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Mathematical methods and exact spacetimes in quadratic gravity

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In Prague, July 22, 2021 David Miškovský

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Abstract: Within this work we are interested in the frame approach to analysis of the field equations in the context of theories of gravity, in particular, the Einstein General Relativity and Quadratic theory of gravity. As the starting point we summarise the least action principle formulation of the General Relativity and introduce the Quadratic gravity as an extension of the classic Einstein–Hilbert action adding quadratic curvature terms. The Quadratic gravity field equations are rewritten into the form separating the Ricci tensor contribution. As a next step we review the Newman–Penrose formalism on a purely geometrical level and discuss employing the field equations constraints. While in the case of General Relativity it is quite trivial, in the Quadratic gravity it becomes much more involved, however, the General Relativity procedure can be followed even here. As an illustration, we formulate the constraints on the gravitational field in the cases of the spherically symmetric spacetimes and so-called *pp*-waves both in the GR as well as Quadratic gravity.

Keywords: Gravitation, General relativity, Quadratic gravity, Newman–Penrose formalism, Bach tensor, Robinson–Trautman geometries, Kundt geometries

Contents

Introduction

Heavenly bodies and their movement perplexed human beings at least from the ancient times. Remains of the most ancient civilisations suggest efforts to understand the night sky and the laws that govern it.

By no means the first very successful attempt to describe the inner workings of our solar system was performed by Ptolemy in the 2nd century AD. By a very careful observation he concluded that all planets follow a circular trajectory with a centre also following a circular trajectory around the Earth. The only object that moved simply in a circular motion was the Sun. Interestingly enough, the only thing that matters for an accurate prediction is the ratio of two diameters assigned to each heavenly body. Absolute values for the diameters are irrelevant as long as the ratio is preserved. If only Ptolemy scaled all the circles with Earth at its centre to match the Sun orbit, the secondary circular orbits with actual planets on them would then orbit around the Sun and even in the correct order. So close was Ptolemy to the discovery of the heliocentric model.

We had to wait quite some time for the heliocentric model since the time of Ptolemy. About 14 hundred years later when the Copernican Revolution took its place. However, for the Catholic Church the model of the universe with the Earth at its centre was very attractive. Because then, with Rome as the centre of the Earth you could say, that the Church was at the centre of everything. Entire universe made only for us with the heavenly sphere slowly rotating around the Earth by the very hand of god. So, a very good reason was needed to persuade the world to adopt the heliocentric model. And that was provided at the court of Rudolf II. Using observations, with accuracy never seen before, made by Danish astronomer Tycho Brahe, brilliant mathematician Johannes Kepler deduced that planets in the solar system actually followed elliptical trajectories with the Sun at one of the focal points. His new model kinematic provided much more accurate predictions and, in fact more elegant description, thus finally winning the argument in favour of the heliocentric system.

Dynamics governed by the gravitational force was deduced from the Kepler laws of motion by an English mathematician, Sir Isaac Newton a bit later. However, even this was not a final piece in the puzzle of unravelling the laws of nature. Our next step in understanding gravity will require a major shift in the perspective which took another three centuries. So, from a point of view that tries to reflect our reality and how it works, now we know that even Newton was wrong. But from an instrumentalist point of view, that focuses on usefulness of a given theory as an instrument to be used, it is one of the most successful theories in the history. It is so simple and elegant that even today, with the knowledge of Einstein's General Relativity, we use Newtonian mechanics in vast majority of cases.

So why was there a need to look for another theory of gravity in the first place? The hints were truly minuscule and easy to miss. But they were there and as measurements improved, they could not be ignored. One of the first hints was the observation of the perihelion precession of Mercury. Precession is a normal phenomenon even in the Newtonian mechanics. And sure enough, most of it was explained by accounting for the gravitational influence of other planets and shape of the Sun. But even then, there were still 43 angular seconds per century left, that could not be explained. Other problems were paradoxes caused by an assumption of instantaneous propagation of information and the notion of absolute space and time. This led Albert Einstein to formulate his Special theory of relativity in 1905, see [1], and ten years later the General theory of relativity, see [2].

The basic idea standing at the ground of this fantastic journey is following. As wave like properties of light become apparent and indisputable, the notion that wave needs a medium to travel through was strong. And thus, Aether was invented. Since Earth would travel through this Aether, we would see variations in the light speed as a result of this movement. But using interferometric measurements none were detected. In a desperate attempt to save the idea of Aether, Hendrik Lorentz came with his famous transformation (Lorentz transformation), that would make Aether basically undetectable. Therefore, it became a redundant term. However, the transformation was correct. Only thing it needed was a genius, bold enough, to take it literally and assume the deformation of space and time itself. All, so that speed of light stays constant regardless of reference frame.

What is so special about constant speed of light? Well easy answer is that it is what we observe. Einstein assumed it from experimental observation. But interestingly enough you do not need to. Transformation from one reference frame to another that is consistent with the principle of inertia, the isotropy of space, the absence of preferred inertial frames, and a group structure will yield Lorentz transformations with $\boldsymbol{c}$ as a velocity scale [3]. So only with these very basic assumptions on the nature of reality around us, we receive equations with constant propagation of information. Value of this constant is determined within the framework of electromagnetism. By creating wave equation from the Maxwell equations, one also receives constant speed of light in vacuum determined by its permittivity and permeability. And this speed corresponds to $\boldsymbol{c}$ mentioned earlier. The speed of causality. But let us continue as Einstein did and constant speed of light will be just one of the assumptions, we build the theory on.

Why not stop at the level of special relativity? It is revolutionary enough right? Well Einstein made another genius observation. Objects in free fall appear to be in inertial frame of reference. So why not another bold assumption and let us take it literally again. In that case, from the point of view of an unfortunate falling observer, all of us are the ones accelerating "upwards". Does not make sense, right? In a three-dimensional Euclidean space, without Earth inflating like a balloon, it certainly does not. But let us roll with it anyway.

General Relativity is built on three principles.

- Principle of equivalence says that we cannot differentiate between gravitational and inertial mass. Now, this is tested with incredible precision.

- Principle of general covariance says that physical laws cannot depended on any reference frame. So, all of them can be written in the tensor from to ensure their invariance.

- Principal of minimal connection says that all physical laws should depend as little as possible on the metric tensor. It is basically the Occam's razor for selecting from multiple ways of rewriting a physical law in the tensor form.

And if Special relativity seemed revolutionary, we would need to completely transform the way we look on the universe to understand General Relativity. Firstly, we combine space and time into a single four-dimensional deformable Lorentz manifold called spacetime. Its geometrical properties, most importantly its curvature, are encoded in a metric tensor $g_{\mu\nu}$. The most important point being, it can be deformed. Moreover, the stage of the universe is now influenced by any matter or energy in it. In other words, objects do not act on each other directly. There is no gravitational force. Instead, they deform spacetime and this deformation in turn influences how object move and behave.

This new point of view is fascinating, but how to describe it? Imagine a pair of massless test particles floating in a space. Other than our two friends, it is an empty space. Each of them has their own so called world line. Each of them is in an inertial frame of reference and in that case, the world line is called a geodesic. In that case geodesics are straight lines, right? Well not necessarily if the spacetime is not flat. All geodesics can even converge to a single (singular) point when talking about an empty universe with a single Schwarzschild black hole. So, returning to our two particles, by measuring the relative acceleration between them as they travel through spacetime, we can tell how it is deformed. Really, this example corresponds to the geodesic deviation where the Riemann curvature tensor comes directly into the game.

In classical mechanics we need the second derivative of the gravitational field to describe its non-homogeneity. So, in analogy, in General Relativity we need the second derivatives of the metric tensor to describe how the spacetime is deformed. These are contained in the Riemann tensor $R_{\alpha\beta\gamma\delta}$, that exactly contains the information about the spacetime curvature. Sources of gravitational field are contained in the energy and momentum tensor $T_{\mu\nu}$. To describe how matter and energy affects the curvature of spacetime we need to find the field equations. Moreover, the energy should be conserved which puts another restriction on the geometric part of the equations. We try all the different combinations of the metric tensor and its first and second derivatives on one side of the equations and the energy momentum tensor on the other side. Since it is a rank 2 tensor, the other side of the equations needs to be as well. Rewriting derivatives in terms of contractions of the Riemann tensor and evaluating constants we would get the famous Einstein field equations [1]

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}, \tag{1}$$

where $R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu}$ is the Ricci tensor, $R = R^{\alpha}{}_{\alpha}$ represents the Ricci scalar, and Λ is a cosmological constant. By solving these equations, we want to find the metric tensor $g_{\mu\nu}$. In fact, it is a very difficult task since we deal with the system of ten non-linear PDEs. Its exact solutions are known only in some specific (likely symmetric) situations [4, 5]. In more realistic situations the perturbative methods or numerical simulations have to be employed. However, in their core the exact solutions have the prominent position.

[1] Here we use geometrical units $c = 1 = G$.

We will investigate this mathematical description and another way of deriving the field equations in more detail in the next chapter. However, before continuing we should verify, if this new point of view does bring something else than considerably more complicated equations than the classical Newtonian mechanics. Short answer is: Yes, it does! First, it does explain the remaining 43 angular seconds per century of the Mercury perihelion shift. Simultaneously and surprisingly, every new (and old) observation so far, i.e., for more than a hundred years, is in compliance with this theory as well. For illustration let us mention a few more experiments:

- It was General Relativity that correctly predicted how much the light rays bend when passing around the Sun. The measurement was made by Sir Arthur Eddington in 1919 [6].
- Expansion of the universe first observed by Edwin Hubble in 1929 can also be explained by the General Relativity cosmological models [7]. This was the crucial experiment for establishing GR as a correct theory.
- General Relativity also predicted a phase shift in photons when moving through gravitational field. This was experimentally verified in 1954 by Daniel Popper in observation of a star called 40 Eridani B [8].
- Another verification of General Relativity was provided by the observation of cosmic microwave background by Arno Penzias and Robert Wilson in 1965 [9].
- Gravitational waves were detected by the LIGO observatory in 2015 when the signal of two black holes merging was observed. This was one of the last directly untested predictions stemming from General Relativity. Verified hundred years since its publishing [10].
- Year 2019 marks capturing of the first photo of a black hole in history. It was a shadow of supermassive black hole located at the centre of galaxy M87. Observed image also corresponds with how it should look like according to General Relativity. It is also a first visual evidence of the existence of black holes [11].

However, except of all the above highly successful experiments there are more theoretical issues which suggest the General Relativity cannot be the final theory of gravity. In the first chapter we review the theory formulated in a more mathematical terms and briefly summarise various possibilities how to extend the Einstein gravitational law. In chapter two of this thesis we will look into the Newman–Penrose formalism and why it is a useful tool for better understanding and description of spacetime geometries. The third chapter focuses on application of the theory from chapter two in theories of gravity, namely General Relativity and Quadratic gravity. Specifically, its field equations and the Bach tensor contribution will be discussed. Subsequently, we will focus on the Robinson–Trautman and Kundt spacetimes in particular their definition, basic properties, and importance as the starting point for discussion of various explicit models in theories of gravity. In the last chapter we will use all the knowledge reviewed and acquired so far in practical applications, i.e., analysis of the particular theories constraints on the spacetime geometry.

1. Variation principles in theory of gravity

The aim of this chapter is to shortly introduce Einstein's General Relativity [1] and how can it be even more elegantly written in the form of a variation principle. Then we will examine possible ways how this theory could be modified and later focus on a specific and very natural class of modifications called Quadratic theories.

1.1 Einstein's General Relativity

As we have already summarised in the introduction, General Relativity is one of the most successful theories in physics. It resists more that hundred years of persistent attempts at proving it could be wrong. And on a more subjective note, we would describe it as a beautifully elegant theory and would not be by far the only one to do so.

More than theory it could be regarded as a heuristic principle that all theories should satisfy. In fact, the name is unfortunate in connection to what general public thinks this theory is about. Despite the popular use of this theory as an argument for how everything is "relative" and depends on a point of view, its inner genius actually lies in invariance. It is the natural assumption that physical laws should not depend on an observer that stands at the core of this theory. In this sense, the form of physical laws is "absolute".

As mentioned earlier, Einstein approached this famously by looking at the spacetime as a single four-dimensional fabric that can be deformed by a matter and energy inside. In this sense gravity, as the source of relative accelerations, is just an effect of deformed four-dimensional spacetime as seen from our three-dimensional existence. In previous chapter we have seen the crown jewel of the whole theory, the Einstein field equations (1).

Moreover, there exists another elegant way how to write these equations (gravitational law) and that is using the variation principle. In general, that means defining a certain action S and then saying that the physical system will evolve in such a way that minimises the action. In other words, the variation of action is zero for the system evolution, namely

$$\delta S = 0. \tag{1.1}$$

For General Relativity, this mystical functional S is called the Einstein–Hilbert action and it has the following form

$$S = \int \mathrm{d}^4x \left[\frac{1}{\mathsf{k}}(R - 2\Lambda) + L_M\right]\sqrt{-g}, \tag{1.2}$$

where $\frac{1}{\mathsf{k}} = \frac{1}{16\pi}$ is a constant, g is a metric determinant, i.e., $g = \det g_{\mu\nu}$, R is the scalar curvature, and Λ stands for the cosmological constant. Everything in square brackets is called Lagrangian density and L_M specifically is a general

[1] A common abbreviation "GR" will be used in this thesis.

Lagrangian density representing matter. So, for vacuum solutions this term would be zero. Inserting this action into the condition (1.1) we would obtain classical field equations (1).

1.2 Modifications of General Relativity

Even though General Relativity is extremely successful and elegant, it is natural to explore its limits. Therefore, why would we need to modify such a beautiful and successful theory? One strong motivation stem from non-renormalizability of this theory that prevents its quantization using classic approach of the QFT perfectly working in the case of remaining fundamental interactions. And this is just one problem of many, where other important questions are related to the cosmological issues connected with presence of dark matter and energy in the observable universe. Simply put, we are aware that we do not possess the whole truth and modification of General Relativity is one way to look for other solutions.

So, what can be modified then? If we look at the Einstein–Hilbert action (1.2), we can see that we integrate in 4 dimensions. So, we could simply modify number of dimensions. For example, for 3 dimensions we would obtain so called Topological gravity, in which geometry is fully determined by the field equations due to the same number of independent components of the Riemann and Ricci tensor, respectively. Vacuum solutions in such theory are maximally symmetric and non-trivial geometries can be obtained by topological identification. This serves as an interesting toy-model for many more realistic scenarios. Or on the other hand we could increase the number of dimensions beyond 4 together with the assumption that we observe only a four-dimensional submanifold. Moreover, setting number of dimensions above 4 together with assumption of the 2nd order field equations would give us class of modifications called Lovelock theories [12]. But there is something special about four dimensions. If we assume that field equations are of the 2nd order and conserved (satisfying Bianchi identities), then the Einstein field equations are unique in four dimensions. But the Einstein–Hilbert action is not. We can add all the Lovelock terms to it that will not affect the resulting field equations. The first of them is the famous Gauss–Bonnet term, namely

$$R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - 4R_{\mu\nu}R^{\mu\nu} + R^2, \tag{1.3}$$

leading to the class of so-called Gauss–Bonnet theories for $D \geq 5$.

Assuming that we focus on vacuum solutions, so $L_M = 0$, the only remaining part that can be modified is the geometric part of the Einstein–Hilbert action.

To summarise, starting with the Einstein–Hilbert action it is very straightforward, without any additional constraints on the order of resulting field equations and with respect to particular situation, to modify

- Number of dimensions D.
- Restriction on geometry, i.e., changing the $R - 2\Lambda$ term.
- Matter content of the spacetime L_M.

Within this thesis we will be interested in the geometric part of the action, allowing for the additional quadratic terms constructed from the curvature tensors.

1.3 Quadratic gravity

Among the theories with modified Einstein–Hilbert action there is a specific class that permits quadratic curvature terms in S. They are called Quadratic theories. In general vacuum case we can write them as

$$S = \int f(R, R_{\mu\nu}R^{\mu\nu}, R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta})\sqrt{-g}\mathrm{d}^D x. \tag{1.4}$$

For simplification we shall use the following notation,

$$\Psi = R_{\mu\nu}R^{\mu\nu} \quad \text{and} \quad \Omega = R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}. \tag{1.5}$$

By applying the condition (1.1) we obtain

$$\begin{aligned} &f_R R_{\mu\nu} - \frac{1}{2} f g_{\mu\nu} + (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu) f_R + 2f_\Psi R_{\mu\alpha}R^\alpha_\nu + 2f_\Omega R_{\alpha\beta\gamma\mu}R^{\alpha\beta\gamma}{}_\nu \\ &+ \Box(f_\Psi R_{\mu\nu}) + g_{\mu\nu}\nabla_\alpha\nabla_\beta(f_\Psi R^{\alpha\beta}) - 2\nabla_\alpha\nabla_\beta(f_\Psi \delta^\beta_\mu R^\alpha{}_\nu + 2f_\Omega R^\alpha{}_{\mu\nu}{}^\beta) = 0, \end{aligned} \tag{1.6}$$

where f_R denotes derivative of f with respect to R and similarly for f with another subscript. The result (1.6) can be further simplified using the Bianchi identities and Leibniz rule. The detail calculation can be found, e.g., in [13].

Pure Quadratic theories, that we shall restrict ourselves to admit only the linear dependence of f on the curvature scalars R, R^2, Ψ, and Ω. In this thesis we will also limit ourselves to 4 dimensions. This means that for general form of f we can choose only 2 out of 3 curvature squares, because we know that the Gauss–Bonnet term (1.3) does not contribute to the integral. We shall also express separately "free" gravitational field hidden in the trace-less part of the Riemann tensor defining the Weyl tensor $C_{\mu\nu\alpha\beta}$ in general dimension

$$\begin{aligned} C_{\mu\nu\alpha\beta} = R_{\mu\nu\alpha\beta} &- \frac{1}{D-2}\left(g_{\mu\alpha}R_{\nu\beta} + g_{\nu\beta}R_{\mu\alpha} - g_{\mu\beta}R_{\nu\alpha} - g_{\nu\alpha}R_{\mu\beta}\right) \\ &+ \frac{1}{(D-1)(D-2)} R\left(g_{\mu\alpha}g_{\nu\beta} - g_{\mu\beta}g_{\nu\alpha}\right), \end{aligned} \tag{1.7}$$

where D is a number of dimensions, so for our case we set $D = 4$.

Our general function f then takes the form of

$$f = \frac{1}{\mathsf{k}}(R - 2\Lambda) - \mathfrak{a}C^2_{\mu\nu\alpha\beta} + \mathfrak{b}R^2, \tag{1.8}$$

where k, $\mathfrak{a}$ and $\mathfrak{b}$ are (arbitrary) constants of the theory. The field equations (1.6) then become

$$\frac{1}{\mathsf{k}}(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu}) - 4\mathfrak{a}B_{\mu\nu} + 2\mathfrak{b}(R_{\mu\nu} - \frac{1}{4}Rg_{\mu\nu} + g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)R = 0, \tag{1.9}$$

where $B_{\mu\nu}$ is the Bach tensor defined as

$$B_{\mu\nu} = (\nabla^\alpha\nabla^\beta - \frac{1}{2}R^{\alpha\beta})C_{\mu\alpha\nu\beta}. \tag{1.10}$$

The Bach tensor is symmetric, trace-less, covariantly constant, and conformaly re-scaled, i.e.,

$$\begin{gathered} B_{\mu\nu} = B_{\nu\mu}, \quad B_{\mu\nu} g^{\mu\nu} = 0, \quad B_{\mu\alpha;\beta} g^{\alpha\beta} = 0, \\ \tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu} \implies \tilde{B}_{\mu\nu} = \Omega^{-2} B_{\mu\nu}. \end{gathered} \tag{1.11}$$

The resulting field equations (1.9) are obviously of the 4th order which opens a completely new land to explore in comparison with General Relativity. Obviously, the trace of the equation (1.9) yields condition for the Ricci scalar, namely

$$R = 6\mathfrak{b}\mathsf{k}\Box R + 4\Lambda. \tag{1.12}$$

As a final part of this chapter, we will rewrite the field equations (1.9) in a bit different form. Reserving explanation for this step at the end of the computation since the reasoning behind it will be much more apparent. Inserting definition of the Bach tensor (1.10) and moving all terms with the Ricci tensor to the beginning of the equation we obtain

$$\begin{gathered} \left(\frac{1}{\mathsf{k}} + 2\mathfrak{b}R\right) R_{\mu\nu} + 2\mathfrak{a} R^{\alpha\beta} C_{\mu\alpha\nu\beta} - \frac{1}{\mathsf{k}}(\frac{1}{2} R g_{\mu\nu} - \Lambda g_{\mu\nu}) - 4\mathfrak{a} \nabla^\alpha \nabla^\beta C_{\mu\alpha\nu\beta} \\ -2\mathfrak{b}(\frac{1}{4} R g_{\mu\nu} - g_{\mu\nu}\Box + \nabla_\mu \nabla_\nu) R = 0, \end{gathered} \tag{1.13}$$

In the second step we denote everything other than the first two terms (explicitly containing the Ricci tensor components) as a separate 2nd rank tensor $Z_{\mu\nu}$, i.e.,

$$\left(\frac{1}{\mathsf{k}} + 2\mathfrak{b}R\right) R_{\mu\nu} + 2\mathfrak{a} R^{\alpha\beta} C_{\mu\alpha\nu\beta} + Z_{\mu\nu} = 0, \tag{1.14}$$

where $Z_{\mu\nu}$ represents

$$Z_{\mu\nu} = -\frac{1}{\mathsf{k}}(\frac{1}{2} R g_{\mu\nu} - \Lambda g_{\mu\nu}) - 4\mathfrak{a} \nabla^\alpha \nabla^\beta C_{\mu\alpha\nu\beta} - 2\mathfrak{b}(\frac{1}{4} R g_{\mu\nu} - g_{\mu\nu}\Box + \nabla_\mu \nabla_\nu) R. \tag{1.15}$$

The form (1.14) separating explicit contribution of the Ricci tensor is important for two reasons. The first one is purely for clarity in following component calculation. However, the second one is more important for the discussion of constraints on the resulting spacetime geometry and will be apparent after the second chapter on the Newman–Penrose formalism. In such a frame approach to classical General Relativity the Ricci tensor components are algebraically constrained in the crucial geometric identities by the Einstein field equations. We would like to explore an analogous approach also in the case of Quadratic gravity.

2. Newman–Penrose formalism

This chapter is an overview of a general tetrad formalism with main focus on one specific case called Newman–Penrose formalism[1]. The content primary source is the first chapter of [14]. Of course, the topic is in detailed covered in many great sources (other than the one already mentioned), for example the original article [15] or a classic textbook [16]. But for this thesis to be easily understandable only with a basic knowledge of introductory course to General Relativity, we would like to present all the definitions that will be used and derive all results we need "from scratch".

2.1 General tetrad formalism

Let us start with a motivation. Assuming you are a confused observer trying to make a sense of the world around you. You have an abstract theory of General Relativity, but that will not help you too much to understand your specific case. A standard procedure is to use the coordinate representation of General Relativity. You pick a clever set of coordinates respecting the symmetry of the problem, find the components of a metric tensor and derive all necessary quantities together with restrictions implied by the field equations. However, you know that the physical reality can be hidden in the coordinate choice and it is still challenging to obtain invariant results.

A different approach can be to erect a set of four basis vectors around you, so called "tetrad", and project all tensors on them. Every fundamental object in General Relativity (like Riemann tensor, Ricci tensor etc.) is now described as a set of scalars. So, in a coordinate invariant form. Of course, it still depends on the initial choice of our tetrad, that can be (as with choice in coordinate system) chosen more or less suitably for a given problem. Generally speaking, respecting a symmetry of the system produces simpler results.

So, let us define a set of four contravariant vectors $e_{(a)}$

$$e_{(a)}{}^{\mu}, \quad \text{where} \quad a = 1, 2, 3, 4. \tag{2.1}$$

The index without parentheses symbolises components of a given vector and the index in parentheses labels the four tetrad vectors.

Next, let us define an "inverse matrix" (or better its components) to (2.1) as

$$e^{(a)}{}_{\mu}, \quad \text{where} \quad a = 1, 2, 3, 4, \tag{2.2}$$

satisfying the following relations

$$e_{(a)}{}^{\mu} e^{(b)}{}_{\mu} = \delta^{(b)}_{(a)}, \qquad e_{(a)}{}^{\mu} e^{(a)}{}_{\nu} = \delta^{\mu}_{\nu}. \tag{2.3}$$

We would also like to lower and rise indices. For the component index it is simply performed by a metric tensor $g_{\mu\nu}$ as

$$e_{(a)\mu} = g_{\mu\nu} e_{(a)}{}^{\nu}, \tag{2.4}$$

[1]We will often use a common abbreviation "NP formalism".

which is also a definition of a covariant vector $e_{(a)\mu}$. And for the vector labelling index we can define a symmetric matrix $\eta_{(a)(b)}$ and its inverse $\eta^{(a)(b)}$ by the following relations

$$\eta_{(a)(b)} e^{(a)}{}_{\mu} = e_{(b)\mu}, \qquad \eta^{(a)(b)} e_{(a)\mu} = e^{(b)}{}_{\mu}, \tag{2.5}$$

and then, of course,

$$\eta^{(a)(b)} \eta_{(b)(c)} = \delta^{(a)}_{(c)}. \tag{2.6}$$

Using these definitions we can also write $g_{\mu\nu}$ and $\eta_{(a)(b)}$ in terms of our basis vectors as

$$e_{(a)\mu} e^{(a)}{}_{\nu} = g_{\mu\nu}, \qquad e_{(a)}{}^{\mu} e_{(b)\mu} = \eta_{(a)(b)}, \tag{2.7}$$

where the second relation can be used as a definition of $\eta_{(a)(b)}$ and relations (2.5) and (2.6) simply follow as its consequence. That way it can be seen more clearly why it makes sense to raise and lower labelling indices, but we did it in way where the analogy with component indices is more apparent.

