g_{\phi\phi}(u^\phi)^2 - 2g_{t\phi}u^t u^\phi\,, \tag{3.12}$$

which after substituting values of the conserved energy and angular momentum

$$\mathcal{E} := -u_t = -g_{tt}u^t \quad \ell := u_\phi = g_{\phi\phi}u^\phi\,, \tag{3.13}$$

is of a general form

$$g_{rr}(u^r)^2 + g_{\theta\theta}(u^\theta)^2 = 2\frac{g^{t\phi}}{g_{tt}g_{\phi\phi}}\mathcal{E}\ell - \frac{1}{g_{tt}}\mathcal{E}^2 - \frac{1}{g_{\phi\phi}}\ell^2 - 1\,. \tag{3.14}$$

3.4 Effective Potential

The method of an effective potential is a valuable technique that can qualitatively describe the motion of a particle. Let us assume the free test particle's mass $m \neq 0$ and the space-time is axially symmetric.

If there are two constants of motion of the system - the energy $\mathcal{E}$ and angular momentum ℓ - the equation for radial and latitudinal velocities is of form (3.14)

$$g_{rr}(u^r)^2 + g_{\theta\theta}(u^\theta)^2 = 2\frac{g^{t\phi}}{g_{tt}g_{\phi\phi}}\mathcal{E}\ell - \frac{1}{g_{tt}}\mathcal{E}^2 - \frac{1}{g_{\phi\phi}}\ell^2 - 1\,. \tag{3.15}$$

If the motion befall in the equatorial plane, i.e. $\theta = \pi/2$, which holds not only for spherically symmetric potentials but also for the equatorial motion in Kerr-Newmann; the equation is reduced to

$$g_{rr}(u^r)^2 = \frac{2g^{t\phi}}{g_{tt}g_{\phi\phi}}\mathcal{E}\ell - \frac{1}{g_{tt}}\mathcal{E}^2 - \frac{1}{g_{\phi\phi}}\ell^2 - 1\,. \tag{3.16}$$

Suppose the term $g^{t\phi} = 0$ (which must be trivially zero for static space-times) and the equation (3.16) is divided by the term g_{rr}. We can then express the equation for radial motion in convenient form[5]

$$(u^r)^2 = -\frac{\mathcal{E}^2}{g_{rr}g_{tt}} - V_{\text{eff}}^2 \,, \tag{3.17}$$

where

$$V_{\text{eff}}^2 := \frac{\ell^2}{g_{\phi\phi}g_{rr}} + \frac{1}{g_{rr}} \,, \tag{3.18}$$

is an abstract variable called an *effective potential.* If the metric has a diagonal form, the metric components are bounded by the following formula

$$g^{\mu\nu} = \frac{1}{g_{\mu\nu}} \,. \tag{3.19}$$

Specifically, if the term

$$-\frac{g^{tt}}{g_{rr}} = -\frac{1}{g_{tt}g_{rr}} = 1 \,, \tag{3.20}$$

which holds e.g. for Schwarzschild or Reissner–Nordström metric, the equation (3.17) is simplified to the known form[6]

$$(u^r)^2 = \mathcal{E}^2 - V_{\text{eff}}^2 \,. \tag{3.21}$$

From (3.17) it is clear that $\mathcal{E}^2 - V_{\text{eff}}^2 \geq 0$ and thus $\mathcal{E}^2 \geq V_{\text{eff}}^2$. The effective potential represents the smallest energy value a particle can have at the exact location.

The motion of the free particle depends on the shape of the effective potential V_{eff}, and the value of conserved energy $\mathcal{E}$ and thus their relation; i.e. the difference $\mathcal{E}^2 - V_{\text{eff}}^2$ tells how big is the magnitude of the velocity going to be. The possible outcomes depend on the effective potential. Qualitatively there are a few remarks which are to be given. If V_{eff} has local extremes, the minimum corresponds to a stable circular orbit and the maximum to an unstable circular orbit. Circular orbits generally depend on the number of occurrences of the extremes and then - to stress once again - on the concrete value of energy. For fixed values of $\mathcal{E}$ and ℓ, the value of r, which corresponds to the maximum effective potential V_{eff} and therefore the unstable circular orbit, will be denoted by R_0. For a homoclinic orbit asymptotically approaching that circular orbit, the second turning point, also called an apocentre, will be labelled as $r_{\max}$. The points are found as roots of the equations $V_{\text{eff}} = 0$ and $\mathrm{d}V_{\text{eff}}/\mathrm{d}r = 0$.[7] The existence of a local maximum of the effective potential can only occur in the relativistic space-time and is its characteristic feature - there always exists an unstable circular orbit. [16]

[5]The term $-\frac{1}{g_{rr}g_{tt}}$ will have a positive sign since the matric's signature is $(-,+,+,+)$.

[6]Even if the term $-\frac{g^{tt}}{g_{rr}}$ does not equal 1, the equation (3.17) can still be successfully used.

[7]The time a particle needs to get to a certain point (e.g. the point of an unstable orbit R_0), is calculated from an integral. This integral, nonetheless, can diverge, and the desired time then tends to infinity. The convergence of the integral generally depends on the shape of the effective potential and can only be examined in concrete cases.

Another case is when the line portraying the value of energy intersects with the function of the effective potential in just one point - if this happens on the right side of the curve, it means the motion of the particle is unbounded, and the particle can escape to infinity. If this happens from the left side, the particle will be attracted to the origin, i.e. the centre of symmetry.
If the value of $\mathcal{E}$ is less than the value of two maxima, the particle's motion will be bounded in the interval defined by those two points.

For the spherically symmetric fields $V_{\text{eff}}(r)$ which satisfy (3.20), the relativistic equation for the radial motion was said to be (3.17). Let us present the well known radial equation, in dependence on the reciprocal radius $u := \frac{1}{r}$, for Schwarzschild black hole

$$\left(\frac{\mathrm{d}u}{\mathrm{d}\theta}\right)^2 = \frac{\mathcal{E}^2 - V_{\text{eff}}^2}{\ell^2} , \quad V_{\text{eff}}^2 = (1 - 2Mu)(1 + \ell^2 u^2) , \tag{3.22}$$

where we used

$$v^r = \frac{\mathrm{d}r}{\mathrm{d}t} \Leftrightarrow \frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\mathrm{d}r}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t} = -\ell\frac{\mathrm{d}u}{\mathrm{d}\theta} . \tag{3.23}$$

To show that formally the relativistic and Newtonian expressions for the effective potential look similar, let us write the equation for the radial motion in Keplerian (central) field with Newtonian potential, which is given by

$$V(r) = -\frac{M}{r} . \tag{3.24}$$

The equation of radial motion in terms of a reciprocal radius $u(\phi) = \frac{1}{r}$ is denoted by

$$\left(\frac{\mathrm{d}u}{\mathrm{d}\phi}\right)^2 = \frac{2(\mathcal{E} - V_{\text{eff}}(u))}{\ell^2} , \tag{3.25}$$

where we used the identity

$$\dot{r} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{u(\phi(t))}\right) = -\frac{1}{u^2}\frac{\mathrm{d}u}{\mathrm{d}\phi}\dot{\phi} = -\frac{1}{u^2}\frac{\mathrm{d}u}{\mathrm{d}\phi}\frac{L}{mr^2} = -\frac{L}{m}\frac{\mathrm{d}u}{\mathrm{d}\phi} . \tag{3.26}$$

From equations (3.22) and (3.25) the similarities in terms of formulation are immediately obvious. The analogy is highly convenient since the use of pseudo-Newtonian potentials in relativity is not unusual (as we will see in the next chapter 4). The graphs that display the curves of the Newtonian and relativistic case are shown in figures 3.2 and 3.1, respectively. The existence of local extremes is observable - there is no local maximum for the Newtonian case. And even though $V_{\text{eff}} \to 0$ as $r \to \infty$ for both cases, there is a difference for $r \to 0$. The relativistic potential $V_{\text{eff}} \to 0$ in contrast with the Newtonian potential, which tends to infinity $V_{\text{eff}} \to \infty$.

Suppose now the metric is not static and hence the term $g^{t\phi} \neq 0$. The equation (3.16) can still be rewritten, so the radial and latitudinal velocities are on one side of an equation and the rest of the terms on another. Qualitatively (3.16) can still be solved in the same manner. Nonetheless, we trivially cannot separate the terms $\mathcal{E}$ and ℓ.

To conclude the results of this section, let us state that the method of effective potential is suitable for both relativistic and non-relativistic examination of the system's behaviour. It is a qualitative method that allows us to compare the conserved value of energy with the shape of an abstract variable - an effective potential. The techniques are generally the same; the task's difficulty depends solely on the symmetry of the field and the shape of the effective potential.

