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ACD model a český kapitálový trh

Katedra pravděpodobnosti a matematické statistiky

Vedoucí diplomové práce: RNDr. Martin Šmíd Ph.D.

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DIPLOMA THESIS

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The ACD Model and the Czech Capital Market

Department of Probability a Mathematical Statistics

Supervisor: RNDr. Martin Šmíd Ph.D.

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Mé poděkování patří především mému školiteli, RNDr. Martinu Šmídovi, Ph.D., za zajímavé téma, vydatnou pomoc v průběhu psaní této práce a v neposlední řadě za trpělivost. Dále bych ráda poděkovala Mgr. Zdeňku Hlávkovi, Ph.D. za pomoc s výpočetními problémy. V neposlední řadě bych ráda zmínila moji rodinu, která mne po celou dobu studia podporovala. Děkuji.

Prohlašuji, že jsem svou diplomovou práci napsala samostatně a výhradně s použitím citovaných pramenů. Souhlasím se zapůjčováním práce.

V Praze dne 7.4.2008 Anna Moravová

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Abstrakt: Tato práce se zabývá autoregresním modelem podmíneněné durace (ACD) a jeho aplikací na data z pražské burzy. ACD model je vhodný pro analýzu časových řad s nestejně rozdělenouou délkou intervalů mezi jednotlivými událostmi. K intervalům přistupujeme jako ke stochastickému procesu. ACD model použijeme k modelování intervalů mezi obchody s akciemi Komerční Banky na pražské burze z roku 2004. Parametry modelu jsou odhadovány metodou maximální věrohodnosti. V další části popíšeme rozšíření ACD modelu - ACD-ACM model. ACM model se používá k modelování diskrétních změn v cenách akcií. Rozdělení změny ceny je podmíněno minulými změnami ceny a dalšími vysvětlujícími proměnými. ACD-ACM model je zde aplikován na data kotací Telecomu z roku 2004. Závěry a výsledky výpočtů z českého kapitálového trhu jsou porovnány s výsledky z NYSE, které prezentovali autoři ACD modelu Engle a Russel ve svých článcích z let 1998 a 2005.
Klíčová slova: ACD, český kapitálový trh, vysokofrekvenční data, bodový proces se značkami

Title: The ACD Model and the Czech Capital Market
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Abstract: This study is concerned with the autoregressive conditional duration model (ACD) and its applications on the data from the Prague Stock exchange. The ACD model is particularly suitable for the analysis of data which arrive at irregular time intervals. We treat the time between events as a stochastic process. We apply the ACD model to model the intervals between the trades with the stock of Komerční Banka at the Prague Stock Exchange in the year 2004. The parameters are estimated by the maximum likelihood method. Further, an extension of the ACD model - the ACD-ACM model - is studied. ACM model is used to model the discrete price changes in the stock prices. The distribution of each price change is considered to be a random variable with distribution conditional on the past price changes and other explanatory variables. The ACD-ACM model is applied to the quote data of the stock of Czech Telecom from the year 2004. The results of the calculations are compared with the results presented by Engle and Russel in their studies from the years 1998 and 2005.
Keywords: Autoregressive conditional duration, Czech Capital Market, high frequency data, marked point process

Chapter 1

Introduction

The availability of high frequency data collections and powerful computational devices were motivations for enhanced research in the field of market microstructure - both theoretical and empirical. Among others new econometric models were developed. [ER98] propose an econometric model which takes into account the irregular spacing of the data. They call their model the Autoregressive Conditional Duration model (ACD). They treat the transaction data as a sequence of arrival times and characteristics associated with the arrival times.

The concept of the ACD model was further developed by [BG00], who introduce the log-ACD model, [GM98] and [Lun98], who propose models based on Burr distribution and Gamma distribution (respectivelly).

[BZ06] apply the log-ACD model to the quote data of 3 stocks traded at the Prague Stock Exchange. They study the information content implicit to the waiting times between market events and they test different market microstructure theories concerning the impact of volume, spread and intensity of trading. Their empirical result is, that only the spread shows a consistent negative impact on the expected durations.

[ER05] proposed further extension of the ACD model. Besides the waiting times between events they study the associated price changes. They decompose the joint distribution of the discrete price changes and time intervals into a product of conditional distribution of price changes and marginal distribution of the waiting times. The distribution of waiting times is modeled by the ACD model and the conditional distribution of the price changes is described by the Autoregressive Conditional Multinomial (ACM) model. The ACM model allows different specifications to capture all possible intertemporal dependencies displayed by high-frequency transaction data.

In this study we present an application of the ACD model and of the ACD-ACM model to the trade and quote data from the Prague Stock Exchange. The study is organized as follows: following the introduction a brief insight in the development of the market microstructure theories and in the trading at the Prague Stock Exchange is presented. In the 2^{nd} chapter the ACD model is introduced. In the next chapter we summarize the fundamental theories which are necessary for the estimation of the coefficients using the ACD model. 4^{th} chapter contains general description of the trade dataset of the KB stock and the model estimates of the ACD model based on the exponential and the Weibull distribution. Chapter 5 introduces the ACM model for the discrete price changes and in chapter 6 the log-ACD-ACM model estimates for the Telecom stock are presented.

1.1 Brief History of Market Microstructure Theories

The first theoretical models explaining the market maker behavior appeared in the 70ies. [Gar86] introduces a monopolistic model. One market maker faces the succession of buy and sell orders. To prevent himself from bancrupcy he sets different buy and sell prices (*bid and ask price*).

In [GM85] a fully new approach was introduced. In their *information based model*, market maker and traders do not possess the same information concerning the traded stock. There are two kinds of traders: *informed and uninformed traders*. Market maker has to deal with both kinds of traders and thus to protect himself from the loss, he sets different bid and ask prices. By observing the duration between the trades or the volume of trades he can learn about the new information. When he sees possible signals of informed trading he adjusts the bid and ask spread to reflect the probability of trading with an informed trader. Following, if an informed trader possesses a good news he has to act quickly to make profit from these superior news. If the news is bad, the informed trader has no motivation to buy and even by selling he has to be careful not to disclose his information. This means that high activity on the market signals good news and low activity signals bad news.

This model was further developed by [EO92]. Easley and O'Hara believe that the clustering of trades contains an important information. Uninformed traders trade for liquidity reasons to rebalance their portfolio and they are supposed to arrive with constant probability. Informed traders enter the market only when they possess (or when they think to possess) a superior information. They buy when the news is good and sell when the news is bad. This behavior shortens the duration between trades. When the duration is long, there is likely to be no new information and the probability of dealing with informed traders is low. As a consequence, the market maker decreases the bid and ask spread. On the contrary, if the durations are short, it signals there may be a superior information and the market maker increases the bid and ask spread.

Different approach is presented by [AP88]. According to their model, the volatility clustering is a consequence of the random clustering of liquidity traders. Hence long duration means the absence of liquidity traders, high fraction of informed traders and as a consequence also higher volatility and increased bid and ask spread.

All the models presented above are based on a market with a monopolistic market maker. Later the differences between stock exchanges with monopolistic market maker (e.g. NYSE) and competing market makers (Prague Stock Exchange, NASDAQ) are explored. [CCS95] observed the intraday patterns of the bid-ask spreads on the market with a single specialist (NYSE) and on the market with competing market makers (NASDAQ). In the first case the bid-ask spread follows the well known U shape (the spread is the highest in the morning, it lowers in the middle of the trading day and raises again before the end of trading), whereas on the market with competing market makers the spread remains stable throughout the day and it even narrows at the end of the trading day. [CCS95] conclude that the price formation process as well as the bid-ask spread dynamics follow different causalities in these two cases.

1.2 The Prague Stock Exchange

[1]

Prague Stock Exchange (PSE) was founded in 1992 and now it is the biggest organiser of the securities market in the Czech Republic. Only licensed dealers - members of the PSE - are entitled to conclude trades. PSE is an electronic exchange, that means that trading is based on an automated processing of instructions to buy and sell securities. The types of trades are Automatic, SPAD, Block trades. In this study we will concentrate on the data from the SPAD.

1.2.1 SPAD (System for Support of the Share and Bond Markets)

The SPAD trading is based on the activities of the market makers and it is divided into two phases, an open phase and a closed phase. The closed phase serves basically to the market makers to clear the trades. The open phase stands from 9:30 - 16:00 CET. During the open phase market makers are obliged to continually quote purchase and sale prices of the issues for which they act as market makers. These quotes must lie within the *allowable spread*, defined by the best quotation increased by 0.5% in each direction. The members of the PSE shall also report all over-the-counter (OTC) trades they perform within 5 minutes during the open phase and within 60 minutes during the closed phase of the SPAD. During the open phase, the OTC trades with SPAD securities are also subject to price limitations.

[1] Source: [Exc]

Chapter 2

The ACD Model

2.1 Background

Consider a stochastic process that is a sequence of arrival times $\{t_0, t_1, \ldots\}$. The counting function $N(t)$ indicates the number of events which have occurred by time t. If there is any characteristics associated with the arrival times, they are called *marks* and the process is then called *marked point process*.[1]

The *conditional intensity process* for t_i, $N(t)$ is defined by

$$\lambda(t|N(t), t_0, t_1, \ldots, t_{N(t)}) = \lim_{\Delta t \to 0} \frac{P(N(t+\Delta t) > N(t)|N(t), t_0, t_1, \ldots, t_{N(t)})}{\Delta t}. \tag{2.1}$$

The conditional intensity[2] process is a *self-exciting* point process, because the past events impact the probability structure of the future events. If the intensity is influenced only by m most recent events we call it the *m-memory self-exciting process.*

There are many ways how to parameterize the conditional intensity. For example the model can be formulated in calender time:

$$\lambda(t|N(t), t_0, t_1, \ldots, t_{N(t)}) = \omega + \sum_{i=1}^{N(t)} \pi_i(t - t_i), \tag{2.2}$$

$$\omega \in \mathbf{R}, \qquad \pi_i \text{ is a function}$$

or based on the intervals between events:

$$\lambda(t|N(t), t_0, t_1, \ldots, t_{N(t)}) = \omega + \sum_{i=1}^{N(t)} \pi_i(t_{N(t)-i+1} - t_{N(t)-i}), \tag{2.3}$$

$$\omega \in \mathbf{R}, \qquad \pi_i \text{ is a function.}$$

[1] A more precise definition can be found for instant in [Dal02]

[2] In this case we consider the intensity to be a function of the condition.

2.2 Specification

The ACD model was introduced by [ER98]. They have suggested a model where the process of the durations x_i between events is specified by:

$$x_i = \varphi_i(x_{i-1}, \dots, x_1; \theta)\epsilon_i \qquad x_i = t_i - t_{i-1}, \qquad i = 1 \dots n \tag{2.4}$$

where φ_i is a function of the past durations and a deterministic parameter θ and where ϵ_i is an i.i.d. sequence of random variables with the density $p(\epsilon, \phi)$ and the expectation $E(\epsilon_i) = 1$.

Given this settings, φ_i is equal to the conditional expectation of the ith duration:

$$E(x_i|x_{i-1}, \dots, x_1; \theta) = \varphi_i(x_{i-1}, \dots, x_1; \theta). \tag{2.5}$$

Let p_0 be the density function of ϵ and S_0 be the associated survival function.[3] Let's define the *baseline hazard* as

$$\lambda_0(t) = \frac{p_0(t)}{S_0(t)}. \tag{2.6}$$

The the *conditional intensity* of the ACD model may be expressed as

$$\lambda(t|N(t), t_1, \dots, t_{N(t)}) = \lambda_0\left(\frac{t - t_{N(t)}}{\varphi_{N(t)+1}}\right)\frac{1}{\varphi_{N(t)+1}}. \tag{2.7}$$

2.3 Parameterization of φ_i

The following parameterization of φ_i was suggested by [ER98]:

$$\varphi_i = \omega + \sum_{j=0}^{p} \alpha_j x_{i-j} + \sum_{j=0}^{q} \beta_j \varphi_{i-j} \tag{2.8}$$

To assure the non-negativity of the expected durations, we set restrictions $\omega > 0$, $\alpha_j \geq 0$, $\beta_j \geq 0$ and to assure the stability we require $\alpha_j + \beta_j < 1$ for all j.

The autoregressive conditional model using the specification (2.8) is called the ACD(p,q). In general, ACD(1,1) through ACD(3,3) seem to be sufficient for practical purposes.

If we want to study the influence of some marks associated with the arrival times, we add them to the specification:

$$\varphi_i = \omega + \sum_{j=0}^{m} \alpha_j x_{i-j} + \sum_{j=0}^{q} \beta_j \varphi_{i-j} + \gamma^T \mathbf{z_i} \tag{2.9}$$

where γ is an r-dimensional parameter vector, $\gamma_j > 0, j = 1 \dots r$, and $\mathbf{z_i}$ is a vector of r nonnegative exogenous variables associated with the i-th arrival time. The variables can be for instance the volume or the spread associated with the last event.

[3] Survival function describes the probability that an event did not happen until the moment t: $S_0(t) = 1 - \int_0^t p_0(s)ds$

[BG00] introduced the log-ACD model, in which we do not need to imply the non-negativity restrictions on the parameters:

$$\log(\varphi_i) = \omega + \sum_{j=0}^{m} \alpha_j \log(x_{i-j}) + \sum_{j=0}^{q} \beta_j \log(\varphi_{i-j}) + \vec{\gamma}_i^T \mathbf{z_i} \tag{2.10}$$

With this specification the non-negativity of the expected durations is guaranteed regardless of the parameter values. This is very helpful if we want to perform empirical testing of the market microstructure theories.

2.4 Distribution of ϵ

[ER98] suggest two possible specifications for the density of ϵ. The simplest one is to assume, that ϵ_i, $i = 1 \dots n$, have the exponential distribution with the parameter $\lambda = 1$. The baseline hazard $\lambda_0(t)$ is then equal to 1 and the conditional intensity is

$$\lambda(t|N(t), t_1, \dots, t_{N(t)}) = \varphi_{N(t)+1}^{-1}. \tag{2.11}$$

This model is called the *exponential ACD model (EACD).*

In the EACD model we actually assume, that the probability that the event $N(t_0)+1$ arrives in the time interval $(t, t+\Delta t),\ t > t_0$ is independent on the distance $t - t_0$ (until the arrival of the event $N(t_0)+1$).