We can now define a "tetrad component" of any vector field V^{μ} as a projection of such a field on our tetrad,

$$V_{(a)} = e_{(a)\mu} V^{\mu}, \qquad V^{(b)} = \eta^{(a)(b)} V_{(a)}. \tag{2.8}$$

The extension of this definition for a general tensor $T^{\mu\ldots\nu}{}_{\alpha\ldots\beta}$ is straightforward, namely

$$T_{(a)\ldots(b)}{}^{(c)\ldots(d)} = e_{(a)\mu} \ldots e_{(b)\nu} e^{(c)\alpha} \ldots e^{(d)\beta} T^{\mu\ldots\nu}{}_{\alpha\ldots\beta}. \tag{2.9}$$

2.1.1 Intrinsic derivative

Let us take a more geometrical point of view and consider basis vectors $\boldsymbol{e}_{(a)}$ as directional derivatives

$$\boldsymbol{e}_{(a)} = e_{(a)}{}^{\mu} \partial_{\mu}. \tag{2.10}$$

So, a derivative with respect to the labelling index is then naturally defined as

$$\Phi_{,(a)} = e_{(a)}{}^{\mu} \Phi_{,\mu}, \tag{2.11}$$

$$\tag{2.12}$$

and

$$\begin{aligned} V_{(a),(b)} &= e_{(b)}{}^{\nu} (e_{(a)}{}^{\mu} V_{\mu})_{,\nu} = e_{(b)}{}^{\nu} (e_{(a)}{}^{\mu} V_{\mu})_{;\nu} \\ &= e_{(b)}{}^{\nu} [e_{(a)}{}^{\mu}{}_{;\nu} V_{\mu} + e_{(a)}{}^{\mu} V_{\mu;\nu}] \\ &= e_{(a)}{}^{\mu} V_{\mu;\nu} e_{(b)}{}^{\nu} + e_{(a)\mu;\nu} e_{(b)}{}^{\nu} e_{(c)}{}^{\mu} V^{(c)}, \end{aligned} \tag{2.13}$$

where ; denotes covariant derivative. Since covariant derivative acting on a scalar is equivalent to regular partial derivative the second step in (2.13) is justified.

Let us now define a key object called "Ricci rotation-coefficients" as

$$\gamma_{(c)(a)(b)} = e_{(c)}{}^{\mu} e_{(a)\mu;\nu} e_{(b)}{}^{\nu}. \tag{2.14}$$

Using this definition, the equation (2.13) takes the form

$$V_{(a),(b)} = e_{(a)}{}^{\mu} V_{\mu;\nu} e_{(b)}{}^{\nu} + \gamma_{(c)(a)(b)} V^{(c)}. \tag{2.15}$$

If we choose the tetrad so that $\eta_{(a)(b)}$ is a constant matrix (which is a nontrivial assumption, thus the following property is valid only for this special case) we immediately see that $\gamma_{(a)(b)(c)}$ is anti-symmetric in the first two indices because

$$0 = (\eta_{(a)(b)})_{,\mu} = (e_{(a)}{}^{\nu} e_{(b)\nu})_{;\mu} \quad \Longrightarrow \quad \gamma_{(c)(a)(b)} = -\gamma_{(a)(c)(b)}. \tag{2.16}$$

With the definition of "Intrinsic derivative" as

$$V_{(a)|(b)} = e_{(a)}{}^{\mu} V_{\mu;\nu} e_{(b)}{}^{\nu}, \tag{2.17}$$

we can rewrite (2.13) using (2.14) and (2.17) as

$$V_{(a)|(b)} = V_{(a),(b)} - \gamma^{(c)}{}_{(a)(b)} V_{(c)}. \tag{2.18}$$

So, the intrinsic derivative is an analogy of the covariant derivative in the labelling indices. This is especially useful for rewriting expressions with covariant derivatives like Bianchi identities in the tetrad components.

2.1.2 Riemann tensor

First, let us find projections of the Riemann tensor on a generic tetrad in terms of the Ricci rotation coefficients. We shall start with the definition of the Riemann tensor as a commutator of two covariant derivatives,

$$e_{(a)\alpha;\mu;\nu} - e_{(a)\alpha;\nu;\mu} = R_{\beta\alpha\mu\nu} e_{(a)}{}^{\beta}. \tag{2.19}$$

The definition (2.14) and the basic properties (2.7) can be used to express covariant derivative $e_{(a)\mu;\nu}$ using the Ricci rotation-coefficients as

$$e_{(a)\mu;\nu} = e^{(c)}{}_{\mu} \gamma_{(c)(a)(b)} e^{(b)}{}_{\nu}. \tag{2.20}$$

Using (2.19) and (2.20) we can write the projection of the Riemann tensor as

$$\begin{aligned}
R_{(a)(b)(c)(d)} =& R_{\beta\alpha\mu\nu} e_{(a)}{}^{\beta} e_{(b)}{}^{\alpha} e_{(c)}{}^{\mu} e_{(d)}{}^{\nu} \\
=& \left[(e_{(a)\alpha;\mu})_{;\nu} - (e_{(a)\alpha;\nu})_{;\mu} \right] e_{(b)}{}^{\alpha} e_{(c)}{}^{\mu} e_{(d)}{}^{\nu} \\
=& \left[(e^{(f)}{}_{\alpha} \gamma_{(f)(a)(g)} e^{(g)}{}_{\mu})_{;\nu} - (e^{(f)}{}_{\alpha} \gamma_{(f)(a)(g)} e^{(g)}{}_{\nu})_{;\mu} \right] e_{(b)}{}^{\alpha} e_{(c)}{}^{\mu} e_{(d)}{}^{\nu} \\
=& \gamma_{(b)}{}^{(f)}{}_{(d)} \gamma_{(f)(a)(c)} + \gamma_{(b)(a)(c),(d)} + \gamma_{(c)}{}^{(g)}{}_{(d)} \gamma_{(b)(a)(g)} \\
& - \gamma_{(b)}{}^{(f)}{}_{(c)} \gamma_{(f)(a)(d)} - \gamma_{(b)(a)(d),(c)} - \gamma_{(d)}{}^{(g)}{}_{(c)} \gamma_{(b)(a)(g)}.
\end{aligned} \tag{2.21}$$

Finally, we get

$$\begin{aligned}
R_{(a)(b)(c)(d)} =& -\gamma_{(a)(b)(c),(d)} + \gamma_{(a)(b)(d),(c)} \\
& + \gamma_{(b)(a)(f)} \left[\gamma_{(c)}{}^{(f)}{}_{(d)} - \gamma_{(d)}{}^{(f)}{}_{(c)} \right] \\
& + \gamma_{(f)(a)(c)} \gamma_{(b)}{}^{(f)}{}_{(d)} - \gamma_{(f)(a)(d)} \gamma_{(b)}{}^{(f)}{}_{(c)}.
\end{aligned} \tag{2.22}$$

Finally, we can easily generalise the definition (2.18) for multiple indices. Particularly, in the case of intrinsic derivative of the Riemann tensor we obtain

$$\begin{aligned}R_{(a)(b)(c)(d)|(f)} =& R_{(a)(b)(c)(d),(f)} \\ &- \eta^{(n)(m)} \Big[\gamma_{(n)(a)(f)} R_{(m)(b)(c)(d)} + \gamma_{(n)(b)(f)} R_{(a)(m)(c)(d)} \\ &+ \gamma_{(n)(c)(f)} R_{(a)(b)(m)(d)} + \gamma_{(n)(d)(f)} R_{(a)(b)(c)(m)}\Big] .\end{aligned} \tag{2.23}$$

We can slightly modify this result by adding a cyclic exchange of the indices to obtain an expression, that can be used to directly calculate the Bianchi identities in any frame formalism, namely

$$\begin{aligned}R_{(a)(b)[(c)(d)|(f)]} =& \frac{1}{6} \sum_{[(c)(d)(f)]} \Big\{ R_{(a)(b)(c)(d),(f)} \\ &- \eta^{(n)(m)} \Big[\gamma_{(n)(a)(f)} R_{(m)(b)(c)(d)} + \gamma_{(n)(b)(f)} R_{(a)(m)(c)(d)} \\ &+ \gamma_{(n)(c)(f)} R_{(a)(b)(m)(d)} + \gamma_{(n)(d)(f)} R_{(a)(b)(c)(m)}\Big]\Big\} .\end{aligned} \tag{2.24}$$

2.2 Defining the Newman–Penrose quantities

In previous section we stated that since we have a "weapon of choice" when it comes to the tetrad vectors it can be advantageous to choose them with respect to a certain symmetry. And since Roger Penrose saw the light-cone structure as the fundamental part of space-time structure, it is not surprising that the tetrad in the Newman–Penrose formalism consists of null vectors. They are traditionally denoted as

$$\boldsymbol{l}, \boldsymbol{n}, \boldsymbol{m}, \bar{\boldsymbol{m}},$$

where $\boldsymbol{l}$, $\boldsymbol{n}$ are real null vectors and $\boldsymbol{m}$, $\bar{\boldsymbol{m}}$ are artificially constructed complex null vectors. And as the notation suggests $\bar{\boldsymbol{m}}$ is a complex conjugate of $\boldsymbol{m}$. The null character of the vectors can be written as

$$\boldsymbol{l} \cdot \boldsymbol{l} = \boldsymbol{n} \cdot \bar{\boldsymbol{n}} = \boldsymbol{m} \cdot \boldsymbol{m} = \bar{\boldsymbol{m}} \cdot \bar{\boldsymbol{m}} = 0. \tag{2.25}$$

Furthermore, as a part of the definition we impose on the tetrad the following orthogonality and normalization conditions,

$$\boldsymbol{l} \cdot \boldsymbol{m} = \boldsymbol{l} \cdot \bar{\boldsymbol{m}} = \boldsymbol{n} \cdot \boldsymbol{m} = \boldsymbol{n} \cdot \bar{\boldsymbol{m}} = 0, \tag{2.26}$$

$$\boldsymbol{l} \cdot \boldsymbol{n} = 1, \tag{2.27}$$

$$\boldsymbol{m} \cdot \bar{\boldsymbol{m}} = -1. \tag{2.28}$$

Finally, freedom in a choice of such null frame is simply given by the Lorentz transformations,

- boost in the plane of null vectors $\boldsymbol{l}$ and $\boldsymbol{n}$ with a real parameter A:

$$l^\mu \mapsto A^2\, l^\mu, \qquad n^\mu \mapsto A^{-2}\, n^\mu, \qquad m^\mu \mapsto m^\mu, \tag{2.29}$$

- rotation in the transverse space of vectors $\boldsymbol{m}$ and $\bar{\boldsymbol{m}}$ encoded in a real parameter χ:

$$l^\mu \mapsto l^\mu, \qquad n^\mu \mapsto n^\mu, \qquad m^\mu \mapsto e^{2\,i\,\chi} m^\mu, \tag{2.30}$$

- null rotation with $\boldsymbol{l}$ fixed given by a complex parameter c:

$$l^\mu \mapsto l^\mu, \qquad m^\mu \mapsto m^\mu + \bar{c}\, l^\mu, \qquad n^\mu \mapsto n^\mu + c\, m^\mu + \bar{c}\, \bar{m}^\mu + |c|^2\, l^\mu, \tag{2.31}$$

- null rotation with $\boldsymbol{n}$ fixed given by a complex parameter d:

$$n^\mu \mapsto n^\mu, \qquad m^\mu \mapsto m^\mu + d\, n^\mu, \qquad l^\mu \mapsto l^\mu + \bar{d}\, m^\mu + d\, \bar{m}^\mu + |d|^2\, n^\mu. \tag{2.32}$$

To express results obtained for a generic tetrad and summarised in the previous section we set

$$\boldsymbol{e}_{(1)} = \boldsymbol{l}, \quad \boldsymbol{e}_{(2)} = \boldsymbol{n}, \quad \boldsymbol{e}_{(3)} = \boldsymbol{m}, \quad \boldsymbol{e}_{(4)} = \bar{\boldsymbol{m}}. \tag{2.33}$$

So, using (2.25)–(2.28) we can write $\eta_{(a)(b)}$ explicitly as

$$\boldsymbol{\eta} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{2.34}$$

Using (2.5) we can now easily find the covariant basis. In correspondence with the geometrical point of view on vectors as directional derivatives, let us define the following

$$\begin{aligned} \boldsymbol{l} \equiv \mathrm{D} = l^\mu \nabla_\mu, \quad & \boldsymbol{n} \equiv \Delta = n^\mu \nabla_\mu, \\ \boldsymbol{m} \equiv \delta = m^\mu \nabla_\mu, \quad & \bar{\boldsymbol{m}} \equiv \bar{\delta} = \bar{m}^\mu \nabla_\mu. \end{aligned} \tag{2.35}$$

It is also customary as a part of this formalism to define special symbols for the Ricci rotation coefficients introduced in (2.14) and we shall call these quantities the spin coefficients,

$$\begin{aligned}
\kappa =& \gamma_{(3)(1)(1)} = m^\mu l_{\mu;\nu} l^\nu = m^\mu \mathrm{D} l_\mu, \\
\sigma =& \gamma_{(3)(1)(3)} = m^\mu l_{\mu;\nu} m^\nu = m^\mu \delta l_\mu, \\
\lambda =& \gamma_{(2)(4)(4)} = n^\mu \bar{m}_{\mu;\nu} \bar{m}^\nu = n^\mu \bar{\delta} \bar{m}_\mu, \\
\nu =& \gamma_{(2)(4)(2)} = n^\mu \bar{m}_{\mu;\nu} n^\nu = n^\mu \Delta \bar{m}_\mu, \\
\rho =& \gamma_{(3)(1)(4)} = m^\mu l_{\mu;\nu} \bar{m}^\nu = m^\mu \bar{\delta} l_\mu, \\
\mu =& \gamma_{(2)(4)(3)} = n^\mu \bar{m}_{\mu;\nu} m^\nu = n^\mu \delta \bar{m}_\mu, \\
\tau =& \gamma_{(3)(1)(2)} = m^\mu l_{\mu;\nu} n^\nu = m^\mu \Delta l_\mu, \\
\pi =& \gamma_{(2)(4)(1)} = n^\mu \bar{m}_{\mu;\nu} l^\nu = n^\mu \mathrm{D} \bar{m}_\mu, \\
\epsilon =& \frac{1}{2}\left(\gamma_{(2)(1)(1)} + \gamma_{(3)(4)(1)}\right) = \frac{1}{2}\left(n^\mu l_{\mu;\nu} l^\nu + m^\mu \bar{m}_{\mu;\nu} l^\nu\right) = \frac{1}{2}\left(n^\mu \mathrm{D} l_\mu + m^\mu \mathrm{D} \bar{m}\right), \\
\gamma =& \frac{1}{2}\left(\gamma_{(2)(1)(2)} + \gamma_{(3)(4)(2)}\right) = \frac{1}{2}\left(n^\mu l_{\mu;\nu} n^\nu + m^\mu \bar{m}_{\mu;\nu} n^\nu\right) = \frac{1}{2}\left(n^\mu \Delta l_\mu + m^\mu \Delta \bar{m}\right), \\
\alpha =& \frac{1}{2}\left(\gamma_{(2)(1)(4)} + \gamma_{(3)(4)(4)}\right) = \frac{1}{2}\left(n^\mu l_{\mu;\nu} \bar{m}^\nu + m^\mu \bar{m}_{\mu;\nu} \bar{m}^\nu\right) = \frac{1}{2}\left(n^\mu \bar{\delta} l_\mu + m^\mu \bar{\delta} \bar{m}\right), \\
\beta =& \frac{1}{2}\left(\gamma_{(2)(1)(3)} + \gamma_{(3)(4)(3)}\right) = \frac{1}{2}\left(n^\mu l_{\mu;\nu} m^\nu + m^\mu \bar{m}_{\mu;\nu} m^\nu\right) = \frac{1}{2}\left(n^\mu \delta l_\mu + m^\mu \delta \bar{m}\right).
\end{aligned} \tag{2.36}$$

2.2.1 Representing the Weyl and Ricci tensors in the NP formalism

As we have already mentioned in the first chapter, the Weyl tensor is a trace-free part of the Riemann tensor defined by (1.7), with $D = 4$ in our case, and its frame components can thus be written as the following expression

$$\begin{aligned} R_{(a)(b)(c)(d)} =& C_{(a)(b)(c)(d)} \\ &- \frac{1}{2}\left(\eta_{(a)(c)}R_{(b)(d)} - \eta_{(b)(c)}R_{(a)(d)} - \eta_{(a)(d)}R_{(b)(c)} + \eta_{(b)(d)}R_{(a)(c)}\right) \\ &+ \frac{1}{6}\left(\eta_{(a)(c)}\eta_{(b)(d)} - \eta_{(a)(d)}\eta_{(b)(c)}\right) R. \end{aligned} \tag{2.37}$$

The above projection of the Weyl tensor on a null tetrad $\boldsymbol{l}$, $\boldsymbol{n}$, $\boldsymbol{m}$, $\bar{\boldsymbol{m}}$ and employing its symmetries we can represent all its independent components by five complex scalars, namely

$$\begin{aligned} \Psi_0 =& - C_{(1)(3)(1)(3)} = -C_{\mu\nu\alpha\beta}l^\mu m^\nu l^\alpha m^\beta, \\ \Psi_1 =& - C_{(1)(2)(1)(3)} = -C_{\mu\nu\alpha\beta}l^\mu n^\nu l^\alpha m^\beta, \\ \Psi_2 =& - C_{(1)(3)(4)(2)} = -C_{\mu\nu\alpha\beta}l^\mu m^\nu \bar{m}^\alpha n^\beta, \\ \Psi_3 =& - C_{(1)(2)(4)(2)} = -C_{\mu\nu\alpha\beta}l^\mu n^\nu \bar{m}^\alpha n^\beta, \\ \Psi_4 =& - C_{(2)(4)(2)(4)} = -C_{\mu\nu\alpha\beta}n^\mu \bar{m}^\nu n^\alpha \bar{m}^\beta. \end{aligned} \tag{2.38}$$

It is also useful to mention that aside from these components of the Weyl tensor (and their complex conjugate which basically means exchange of $\boldsymbol{m}$ and $\bar{\boldsymbol{m}}$ vectors, i.e., $3 \leftrightarrow 4$) other non-zero components are

$$\begin{aligned} C_{(1)(3)(3)(4)} =& \Psi_1, \\ C_{(2)(4)(4)(3)} =& \Psi_3, \\ C_{(1)(2)(1)(2)} =& C_{(3)(4)(3)(4)} = -\left(\Psi_2 + \bar{\Psi_2}\right), \\ C_{(1)(2)(3)(4)} =& \left(\Psi_2 - \bar{\Psi_2}\right). \end{aligned} \tag{2.39}$$

It is also standard to denote frame projections of the Ricci tensor as

$$\begin{aligned} \Phi_{00} &= -\frac{1}{2}R_{(1)(1)} = -\frac{1}{2}R_{\mu\nu}l^\mu l^\nu, \\ \Phi_{01} &= -\frac{1}{2}R_{(1)(3)} = -\frac{1}{2}R_{\mu\nu}l^\mu m^\nu, \\ \Phi_{10} &= -\frac{1}{2}R_{(1)(4)} = -\frac{1}{2}R_{\mu\nu}l^\mu \bar{m}^\nu, \\ \Phi_{11} &= -\frac{1}{4}\left(R_{(1)(2)} + R_{(3)(4)}\right) = -\frac{1}{4}\left(R_{\mu\nu}l^\mu n^\nu + R_{\mu\nu}m^\mu \bar{m}^\nu\right), \\ \Phi_{02} &= -\frac{1}{2}R_{(3)(3)} = -\frac{1}{2}R_{\mu\nu}m^\mu m^\nu, \\ \Phi_{20} &= -\frac{1}{2}R_{(4)(4)} = -\frac{1}{2}R_{\mu\nu}\bar{m}^\mu \bar{m}^\nu, \\ \Phi_{12} &= -\frac{1}{2}R_{(2)(3)} = -\frac{1}{2}R_{\mu\nu}n^\mu m^\nu, \\ \Phi_{21} &= -\frac{1}{2}R_{(2)(4)} = -\frac{1}{2}R_{\mu\nu}n^\mu \bar{m}^\nu, \\ \Phi_{22} &= -\frac{1}{2}R_{(2)(2)} = -\frac{1}{2}R_{\mu\nu}n^\mu n^\nu. \end{aligned} \tag{2.40}$$

Finally, the Ricci scalar, as a trace of the Ricci tensor, takes the form

$$R = 2\left(R_{(1)(2)} - R_{(3)(4)}\right). \tag{2.41}$$

2.2.2 Geometric constrains on the frame components

So far, we have introduced tetrad formalism and projections of all essential objects on the tetrad. But we still need to cover a vital part of the NP formalism. Namely, constrains stemming from objects defined so far and their properties. In particular, commutation relations of the basis vectors and in consequence of four directional derivatives they represent, the Ricci identities and the Bianchi identities. All these constrains are purely geometrical. The Ricci identities are simply equations defining the Riemann tensor (2.19) and the Bianchi identities come from the covariant derivative of the Riemann tensor. They must be satisfied regardless of other constrains for any theory to be consistent.

Commutation relations

The Lie bracket is in general an essential object, when talking about geometry and structure in any given theory.

Starting with the Lie bracket of two covariant vectors defined in (2.33) we get

$$\begin{aligned}
[\boldsymbol{e}_{(a)}, \boldsymbol{e}_{(b)}]f =&[e_{(a)}{}^{\mu}\nabla_{\mu}, e_{(b)}{}^{\nu}\nabla_{\nu}]f \\
=&e_{(a)}{}^{\mu}\nabla_{\mu}(e_{(b)}{}^{\nu}\nabla_{\nu}f) - e_{(b)}{}^{\nu}\nabla_{\nu}(e_{(a)}{}^{\mu}\nabla_{\mu}f) \\
=&(e_{(a)}{}^{\nu}e_{(b)\mu;\nu} - e_{(b)}{}^{\nu}e_{(a)\mu;\nu})\nabla^{\mu}f \\
=&(e_{(a)}{}^{\nu}e^{(f)}{}_{\mu}\gamma_{(f)(b)(g)}e^{(g)}{}_{\nu} - e_{(b)}{}^{\nu}e^{(f)}{}_{\mu}\gamma_{(f)(a)(g)}e^{(g)}{}_{\nu})\nabla^{\mu}f \\
=&(\gamma_{(a)(f)(b)} - \gamma_{(b)(f)(a)})e^{(f)\mu}\nabla_{\mu}f \\
=&(\gamma_{(a)(f)(b)} - \gamma_{(b)(f)(a)})\boldsymbol{e}^{(f)}f.
\end{aligned} \tag{2.42}$$

As mentioned before, they define directional derivatives (2.35) and we used this fact in the first step. In the second step we rewrite the Lie bracket according to its definition and in the third step we have rewritten covariant derivatives in terms of the Ricci rotation coefficients according to (2.20). After a few simple manipulations we can express the Lie bracket of two vectors as

$$[\boldsymbol{e}_{(a)}, \boldsymbol{e}_{(b)}] = (\gamma_{(a)(f)(b)} - \gamma_{(b)(f)(a)})\boldsymbol{e}^{(f)}. \tag{2.43}$$

As an illustration we will calculate such Lie bracket for one pair of the vectors, namely $\boldsymbol{e}_{(1)}$ and $\boldsymbol{e}_{(2)}$,

$$\begin{aligned}
[\boldsymbol{e}_{(1)}, \boldsymbol{e}_{(2)}] =&\mathrm{D}\Delta - \Delta\mathrm{D} \\
=&(\gamma_{(1)(f)(2)} - \gamma_{(2)(f)(1)})\boldsymbol{e}^{(f)} \\
=& - \gamma_{(2)(1)(1)}\boldsymbol{e}^{(1)} + \gamma_{(1)(2)(2)}\boldsymbol{e}^{(2)} + (\gamma_{(1)(3)(2)} - \gamma_{(2)(3)(1)})\boldsymbol{e}^{(3)} \\
&+ (\gamma_{(1)(4)(2)} - \gamma_{(2)(4)(1)})\boldsymbol{e}^{(4)} \\
=& - (\epsilon + \bar{\epsilon})\Delta - (\gamma + \bar{\gamma})\mathrm{D} - (-\tau - \bar{\pi})\bar{\delta} - (-\bar{\tau} - \pi)\delta.
\end{aligned} \tag{2.44}$$

Repeating this process for all possible combinations of the basis vectors we will obtain full set of commutation relations

$$\Delta \mathrm{D} - \mathrm{D}\Delta = (\gamma + \bar{\gamma})\,\mathrm{D} + (\epsilon + \bar{\epsilon})\,\Delta - (\bar{\tau} + \pi)\,\delta - (\tau + \bar{\pi})\,\bar{\delta}, \tag{2.45}$$

$$\delta \mathrm{D} - \mathrm{D}\delta = (\bar{\alpha} + \beta - \bar{\pi})\,\mathrm{D} + \kappa\Delta - (\bar{\rho} + \epsilon - \bar{\epsilon})\,\delta - \sigma\bar{\delta}, \tag{2.46}$$

$$\delta\Delta - \Delta\delta = -\bar{\nu}\mathrm{D} + (\tau - \bar{\alpha} - \beta)\,\Delta + (\mu - \gamma + \bar{\gamma})\,\delta + \bar{\lambda}\bar{\delta}, \tag{2.47}$$

$$\bar{\delta}\delta - \delta\bar{\delta} = (\bar{\mu} - \mu)\,\mathrm{D} + (\bar{\rho} - \rho)\,\Delta + \left(\alpha - \bar{\beta}\right)\delta + (\beta - \bar{\alpha})\,\bar{\delta}, \tag{2.48}$$

and thus, the first set of conditions for the geometry and spin coefficients is known.