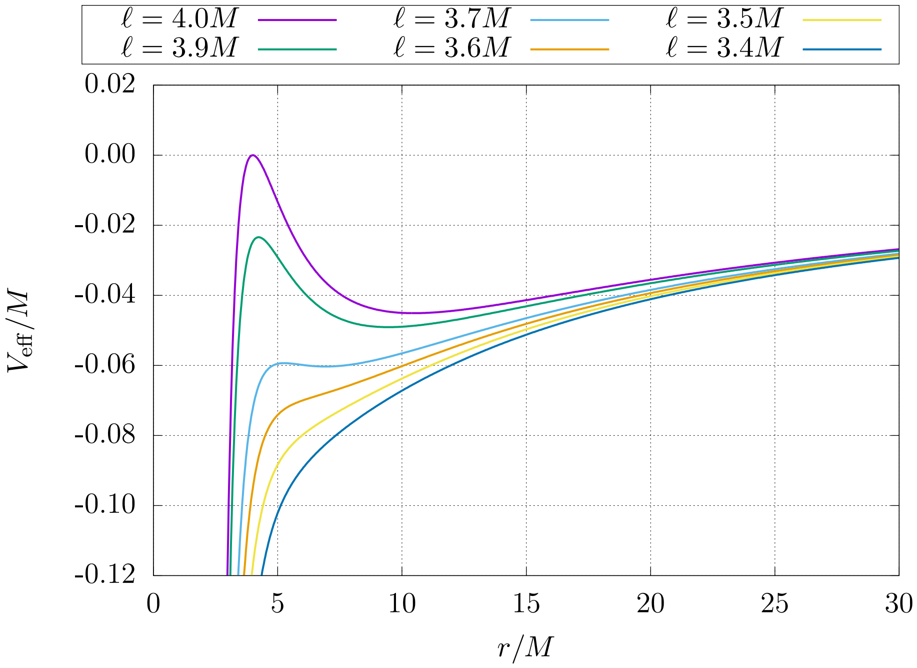

Figure 3.1: The example of the radial shape of the effective potential of a pseudo-Newtonian potential mimicking the relativistic case. As opposed to the classical Newtonian picture (see Fig. 3.2), there are visible both extremes - a maximum and a minimum - for certain combinations of parameters M and ℓ. Since the graph is plotted for Paczyński-Wiita (to be introduced in 4) potential, the condition $\ell > \sqrt{27/2}M$ must hold for the extremes in order to exist. The value $M = 1$ is fixed, and the values of ℓ vary so the case with no maximum, and hence no unstable orbit, can be observed.

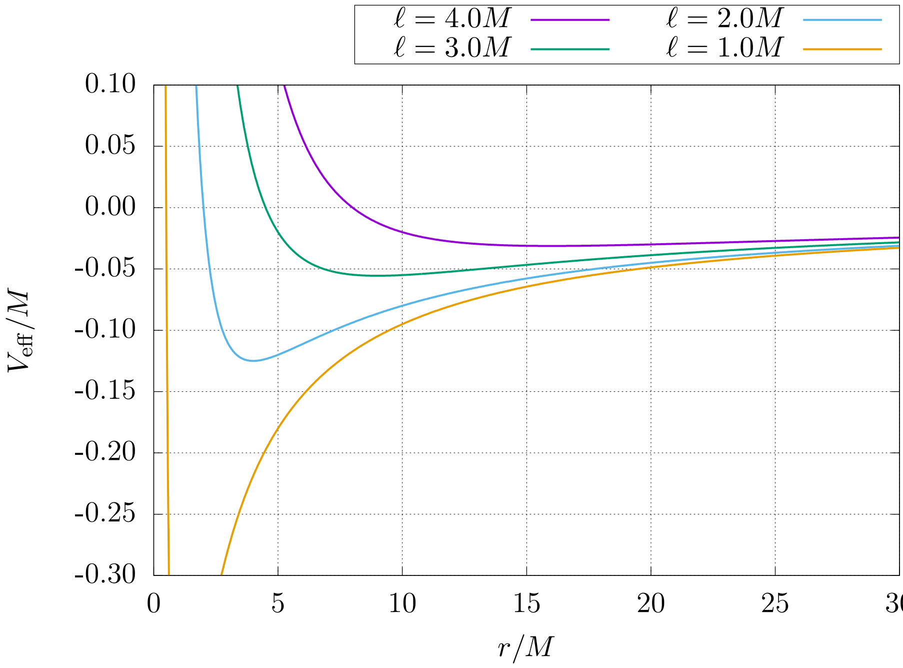

Figure 3.2: An example of the Newtonian effective potential for different values of the conserved angular momentum ℓ, where the mass of the centre of symmetry is set to be $M = 1$. Since the energy $\mathcal{E}$ is conserved, its given constant value is the same for every r and there exist points of turnover denoted by the intersection of the horizontal line which represents the constant $\mathcal{E}$ with the curve of the effective potential $V_{\text{eff}} = \mathcal{E}$, which means $\mathrm{d}u/\mathrm{d}\phi = 0$, i.e. $\dot{r} = 0$. The number of turnover points depends on the constant value of conserved energy.
For $\mathcal{E} > 0$, there exists one turnover point, and the motion is unbounded, i.e. hyperbolic. For $\mathcal{E} = 0$, the motion is parabolic, and it is the motion between the two different dynamical behaviours. Finally, if $\mathcal{E} < 0$, the motion is elliptical and bounded between two turnover points. There also exists a unique circular motion for the minimum of V_{eff}.
Bellow the effective potential curve graph, there lies a forbidden region $V_{\text{eff}} > E$. The Newtonian case's only possible extreme is the minimum effective potential, corresponding to the stable circular orbit. Although the extreme might not occur for every combination of parameters.

4. Pseudo-Newtonian Potentials

Some of the features of Schwarzschild space-time is possible to obtain without using general relativity. Firstly this was suggested by Paczyński, and Wiita [27]. Their reproduction of this specific space-time was done by introducing a new potential.

Potentials that describe the general but symmetric space-times in means of interacting forces in flat space-time are now called pseudo-Newtonian. They are adjustments of Newtonian potential which has a known form $V_{NW} = -\frac{M}{r}$ with M being the mass of the centre of symmetry. This method is used because general relativistic calculations can be practically non-applicable in strong fields, or relativistic interactions might not necessarily happen in weak fields. They are used to reproduce the structure of circular orbits in relativistic space-times [28]. Different potentials can mimic some black hole's features qualitatively well, while other characteristics remain misinterpreted.; this is observed by using the phase space portrait. While examining the dynamics of massive free test particles, it is convenient to compare their dynamical portraits and their changes in dependence on outer parameters. So far the Paczyński-Wiita potential

$$V_{\mathrm{PW}} = -\frac{M}{r-2M} , \tag{4.1}$$

where the $r = 2M$ represents the black hole's event horizon, has been the most effective in reproducing the stable circular orbit [29]. The list of other commonly used pseudo-Newtonian potentials can be found in [28]. Another approach of verifying their relevance to the black hole space-time - specifically, the Paczyński-Wiita potential, is also to be found there.[1]

The pseudo-Newtonian potentials suggested in [30] were used in this chapter as a replacement to the classical description of the space-time around the static black-hole. Their qualitative properties were already verified numerically; these computations were made to find the appropriate homoclinic orbits analytically. A concrete method for finding the orbit was previously suggested in [24] and the same approach was used here. The general form of the effective potential is

$$V_{\mathrm{eff}} = V + \frac{\ell^2}{2r^2} , \tag{4.2}$$

where V stands for the pseudo-Newtonian potential.

4.1 Logarithmic Potential

For the logarithmic potential (proposed in [30])

$$V_{\ln} = \frac{1}{3}\ln\left(1 - \frac{3M}{r}\right) , \tag{4.3}$$

[1] Although Paczyński-Wiita potential has been used the most to study the accretion discs - it produces the least satisfying result for the radiated spectra of the source in a circular orbit. [28].

the equation for radial velocity is denoted by

$$\frac{1}{2}(v^r)^2 = \mathcal{E} - V_{\text{eff}} \ , \tag{4.4}$$

where

$$V_{\text{eff}} := \frac{\ell^2}{2r^2} + \frac{1}{3}\ln\left(1 - \frac{3M}{r}\right) . \tag{4.5}$$

Since the value of (4.4) must be greater than zero, the effective potential V_{eff} must be smaller than the energy $\mathcal{E}$. The fixed points are found as extremal points of $V_{\text{eff}} = 0$. The equation $\mathrm{d}V_{\text{eff}}/\mathrm{d}r = 0$ has two roots. The smaller root denoted by R_0 corresponds to the local maximum of the potential and represents the unstable circular orbit and the second root denoted by r_{min} corresponds to the local minimum and represents the stable orbit.