A more flexible option, also suggested by [ER98], is to apply the Weibull distribution as the density of ϵ_i. The density of the Weibull distribution is:

$$w(x; \kappa, \gamma) = \gamma \kappa^\gamma x^{\gamma-1} \exp\left\{-(\kappa x)^\gamma\right\}. \tag{2.12}$$

To assume, that a x has Weibull distribution, is equivalent to the assumption, that x^γ has the exponential distribution. So, if γ equals 1, the Weibull distribution is equivalent to the exponential distribution with the parameter κ. In the ACD model a necessary condition for the distribution of ϵ_i is a unit mean, whereas the mean of the Weibull distribution equals $\kappa^{-\gamma}\Gamma\left(1+1/\gamma\right)$. So we define ϵ_i to have the Weibull distribution divided by the mean, where κ is equal to 1. Then the density of ϵ_i is:

$$w^*(u; 1, \gamma) = \gamma\Gamma\left(1+1/\gamma\right)^\gamma u^{\gamma-1} \exp\left\{-\left(\Gamma\left(1+1/\gamma\right) u\right)^\gamma\right\} \tag{2.13}$$

The survival function derived from the transformed Weibull distribution $w^*(u; 1, \gamma)$ equals $S_0 = e^{-(\Gamma(1+1/\gamma)u)^\gamma}$ and the relevant baseline hazard function equals

$$\lambda_0(u) = \gamma\Gamma\left(1+1/\gamma\right)^\gamma u^{\gamma-1}. \tag{2.14}$$

The baseline hazard is now either increasing or decreasing. Long durations are more likely in case that γ is smaller than 1 or less likely if γ is greater than 1. The conditional intensity equals:

$$\lambda(t|x_{N(t)}, \dots, x_1) = \gamma \left(\frac{\Gamma\left(1+1/\gamma\right)}{\varphi_{N(t)+1}}\right)^\gamma (t - t_{N(t)})^{\gamma-1}. \tag{2.15}$$

This model is called the *Weibull-ACD model (WACD)*.

[Lun98] introduced a modification of the ACD model using the generalized gamma distribution and [GM98] studied a model based on the Burr distribution. These models are more flexible because the shapes of their hazard functions can be constant, monotonic as well as U-shaped. The exponential and Weibull distributions are special cases of those distributions.

2.5 The Log-likelihood Function

Now we can derive the log-likelihood functions for the EACD and the WACD model.

Let ϵ_i be i.i.d with exponential distribution and parameter $\lambda = 1$. The conditional density of x_i, $x_i = \varphi_i \epsilon_i$, is

$$f_i(x) = e^{-\frac{x}{\varphi_i}} \frac{1}{\varphi_i}, \tag{2.16}$$

hence the log-likelihood function for EACD model is equal to

$$L(\theta) = -\sum_{i=1}^{N(T)} \left\{ \log(\varphi_i) + \frac{x_i}{\varphi_i} \right\}. \tag{2.17}$$

Now let ϵ_i be i.i.d with Weibull distribution w^*. The density of x_i is

$$f_i(x) = \gamma x^{\gamma-1} \left(\frac{\Gamma\left(1 + 1/\gamma\right)}{\varphi_i} \right)^{\gamma} \exp\left\{ - \left(\frac{\Gamma\left(1 + 1/\gamma\right) x}{\varphi_i} \right)^{\gamma} \right\} \tag{2.18}$$

and the log-likelihood function for the WACD model equals

$$L(\theta) = \sum_{i=1}^{N(T)} \log\left(\frac{\gamma}{x_i}\right) + \gamma \log\left(\frac{\Gamma(1 + 1/\gamma)x_i}{\varphi_i}\right) - \left(\frac{\Gamma(1 + 1/\gamma)x_i}{\varphi_i}\right)^{\gamma}. \tag{2.19}$$

Chapter 3

MLE

To obtain the estimates of the parameters in the ACD model the multivariate maximum likelihood method is used. In this section we would like to approach some theoretical issues concerning the MLE and its approximation.

The objective is to find the maximum of the log-likelihood function $L(\theta)$. Assuming $L(\theta)$ is differentiable, we define the score function as a vector of the first derivatives

$$S(\theta) = \frac{\partial}{\partial \theta} L(\theta), \tag{3.1}$$

so the MLE $\hat{\theta}$ is the solution of the equation $S(\theta) = 0$.

We further define the observed Fisher information matrix $I(\theta)$ as

$$I(\theta) = -\frac{\partial^2}{\partial \theta^2} L(\theta) \tag{3.2}$$

and the expected Fisher information matrix $J_n(\theta) = E_\theta I(\theta)$. Under usual regularity conditions[1] it holds that $I(\theta) = J_n(\theta)$ and $J_n(\theta) = nJ(\theta)$.

In the next three sections we approach the problem of the consistency of the estimates. In the ACD model we assume that ϵ_i are i.i.d. with exponential or Weibull distribution, in reality we usually see, that the residuals are neither i.i.d. nor do they follow the assumed distributions. Using the results of [GMT84] and [LH94] we will show, that MLE estimations of the EACD model and the WACD(1,1) model are consistent.

3.1 MLE under a Wrong Model

In this section we would like to investigate the robustness of our estimates against a wrong model. Let $\hat{\theta}$ be a consistent MLE estimate of the vector θ_0 based on an assumed model $f_\theta(x)$. Let's assume that we do not know the right model. We define:

$$\Gamma \equiv E\left(\frac{\partial \log f_\theta(X)}{\partial \theta}\right)\left(\frac{\partial \log f_\theta(X)}{\partial \theta'}\right)|_{\theta=\theta_0} \tag{3.3}$$

[1]The regularity conditions can be found i.e. in [And02].

and, similarly as in (3.2),

$$I \equiv -E\frac{\partial^2 \log f_\theta(X)}{\partial\theta\partial\theta'}|_{\theta=\theta_0}. \tag{3.4}$$

The expected value is taken with respect to the true but unknown distribution. If $f_\theta(x)$ is the true model, the matrices (3.3) and (3.4) are equal.

The following theorem from [Paw01] describes the asymptotic behavior of the MLE estimates under the assumption of a wrong model.

Theorem 3.1.1 *Let $x_1, \dots x_n$ be an iid. sample. Under usual regularity conditions* [2]

$$\sqrt{n}(\hat{\theta} - \theta_0) \xrightarrow{d} N(0, I^{-1}\Gamma I^{-1}). \tag{3.5}$$

Unfortunately this helpful theorem cannot be applied in the case of the EACD and the WACD model because the residuals ϵ^i are not likely to be i.i.d. In the next three sections we will search for a generalization of this theorem.

3.2 The Pseudo Maximum Likelihood Estimation

[GMT84] have studied the properties of an estimator obtained by maximizing a log-likelihood function associated with a family, which does not necessarily contain the true probability distribution. They call this method the *pseudo maximum likelihood method.*

[GMT84] show that the log-likelihood functions based on a linear exponential family give consistent and asymptotically normal estimators of the parameters:

Definition 3.2.1 *A family of probability measures on $\mathbb{R}^m$, indexed by parameter $\chi \in \mathcal{M} \subset \mathbb{R}^m$ is called linear exponential, if:*

- *every element of the family has a density function with respect to a given measure $\nu(du)$ and this density can be written as*

$$l(u,\chi) = \exp\left\{A(\chi) + B(u) + C(\chi)u\right\}, \qquad u \in \mathbb{R}^m, \tag{3.6}$$

where $A(\chi)$ and $B(u)$ are scalars and $C(\chi)$ is a row vector of size m;

- *χ is the mean of the distribution distribution, whose density is equal to $l(u,\chi)$.*

We estimate θ_0 in the model:

$$y_t = f(x_t, \theta_0) + e_t, \tag{3.7}$$

where $\theta_0 \in \Theta \subset \mathbb{R}^k, x_t \in \mathbb{R}^p, y_t \in \mathbb{R}^m$ and $e_t \in \mathbb{R}^k$. Let's assume that the conditional distribution of $e_1, \dots, e_T$ given $x_1, \dots, x_T$ is equal to the product of the conditional distributions $\mathcal{L}(e_t, x_t)$, where $\mathcal{L}(e_t, x_t = x) = \mathcal{L}(e_s, x_s = x)$, $t \neq s$. The true unknown

[2] [And02]

conditional distribution of y_t given x_t, will be denoted as $\lambda_0(x_t, \theta_0)$. Further assume that $E(y_t|x_t) = f(x_t, \theta_0)$ and that conditional variance matrix $\Omega_0(x_t)$ exists for all x_t. Under these assumptions [GMT84] prove that the estimates of θ_0 obtained by maximizing of

$$\sum_{t=1}^{T} \log l(y_t, f(x_t, \theta)), \tag{3.8}$$

where $l(u, \chi)$ is a family of probability distribution functions, are strongly consistent.

Further they prove following theorem:

Theorem 3.2.2 *If $(l(u, \chi), \chi \in \mathcal{M})$ is a linear exponential family, the pseudo maximum likelihood estimator $\hat{\theta}_T$ is such that:*

$$\sqrt{T}(\hat{\theta}_T - \theta_0) \xrightarrow{d} N(0, I^{-1}\Gamma I^{-1}) \tag{3.9}$$

where I and Γ are matrices (3.3) and (3.4), respectively (Section 3.1).

To apply their approach for the EACD(p,q) model we need to realize two things:

1. The results of [GMT84] remain valid even if we consider a slightly different initial model:

$$y_t = f(x_t, \theta_0)e_t. \tag{3.10}$$

Proof: In their study, [GMT84] do not use the form (3.7) of the model in any of the proofs. Thus we may consider the model (3.7) to be only illustrative. If we use another form of the model, the results presented by [GMT84] remain valid as long as the log-likelihood function has the appropriate form.

□

2. For the exponential ACD model $l(u, \chi)$ takes following form:

$$l(x_i, \varphi_i) = e^{-\frac{x_i}{\varphi_i}} \frac{1}{\varphi_i} = e^{-\frac{x_i}{\varphi_i} - \log(\varphi_i)}. \tag{3.11}$$

Thus the likelihood function of the exponential ACD model belongs to the linear exponential family, where $A(\varphi_i) = -\log(\varphi_i)$ and $C(\varphi_i) = -\frac{1}{\varphi_i}$. This linear exponential family is also known as the *Gamma exponential family*.

From the statements 1 and 2 above we may conclude, that the estimates of the parameters in the EACD(p,q) model are consistent and asymptotically normal.

The Weibull distribution belongs to the exponential family, but not to the linear exponential family and thus we cannot apply the results obtained by [GMT84]. In the next two sections we will study a different approach to this problem, which can be used for the Weibull ACD(1,1) model.

3.3 Consistency of the EACD(1,1) Based on [LH94]

[ER98] point out the analogy between the ACD model and the GARCH model. The ARCH model was originally introduced by [Eng82] and the generalized ARCH (GARCH) by [Bol86]. [LH94] proved the consistency of the quasi-maximum likelihood[3] estimates in the GARCH(1,1) model. Using the analogy between the ACD and GARCH model, [ER98] derive the consistency of the quasi-maximum likelihood estimates in the EACD(1,1) model.

In the first part of this section we summarize the results from [LH94]. In the second part we discuss a theorem from [ER98]. In this theorem they prove that the EACD(1,1) model can be linked to the results obtained by [LH94].

[LH94] consider an observed sequence y_t such that

$$y_t = \gamma_0 + \epsilon_t, \qquad t = 1, \ldots, n, \tag{3.12}$$

where $E(\epsilon_t|I_{t-1}) = 0$ and $I_t = \sigma(\epsilon_t, \epsilon_{t-1}, \ldots)$.

h_t is defined as the conditional variance of y_t:

$$h_t \equiv E(\epsilon_t^2|I_{t-1}) \tag{3.13}$$

and we assume, that h_t follows the GARCH(1,1) process:

$$h_t = \omega(1-\beta) + \alpha\epsilon_{t-1}^2 + \beta h_{t-1} \text{ a.s.}, \tag{3.14}$$

where $\theta \equiv [\gamma, \omega, \alpha, \beta]'$ and $h_1(\theta) = \omega$, $\theta \in \Theta$, Θ is a compact parameter space.

Further we define the rescaled variable

$$z_t = \frac{\epsilon_t}{h_t^{1/2}} \tag{3.15}$$

From the definitions of z_t and h_t, $E(z_t|I_{t-1}) = 0$ a.s. and $E(z_t^2|I_{t-1}) = 1$ a.s.

The estimation of GARCH is frequently done under the assumption, that z_t is *i.i.d.* with $N(0,1)$ distribution. In that case, the log-likelihood takes the form (ignoring constants)

$$L_n(\theta) = \frac{1}{2n}\sum_{t=1}^{n} l_t(\theta) \qquad l_t(\theta) = -\left(\log h_t(\theta) + \frac{e_t^2}{h_t(\theta)}\right). \tag{3.16}$$

However, in reality we do not know the correct density of z_t. Thus we refer to the likelihood as a quasi-likelihood.

The objective of [LH94] was to prove, that under following assumptions, the log-likelihood function defined in (3.16) will consistently estimate the parameters of the GARCH(1,1) model even if the random variable z_t is neither Gaussian nor *i.i.d.*.

The assumptions stated by [LH94] are:

[3] In contrast to the full likelihood, in the quasi-maximum likelihood framework we do not specify the probability structure, but only the mean and variance function. The study of [LH94] is based on the assumption, that the mean and variance functions of the model is specified correctly, and then the Gaussian likelihood is used as a vehicle to estimate the parameters.

Assumptions 3.3.1 *(i) z_t is strictly stationary and ergodic and z_t^2 is nondegenerate.*

(ii) $sup_t E(\log(\beta + \alpha z_t^2)|I_{t-1}) < 0$ a.s.

(iii) $E(z_t^4|I_{t-1}) \leq K \leq \infty$ a.s. (uniformly bounded conditional fourth moment of z_t)

(iv) θ lies in the interior of Θ

[LH94] prove that following theorem holds:

Theorem 3.3.2 *Under Assumptions 3.3.1*

$$\hat{\theta}_n \xrightarrow{p} \theta_0, \tag{3.17}$$

where $\hat{\theta}_n = argmax_{\theta \in \Theta} L_n(\theta)$.

They further present a result concerning the asymptotic normality of the quasi-MLE:

Theorem 3.3.3 *Under Assumptions 3.3.1*

$$\sqrt{n}(\hat{\theta}_n - \theta_0) \xrightarrow{D} N(0, I^{-1}\Gamma I^{-1}) \tag{3.18}$$

where I and Γ are matrices (3.3) and (3.4), respectively (Section 3.1).

The following equation provides a consistent estimate of the matrix Γ:

$$\hat{\Gamma}_n(\theta) = \frac{1}{n}\sum_{t=1}^{n} \partial l_t(\theta)\partial l_t(\theta)', \qquad \hat{\Gamma}_n = \hat{\Gamma}_n(\hat{\theta}_n) \tag{3.19}$$

Corollary to [LH94], [ER98] link the EACD(1,1) model to the GARCH(1,1) model:

Lemma 3.3.4 *Consider a model*

$$x_i = \varphi_i \epsilon_i. \tag{3.20}$$

where $E(x_i|I_{i-1}) = \varphi_i$:

$$\varphi_i = \omega + \alpha x_{i-1} + \beta \varphi_{i-1}. \tag{3.21}$$

and for ϵ_i it holds:

(i) ϵ_i is strictly stationary and ergodic, nondegenerate

(ii) $sup_i E[\log(\beta + \alpha\epsilon_i|I_{i-1})] < 0$ a.s.