Ricci identities

Famous Ricci identities, that provide another set of constrains can be obtained by writing various nonzero components of the Riemann tensor in the form of (2.22) and by use of notation for the Ricci spin coefficients (2.36) and the projections of the Weyl and Ricci tensors (2.38), (2.40) and the Ricci scalar (2.41).

As an illustration let us also explicitly derive one of the Ricci identities stemming from the $R_{(1)(3)(1)(3)}$ component of the projected Riemann tensor. We have two equations to be employed. Starting with (2.22) we will explicitly express this component in terms of the Ricci rotation coefficients and rewrite these as the spin coefficients, i.e.,

$$\begin{aligned}
R_{(1)(3)(1)(3)} =& -\gamma_{(1)(3)(1),(3)} + \gamma_{(1)(3)(3),(1)} \\
&+ \gamma_{(3)(1)(f)}\gamma_{(1)(g)(3)}\eta^{(g)(f)} - \gamma_{(3)(1)(f)}\gamma_{(3)(g)(1)}\eta^{(g)(f)} \\
&+ \gamma_{(f)(1)(1)}\gamma_{(3)(g)(3)}\eta^{(g)(f)} - \gamma_{(f)(1)(3)}\gamma_{(3)(g)(1)}\eta^{(g)(f)} \\
=&\delta\kappa - \mathrm{D}\sigma \\
&+ \gamma_{(3)(1)(1)}\gamma_{(1)(2)(3)} - \gamma_{(3)(1)(4)}\gamma_{(1)(3)(3)} - \gamma_{(3)(1)(3)}\gamma_{(1)(4)(3)} \\
&- \gamma_{(3)(1)(2)}\gamma_{(3)(1)(1)} - \gamma_{(3)(1)(1)}\gamma_{(3)(2)(1)} + \gamma_{(3)(1)(3)}\gamma_{(3)(4)(1)} \\
&+ \gamma_{(2)(1)(1)}\gamma_{(3)(1)(3)} - \gamma_{(3)(1)(1)}\gamma_{(3)(4)(3)} \\
&- \gamma_{(2)(1)(3)}\gamma_{(3)(1)(1)} + \gamma_{(3)(1)(3)}\gamma_{(3)(4)(1)} \\
=&\delta\kappa - \mathrm{D}\sigma \\
&+ \kappa(-\bar{\alpha} - \beta - \tau - \bar{\pi} + \bar{\alpha} - \beta - \bar{\alpha} - \beta) \\
&+ \sigma(\rho + \bar{\rho} + \epsilon - \bar{\epsilon} + \epsilon + \bar{\epsilon} + \epsilon - \bar{\epsilon}) \\
=&\delta\kappa - \mathrm{D}\sigma + \kappa(\bar{\pi} - \tau - 3\beta - \bar{\alpha}) + \sigma(3\epsilon - \bar{\epsilon} + \rho + \bar{\rho}).
\end{aligned} \tag{2.49}$$

Furthermore, the second equation we need to use is the projected definition of the Weyl tensor (2.37). Again, for the $R_{(1)(3)(1)(3)}$ component we obtain

$$\begin{aligned}
R_{(1)(3)(1)(3)} =& C_{(1)(3)(1)(3)} \\
&- \frac{1}{2}\left(\cancel{\eta_{(1)(1)}}R_{(3)(3)} - \cancel{\eta_{(3)(1)}}R_{(1)(3)} - \cancel{\eta_{(1)(3)}}R_{(3)(1)} + \cancel{\eta_{(3)(3)}}R_{(1)(1)}\right) \\
&+ \frac{1}{6}\left(\cancel{\eta_{(1)(1)}}\cancel{\eta_{(3)(3)}} - \cancel{\eta_{(1)(3)}}\cancel{\eta_{(3)(1)}}\right)R \\
=& -\Psi_0.
\end{aligned} \tag{2.50}$$

Combining our findings in equations (2.49) and (2.50) we can write the first Ricci identity as

$$\mathrm{D}\sigma - \delta\kappa = \sigma(3\epsilon - \bar{\epsilon} + \rho + \bar{\rho}) + \kappa(\bar{\pi} - \tau - 3\beta - \bar{\alpha}) + \Psi_0. \tag{2.51}$$

Following this procedure for all possible combinations of indices we obtain the full set of Ricci identities, namely

$$\mathrm{D}\sigma - \delta\kappa = \sigma\left(3\epsilon - \bar{\epsilon} + \rho + \bar{\rho}\right) + \kappa\left(\bar{\pi} - \tau - 3\beta - \bar{\alpha}\right) + \Psi_0, \tag{2.52}$$

$$\mathrm{D}\rho - \bar{\delta}\kappa = \left(\rho^2 + \sigma\bar{\sigma}\right) + \rho\left(\epsilon + \bar{\epsilon}\right) - \bar{\kappa}\tau - \kappa\left(3\alpha + \bar{\beta} - \pi\right) + \Phi_{00}, \tag{2.53}$$

$$\begin{aligned}\mathrm{D}\tau - \Delta\kappa =& \rho\left(\tau + \bar{\pi}\right) + \sigma\left(\bar{\tau} + \pi\right) + \tau\left(\epsilon - \bar{\epsilon}\right) - \kappa\left(3\gamma + \bar{\gamma}\right) \\ & + \Psi_1 + \Phi_{01},\end{aligned} \tag{2.54}$$

$$\mathrm{D}\alpha - \bar{\delta}\epsilon = \alpha\left(\rho + \bar{\epsilon} - 2\epsilon\right) + \beta\bar{\sigma} - \bar{\beta}\epsilon - \kappa\lambda - \bar{\kappa}\gamma + \pi(\epsilon + \rho) + \Phi_{10}, \tag{2.55}$$

$$\mathrm{D}\beta - \delta\epsilon = \sigma(\alpha + \pi) + \beta\left(\bar{\rho} - \bar{\epsilon}\right) - \kappa(\mu + \gamma) - \epsilon\left(\bar{\alpha} - \bar{\pi}\right) + \Psi_1, \tag{2.56}$$

$$\begin{aligned}\mathrm{D}\gamma - \Delta\epsilon =& \alpha\left(\tau + \bar{\pi}\right) + \beta\left(\bar{\tau} + \pi\right) - \gamma\left(\epsilon + \bar{\epsilon}\right) - \epsilon\left(\gamma + \bar{\gamma}\right) + \tau\pi \\ & - \nu\kappa + \Psi_2 + \Phi_{11} - \frac{1}{24}R,\end{aligned} \tag{2.57}$$

$$\mathrm{D}\lambda - \bar{\delta}\pi = \left(\rho\lambda + \bar{\sigma}\mu\right) + \pi(\pi + \alpha - \beta) - \nu\bar{\kappa} - \lambda\left(3\epsilon - \bar{\epsilon}\right) + \Phi_{20}, \tag{2.58}$$

$$\mathrm{D}\mu - \delta\pi = \left(\bar{\rho}\mu + \sigma\lambda\right) + \pi\left(\bar{\pi} - \bar{\alpha} + \beta\right) - \mu\left(\epsilon + \bar{\epsilon}\right) - \nu\kappa + \Psi_2 + \frac{1}{12}R, \tag{2.59}$$

$$\begin{aligned}\mathrm{D}\nu - \Delta\pi =& \mu\left(\pi + \bar{\tau}\right) + \lambda\left(\bar{\pi} + \tau\right) + \pi\left(\gamma - \bar{\gamma}\right) - \nu\left(3\epsilon + \bar{\epsilon}\right) \\ & + \Psi_3 + \Phi_{21},\end{aligned} \tag{2.60}$$

$$\Delta\lambda - \bar{\delta}\nu = -\lambda\left(\mu + \bar{\mu} + 3\gamma - \bar{\gamma}\right) + \nu\left(3\alpha + \bar{\beta} + \pi - \bar{\tau}\right) - \Psi_4, \tag{2.61}$$

$$\begin{aligned}\delta\rho - \bar{\delta}\sigma =& \rho\left(\bar{\alpha} + \beta\right) - \sigma\left(3\alpha - \bar{\beta}\right) + \tau\left(\rho - \bar{\rho}\right) + \kappa\left(\mu - \bar{\mu}\right) \\ & - \Psi_1 + \Phi_{01},\end{aligned} \tag{2.62}$$

$$\begin{aligned}\delta\alpha - \bar{\delta}\beta =& (\mu\rho - \lambda\sigma) + \alpha\bar{\alpha} + \beta\bar{\beta} - 2\alpha\beta + \gamma\left(\rho - \bar{\rho}\right) + \epsilon\left(\mu - \bar{\mu}\right) \\ & - \Psi_2 + \Phi_{11} + \frac{1}{24}R,\end{aligned} \tag{2.63}$$

$$\begin{aligned}\delta\lambda - \bar{\delta}\mu =& \nu\left(\rho - \bar{\rho}\right) + \pi\left(\mu - \bar{\mu}\right) + \mu\left(\alpha + \bar{\beta}\right) + \lambda\left(\bar{\alpha} - 3\beta\right) \\ & - \Psi_3 + \Phi_{21},\end{aligned} \tag{2.64}$$

$$\delta\nu - \Delta\mu = \left(\mu^2 + \lambda\bar{\lambda}\right) + \mu\left(\gamma + \bar{\gamma}\right) - \bar{\nu}\pi + \nu\left(\tau - 3\beta - \bar{\alpha}\right) + \Phi_{22}, \tag{2.65}$$

$$\delta\gamma - \Delta\beta = \gamma\left(\tau - \bar{\alpha} - \beta\right) + \mu\tau - \sigma\nu - \epsilon\bar{\nu} - \beta\left(\gamma - \bar{\gamma} - \mu\right) + \alpha\bar{\lambda} + \Phi_{12}, \tag{2.66}$$

$$\delta\tau - \Delta\sigma = \left(\mu\sigma + \bar{\lambda}\rho\right) + \tau\left(\tau + \beta - \bar{\alpha}\right) - \sigma\left(3\gamma - \bar{\gamma}\right) - \kappa\bar{\nu} + \Phi_{02}, \tag{2.67}$$

$$\Delta\rho - \bar{\delta}\tau = -\left(\rho\bar{\mu} + \sigma\lambda\right) + \tau\left(\bar{\beta} - \alpha - \bar{\tau}\right) + \rho\left(\gamma + \bar{\gamma}\right) + \nu\kappa - \Psi_2 - \frac{1}{12}R, \tag{2.68}$$

$$\Delta\alpha - \bar{\delta}\gamma = \nu(\rho + \epsilon) - \lambda(\tau + \beta) + \alpha\left(\bar{\gamma} - \bar{\mu}\right) + \gamma\left(\bar{\beta} - \bar{\tau}\right) - \Psi_3. \tag{2.69}$$

Bianchi identities

The last purely geometrical constrains are hidden in the first and second Bianchi identities. The first Bianchi identities come from the cyclic exchange of indices in the covariant derivative of the Riemann tensor, namely

$$R_{\mu\nu[\alpha\beta;\gamma]} = \frac{1}{3}(R_{\mu\nu\alpha\beta;\gamma} + R_{\mu\nu\beta\gamma;\alpha} + R_{\mu\nu\gamma\alpha\beta}) = 0. \tag{2.70}$$

Projecting these relations, we can make use of the equation (2.24) to express them in the NP formalism. The second Bianchi identities are obtained by contracting the first set of identities, i.e.,

$$
\begin{aligned}
&g^{\mu\alpha}g^{\nu\beta}(R_{\mu\nu\alpha\beta;\gamma}+R_{\mu\nu\beta\gamma;\alpha}+R_{\mu\nu\gamma\alpha;\beta})\\
&=g^{\nu\beta}(R_{\nu\beta;\gamma}+g^{\mu\alpha}R_{\mu\nu\beta\gamma;\alpha}-R_{\nu\gamma;\beta})\\
&=R_{;\gamma}-g^{\mu\alpha}R_{\mu\gamma;\alpha}-g^{\nu\beta}R_{\nu\gamma;\beta}\\
&=R_{;\gamma}-2R^{\mu}{}_{\gamma;\mu}=0.
\end{aligned}
\tag{2.71}
$$

Projecting the second Bianchi identities on the null tetrad yields

$$
\begin{aligned}
&e_{(a)}{}^{\gamma}R_{;\gamma}-2e_{(a)}{}^{\gamma}g^{\mu\nu}R_{\nu\gamma;\mu}\\
&=R_{|(a)}-2e_{(a)}{}^{\gamma}e_{(b)}{}^{\mu}e^{(b)\nu}R_{\nu\gamma;\mu}\\
&=R_{|(a)}-2\eta^{(b)(c)}R_{(c)(a)|(b)}=0.
\end{aligned}
\tag{2.72}
$$

The full set of independent equations consists of 8 complex equations from the first set of Bianchi identities,

$$
\begin{aligned}
&R_{(1)(3)[(1)(3)|(4)]}=0, \quad R_{(1)(3)[(2)(1)|(4)]}=0, \quad R_{(1)(3)[(1)(3)|(2)]}=0,\\
&R_{(1)(3)[(4)(3)|(2)]}=0, \quad R_{(4)(2)[(1)(3)|(4)]}=0, \quad R_{(4)(2)[(2)(1)|(4)]}=0,\\
&R_{(4)(2)[(1)(3)|(2)]}=0, \quad R_{(4)(2)[(4)(3)|(2)]}=0,
\end{aligned}
\tag{2.73}
$$

and two real and one complex equation from the second set of Bianchi identities,

$$
\begin{aligned}
&R_{(1)(1)|(2)}+R_{(3)(4)|(1)}-R_{(1)(3)|(4)}-R_{(1)(4)|(3)}=0,\\
&R_{(2)(2)|(1)}+R_{(3)(4)|(2)}-R_{(2)(3)|(4)}-R_{(2)(4)|(3)}=0,\\
&R_{(3)(3)|(4)}+R_{(1)(2)|(3)}-R_{(3)(1)|(2)}-R_{(3)(2)|(1)}=0,
\end{aligned}
\tag{2.74}
$$

giving us the total of 20 independent Bianchi identities.

For illustration let us explicitly express the first equation in (2.73), namely

$$
R_{(1)(3)(1)(3)|(4)}+R_{(1)(3)(3)(4)|(1)}+R_{(1)(3)(4)(1)|(3)}=0.
\tag{2.75}
$$

Rewriting all three projections of the Riemann tensor (firstly without the derivative) using (2.37) we obtain

$$
\begin{aligned}
R_{(1)(3)(1)(3)}=&C_{(1)(3)(1)(3)}\\
&-\frac{1}{2}\left(\cancel{\eta_{(1)(1)}}R_{(3)(3)}-\cancel{\eta_{(3)(1)}}R_{(1)(3)}-\cancel{\eta_{(1)(3)}}R_{(3)(1)}+\cancel{\eta_{(3)(3)}}R_{(1)(1)}\right)\\
&+\frac{1}{6}\left(\cancel{\eta_{(1)(1)}}\cancel{\eta_{(3)(3)}}-\cancel{\eta_{(1)(3)}}\cancel{\eta_{(3)(1)}}\right)R\\
=&C_{(1)(3)(1)(3)},
\end{aligned}
\tag{2.76}
$$

$$
\begin{aligned}
R_{(1)(3)(3)(4)}=&C_{(1)(3)(3)(4)}\\
&-\frac{1}{2}\left(\cancel{\eta_{(1)(3)}}R_{(3)(4)}-\cancel{\eta_{(3)(3)}}R_{(1)(4)}-\cancel{\eta_{(1)(4)}}R_{(3)(3)}+\eta_{(3)(4)}R_{(1)(3)}\right)\\
&+\frac{1}{6}\left(\cancel{\eta_{(1)(3)}}\eta_{(3)(4)}-\cancel{\eta_{(1)(4)}}\cancel{\eta_{(3)(3)}}\right)R\\
=&C_{(1)(3)(3)(4)}+\frac{1}{2}R_{(1)(3)},
\end{aligned}
\tag{2.77}
$$

$$
\begin{aligned}
R_{(1)(3)(4)(1)} =& \cancel{C_{(1)(3)(4)(1)}} \\
& - \frac{1}{2}\left(\cancel{\eta_{(1)(4)}} R_{(3)(1)} - \eta_{(3)(4)} R_{(1)(1)} - \cancel{\eta_{(1)(1)}} R_{(3)(4)} + \cancel{\eta_{(3)(1)}} R_{(1)(4)}\right) \\
& + \frac{1}{6}\left(\cancel{\eta_{(1)(4)}}\,\cancel{\eta_{(3)(1)}} - \cancel{\eta_{(1)(1)}}\eta_{(3)(4)}\right) R \\
=& -\frac{1}{2} R_{(1)(1)}.
\end{aligned} \tag{2.78}
$$

Therefore, the equation (2.75) takes the form

$$
C_{(1)(3)(1)(3)|(4)} + C_{(1)(3)(3)(4)|(1)} + \frac{1}{2} R_{(1)(3)|(1)} - \frac{1}{2} R_{(1)(1)|(3)} = 0. \tag{2.79}
$$

We have already explicitly written how the intrinsic derivative acts on the Riemann tensor (2.23). For projections of the Weyl tensor it will be exactly the same, i.e.,

$$
\begin{aligned}
C_{(1)(3)(1)(3)|(4)} =& C_{(1)(3)(1)(3),(4)} \\
& - \eta^{(n)(m)}\left[\gamma_{(n)(1)(4)} C_{(m)(3)(1)(3)} + \gamma_{(n)(3)(4)} C_{(1)(m)(1)(3)} \right. \\
& \left. + \gamma_{(n)(1)(4)} C_{(1)(3)(m)(3)} + \gamma_{(n)(3)(4)} C_{(1)(3)(1)(m)}\right] \\
=& -\bar{\delta}\Psi_0 - \gamma_{(2)(1)(4)} C_{(1)(3)(1)(3)} + \gamma_{(3)(1)(4)} C_{(4)(3)(1)(3)} \\
& - \gamma_{(1)(3)(4)} C_{(1)(2)(1)(3)} + \gamma_{(4)(3)(4)} C_{(1)(3)(1)(3)} \\
& - \gamma_{(2)(1)(4)} C_{(1)(3)(1)(3)} + \gamma_{(3)(1)(4)} C_{(1)(3)(4)(3)} \\
& - \gamma_{(1)(3)(4)} C_{(1)(3)(1)(2)} + \gamma_{(4)(3)(4)} C_{(1)(3)(1)(3)} \\
=& -\bar{\delta}\Psi_0 + 4\alpha\Psi_0 - 4\rho\Psi_1,
\end{aligned} \tag{2.80}
$$

$$
\begin{aligned}
C_{(1)(3)(3)(4)|(1)} =& C_{(1)(3)(3)(4),(1)} \\
& - \eta^{(n)(m)}\left[\gamma_{(n)(1)(1)} C_{(m)(3)(3)(4)} + \gamma_{(n)(3)(1)} C_{(1)(m)(3)(4)} \right. \\
& \left. + \gamma_{(n)(3)(1)} C_{(1)(3)(m)(4)} + \gamma_{(n)(4)(1)} C_{(1)(3)(3)(m)}\right] \\
=& \mathrm{D}\Psi_1 - \gamma_{(2)(1)(1)} C_{(1)(3)(3)(4)} + \gamma_{(3)(1)(1)} C_{(4)(3)(3)(4)} \\
& - \gamma_{(1)(3)(1)} C_{(1)(2)(3)(4)} + \gamma_{(4)(3)(1)} C_{(1)(3)(3)(4)} \\
& - \gamma_{(1)(3)(1)} C_{(1)(3)(2)(4)} + \gamma_{(4)(3)(1)} C_{(1)(3)(3)(4)} \\
& - \gamma_{(2)(4)(1)} C_{(1)(3)(3)(1)} + \gamma_{(3)(4)(1)} C_{(1)(3)(3)(4)} \\
=& \mathrm{D}\Psi_1 - 2\epsilon\Psi_1 + 3\kappa\Psi_2 - \pi\Psi_0.
\end{aligned} \tag{2.81}
$$

Intrinsic derivative acting on the Ricci tensor is even more simple, namely

$$
R_{(a)(b)|(c)} = R_{(a)(b),(c)} - \eta^{(n)(m)}\left[\gamma_{(n)(a)(c)} R_{(m)(b)} + \gamma_{(n)(b)(c)} R_{(a)(m)}\right]. \tag{2.82}
$$

So, expressing the remaining two parts of the equation (2.79) we thus obtain

$$
\begin{aligned}
R_{(1)(3)|(1)} =& R_{(1)(3),(1)} - \eta^{(n)(m)}\left[\gamma_{(n)(1)(1)} R_{(m)(3)} + \gamma_{(n)(3)(1)} R_{(1)(m)}\right] \\
=& -2\mathrm{D}\Phi_{01} - \gamma_{(2)(1)(1)} R_{(1)(3)} + \gamma_{(3)(1)(1)} R_{(4)(3)} + \gamma_{(4)(1)(1)} R_{(3)(3)} \\
& - \gamma_{(1)(3)(1)} R_{(1)(2)} - \gamma_{(2)(3)(1)} R_{(1)(1)} + \gamma_{(4)(3)(1)} R_{(1)(3)} \\
=& -2\mathrm{D}\Phi_{01} + 4\epsilon\Phi_{01} - 4\kappa\Phi_{11} - 2\bar{\kappa}\Phi_{02} + 2\bar{\pi}\Phi_{00},
\end{aligned} \tag{2.83}
$$

and

$$
\begin{aligned}
R_{(1)(1)|(3)} =& R_{(1)(1),(3)} - \eta^{(n)(m)}\left[\gamma_{(n)(1)(3)}R_{(m)(1)} + \gamma_{(n)(1)(3)}R_{(1)(m)}\right] \\
=& -2\delta\Phi_{00} - \gamma_{(2)(1)(3)}R_{(1)(1)} + \gamma_{(3)(1)(3)}R_{(4)(1)} + \gamma_{(4)(1)(3)}R_{(3)(1)} \\
& - \gamma_{(2)(1)(3)}R_{(1)(1)} + \gamma_{(3)(1)(3)}R_{(1)(4)} + \gamma_{(4)(1)(3)}R_{(1)(3)} \\
=& -2\delta\Phi_{00} + 4(\bar{\alpha} + \beta)\Phi_{00} - 4\sigma\Phi_{10} - 4\bar{\rho}\Phi_{01}.
\end{aligned} \tag{2.84}
$$

Finally, combining everything together the equation (2.79) now takes the form,

$$
\begin{aligned}
0 =& -\bar{\delta}\Psi_0 + \mathrm{D}\Psi_1 + (4\alpha - \pi)\Psi_0 - 2(2\rho + \epsilon)\Psi_1 + 3\kappa\Psi_2 \\
& - \mathrm{D}\Phi_{01} + \delta\Phi_{00} + 2(\epsilon + \bar{\rho})\Phi_{01} + 2\sigma\Phi_{10} - 2\kappa\Phi_{11} - \bar{\kappa}\Phi_{02} + \bar{\pi}\Phi_{00} \\
& + (\bar{\pi} - 2\bar{\alpha} - 2\beta)\Phi_{00}.
\end{aligned} \tag{2.85}
$$