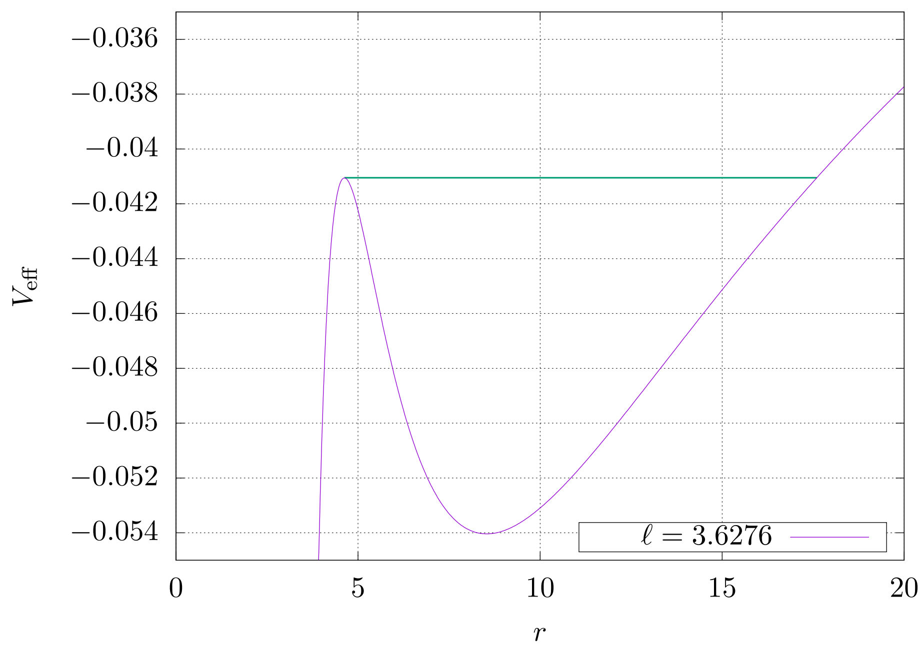

Figure 4.1: A radial shape of the effective potential with corresponding homoclinic orbit for logarithmic pseudo-Newtonian potential. Both local extremes are visible in the graph. The minimum corresponds to the stable circular orbit; the maximum corresponds to the unstable circular orbit. The homoclinic orbit's energy level is drawn in blue, and it connects the maximal value at R_0 with the apocentre at $r_{\max}$. The graph is plotted for the values $M = 1$, $\ell = 3.6276M$.

From $\mathrm{d}V_{\text{eff}}/\mathrm{d}r = 0$ the equations for R_0 and ℓ^2, respectively, are expressed. Those expressions correspond to the unstable circular orbit.

$$R_0 = \frac{\ell}{2M}\left(\ell - \sqrt{\ell^2 - 12M^2}\right) , \tag{4.6}$$

$$\ell^2 = \frac{MR_0^2}{R_0 - 3M} . \tag{4.7}$$

The energy level $\mathcal{E}_0$ equals to the value of the effective potential V_{eff} at R_0. Its value is therefore obtained by substituting the (4.6) into the expression for V_{eff}

$$\mathcal{E}_0 := \frac{M}{2R_0}\frac{1}{1-\frac{3M}{R_0}} + \frac{1}{3}\ln\left(1-\frac{3M}{R_0}\right) \tag{4.8}$$

For further computations it is convenient to rewrite (4.4) and (4.5) in terms of the reciprocal radius $u := \frac{1}{r}$

$$\frac{1}{2}(v^r)^2 = \mathcal{E} - V_{\text{eff}}(u)\,, \tag{4.9}$$

$$V_{\text{eff}}(u) = \frac{\ell^2 u^2}{2} + \frac{1}{3}\ln(1-3Mu) \tag{4.10}$$

By the same procedure the roots of $\mathrm{d}V_{\text{eff}}/\mathrm{d}u$ are determined. U_0 thus coincides with the previously calculated $1/R_0$ and is of form

$$U_0 = \frac{\ell + \sqrt{\ell^2 - 12M^2}}{6M\ell}\,. \tag{4.11}$$

After rewriting the radial velocity in terms of the reciprocal radius ([24])

$$v^r = \frac{\mathrm{d}r}{\mathrm{d}t} \Leftrightarrow \frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\mathrm{d}r}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t} = -\ell\frac{\mathrm{d}u}{\mathrm{d}\theta}\,, \tag{4.12}$$

the equation for radial motion is given by

$$\left(\frac{\mathrm{d}u}{\mathrm{d}\theta}\right)^2 = \frac{2}{\ell^2}\left[\mathcal{E} - \frac{\ell^2u^2}{2} - \frac{1}{3}\ln(1-3Mu)\right]\,. \tag{4.13}$$

Substituting $\mathcal{E}_0$ from (4.8) - in terms of the reciprocal radius u - into the equation (4.13) can be rewritten into

$$\left(\frac{\mathrm{d}u}{\mathrm{d}\theta}\right)^2 = \frac{1}{\ell^2}\left[K - \ell^2u^2 - \frac{2}{3}\ln(1-3Mu)\right]\,, \tag{4.14}$$

where

$$K := \frac{MU_0}{1-3MU_0} + \frac{2}{3}\ln(1-3MU_0)\,, \tag{4.15}$$

is a constant term. Equation (4.14) is not possible to rewrite into an integrable form and hence impossible to solve analytically.

4.2 Paczyński-Wiita Potential

Analogical method for finding the homoclinic orbit is applied to the Paczyński-Wiita potential

$$V_{\text{PW}} = -\frac{M}{r-2M}. \tag{4.16}$$

The equation for radial velocity reads (for illustration we have plotted different values of the effective potential into Fig. 3.1)

$$\frac{1}{2}(v^r)^2 = \mathcal{E} - \left[\frac{\ell^2}{2r^2} - \frac{M}{r-2M}\right] =: \mathcal{E} - V_{\text{eff}} \ , \tag{4.17}$$

which after differentiation with respect to r has the form

$$\frac{\mathrm{d}V_{\text{eff}}}{\mathrm{d}r} = -\frac{\ell^2}{r^3} + \frac{M}{(r-2M)^2} = 0 \ . \tag{4.18}$$

Lastly it is possible to express ℓ^2 as

$$\ell^2 = \frac{r^3 M}{(r-2M)^2} \ , \tag{4.19}$$

from the equation (4.18) .

Calculating the roots of the cubic equation (4.18) with respect to r is analytically possible by using Cardano's formula. The roots are non-trivial and generally imaginary. They determine the shape of the curve of the effective potential. We will display the generic form of all three roots R_1, R_2 and R_0 (the root R_0 corresponds to the unstable circular orbit

$$R_1 = \frac{1}{3M}\left[\ell^2 - \alpha_-\right] \ , \tag{4.20}$$

$$R_2 = \frac{1}{3M}\left[\ell^2 - \frac{\alpha_-}{2} + \frac{\sqrt{3}i}{2}\alpha_+\right] \ , \tag{4.21}$$

$$R_0 = \frac{1}{3M}\left[\ell^2 - \frac{\alpha_-}{2} - \frac{\sqrt{3}i}{2}\alpha_+\right] \ , \tag{4.22}$$

where

$$\alpha_\pm = \gamma \pm \frac{\ell^2(12M^2 - \ell^2)}{\gamma} \ , \tag{4.23}$$

$$\beta = 27M^2 - 2\ell^2 \ , \tag{4.24}$$

$$\gamma = \left[\ell^2\left(6\sqrt{3}M^3\sqrt{\beta} + 54M^4 - 18M^2 l^2 + \ell^4\right)\right]^{\frac{1}{3}} \ . \tag{4.25}$$

The character of these solutions is determined by the sign of the term β.
If $\ell < \sqrt{\frac{27}{2}}M \approx 3.674M$, we obtain one real root,[2] which does not correspond to the unstable orbit and is of no physical meaning, therefore has no meaning in our calculations. If $\ell > \sqrt{\frac{27}{2}}M$, γ is a complex number.[3] Moreover, there are three real roots of the equation (4.18). Two correspond to the local minima of the effective potential, and one corresponds to the local maximum. The root

[2]There are three roots in total but two of them are imaginary and since we are in search of the extremes of the function, the generally complex number are no use for us.

[3]Since we operate with the physical meaning and are used and able to display mainly two-dimensional graphs, the imaginary parts of the roots are not taken into account. Nonetheless, they still might carry important information about the function itself, although it is not in our power to examine this property. Mathematically, the complex part of each root signifies at which point in the complex plane the function equals zero.

R_0 correlates with the maximum and hence coincide with the requested unstable orbit.

Which root corresponds to the unstable circular orbit one can determine from a graph (see fig. 3.1). The smallest real-valued part belongs to R_2 from (4.20) and corresponds to the stable "artifical" minimum lying below $r = 2M$. Secondly, the middle root R_0 represents the unstable circular orbit (and its conjugate homoclinic orbit) at the potential maximum. The largest root R_1 represents another stable orbit, this time above R_0, at a potential minimum. The apocentre is a point $r_{\rm max} > R_0$ where the effective potential $V_{\rm eff}$ coincides with the energy $\mathcal{E}_0$ corresponding to R_0 i.e. $\mathcal{E}_0 = V_{\rm eff}$. It is estimated that the homoclinic orbit is located in the interval $u \in (R_0, r_{\rm max})$.