(iii) ϵ_i has bounded conditional second moments.

(iv) $\theta = [\omega, \alpha, \beta]'$ is in the interior of Θ

(v) $\varphi_1 = \omega/(1-\beta)$ (a start-up condition)

then the maximizer of L,

$$L(\theta) = -\sum_{i=1}^{N(T)} \left\{ \log(\varphi_i) + \frac{x_i}{\varphi_i} \right\} \tag{3.22}$$

will be consistent estimator of θ_0 and asymptotically normal with the covariance matrix given in (3.18).

Sketch of the proof: Let's define:

$$y_i = d_i \sqrt{x_i}, \tag{3.23}$$

where d_i is independent of x_i and it is *i.i.d.* and equal to 1 with probability 0.5 and to -1 with probability 0.5.

Then the expected value of y_i is $\gamma_0 = E(y_i) = 0$ and the variance of y_i equals φ_i. Thus the random variable $y_i/\sqrt{\varphi_i}$ is analogous to z_i defined in (3.15).

If we substitute (3.23) in the log-likelihood given in (3.22) we receive the Gaussian likelihood presented in (3.16).

The last step to prove the consistency of the quasi-maximum likelihood estimates is to ensure, that $y_i/\sqrt{\varphi_i}$ will meet the Assumptions 3.3.1 requested by [LH94]. This maintain the conditions (i) through (v) in Lemma 3.3.4.

Assumption 3.3.1 (i) follows from the condition (i) in Lemma 3.3.4 because the square root of a strictly stationary, ergodic and nondegenerate positive random variable is also strictly stationary, ergodic and nondegenerate. Assumptions 3.3.1 (ii), (iii) and (iv) are equivalent to the conditions (ii), (iii) and (iv) of Lemma 3.3.4.

□

Theorem 3.3.5 *Let's consider the EACD(1,1) model, where ϵ_i is not i.i.d. with exponential distribution, but it is:*

(i) strictly stationary, ergodic and nondegenerate

(ii) $E\epsilon_i = 1$

(iii) ϵ_i has bounded conditional second moments

(iv) $\varphi_1 = \omega/(1-\beta)$.

Then the estimates of the parameters obtained by maximizing the log-likelihood function given in (2.17) are consistent and asymptotically normal quasi-maximum likelihood estimates.

Proof: Theorem 3.3.5 follows from the fact that the EACD(1,1) model fulfills the conditions given in Lemma 3.3.4.

□

The great advantage of this result is that it covers the cases, when ϵ_i is not i.i.d.

3.4 The Consistency of the Quasi-Maximum Likelihood Estimates for the WACD(1,1) Model

[ER98] expect that the result from previous section can not be easily extended above EACD(1,1) but they express a believe that similar results will be true for higher order models.

Here we would like to present a proof that similar results as in the previous section are also valid for the Weibull ACD(1,1) model. The following lemma is a modification of Lemma 3.3.4:

Lemma 3.4.1 *Consider a model*

$$x_i = \varphi_i \epsilon_i. \tag{3.24}$$

where $E(x_i|I_{i-1}) = \varphi_i$,

$$\varphi_i = \omega + \alpha x_{i-1} + \beta \varphi_{i-1}. \tag{3.25}$$

and for ϵ_i *it holds:*

(i) $\epsilon_i = x_i/\varphi_i$ *is strictly stationary and ergodic, nondegenerate*

(ii) $sup_i E[\log(\beta + \alpha\Gamma(1+1/\gamma)^\gamma \epsilon_i^\gamma | I_{i-1})] < 0$ *a.s.*

(iii) ϵ_i^γ *has bounded conditional second moments*

(iv) $\theta_0 = [\omega_0, \alpha_0, \beta_0]'$ *is the interior of* Θ

(v) $\varphi_1 = \omega/(1-\beta)$

(vi) $0 < \gamma \leq K < \infty$.

Then the maximizer of L,

$$L(\theta) = \sum_{i=1}^{N(T)} \log\left(\frac{\gamma}{x_i}\right) + \gamma \log\left(\frac{\Gamma(1+1/\gamma)x_i}{\varphi_i}\right) - \left(\frac{\Gamma(1+1/\gamma)x_i}{\varphi_i}\right)^\gamma \tag{3.26}$$

will be consistent quasi-maximum likelihood estimator of θ_0 *and asymptotically normal with the robust covariance matrix given in 3.18.*

Proof: Let's denote

$$\rho_i = \frac{\Gamma(1+1/\gamma)^\gamma}{\varphi_i^{\gamma-1}}.$$

We define a random process $y_i, i = 1 \ldots n$

$$y_i = d_i \sqrt{\rho_i x_i^\gamma}. \tag{3.27}$$

The expected value of y_i equals 0 and the variance equals:

$$var(y_i) = E(y_i)^2 = \rho_i E(x_i)^\gamma. \tag{3.28}$$

$$E(x_i)^\gamma = \int_0^\infty x_i^\gamma \gamma x_i^{\gamma-1} \left(\frac{\Gamma(1+1/\gamma)}{\varphi_i}\right)^\gamma \exp\left\{-\left(\frac{\Gamma(1+1/\gamma)\, x_i}{\varphi_i}\right)^\gamma\right\} dx_i$$

We substitute $u = \left(\frac{\Gamma(1+1/\gamma)x_i}{\varphi_i}\right)^\gamma$ and we receive:

$$\begin{aligned} E(x_i)^\gamma &= \left(\frac{\varphi_i}{\Gamma(1+1/\gamma)}\right)^\gamma \int_0^\infty u \exp\{-u\}\, du \\ &= \left(\frac{\varphi_i}{\Gamma(1+1/\gamma)}\right)^\gamma \left\{\left[-ue^{(-u)}\right]_0^\infty + \int_0^\infty e^{(-u)} du\right\} \\ &= \left(\frac{\varphi_i}{\Gamma(1+1/\gamma)}\right)^\gamma \end{aligned}$$

Thus

$$var(y_i) = \varphi_i.$$

So the expected value as well as the variance of $y_i/\sqrt{\varphi_i}$ are equal to the expected value and variance of z_i.

Further we need to verify that if we substitute y_i from (3.27) in the density function of the Weibull distribution (with mean 1) we receive the quasi log-likelihood formula from [LH94] given in (3.16).

The density function of Weibull distribution with mean 1 is (Equation 2.18):

$$f(x_i) = \gamma x_i^{\gamma-1} \left(\frac{\rho_i}{\varphi_i}\right) \exp\left\{-\left(\frac{\rho_i x_i^\gamma}{\varphi_i}\right)\right\} \tag{3.29}$$

and the inverse function to (3.27) and its derivative equal:

$$x_i = \left(\frac{y_i^2}{\rho_i}\right)^{\frac{1}{\gamma}} \qquad \frac{\partial x_i}{\partial y_i} = \frac{2}{\gamma\ \rho_i^{1/\gamma}}\ y_i^{\frac{2-\gamma}{\gamma}}. \tag{3.30}$$

We can see that d_i disappears from the equation.

We substitute (3.30) into (3.29) and we receive the density function of y_i:

$$g(y_i) \;=\; 2y_i \exp\left\{\frac{y_i^2}{\varphi_i}\right\} \frac{1}{\varphi_i}$$

The log-likelihood function derived from this density (ignoring constants) equals:

$$\tilde{L}(\theta) = -\sum_{i=1}^{n} \left(\log(\varphi_i) + \frac{y_i^2}{\varphi_i}\right). \tag{3.31}$$

and this is equal to the log-likelihood given in (3.16).

Finally we need to prove, that y_i meets Assumptions 3.3.1.

The assumption (i) follows from realizing, that finite positive power of a strictly stationary, ergodic and nondegenerate positive random variable is also strictly stationary, ergodic and nondegenerate. If we multiple this strictly stationary, ergodic and nondegenerate random variable by a constant it will keep these characteristics.

To fulfill the assumption (ii), following equation must hold:

$$sup_i E[\log(\beta + \alpha\,(y_i/\sqrt{\varphi_i})^2\,|I_{i-1})] < 0.$$

We substitute (3.27) and we receive:

$$sup_i E[\log(\beta + \alpha\Gamma(1+1/\gamma)^\gamma \left(\frac{x_i}{\varphi_i}\right)^\gamma |I_{i-1})] = sup_i E[\log(\beta + \alpha\Gamma(1+1/\gamma)^\gamma \epsilon_i^\gamma |I_{i-1})]. \quad (3.32)$$

From the assumption (iii) we have: $E(y_i^4|I_{i-1}) < K < \infty$.

$$\begin{aligned} E(y_i^4|I_{i-1}) &= E\left[(\rho_i\; x_i^\gamma)^2\,|I_{i-1}\right] \\ &= E\left[(\rho_i\,(\varphi_i\epsilon_i)^\gamma)^2\,|I_{i-1}\right] \\ &= \Gamma(1+1/\gamma)^{2\gamma}\varphi_i^2 E(\epsilon_i^{2\gamma}|I_{i-1}) \end{aligned}$$

Thus $E(\epsilon_i^{2\gamma}|I_{i-1}) < K' < \infty \rightarrow E(y_i^4|I_{i-1}) < K < \infty$.

Assumptions 3.3.1 (iv) and (v) are equivalent to the conditions (iv) and (v) of the Lemma.

□

Theorem 3.4.2 *Let's consider the WACD(1,1) model, where ϵ_i is not i.i.d. with Weibull distribution, but it is:*

(i) strictly stationary, ergodic and nondegenerate

(ii) $E\epsilon_i^\gamma \leq \frac{1-\beta}{\alpha\;\Gamma(1+1/\gamma)^\gamma}$

(iii) ϵ_i^γ *has bounded conditional second moments*

(iv) $\varphi_1 = \omega/(1-\beta)$.

Then the estimates of the parameters obtained by maximizing the log-likelihood function given in (2.19) are consistent and asymptotically normal quasi-maximum likelihood estimates.

Proof: We want to apply Lemma 3.4.1. The conditions of the Lemma 3.4.1, except for the condition (ii), follow trivially from the conditions of Theorem 3.4.2 and the definition of the WACD(1,1) model. To prove the condition (ii) we use the Jensen's inequality:

$$sup_i E\left[\log(\beta + \alpha\Gamma(1+1/\gamma)^\gamma \epsilon_i^\gamma)|I_{i-1}\right] \leq sup_i \left\{\log E\left[\beta + \alpha\Gamma(1+1/\gamma)^\gamma\epsilon_i^\gamma|I_{i-1}\right]\right\}. \quad (3.33)$$

Further, from the condition (ii) of Theorem 3.4.2 follows:

$$E\left[\beta + \alpha\Gamma(1+1/\gamma)^\gamma\epsilon_i^\gamma|I_{i-1}\right] = \beta + \alpha\Gamma(1+1/\gamma)^\gamma E(\epsilon_i^\gamma|I_{i-1}) \leq 1,$$

which, in the combination with (3.33), verifies (ii) of Lemma 3.4.1.

□

Chapter 4

Modeling of the Intervals Between Trades of KB Stock Using the ACD Model

4.1 Data Description

The study is based on the data extracted from the Trade and Quote dataset provided by the Prague Stock Exchange. The period lasts from 5th January 2000 to 12th November 2004. The data concern twelve securities, which were traded in SPAD during this period. There are two datasets: intra-day trade data and intra-day quote data. The first dataset contains detailed information on every single trade; the time and date of the trade, the price, the volume, the number of shares sold and the information whether the trade was conducted within the open phase of the SPAD or not. The latter dataset records the quote process; the date and time of the quote posted, the relevant bid and ask price and the instantaneous allowable spread.

We use the ACD model to describe the process of the waiting times (further in this study we will call them durations) between the trades of Komerční Banka stock in the period from 2nd January 2004 to 12th November 2004. We delete the trades, which were performed during the closed phase of SPAD and if there are multiple trades within one second, we consider them to be only one trade. After these adjustments we are left with 17319 trades.

We ignore the overnight waiting times and we calculate the durations between trades. The minimum time between two trades is 1 second, maximum duration is 16900 (4 hours, 41 minutes and 40 seconds). The average duration equals 297 seconds (4 minutes and 57 seconds).

In Figure 4.1 a histogram of the durations is plotted.

The ACD model is proposed especially for correlated durations. Table 4.1 contains autocorrelations and partial autocorrelations of the durations. We use the Ljung-Box statistic[1] to test the hypothesis, that the first 10 autocorrelations are equal to zero. The

[1] The Ljung-Box statistic was introduced by [LB78]. They refined the portmanteau test of white noise,

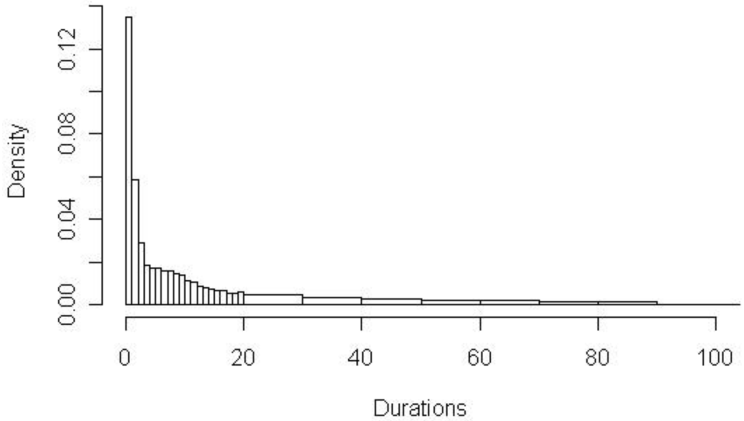


Figure 4.1: Histogram of the durations between trades.

	Lag 1	Lag 2	Lag 3	Lag 4	Lag 5	Lag 6	Lag 7	Lag 8	Lag 9	Lag 10
ACF	0.130	0.046	0.049	0.048	0.026	0.014	0.026	0.009	0.001	0.017
PACF	0.130	0.030	0.041	0.036	0.013	0.005	0.019	-0.001	-0.004	0.014

Table 4.1: Autocorrelations and partial autocorrelations of the durations between trades.

5% critical value equals 18.3, so we reject the null hypothesis with the test statistic equal to 443.4 and the corresponding $p - value < 2.2e^{-16}$.