Following this procedure, it is straightforward to write the full set of the first Bianchi identities,

$$
\begin{aligned}
0 =& -\bar{\delta}\Psi_0 + \mathrm{D}\Psi_1 + (4\alpha - \pi)\Psi_0 - 2(2\rho + \epsilon)\Psi_1 + 3\kappa\Psi_2 \\
& - \mathrm{D}\Phi_{01} + \delta\Phi_{00} + 2\left(\epsilon + \bar{\rho}\right)\Phi_{01} + 2\sigma\Phi_{10} - 2\kappa\Phi_{11} - \bar{\kappa}\Phi_{02} \\
& + \left(\bar{\pi} - 2\bar{\alpha} - 2\beta\right)\Phi_{00},
\end{aligned} \tag{2.86}
$$

$$
\begin{aligned}
0 =& +\bar{\delta}\Psi_1 - \mathrm{D}\Psi_2 - \lambda\Psi_0 + 2(\pi - \alpha)\Psi_1 + 3\rho\Psi_2 - 2\kappa\Psi_3 \\
& + \bar{\delta}\Phi_{01} - \Delta\Phi_{00} - 2\left(\alpha + \bar{\tau}\right)\Phi_{01} + 2\rho\Phi_{11} + \bar{\sigma}\Phi_{02} \\
& - \left(\bar{\mu} - 2\gamma - 2\bar{\gamma}\right)\Phi_{00} - 2\tau\Phi_{10} - \frac{1}{12}\mathrm{D}R,
\end{aligned} \tag{2.87}
$$

$$
\begin{aligned}
0 =& -\bar{\delta}\Psi_2 + \mathrm{D}\Psi_3 + 2\lambda\Psi_1 - 3\pi\Psi_2 + 2(\epsilon - \rho)\Psi_3 + \kappa\Psi_4 \\
& - \mathrm{D}\Phi_{21} + \delta\Phi_{20} + 2\left(\bar{\rho} - \epsilon\right)\Phi_{21} - 2\mu\Phi_{10} + 2\pi\Phi_{11} - \bar{\kappa}\Phi_{22} \\
& - \left(2\bar{\alpha} - 2\beta - \bar{\pi}\right)\Phi_{20} - \frac{1}{12}\bar{\delta}R,
\end{aligned} \tag{2.88}
$$

$$
\begin{aligned}
0 =& +\bar{\delta}\Psi_3 - \mathrm{D}\Psi_4 - 3\lambda\Psi_2 + 2(2\pi + \alpha)\Psi_3 - (4\epsilon - \rho)\Psi_4 \\
& - \Delta\Phi_{20} + \bar{\delta}\Phi_{21} + 2\left(\alpha - \bar{\tau}\right)\Phi_{21} + 2\nu\Phi_{10} + \bar{\sigma}\Phi_{22} - 2\lambda\Phi_{11} \\
& - \left(\bar{\mu} + 2\gamma - 2\bar{\gamma}\right)\Phi_{20},
\end{aligned} \tag{2.89}
$$

$$
\begin{aligned}
0 =& -\Delta\Psi_0 + \delta\Psi_1 + (4\gamma - \mu)\Psi_0 - 2(2\tau + \beta)\Psi_1 + 3\sigma\Psi_2 \\
& - \mathrm{D}\Phi_{02} + \delta\Phi_{01} + 2\left(\bar{\pi} - \beta\right)\Phi_{01} - 2\kappa\Phi_{12} - \bar{\lambda}\Phi_{00} + 2\sigma\Phi_{11} \\
& + \left(\bar{\rho} + 2\epsilon - 2\bar{\epsilon}\right)\Phi_{02},
\end{aligned} \tag{2.90}
$$

$$
\begin{aligned}
0 =& -\Delta\Psi_1 + \delta\Psi_2 + \nu\Psi_0 + 2(\gamma - \mu)\Psi_1 - 3\tau\Psi_2 + 2\sigma\Psi_3 \\
& + \Delta\Phi_{01} - \bar{\delta}\Phi_{02} + 2\left(\bar{\mu} - \gamma\right)\Phi_{01} - 2\rho\Phi_{12} - \bar{\nu}\Phi_{00} + 2\tau\Phi_{11} \\
& + \left(\bar{\tau} - 2\bar{\beta} + 2\alpha\right)\Phi_{02} + \frac{1}{12}\delta R,
\end{aligned} \tag{2.91}
$$

$$
\begin{aligned}
0 =& -\Delta\Psi_2 + \delta\Psi_3 + 2\nu\Psi_1 - 3\mu\Psi_2 + 2(\beta - \tau)\Psi_3 + \sigma\Psi_4 \\
& - \mathrm{D}\Phi_{22} + \delta\Phi_{21} + 2\left(\bar{\pi} + \beta\right)\Phi_{21} - 2\mu\Phi_{11} - \bar{\lambda}\Phi_{20} + 2\pi\Phi_{12} \\
& + \left(\bar{\rho} - 2\epsilon - 2\bar{\epsilon}\right)\Phi_{22} - \frac{1}{12}\Delta R,
\end{aligned} \tag{2.92}
$$

$$
\begin{aligned}
0 =& -\Delta\Psi_3 + \delta\Psi_4 + 3\nu\Psi_2 - 2(\gamma + 2\mu)\Psi_3 - (\tau - 4\beta)\Psi_4 \\
& + \Delta\Phi_{21} - \bar{\delta}\Phi_{22} + 2\left(\bar{\mu} + \gamma\right)\Phi_{21} - 2\nu\Phi_{11} - \bar{\nu}\Phi_{20} + 2\lambda\Phi_{12} \\
& + \left(\bar{\tau} - 2\alpha - 2\bar{\beta}\right)\Phi_{22}.
\end{aligned} \tag{2.93}
$$

Furthermore, the second Bianchi identities in the NP formalism take the form

$$\begin{aligned}
&\bar{\delta}\Phi_{01} + \delta\Phi_{10} - D\left(\Phi_{11} + \frac{R}{8}\right) - \Delta\Phi_{00} \\
&\quad = \bar{\kappa}\Phi_{12} + \kappa\Phi_{21} + (2\alpha + 2\bar{\tau} - \pi)\,\Phi_{01} + (2\bar{\alpha} + 2\tau - \bar{\pi})\,\Phi_{10} \\
&\quad - 2\,(\rho + \bar{\rho})\,\Phi_{11} - \bar{\sigma}\Phi_{02} - \sigma\Phi_{20} + [\mu + \bar{\mu} - 2\,(\gamma + \bar{\gamma})]\,\Phi_{00}, \qquad (2.94)\\
&\bar{\delta}\Phi_{12} + \delta\Phi_{21} - \Delta\left(\Phi_{11} + \frac{R}{8}\right) - D\Phi_{22} \\
&= -\nu\Phi_{01} - \bar{\nu}\Phi_{10} + \left(\bar{\tau} - 2\bar{\beta} - 2\pi\right)\Phi_{12} + (\tau - 2\beta - 2\bar{\pi})\,\Phi_{21} \\
&\quad + 2\,(\mu + \bar{\mu})\,\Phi_{11} - (\rho + \bar{\rho} - 2\epsilon - 2\bar{\epsilon})\,\Phi_{22} + \lambda\Phi_{02} + \bar{\lambda}\Phi_{20}, \qquad (2.95)\\
&\delta\left(\Phi_{11} - \frac{R}{8}\right) - D\Phi_{12} - \Delta\Phi_{01} + \bar{\delta}\Phi_{02} \\
&\quad = \kappa\Phi_{22} - \bar{\nu}\Phi_{00} + \left(\bar{\tau} - \pi + 2\alpha - 2\bar{\beta}\right)\Phi_{02} - \sigma\Phi_{21} + \bar{\lambda}\Phi_{10} \\
&\quad + 2\,(\tau - \bar{\pi})\,\Phi_{11} - (2\rho + \bar{\rho} - 2\bar{\epsilon})\,\Phi_{12} + (2\bar{\mu} + \mu - 2\gamma)\,\Phi_{01}. \qquad (2.96)
\end{aligned}$$

2.2.3 Interpretation of the NP quantities

This section briefly summarises the geometric interpretation of specific spin coefficients related to the geodesic motion. In the second part we outline a possible way how to investigate the Weyl and Ricci tensor components influence on the relative motion of free test particles. Other than the main source for the whole chapter [14] another source for this specific section is [17].

Optical scalars

When it comes to the discussion on physical meaning of the spin coefficients, it is useful to start by rewriting the covariant derivative acting on $\boldsymbol{l}$ in terms of the Ricci rotation coefficients in correspondence with (2.20), i.e.,

$$l_{\mu;\nu} = e^{(a)}{}_{\mu}\gamma_{(a)1(b)}e^{(b)}{}_{\nu}. \qquad (2.97)$$

In explicit form we obtain the following decomposition

$$\begin{aligned}
l_{\mu;\nu} = &(\epsilon + \bar{\epsilon})\,l_\mu n_\nu + (\gamma + \bar{\gamma})\,l_\mu l_\nu - (\bar{\alpha} + \beta)\,l_\mu \bar{m}_\nu - \left(\alpha + \bar{\beta}\right) l_\mu m_\nu \\
&- \kappa\bar{m}_\mu n_\nu - \bar{\kappa} m_\mu n_\nu + \sigma\bar{m}_\mu \bar{m}_\nu + \bar{\sigma} m_\mu m_\nu \\
&- \tau\bar{m}_\mu l_\nu - \bar{\tau} m_\mu l_\nu + \rho\bar{m}_\mu m_\nu + \bar{\rho} m_\mu \bar{m}_\nu. \qquad (2.98)
\end{aligned}$$

Projecting this result on the same null vector $\boldsymbol{l}$ will result in

$$l_{\mu;\nu}l^{\nu} = (\epsilon + \bar{\epsilon})\,l_\mu - \kappa\bar{m}_\mu - \bar{\kappa} m_\mu. \qquad (2.99)$$

If the left-hand side of the above equation is proportional to $\boldsymbol{l}$ itself (up to an arbitrary scalar function),vector $\boldsymbol{l}$ would, by definition, generate congruence of null geodesics. We can see that this is true if, and only if, $\kappa = 0$.

Stronger condition would be to set left hand side of (2.99) to zero and in that case, we would say that the geodesics are in addition affinely parametrized. To the condition above, we would also need to add $\epsilon = 0$.

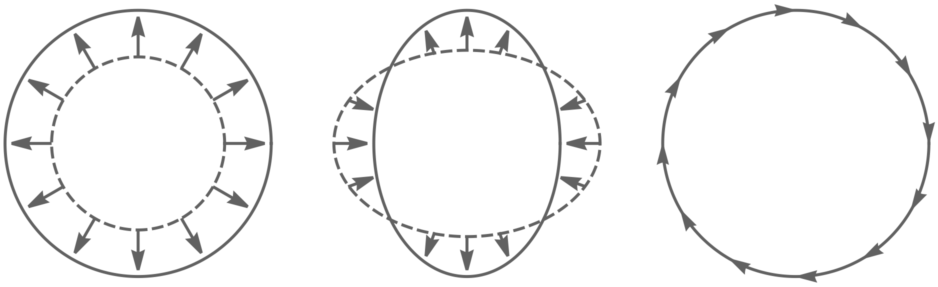

Figure 2.1: Expansion, shear and twist in the transverse projection.

Analogous results can be obtained for the second null vector $\boldsymbol{n}$ and its covariant derivative components.

Let us now continue with the affinely parametrized congruence of null geodesics. In other words, we assume that $\kappa = 0$ and $\epsilon = 0$ and from the equation (2.98) we thus immediately obtain

$$\begin{aligned} l_{\mu;\nu} = &(\gamma + \bar{\gamma})\, l_\mu l_\nu - (\bar{\alpha} + \beta)\, l_\mu \bar{m}_\nu - \left(\alpha + \bar{\beta}\right) l_\mu m_\nu - \tau \bar{m}_\mu l_\nu \\ &+ \sigma \bar{m}_\mu \bar{m}_\nu + \bar{\sigma} m_\mu m_\nu + \rho \bar{m}_\mu m_\nu + \bar{\rho} m_\mu \bar{m}_\nu - \bar{\tau} m_\mu l_\nu . \end{aligned} \tag{2.100}$$

By simple manipulations with (2.100) we can derive the following relations

$$\frac{1}{2} l^\mu{}_{;\mu} = -\frac{1}{2}\left(\rho + \bar{\rho}\right) = \theta, \tag{2.101}$$

$$\frac{1}{2} l_{[\mu;\nu]} l^{\mu;\nu} = -\frac{1}{4}\left(\rho - \bar{\rho}\right)^2 = \omega^2, \tag{2.102}$$

$$\frac{1}{2} l_{(\mu,\nu)} l^{\mu;\nu} = \theta^2 + |\sigma|^2, \tag{2.103}$$

defining three optical scalars, namely expansion θ, twist ω, and shear σ. The logic behind these names used by R. Sachs will become clear momentarily. Obviously, an alternative definition of expansion and twist is simply

$$\theta = -\operatorname{Re}\rho \quad \text{and} \quad \omega = \operatorname{Im}\rho. \tag{2.104}$$

For a physical interpretation of the optical scalars we can imagine a bundle of light rays travelling through the space in a circular formation. Behaviour of such a congruence, i.e., if the light rays expand into a circle with large diameter, rotate around the centre of this circle or if they are deformed into an ellipse, can be directly seen from the change of $\boldsymbol{l}$ in the orthogonal transverse direction $\boldsymbol{m}$ as it propagates, i.e.,

$$l_{\mu;\nu} m^j = -\gamma_{1(b)3} e^{(b)} = (\bar{\alpha} + \beta)\, l_\mu - \bar{\rho} m_\mu - \sigma \bar{m}_\mu . \tag{2.105}$$

For visualisation of such transverse deformations of the null geodesic congruence see figures 2.1 and 2.2.

Weyl and Ricci components

To investigate geometrical influence and physical meaning of the Weyl and Ricci tensor components (2.38) and (2.40) it is natural to employ the equation of

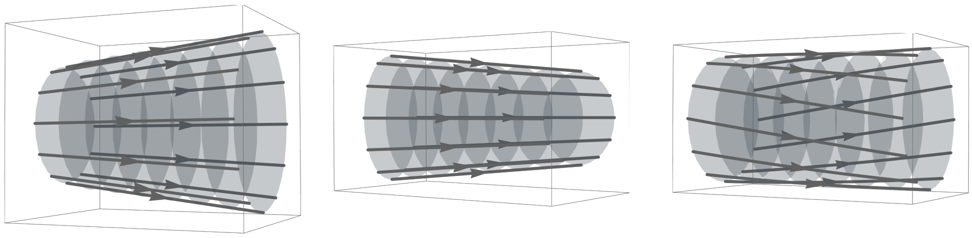

Figure 2.2: Expansion, shear and twist in space picture.

geodesic deviation. Its coordinate form can be written as

$$\frac{\mathrm{D}^2 Z^\mu}{\mathrm{d}\tau^2} = R^\mu{}_{\alpha\beta\nu} u^\alpha u^\beta Z^\nu, \tag{2.106}$$

where

$$\frac{\mathrm{D}^2 Z^\mu}{\mathrm{d}\tau^2} = \left(Z^\mu{}_{;\gamma} u^\gamma\right)_{;\delta} u^\delta = Z^\mu{}_{;\gamma\delta} u^\gamma u^\delta, \tag{2.107}$$

and Z^μ represents components of the separation vector in the congruence and u^α are components of the fiducial time-like observer 4-velocity. To get a coordinate independent information it is natural to project this equation onto an orthonormal frame $\boldsymbol{e}_{(a)}$ with the time-like vector identification $\boldsymbol{e}_{(0)} = \boldsymbol{u}$, i.e., $\boldsymbol{e}_{(a)} \cdot \boldsymbol{e}_{(b)} = \eta_{(a)(b)}$. Obviously, the $\boldsymbol{e}_{(0)}$ projection is trivial, namely

$$\frac{\mathrm{d}^2 Z^{(0)}}{\mathrm{d}\tau^2} = -u_\mu \frac{\mathrm{D}^2 Z^\mu}{\mathrm{d}\tau^2} = -R_{\mu\alpha\beta\nu} u^\mu u^\alpha u^\beta Z^\nu = 0, \tag{2.108}$$

and we have to discuss only the remaining three components,

$$\ddot{Z}^{(\mathrm{i})} = R^{(\mathrm{i})}{}_{(0)(0)(\mathrm{j})} Z^{(\mathrm{j})}, \tag{2.109}$$

where $\mathrm{i}, \mathrm{j} = 1, 2, 3$. Moreover, the Riemann tensor can be decomposed as

$$R_{(\mathrm{i})(0)(0)(\mathrm{j})} = C_{(\mathrm{i})(0)(0)(\mathrm{j})} + \frac{1}{2}\left(R_{(\mathrm{i})(\mathrm{j})} - \delta_{\mathrm{ij}} R_{(0)(0)}\right) - \frac{1}{6} R \delta_{\mathrm{ij}}. \tag{2.110}$$

Finally, we can relate the orthonormal interpretation frame with the null frame introduced within the NP formalism,

$$\begin{aligned} \boldsymbol{e}_{(0)} &= \frac{1}{\sqrt{2}}(\boldsymbol{l} + \boldsymbol{n}), \qquad & \boldsymbol{e}_{(1)} &= \frac{1}{\sqrt{2}}(\boldsymbol{l} - \boldsymbol{n}), \\ \boldsymbol{e}_{(2)} &= \frac{1}{\sqrt{2}}(\boldsymbol{m} + \bar{\boldsymbol{m}}), \qquad & \boldsymbol{e}_{(3)} &= \frac{1}{i\sqrt{2}}(\boldsymbol{m} - \bar{\boldsymbol{m}}), \end{aligned} \tag{2.111}$$

and rewrite the crucial expression (2.110) encoding the test particles relative acceleration in terms of the Weyl tensor (2.38) and Ricci tensor (2.40) null frame components. Based on such an analysis, we should be in principle able to connect these frame components with particular relative motion in the geodesic congruence. Moreover, in a given theory of gravity we can also employ the field equation to separate the matter influence and free gravitation field. Detailed discussion of the GR case can be found in [18, 19].

3. Newman–Penrose formalism in theories of gravity

This chapter explores application of the NP formalism geometric constraints in particular theories of gravity. In the first part we will look at the NP formalism in classical General Relativity with a simple example of Schwarzschild solution. Subsequently, a section applying the NP quantities in the context of Quadratic gravity will follow. As we know the field equations take a more complicated form in such a case. Especially complicated new object entering the game is the Bach tensor. As a main goal here we explicitly compute all independent components of the frame projections and formulate the procedure how to combine the geometric and Quadratic gravity field equations constraints.

3.1 General Relativity

In the case of Einstein's theory, the field equations (1) can be written in the form,

$$R_{\mu\nu} = \Lambda g_{\mu\nu} + 8\pi(T_{\mu\nu} - \frac{1}{2}T g_{\mu\nu}), \tag{3.1}$$

which explicitly relates the Ricci tensor with a matter content of the spacetime. It is thus obvious that the frame projections of the field equations represent just algebraic constrains between the Ricci and energy-momentum tensor components. Therefore, in the case of General Relativity employing the field equations into the frame formalism is very straightforward. For the vacuum spacetimes with $\Lambda = 0$ the simple condition of vanishing Ricci tensor has to enter the geometric identities of the previous chapter.

We believe that the best approach for understanding how the NP formalism can be used in the General Relativity is to go through an explicit example. This section summarises example from [20].

Example: Vacuum spherical spacetime

We start with general line element with two unspecified functions $u(r,t)$ and $v(r,t)$, namely

$$ds^2 = e^{2v}dt^2 - e^{2u}dr^2 - r^2\left(d\theta^2 + \sin\theta^2\theta d\phi^2\right). \tag{3.2}$$

The matrix form of this metric is

$$g_{\mu\nu} = \begin{pmatrix} e^{2v} & 0 & 0 & 0 \\ 0 & -e^{2u} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin\theta^2 \end{pmatrix}. \tag{3.3}$$

When it comes to the null tetrad we have a freedom of its choice. This choice will greatly influence all the following calculations and simplicity of the problem formulation. The only conditions that need to be satisfied are (2.28). For this particular metric its natural to choose the following tetrad

$$l^\mu = \begin{pmatrix} \frac{1}{\sqrt{2}}e^{-v} \\ \frac{1}{\sqrt{2}}e^{-u} \\ 0 \\ 0 \end{pmatrix}, \quad n^\mu = \begin{pmatrix} \frac{1}{\sqrt{2}}e^{-v} \\ -\frac{1}{\sqrt{2}}e^{-u} \\ 0 \\ 0 \end{pmatrix}, \quad m^\mu = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{\sqrt{2}r} \\ \frac{i}{\sqrt{2}r\sin\theta} \end{pmatrix}, \quad \bar{m}^\mu = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{\sqrt{2}r} \\ \frac{-i}{\sqrt{2}r\sin\theta} \end{pmatrix}. \tag{3.4}$$

Now, we can compute the spin coefficients (2.36). The only nonzero coefficients are

$$\epsilon = \frac{1}{2\sqrt{2}}(e^{-u}v_{,r} + e^{-v}u_{,t}), \tag{3.5}$$

$$\gamma = \frac{1}{2\sqrt{2}}(e^{-u}v_{,r} - e^{-v}u_{,t}), \tag{3.6}$$

$$\alpha = -\frac{\cot\theta}{2\sqrt{2}r}, \tag{3.7}$$

$$\beta = -\alpha = \frac{\cot\theta}{2\sqrt{2}r}, \tag{3.8}$$

$$\rho = -\frac{e^{-u}}{\sqrt{2}r}, \tag{3.9}$$

$$\mu = \rho = -\frac{e^{-u}}{\sqrt{2}r}. \tag{3.10}$$

$$\tag{3.11}$$

Before we substitute these expressions into the Ricci identities (2.52)–(2.69) it is useful to realise that ϵ, γ, ρ, and μ are functions of r and t only. That means that derivatives δ and $\bar{\delta}$ acting on these spin coefficients yield zero since they are defined by m^μ and $\bar{m}^\mu$. So according to the definition (2.35) only θ and ϕ components of the covariant derivative are present in the δ and $\bar{\delta}$. To summarise,

$$0 = \delta\epsilon = \bar{\delta}\epsilon = \delta\gamma = \bar{\delta}\gamma = \delta\rho = \bar{\delta}\rho = \delta\mu = \bar{\delta}\mu. \tag{3.12}$$

Keeping only the non-trivial terms in the Ricci identities (2.52)–(2.69) we obtain

$$0 = \Psi_0 = \Psi_1 = \Psi_3 = \Psi_4, \tag{3.13}$$

$$0 = \Phi_{10} = \Phi_{01} = \Phi_{20} = \Phi_{02} = \Phi_{12} = \Phi_{21}, \tag{3.14}$$

$$\mathrm{D}\rho = \rho^2 + 2\epsilon\rho + \Phi_{00}, \tag{3.15}$$

$$\Delta\rho = -\rho^2 - 2\gamma\rho - \Phi_{22}, \tag{3.16}$$

$$\mathrm{D}\gamma - \Delta\epsilon = -4\epsilon\gamma + \Psi_2 - \frac{R}{24} + \Phi_{11}, \tag{3.17}$$

$$\delta\alpha + \bar{\delta}\alpha = 4\alpha^2 + \rho^2 - \Psi_2 + \frac{R}{24} + \Phi_{11}, \tag{3.18}$$

$$0 = -4\epsilon\rho + \Psi_2 + 2\frac{R}{24} - \Phi_{00}, \tag{3.19}$$

$$0 = -4\gamma\rho + \Psi_2 + 2\frac{R}{24} - \Phi_{22}. \tag{3.20}$$

As we can see a lot of projected components of the Ricci tensor are zero automatically. This follows from the metric ansatz and the geometric constrains without employing any particular theory field equations. In the vacuum General Relativity with the vanishing cosmological constant all components of the Ricci tensor, and the Ricci scalar, have to be zero due to the Einstein field equations. Therefore, in addition we must set the remaining non-vanishing Ricci tensor components Φ as well as the scalar curvature R to zero to fulfil the field equations. The remaining nontrivial constrains take the form

$$\begin{aligned} \mathrm{D}\rho =& \rho^2 + 2\epsilon\rho, && (3.21)\\ \Delta\rho =& -\rho^2 - 2\gamma\rho, && (3.22)\\ \mathrm{D}\gamma - \Delta\epsilon =& -4\epsilon\gamma + \Psi_2, && (3.23)\\ \delta\alpha + \bar{\delta}\alpha =& 4\alpha^2 + \rho^2 - \Psi_2, && (3.24)\\ 0 =& -4\epsilon\rho + \Psi_2, && (3.25)\\ 0 =& -4\gamma\rho + \Psi_2. && (3.26) \end{aligned}$$

Now we need to solve these field equations. Let us start with algebraic equations (3.25) and (3.26) that give us obviously equality of the scalars γ and ϵ, i.e.,

$$\gamma = \epsilon. \tag{3.27}$$

Combining the equations (3.21) and (3.22) and substituting the equality above yields,

$$0 = (\mathrm{D} + \Delta)\rho = \sqrt{2}e^{-v}\rho_{,t} = -\frac{\sqrt{2}e^{-v-u}}{\sqrt{2}r}u_{,t},$$

which implies

$$u_{,t} = 0. \tag{3.28}$$

Furthermore, equipped with the constraint (3.28) the equation (3.21) will give us another useful relation, namely

$$\begin{aligned} & \mathrm{D}\rho = \rho^2 + 2\epsilon\rho, \\ \Longrightarrow \quad & \frac{e^{-u}}{\sqrt{2}}\rho_{,r} = \frac{e^{-2u}}{2r^2} - 2\frac{e^{-2u}v_{,r}}{4r}, \\ \Longrightarrow \quad & \frac{e^{-2u}}{2r^2} + \frac{e^{-2u}}{2r}u_{,r} = \frac{e^{-2u}}{2r^2} - \frac{e^{-2u}}{2r}v_{,r}, \end{aligned}$$

resulting in

$$(u + v)_{,r} = 0. \tag{3.29}$$

Substitution for Ψ_2 from (3.25) into the equation (3.24) yields

$$
\begin{aligned}
& \delta\alpha + \bar{\delta}\alpha = 4\alpha^2 + \rho^2 - 4\epsilon\rho, \\
\Longrightarrow \quad & \frac{\sqrt{2}}{r}\alpha_{,\theta} = 4\frac{\cos\theta^2}{8r^2\sin\theta^2} + \frac{e^{-2u}}{2r^2} + \frac{e^{-2u}}{r}v_{,r}, \\
\Longrightarrow \quad & \frac{1}{2r^2\sin\theta^2} = \frac{\cos\theta^2}{2r^2\sin\theta^2} + \frac{e^{-2u}}{2r^2} - \frac{e^{-2u}}{r}u_{,r}, \\
\Longrightarrow \quad & \frac{1}{2r^2} = \frac{e^{-2u}}{2r^2}(1 - 2ru_{,r}), \\
\Longrightarrow \quad & e^{2u} = 1 - 2ru_{,r}, \\
\Longrightarrow \quad & \frac{u_{,r}}{1 - e^{2u}} = \frac{1}{2r}, \\
\Longrightarrow \quad & \frac{e^{2u}}{1 - e^{2u}} = rd,
\end{aligned}
$$

where d is an integration constant. Thus, we obtain the metric coefficient e^{2u} in the form

$$
e^{2u} = \frac{1}{1 + \frac{1}{rd}} \tag{3.30}
$$

Finally, previous result in combination with the equation (3.29) provides

$$
v_{,r} = -\frac{1 - e^{2u}}{2r} = \frac{1}{2r(1 + rd)},
$$

So, the second metric coefficient e^{2v} takes the form

$$
e^{2v} = \frac{rc}{1 + rd}f(t)^2 = \frac{1}{\frac{d}{c} + \frac{1}{rc}}f(t)^2, \tag{3.31}
$$

where c is another integration constant. After coordinate transformation rescaling the coordinate t, namely

$$
dt' = f(t)dt, \tag{3.32}
$$

and choosing the previously introduced constants to be $c = d = -\frac{1}{2M}$ we arrive at the classic Schwarzschild metric,

$$
ds^2 = \left(1 - \frac{2M}{r}\right)dt^2 - \left(1 - \frac{2M}{r}\right)^{-1}dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\theta^2\right). \tag{3.33}
$$

We hope that now it is much more apparent how can one use the NP formalism in practical calculations within General Relativity. However, before we can employ similar formalism and procedure in Quadratic Gravity, we will need to introduce a few more specific objects in the NP formalism, and discuss constraints following the field equations.