The energy $\mathcal{E}_0(R_0)$ is obtained by substituting (4.19) into the expression for the effective potential $V_{\rm eff} = \dfrac{\ell^2}{2r^2} - \dfrac{M}{r-2M}$ at R_0

$$\begin{aligned}\mathcal{E}_0 &= \frac{R_0 M}{2(R_0-2M)^2} - \frac{M}{R_0-2M} \\ &= \frac{M(4M-R_0)}{2(R_0-2M)^2}\,.\end{aligned} \tag{4.26}$$

The equation for radial motion can be rewritten in terms of the reciprocal radius $u = 1/r$ which leads to

$$\left(\frac{\mathrm{d}u}{\mathrm{d}\theta}\right)^2 = \frac{2}{\ell^2}\left(\mathcal{E} - \frac{\ell^2 u^2}{2} + \frac{Mu}{1-2Mu}\right)\,, \tag{4.27}$$

where $U_0 = 1/R_0$ and $u_{\rm max} = 1/r_{\rm max}$. Thereupon the expression for the energy $\mathcal{E}_0$ one can rewrite as

$$\mathcal{E}_0(u) = -\frac{MU_0(1-4MU_0)}{2(1-2MU_0)^2} \tag{4.28}$$

The value of the apocentre $r_{\rm max}$ is found by substituting ℓ^2 from (4.19) into $V_{\rm eff} = \mathcal{E}_0$

$$\begin{aligned}\frac{MR_0^3}{2r^2(R_0-2M)^2} - \frac{M}{r-2M} &= \frac{M(4M-R_0)}{2(R_0-2M)^2}\,, \\ r_{\rm max} &= \frac{2MR_0}{R_0-4M}\,,\end{aligned} \tag{4.29}$$

which rewritten in terms of the reciprocal radius reads

$$\begin{aligned}\frac{u^2}{U_0} - \frac{2u(1-2MU_0)^2}{1-2Mu} &= 4MU_0 - 1\,, \\ u_{\rm max} &= \frac{1-4MU_0}{2M}\,.\end{aligned} \tag{4.30}$$

The equation for the homoclinic orbit can be found by solving the equation (4.27) after substituting ℓ^2 from (4.19). It must have the form

$$\left(\frac{\mathrm{d}u}{\mathrm{d}\theta}\right)^2 = \mathcal{C}(U_0-u)^2(u-u_{\rm max})\,. \tag{4.31}$$

The term $\mathcal{C}$ is found by direct comparison of $\mathcal{C}u^3$ with the u^3 term in the equation (4.27) and it is of form

$$\mathcal{C} = \frac{2M}{1-2Mu} \, . \tag{4.32}$$

The implicit equation for the homoclinic orbit thus reads

$$\theta(u) = \frac{2\sqrt{2}}{\omega}\left[A \ln C + B(\ln D - 2\ln 2)\right] \, , \tag{4.33}$$

where

$$\begin{aligned}
\upsilon :=& 4MU_0 - 1 \, , \\
\sigma :=& u\upsilon + 2M \, , \\
\omega :=& U_0\upsilon + 2M \, , \\
\kappa :=& 2Mu - 1 \, , \\
\lambda :=& 2MU_0 - 1 \, , \\
A :=& -\frac{\sqrt{M}}{4}\sqrt{32M^2U_0^3(MU_0-1) + 4M^2U_0(4MU_0-3) + (2M + 5MU_0^2 - U_0)} \, , \\
B :=& M\sqrt{2}\left(MU_0^2 + \frac{1}{2}M - \frac{1}{4}U_0\right) \, , \\
C :=& \frac{1}{u-U_0}\left\{2\sqrt{\sigma\omega\kappa\lambda} + (u+U_0) + 4M\left[U_0^2(4u-1) - U_0(2u-1) + (u-1)\right]\right\} \, , \\
D :=& 4u\sqrt{M\upsilon} + \sqrt{\frac{2}{M\upsilon}}\left[4M(M-U_0)+1\right] + 4\sqrt{\sigma\kappa} \, .
\end{aligned}$$

4.3 Comparison of Homoclinic Orbits

The spatial shapes of homoclinic orbits were plotted for both pseudo-Newtonian potentials (see Figures 4.2 and 4.1) and the relativistic Schwarzschild case (Fig. 4.3). The homoclinic orbits were plotted only for the part which lays in the interval $-3\pi \leq \theta \leq 3\pi$. The orbits were chosen to have approximately the same value of apocentre $r_{\max} = 17.6$. The values of ℓ which correspond to this $r_{\max}$ are $\ell = 3.6276$ for the logarithmic potential, $\ell = 3.8151$ for Paczyński-Wiita potential and $\ell = 3.669$ for Schwarzschild potential. From a further distance, the graphs are very similar; however, if closely observed, we can see that the circular orbit is clearest in the case of Paczyński-Wiita. This can mean that the homoclinic orbit corresponding to the logarithmic potential asymptotically whirls to the unstable circular orbit more successfully than the one for Paczyński-Wiita potential. This is an exciting observation since, according to, for instance, [28]; Paczyński-Wiita potential was considered to be the best approximation for the calculations of the unstable orbits.

The particular homoclinic orbit depends on the choice of $\mathcal{E}$ (or its corresponding value of ℓ). This orbit is fixed, and since the Melnikov function is calculated solely for one orbit at a time, it is possible to evaluate the Melnikov integral.

We shall now present the equations of homoclinic orbits calculated in [24]. Their displayed form was slightly modified to simplify the equations. The equation (4.34) refers to the homoclinic orbit in Schwarzschild metric, (4.35) represents

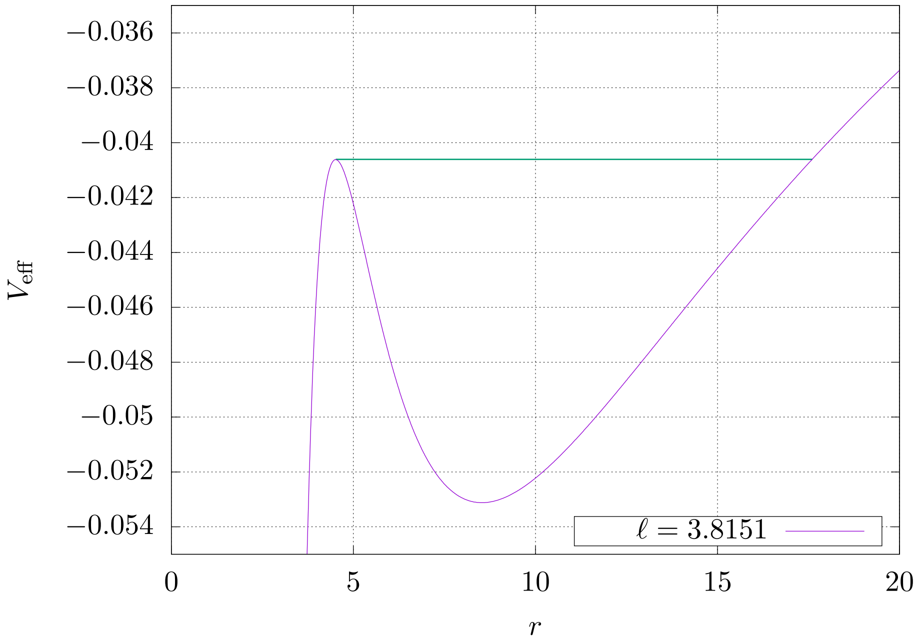

Figure 4.2: A radial shape of the effective potential with corresponding homoclinic orbit for Paczyński-Wiita pseudo-Newtonian potential. The local extremes are displayed in the graph. The minimum corresponds to the stable circular orbit; the maximum corresponds to the unstable circular orbit. The homoclinic orbit's energy level is drawn in green, and it connects the maximal value at R_0 with the apocentre at $r_{\rm max}$. The graph is plotted for the values $M = 1$, $\ell = 3.8151M$.

homoclinic orbit for the extreme Reissner-Nordström black hole and (4.36) denotes the homoclinic orbit for Nowak-Wagoner pseudo-Newtonian potential. The corresponding graphs which display the shapes of the effective potentials and appropriate homoclinic orbits are to be found in [24] too. Let us state that the shape of the pseudo-Newtonian potential resembles the relativistic case at least by the displayed curve.