4.2 Daily Pattern of the Transaction Durations

In the previous studies of this topic ([ER98], [BZ06]) it was shown, that the durations between trades exhibit strong diurnal pattern. The trading activity is high at the beginning of the open phase, becomes lower in the middle of the day and raises again at the end of the trading. The rise after 15:00 in the afternoon may be also connected with the opening of the stock markets in the USA.

Therefore we can view the duration process as if it consisted of two components; deterministic component specified by the time-of-day effect and a stochastic component described by the ACD model. Following [ER98], we will define the time-of-day effect as

which was introduced in [BP70]. They present the statistic

$$Q_m = n(n+2)\sum_{k=1}^{m}(n-k)^{-1}r_k^2,$$

where

$$r_k = \frac{1/n\sum_{t=k+1}^{n}(\epsilon_t - \bar{\epsilon})(\epsilon_{t-k} - \bar{\epsilon})}{1/n\sum_{t=1}^{n}(\epsilon_t - \bar{\epsilon})^2}.$$

They show that for white noise sequence ϵ_i, where n is large enough and $m << n$, the Q_m statistic has approximately χ^2_m distribution.

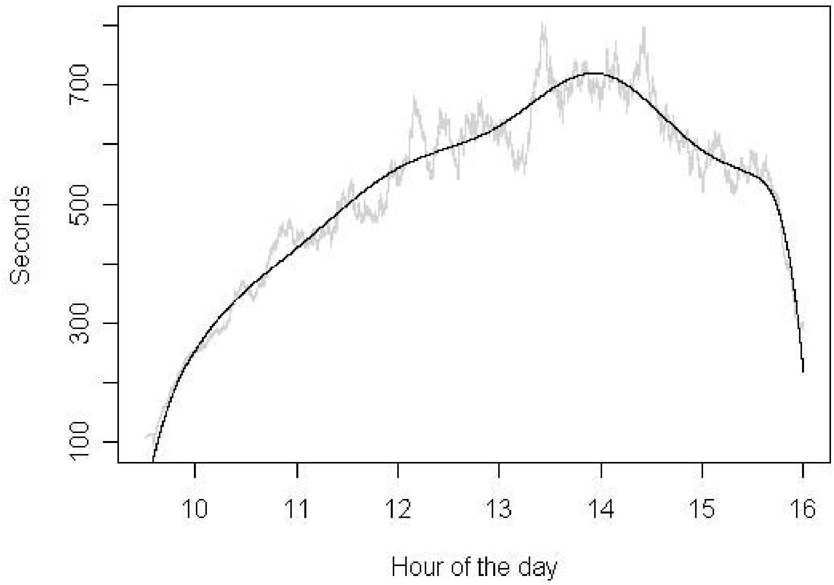


Figure 4.2: The estimate of a daily pattern for transaction durations.

a multiplicative component:

$$\tilde{x_i} = x_i \phi_i \tag{4.1}$$

where $\tilde{x}_i$ are the original raw durations between trades, x_i are diurnally adjusted durations and ϕ_i is a function describing the time-of-day effect.

The time-of-day function can be either included in the model and the parameters can be estimated simultaneously with the parameters of the ACD model or it can be removed before the estimation of the ACD. [ER98] perform the joint estimation of the parameters and diurnal effect but they report, that the results are very similar for both procedures. The majority of the empirical studies (i.e. [BG00], [BZ06]) use the two-step procedure.

In this part of the study we follow the two-step procedure as it was described by [BG00]. We calculate the smoothed durations for every time point t of the trading day as follows: we average the durations over $t \pm 5$ minute interval and then we smooth the time-of-day function by a cubic spline.

In Figure 4.2 the time-of-day function ϕ is plotted. The gray line designates the durations averaged over $t \pm 5$ minutes intervals, the solid black line plots the cubic spline with 19 equidistant nods. This picture illustrates the general pattern (inverted U shape), the daily patterns of the durations may differ quite strongly on daily or weekly basis.

Finally we calculate the diurnally adjusted durations. The minimum of the adjusted durations is 0.00197, average lies at 0.988 and maximum at 117.8. In Figure 4.3 the histogram of the diurnally adjusted durations is plotted.

In Table 4.2 the autocorrelations and partial autocorrelations of the adjusted durations are recorded. We calculate the Ljung-Box statistic of the null hypothesis that the first 10 autocorrelations are equal to zero. The hypothesis is rejected with the test statistic equal to 316 and the corresponding $p-value < 2.2 \times 10^{-16}$. The autocorrelations of the

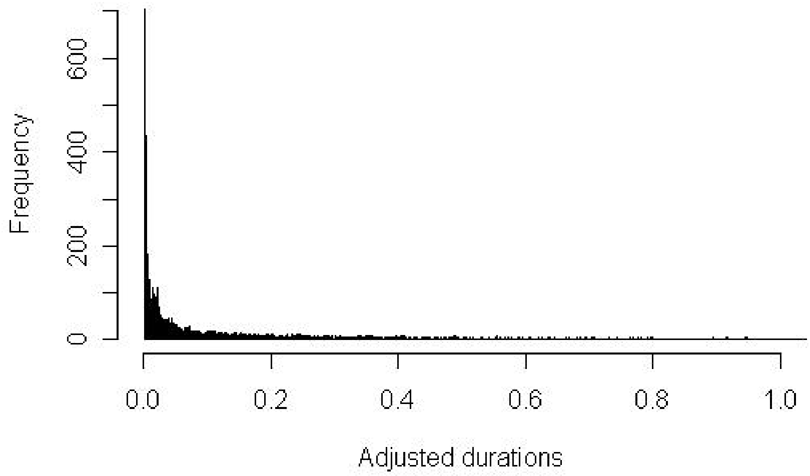


Figure 4.3: Histogram of the diurnally adjusted durations.

	Lag 1	Lag 2	Lag 3	Lag 4	Lag 5	Lag 6	Lag 7	Lag 8	Lag 9	Lag 10
ACF	0.110	0.036	0.031	0.037	0.027	0.013	0.031	0.016	0.007	0.015
PACF	0.110	0.024	0.025	0.030	0.019	0.005	.027	0.008	0.001	0.012

Table 4.2: Autocorrelations and partial autocorrelations of the diurnally adjusted durations.

adjusted durations are lower than by the raw durations, but they are still significantly different from zero, which signals that the autocorrelation of the raw durations are not only a result of the daily factor.

4.3 The Exponential ACD Model of the Adjusted Durations

We want to perform the maximum likelihood estimation of the EACD(1,1) and EACD(2,2) models. For the maximization of the likelihood function the BHHH algorithm[2] with analytical derivatives is used.

In the case of the EACD(2,2) model the estimate of the parameter α_2 does not significantly differ from zero. Therefore we omit the parameter α_2 in the model and we perform the estimation of the parameters of the EACD(1,2).

In Table 4.3 the estimated parameters are listed.

The sums $\alpha_1 + \beta_1 = 0.5536$ for the EACD(1,1) model and $\alpha_1 + \beta_1 + \beta_2 = 0.6089$ for the EACD(1,2) are low in comparison to the results obtained by [ER98], which indicates that in this model the past durations and past expected durations have rather low impact on the present durations.

[2]The BHHH algorithm was introduced by [BHHH74].

EACD (1,1)			
	ω	α_1	β_1
Estimate	0.4489	0.2855	0.2678
p-value	0.00042	1.79×10^{-8}	0.162

EACD (1,2)				
	ω	α_1	β_1	β_2
Estimate	0.4324	0.2852	0.2233	0.1004
p-value	0.0235	1.01×10^{-06}	0.1864	0.2858

Table 4.3: Parameter estimates for the exponential ACD(1,1) and exponential ACD(1,2) model. *Notes:* $\varphi_i = \omega + \alpha_1 x_{i-1} + \beta_1 \varphi_{i-1}$ *in ACD(1,1) resp.* $\varphi_i = \omega + \alpha_1 x_{i-1} + \beta_1 \varphi_{i-1} + \beta_2 \varphi_{i-2}$ *in ACD(1,2),* ω *is positive,* α_1*,* β_1 *and* β_2 *are positive and* $\in (0;1)$.

Bellow each parameter the appropriate $p - value$ for the test of H_0*: parameter is equal to 0* is presented. The test is performed using the normal approximation with the covariance matrix $I^{-1}\Gamma I^{-1}$ defined in Section 3.1. The consistency and asymptotic normality for the EACD estimates, when the residuals ϵ_i are not i.i.d. with exponential distribution, was discussed in Section 3.2 and for the EACD(1,1) additionally in Section 3.3. For the matrix Γ we use the approximation (3.19) presented by [LH94].

In Picture 4.4 the raw durations, their estimates and the time-of-day effect for one day are plotted. As it is mentioned above, the daily pattern of the durations may differ heavily from the time-of-day function. This day is one of the days with rather higher number of trades, therefore the time-of-day function is generally higher than the durations.

We test the hypothesis H_0: EACD(1,1) against H_1: EACD(1,2). The likelihood ratio statistic equals 56 and it has asymptotically χ^2_1 distribution. Thus we reject H_0 with p-value equal to 3.5×10^{-13}.

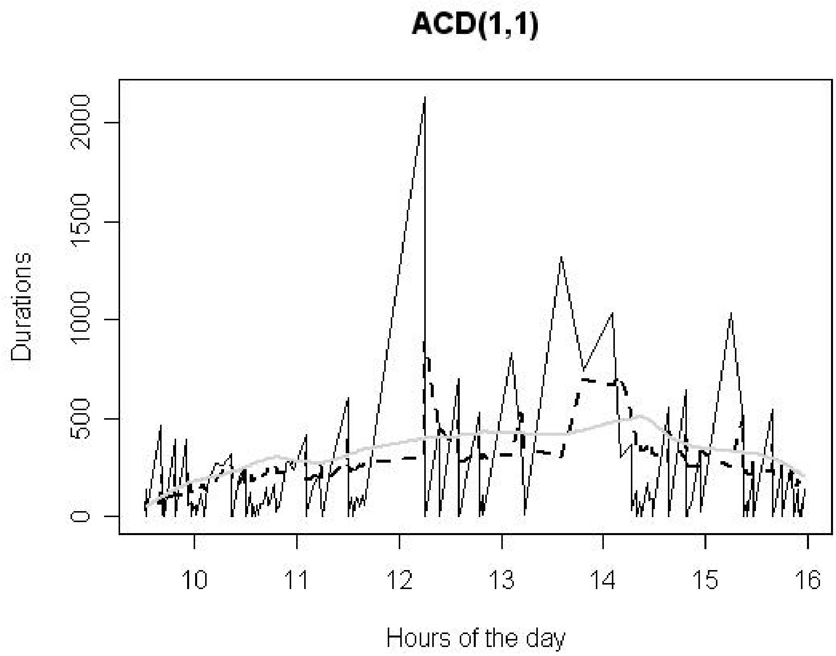
4.3.1 Standardized Durations

The ACD model is supposed to capture the intertemporal correlations of the durations (or diurnally adjusted durations). The model assumes that the stochastic information of the model (in our model described as ϵ_i) is i.i.d. To test this assumption let's define the *standardized durations*:

$$\hat{\epsilon}_i = \frac{x_i}{\hat{\varphi}_i} = \frac{\tilde{x}_i}{\hat{\varphi}_i \phi_i} \tag{4.2}$$

where x_i ($\tilde{x}_i$) are diurnally adjusted durations (raw durations), φ_i are the estimations of the expected durations and ϕ_i is the time-of-day function. The statistics about the standardized durations are in Table 4.4.

We perform the Ljung-Box test with 10 lags to test the hypothesis, that the autocorrelations are equal to zero. The Ljung-Box test statistic for the standardized residuals from the EACD(1,1) model equals 53.3 with the corresponding $p - value$ equal to 6.676×10^{-8} and for the standardized residuals from the EACD(1,2) model it is 55.9,



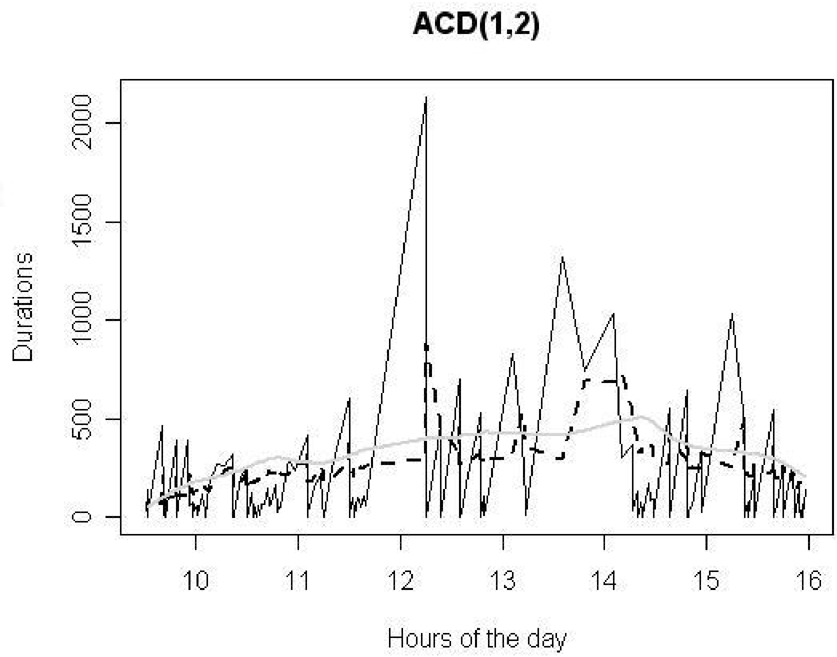


Figure 4.4: Plot of the estimated durations on 10.5.2004. The solid line shows the raw durations, dashed line are the estimated durations and the gray line presents the time-of-day effect.

	EACD (1,1)	EACD (1,2)
Mean	0.9999	1.004
Standard deviation	2.640	2.667
Ljung box (lag 10)	53.3	55.9

Table 4.4: Statistic from the standardized durations.

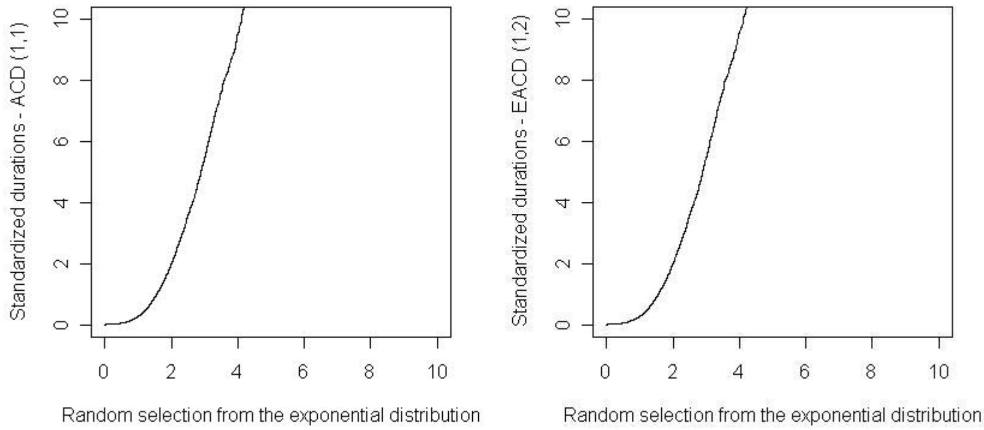


Figure 4.5: The QQ-Plots of the standardized durations from the EACD (1,1) resp. EACD (1,2) model and a random selection from exponential distribution.