3.2 Quadratic gravity

In this section we will explicitly express the field equations of Quadratic gravity (1.9) using the NP scalars and present possible way how to employ corresponding restrictions withing geometric constraints of the second chapter.

We are motivated by the classical General Relativity approach eliminating the Ricci tensor components. The crucial observation is that the Quadratic Gravity field equation does not contain derivatives of the Ricci tensor. It should be possible to follow the same strategy.

In the first chapter we have rewritten the field equations of Quadratic gravity into the form (1.14) by introducing a new tensor $Z_{\mu\nu}$. Our motivation was to explicitly separate the Ricci tensor contribution. Projecting this set of equations onto the NP null frame yields

$$
\begin{aligned}
0 =& -4\mathfrak{a}(\Phi_{20}\Psi_0 + \Phi_{02}\bar{\Psi}_0 - 2\Phi_{10}\Psi_1 - 2\Phi_{01}\bar{\Psi}_1 + \Phi_{00}\Psi_2 + \Phi_{00}\bar{\Psi}_2) \\
&+ \frac{2\Phi_{00}}{\mathsf{k}} + 4\mathfrak{b}R\Phi_{00} - Z_{(1)(1)}, && (3.34) \\
0 =& -4\mathfrak{a}(\Phi_{21}\Psi_1 + \Phi_{12}\bar{\Psi}_1 - 2\Phi_{11}(\Psi_2 + \bar{\Psi}_2) + \Phi_{01}\Psi_3 + \Phi_{10}\bar{\Psi}_3) \\
&+ \frac{1}{\mathsf{k}}(2\Phi_{11} - \frac{R}{4}) + 2\mathfrak{b}R(2\Phi_{11} - \frac{R}{4}) - Z_{(1)(2)}, && (3.35) \\
0 =& -4\mathfrak{a}(\Phi_{21}\Psi_0 - 2\Phi_{11}\Psi_1 + \Phi_{02}\bar{\Psi}_1 + \Phi_{01}\Psi_2 - 2\Phi_{01}\bar{\Psi}_2 + \Phi_{00}\bar{\Psi}_3) \\
&+ \frac{2\Phi_{01}}{\mathsf{k}} + 4\mathfrak{b}R\Phi_{01} - Z_{(1)(3)}, && (3.36) \\
0 =& -4\mathfrak{a}\Big(\Phi_{22}(\Psi_2 + \bar{\Psi}_2) - 2\Phi_{12}\Psi_3 - 2\Phi_{21}\bar{\Psi}_3 + \Phi_{02}\Psi_4 + \Phi_{20}\bar{\Psi}_4\Big) \\
&+ \frac{2\Phi_{22}}{\mathsf{k}} + 4\mathfrak{b}R\Phi_{22} - Z_{(2)(2)}, && (3.37) \\
0 =& -4\mathfrak{a}(\Phi_{22}\Psi_1 - 2\Phi_{12}\Psi_2 + \Phi_{12}\bar{\Psi}_2 + \Phi_{02}\Psi_3 - 2\Phi_{11}\bar{\Psi}_3 + \Phi_{10}\bar{\Psi}_4) \\
&+ \frac{2\Phi_{12}}{\mathsf{k}} + 4\mathfrak{b}R\Phi_{12} - Z_{(2)(3)}, && (3.38) \\
0 =& -4\mathfrak{a}(\Phi_{22}\Psi_0 - 2\Phi_{12}\Psi_1 + \Phi_{02}\Psi_2 + \Phi_{02}\bar{\Psi}_2 - 2\Phi_{01}\bar{\Psi}_3 + \Phi_{00}\bar{\Psi}_4) \\
&+ \frac{2\Phi_{02}}{\mathsf{k}} + 4\mathfrak{b}R\Phi_{02} - Z_{(3)(3)}, && (3.39) \\
0 =& -4\mathfrak{a}(\Phi_{21}\Psi_1 + \Phi_{12}\bar{\Psi}_1 - 2\Phi_{11}\Psi_2 - 2\Phi_{11}\bar{\Psi}_2 + \Phi_{01}\Psi_3 + \Phi_{10}\bar{\Psi}_3) \\
&+ \frac{1}{\mathsf{k}}(2\Phi_{11} + \frac{R}{4}) + 2\mathfrak{b}R(2\Phi_{11} + \frac{R}{4}) - Z_{(3)(4)} && (3.40)
\end{aligned}
$$

where components of the Weyl and Ricci tensors are defined by (2.38) and (2.40) respectively, and e.g., $Z_{(1)(1)} = Z_{\mu\nu}l^\mu l^\nu$ and $Z_{(2)(3)} = Z_{\mu\nu}n^\mu m^\nu$ etc. In fact, this is the crucial set of equations where all the Ricci tensor components are explicitly listed and one can in principle solve the algebraic system of equations to obtain these components. Then, they can be substituted into the geometric identities of the previous chapter.

To be fully explicit, in the next step we express all relevant projections of the $Z_{\mu\nu}$ tensor, i.e.,

$$
\begin{aligned}
Z_{(1)(1)} =& -4\mathfrak{a}B^Z_{(1)(1)} + 2\mathfrak{b}\Big((\epsilon + \bar{\epsilon})\mathrm{D}R - \mathrm{DD}R - \bar{\kappa}\delta R - \kappa\bar{\delta}R\Big), && (3.41) \\
Z_{(1)(2)} =& -\frac{1}{\mathsf{k}}(\frac{R}{2} - \Lambda) - 4\mathfrak{a}B^Z_{(1)(2)} + 2\mathfrak{b}\Big(-\frac{1}{4}R^2 - (\gamma + \bar{\gamma} - \mu - \bar{\mu})\mathrm{D}R \\
&- \rho\Delta R - \bar{\rho}\Delta R + \Delta\mathrm{D}R + \alpha\delta R - \bar{\beta}\delta R + \bar{\tau}\delta R - \delta\bar{\delta}R \\
&+ \bar{\alpha}\bar{\delta}R - \beta\bar{\delta}R + \tau\bar{\delta}R - \bar{\delta}\delta R\Big), && (3.42)
\end{aligned}
$$

$$Z_{(1)(3)} = -4\mathfrak{a}B^Z_{(1)(3)} + 2\mathfrak{b}(\bar{\pi}\mathrm{D}R - \mathrm{D}\delta R - \kappa\Delta R + \epsilon\delta R - \bar{\epsilon}\delta R), \tag{3.43}$$

$$Z_{(2)(2)} = -4\mathfrak{a}B^Z_{(2)(2)} + 2\mathfrak{b}\Big((\gamma + \bar{\gamma})\Delta R + \Delta\Delta R - \nu\delta R - \bar{\nu}\bar{\delta}R\Big), \tag{3.44}$$

$$Z_{(2)(3)} = -4\mathfrak{a}B^Z_{(2)(3)} + 2\mathfrak{b}(\bar{\nu}\mathrm{D}R - \tau\Delta R - \Delta\delta R + \gamma\delta R - \bar{\gamma}\delta R), \tag{3.45}$$

$$Z_{(3)(3)} = -4\mathfrak{a}B^Z_{(3)(3)} + 2\mathfrak{b}(\bar{R}\mathrm{D}R - \sigma\Delta R - \bar{\alpha}\delta R + \beta\delta R - \delta\delta R), \tag{3.46}$$

$$\begin{aligned} Z_{(3)(4)} =& \frac{1}{\mathsf{k}}(\frac{R}{2} - \Lambda) - 4\mathfrak{a}B^Z_{(3)(4)} + 2\mathfrak{b}\Big(\frac{1}{4}R^2 + (\gamma + \bar{\gamma} - \bar{\mu})\mathrm{D}R \\ & - \mathrm{D}\Delta R - \epsilon\Delta R - \bar{\epsilon}\Delta R + \rho\Delta R - \Delta\mathrm{D}R - \alpha\delta R + \bar{\beta}\delta R \\ & + \pi\delta R - \bar{\tau}\delta R + \bar{\pi}\bar{\delta}R - \tau\bar{\delta}R + \bar{\delta}\delta R\Big). \end{aligned} \tag{3.47}$$

where $B^Z_{(a)(b)} = B^Z_{\mu\nu}e_{(a)}^{\ \mu}e_{(b)}^{\ \nu}$ with $B^Z_{\mu\nu}$ representing "Ricci-independent" part of the Bach tensor corresponding to the second covariant derivative of the Weyl tensor, namely

$$B^Z_{\mu\nu} = \nabla^\alpha\nabla^\beta C_{\mu\alpha\nu\beta}. \tag{3.48}$$

This is thus the last piece which has to be expressed using the spin coefficients.

Here we make a small natural sidestep. Instead of calculating only the projection of $B^Z_{\mu\nu}$ part, we express full Bach tensor and identify this splitting. The same procedure as within all previous projections is now applied to the definition of Bach tensor (1.10), namely

$$\begin{aligned} B_{(a)(b)} =& C_{(a)(p)(b)(r),(q),(s)}\eta^{(p)(q)}\eta^{(r)(s)} - \frac{1}{2}R^{(c)(d)}C_{(a)(c)(b)(d)} \\ & - \Big(\gamma^{(m)(c)}_{\ \ (a)}C_{(m)(c)(b)(d)}\Big)^{,(d)} - \Big(\gamma^{(m)(c)}_{\ \ (c)}C_{(a)(m)(b)(d)}\Big)^{,(d)} \\ & - \Big(\gamma^{(m)(c)}_{\ \ (d)}C_{(a)(c)(b)(m)}\Big)^{,(d)} - \Big(\gamma^{(m)(c)}_{\ \ (b)}C_{(a)(c)(m)(d)}\Big)^{,(d)} \\ & - \gamma^{(m)(d)}_{\ \ (a)}C_{(m)(c)(b)(d)}^{\ \ ,(c)} - \gamma^{(m)(d)}_{\ \ (b)}C_{(a)(c)(m)(d)}^{\ \ ,(c)} \\ & - \gamma^{(d)(m)}_{\ \ (m)}C_{(a)(c)(b)(d)}^{\ \ ,(c)} \\ & + \Big(\gamma^{(n)(d)}_{\ \ (a)}\gamma^{(m)(c)}_{\ \ (n)} + \gamma^{(m)(d)}_{\ \ (a)}\gamma^{(c)(n)}_{\ \ (n)} + \gamma^{(m)(n)}_{\ \ (a)}\gamma^{(d)(c)}_{\ \ (n)} + \gamma^{(m)(c)}_{\ \ (a)}\gamma^{(d)(n)}_{\ \ (n)}\Big)C_{(m)(c)(b)(d)} \\ & + \Big(\gamma^{(m)(d)}_{\ \ (b)}\gamma^{(c)(m)}_{\ \ (m)} + \gamma^{(n)(d)}_{\ \ (b)}\gamma^{(m)(c)}_{\ \ (n)} + \gamma^{(m)(n)}_{\ \ (b)}\gamma^{(d)(c)}_{\ \ (n)} + \gamma^{(m)(c)}_{\ \ (b)}\gamma^{(d)(n)}_{\ \ (n)}\Big)C_{(a)(c)(m)(d)} \\ & + \Big(\gamma^{(m)(c)}_{\ \ (a)}\gamma^{(n)(d)}_{\ \ (b)} + \gamma^{(n)(c)}_{\ \ (a)}\gamma^{(m)(d)}_{\ \ (b)}\Big)C_{(n)(c)(m)(d)} \\ & + \Big(\gamma^{(d)(n)}_{\ \ (n)}\gamma^{(c)(m)}_{\ \ (m)} + \gamma^{(m)(n)}_{\ \ (n)}\gamma^{(d)(c)}_{\ \ (m)}\Big)C_{(a)(c)(b)(d)} \end{aligned} \tag{3.49}$$

The evaluation of particular components then gives following relations,

$$
\begin{aligned}
B_{(1)(1)} =& - \Phi_{20}\Psi_0 - \Phi_{02}\bar{\Psi}_0 + 2\Phi_{10}\Psi_1 + 2\Phi_{01}\bar{\Psi}_1 - \Phi_{00}\Psi_2 - \Phi_{00}\bar{\Psi}_2 + B^{Z}_{(1)(1)} \\
=& - \Phi_{20}\Psi_0 + 2\Phi_{10}\Psi_1 - \Phi_{00}\Psi_2 \\
& - \bar{\delta}\bar{\delta}\Psi_0 + \mathrm{D}\bar{\delta}\Psi_1 + \bar{\delta}\mathrm{D}\Psi_1 - \mathrm{D}\mathrm{D}\Psi_2 \\
& - \lambda\mathrm{D}\Psi_0 - \bar{\sigma}\Delta\Psi_0 - (-7\alpha - \bar{\beta} + 2\pi)\bar{\delta}\Psi_0 \\
& - (5\alpha + \bar{\beta} - 3\pi)\mathrm{D}\Psi_1 + \bar{\kappa}\Delta\Psi_1 + \bar{\sigma}\delta\Psi_1 - (3\epsilon + \bar{\epsilon} + 7\rho)\bar{\delta}\Psi_1 \\
& - (-\epsilon - \bar{\epsilon} - 6\rho)\mathrm{D}\Psi_2 - \bar{\kappa}\delta\Psi_2 + 5\kappa\bar{\delta}\Psi_2 \\
& - 4\kappa\mathrm{D}\Psi_3 \\
& - \Psi_0(12\alpha^2 + 4\alpha\bar{\beta} - \epsilon\lambda - \bar{\epsilon}\lambda + \bar{\kappa}\nu - 7\alpha\pi - \bar{\beta}\pi + \pi^2 - 3\lambda\rho - 4\gamma\bar{\sigma} \\
& \quad + \mu\bar{\sigma} + \mathrm{D}\lambda - 4\bar{\delta}\alpha + \bar{\delta}\pi) \\
& + 2\Psi_1(4\alpha\epsilon + \bar{\beta}\epsilon + \alpha\bar{\epsilon} - \gamma\bar{\kappa} - 2\kappa\lambda + \bar{\kappa}\mu - 2\epsilon\pi - \bar{\epsilon}\pi + 9\alpha\rho + 2\bar{\beta}\rho \\
& \quad - 5\pi\rho - \beta\bar{\sigma} - 2\bar{\sigma}\tau - \mathrm{D}\alpha + \mathrm{D}\pi - \bar{\delta}\epsilon - 2\bar{\delta}\rho) \\
& - 3\Psi_2(3\alpha\kappa + \bar{\beta}\kappa - 3\kappa\pi + \epsilon\rho + \bar{\epsilon}\rho + 3\rho^2 - \sigma\bar{\sigma} - \bar{\kappa}\tau - \mathrm{D}\rho - \bar{\delta}\kappa) \\
& - 2\Psi_3(\epsilon\kappa - \bar{\epsilon}\kappa - 5\kappa\rho + \bar{\kappa}\sigma + \mathrm{D}\kappa) \\
& - 2\Psi_4\kappa^2 \\
& + c.c.,
\end{aligned} \tag{3.50}
$$

$$
\begin{aligned}
B_{(1)(2)} =& - \Phi_{21}\Psi_1 - \Phi_{12}\bar{\Psi}_1 + 2\Phi_{11}(\Psi_2 + \bar{\Psi}_2) - \Phi_{01}\Psi_3 - \Phi_{10}\bar{\Psi}_3 + B^{Z}_{(1)(2)} \\
=& - \Phi_{21}\Psi_1 + 2\Phi_{11}\Psi_2 - \Phi_{01}\Psi_3 \\
& - \bar{\delta}\Delta\Psi_1 + \mathrm{D}\Delta\Psi_2 + \bar{\delta}\delta\Psi_2 - \mathrm{D}\delta\Psi_3 \\
& + \lambda\Delta\Psi_0 + \nu\bar{\delta}\Psi_0 \\
& - 2\nu\mathrm{D}\Psi_1 - (-\alpha + \bar{\beta} + 2\pi)\Delta\Psi_1 - \lambda\delta\Psi_1 - (-2\gamma + 2\mu - \bar{\mu})\bar{\delta}\Psi_1 \\
& - (-3\mu + \bar{\mu})\mathrm{D}\Psi_2 - (-\epsilon - \bar{\epsilon} + 2\rho)\Delta\Psi_2 - (\alpha - \bar{\beta} - 2\pi)\delta\Psi_2 \\
& \quad - (\bar{\pi} + 3\tau)\bar{\delta}\Psi_2 \\
& - (2\beta - \bar{\pi} - 2\tau)\mathrm{D}\Psi_3 + \kappa\Delta\Psi_3 - (\epsilon + \bar{\epsilon} - 2\rho)\delta\Psi_3 + 2\sigma\bar{\delta}\Psi_3 \\
& - \sigma\mathrm{D}\Psi_4 - \kappa\delta\Psi_4 \\
& - \Psi_0(4\gamma\lambda - \lambda\mu + \lambda\bar{\mu} + \alpha\nu - \bar{\beta}\nu - 2\nu\pi - \bar{\delta}\nu) \\
& - \Psi_1(2\alpha\gamma - 2\bar{\beta}\gamma - 2\beta\lambda - 2\alpha\mu + 2\bar{\beta}\mu + 2\alpha\bar{\mu} + 2\epsilon\nu + 2\bar{\epsilon}\nu \\
& \quad - 4\gamma\pi + 4\mu\pi - 2\bar{\mu}\pi - 2\lambda\bar{\pi} - 4\nu\rho - 4\lambda\tau + 2\mathrm{D}\nu - 2\bar{\delta}\gamma + 2\bar{\delta}\mu) \\
& + 3\Psi_2(\epsilon\mu + \bar{\epsilon}\mu - \kappa\nu - \pi\bar{\pi} - 2\mu\rho + \bar{\mu}\rho - \lambda\sigma + \alpha\tau - \bar{\beta}\tau - 2\pi\tau \\
& \quad + \mathrm{D}\mu - \bar{\delta}\tau) \\
& - 2\Psi_3(\beta\epsilon + \beta\bar{\epsilon} - \gamma\kappa - 2\kappa\mu + \kappa\bar{\mu} - \epsilon\bar{\pi} - 2\beta\rho + \bar{\pi}\rho + \alpha\sigma - \bar{\beta}\sigma \\
& \quad - 2\pi\sigma - \epsilon\tau - \bar{\epsilon}\tau + 2\rho\tau + \mathrm{D}\beta - \mathrm{D}\tau - \bar{\delta}\sigma) \\
& - \Psi_4(4\beta\kappa - \kappa\bar{\pi} + \epsilon\sigma + \bar{\epsilon}\sigma - 2\rho\sigma - \kappa\tau + \mathrm{D}\sigma) \\
& + c.c.,
\end{aligned} \tag{3.51}
$$

$$
\begin{aligned}
B_{(1)(3)} =& -\Phi_{21}\Psi_0 + 2\Phi_{11}\Psi_1 - \Phi_{01}\Psi_2 - \Phi_{02}\bar{\Psi}_1 + 2\Phi_{01}\bar{\Psi}_2 - \Phi_{00}\bar{\Psi}_3 + B^Z_{(1)(3)} \\
=& -\Phi_{21}\Psi_0 + 2\Phi_{11}\Psi_1 - \Phi_{01}\Psi_2 \\
& -\bar{\delta}\Delta\Psi_0 + \mathrm{D}\Delta\Psi_1 + \bar{\delta}\delta\Psi_1 - \mathrm{D}\delta\Psi_2 \\
& -\nu\mathrm{D}\Psi_0 - (-3\alpha + \bar{\beta} + \pi)\Delta\Psi_0 - (-4\gamma + \mu - \bar{\mu})\bar{\delta}\Psi_0 \\
& -(2\gamma - 2\mu + \bar{\mu})\mathrm{D}\Psi_1 - (\epsilon - \bar{\epsilon} + 3\rho)\Delta\Psi_1 - (3\alpha - \bar{\beta} - \pi)\delta\Psi_1 \\
& \quad - (2\beta + \bar{\pi} + 4\tau)\bar{\delta}\Psi_1 \\
& -(-\bar{\pi} - 3\tau)\mathrm{D}\Psi_2 + 2\kappa\Delta\Psi_2 - (-\epsilon + \bar{\epsilon} - 3\rho)\delta\Psi_2 + 3\sigma\bar{\delta}\Psi_2 \\
& -2\sigma\mathrm{D}\Psi_3 - 2\kappa\delta\Psi_3 \\
& -\Psi_0(12\alpha\gamma - 4\bar{\beta}\gamma - 3\alpha\mu + \bar{\beta}\mu + 4\alpha\bar{\mu} - \epsilon\nu + \bar{\epsilon}\nu - 4\gamma\pi \\
& \quad + \mu\pi - \bar{\mu}\pi - \lambda\bar{\pi} - 3\nu\rho + \mathrm{D}\nu - 4\bar{\delta}\gamma + \bar{\delta}\mu) \\
& +2\Psi_1(3\alpha\beta - \beta\bar{\beta} + \gamma\epsilon - \gamma\bar{\epsilon} - \epsilon\mu + \bar{\epsilon}\mu + \epsilon\bar{\mu} - 2\kappa\nu - \beta\pi + \alpha\bar{\pi} - \pi\bar{\pi} \\
& \quad + 3\gamma\rho - 3\mu\rho + 2\bar{\mu}\rho + 6\alpha\tau - 2\bar{\beta}\tau - 2\pi\tau - \mathrm{D}\gamma + \mathrm{D}\mu - \bar{\delta}\beta - 2\bar{\delta}\tau) \\
& +3\Psi_2(2\kappa\mu - \kappa\bar{\mu} - \bar{\pi}\rho - 3\alpha\sigma + \bar{\beta}\sigma + \pi\sigma - \epsilon\tau + \bar{\epsilon}\tau - 3\rho\tau + \mathrm{D}\tau + \bar{\delta}\sigma) \\
& -2\Psi_3(2\beta\kappa - \kappa\bar{\pi} - \epsilon\sigma + \bar{\epsilon}\sigma - 3\rho\sigma - 2\kappa\tau + \mathrm{D}\sigma) \\
& -2\Psi_4\kappa\sigma \\
& -\Phi_{02}\bar{\Psi}_1 + 2\Phi_{01}\bar{\Psi}_2 - \Phi_{00}\bar{\Psi}_3 \\
& -\delta\delta\bar{\Psi}_1 + \delta\mathrm{D}\bar{\Psi}_2 + \mathrm{D}\delta\bar{\Psi}_2 - \mathrm{D}\mathrm{D}\bar{\Psi}_3 \\
& +2\bar{\lambda}\delta\bar{\Psi}_0 \\
& -3\bar{\lambda}\mathrm{D}\bar{\Psi}_1 - \sigma\Delta\bar{\Psi}_1 + (3\bar{\alpha} + \beta - 4\bar{\pi})\delta\bar{\Psi}_1 \\
& +(-\bar{\alpha} - \beta + 5\bar{\pi})\mathrm{D}\bar{\Psi}_2 + \kappa\Delta\bar{\Psi}_2 + (-\epsilon + \bar{\epsilon} - 5\bar{\rho})\delta\bar{\Psi}_2 + \sigma\bar{\delta}\bar{\Psi}_2 \\
& +(\epsilon - 3\bar{\epsilon} + 4\bar{\rho})\mathrm{D}\bar{\Psi}_3 + 3\bar{\kappa}\delta\bar{\Psi}_3 - \kappa\bar{\delta}\bar{\Psi}_3 \\
& -2\bar{\kappa}\mathrm{D}\bar{\Psi}_4 \\
& +\bar{\Psi}_0(-5\bar{\alpha}\bar{\lambda} - \beta\bar{\lambda} + 3\bar{\lambda}\bar{\pi} + \bar{\nu}\sigma + \delta\bar{\lambda}) \\
& -2\bar{\Psi}_1(\bar{\alpha}^2 + \bar{\alpha}\beta - \epsilon\bar{\lambda} + \kappa\bar{\nu} - 3\bar{\alpha}\bar{\pi} - \beta\bar{\pi} + 2\bar{\pi}^2 - 4\bar{\lambda}\bar{\rho} - \bar{\gamma}\sigma + \bar{\mu}\sigma \\
& \quad + \mathrm{D}\bar{\lambda} - \delta\bar{\alpha} + \delta\bar{\pi}) \\
& -3\bar{\Psi}_2(2\bar{\kappa}\bar{\lambda} - \kappa\bar{\mu} + \epsilon\bar{\pi} - \bar{\epsilon}\bar{\pi} - \bar{\alpha}\bar{\rho} - \beta\bar{\rho} + 4\bar{\pi}\bar{\rho} + \sigma\bar{\tau} - \mathrm{D}\bar{\pi} + \delta\bar{\rho}) \\
& +2\bar{\Psi}_3(\epsilon\bar{\epsilon} - \bar{\epsilon}^2 - \bar{\beta}\kappa - \beta\bar{\kappa} + 4\bar{\kappa}\bar{\pi} - \epsilon\bar{\rho} + 3\bar{\epsilon}\bar{\rho} - 2\bar{\rho}^2 + \sigma\bar{\sigma} + \kappa\bar{\tau} - \mathrm{D}\bar{\epsilon} \\
& \quad + \mathrm{D}\bar{\rho} + \delta\bar{\kappa}) \\
& +\bar{\Psi}_4(\epsilon\bar{\kappa} - 5\bar{\epsilon}\bar{\kappa} + 3\bar{\kappa}\bar{\rho} - \kappa\bar{\sigma} - \mathrm{D}\bar{\kappa}),
\end{aligned}
\tag{3.52}
$$