$$u_0(\theta) = u_{max} - c_1 \tanh^2\left(\frac{1}{2}\theta\sqrt{2Mc_1}\right) , \tag{4.34}$$

$$u_0(\theta) = U_0 \ + \frac{2c_1c_2}{c_1 - c_2 - (c_1 + c_2)\cosh(M\theta\sqrt{c_1c_2})} , \tag{4.35}$$

$$u_0(\theta) = u_{max} + c_1 \tanh^2\left(\frac{M}{\ell}\theta\sqrt{6Mc_1}\right) , \tag{4.36}$$

where

$$c_1 := U_0 - u_{max} ,$$
$$c_2 := u_{min} - U_0 ,$$

are constants. The values $U_0 := 1/R_0$ and $u_{max} := 1/r_{\max}$ are the reciprocal radii. For the Reissner-Nordström equation then holds

$$u_{max/min} = \frac{1}{M} - U_0 \mp \sqrt{\frac{U_0}{M}}\,. \tag{4.37}$$

The equations for homoclinic orbits in their explicit form as expressed above cannot be directly compared with our conclusions, after all. We have attained only the implicit form for Paczyński-Wiita pseudo-Newtonian potential (**??**) and an analytically unsolvable differential equation (4.14) for the logarithmic pseudo-Newtonian potential. Even though the logarithmic potential was not calculated, the portraying of its appropriate homoclinic orbit had shown that it is an excellent approximation (at least for the chosen set of values of $\mathcal{E}$ and M).[4]

[4]Originally, a numerical mistake was made throughout the calculations of the homoclinic orbit for Paczyński-Wiita pseudo-Newtonian potential, i.e. instead of calculating the equation (4.27), the equation

$$\left(\frac{\mathrm{d}u}{\mathrm{d}\theta}\right)^2 = \frac{1}{\ell^2}\left(2\mathcal{E} - \frac{\ell^2 u^2}{2} + \frac{Mu}{1-2Mu}\right)\,, \tag{4.38}$$

was successively computed. Despite the numerical mistake, the results very accurately corresponded to the homoclinic orbit calculated for the Schwarzschild space-time.

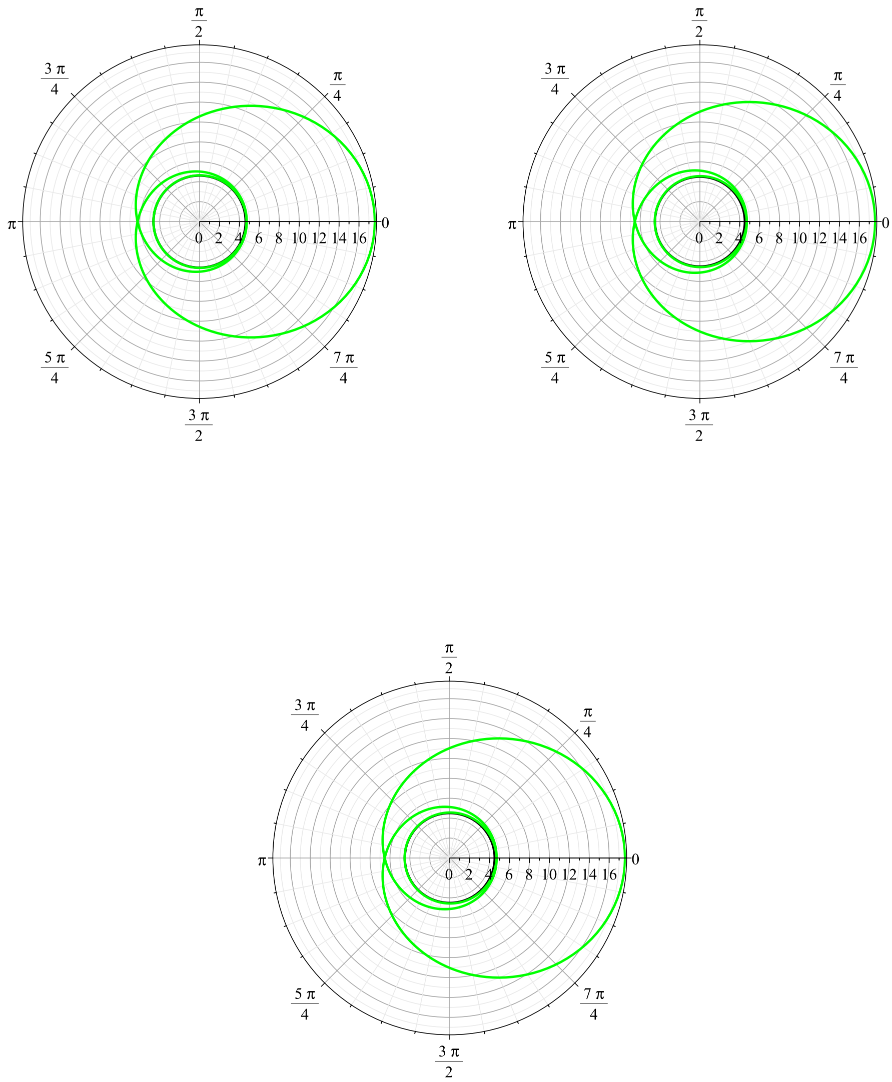

Figure 4.3: Homoclinic orbits for logarithmic (top left), Paczyński-Wiita (top right) and exact Schwarzschild (bottom) effective potentials.The spatial shapes of homoclinic orbits are displayed in green, the appropriate unstable circular orbits as the future and past infinite asymptotes of the homoclinic orbits are drawn in black. The graphs are plotted for fixed value of $M = 1$ and distinct values of ℓ for each case. Specifically, $\ell = 3.6276$ for logarithmic potential, $\ell = 3.8151$ for Paczyński-Wiita potential and $\ell = 3.669$ for Schwarzschild field. The graphs were kindly made by O. Semerák.

5. Black Hole Fields with an External Source

This chapter (based on [26], [19], [31], [32], [30], [33], [24]) shall concentrate on examining the field of stationary, symmetrical black holes with an additional source. Explicitly we will be interested in sources with a disc or a ring shape. Furthermore, we will assume the sources are with no charges and currents and without a radial pressure. We shall simplify the geodesic equations for some bounded motions for the Bach-Weyl ring and express the equations of geodesic for a generic source in the field of a Schwarzschild black hole. Secondly, we will investigate the geodesic motion in generic circular-space time and find its homoclinic orbit.

5.1 Weyl Fields

The metric outside a static an axially symmetric source surrounded by a ring or a disc can in Weyl coordinates (t, ϕ, ρ, z) be expressed as

$$\mathrm{d}s^2 = -e^{2\nu}\mathrm{d}t^2 + \rho^2 e^{-2\nu}\mathrm{d}\phi^2 + e^{2\zeta-2\nu}(\mathrm{d}\rho^2 + \mathrm{d}z^2)\,, \tag{5.1}$$

where ν and ζ are unknown functions which only depend on z and ρ. ν can be understood as a gravitational potential which satisfies the Laplace equation and superposes linearly while ζ is a function which can be found by quadrature [19]. Since the energy and angular momentum are conserved, i.e. $u_t = -\mathcal{E}$ and $u_\phi = \ell$, the geodesic equations read

$$\frac{\mathrm{d}u^t}{\mathrm{d}\tau} = -\frac{2\mathcal{E}}{e^{2\nu}}\left(\nu_{,\rho}u^\rho + \nu_{,z}u^z\right)\,, \tag{5.2}$$

$$\frac{\mathrm{d}u^\phi}{\mathrm{d}\tau} = \frac{2e^{2\nu}\ell}{\rho^3}\left[\rho(\nu_{,\rho}u^\rho + \nu_{,z}u^z) - u^\rho\right]\,, \tag{5.3}$$

$$\begin{aligned}\frac{\mathrm{d}u^\rho}{\mathrm{d}\tau} =& -\frac{\mathcal{E}^2\nu_{,\rho}}{e^{2\zeta}} + \frac{\ell^2 e^{4\nu}}{\rho^3 e^{2\zeta}}(1-\rho\nu_{,\rho}) + (\nu_{,\rho}-\zeta_{,\rho})\left[(u^\rho)^2 - (u^z)^2\right] \\ &+ 2\left(\nu_{,z}-\zeta_{,z}\right)u^z u^\rho\,,\end{aligned} \tag{5.4}$$

$$\begin{aligned}\frac{\mathrm{d}u^z}{\mathrm{d}\tau} =& -\frac{\mathcal{E}^2\nu_{,z}}{e^{2\zeta}} - \frac{\ell^2 e^{4\nu}\nu_{,z}}{\rho^2 e^{2\zeta}} - (\nu_{,z}-\zeta_{,z})\left[(u^\rho)^2 - (u^z)^2\right] \\ &+ 2\left(\nu_{,\rho}-\zeta_{,\rho}\right)u^z u^\rho\,,\end{aligned} \tag{5.5}$$

where $\zeta,\rho = \rho\left[(\nu_{,\rho})^2 - (\nu_{,z})^2\right]$ $\zeta,z = 2\rho\nu_\rho\nu_z$. The thin Bach-Weyl ring lies in the equatorial plane of the black hole and will serve as the external source. The functions ν, ζ for the Bach-Weyl ring solution have the following form [19]

$$\nu = -\frac{2\mathcal{M}K(k)}{\pi l_2}\,, \tag{5.6}$$

$$\zeta = \frac{\mathcal{M}^2 k^4}{4\pi b^2\rho}\left[(\rho+b)(-K^2 + 4k'^2 K\dot{K} + 4k^2k'^2\dot{K}^2) - 4\rho k^2 k'^2(k'^2+2)\dot{K}^2\right]\,, \tag{5.7}$$

where $\mathcal{M}$ is the mass of the ring and b is Weyl's radius, $K(k)$ is a complete first-kind Legendre elliptic integral given by

$$K(k) = \int_0^{\pi/2} \frac{\mathrm{d}\phi}{\sqrt{1-k^2\sin^2\phi}} \,,$$

and

$$\dot{K} := \frac{\mathrm{d}K}{\mathrm{d}(k^2)} \quad k'^2 = \frac{(l_1)^2}{(l_2)^2} \quad k^2 = 1 - k'^2 = \frac{4\rho b}{(l_2)^2} \,,$$
$$l_{1,2} := \sqrt{(\rho \mp b)^2 + z^2} \,.$$