$p - value = 2.086 \times 10^{-8}$. This is much less than the original values for the diurnally adjusted durations (316), but it is still much higher than the 5% critical value of the χ^2_{10} distribution (18.3). This shows that the model was not fully successful in capturing the intertemporal dependencies.

We performed the calculation under the assumption, that the standardized residuals have exponential distribution with parameter $\lambda = 1$. From the qq-plot in Figure 4.5 we see, that the standardized durations do not seem to have exponential distribution. Further evidence comes from the high values of standard deviations of the standardized durations.

4.4 The Weibull ACD Model of the Adjusted Durations

The maximum likelihood estimation of the WACD(1,1) and the WACD(1,2) is performed using the BHHH algorithm with analytical derivatives. The WACD(1,2) model is presented instead of the WACD(2,2) model because the estimates of α_2 in the WACD(2,2) model do not differ significantly from zero. In Table 4.5 the estimated parameters are listed. The consistency and the asymptotic normality of the WACD(1,1) model was proved in Theorem 3.4.2. In the case of the WACD(1,2) model the consistency and asymptotic normality has not been proved until now so the results should be taken only as informative.

WACD (1,1)				
	ω	α_1	β_1	γ
Estimate	0.4956	0.5870	0.04832	0.4444
p-value	0	7.9×10^{-16}	0.073	0

WACD (1,2)					
	ω	α_1	β_1	β_2	γ
Estimate	0.4883	0.5872	0.04123	0.01295	0.4445
p-value	0	1.3×10^{-15}	0.0951	0.0234	0

Table 4.5: Parameter estimates for the WACD(1,1) and the WACD(1,2) model. *Notes:* $\varphi_i = \omega + \alpha_1 x_{i-1} + \beta_1 \varphi_{i-1}$ *(WACD(1,1)) resp.* $\varphi_i = \omega + \alpha_1 x_{i-1} + \beta_1 \varphi_{i-1} + \beta_2 \varphi_{i-2}$ *(WACD(1,2)),* ω *is positive,* α_1*,* α_2*,* β_1 *and* β_2 *are positive and* $\in (0;1)$.

	WACD (1,1)		**WACD (1,2)**	
	$\hat{\epsilon}$	$\hat{\epsilon}^{\gamma}$	$\hat{\epsilon}$	$\hat{\epsilon}^{\gamma}$
Mean	1.1467	0.6597	1.146	0.660
Standard deviation	3.235	0.7279	3.223	0.728
Ljung box (lag 10)	96.2	36.6	91.5	36.01

Table 4.6: Statistic from the standardized durations and transformed standardized durations.

From Table 4.5 we see that the values of the estimated parameters in the WACD models are different from the values in the EACD model. The high value of the estimate of α_1 signals strong influence of the previous durations on the present durations. The estimated value of γ is smaller than 1, thus the baseline hazard function is decreasing.

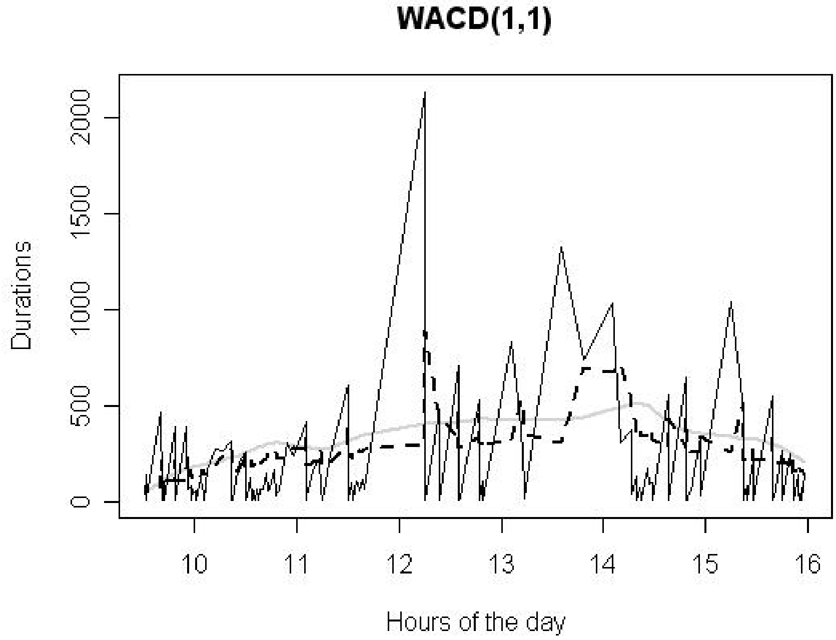
The WACD model is equivalent to the EACD model with γ equal to one. We use the likelihood ratio to test $H_0 : \gamma = 1$ (EACD(1,1) or EACD(1,2)) against $H_1 : \gamma \neq 1$ (WACD(1,1) or WACD(1,2)). The test strongly rejects the null hypothesis in both cases.

4.4.1 Standardized Durations

If the Weibull specification is correct, then if we raise the standardized durations $\hat{\epsilon_i}$ defined in 4.2 to the power γ the obtained process should be i.i.d with exponential distribution, where $\lambda = 1$.

In Table 4.6 the mean, standard deviation and the Ljung-Box test statistic for the standardized durations $\hat{\epsilon_i}$ and the transformed standardized durations $\hat{\epsilon_i}^{\gamma}$ are presented. We can see, that neither the mean nor the standard deviation of $\hat{\epsilon_i}^{\gamma}$ are at least close to 1 and thus $\hat{\epsilon_i}^{\gamma}$ is not a random sample from exponential distribution.

Similar as [ER98], we conclude that neither the EACD nor the WACD model has successfully estimated the distribution of the residuals. Lately, models based on other distributions with non-monotonic hazard functions were presented. For a good review of these models see [Vuo06]. These models outperform the exponential ACD and the Weibull



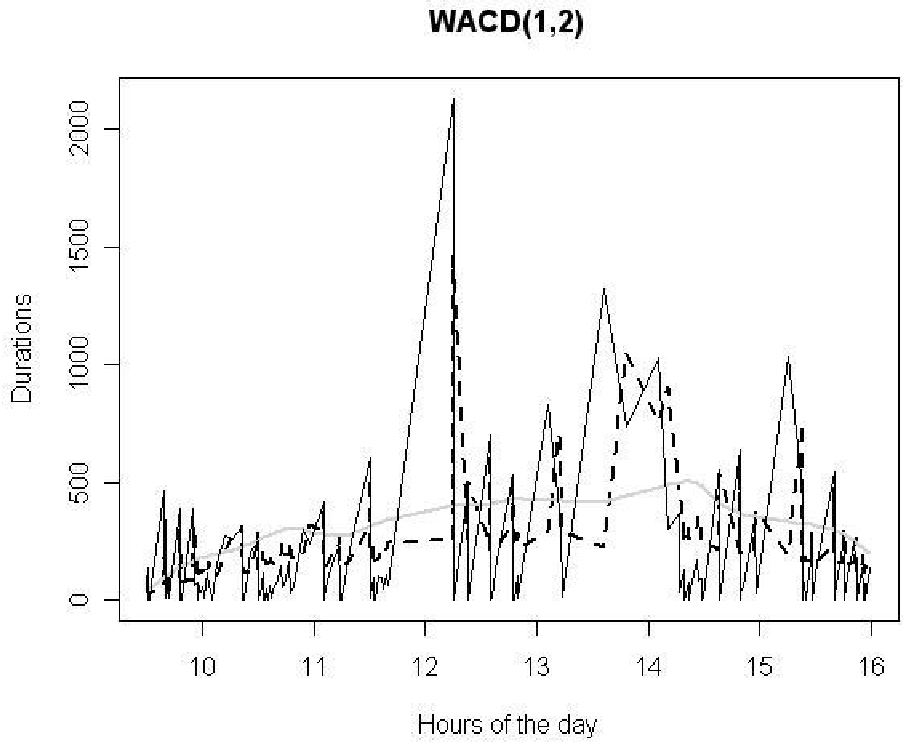


Figure 4.6: Plot of the estimated durations on 10.5.2004. The solid line shows the raw durations, dashed line are the durations estimated with WACD(1,1) (upper plot) and WACD(1,2) (lower plot) and the gray line presents the time of day function.

ACD model and they show the way for the future research.

Chapter 5

The ACM Model

The goal of the ACM model is to develop an econometric model for discrete-valued, irregularly-spaced time series data. The model estimates the joint distribution of the discrete price changes y_i and the durations $\tau_i = t_i - t_{i-1}$ between them conditional on the past durations and price changes and other explanatory variables such as volume, spread e.g.

Let p_i be the price associated with the ith transaction and $y_i = p_i - p_{i-1}$ the corresponding price change. The price changes y_i are discrete and they can take k unique values.

Let's denote the conditional density as follows:

$$f(y_i, \tau_i | \mathbf{y^{i-1}}, \boldsymbol{\tau}^{\mathbf{i-1}}), \tag{5.1}$$

where $\mathbf{y^{i-1}} = (y_{i-1}, ..., y_1)$ and $\boldsymbol{\tau}^{\mathbf{i-1}} = (\tau_{i-1}, ..., \tau_1)$.

[ER05] operate with the decomposition of the joint conditional density into a product of the conditional density of the price changes y_i and marginal density of the durations τ_i.

$$f(y_i, \tau_i | \mathbf{y^{i-1}}, \boldsymbol{\tau}^{\mathbf{i-1}}) = g(y_i | \mathbf{y^{i-1}}, \boldsymbol{\tau}^{\mathbf{i}}) q(\tau_i | \mathbf{y^{i-1}}, \boldsymbol{\tau}^{\mathbf{i-1}}) \tag{5.2}$$

Thanks to the decomposition (5.2) we can study the price changes and the durations separately.

The instantaneous probability that the ith event exits to state Y, given the duration τ since the last event, is called the hazard function. The hazard function is defined by:

$$\theta_i(y, \tau) = \lim_{\Delta t \to 0} \frac{P(y_i = Y, \tau \leq \tau_i \leq \tau + \Delta t | \tau_i \geq \tau, \mathbf{y^{i-1}}, \boldsymbol{\tau}^{\mathbf{i-1}})}{\Delta t} \tag{5.3}$$

For small values of Δt Equation 5.3 is approximately equal to the probability, that the process exits to state Y in the time period $[\tau, \tau + \Delta t]$, given that there is no event by the duration τ since the last event. Thus the hazard function can be obtained from (5.2) by dividing the marginal density of the durations by the probability, that the event has not occurred by the time τ:

$$\theta_i(y, \tau) = \kappa(\tau | \mathbf{y^{i-1}}, \boldsymbol{\tau}^{\mathbf{i-1}}) g(y | \mathbf{y^{i-1}}, \boldsymbol{\tau}^{\mathbf{i}}) \tag{5.4}$$

where

$$\kappa(\tau|\mathbf{y^{i-1}}, \tau^{\mathbf{i-1}}) = \frac{q(\tau_i|\mathbf{y^{i-1}}, \tau^{\mathbf{i-1}})}{1 - \int_0^\tau q(s|\mathbf{y^{i-1}}, \tau^{\mathbf{i-1}})ds}$$

is the hazard function associated with the distribution of the durations.

For the modeling of $q(.)$ we will use the ACD model, the specification of $g(.)$ will be discussed in the next section.

5.1 The Specification of $g(.)$

Let k denote the number of states which can be taken by the random variable y_i.

Let $\tilde{\mathbf{x}}_\mathbf{i}$ be a k-dimensional unit vector. The vector $\tilde{\mathbf{x}}_\mathbf{i}$ has 1 at the jth position, if the ith event y_i takes the j-th state. At other positions of the vector $\tilde{\mathbf{x}}_\mathbf{i}$ there is a zero.

Let $\tilde{\pi}_\mathbf{i}$ denote the k-dimensional vector of conditional probabilities associated with the i-th event: the j-th position of $\tilde{\pi}_\mathbf{i}$ describes the conditional probability, that the i-th event takes the j-th state (that is the probability, that the j-th element of $\tilde{\mathbf{x}}_\mathbf{i}$ will equal 1).

The following equation links these two vectors:

$$\tilde{\pi}_\mathbf{i} = \mathbf{P}\tilde{\mathbf{x}}_\mathbf{i-1} \tag{5.5}$$

where $\mathbf{P}$ is a $(k \times k)$-dimensional transition matrix.

A transition matrix must satisfy:

1. all elements are non-negative
2. each column must sum to unity.

For the first order Markov chain, the transition matrix is constant. In more general settings, the matrix $\mathbf{P}$ can be conditional on past events and their arrival times and it may vary with the information available at the time t_{i-1}. To satisfy the restrictions 1. and 2. in the case of a time-varying conditional transition matrix [ER05] use the inverse logistic transformation.

Let $\tilde{\pi}_{im}$ and $\tilde{x}_{ij}$ denote the m-th and j-th element of the vectors $\tilde{\pi}_\mathbf{i}$ and $\tilde{\mathbf{x}}_\mathbf{i}$ respectively. We denote:

$$\mathbf{h}(\pi_\mathbf{i}) = log \begin{pmatrix} \frac{\tilde{\pi}_{i1}}{\tilde{\pi}_{ik}} \\ \vdots \\ \frac{\tilde{\pi}_{\mathbf{i(k-1)}}}{\tilde{\pi}_{ik}} \end{pmatrix} = log \left(\frac{\pi_\mathbf{i}}{1 - \mathbf{1}' \, \pi_\mathbf{i}} \right) \tag{5.6}$$

where $\pi_\mathbf{i}$ is a $(k-1)$-dimensional vector created from $\tilde{\pi}_\mathbf{i}$ by erasing the kth element of the vector. (From here on, letters with tildes $(\tilde{\mathbf{x}}_\mathbf{i}, \tilde{\pi}_\mathbf{i})$ will describe k-dimensional vectors and letters without tilde $(\mathbf{x}_\mathbf{i}, \pi_\mathbf{i})$ will describe $(k-1)$-dimensional vectors obtained through erasing one of the elements of the vector.)