$$
\begin{aligned}
B_{(2)(3)} = &- \Phi_{22}\Psi_1 + 2\Phi_{12}\Psi_2 - \Phi_{02}\Psi_3 - \Phi_{12}\bar{\Psi}_2 + 2\Phi_{11}\bar{\Psi}_3 - \Phi_{10}\bar{\Psi}_4 + B^{Z}_{(2)(3)} \\
= &- \Phi_{22}\Psi_1 + 2\Phi_{12}\Psi_2 - \Phi_{02}\Psi_3 \\
&- \Delta\Delta\Psi_1 + \Delta\delta\Psi_2 + \delta\Delta\Psi_2 - \delta\delta\Psi_3 \\
&+ 2\nu\Delta\Psi_0 \\
&- (-3\gamma + \bar{\gamma} + 4\mu)\Delta\Psi_1 - 3\nu\delta\Psi_1 + \bar{\nu}\bar{\delta}\Psi_1 \\
&- \bar{\nu}\mathrm{D}\Psi_2 - (-\bar{\alpha} - \beta + 5\tau)\Delta\Psi_2 - (\gamma - \bar{\gamma} - 5\mu)\delta\Psi_2 - \bar{\lambda}\bar{\delta}\Psi_2 \\
&+ \bar{\lambda}\mathrm{D}\Psi_3 + 3\sigma\Delta\Psi_3 - (\bar{\alpha} + 3\beta - 4\tau)\delta\Psi_3 \\
&- 2\sigma\delta\Psi_4 \\
&- \Psi_0(5\gamma\nu - \bar{\gamma}\nu - 3\mu\nu + \lambda\bar{\nu} - \Delta\nu) \\
&- 2\Psi_1(\gamma^2 - \gamma\bar{\gamma} - \lambda\bar{\lambda} - 3\gamma\mu + \bar{\gamma}\mu + 2\mu^2 + \bar{\alpha}\nu + \alpha\bar{\nu} - \bar{\nu}\pi - 4\nu\tau \\
&\quad - \Delta\gamma + \Delta\mu + \delta\nu) \\
&- \Psi_2(-3\bar{\alpha}\mu - 3\beta\mu + 3\bar{\lambda}\pi - 3\bar{\nu}\rho + 6\nu\sigma - 3\gamma\tau + 3\bar{\gamma}\tau + 12\mu\tau \\
&\quad + 3\Delta\tau - 3\delta\mu) \\
&- 2\Psi_3(\bar{\alpha}\beta + \beta^2 - \epsilon\bar{\lambda} + \kappa\bar{\nu} + \bar{\lambda}\rho - \bar{\gamma}\sigma - 4\mu\sigma - \bar{\alpha}\tau - 3\beta\tau + 2\tau^2 \\
&\quad - \Delta\sigma + \delta\beta - \delta\tau) \\
&- \Psi_4\Big(-\kappa\bar{\lambda} + \sigma(\bar{\alpha} + 5\beta - 3\tau) + \delta\sigma\Big) \\
&- \Phi_{12}\bar{\Psi}_2 + 2\Phi_{11}\bar{\Psi}_3 - \Phi_{10}\bar{\Psi}_4 \\
&+ \Delta\mathrm{D}\bar{\Psi}_3 - \Delta\delta\bar{\Psi}_2 - \bar{\delta}\mathrm{D}\bar{\Psi}_4 + \bar{\delta}\delta\bar{\Psi}_3 \\
&+ 2\bar{\lambda}\Delta\bar{\Psi}_1 + 2\bar{\nu}\delta\bar{\Psi}_1 \\
&- 2\bar{\nu}\mathrm{D}\bar{\Psi}_2 + (-3\bar{\pi} - \tau)\Delta\bar{\Psi}_2 + (\gamma - \bar{\gamma} - 3\bar{\mu})\delta\bar{\Psi}_2 - 3\bar{\lambda}\bar{\delta}\bar{\Psi}_2 \\
&+ (-\gamma + \bar{\gamma} + 3\bar{\mu})\mathrm{D}\bar{\Psi}_3 + (2\bar{\epsilon} + \rho - 2\bar{\rho})\Delta\bar{\Psi}_3 + (-\alpha + 3\bar{\beta} - \bar{\tau})\delta\bar{\Psi}_3 \\
&\quad + (2\bar{\alpha} + 4\bar{\pi} + \tau)\bar{\delta}\bar{\Psi}_3 \\
&+ (\alpha - 3\bar{\beta} + \bar{\tau})\mathrm{D}\bar{\Psi}_4 + \bar{\kappa}\Delta\bar{\Psi}_4 + (-4\bar{\epsilon} - \rho + \bar{\rho})\bar{\delta}\bar{\Psi}_4 \\
&- 2\bar{\Psi}_0\bar{\lambda}\bar{\nu} \\
&+ 2\bar{\Psi}_1(-\gamma\bar{\lambda} + \bar{\gamma}\bar{\lambda} + 3\bar{\lambda}\bar{\mu} - 2\bar{\alpha}\bar{\nu} + 2\bar{\nu}\bar{\pi} + \bar{\nu}\tau + \Delta\bar{\lambda}) \\
&+ \bar{\Psi}_2(3\alpha\bar{\lambda} - 9\bar{\beta}\bar{\lambda} + 3\gamma\bar{\pi} - 3\bar{\gamma}\bar{\pi} - 9\bar{\mu}\bar{\pi} - 3\bar{\nu}\rho + 6\bar{\nu}\bar{\rho} \\
&\quad - 3\bar{\mu}\tau + 3\bar{\lambda}\bar{\tau} - 3\Delta\bar{\pi} - 3\bar{\delta}\bar{\lambda}) \\
&- 2\bar{\Psi}_3(\alpha\bar{\alpha} - 3\bar{\alpha}\bar{\beta} + \gamma\bar{\epsilon} - \bar{\gamma}\bar{\epsilon} - 3\epsilon\bar{\mu} + 2\bar{\kappa}\bar{\nu} + 2\alpha\bar{\pi} - 6\bar{\beta}\bar{\pi} - \bar{\gamma}\rho - 2\bar{\mu}\rho \\
&\quad - \gamma\bar{\rho} + \bar{\gamma}\bar{\rho} + 3\bar{\mu}\bar{\rho} - \bar{\beta}\tau + \bar{\alpha}\bar{\tau} + 2\bar{\pi}\bar{\tau} + \tau\bar{\tau} - \Delta\bar{\epsilon} + \Delta\bar{\rho} - \bar{\delta}\bar{\alpha} - 2\bar{\delta}\bar{\pi}) \\
&+ \bar{\Psi}_4(4\alpha\bar{\epsilon} - 12\bar{\beta}\bar{\epsilon} - \gamma\bar{\kappa} + \bar{\gamma}\bar{\kappa} + 3\bar{\kappa}\bar{\mu} - 4\bar{\beta}\rho - \alpha\bar{\rho} + 3\bar{\beta}\bar{\rho} + \bar{\sigma}\tau + 4\bar{\epsilon}\bar{\tau} \\
&\quad + \rho\bar{\tau} - \bar{\rho}\bar{\tau} + \Delta\bar{\kappa} - 4\bar{\delta}\bar{\epsilon} + \bar{\delta}\bar{\rho}), \qquad (3.53)
\end{aligned}
$$

$$
\begin{aligned}
B_{(3)(3)} =& - \Phi_{22}\Psi_0 + 2\Phi_{12}\Psi_1 - \Phi_{02}\Psi_2 - \Phi_{02}\bar{\Psi}_2 + 2\Phi_{01}\bar{\Psi}_3 - \Phi_{00}\bar{\Psi}_4 + B^Z_{(3)(3)} \\
=& - \Phi_{22}\Psi_0 + 2\Phi_{12}\Psi_1 - \Phi_{02}\Psi_2 \\
& - \Delta\Delta\Psi_0 + \Delta\delta\Psi_1 + \delta\Delta\Psi_1 - \delta\delta\Psi_2 \\
& - (-7\gamma + \bar{\gamma} + 2\mu)\Delta\Psi_0 - \nu\delta\Psi_0 + \bar{\nu}\bar{\delta}\Psi_0 \\
& - \bar{\nu}\mathrm{D}\Psi_1 - (-\bar{\alpha} + 3\beta + 7\tau)\Delta\Psi_1 - (5\gamma - \bar{\gamma} - 3\mu)\delta\Psi_1 - \bar{\lambda}\bar{\delta}\Psi_1 \\
& + \bar{\lambda}\mathrm{D}\Psi_2 + 5\sigma\Delta\Psi_2 - (\bar{\alpha} - \beta - 6\tau)\delta\Psi_2 \\
& - 4\sigma\delta\Psi_3 \\
& - \Psi_0(12\gamma^2 - 4\gamma\bar{\gamma} - \lambda\bar{\lambda} - 7\gamma\mu + \bar{\gamma}\mu + \mu^2 + \bar{\alpha}\nu - \beta\nu + 4\alpha\bar{\nu} - \bar{\nu}\pi \\
& \quad - 3\nu\tau - 4\Delta\gamma + \Delta\mu + \delta\nu) \\
& + 2\Psi_1(-\bar{\alpha}\gamma + 4\beta\gamma - \beta\bar{\gamma} + \alpha\bar{\lambda} + \bar{\alpha}\mu - 2\beta\mu + \epsilon\bar{\nu} - \bar{\lambda}\pi + 2\bar{\nu}\rho \\
& \quad - 2\nu\sigma + 9\gamma\tau - 2\bar{\gamma}\tau - 5\mu\tau - \Delta\beta - 2\Delta\tau - \delta\gamma + \delta\mu) \\
& - 3\Psi_2(\kappa\bar{\nu} + \bar{\lambda}\rho + 3\gamma\sigma - \bar{\gamma}\sigma - 3\mu\sigma - \bar{\alpha}\tau + \beta\tau + 3\tau^2 - \Delta\sigma - \delta\tau) \\
& - 2\Psi_3\Big(-\kappa\bar{\lambda} + \sigma(\bar{\alpha} + \beta - 5\tau) + \delta\sigma\Big) \\
& - 2\Psi_4\sigma^2 \\
& - \Phi_{02}\bar{\Psi}_2 + 2\Phi_{01}\bar{\Psi}_3 - \Phi_{00}\bar{\Psi}_4 \\
& - \mathrm{D}\mathrm{D}\bar{\Psi}_4 + \mathrm{D}\delta\bar{\Psi}_3 + \delta\mathrm{D}\bar{\Psi}_3 - \delta\delta\bar{\Psi}_2 \\
& + 4\bar{\lambda}\delta\bar{\Psi}_1 \\
& - 5\bar{\lambda}\mathrm{D}\bar{\Psi}_2 - \sigma\Delta\bar{\Psi}_2 + (-\bar{\alpha} + \beta - 6\bar{\pi})\delta\bar{\Psi}_2 \\
& + (3\bar{\alpha} - \beta + 7\bar{\pi})\mathrm{D}\bar{\Psi}_3 + \kappa\Delta\bar{\Psi}_3 + (-\epsilon + 5\bar{\epsilon} - 3\bar{\rho})\delta\bar{\Psi}_3 + \sigma\bar{\delta}\bar{\Psi}_3 \\
& + (\epsilon - 7\bar{\epsilon} + 2\bar{\rho})\mathrm{D}\bar{\Psi}_4 + \bar{\kappa}\delta\bar{\Psi}_4 - \kappa\bar{\delta}\bar{\Psi}_4 \\
& - 2\bar{\Psi}_0\bar{\lambda}^2 \\
& + 2\bar{\Psi}_1(-\bar{\alpha}\bar{\lambda} - \beta\bar{\lambda} + 5\bar{\lambda}\bar{\pi} + \bar{\nu}\sigma + \delta\bar{\lambda}) \\
& + 3\bar{\Psi}_2(\epsilon\bar{\lambda} - 3\bar{\epsilon}\bar{\lambda} - \kappa\bar{\nu} - \bar{\alpha}\bar{\pi} + \beta\bar{\pi} - 3\bar{\pi}^2 + 3\bar{\lambda}\bar{\rho} - \bar{\mu}\sigma - \mathrm{D}\bar{\lambda} - \delta\bar{\pi}) \\
& - 2\bar{\Psi}_3(\bar{\alpha}\epsilon - 4\bar{\alpha}\bar{\epsilon} + \beta\bar{\epsilon} - \bar{\gamma}\kappa + 2\bar{\kappa}\bar{\lambda} - 2\kappa\bar{\mu} + 2\epsilon\bar{\pi} - 9\bar{\epsilon}\bar{\pi} + 2\bar{\alpha}\bar{\rho} - \beta\bar{\rho} \\
& \quad + 5\bar{\pi}\bar{\rho} - \bar{\beta}\sigma + \sigma\bar{\tau} - \mathrm{D}\bar{\alpha} - 2\mathrm{D}\bar{\pi} - \delta\bar{\epsilon} + \delta\bar{\rho}) \\
& + \bar{\Psi}_4(4\epsilon\bar{\epsilon} - 12\bar{\epsilon}^2 - 4\bar{\beta}\kappa + \bar{\alpha}\bar{\kappa} - \beta\bar{\kappa} + 3\bar{\kappa}\bar{\pi} - \epsilon\bar{\rho} + 7\bar{\epsilon}\bar{\rho} - \bar{\rho}^2 + \sigma\bar{\sigma} \\
& \quad + \kappa\bar{\tau} - 4\mathrm{D}\bar{\epsilon} + \mathrm{D}\bar{\rho} + \delta\bar{\kappa}),
\end{aligned} \tag{3.54}
$$

$$
\begin{aligned}
B_{(2)(2)} =& - \Phi_{22}(\Psi_2 + \bar{\Psi}_2) + 2\Phi_{12}\Psi_3 + 2\Phi_{21}\bar{\Psi}_3 - \Phi_{02}\Psi_4 - \Phi_{20}\bar{\Psi}_4 + B^Z_{(2)(2)} \\
=& - \Phi_{22}\Psi_2 + 2\Phi_{12}\Psi_3 - \Phi_{02}\Psi_4 \\
& - \Delta\Delta\Psi_2 + \Delta\delta\Psi_3 + \delta\Delta\Psi_3 - \delta\delta\Psi_4 \\
& + 4\nu\Delta\Psi_1 \\
& - (\gamma + \bar{\gamma} + 6\mu)\Delta\Psi_2 - 5\nu\delta\Psi_2 + \bar{\nu}\bar{\delta}\Psi_2 \\
& - \bar{\nu}\mathrm{D}\Psi_3 - (-\bar{\alpha} - 5\beta + 3\tau)\Delta\Psi_3 - (-3\gamma - \bar{\gamma} - 7\mu)\delta\Psi_3 - \bar{\lambda}\bar{\delta}\Psi_3 \\
& + \bar{\lambda}\mathrm{D}\Psi_4 + \sigma\Delta\Psi_4 - (\bar{\alpha} + 7\beta - 2\tau)\delta\Psi_4 \\
& - 2\Psi_0\nu^2 \\
& - 2\Psi_1(\gamma\nu - \bar{\gamma}\nu - 5\mu\nu + \lambda\bar{\nu} - \Delta\nu) \\
& - 3\Psi_2(-\lambda\bar{\lambda} + \gamma\mu + \bar{\gamma}\mu + 3\mu^2 + \bar{\alpha}\nu + 3\beta\nu - \bar{\nu}\pi - 3\nu\tau + \Delta\mu + \delta\nu) \\
& + 2\Psi_3(\bar{\alpha}\gamma + 4\beta\gamma + \beta\bar{\gamma} - \alpha\bar{\lambda} + 2\bar{\alpha}\mu + 9\beta\mu - \epsilon\bar{\nu} - 2\bar{\lambda}\pi + \bar{\nu}\rho \\
& \quad - 2\nu\sigma - 2\gamma\tau - \bar{\gamma}\tau - 5\mu\tau + \Delta\beta - \Delta\tau + \delta\gamma + 2\delta\mu) \\
& - \Psi_4(4\bar{\alpha}\beta + 12\beta^2 - 4\epsilon\bar{\lambda} + \kappa\bar{\nu} + \bar{\lambda}\rho - \gamma\sigma - \bar{\gamma}\sigma - 3\mu\sigma - \bar{\alpha}\tau - 7\beta\tau \\
& \quad + \tau^2 - \Delta\sigma + 4\delta\beta - \delta\tau) \\
& + c.c.,
\end{aligned}
\tag{3.55}
$$

where *c.c.* denotes complex conjugation. It is quite natural that some terms have this part and some do not. For example

$$\bar{B}_{(1)(2)} = B_{(1)(2)}, \tag{3.56}$$

since $\boldsymbol{l}$ and $\boldsymbol{n}$ are real null vectors. On the other hand

$$\bar{B}_{(1)(3)} = B_{(1)(4)}, \tag{3.57}$$

since $\boldsymbol{m}$ is a complex vector. Therefore $B_{(1)(4)}$ is not written explicitly since it is contained in the $B_{(1)(3)}$ component. And similarly with other components. Bach tensor is also trace-less, which projected to the null tetrad translates to the condition

$$B_{(1)(2)} = B_{(3)(4)}. \tag{3.58}$$

Therefore $B_{(3)(4)}$ is not expressed explicitly either.

4. Examples within the Robinson–Trautman and Kundt classes

In this last chapter we would like to illustrate the presented general frame approach to the Quadratic gravity applied to the analysis of specific geometrically defined situations. Our main aim here is to derive the corresponding constraints on the spacetime geometry and identify the most elementary pieces entering such conditions.

4.1 Robinson–Trautman and Kundt geometries

The Robinson–Trautman and Kundt families [4, 5] are invariantly defined in terms of the optical scalars describing null affinely parameterised geodesic congruence, see section 2.2.3. They can thus serve as a suitable toy models for a comparison of specific spacetimes restricted by different theories of gravity. In particular, both classes represent manifolds admitting a congruence with vanishing twist and shear. However, the Robinson–Trautman class has a non-trivial expansion while within the Kundt class all optical scalars are zero, i.e.,

- Robinson–Trautman class $\Leftrightarrow$ $\omega = \sigma = 0$ and $\Theta \neq 0$ (see e.g., [21, 22, 4, 5, 23]),
- Kundt class $\Leftrightarrow$ $\omega = \sigma = \Theta = 0$ (see e.g., [24, 25, 4, 5, 26]).

These properties of optical scalars are closely related to the existence of naturally adapted coordinates and line element. In particular, the restrictions implied by the vanishing twist of $\boldsymbol{l}$, $\omega = 0$, can be expressed in the form $l_{[\mu;\nu]} = 0$ which due to the Frobenius theorem (see e.g., [4]) guarantees existence of the null foliation with $\boldsymbol{l}$ being its normal (tangent). Introducing a suitable set of coordinates $(u,\, r,\, x^i)$ we can write the line element in the form

$$ds^2 = -g_{ij}(u,r,x)\, dx^i dx^j + 2g_{ui}(u,r,x)\, dx^i du + 2du dr + g_{uu}(u,r,x)\, du^2\,, \quad (4.1)$$

where the coordinate u labels null hypersurface identified by $u = \text{const}$, the coordinate r represents an affine parameter along the non-twisting null congruence, i.e., for generator of the non-twisting null congruence we can write $\boldsymbol{l} = \partial_r$, and finally, x^i stands for a pair of spatial coordinates covering the transverse two-dimensional Riemannian space with the metric $g_{ij}(u,r,x)$. Moreover, the shear-free condition $\sigma = 0$ restricts the r-dependence of this metric into the form

$$g_{ij} = V^2(u,r,x)\, g_{ij}(u,x)\,, \qquad \text{where} \qquad V = \exp\Big(\int \Theta(u,r,x)\, dr\Big)\,, \quad (4.2)$$

see e.g., [23]. Finally, in the Kundt case with $\Theta = 0$ we effectively have $V = 1$ and the spatial metric is thus r-independent, i.e., $g_{ij} = g_{ij}(u,x)$.

4.2 Spherically symmetric spacetimes

In the class of expanding Robinson–Trautman geometries the spherically symmetric line element can be identified as

$$ds^2 = H(r)du^2 + 2dudr - V(r)^2\left(d\theta^2 + \sin\theta^2\theta d\phi^2\right), \tag{4.3}$$

see e.g., [5]. The matrix form of this line element simply becomes

$$g_{\mu\nu} = \begin{pmatrix} H & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -V^2 & 0 \\ 0 & 0 & 0 & -V^2\sin\theta^2 \end{pmatrix}, \tag{4.4}$$

and the natural null frame takes the form

$$l^\mu = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad n^\nu = \begin{pmatrix} 1 \\ -\frac{H}{2} \\ 0 \\ 0 \end{pmatrix}, \quad m^\nu = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{\sqrt{2}V)} \\ \frac{i\csc\theta}{\sqrt{2}V} \end{pmatrix}, \quad \bar{m}^\nu = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{\sqrt{2}V} \\ \frac{-i\csc\theta}{\sqrt{2}V} \end{pmatrix}. \tag{4.5}$$

In this case, the only nonzero spin coefficients are

$$\rho = -\frac{V'}{V}, \tag{4.6}$$

$$\alpha = -\frac{\cot\theta}{2\sqrt{2}V}, \tag{4.7}$$

$$\beta = -\alpha = \frac{\cot\theta}{2\sqrt{2}V}, \tag{4.8}$$

$$\mu = -\frac{HV'}{2V} = \frac{H}{2}\rho, \tag{4.9}$$

$$\gamma = \frac{H'}{4}, \tag{4.10}$$

and the Ricci identities (2.52)–(2.69) now take the form

$$0 = \Psi_0 = \Psi_1 = \Psi_3 = \Psi_4, \tag{4.11}$$

$$0 = \Phi_{01} = \Phi_{10} = \Phi_{02} = \Phi_{20} = \Phi_{12} = \Phi_{21}, \tag{4.12}$$

$$\mathrm{D}\rho = \rho^2 + \Phi_{00}, \tag{4.13}$$

$$\Delta\mu = -\mu^2 - 2\gamma\mu - \Phi_{22}, \tag{4.14}$$

$$\mathrm{D}\gamma = \Psi_2 - \frac{R}{24} + \Phi_{11}, \tag{4.15}$$

$$\mathrm{D}\mu = \mu\rho + \Psi_2 + \frac{R}{12}, \tag{4.16}$$

$$(\delta + \bar{\delta})\alpha = \mu\rho + 4\alpha^2 - \Psi_2 + \Phi_{11} + \frac{R}{24}. \tag{4.17}$$

Obviously, the geometries (4.3) are of algebraic Weyl type D and the frame (4.5) corresponds to the principal null directions. As the next step we explicitly evaluate expressions (4.11)–(4.17). Before computing the first one let us look at action of derivative D on the scalar ρ,

$$\mathrm{D}\rho = \rho' = -\frac{V''}{V} + \left(\frac{V'}{V}\right)^2 = -\frac{V''}{V} + \rho^2. \tag{4.18}$$

Substituting this result into the equation (4.13) we get

$$\Phi_{00} = -\frac{V''}{V}. \tag{4.19}$$

Evaluating the $\Delta\mu$ term for its use in the second equation yields

$$\begin{aligned}\Delta\mu =& -\frac{H}{2}\mu' = -\frac{H}{2}\left(-2\frac{H'}{4}\frac{V'}{V} - \frac{HV''}{2V} + \frac{H}{2}\left(\frac{V'}{V}\right)^2\right) \\ =& -2\frac{H}{2}\rho\gamma - \left(\frac{H}{2}\right)^2\Phi_{00} - \left(\frac{H}{2}\rho\right)^2 \\ =& -2\mu\gamma - \left(\frac{H}{2}\right)^2\Phi_{00} - \mu^2, \end{aligned} \tag{4.20}$$

where the equation (4.19) was employed in the third step. Subsequently, evaluating the equation (4.14) using this result we obtain a condition

$$\Phi_{22} = \frac{1}{4}H^2\Phi_{00}. \tag{4.21}$$

Furthermore, evaluating the equation (4.15) gives us

$$\gamma' = \frac{H''}{4} = \Psi_2 - \frac{R}{24} + \Phi_{11},$$

and thus

$$H'' = 4\Psi_2 - \frac{R}{6} + 4\Phi_{11}. \tag{4.22}$$

Similarly to the $\Delta\mu$ term, we can calculate the $\mathrm{D}\mu$ term, i.e.,

$$\mathrm{D}\mu = \mu' = 2\gamma\rho + \frac{H}{2}\Phi_{00} + \frac{H}{2}\rho^2. \tag{4.23}$$