Interesting orbits in this settings are the ones lying in the meridional plane, i.e. the plane which is perpendicular to the equatorial plane. On that account $\ell = u_\phi = 0$ and therefore $\phi =$constant.
From four-velocity normalization (3.7) we obtain

$$e^{2\zeta-2\nu}[(u^\rho)^2 + (u^z)^2] = -1 + e^{-2\nu}\mathcal{E}^2 - \frac{e^{2\nu}}{\rho^2}\ell^2 \,, \tag{5.8}$$

$$e^{2\zeta}[(u^\rho)^2 + (u^z)^2] = \mathcal{E}^2 - e^{2\nu}\left(1 + \frac{e^{2\nu}\ell^2}{\rho^2}\right) = \mathcal{E}^2 - \mathrm{V_{eff}}^2 \,, \tag{5.9}$$

which after simplifying for the meridonial plane reads

$$e^{2\zeta-2\nu}\left[(u^\rho)^2 + (u^z)^2\right] = \mathcal{E}^2 - e^{2\nu} \,. \tag{5.10}$$

There are four different cases of orbits being considered. In each of them, the equations are significantly simplified. After the appropriate conditions are substituted, it is possible to express one of the four-velocity components from its normalization. This term can then be substituted back into the equations of geodesic motion, which are further simplified and the equations can be expressed solely in terms of ν and ζ. However, except the last case, the equations hold exactly at one specific point.

- $\boxed{u^\rho = 0}$
  Orbit that crosses the equatorial plane at exactly perpendicular plane must satisfy $z = 0$ and $u^\rho = 0$. Generally this works only for the circular equatorial geodesics alongside the axis. $u^\rho = 0$ holds concretely at one point. The expression for the motion in z direction has thus the form

  $$e^{2\zeta}(u^z)^2 = \mathcal{E}^2 - e^{2\nu} \,. \tag{5.11}$$

  After expressing the term u^z and by substituting it into the equations for geodesic motion, they are of form

  $$\frac{\mathrm{d}u^t}{\mathrm{d}\tau} = -\frac{2\mathcal{E}}{e^{2\nu+\zeta}}\nu_{,z}\sqrt{\mathcal{E}^2 - e^{2\nu}} \,, \tag{5.12}$$
  $$\frac{\mathrm{d}u^\rho}{\mathrm{d}\tau} = -\frac{\mathcal{E}^2\nu_{,\rho}}{e^{2\zeta}} - e^{-2\zeta}(\nu_{,\rho} - \zeta_{,\rho})(\mathcal{E}^2 - e^{2\nu}) \,, \tag{5.13}$$
  $$\frac{\mathrm{d}u^z}{\mathrm{d}\tau} = -\frac{\mathcal{E}^2\nu_{,z}}{e^{2\zeta}} + e^{-2\zeta}(\nu_{,z} - \zeta_{,z})(\mathcal{E}^2 - e^{2\nu}) \,. \tag{5.14}$$

- $\boxed{u^z = 0,\ \rho = 0}$
  This orbit crosses the axis at a perpendicular way, i.e. $u^z = 0$ and $\rho = 0$ the equation for the motion at one point when ρ direction reads

$$e^{2\zeta}(u^\rho)^2 = \mathcal{E}^2 - e^{2\nu}\,. \tag{5.15}$$

  Both conditions work at solely one point. The condition $u^z = 0$ holds for any motion at the equatorial plane. Consequently because $\zeta = 0$, $\zeta_{,\rho} = 0$, $\zeta_{,z} = 0$ and $\nu_{,\rho} = 0$ and the term u^ρ is expressed from (5.15), the geodesic equations are simplified as follows

$$\frac{\mathrm{d}u^t}{\mathrm{d}\tau} = 0\,, \tag{5.16}$$

$$\frac{\mathrm{d}u^\rho}{\mathrm{d}\tau} = 0\,, \tag{5.17}$$

$$\frac{\mathrm{d}u^z}{\mathrm{d}\tau} = -\,\nu_{,z}\left[\mathcal{E}^2(1+e^{-2\zeta}) - e^{2\nu-2\zeta}\right]\,. \tag{5.18}$$

- $\boxed{\zeta_{,z} = 0,\ \nu_{,z} = 0}$
  If the crossing of the equatorial plane is general, the four-velocity normalization cannot be simplified and remains of form (5.10). However, this expression once again holds only for one point and generally does not have to be satisfied across the disc. Nonetheless, it is possible to express u^z as a function of u^ρ which equals

$$(u^z)^2 = e^{-2\zeta}(\mathcal{E}^2 - e^{2\nu}) - (u^\rho)^2\,. \tag{5.19}$$

  This can be substituted into the equations of motions

$$\frac{\mathrm{d}u^t}{\mathrm{d}\tau} = -\,\frac{2\mathcal{E}}{e^{2\nu}}\nu_{,\rho}u^\rho\,, \tag{5.20}$$

$$\frac{\mathrm{d}u^\rho}{\mathrm{d}\tau} = -\,\frac{\mathcal{E}^2}{e^{2\zeta}}\nu_{,\rho} + (\nu_{,\rho} - \zeta_{,\rho})\left[2(u^\rho)^2 - e^{-2\zeta}(\mathcal{E}^2 - e^{2\nu})\right]\,, \tag{5.21}$$

$$\frac{\mathrm{d}u^z}{\mathrm{d}\tau} = 2(\nu_{,\rho} - \zeta_{,\rho})u^\rho\sqrt{e^{-2\zeta}(\mathcal{E}^2 - e^{2\nu}) - (u^\rho)^2}\,. \tag{5.22}$$

- $\boxed{u^z = 0}$
  The motion which is zero in u^z direction is generalization of the second case, where the orbit crossed an axis in a perpendicular way. Since the equation (5.15) is of the same form, the geodesic equations are rewritten using the expressed term for u^ρ.

$$\frac{\mathrm{d}u^t}{\mathrm{d}\tau} = -\,\frac{2\mathcal{E}}{e^{2\nu}}\nu_{,\rho}e^{-\zeta}\sqrt{\mathcal{E}^2 - e^{2\nu}}\,, \tag{5.23}$$

$$\frac{\mathrm{d}u^\rho}{\mathrm{d}\tau} = -\,\frac{\mathcal{E}^2\nu_{,\rho}}{e^{2\zeta}} + e^{-2\zeta}(\nu_{,\rho} - \zeta_{,\rho})(\mathcal{E}^2 - e^{2\nu})\,, \tag{5.24}$$

$$\frac{\mathrm{d}u^z}{\mathrm{d}\tau} = -\,\frac{\mathcal{E}^2\nu_{,z}}{e^{2\zeta}} - e^{-2\zeta}(\nu_{,z} - \zeta_{,z})(\mathcal{E}^2 - e^{2\nu})\,. \tag{5.25}$$

In all four cases, the equations for geodesic motion have been modified, however, since the functions ν and ζ are rather difficult expressions it is not easily possible to solve the final equations analytically. Hence, it is unnecessary to try to simplify the general form of the equations since the exact calculations shall differ in dependence on the form of ν and ζ. Some of the explicit solutions - specifically for the Bach-Weyl ring - can be found in [19]. The equations could although be used in further calculations for specific external sources.