We define:

$$\mathbf{h}(\pi_\mathbf{i}) = \mathbf{P}^* \mathbf{x}_\mathbf{i-1} + \mathbf{c} \tag{5.7}$$

where $\mathbf{P}^*$ is a $(k-1)\times(k-1)$-dimensional matrix and $\mathbf{c}$ is a $(k-1)$-dimensional vector.

Given (5.7) the vector $\tilde{\pi}_\mathbf{i}$ can be recovered from any values of $\mathbf{P}^*$ and $\mathbf{c}$ by

$$\pi_\mathbf{i} = \frac{\exp(\mathbf{P}^*\mathbf{x_{i-1}} + \mathbf{c})}{1 + \mathbf{1}'\exp(\mathbf{P}^*\mathbf{x_{i-1}} + \mathbf{c})}, \tag{5.8}$$

where $\exp(\mathbf{P}^*)$ is interpreted as a matrix with the m, n element equal to $\exp(P_{m,n})$.

From this expression we can see that all probabilities $\tilde{\pi}_{im}$ including the probability for the kth state ($\tilde{\pi}_{ik} = 1 - \mathbf{1}'\pi_\mathbf{i}$) are positive and sum to unity.

From (5.7) we also obtain the expression for the transition matrix $\mathbf{P}$:

$$P_{mn} = \frac{\exp(P^*_{mn} + c_m)}{1 + \sum_{j=1}^{k-1}\exp(P^*_{jn} + c_j)}, \qquad m, n = 1\ldots(k-1) \tag{5.9}$$

$$P_{kn} = 1 - \sum_{j=1}^{k-1} P_{jn}, \qquad n = 1\ldots(k-1) \tag{5.10}$$

$$P_{mk} = \frac{\tilde{\pi}_\mathbf{i} - \sum_{j=1}^{k-1} P_{n,j}\tilde{\mathbf{x}}_\mathbf{j}}{\tilde{\mathbf{x}}_\mathbf{k}}, \qquad m = 1\ldots k \tag{5.11}$$

where (5.10) is obtained from condition 2. and (5.11) comes from Equation 5.5:

All elements of $\mathbf{P}$ are positive and the columns sum to unity. It follows that if we estimate the matrix $\mathbf{P}^*$ and the vector $\mathbf{c}$ we do not need to imply any restrictions on the parameters.

By generalizing (5.7) we receive a more elaborative dynamic structure with the dependence on richer information. In the following definition we generalize the transition matrix from a time-invariant one into a time-varying conditional transition matrix.

Definition 5.1.1 *The Autoregressive Conditional Model ACM(p,q,r) is given by*

$$\mathbf{h}(\pi_\mathbf{i}) = \sum_{\mathbf{j=1}}^{\mathbf{p}} \mathbf{A_j}(\mathbf{x_{i-j}} - \pi_{\mathbf{i-j}}) + \sum_{\mathbf{j=1}}^{\mathbf{q}} \mathbf{B_j}\mathbf{h}(\pi_{\mathbf{i-j}}) + \mathbf{C}\mathbf{z_i} \tag{5.12}$$

where $\mathbf{A_j}$, $\mathbf{B_j}$ *denote* $(k-1)\times(k-1)$*-dimensional parameter matrices with time subscripts,* $\mathbf{B_j}$ *is a diagonal matrix,*

$\mathbf{z_i}$ *is an* $(r+1)$*-dimensional vector,*

$\mathbf{C}$ *denotes a* $(k-1)\times(r+1)$*-dimensional parameter matrix.*

The specification in (5.12) describes the transition probabilities of the random variable y_i. The vector $\mathbf{z_i}$ contains 1 in the first element to form a constant and r explanatory variables such as duration, spread, volume of the past trades etc.

From Definition 5.1.1 it is apparent how the history impacts the transition probabilities. The $(k-1)$-dimensional vector of probabilities $\pi_{\mathbf{i}}$ is obtained from:

$$\pi_{\mathbf{i}} = \left[exp\left(\sum_{j=1}^{p}\mathbf{A_j}(\mathbf{x_{i-j}}-\pi_{\mathbf{i-j}})+\sum_{j=1}^{q}\mathbf{B_j}\mathbf{h}(\pi_{\mathbf{i-j}})+\mathbf{C}\mathbf{z_i}\right)\right] \times\left[1+\mathbf{1}'exp\left(\sum_{j=1}^{p}\mathbf{A_j}(\mathbf{x_{i-j}}-\pi_{\mathbf{i-j}})+\sum_{j=1}^{q}\mathbf{B_j}\mathbf{h}(\pi_{\mathbf{i-j}})+\mathbf{C}\mathbf{z_i}\right)\right]^{-1}, \quad (5.13)$$

and the kth probability is recovered from the condition, that all elements mus sum to unity. At the time $i-1$ knowing all past $\mathbf{x}$ and π we can calculate the $\pi_{\mathbf{i}}$. Consequently given some starting values we can construct a full sequence of the probabilities π from the observations $\mathbf{x}$. From this the likelihood function and its numerical derivatives can be evaluated.

5.2 The Log-Likelihood Function for the ACD-ACM Model

Given initial conditions we can calculate the entire path of the $\pi_{\mathbf{i}}$. Hence the likelihood can be constructed as a product of the conditional densities. The log-likelihood of the ACM model is then expressed as

$$L=\sum_{i=1}^{N}\sum_{j=1}^{k}\tilde{x}_{ij}\log(\tilde{\pi}_{ij})=\sum_{i=1}^{N}\tilde{\mathbf{x}}_{\mathbf{i}}'\log(\tilde{\pi}_{\mathbf{i}}) \quad (5.14)$$

where $\tilde{x}_{ij}$ is the jth element of $\tilde{\mathbf{x}}_{\mathbf{i}}$, $\tilde{\pi}_{ij}$ denotes the jth element of $\tilde{\pi}_{\mathbf{i}}$ and N denotes the number of observations. Under the usual regularity conditions[1] the parameter estimates are consistent and asymptotically normal.

Each of the likelihood functions depend on different parameters, thus the estimation of the ACD and ACM parameters may be done separately by maximizing two log-likelihood functions: the ACM log-likelihood given in (5.14) and $\sum_{i=1}^{N}q(\tau_i|\mathbf{y^{i-1}},\tau^{\mathbf{i-1}})$, which is specified by the ACD model.

5.3 Model Diagnostic

We denote the conditional variance matrix of $\mathbf{x_i}$ as $\mathbf{V_i}$:

$$\mathbf{V_i}=\mathrm{Var}(\mathbf{x_i}|I^{i-1}) \quad (5.15)$$

where I^{i-1} is the information available at the time $i-1$.

$$\mathbf{V_i}=E(\mathbf{x_i}\mathbf{x_j}')-E(\mathbf{x_i})E(\mathbf{x_j}')=diag(\pi_{\mathbf{i}})-\pi_{\mathbf{i}}\pi_{\mathbf{j}}' \quad (5.16)$$

[1] [And02]

If we substitute $\pi_{\mathbf{i}}$ with $\hat{\pi}_{\mathbf{i}}$ we obtain the sample variance matrix.

[ER05] suggest following model diagnostic tests: Let's consider a sequence of errors

$$\mathbf{v}_{\mathbf{i}}^{*} = \mathbf{x}_{\mathbf{i}} - \pi_{\mathbf{i}}. \tag{5.17}$$

$\mathbf{v}_{\mathbf{i}}^{*}$ forms a heteroscedastic difference sequence, whose conditional variance matrix equals $\mathbf{V}_{\mathbf{i}}$.

Standardized errors are constructed by premultiplying $\mathbf{v}_{\mathbf{i}}^{*}$ by the Cholesky decomposition of the conditional variance matrix. The Cholesky decomposition matrix $\mathbf{L}_{\mathbf{i}}$ is a lower triangular matrix, for which holds: $\mathbf{V}_{\mathbf{i}} = \mathbf{L}_{\mathbf{i}}\mathbf{L}_{\mathbf{i}}^{\mathbf{T}}$.

Thus the standardized errors equal:

$$\mathbf{v}_{\mathbf{i}} = \mathbf{L}_{\mathbf{i}}^{-\mathbf{1}}\mathbf{v}_{\mathbf{i}}^{*}. \tag{5.18}$$

$\mathbf{v}_{\mathbf{i}}$ should be uncorrelated with the past and its variance matrix should be equal to the $(k-1) \times (k-1)$ identity matrix. The series of the sample standardized residuals $\hat{\mathbf{v}}_{\mathbf{i}}$ are constructed using the parameter estimates: $\hat{\mathbf{v}}_{\mathbf{i}} = \mathbf{x}_{\mathbf{i}} - \hat{\pi}_{\mathbf{i}}$. To see how good does the ACM model capture the intertemporal correlations of the data we test whether the sequence of the sample standardized residuals $\hat{\mathbf{v}}_{\mathbf{i}}$ is uncorrelated.

Chapter 6

Modeling of the ACM-ACD process

6.1 Data

Using the ACM-ACD model we will analyze the intra-day quote data of the Telecom stock. The period analyzed lasts from 2nd January 2004 to 17th November 2004. In this period the trading activity was higher than in the prior years.

First we have perform minor adjustments of the data. If there are 2 quotes within one second, only the better one is chosen for the model. After these adjustments we are left with 25078 individual events in the sample.

We ignore the overnight waiting times and we calculate the durations t_i between quotes. The average waiting time in the observed period is 205 seconds (6 minutes and 25 seconds), the minimum duration is 1 second and the maximum duration is 13670 seconds (3 hours, 47 minutes and 50 seconds).

The mid-price p_i is obtained by averaging the bid and ask price. The average mid price in the observed period is 323.7 Kč. We define the price change Δp_i as the change in the mid-price: $\Delta p_i = p_i - p_{i-1}$. The minimum price change in the observed period is 0 Kč, the maximum positive change is 6.55 Kč and the maximum negative change is 18.7 Kč.

The ACM model is based on the discrete price changes. From the histogram in Figure 6.1 we can see, that the price changes of ±0.5 Kč, ±0.25 Kč and 0 Kč are the most frequent ones, although all other possible values are represented as well. For the purpose of the ACM analysis we divide the price changes into 5 groups. The instant price change is represented by the state vector $\mathbf{x_i}$:

$$\mathbf{x_i} = \begin{cases} [1,0,0,0] & \text{if} \quad \Delta p_i \leq -0.5 \\ [0,1,0,0] & \text{if} \quad \Delta p_i \in (-0.5; -0.25] \\ [0,0,0,0] & \text{if} \quad \Delta p_i \in (-0.25; 0.25) \\ [0,0,1,0] & \text{if} \quad \Delta p_i \in [0.25; 0.5) \\ [0,0,0,1] & \text{if} \quad \Delta p_i \geq 0.5 \end{cases} \tag{6.1}$$

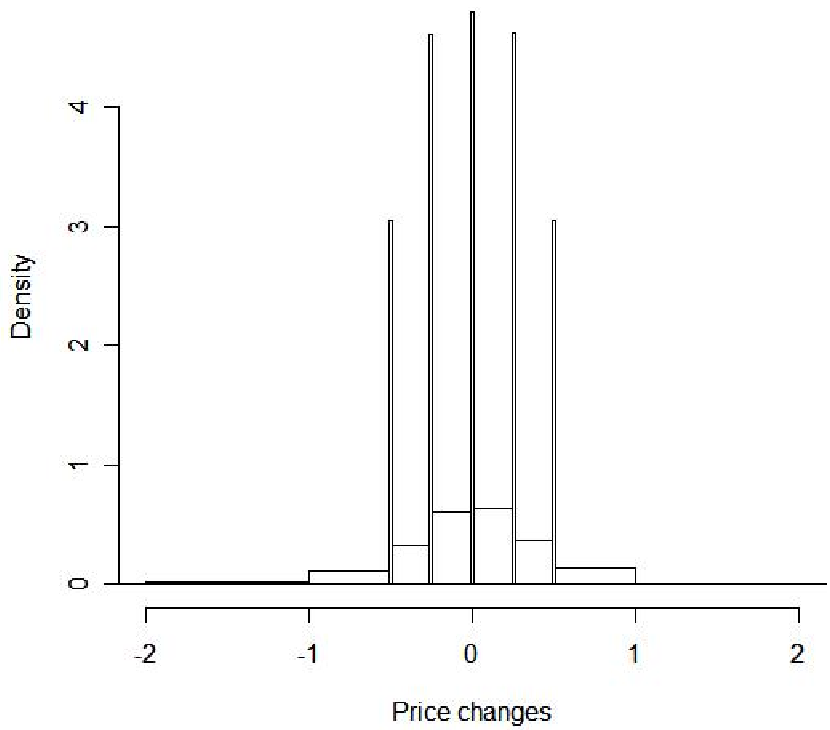


Figure 6.1: Histogram of the price changes.

6.2 The Sample Cross-Correlations of the State Vector Elements

To analyze the inter-temporal dependencies between the price changes we calculate the sample cross-correlations C_s of the state vector $\mathbf{x_i}$. Denoting the mean of $\mathbf{x_i}$ as $\bar{\mathbf{x}}$ the sth sample cross-correlation matrix is calculated as follows:

$$C_s = \frac{N-1}{N-s-1}\left[\sum_{i=1}^{N}(\mathbf{x_i}-\bar{\mathbf{x}})(\mathbf{x_i}-\bar{\mathbf{x}})'\right]^{-1}\left[\sum_{i=s+1}^{N}(\mathbf{x_i}-\bar{\mathbf{x}})(\mathbf{x_{i-s}}-\bar{\mathbf{x}})'\right] \quad (6.2)$$

Following [BT81] the sample cross-correlations are presented in the matrix form in Table 6.1. We test the hypothesis whether the cross-correlations are equal to zero. The signs '+' or '-' designate that the positive and negative cross-correlations (respectively) have exceeded the 5% significance level ($1.96 \times N^{-1/2}$) and '.' designates, that the value is within the 5% confidence interval.