Now evaluating the equation (4.16) yields

$$2\gamma\rho + \frac{H}{2}\Phi_{00} + \frac{H}{2}\rho^2 = \frac{H}{2}\rho^2 + \Psi_2 + \frac{R}{12},$$

that is

$$H'V' = -2V\Psi_2 - \frac{VR}{6} + HV\Phi_{00}. \tag{4.24}$$

For computation of the last Ricci equation we need the following derivative

$$(\delta + \bar{\delta})\alpha = \frac{\sqrt{2}}{V}\alpha_{,\theta} = \frac{\sqrt{2}}{V}\frac{1}{2\sqrt{2}V\sin\theta^2} = \frac{1}{2V^2\sin\theta^2}. \tag{4.25}$$

Finally, the equation (4.17) can be expressed as

$$\frac{1}{2V^2\sin\theta^2} = \frac{H}{2}\left(\frac{V'}{V}\right)^2 + \frac{\cos\theta^2}{2V^2\sin\theta^2} - \Psi_2 + \Phi_{11} + \frac{R}{24}, \tag{4.26}$$

which can be further simplified to the form

$$\frac{1}{2V^2} = \frac{1}{2V^2}H(V')^2 - \Psi_2 + \Phi_{11} + \frac{R}{24}. \tag{4.27}$$

To summarise, the compete set the Ricci identities becomes

$$V'' = -V\Phi_{00}, \tag{4.28}$$
$$H^2\Phi_{00} = 4\Phi_{22}, \tag{4.29}$$
$$H(V')^2 = 1 + 2V^2\Psi_2 - 2V^2\Phi_{11} - V^2\frac{R}{12}, \tag{4.30}$$
$$H'V' = -2V\Psi_2 - \frac{VR}{6} + HV\Phi_{00}, \tag{4.31}$$
$$H'' = 4\Psi_2 - \frac{R}{6} + 4\Phi_{11}. \tag{4.32}$$

where the scalar curvature R is given by $R = (-2+2H(V')^2+V^2H''+4V(H'V'+HV''))/V^2$. The Bianchi identities (2.93) and (2.96) form the following set of constraints

$$\frac{1}{24}\mathrm{D}R + \mathrm{D}\Phi_{11} - \mathrm{D}\Psi_2 = -\Phi_{00}\mu + 2\Phi_{11}\rho - 3\Psi_2\rho, \tag{4.33}$$
$$\frac{1}{12}\Delta R + \mathrm{D}\Phi_{22} + \Delta\Psi_2 = -2\Phi_{11}\mu - 3\Psi_2\mu + \Phi_{22}\rho, \tag{4.34}$$
$$\frac{1}{12}\mathrm{D}R + \Delta\Phi_{00} + \mathrm{D}\Psi_2 = 4\Phi_{00}\gamma - \Phi_{00}\mu + 2\Phi_{11}\rho + 3\Psi_2\rho, \tag{4.35}$$
$$\frac{1}{24}\Delta R + \Delta\Phi_{11} - \Delta\Psi_2 = -2\Phi_{11}\mu - 3\Psi_2\mu + \Phi_{22}\rho. \tag{4.36}$$

GR constraints and the Schwarzschild solution

In General Relativity the Ricci tensor and the scalar curvature are zero in the vacuum spacetime with a vanishing cosmological constant. It directly stems from the Einstein field equations. Therefore, we can set to zero the remaining non-trivial projections, i.e.,

$$0 = \Phi_{00} = \Phi_{11} = \Phi_{22} = R. \tag{4.37}$$

The NP constraints now take a very simple form, namely

$$V'' = 0, \tag{4.38}$$
$$H(V')^2 = 1 + 2V^2\Psi_2, \tag{4.39}$$
$$H'V' = -2V\Psi_2, \tag{4.40}$$
$$H'' = 4\Psi_2. \tag{4.41}$$

Obviously, the solution to the equation (4.38) is simply $V = ar + b$ where a and b are integration constants. Since we want to obtain a classic form of the Schwarzschild metric at the end it is clear that the constants need to be $a = 1$ and $b = 0$ which leaves us with

$$V = r. \tag{4.42}$$

This constant fixing is allowed due to geometrical meaning of the r coordinate being an affine parameter with freedom in its rescaling. Substituting for the V function into the remaining equations yields

$$\begin{aligned} H =& 1 + 2r^2\Psi_2, && (4.43)\\ H' =& -2r\Psi_2, && (4.44)\\ H'' =& 4\Psi_2. && (4.45)\end{aligned}$$

Last two expressions (4.44) and (4.45) combined together give us equation for the metric function H, namely $H' = -\frac{r}{2}H''$, which can be integrated as

$$H' = \frac{c}{r^2}. \tag{4.46}$$

Inserting this result into the first expression (4.43) provides the explicit form of the metric function $H(r)$, namely

$$H = 1 - rH' = 1 - r\frac{c}{r^2}. \tag{4.47}$$

Finally, denoting the integration constant as $c \equiv 2M$ we arrive at the Schwarzschild metric in the classical form,

$$ds^2 = \left(1 - \frac{2M}{r}\right) du^2 + 2dudr - r^2\left(d\theta^2 + \sin\theta^2\theta d\phi^2\right). \tag{4.48}$$

Constraints of the Quadratic Gravity

In the case of Quadratic gravity with an artificial assumption of the vanishing scalar curvature, corresponding to the previously studied cases [13, 27, 28, 29, 30], the projected field equations (3.40) with $R = 0$ imply the following constraints

$$\begin{aligned} 0 =& \Phi_{00} - c\Phi_{00}\Psi_2 - 9c\Psi_2\rho^2 + 6c\rho \mathrm{D}\Psi_2 + 3c\Psi_2\mathrm{D}\rho - c\mathrm{DD}\Psi_2, && (4.49)\\ 0 =& \Phi_{11} + 2c\Phi_{11}\Psi_2 - 3c\Psi_2\mu\rho + 2c\mu\mathrm{D}\Psi_2 + 3c\Psi_2\mathrm{D}\mu - 2c\rho\Delta\Psi_2 + \mathrm{D}\Delta\Psi_2, && (4.50)\\ 0 =& \Phi_{22} - c\Phi_{22}\Psi_2 - 6c\Psi_2\gamma\mu - 9c\Psi_2\mu^2 - 2c\gamma\Delta\Psi_2 - 6c\mu\Delta\Psi_2 \\ &- 3c\Psi_2\Delta\mu - c\Delta\Delta\Psi_2, && (4.51)\end{aligned}$$

where $c = 4\mathfrak{a}\mathsf{k}$. The Ricci tensor components can be easily expressed, i.e.,

$$\begin{aligned} \Phi_{00} =& \frac{c}{1 - c\Psi_2}(9\Psi_2\rho^2 - 6\rho\mathrm{D}\Psi_2 - 3\Psi_2\mathrm{D}\rho + \mathrm{DD}\Psi_2), && (4.52)\\ \Phi_{11} =& \frac{c}{1 + 2c\Psi_2}(3\Psi_2\mu\rho - 2\mu\mathrm{D}\Psi_2 - 3\Psi_2\mathrm{D}\mu + 2\rho\Delta\Psi_2 - \mathrm{D}\Delta\Psi_2), && (4.53)\\ \Phi_{22} =& \frac{c}{1 - c\Psi_2}(6\Psi_2\gamma\mu + 9\Psi_2\mu^2 + 2\gamma\Delta\Psi_2 + 6\mu\Delta\Psi_2 + 3\Psi_2\Delta\mu + \Delta\Delta\Psi_2). && (4.54)\end{aligned}$$

Subsequently, combined with the above geometric identities, this should lead to the constrains explicitly derived in [13] which are quite complicated, and we thus focus on a more elegant approach in the next section.

4.3 Spherically symmetric spacetimes in the conformal-to-Kundt form

This section represents a preliminary sketch of the discussion related to the spherically symmetric solution, however, in a more unusual form. Employing a suitable coordinate transformation, see [31], the Robinson–Trautman metric (4.3) can be rewritten in the conformal-to-Kundt form

$$ds^2 = \Omega(r)^2 \left(H(r)du^2 + 2dudr - d\theta^2 - \sin\theta^2\theta d\phi^2\right), \tag{4.55}$$

where the expanding character of the spacetime is encoded in the r-dependence of the conformal factor Ω. To be precise the r coordinate and metric function $H(r)$ are different from those in metric (4.3) Its matrix form becomes

$$g_{\mu\nu} = \begin{pmatrix} H\Omega^2 & \Omega^2 & 0 & 0 \\ \Omega^2 & 0 & 0 & 0 \\ 0 & 0 & -\Omega^2 & 0 \\ 0 & 0 & 0 & -\Omega^2\sin\theta^2 \end{pmatrix}. \tag{4.56}$$

The null tetrad can be obtained as a trivial modification of the one used in the previous case, namely

$$l^\mu = \begin{pmatrix} 0 \\ \frac{1}{\Omega} \\ 0 \\ 0 \end{pmatrix}, \quad n^\nu = \begin{pmatrix} \frac{1}{\Omega} \\ -\frac{H}{2\Omega} \\ 0 \\ 0 \end{pmatrix}, \quad m^\nu = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{\sqrt{2}\Omega)} \\ \frac{i\csc\theta}{\sqrt{2}\Omega} \end{pmatrix}, \quad \bar{m}^\nu = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{\sqrt{2}\Omega} \\ \frac{-i\csc\theta}{\sqrt{2}\Omega} \end{pmatrix}. \tag{4.57}$$

The only nonzero spin coefficients are

$$\epsilon = \frac{\Omega'}{2\Omega^2}, \tag{4.58}$$

$$\rho = -\frac{\Omega'}{\Omega^2}, \tag{4.59}$$

$$\alpha = -\frac{\cot\theta}{2\sqrt{2}\Omega}, \tag{4.60}$$

$$\beta = -\alpha = \frac{\cot\theta}{2\sqrt{2}\Omega}, \tag{4.61}$$

$$\mu = -\frac{H\Omega'}{2\Omega^2} = \frac{H}{2}\rho, \tag{4.62}$$

$$\gamma = \frac{\Omega H' + \Omega' H}{4\Omega^2}, \tag{4.63}$$

and the NP form of the Ricci identities (2.52)–(2.69) now becomes

$$0 = \Psi_0 = \Psi_1 = \Psi_3 = \Psi_4, \tag{4.64}$$
$$0 = \Phi_{01} = \Phi_{10} = \Phi_{02} = \Phi_{20} = \Phi_{12} = \Phi_{21}, \tag{4.65}$$
$$\mathrm{D}\rho = \rho(2\epsilon + \rho) + \Phi_{00}, \tag{4.66}$$
$$\Delta\mu = -\mu^2 - 2\gamma\mu - \Phi_{22}, \tag{4.67}$$
$$\mathrm{D}\gamma - \Delta\epsilon = -4\gamma\epsilon + \Psi_2 - \frac{R}{24} + \Phi_{11}, \tag{4.68}$$
$$\mathrm{D}\mu = \mu\rho - 2\epsilon\mu + \Psi_2 + \frac{R}{12}, \tag{4.69}$$
$$(\delta + \bar{\delta})\alpha = \mu\rho + 4\alpha^2 - \Psi_2 + \Phi_{11} + \frac{R}{24}. \tag{4.70}$$

As in the previous case, before evaluating the first equation let us look at the derivative D acting on ρ,

$$\mathrm{D}\rho = \frac{1}{\Omega}\rho' = -\frac{\Omega''}{\Omega^3} + 2\frac{(\Omega')^2}{\Omega^4}. \tag{4.71}$$

Using this result in the equation (4.66) gives us

$$\Omega\Omega'' - 2(\Omega')^2 = -\Omega^4\Phi_{00}. \tag{4.72}$$

Evaluating the $\Delta\mu$ term for its use in the second equation yields

$$\begin{aligned} \Delta\mu &= -\frac{H}{2\Omega}\mu' = -\frac{H}{2\Omega}\left(\frac{H'}{2}\rho + \frac{H}{2}\Phi_{00}\Omega\right) \\ &= -\frac{HH'}{4\Omega}\rho - \left(\frac{H}{2}\right)^2\Phi_{00}, \end{aligned} \tag{4.73}$$

where the equation (4.72) was used in the third step. Subsequently, using this result calculation of the equation (4.67) leads to

$$H^2\Phi_{00} = 4\Phi_{22}. \tag{4.74}$$

Furthermore, evaluating the equation (4.68) gives us

$$(\Omega H)'' = \Omega^3 H\Phi_{00} + 4\Psi_2\Omega^3 - \frac{R\Omega^3}{6} + 4\Phi_{11}\Omega^3. \tag{4.75}$$

In analogy with the $\Delta\mu$ term we can calculate the $\mathrm{D}\mu$ derivative, i.e.

$$\begin{aligned} \mathrm{D}\mu &= \frac{1}{\Omega}\mu' = \frac{1}{\Omega}\left(\frac{H'}{2}\rho - \frac{H}{2}\frac{\Omega''}{\Omega^2} + H\frac{(\Omega')^2}{\Omega^3}\right) \\ &= -\frac{H'\Omega'}{2\Omega^3} - \frac{H\Omega''}{2\Omega^3} + \frac{H(\Omega')^2}{\Omega^4}. \end{aligned} \tag{4.76}$$

With the above expression in hand we evaluate the equation (4.69),

$$(H\Omega')' = -2\Omega^3\Psi_2 - \Omega^3\frac{R}{6}. \tag{4.77}$$

For the last Ricci identity, we need to evaluate the following relation

$$(\delta + \bar{\delta})\alpha = \frac{\sqrt{2}}{\Omega}\alpha_{,\theta} = \frac{\sqrt{2}}{\Omega}\frac{1}{2\sqrt{2}\Omega \sin\theta^2} = \frac{1}{2\Omega^2 \sin\theta^2}, \tag{4.78}$$

which combined with the equation (4.70) gives

$$H(\Omega')^2 = \Omega^2 + 2\Omega^4\Psi_2 - 2\Omega^4\Phi_{11} - \Omega^4\frac{R}{12}. \tag{4.79}$$

The Ricci scalar is $R = (\Omega(H'' - 2) + 6(H'\Omega' + H\Omega''))/\Omega^3$. We thus have the complete set of nontrivial Ricci identities. What remains is to express the Bianchi identities (2.93)–(2.96). This gives the following constraints,

$$\frac{1}{24}\mathrm{D}R + \mathrm{D}\Phi_{11} - \mathrm{D}\Psi_2 = -\Phi_{00}\mu + 2\Phi_{11}\rho - 3\Psi_2\rho, \tag{4.80}$$

$$\frac{1}{12}\Delta R + \mathrm{D}\Phi_{22} + \Delta\Psi_2 = -2\Phi_{11}\mu - 3\Psi_2\mu + \Phi_{22}\rho - 4\Phi_{22}\epsilon, \tag{4.81}$$

$$\frac{1}{12}\mathrm{D}R + \Delta\Phi_{00} + \mathrm{D}\Psi_2 = 4\Phi_{00}\gamma - \Phi_{00}\mu + 2\Phi_{11}\rho + 3\Psi_2\rho, \tag{4.82}$$

$$\frac{1}{24}\Delta R + \Delta\Phi_{11} - \Delta\Psi_2 = -2\Phi_{11}\mu + 3\Psi_2\mu + \Phi_{22}\rho. \tag{4.83}$$

These geometric conditions have to be further combined with those implied by the particular theory field equations.

Constraints of the Quadratic Gravity

Finally, for the subclass of geometries with vanishing Ricci scalar discussed using the conformal-to Kundt metric form in [27, 28, 29, 30] the Quadratic gravity field equation gives

$$0 = -\Phi_{00} + c\Phi_{00}\Psi_2 + c(6\Psi_2\epsilon\rho + 9\Psi_2\rho^2 - 2\epsilon\mathrm{D}\Psi_2 - 6\rho\mathrm{D}\Psi_2 - 3\Psi_2\mathrm{D}\rho + \mathrm{D}\mathrm{D}\Psi_2), \tag{4.84}$$

$$0 = -\Phi_{11} - 2c\Phi_{11}\Psi_2 + c(-6\Psi_2\epsilon\mu + 3\Psi_2\mu\rho - 2\mu\mathrm{D}\Psi_2 - 3\Psi_2\mathrm{D}\mu - \mathrm{D}\Delta\Psi_2 - 2\epsilon\Delta\Psi_2 + 2\rho\Delta\Psi_2), \tag{4.85}$$

$$0 = -\Phi_{22} + c\Phi_{22}\Psi_2 + c(6\Psi_2\gamma\mu + 9\Psi_2\mu^2 + 2\gamma\Delta\Psi_2 + 6\mu\Delta\Psi_2 + 3\Psi_2\Delta\mu + \Delta\Delta\Psi_2), \tag{4.86}$$

where $c = 4\mathfrak{a}\mathsf{k}$. This set of equations can be solved with respect to the Ricci tensor components which can be subsequently combined with the geometric identities. In particular

$$\Phi_{00} = \frac{c}{1 - c\Psi_2}(6\Psi_2\rho^2 - 5\rho\mathrm{D}\Psi_2 - 3\Psi_2\mathrm{D}\rho + \mathrm{D}\mathrm{D}\Psi_2), \tag{4.87}$$

$$\Phi_{11} = \frac{c}{1 + 2c\Psi_2}(6\Psi_2\mu\rho - 2\mu\mathrm{D}\Psi_2 - 3\Psi_2\mathrm{D}\mu - \mathrm{D}\Delta\Psi_2 + 3\rho\Delta\Psi_2), \tag{4.88}$$

$$\Phi_{22} = \frac{c}{1 - c\Psi_2}(6\Psi_2\gamma\mu + 9\Psi_2\mu^2 + 2\gamma\Delta\Psi_2 + 6\mu\Delta\Psi_2 + 3\Psi_2\Delta\mu + \Delta\Delta\Psi_2), \tag{4.89}$$

where we used the simple observation that $-2\epsilon = \rho$ to simplify the expressions. Inserting (4.87) into the only Ricci identity containing Φ_{00}, namely (4.66), yields

$$\begin{aligned} \mathrm{D}\rho &= \frac{c}{1 - c\Psi_2}(6\Psi_2\rho^2 - 5\rho\mathrm{D}\Psi_2 - 3\Psi_2\mathrm{D}\rho + \mathrm{D}\mathrm{D}\Psi_2), \\ \mathrm{D}\rho &= c(6\Psi_2\rho^2 - 5\rho\mathrm{D}\Psi_2 - 2\Psi_2\mathrm{D}\rho + \mathrm{D}\mathrm{D}\Psi_2), \\ \Omega\Omega'' - 2(\Omega')^2 &= -\frac{1}{3}\mathsf{ak}H''''. \end{aligned} \tag{4.90}$$

Combining Ricci identities (4.69) and (4.70) with $R = 0$ provides an equation for Φ_{11} without Ψ_2

$$\mathrm{D}\mu + (\delta + \bar{\delta})\alpha - 4\alpha^2 - 3\mu\rho = \Phi_{11}, \tag{4.91}$$

and substituting Φ_{11} for (4.88) gives

$$\begin{aligned} \mathrm{D}\mu + (\delta + \bar{\delta})\alpha - 4\alpha^2 - 3\mu\rho &= \frac{c}{1 + 2c\Psi_2}(6\Psi_2\mu\rho - 2\mu\mathrm{D}\Psi_2 - 3\Psi_2\mathrm{D}\mu \\ &\quad - \mathrm{D}\Delta\Psi_2 + 3\rho\Delta\Psi_2) \\ 3H(\Omega')^2 + \Omega\Omega'H' - \Omega^2 &= -\frac{1}{3}\mathsf{ak}(H'''H - \frac{1}{2}(H'')^2 + 2). \end{aligned} \tag{4.92}$$

Equations (4.90) and (4.92) for $H \to -H$ (because of slightly different metric form) restore the explicit constraints of [27] solved in the form of power series.

4.4 *pp*-wave geometries

The well-known *pp*-wave spacetimes are defined as manifolds admitting covariantly constant null vector field. Due to the definition of optical scalars they thus belong to the class of Kundt geometries. The simplest member of this family can be written as

$$ds^2 = H(u, x, y)du^2 + 2dudr - dx^2 - dy^2. \tag{4.93}$$

where we deal with the flat transverse space and the non-trivial curvature localised on the support in the u coordinate of the unknown metric function $H(u, x, y)$. It is useful to write down the matrix form of this metric,

$$g_{\mu\nu} = \begin{pmatrix} H & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{4.94}$$

The choice of the suitable null tetrad is almost obvious and such a frame takes the form

$$l^\mu = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad n^\nu = \begin{pmatrix} 1 \\ -\frac{H}{2} \\ 0 \\ 0 \end{pmatrix}, \quad m^\nu = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix}, \quad \bar{m}^\nu = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} \end{pmatrix}. \tag{4.95}$$

where $\boldsymbol{l}$ is degenerated principal null direction and the geometry is of Weyl type N. The simplicity of this geometry is illustrated by only one nonzero spin coefficient,

$$\nu = \frac{1}{2\sqrt{2}}(H_{,x} - iH_{,y}). \tag{4.96}$$

Therefore, the NP constraints (2.52)–(2.69), i.e., the Ricci identities, take a very simple form

$$
\begin{aligned}
0 =& \Psi_0 = \Psi_1 = \Psi_2 = \Psi_3, && (4.97)\\
0 =& \Phi_{00} = \Phi_{10} = \Phi_{01} = \Phi_{11} = \Phi_{20} = \Phi_{02} = \Phi_{12} = \Phi_{21} = R, && (4.98)\\
\delta\nu =& \Phi_{22}, && (4.99)\\
\bar{\delta}\nu =& \Psi_4, && (4.100)
\end{aligned}
$$

which explicitly proves that the ansatz metric (4.93) is of the Weyl and Ricci type N, respectively. The first equation that we shall focus on is (4.99). After simple manipulations we arrive at the explicit expression for the Ricci tensor component,

$$\delta\nu = \frac{1}{\sqrt{2}}(\nu_{,x} + i\nu_{,y}) = \Phi_{22},$$

that is

$$\Phi_{22} = \frac{1}{4}(H_{,xx} + H_{,yy}). \tag{4.101}$$

Shifting our attention to the second non-trivial equation (4.100) we obtain the Weyl tensor component Ψ_4 as

$$\bar{\delta}\nu = \frac{1}{\sqrt{2}}(\nu_{,x} - i\nu_{,y}) = \Psi_4,$$

which becomes

$$\Psi_4 = \frac{1}{4}(H_{,xx} - 2iH_{,xy} - H_{,yy}). \tag{4.102}$$

Subsequently, the Bianchi identities (2.93) and (2.96) are reduced to the following equation

$$\delta\Psi_4 = \bar{\delta}\Phi_{22}, \tag{4.103}$$

which on the coordinate level, using (4.101) and (4.102), yields

$$\delta(H_{,xx} - 2iH_{,xy} - H_{,yy}) = \bar{\delta}(H_{,xx} + H_{,yy}),$$

and it can be easily verified that this condition is identically satisfied,

$$
\begin{aligned}
&H_{,xxx} - 2iH_{,xxy} - H_{,xyy} + iH_{,xxy} + 2H_{,xyy}\\
&\quad - iH_{,yyy} - H_{,xxx} - H_{,xyy} + iH_{,xxy} + iH_{,yyy} = 0.
\end{aligned} \tag{4.104}
$$

pp-waves in General Relativity

In vacuum General Relativity the only non-trivial constrain implied by the Einstein equations is

$$\Phi_{22} = 0, \tag{4.105}$$

which on the coordinates level, and employing (4.101), takes the form of flat Laplace equation for the metric function H, namely

$$H_{,xx} + H_{,yy} = 0. \tag{4.106}$$

Its explicit solutions for various settings can be found e.g., in [5]. Here, as an explicit example, we mention the axially symmetric case. In such a situation (4.106) becomes

$$H_{,\zeta\zeta} + \frac{1}{\zeta} H_{,\zeta} = 0, \tag{4.107}$$

with ζ being a radial distance, i.e., $\zeta^2 = x^2 + y^2$. The solution can be then written as

$$H(u, x, y) = -2p(u) \ln \zeta, \tag{4.108}$$

where $p(u)$ is a general profile function.

pp-waves in Quadratic Gravity

In the Quadratic gravity we get a more involved condition on the Ricci tensor component implied by the field equations. In particular, for the metric (4.93) we obtain that the cosmological constant has to be vanishing, $\Lambda = 0$, and the Ricci tensor has to satisfy

$$\Phi_{22} = 2\mathsf{ak}(\delta\delta\Psi_4 + \bar{\delta}\bar{\delta}\bar{\Psi}_4). \tag{4.109}$$

To analyse this equation let us calculate the $\delta\delta\Psi_4$ term, i.e.,

$$\begin{aligned}
\delta\delta\Psi_4 =& \frac{1}{4}\delta\delta(H_{,xx} - 2iH_{,xy} - H_{,yy}),\\
=& \frac{1}{4\sqrt{2}}\delta(H_{,xxx} - 2iH_{,xxy} - H_{,xyy} + iH_{,xxy} + 2H_{,xyy} - iH_{,yyy})\\
=& \frac{1}{8}(H_{,xxxx} - 2iH_{,xxxy} - H_{,xxyy} + iH_{,xxxy} + 2H_{xxyy} - iH_{,xyyy}\\
&+ iH_{,xxxy} + 2H_{,xxyy} - iH_{,xyyy} - H_{,xxyy} + 2iH_{xyyy} + H_{,yyyy}),
\end{aligned} \tag{4.110}$$

and similarly, the $\bar{\delta}\bar{\delta}\bar{\Psi}_4$ term gives us

$$\begin{aligned}
\bar{\delta}\bar{\delta}\bar{\Psi}_4 =& \frac{1}{4}\bar{\delta}\bar{\delta}(H_{,xx} + 2iH_{,xy} - H_{,yy})\\
=& \frac{1}{4\sqrt{2}}\bar{\delta}(H_{,xxx} + 2iH_{,xxy} - H_{,xyy} - iH_{,xxy} + 2H_{,xyy} + iH_{,yyy}),\\
=& \frac{1}{8}(H_{,xxxx} + 2iH_{,xxxy} - H_{,xxyy} - iH_{,xxxy} + 2H_{xxyy} + iH_{,xyyy}\\
&- iH_{,xxxy} + 2H_{,xxyy} + iH_{,xyyy} - H_{,xxyy} - 2iH_{xyyy} + H_{,yyyy}).
\end{aligned} \tag{4.111}$$

Combining the constants into one $c \equiv 2\mathsf{ak}$ and using the above relations we obtain

$$\Phi_{22} = c(\delta\delta\Psi_4 + \bar{\delta}\bar{\delta}\bar{\Psi}_4) = \frac{c}{4}(H_{,xxxx} + 2H_{xxyy} + H_{,yyyy}). \tag{4.112}$$

Substituting this result into the equation (4.101) yields

$$H_{,xx} + H_{,yy} + c(H_{,xxxx} + 2H_{xxyy} + H_{,yyyy}) = 0, \tag{4.113}$$

which can be rewritten as

$$\Big(H + c(H_{,xx} + H_{,yy})\Big)_{,xx} + \Big(H + c(H_{,xx} + H_{,yy})\Big)_{,yy}. \tag{4.114}$$

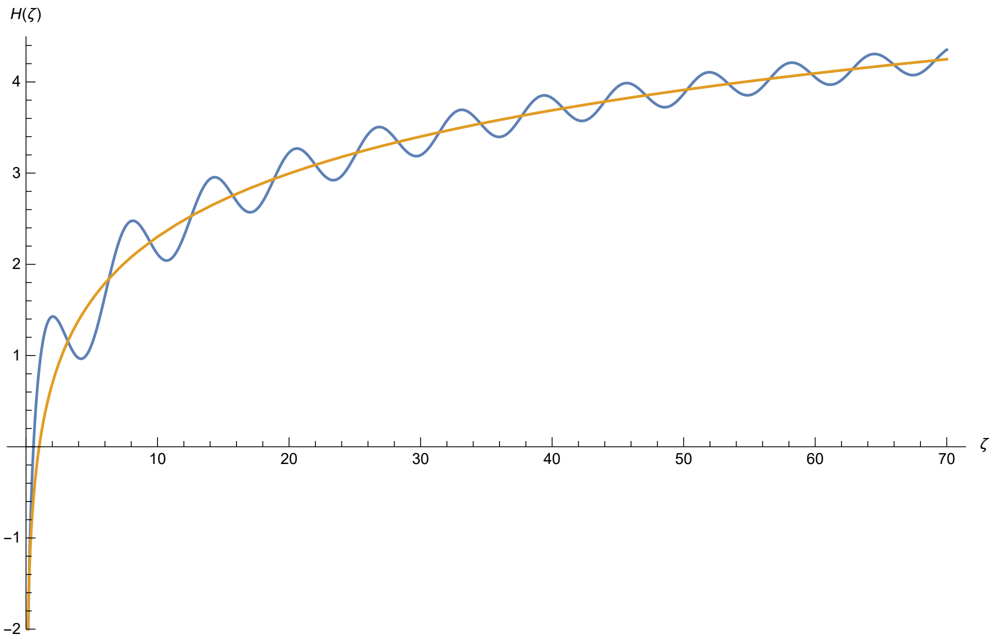

Figure 4.1: Plot of the metric function H (blue) for the case of *pp*-waves in Quadratic Gravity with the following values of parameters $c = 1$, $d = 1$, $e = 1$, $p = -1/2$. The GR solution $\ln(\zeta)$ is also indicated (orange).