5.2 Schwarzschild Space-time

The metric for a Schwarzschild black hole with an additional static and axially symmetric thin source with no radial pressure can be expressed in Schwarzschild coordinates (t, r, θ, ϕ) and is of form

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right) e^{2\hat{\nu}}\, \mathrm{d}t^2 + \frac{e^{2\hat{\zeta}-2\hat{\nu}}}{1 - \frac{2M}{r}}\, \mathrm{d}r^2 + r^2 e^{-2\hat{\nu}} \left(e^{2\hat{\zeta}}\, \mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2\right) , \quad (5.26)$$

where $\hat{\nu}(r, \theta)$ is the potential of the external source and $\hat{\zeta}(r, \theta)$ is given by the difference of two functions $\zeta_{1,2}$ that correspond to the black hole field and the field of the additional source, respectively [19]. We will now consider the conserved quantities $u_t = -\mathcal{E}$ and $\ell = u_\phi$. Except from those constants of motion we shall not state other conditions on the generic functions ν and ζ. The Christoffel symbols are calculated from the equation (3.3) and the geodesic equations are later computed from (3.1). The Christoffel symbols are hence of the form as listed bellow

$$\Gamma^t_{\ tt} = \Gamma^\phi_{\ \phi\phi} = 0 = \Gamma^t_{\ t\phi} , \quad (5.27)$$

$$\Gamma^r_{\ rr} = \hat{\zeta}_{,r} - \hat{\nu}_{,r} - \frac{M}{r(r-2M)} , \quad (5.28)$$

$$\Gamma^\theta_{\ \theta\theta} = \hat{\zeta}_{,\theta} - \hat{\nu}_{,\theta} , \quad (5.29)$$

$$\Gamma^t_{\ tr} = \hat{\nu}_{,r} + \frac{M}{r(r-2M)} , \quad (5.30)$$

$$\Gamma^t_{\ t\theta} = \hat{\nu}_{,\theta} , \quad (5.31)$$

$$\Gamma^\phi_{\ \phi\theta} = \frac{\cos\theta}{\sin\theta} - \hat{\nu}_{,\theta} , \quad (5.32)$$

$$\Gamma^\phi_{\ \phi r} = \frac{1}{r} - \hat{\nu}_{,r} , \quad (5.33)$$

$$\Gamma^r_{\ r\theta} = \hat{\zeta}_{,\theta} - \hat{\nu}_{,\theta} , \quad (5.34)$$

$$\Gamma^r_{\ \theta\theta} = (r-2M)\left[r(\hat{\nu}_{,r} + \zeta_{,r}) - 1\right] , \quad (5.35)$$

$$\Gamma^r_{\ \phi\phi} = (r-2M)\, e^{-2\hat{\zeta}} \sin^2\theta\, (r\hat{\nu}_{,r} - 1) , \quad (5.36)$$

$$\Gamma^r_{\ tt} = e^{4\hat{\nu}-2\hat{\zeta}} (r-2M) \frac{r(r-2M)\, \hat{\nu_{,r}} + M}{r^3} , \quad (5.37)$$

$$\Gamma^\theta_{\ \phi\phi} = e^{-2\hat{\zeta}} \left[\sin^2\theta \hat{\nu}_{,\theta} - \sin\theta\cos\theta\right] , \quad (5.38)$$

$$\Gamma^\theta_{\ tt} = e^{4\hat{\nu}-2\hat{\zeta}} \frac{r-2M}{r^3} \hat{\nu}_{,\theta} , \quad (5.39)$$

$$\Gamma^\theta_{\ rr} = \frac{\hat{\nu}_{,\theta} - \hat{\zeta}_{,\theta}}{r(r-2M)} , \quad (5.40)$$

$$\Gamma^\theta_{\ \theta r} = \frac{1}{r} + \hat{\zeta}_{,r} - \hat{\nu}_{,r} . \quad (5.41)$$

The appropriate components of Christoffel symbols are used to express the geodesic equations

$$
\begin{aligned}
\frac{\mathrm{d}u^t}{\mathrm{d}\tau} =& -2\mathcal{E}e^{-2\hat{\nu}}\frac{r}{r-2M}\left[\hat{\nu}_{,\theta}u^\theta + \left(\hat{\nu}_{,r} + \frac{M}{r(r-2M)}\right)u^r\right] , && (5.42)\\
\frac{\mathrm{d}u^\phi}{\mathrm{d}\tau} =& \frac{2e^{2\hat{\nu}}\ell}{r^2\sin^2\theta}\left[(\hat{\nu}_{,\theta}u^\theta) - \frac{\cos\theta}{\sin\theta})u^\theta + (\hat{\nu}_{,r} - \frac{1}{r})u^r\right] , && (5.43)\\
\frac{\mathrm{d}u^r}{\mathrm{d}\tau} =& (\hat{\nu}_{,\theta} - \hat{\zeta}_{,r})\left[r(2M-r)(u^\theta)^2 + (u^r)^2\right] + 2(\hat{\zeta}_{,\theta} - \hat{\nu}_{,\theta})u^r u^\theta && (5.44)\\
&+ \frac{M}{r(r-2M)}(u^r)^2 + (r-2M)(u^\theta)^2\\
&+ \ell^2 e^{4\hat{\nu}-2\hat{\zeta}}(2M-r)\frac{r\hat{\nu}_{,r}-1}{r^4\sin^4\theta} - \mathcal{E}^2e^{-2\hat{\zeta}}\left[\hat{\nu}_{,r} + \frac{M}{r(r-2M)}\right] ,\\
\frac{\mathrm{d}u^\theta}{\mathrm{d}\tau} =& \left(\hat{\nu}_{,\theta} - \hat{\zeta}_{,\theta}\right)\left[(u^\theta)^2 - \frac{(u^r)^2}{r(r-2M)}\right] - 2\left(\frac{1}{r} + \hat{\zeta}_{,r} - \hat{\nu}_{,r}\right)u^r u^\theta && (5.45)\\
&+ \frac{e^{4\hat{\nu}-2\hat{\zeta}}\ell^2}{r^4}\left(\frac{\cos\theta}{\sin^3\theta} - \frac{\hat{\nu}_{,\theta}}{\sin^2\theta}\right) - \frac{e^{-2\hat{\zeta}}\mathcal{E}^2\hat{\nu}_{,\theta}}{r(r-2M)} .
\end{aligned}
$$

Furthermore, one can obtain the equation for the radial and latitudinal velocities from the four-velocity normalization (3.7) and the metric (5.26),

$$
e^{2\hat{\zeta}}\left[(u^r)^2 + r(r-2M)(u^\theta)^2\right] = \mathcal{E}^2 - \mathrm{V_{eff}}^2 , \tag{5.46}
$$

where

$$
V_{\mathrm{eff}}^2 = \left(1 - \frac{2M}{r}\right)\left[1 + \frac{e^{2\hat{\nu}}\ell^2}{r^2\sin^2\theta}\right]e^{2\hat{\nu}} , \tag{5.47}
$$

is the effective potential for Schwarzschild black hole with external source. According to [20] the method of effective potential is used at the equatorial plane i. e. $\theta = \pi/2$ and therefore $\dot{\theta} = 0$. The equation (5.46) thus represents just the radial motion and reduces to

$$
e^{2\hat{\zeta}}(u^r)^2 = \mathcal{E}^2 - \left(1 - \frac{2M}{r}\right)\left(1 + \frac{e^{2\hat{\nu}}\ell^2}{r^2}\right)e^{2\hat{\nu}} . \tag{5.48}
$$

The maximum of the potential is given by $\mathrm{d}V_{\mathrm{eff}}/\mathrm{d}t = 0$ from which we can express

$$
\ell^2 = \frac{2Mr^3 - r^4\hat{\nu}_{,r} - Mr}{r^2e^{2\hat{\nu}}\hat{\nu}_{,r} - e^{2\hat{\nu}}r + 3Me^{2\hat{\nu}} + e^{2\hat{\nu}}r^2 - 4Me^{2\hat{\nu}}\hat{\nu}_{,r}} . \tag{5.49}
$$

The equation (5.49) can be substituted into (5.48) where $\mathcal{E}^2 = V_{\mathrm{eff}}^2$, which leads to

$$
\mathcal{E}^2 = \left(1 - \frac{2M}{R_0}\right)\left(1 + \frac{e^{2\hat{\nu}}}{R_0}\frac{2MR_0^2 - R_0^3\hat{\nu}_{,r} - M}{R_0^2e^{2\hat{\nu}}\hat{\nu}_{,r} - e^{2\hat{\nu}}R_0 + 3Me^{2\hat{\nu}} + e^{2\hat{\nu}}R_0^2 - 4Me^{2\hat{\nu}}\hat{\nu}_{,r}}\right) , \tag{5.50}
$$

where the value of the root R_0 once again corresponds to the unstable orbit and its conjugate homoclinic orbit.