From Table 6.1 it is apparent that there are significant inter-temporal dependencies between the price changes. The upper left and lower right quadrants signal continuation of the price changes and the upper right and lower left quadrants correspond to reversals of the changes. For lag 1 we can see that the extreme price changes (±0.5 Kč) are likely to be reversed but smaller price changes tend to be continued. Beyond the first lag, the main diagonal correlations are generally positive signaling that the started trend is likely to be

$$
\begin{array}{ccccc}
s = & 1 & 2 & 3 & 4 \\
 & \begin{bmatrix} - & + & - & + \\ . & + & - & . \\ . & - & + & . \\ + & - & + & - \end{bmatrix}
 & \begin{bmatrix} + & + & . & . \\ - & + & - & - \\ - & - & + & - \\ . & - & + & + \end{bmatrix}
 & \begin{bmatrix} + & + & - & + \\ . & + & - & - \\ - & - & + & . \\ + & - & + & + \end{bmatrix}
 & \begin{bmatrix} + & . & - & . \\ . & + & - & - \\ - & - & + & . \\ . & - & . & + \end{bmatrix} \\
s = & 5 & 6 & 7 & 8 \\
 & \begin{bmatrix} + & . & - & + \\ + & . & - & - \\ - & - & + & . \\ + & - & . & + \end{bmatrix}
 & \begin{bmatrix} + & . & - & . \\ . & + & - & - \\ - & . & + & . \\ + & - & . & + \end{bmatrix}
 & \begin{bmatrix} + & . & - & + \\ + & . & - & - \\ - & - & + & . \\ + & - & . & + \end{bmatrix}
 & \begin{bmatrix} + & . & - & . \\ . & + & - & - \\ - & . & + & . \\ . & - & . & + \end{bmatrix} \\
s = & 9 & 10 & & \\
 & \begin{bmatrix} + & . & . & + \\ . & . & - & . \\ - & . & . & . \\ + & . & . & + \end{bmatrix}
 & \begin{bmatrix} + & . & . & + \\ . & + & - & . \\ - & . & + & . \\ + & - & . & + \end{bmatrix}
 & &
\end{array}
$$

Table 6.1: Tiao-Box representation of the sample cross-correlations of the state vector x_i.

followed. Significantly positive values in the corners signal, that the bigger price changes are likely to be followed by bigger price changes in either directions whereas smaller price changes are not very likely to be reversed.

The Tiao-Box plot presented by [ER05] is different from ours. We suppose that the main reason is that in their study Engle and Russel operate with the trades dataset and not with the quotes. In the case of the trade process the Tiao-Box plot is strongly influenced by the fact, that both (buy and sell) prices are recorded. Thus many of the price changes are caused by the difference between the buy and the sell price and not by any change on the market.

Finally let's turn our interest to the symmetry of the sample cross-correlation matrices. In most of the matrices we can see that the signs of the correlations reflected through the center of the matrices are the same. This signals that there may be some symmetry in the price process. We discuss this issue later in this study in Section 6.5.

6.3 The Diurnal Effect in the Price Changes Process

As shown in the study [ER98] the durations between trades as well as the durations between price changes follow the inverted U-shape. [BZ06] show, that the durations of the quote process of the stocks traded at the Prague Stock Exchange exhibit similar shape. Surprisingly [ER05] do not observe any daily pattern in the behavior of the state vector $\mathbf{x_i}$.

To see if the state vectors $\mathbf{x_i}$ follow any periodic pattern through the day, we treat each of the elements of $\mathbf{x_i}$ as a univariate time series. We place nodes at each hour of trading. We denote the nodes $u_0, \dots, u_7$, where u_0 and u_7 equal the beginning of trading at 9:30 and the end of trading at 16:00, respectively. The last time interval $[u_6, u_7)$ lasts only 30 minutes. We fit a linear spline in the time of the day by a logistic regression model.

We define θ_{ij}, $i = 1 \dots n, j = 1 \dots 4$ to be the probability that the ith element of the univariate time series $x_{.j}$ equals 1. The logistic regression model takes following form:

$$\log\left(\frac{\theta_{ij}}{1-\theta_{ij}}\right) = d_0 + d_1 I_1(x_{ij}) + d_2 I_2(x_{ij}) + d_3 I_3(x_{ij}) + d_4 I_4(x_{ij}) + d_5 I_5(x_{ij}) + d_6 I_6(x_{ij}) \quad (6.3)$$

where $d_0 \dots d_6$ are the estimated parameters and $I_k(x_{ij}), k = 1 \dots 6$ are functions, which indicate, whether the ith event happened during the kth time interval $[u_{k-1}, u_k)$. Given the observed data, the likelihood function for the logistic regression model equals:

$$L(d_0, \dots, d_6) = \prod_{i=1}^{n} {\theta_{ij}}^{x_{ij}} (1 - \theta_{ij})^{1-x_{ij}}. \quad (6.4)$$

The results of the logistic regression are listed in Table 6.2. If there is no diurnal effect, the coefficients d_1 through d_6 should be close to zero. We test the hypothesis that the coefficients d_1 through d_6 are equal to zero using the likelihood ratio test. The test statistic has χ^2 distribution with 6 degrees of freedom. In Table 6.2 the estimated coefficients, value of the test statistic and the appropriate $p - value$ are listed.

	d_0	d_1	d_2	d_3	d_4	d_5	d_6
$x_{.1}$	-2.073	0.342	0.284	0.216	0.298	0.192	0.174
(p-value)	$(< 2 \cdot 10^{-16})$	$(1.2 \cdot 10^{-7})$	$(7.5 \cdot 10^{-5})$	(0.004)	$(8.4 \cdot 10^{-5})$	(0.013)	(0.018)
		LR statistic:	33.5			p-value:	8.4×10^{-6}
$x_{.2}$	-1.630	-0.013	0.110	0.041	-0.062	0.033	0.040
(p-value)	$(< 2 \cdot 10^{-16})$	(0.816)	(0.079)	(0.532)	(0.366)	(0.624)	(0.539)
		LR-statistic:	8.4			p-value:	0.2
$x_{.3}$	-1.425	-0.141	-0.157	-0.094	-0.081	-0.225	-0.162
(p-value)	$(< 2 \cdot 10^{-16})$	(0.009)	(0.011)	(0.140)	(0.213)	(0.001)	(0.010)
		LR-statistic:	14.7			p-value:	0.023
$x_{.4}$	-2.001	0.352	0.165	0.217	0.194	0.062	0.015
(p-value)	$(< 2 \cdot 10^{-16})$	$(1.9 \cdot 10^{-8})$	(0.02)	(0.003)	(0.010)	(0.425)	(0.832)
		LR-statistic:	50.6			p-value:	3.5×10^{-9}

Table 6.2: Estimation of the diurnal factor of the state vector using the logistic regression.

For comparison we also test the diurnal factor of the durations. The durations are fitted by least squares to a linear spline. Nodes are placed at each hour of the trading with the last interval lasting only half an hour. In Table 6.3 the values of the estimated parameters are presented. An F-statistic with 6 and 25071 degrees of freedom is provided to assess the null hypothesis that the parameters $d_1, \dots, d_6$ are equal to zero - in other words that there is no diurnal effect.

From Table 6.2 we see that for the small upward and downward price change we reject the diurnal effect on the 1% level but for the big upward and downward price change, the p-value of the test is very small. In both cases the parameter d_1 has the highest value and the smallest p-value thus we may assume, that the diurnal effect has the most influence at the beginning of the day. The reason for that may be, that in the morning the market-makers are correcting the prices to reflect the changes and trades which happened overnight. Thus big price changes are more likely to happen.

	d_0	d_1	d_2	d_3	d_4	d_5	d_6
Durations	155.211	-40.771	52.246	85.162	123.178	130.561	105.476
(t-statistic)	(15.961)	(-3.334)	(3.822)	(5.936)	(8.422)	(8.909)	(7.582)
R^2:	0.0125		F statistic	131.6525		p-value:	$(< 2 \cdot 10^{-16})$

Table 6.3: Estimation of the diurnal factor of the quote durations using least square fitting.

The smallest $p-value$ for the test of the diurnal effect has the duration process (Table 6.3). We follow [ER05] and we ignore the diurnal effect of the state vector in the further analyzis but we will calculate with the diurnal effect in the process of durations.

6.4 Selection of the Model

The state vector $\tilde{\mathbf{x}}_\mathbf{i}$, which will be substituted in the log-likelihood function equals:

$$\tilde{\mathbf{x}}_\mathbf{i} = \begin{cases} [1,0,0,0,0] & \text{if} \quad \Delta p_i \leq -0.5 \\ [0,1,0,0,0] & \text{if} \quad \Delta p_i \in (-0.5; -0.25] \\ [0,0,1,0,0] & \text{if} \quad \Delta p_i \in (-0.25; 0.25) \\ [0,0,0,1,0] & \text{if} \quad \Delta p_i \in [0.25; 0.5) \\ [0,0,0,0,1] & \text{if} \quad \Delta p_i \geq 0.5 \end{cases} \tag{6.5}$$

Complicated recursive structure of the ACM log-likelihood function hinders from using analytical derivatives. In combination with a large sample size and a large number of parameters to be estimated, the calculations become very time consuming. Therefore we follow the 'simple to general' model selection procedure. For the selection of the model we use a sample of 1000 observations.

In Table 6.4 the results of the comparison of the nested models using the likelihood ratio statistic are listed. The table shows that we do not reject ACM(2,2) in favor of ACM(3,2) but we have to reject ACM(2,2) in favor of ACM(2,3). Then we do not reject ACM(2,3) in favor of ACM(3,3). Thus we choose the ACM(2,3) model for the modeling of the price dynamics.

In case of the ACD model we reject ACD(1,1) in favor of ACD(1,2) and then the ACD(1,2) is not rejected for ACD(2,2). We can see that there is almost no improvement of the likelihood function from ACD(1,2) to ACD(2,2).

6.5 Symmetry in Price Dynamics

Let us define matrix $\mathbf{Q}$ to be a rotated identity matrix:

$$\mathbf{Q} = \begin{bmatrix} 0 & & & 1 \\ & & \cdot & \\ & \cdot & & \\ \cdot & & & \\ 1 & & & 0 \end{bmatrix}. \tag{6.6}$$

ACM(p,q): $\mathbf{h}(\pi_i) = \mu + \sum_{j=1}^{p} \mathbf{A_j x_{i-j}} + \sum_{j=1}^{q} \mathbf{C_j h}(\pi_{\mathbf{i-j}})$.					
H_0	H_1	LR	df	5% critical value of χ^2_{df}	1% critical value of χ^2_{df}
ACM(1,1)	ACM(2,1)	59.9	16	26.3	31.1
ACM(1,1)	ACM(1,2)	59.8	4	9.5	13.3
ACM(2,1)	ACM(2,2)	49.6	4	9.5	13.3
ACM(1,2)	ACM(2,2)	49	16	26.3	31.1
ACM(2,2)	ACM(2,3)	16	4	9.5	13.3
ACM(2,2)	ACM(3,2)	6	16	26.3	31.1
ACM(2,3)	ACM(3,3)	25	16	26.3	31.1

ACD(m,n): $\varphi_i = \omega + \sum_{j=1}^{m} \alpha_j t_{i-j} + \sum_{j=1}^{n} \beta_j \varphi_{i-j} + \gamma \mathbf{T}_{i-1}$					
H_0	H_1	LR	df	5% critical value of χ^2_{df}	1% critical value of χ^2_{df}
ACD(1,1)	ACD(2,2)	16	2	6.0	9.2
ACD(1,1)	ACD(1,2)	14.66	1	3.8	6.6
ACD(1,2)	ACD(2,2)	1.34	1	3.8	6.6

Table 6.4: Results of the likelihood ratio test for ACM and ACD models

Based on [ER05] we define the symmetry in price dynamics as follows:

Definition 6.5.1 *For the ACM(p,q) model we say, that the transaction price process is dynamic-symmetric for prices if for the matrices* $\mathbf{A_i}$ *and* $\mathbf{B_j}$ *holds:*

$$A_i Q = Q A_i, \qquad \textit{resp.} \qquad B_j Q = Q B_j, \qquad i = 1 \dots p,\ j = 1 \dots q. \tag{6.7}$$

Further we say that the transaction price process is dynamic-symmetric for the j*-th element of* z_i *if the corresponding* j*-th column of the matrix* $\mathbf{C}$ *is a symmetric vector.*

The matrices $\mathbf{A_i}$ and $\mathbf{B_i}$ which hold (6.7) are called *response symmetric*. [ER05] prove that if the ACM model is dynamic-symmetric for price changes and for all elements of $\mathbf{z_i}$, the mirror image history of price changes and $\mathbf{z_i}$ produces a mirror image transition probabilities:

$$\mathbf{Q}\pi_i(\mathbf{x_{i-1}}, \mathbf{x_{i-2}}, \dots, \mathbf{z_i}) = \pi_i(\mathbf{Qx_{i-1}}, \mathbf{Qx_{i-2}}, \dots, \mathbf{Qz_i}) \tag{6.8}$$

Now we can test the hypothesis that the ACM(2,3) model is dynamic-symmetric for prices and for the vector of constants μ. On the dynamic-symmetric ACM(2,3) model we imply following restrictions:

The vector of constants:

$$\mu = (\mu_1, \mu_2, \mu_2, \mu_1)' \tag{6.9}$$

The matrix $\mathbf{A_i}$ equals:

$$\mathbf{A_i} = \begin{bmatrix} a^i_{11} & a^i_{12} & a^i_{13} & a^i_{14} \\ a^i_{21} & a^i_{22} & a^i_{23} & a^i_{24} \\ a^i_{24} & a^i_{23} & a^i_{22} & a^i_{21} \\ a^i_{14} & a^i_{13} & a^i_{12} & a^i_{11} \end{bmatrix}, \qquad i = 1,\ 2. \tag{6.10}$$

And the diagonal matrix $\mathbf{B_j}$ equals:

$$\mathbf{B_j} = \begin{bmatrix} b^j_{11} & 0 & 0 & 0 \\ 0 & b^j_{22} & 0 & 0 \\ 0 & 0 & b^j_{22} & 0 \\ 0 & 0 & 0 & b^j_{11} \end{bmatrix} \qquad j = 1,\ 2,\ 3. \tag{6.11}$$

We test the hypothesis H_0: dynamic-symmetric ACM(2,3) against H_1: unrestricted ACM(2,3). The likelihood ratio statistic equals 23.8 with 24 degrees of freedom and so we do not reject H_0 neither on 1% level with critical value of 43 nor on 5% level with critical value of 36.4. Thus for testing further hypothesis we will use the symmetric ACM(2,3) model.

6.6 Explanatory Variables in the ACD-ACM Model

Prior to the estimation we include the influence of the durations in the ACM model. We add the logarithm of the past durations to the ACM model. The likelihood ratio test suggest, that 2 lags of log-durations should be added to the ACM model. If we add the third lag, the improvement of the log-likelihood function is negligible.

Thus we estimate the parameters of the ACM(2,3) model in the following form:

$$\begin{aligned} \mathbf{h}(\pi_i) &= \mu + \mathbf{A_1}(\mathbf{x_{i-1}} - \pi_{\mathbf{i-1}}) + \mathbf{A_2}(\mathbf{x_{i-2}} - \pi_{\mathbf{i-2}}) \\ &\quad + \mathbf{B_1 h}(\pi_{\mathbf{i-1}}) + \mathbf{B_2 h}(\pi_{\mathbf{i-2}}) + \mathbf{B_3 h}(\pi_{\mathbf{i-3}}) + \mathbf{d_1}\tau_{\mathbf{i-1}} + \mathbf{d_2}\tau_{\mathbf{i-2}} \end{aligned} \tag{6.12}$$

where $\mathbf{A_i}$, $\mathbf{B_j}$ denote (4×4)-dimensional response symmetric parameter matrices, $\mathbf{B_j}$ are diagonal matrices, μ denotes a parameter vector of length 4, representing the constant in the model and $\mathbf{d_i}$ are parameter vectors with 4 elements, linking the influence of the past durations.