Furthermore, this can be re-expressed in a different form, namely

$$G_{,xx} + G_{,yy} = 0, \tag{4.115}$$

where we have defined the function G as

$$G \equiv H + c(H_{,xx} + H_{,yy}). \tag{4.116}$$

In this form we can immediately see, that all solutions to General Relativity are automatically solutions to the Quadratic gravity as well. This is well-known fact which in our case can be observed from

$$H_{,xx} + H_{,yy} = 0 \qquad \Longrightarrow \qquad G = H. \tag{4.117}$$

The Quadratic gravity solution H can be found as a solution to the Helmholtz-like equation $G = f$ where the function f must be solution to the Laplace's equation, see e.g., [32]. In the axially symmetric case we thus get

$$H(u, x, y) = -2p(u) \left[dJ_0\left(\frac{\zeta}{\sqrt{c}}\right) + eY_0\left(\frac{\zeta}{\sqrt{c}}\right) + \ln\zeta \right], \tag{4.118}$$

where d, e are general constants and J_α, Y_α are Bessel functions of the first and second kind respectively. It is natural, but not necessary, to choose common profile $p(u)$ for all parts of H. The plot of metric function H for a specific value of parameters $c = 1$, $d = 1$, $e = 1$, $p = -1/2$ is in the figure 4.1.

Conclusion

Within the thesis we have been mainly interested in the frame approach to the analysis of the field equations in the context of theories of gravity. In particular, we have tried to extend the famous Newman–Penrose formalism for the case of so-called Quadratic theory of gravity.

The thesis consists of introduction, four chapters, conclusion, one appendix, and list of references. In the introduction we have very briefly summarised history of the gravity descriptions ending with the Albert Einstein General Relativity and a list of some successful experimental tests of this theory formed during last century.

In the first chapter we have described an elegant mathematical formulation of the General Relativity in terms of the least action principle which very naturally allows for various theory modifications. The basic ways in which the Einstein theory can be modified are outlined as well. We have introduced a concept of the Quadratic gravity extending the classic Einstein–Hilbert action by adding quadratic curvature terms and representing thus the higher order correction of any final purely geometrical theory. Moreover, we have rewritten the Quadratic gravity field equations in such a way that the contribution of the Ricci tensor coordinate components is separated. This is crucial for the upcoming discussion.

In the chapter two we have reviewed the most important parts of a general frame formalism and used these results to present detailed summary of the Newman–Penrose formalism where the null frame comes into the game. All these concepts are based on purely geometrical identities without any assumptions given by particular field equations. In the last part of this chapter we have also mentioned ways how to interpret specific spin coefficients and tensor components.

The third chapter employs the field equations. While in the case of General Relativity it is quite trivial, in the Quadratic gravity it becomes much more involved. Here we use the fact that even in the Quadratic gravity the field equations dependence on the Ricci tensor is only linear and there are no Ricci tensor derivatives. This enables to follow the General Relativity procedure and eliminate the Ricci tensor contribution from the geometrical identities of the second chapter.

As an illustration, in the last chapter we formulate the constraints on the gravitational field for important situations corresponding to the spherically symmetric spacetimes and so-called *pp*-waves both in the General Relativity as well as Quadratic gravity.

Finally, in the appendix we briefly review the spinor formalism to be able to compare some of our calculations with already published results.

Within the topic of the thesis there still remain many open questions which should be addressed in the further work. Definitely the discussion of particular examples should be more detailed and we hope to obtain new non-trivial results using the frame formulation of constraints. In particular, situations with the algebraically special Weyl and Ricci tensors and geometrically privileged null geodesic congruences are very promising to be studied.

A. Spinor formalism and the Bach tensor components

In this appendix we want to compare our results obtained in the third chapter of this thesis with those published in [33] using so called Geroch–Held–Penrose (GHP) formalism. For this purpose, let us first briefly introduce the spinor calculus. Then we will list all the independent components of the Bach tensor in the GHP formalism presented in [33], and employed in the context of the conformal Weyl gravity, together with relation to our expressions.

Spinor formalism

To represent tensors in the spinor form, we need to make two modifications. Firstly, the abstract indices will now appear in pairs of a primed and unprimed index. The second modification is that we need an operation of complex conjugation acting on a general spinor defined as

$$\kappa^{\overline{AB'}} = \overline{\kappa}^{A'B}. \tag{A.1}$$

The metric tensor in the spinor calculus can be expressed as

$$g_{ab} = \epsilon_{AB}\epsilon_{A'B'}, \tag{A.2}$$

with the following condition

$$\epsilon_{AB} = -\epsilon_{BA}, \quad \epsilon_{A'B'} = -\epsilon_{B'A'}, \tag{A.3}$$

where ϵ_{AB} is the complex conjugate of $\epsilon_{A'B'}$. In the spinor calculus, the quantity (A.3) will play a similar role as the metric tensor. It is defined by the following relations

$$\epsilon_{AB}\kappa^A = \kappa_B, \tag{A.4}$$
$$\epsilon^{AB}\kappa_B = \kappa^A, \tag{A.5}$$
$$\epsilon^{AB}\epsilon_{CB} = \epsilon_C{}^A, \tag{A.6}$$
$$\epsilon^{AB} = -\epsilon^{BA}. \tag{A.7}$$

Now in analogy with the tetrad defined in the second chapter we shall define normalised dyad in the spinor calculus, namely

$$\epsilon^A_{\boldsymbol{A}} = (o^A, \iota^A), \tag{A.8}$$

satisfying the following relations

$$\epsilon_{AB}o^A o^B = o_A o^A = 0, \tag{A.9}$$
$$\epsilon_{AB}\iota^A \iota^B = \iota_A \iota^A = 0, \tag{A.10}$$
$$\epsilon_{AB}o^A \iota^B = o_A \iota^A = 1. \tag{A.11}$$

So now we can write the null tetrad in the spinor form using the normalised dyad

$$\boldsymbol{l} = o^A o^{A'}, \quad \boldsymbol{n} = \iota^A \iota^{A'}, \quad \boldsymbol{m} = o^A \iota^{A'}, \quad \bar{\boldsymbol{m}} = \iota^A o^{A'}, \tag{A.12}$$

and using (A.12) we can rewrite derivatives (2.35) in the spinor formalism as

$$D = o^A o^{A'} \nabla_{AA'}, \quad D' = \iota^A \iota^{A'} \nabla_{AA'}, \quad \delta = o^A \iota^{A'} \nabla_{AA'}, \quad \delta' = \iota^A o^{A'} \nabla_{AA'}. \tag{A.13}$$

Moreover, in analogy with the spin coefficients (2.36) we define the corresponding quantities also in the spinor formalism as

$$\begin{array}{lll} \kappa = o^A D o_A, & \gamma' = -\iota^A D o_A, & \tau' = -\iota^A D \iota_A, \\ \rho = o^A \delta' o_A, & \beta' = -\iota^A \delta' o_A, & \sigma' = -\iota^A \delta' \iota_A, \\ \sigma = o^A \delta o_A, & \beta = \iota^A \delta o_A, & \rho' = -\iota^A \delta \iota_A, \\ \tau = o^A D' o_A, & \gamma = \iota^A D' o_A, & \kappa' = -\iota^A D' \iota_A. \end{array} \tag{A.14}$$

Here, we used the standard notation for the spinor formalism. Unprimed coefficients correspond to their counterparts in the NP formalism and primed coefficients can be written in the NP formalism notation by the following substitutions

$$\sigma' = -\lambda, \quad \kappa' = -\nu, \quad \rho' = -\mu, \quad \tau' = -\pi, \quad \gamma' = -\epsilon, \quad \beta' = -\alpha. \tag{A.15}$$

These can be easily derived from (A.14) by using the definitions (A.12) and (A.13).

In the GHP formalism the most general change of spin-frame which leaves two null directions invariant looks like

$$o^A \to \lambda o^A, \qquad \iota^A \to \lambda^{-1} \iota^A, \tag{A.16}$$

where λ is a general complex scalar field. For prime indices we would write complex conjugate $\bar{\lambda}$ in the relations above instead.

For a general scalar this transformation takes the form

$$\eta \to \lambda^p \bar{\lambda}^q \eta, \tag{A.17}$$

introducing the weights p, q and η is thus called weighted scalar of type (p, q).

Let us introduce the following operators motivated by the GHP formalism that allow for a more compact notation within this formalism,

$$þ = D + p\gamma' + q\bar{\gamma}' \qquad (1, 1), \tag{A.18}$$

$$þ' = D' - p\gamma - q\bar{\gamma} \qquad (-1, -1), \tag{A.19}$$

$$ð = \delta - p\beta + q\bar{\beta}' \qquad (1, -1), \tag{A.20}$$

$$ð' = \delta' + p\beta' - q\bar{\beta} \qquad (-1, 1), \tag{A.21}$$

where the brackets denote the weight of these operators.

Finally, we will also need to know the weights of the following scalars

$$\begin{array}{ll} \Psi_r & (4 - 2r, 0), \\ \Phi_{rt} & (2 - 2r, 2 - 2t), \\ \Lambda & (1, 1), \\ \kappa & (3, 1), \\ \rho & (1, 1), \\ \sigma & (3, -1), \\ \tau & (1, -1), \\ \beta & (1, -1), \\ \gamma & (-1, -1). \end{array} \tag{A.22}$$

As an example, let us rewrite the following term from the GHP formalism to the NP formalism,

$$(ð' - 2\tau')\Psi_1 = \delta'\Psi_1 + 2\beta'\Psi_1 - 2\tau'\Psi_1. \tag{A.23}$$

According to (A.22) the Ψ_1 has the weight $(2,0)$. Substitution of this pair of integers for p and q into the definition of $ð'$ operator (A.21) provides the result above. All three terms that we obtained have the weight of $(1,1)$. Using this knowledge, the action of another operators on these terms is straightforward.

Bach tensor in the GHP formalism

In the paper [33] studying the frame approach to the conformal Weyl gravity the Bach tensor components, as its key ingredient, were presented using the compact GHP notation. They were presented in the form

$$\begin{aligned}\frac{1}{2}B_{00} =& (þ - 3\rho)\left[(ð' - 2\tau')\Psi_1 - (þ - 3\rho)\Psi_2 + \sigma'\Psi_0 - 2\kappa\Psi_3\right]\\ &+ (ð' - \tau')\left[(þ - 4\rho)\Psi_1 - (ð' - \tau')\Psi_0 + 3\kappa\Psi_2\right]\\ &+ 2\kappa\left[(ð' - 3\tau')\Psi_2 - (þ - 2\rho)\Psi_3 + 2\sigma'\Psi_1 - \kappa\Psi_4\right]\\ &+ \bar{\kappa}\left[(þ' - 2\rho')\Psi_1 - (ð - 3\tau)\Psi_2 + \kappa'\Psi_0 - 2\sigma\Psi_3\right]\\ &+ \bar{\sigma}\left[(ð - 4\tau)\Psi_1 - (þ' - \rho')\Psi_0 + 3\sigma\Psi_2\right] - \Phi_{20}\Psi_0 + 2\Phi_{10}\Psi_1 - \Phi_{00}\Psi_2,\end{aligned} \tag{A.24}$$

$$\begin{aligned}\frac{1}{2}B_{02} =& (ð - 3\tau)\left[(þ' - 2\rho')\Psi_1 - (ð - 3\tau)\Psi_2 + \kappa'\Psi_0 - 2\sigma\Psi_3\right]\\ &+ (þ' - \rho')\left[(ð - 4\tau)\Psi_1 - (þ' - \rho')\Psi_0 + 3\sigma\Psi_2\right]\\ &+ 2\sigma\left[(þ' - 3\rho')\Psi_2 - (ð - 2\tau)\Psi_3 + 2\kappa'\Psi_1 - \sigma\Psi_4\right]\\ &+ \bar{\sigma}'\left[(ð' - 2\tau')\Psi_1 - (þ - 3\rho)\Psi_2 + \sigma'\Psi_0 - 2\kappa\Psi_3\right]\\ &+ \bar{\kappa}'\left[(þ - 4\rho)\Psi_1 - (ð' - \tau')\Psi_0 + 3\kappa\Psi_2\right] - \Phi_{22}\Psi_0 + 2\Phi_{12}\Psi_1 - \Phi_{02}\Psi_2,\end{aligned} \tag{A.25}$$

$$\begin{aligned}\frac{1}{2}B_{20} =& (ð' - 3\tau')\left[(þ - 2\rho)\Psi_3 - (ð' - 3\tau')\Psi_2 + \kappa\Psi_4 - 2\sigma'\Psi_1\right]\\ &+ (þ - \rho)\left[(ð' - 4\tau')\Psi_3 - (þ - \rho)\Psi_4 + 3\sigma'\Psi_2\right]\\ &+ 2\sigma'\left[(þ - 3\rho)\Psi_2 - (ð' - 2\tau')\Psi_1 + 2\kappa\Psi_3 - \sigma'\Psi_0\right]\\ &+ \bar{\sigma}\left[(ð - 2\tau)\Psi_3 - (þ' - 3\rho')\Psi_2 + \sigma\Psi_4 - 2\kappa'\Psi_1\right]\\ &+ \bar{\kappa}\left[(þ' - 4\rho')\Psi_3 - (ð - \tau)\Psi_4 + 3\kappa'\Psi_2\right] - \Phi_{00}\Psi_4 + 2\Phi_{10}\Psi_3 - \Phi_{20}\Psi_2,\end{aligned} \tag{A.26}$$

$$\begin{aligned}\frac{1}{2}B_{22} =& (þ' - 3\rho')\left[(ð - 2\tau)\Psi_3 - (þ' - 3\rho')\Psi_2 + \sigma\Psi_4 - 2\kappa'\Psi_1\right]\\ &+ (ð - \tau)\left[(þ' - 4\rho')\Psi_3 - (ð - \tau)\Psi_4 + 3\kappa'\Psi_2\right]\\ &+ 2\kappa'\left[(ð - 3\tau)\Psi_2 - (þ' - 2\rho')\Psi_1 + 2\sigma\Psi_3 - \kappa'\Psi_0\right]\\ &+ \bar{\kappa}'\left[(þ - 2\rho)\Psi_3 - (ð' - 3\tau')\Psi_2 + \kappa\Psi_4 - 2\sigma'\Psi_1\right]\\ &+ \bar{\sigma}'\left[(ð' - 4\tau')\Psi_3 - (þ - \rho)\Psi_4 + 3\sigma'\Psi_2\right] - \Phi_{02}\Psi_4 + 2\Phi_{12}\Psi_3 - \Phi_{22}\Psi_2,\end{aligned} \tag{A.27}$$

$$
\begin{aligned}
\frac{1}{2}B_{01} =&(\text{þ} - 3\rho)\left[(\text{þ}' - 2\rho')\,\Psi_1 - (\eth - 3\tau)\Psi_2 + \kappa'\Psi_0 - 2\sigma\Psi_3\right] \\
&+ (\eth' - \tau')\left[(\eth - 4\tau)\Psi_1 - (\text{þ}' - \rho')\,\Psi_0 + 3\sigma\Psi_2\right] \\
&+ 2\kappa\left[(\text{þ}' - 3\rho')\,\Psi_2 - (\eth - 2\tau)\Psi_3 + 2\kappa'\Psi_1 - \sigma\Psi_4\right] \\
&+ \bar{\tau}'\left[(\eth' - 2\tau')\,\Psi_1 - (\text{þ} - 3\rho)\Psi_2 + \sigma'\Psi_0 - 2\kappa\Psi_3\right] \\
&+ \bar{\rho}'\left[(\text{þ} - 4\rho)\Psi_1 - (\eth' - \tau')\,\Psi_0 + 3\kappa\Psi_2\right] - \Phi_{21}\Psi_0 + 2\Phi_{11}\Psi_1 - \Phi_{01}\Psi_2,
\end{aligned}
\tag{A.28}
$$

$$
\begin{aligned}
\frac{1}{2}B_{21} =& (\text{þ}' - 3\rho')\left[(\text{þ} - 2\rho)\Psi_3 - (\eth' - 3\tau')\,\Psi_2 + \kappa\Psi_4 - 2\sigma'\Psi_1\right] \\
&+ (\eth - \tau)\left[(\eth' - 4\tau')\,\Psi_3 - (\text{þ} - \rho)\Psi_4 + 3\sigma'\Psi_2\right] \\
&+ 2\kappa'\left[(\text{þ} - 3\rho)\Psi_2 - (\eth' - 2\tau')\,\Psi_1 + 2\kappa\Psi_3 - \sigma'\Psi_0\right] \\
&+ \bar{\tau}\left[(\eth - 2\tau)\Psi_3 - (\text{þ}' - 3\rho')\,\Psi_2 + \sigma\Psi_4 - 2\kappa'\Psi_1\right] \\
&+ \bar{\rho}\left[(\text{þ}' - 4\rho')\,\Psi_3 - (\eth - \tau)\Psi_4 + 3\kappa'\Psi_2\right] - \Phi_{01}\Psi_4 + 2\Phi_{11}\Psi_3 - \Phi_{21}\Psi_2,
\end{aligned}
\tag{A.29}
$$

$$
\begin{aligned}
\frac{1}{2}B_{10} =&(\text{þ} - 2\rho)\left[(\eth' - 3\tau')\,\Psi_2 - (\text{þ} - 2\rho)\Psi_3 + 2\sigma'\Psi_1 - \kappa\Psi_4\right] \\
&+ (\eth' - 2\tau')\left[(\text{þ} - 3\rho)\Psi_2 - (\eth' - 2\tau')\,\Psi_1 - \sigma'\Psi_0 + 2\kappa\Psi_3\right] \\
&+ \kappa\left[(\eth' - 4\tau')\,\Psi_3 - (\text{þ} - \rho)\Psi_4 + 3\sigma'\Psi_2\right] \\
&+ \bar{\kappa}\left[(\text{þ}' - 3\rho')\,\Psi_2 - (\eth - 2\tau)\Psi_3 + 2\kappa'\Psi_1 - \sigma\Psi_4\right] \\
&+ \bar{\sigma}\left[(\eth - 3\tau)\Psi_2 - (\text{þ}' - 2\rho')\,\Psi_1 - \kappa'\Psi_0 + 2\sigma\Psi_3\right] - \Phi_{20}\Psi_1 + 2\Phi_{10}\Psi_2 - \Phi_{00}\Psi_3,
\end{aligned}
\tag{A.30}
$$

$$
\begin{aligned}
\frac{1}{2}B_{12} =& (\text{þ}' - 2\rho')\left[(\eth - 3\tau)\Psi_2 - (\text{þ}' - 2\rho')\,\Psi_1 + 2\sigma\Psi_3 - \kappa'\Psi_0\right] \\
&+ (\eth - 2\tau)\left[(\text{þ}' - 3\rho')\,\Psi_2 - (\eth - 2\tau)\Psi_3 - \sigma\Psi_4 + 2\kappa'\Psi_1\right] \\
&+ \kappa'\left[(\eth - 4\tau)\Psi_1 - (\text{þ}' - \rho')\,\Psi_0 + 3\sigma\Psi_2\right] \\
&+ \bar{\kappa}'\left[(\text{þ} - 3\rho)\Psi_2 - (\eth' - 2\tau')\,\Psi_1 + 2\kappa\Psi_3 - \sigma'\Psi_0\right] \\
&+ \bar{\sigma}'\left[(\eth' - 3\tau')\,\Psi_2 - (\text{þ} - 2\rho)\Psi_3 + 2\sigma'\Psi_1 - \kappa\Psi_4\right] - \Phi_{02}\Psi_3 + 2\Phi_{12}\Psi_2 - \Phi_{22}\Psi_1,
\end{aligned}
\tag{A.31}
$$

$$
\begin{aligned}
\frac{1}{2}B_{11} =&(\text{þ} - 2\rho)\left[(\text{þ}' - 3\rho')\,\Psi_2 - (\eth - 2\tau)\Psi_3 + 2\kappa'\Psi_1 - \sigma\Psi_4\right] \\
&+ (\eth' - 2\tau')\left[(\eth - 3\tau)\Psi_2 - (\text{þ}' - 2\rho')\,\Psi_1 - \kappa'\Psi_0 + 2\sigma\Psi_3\right] \\
&+ \kappa\left[(\text{þ}' - 4\rho')\,\Psi_3 - (\eth - \tau)\Psi_4 + 3\kappa'\Psi_2\right] \\
&+ \sigma'\left[(\eth - 4\tau)\Psi_1 - (\text{þ}' - \rho')\,\Psi_0 + 3\sigma\Psi_2\right] \\
&+ \bar{\rho}'\left[(\text{þ} - 3\rho)\Psi_2 - (\eth' - 2\tau')\,\Psi_1 + 2\kappa\Psi_3 - \sigma'\Psi_0\right] - \Phi_{21}\Psi_1 + 2\Phi_{11}\Psi_2 - \Phi_{01}\Psi_3,
\end{aligned}
\tag{A.32}
$$

To compare these results with those derived in our third chapter we have to employ relations (A.18)–(A.23). Moreover, the [33] scalars must be combined as

$$B^{NP}_{(1)(1)} = \frac{1}{2}\left(B_{00} + \bar{B}_{00}\right), \tag{A.33}$$

$$B^{NP}_{(1)(2)} = \frac{1}{2}\left(B_{11} + \bar{B}_{11}\right), \tag{A.34}$$

$$B^{NP}_{(1)(3)} = \frac{1}{2}\left(B_{01} + \bar{B}_{10}\right), \tag{A.35}$$

$$B^{NP}_{(2)(2)} = \frac{1}{2}\left(B_{22} + \bar{B}_{22}\right), \tag{A.36}$$

$$B^{NP}_{(2)(3)} = \frac{1}{2}\left(B_{12} + \bar{B}_{21}\right), \tag{A.37}$$

$$B^{NP}_{(3)(3)} = \frac{1}{2}\left(B_{02} + \bar{B}_{20}\right), \tag{A.38}$$

where NP denotes scalars of the third chapter.

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