The general form (5.46) is after substituting $\ell = 0$ reduced to

$$e^{2\hat{\zeta}}\left[(u^r)^2 + r(r-2M)(u^\theta)^2\right] = \mathcal{E}^2 - \left(1 - \frac{2M}{r}\right)e^{2\hat{\nu}} . \tag{5.51}$$

In cases where $u^r = 0$, the latitudinal component is thus of the form

$$e^{2\hat{\zeta}} r^2 (u^\theta)^2 = \frac{\mathcal{E}^2}{1 - \frac{2M}{r}} - e^{2\hat{\nu}} . \tag{5.52}$$

5.3 Kerr Space-time

The metric function for circular space-time as stated in (3.6) has the form

$$\mathrm{d}s^2 = -e^{2\nu}\,\mathrm{d}t^2 + B^2 r^2 \sin^2\theta e^{-2\nu}(\mathrm{d}\phi - \omega\,\mathrm{d}t)^2 + e^{2\zeta-2\nu}(\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2). \tag{5.53}$$

The well known Kerr metric is the solution for uncharged, rotating black hole and in Boyer-Lindquist coordinates reads

$$\begin{aligned} \mathrm{d}s^2 = &-\left(1 - \frac{2Mr}{\Sigma}\right)\mathrm{d}t^2 - \frac{4Mar\sin^2\theta}{\Sigma}\,\mathrm{d}t\mathrm{d}\phi + \\ &+ \sin^2\theta\left(r^2 + a^2 + \frac{2Ma^2 r\sin^2\theta}{\Sigma}\right)\mathrm{d}\phi^2 + \frac{\Sigma}{\Delta}\,\mathrm{d}r^2 + \Sigma\,\mathrm{d}\theta^2 , \end{aligned} \tag{5.54}$$

where M denotes the mass of the centre of gravity, a is the spin angular momentum per unit mass and

$$\Sigma := r^2 + a^2\cos^2\theta , \quad \Delta := r^2 - 2Mr + a^2 . \tag{5.55}$$

The Kerr metric is stationary and axially symmetric and the motion alongside geodesics conserves orbital energy E, axial angular momentum L_z, the Carter constant Q [34] and the rest mass μ (more to be found in [21]). The parameter a is connected with rotation since, for $a = 0$, the metric reduces into the Schwarzschild form.[1] The metric is not static, and hence it must contain the non-diagonal term $g_{t\phi}$. From the section 3.4 we thus know that in the generalized form of the effective potential, the conserved terms $\mathcal{E}$ and ℓ cannot be separated.

5.4 Homoclinic Orbits in Circular Space-times

The procedure of finding a homoclinic orbit can be done by calculating the Hamiltonian for a relativistic non-spinning free particle of mass m

$$H = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu . \tag{5.56}$$

The Hamiltonian is then derived specifically for the Kerr metric and, by using linearization and canonical transformation, is rewritten in terms of action-angle variables. The Hamiltonian equations are hence expressed in a more convenient form. For the full description of this process, we shall refer to the article [21] where the procedure is described in detail.

[1] More features of the Kerr metric can be found in [26].

In addition, one can obtain the homoclinic orbit for the circular space-times (and precisely the Kerr metric, which was done in [20]) from the generalization of the effective potential method.

Assuming that the motion takes place in the equatorial plane $\theta = \pi/2$,[2] from (3.13) together with the four-velocity normalization

$$g^{\mu\nu} u_\mu u_\nu = -1 \,, \tag{5.57}$$

the radial motion equation can be derived in the following form

$$\begin{aligned} e^{2\zeta}(u^r)^2 =& - e^{2\nu} + (\mathcal{E} - \omega\ell)^2 - \frac{e^{4\nu}\ell^2}{B^2 r^2} \\ =& (\mathcal{E} - \omega\ell)^2 - N^2\left(1 + \frac{N^2\ell^2}{r^2}\right) := \texttt{RHS} \,, \end{aligned} \tag{5.58}$$

where we have substituted $\theta = \pi/2$, $B = 1$ and $N^2 = e^{2\nu}$. It is eventually possible to express the equations for ℓ and $\mathcal{E}$ from finding roots of the equation (5.58) accompanying by finding the roots of its derivative with respect to r. From this conditions, i.e. $\texttt{RHS} = 0$ and $\texttt{RHS}_{,r} = 0$, one shall get two equations

$$r^2(\mathcal{E} - \omega\ell)^2 = r^2 N^2 + N^4\ell^2 \,, \tag{5.59}$$

$$2r(\mathcal{E} - \omega\ell)^2 - 2r^2 l\omega_{,r}(\mathcal{E} - \omega\ell) = 2rN^2 + 2r^2 N N_{,r} + 4N^3 N_{,r}\ell^2 \,. \tag{5.60}$$

The aim is to find the value of R_0, which corresponds to the unstable circular orbit. This can be done by expressing ℓ and $\mathcal{E}$ from (5.59) and (5.60) at this exact point.

Thus it is possible to substitute the term $(\mathcal{E} - \omega\ell)^2$ from (5.59) to (5.60) and then express ℓ^2 as

$$\frac{2N^2\ell^2}{r^2} = - \frac{D - 2rN^3 N_{,r} \pm r^2\omega_{,r}\sqrt{D}}{D - N^4} \,, \tag{5.61}$$

where

$$D := r^4\omega_{,r}^2 + 4rN^2 N_{,r}(N - rN_{,r}) \,. \tag{5.62}$$

Substituting equation (5.61) back into (5.59) gives

$$(\mathcal{E} - \omega\ell)^2 = \frac{N^2}{2(D - N^4)}\left(D - 2N^3(N - rN_{,r}) \mp r^2\omega_{,r}\sqrt{D}\right) \,. \tag{5.63}$$

Finally, by solving the equation (5.63) for $\mathcal{E}$ we obtain the expression

$$\mathcal{E}_{1,2} = \ell(\omega \pm \sqrt{C}) \,, \tag{5.64}$$

where

$$C := \omega^2 + \frac{N^2}{2(D - N^4)}\left[D - 2N^3(N + R_0 N_{,r}) \mp R_0^2\omega_{,r}\sqrt{D}\right] \,. \tag{5.65}$$

[2] This is not a general case. The motion in the equatorial plane is chosen to simplify the generic equations and to illustrate the behaviour in one specific case.

Now we have found the values of $\mathcal{E}$ and ℓ at the point R_0, which corresponds to the unstable orbit and is a double root of (5.59).
If the expressions for ℓ and $\mathcal{E}$ are now substituted into the equation (5.59) and the equation is solved for r one should get the r_{max} root, a simple root of the equation. The equation for finding the root r has the form

$$r^2 = \frac{N^4 \ell^2}{(\mathcal{E} - \omega \ell)^2 - N^2} \, . \tag{5.66}$$

If one wants to obtain the root corresponding to the apocentre, they have to substitute $\mathcal{E}$ from equation (5.64) after which follows the substitution of ℓ from (5.61). Since both those terms are already constant, the resulted value of r_{max} is trivially derived from (5.66) (although the expression does not have to be trivial at all). The above-described method is a general procedure of finding the roots that coincide with the unstable circular orbit and the homoclinic orbit.

To conclude this chapter - which might seem slightly inconsistent - that is because we have tried to simplify the generic equation for various space-times. The geodesic equations which represent the dynamical system in general relativity were studied and adjusted. The fact we have studied them in different fields is minor when compared to their dynamical properties. Consequently, distinct conditions were applied to simplify the equations yet more. This approach was sometimes successful, and the geodesic equations were significantly modified. However, we have to stress that most of the functions we have calculated with were unknown, and thus we cannot state if the equations of geodesic motion will be solvable after substituting specific functions. Finally, we would like to stress that even though we have not obtained any results for concrete space-times, we have significantly simplified the general forms, which might be helpful in possible future calculations.

Conclusion

Unlike many other mathematical theories, chaos theory deals with the dynamics behaviour without assuming the knowledge of everything. It admits the inaccuracy of unique solutions and leaves the results open to interpretation.

Homoclinic orbits seem to play a crucial role in examining chaotic behaviour in nonlinear relativistic systems, yet the analytical approach of their research is not always successful. The presence of an external source disrupts the complete integrability of the geodesic equations around an isolated black hole. The nature of the resulting space-time is examined in terms of homoclinic orbits and their close neighbourhood. The mathematical prerequisites for understanding the commonly used terms in classical chaos theory were defined; furthermore, the homoclinic orbits and their relevant terms were thoroughly characterized. The homoclinic orbits were analytically computed and graphically displayed for Paczyński-Wiita and logarithmic pseudo-Newtonian potentials. Since they simulated the space-time of the Schwarzschild black hole, they were compared with its appropriate homoclinic orbit. These equations could successively be used to calculate the Melnikov integral from which the chaotic behaviour can be revealed.

The equations of geodesic motion were calculated for generic circular space-time. The point of local maximum corresponding to the unstable orbit was found by using the effective potential method. Substituting concrete conditions, e.g. the field being static and axially symmetric with Bach-Weyl ring as the external source, the equations of geodesic motion were simplified to a great extent. Finally, the equations of motion were derived for the Schwarzschild field with an additional general source. Although the geodesic equations do not have any immediate utilization, they could be potentially helpful with simplifying the calculations of the dynamical behaviour for a concrete metric that satisfies the initial symmetry of the problem.

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