Further we need to estimate the $q(\tau_i|\mathbf{y^{i-1}}, \tau^{\mathbf{i-1}})$. We use the exponential log-ACD model. In the log-ACD model has been chosen because we do not need to impose any restrictions on the parameters. Thus it is more suitable for adding further explanatory variables into the model.

In order to improve the flexibility of the model we add the time-of-day function as an explanatory variable to the model. We use following procedure: at first, we estimate the time-of-day function using the same approach as in Section 4.2. We denote this time-of-day function as $T(t)$. The appropriate values of $T(t)$ are added to the model with the weight γ. We estimate the parameter γ jointly with other parameters in the model. The estimated value of the parameter γ shows, to which extent is the diurnal factor responsible for the changes in the durations throughout the trading day.

Further we add the past price changes as explanatory variables to the log-ACD model. The likelihood ratio test suggests that every lag of price changes brings a significant improvement of the log-likelihood function. Anyways adding of too many past lags of the price changes to the model would result in a very high number of parameters in the

$$
s = \quad
\overset{1}{\begin{bmatrix} + & - & \cdot & - \\ \cdot & + & - & \cdot \\ \cdot & - & + & \cdot \\ \cdot & + & \cdot & + \end{bmatrix}}
\overset{2}{\begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}}
\overset{3}{\begin{bmatrix} \cdot & - & + & \cdot \\ \cdot & - & \cdot & \cdot \\ \cdot & - & \cdot & \cdot \\ \cdot & \cdot & - & \cdot \end{bmatrix}}
\overset{4}{\begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ + & \cdot & \cdot & \cdot \end{bmatrix}}
$$

$$
s = \quad
\overset{5}{\begin{bmatrix} \cdot & - & - & \cdot \\ \cdot & \cdot & - & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & + \end{bmatrix}}
\overset{6}{\begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & + & - & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}}
\overset{7}{\begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & + \end{bmatrix}}
\overset{8}{\begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}}
$$

$$
s = \quad
\overset{9}{\begin{bmatrix} + & \cdot & + & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}}
\overset{10}{\begin{bmatrix} + & \cdot & \cdot & + \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}}
$$

Table 6.5: Tiao-Box representation of the sample cross-correlations of the standardized residual errors $\hat{\mathbf{v}}_\mathbf{i}$.

model. Subsequently the estimation would become very time consuming. Therefore we add just 2 lags of the past price changes to the model.

After these adjustments the log-ACD model has the following form:

$$\log(\varphi_i) = \omega + \alpha_1 \log(\tau_{i-1}) + \beta_1 \log(\varphi_{i-1}) + \beta_2 \log(\varphi_{i-2}) + \gamma T(t_{i-1}) + \delta_\mathbf{1}\mathbf{x}_{(\mathbf{i-1})} + \delta_\mathbf{2}\mathbf{x}_{(\mathbf{i-2})} \quad (6.13)$$

where ω, α_1, β_j and γ are scalar parameters and $\delta_\mathbf{j}$ are parameter vectors of length 4.

6.6.1 Model Diagnostics

Before discussing the parameter estimates, we will examine the model diagnostics.

We calculate the standardized errors $\hat{\mathbf{v}}_\mathbf{i}$ and their sample cross-correlations. The Tiao-Box representation of the sample cross-correlations is shown in the table 6.5. The cross-correlations which are significant at the 5% level (exceeding $1.96\sqrt{N}$) are denoted by "+" and "-". The results are presented in Table 6.5. From the table we can see that many of the significant correlations were eliminated. Anyways the first lag sample cross-correlations of the standardized errors are still significantly positive.

The standardized durations $\hat{\epsilon}_i$ from the log-ACD are calculated from Equation 4.2 (Section 4.3.1). $\hat{\epsilon}_i$ should be distributed as i.i.d. and unit exponential. The Ljung-Box test of the null hypothesis, that the first 10 lags of $\hat{\epsilon}$ are uncorrelated has a p-value equal to 0.02. Thus on 1% level we do not reject the hypothesis that $\hat{\epsilon}_i$ are uncorrelated. We can say that we were successful in eliminating the intertemporal dependencies from the data. This may be due to the added explanatory variables or due to a different approach to the diurnal factor in comparison to the procedure presented in Section 4.2.

In Table 6.6 statistics from the standardized residuals are presented. The high value of standard error signals, that the $\hat{\epsilon}_i$ is not a random sample from unit exponential distribution.

	EACD (1,2)
Mean	1.000
Standard deviation	2.590
Ljung box (lag 10)	21

Table 6.6: Statistic from the standardized durations.

6.6.2 Interpretation of the Results

Finally we turn to the interpretation of the results. The estimated parameters are listed in Table 6.7.

Particularly high coefficients in matrices $\mathbf{A_i}$ show a strong impact of the past price changes on contemporaneous ones. The values in $\mathbf{A_2}$ are higher then the values in $\mathbf{A_1}$, which designates, that the market reacts with protracted lag on the past price changes.

The sums of j-th diagonal elements of matrices $\mathbf{B_i}$ characterize the persistence of the j-th state. The persistence is the tendency for continuation in the price process - if the price change is likely to be followed by a price change of the same magnitude. The values of the sums are rather moderate: 0.481 for the bigger price changes and 0.69 for the smaller price changes. This signals low persistence of the price changes.

Further we would like to investigate about the duration$\times$price change relationship. From the vector δ_i in the ACD model we can see that large negative changes in the prices are associated with shorter durations. This assumption is supported by the values of vectors d_i in the ACM model. We may conclude that large negative price changes result in higher activity in the market.

In the ACD model the high value of $\beta_1 + \beta_2$ signals strong persistence of the waiting times between quotes. Further interesting result is the small value of γ which equals 0.323. This shows, that the impact of the time of day function is not so strong as we assume when remove the diurnal factor prior to the estimation of the parameters.

ACM(2,3)									
c	-0.129	-0.660	-0.660	-0.129					
	(0.340)	(0.167)							
$\mathbf{A_1}$	0.194	1.827	-1.053	1.938	$\mathbf{A_2}$	0.883	-0.679	1.217	-0.515
	(0.130)	(0)	(0.154)	(0)		(0.467)	(0.234)	(0.127)	(0.292)
	0.051	0.071	-0.343	0.483		0.503	1.371	0.742	0.447
	(0.112)	(0.013)	(0)	(0.00004)		(0.012)	(0)	(0.012)	(0.212)
	0.483	-0.343	0.071	0.051		0.447	0.742	1.371	0.503
	1.938	-1.053	1.827	0.194		-0.515	1.217	-0.679	0.883
$\mathbf{B_1}$	0.312	0	0	0	$\mathbf{B_2}$	0.321	0	0	0
	(0.376)					(0.486)			
	0	0.617	0	0		0	0.084	0	0
		(0.382)					(0.016)		
	0	0	0.617	0		0	0	0.084	0
	0	0	0	0.312		0	0	0	0.321
$\mathbf{B_3}$	-0.152	0	0	0		$\mathbf{d_1}$	-0.003	$\mathbf{d_2}$	0.001
	(0.418)						(0.470)		(0.479)
	0	-0.011	0	0			-0.028		0.037
		(0.223)					(0.383)		(0.293)
	0	0	-0.011	0			0.082		-0.023
							(0.252)		(0.388)
	0	0	0	-0.152			-0.014		-0.012
							(0.471)		(0.477)
ACD(1,2)									
ω	α_1	β_1	β_2	γ		$\boldsymbol{\delta_1}$	0.133	$\boldsymbol{\delta_1}$	0.191
							(0.306)		(0.285)
-0.537	0.206	0.491	0.152	0.323			0.125		-0.209
(0.424)	(0.004)	(0.043)	(0.047)	(0.039)			(0.286)		(0.178)
							0.023		-0.117
							(0.451)		(0.291)
							-0.050		-0.332
							(0.445)		(0.102)

Table 6.7: Parameter estimates for the ACD(1,2)-ACM(2,3) model. The numbers in brackets represent the $p - value$ of the test for zero of the estimated parameters.

Chapter 7

Conclusion

In this study we applied the ACD model and the ACD-ACM model to the stocks traded at the Prague Stock Exchange. The first half of the study was concentrated on the exponential and the Weibull ACD model. We used several approaches to generalize the asymptotic properties of the estimates. Based on the study [GMT84] we derived the asymptotic properties of the EACD(p,q) model. Corollary to [ER98] we used the analogy with the GARCH model to prove the consistency of the quasi-maximum likelihood estimates of the Weibull ACD(1,1) model.

We applied the EACD and WACD model to the market data of the Komerční Banka stock. We used a two step procedure; first we removed the diurnal factor from the data and then we applied the ACD model. The removement of the diurnal factor did not reduce the large value of the Ljung-Box statistics associated with the waiting times between the transactions. This signals that the large autocorrelations of the durations cannot be explained by the diurnal factor. In contrary, the standardized residuals obtained from the EACD and the WACD model have much lower values of the Ljung-Box statistics, which shows that in both cases the ACD model did a good job in removing the intertemporal dependencies of the durations. The large values of the standard deviation showed that the standardized residuals did not have the assumed exponential or Weibull distribution. Hence neither the EACD model nor the WACD model are a fully appropriate model and in the future research, different distributions should be tested.

In the second half of the study we applied the ACD-ACM model to the quote data of the Telecom stock at the Prague Stock Exchange. Thanks to the assumptions we made we could estimate the price changes and the arrival times separately. We used the autoregressive conditional multinomial model (ACM) to model the price changes and the ACD for the marginal distribution of the durations.

We divided the price changes into 5 groups and we used an autoregressive model to estimate the price transition probabilities. The sample cross-correlation matrix indicated that there were significant intertemporal dependencies between the price changes. The simple to general model selection procedure suggested to use the ACM(2,3) model, which was dynamic-symmetric for price changes. The sample cross-correlations of the standardized residuals were much less significant than the sample cross-correlations of the original price changes, which signaled that the model was successful in capturing the intertemporal dependencies in the price changes. From the values of the parameters we concluded that large negative price changes were associated with shorter durations, but, as we were concentrated only on one stock, we can not draw any general judgments.

To model the marginal distributions of the durations we used the log-ACD model. The likelihood ratio tests suggested that the log-EACD(1,2) model was suitable for the analysis. Generally we may conclude that the log-ACD model did a very good job in reducing the large value of the Ljung Box statistic.

We consider the ACD-ACM model to be a good vehicle to test the market microstructure hypotheses. It can be used to test the influence of the volume, the spread etc. on the durations and on the price changes. The explanatory variables can be easily added to the model and due to the straightforwardness of the model, it is easy to interpret the results.

Bibliography

[And02] J. Anděl. *Základy matematické statistiky*. Univerzita Karlova v Praze, 2002.

[AP88] A. R. Admati and P. Pfeiderer. A theory of intraday patterns: Volume and price variability. *The Review of Financial Studies*, 1:3–40, 1988.

[BG00] L. Bauwens and P. Giot. The logarithmic acd model: An application to the bid-ask quote process of three nyse stocks. *Annales D'Économie et Statistique*, 60:117–149, 2000.

[BHHH74] E. Berndt, B. Hall, R. Hall, and J. Hausman. Estimation and inference in nonlinear structural models. *Annals of Social Measurement*, 3:653–665, 1974.

[Bol86] T. Bollersev. Generalized autoregressive conditional heteroskedsticity. *Journal of Econometrics*, 31:307–327, 1986.

[BP70] G.E.P. Box and D.A. Pierce. Distribution of residual autocorrelations in autoregressive-integrated moving average time series models. *Journal of the American Statistical Association*, 15:1509–1526, 1970.

[BT81] G. Box and G. Tiao. Modeling multiple time series with applications. *Journal of the American Statistical Association*, 76:802–816, 1981.

[BZ06] V. Bubák and F. Zikeš. Trading intesity and intraday volatility on the prague stock exchange: Evidence from an autoregressive conditional duration model. *Finance a úvěr - Czech Journal of Economics and Science*, 56:223–244, 2006.

[CCS95] K.C. Chan, W.G. Christie, and P.H. Shultz. Market structure and the intraday pattern of bid-ask spreads for nasdaq securities. *Journal of Business*, 68:35–60, 1995.

[Dal02] J. P. Daley. *An Introduction to the Theory of Point Processes*. Oxford University Press, 2002.

[Eng82] R.F. Engle. Autoregressive conditional heteroskedasticity with estimates of the variance of united kingdom inflation. *Econometrica*, 50:987–1007, 1982.

[EO92] D. Easley and M. O'Hara. Time and the process of security price adjustment. *The Journal of Finance*, 19:69–90, 1992.

[ER98] R. Engle and J. Russel. Autoregressive conditional duration: A new model for irregularly spaced data. *Econometrica*, 66:1127–1162, 1998.

[ER05] R. Engle and J. Russel. A discrete-state continuos-time model of financial transaction prices and times: The autoregressice conditional multinomial-autoregressive conditional duration model. *Journal of Business and Economic Statistics*, 23:166–180, 2005.

[Exc] Prague Stock Exchange.

[Gar86] M. Garman. Market microstructure. *Journal of Financial Economics*, 3:257–275, 1986.

[GM85] L. Glosten and P. Milgrom. Bid, ask, and the transaction prices in a specialist market with heterogenously informed traders. *Journal of Financial Economics*, 13:71–100, 1985.

[GM98] Grammig and Maurer. Autoregressive conditional duration: A new model for irregularly spaced data. *Econometrica*, 66:1127–1162, 1998.

[GMT84] C. Gourieroux, A. Monfort, and A. Trognon. Pseudo maximum likelihood methods: Theory. *Econometrica*, 52:681–700, 1984.

[LB78] G.M. Ljung and G.E.P. Box. On a measure of lack of fit of fit in time series models. *Biometrika*, 65:297–303, 1978.

[LH94] S.-W. Lee and B.E. Hansen. Asymoptotic theory for the garch(1,1) quasi-maximum likelihood estimator. *Econometric Theory*, 10:29–52, 1994.

[Lun98] Lunde. Autoregressive conditional duration: A new model for irregularly spaced data. *Econometrica*, 66:1127–1162, 1998.

[Paw01] Y. Pawitan. *In All Likelihood - Statistical Modelling and Inference Using Likelihood*. Claredon Press Oxford, 2001.

[Vuo06] T. Vuorenmaa. A qweibull autoregressive conditional duration model and threshold dependence. *University of Helsinki, RUESG and HECER*, 117:1–61, 2006.