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Homomorphisms into Unary Algebras

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I declare that I carried out this bachelor thesis independently, and only with the cited sources, literature and other professional sources. It has not been used to obtain another or the same degree.

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In Prague, 21st July 2021 Jiří Škrobánek m. p.

I would like to thank my supervisor doc. Libor Barto for his invaluable advice.

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Abstract: This thesis studies the computational complexity of constraint satisfaction problem (CSP) over structures with unary operations (unary algebras). We concentrate on a special class of such CSPs, so called reversing problems. We present a new proof of complexity classification for reversing problems, which uses the algebraic approach based on studying polymorphisms. We show that some reversing problems admit near unanimity polymorphisms (and are therefore solvable in polynomial time) while the remaining reversing problems do not admit weak near unanimity polymorphisms (and are therefore NP-complete).

Keywords: Constraint Satisfaction Problem, Weak Near Unanimity, Homomorphism, Reversing Problem

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Introduction

Many important computational problems can be phrased as constraint satisfaction problems (CSPs) over relational structures. This includes, for instance, various Boolean satisfiability problems or graph coloring problems.

The most important existing result (due to Zhuk [1] and Bulatov [2]) is probably the so called CSP dichotomy theorem, which states that for every finite relational structure, its CSP is either NP-complete or solvable in polynomial time.

CSPs over structures that are not purely relational have not been explored so thoroughly. Some investigation was carried out by Feder, Madelaine, and Stewart [3]. Among other results, they managed to show that the CSP over any relational structure can be converted to the CSP over an algebra with two unary operations. This fact motivated Feder et al. to examine more closely special cases of these algebras.

Significant attention was devoted by Feder et al. to the study of the so called *reversing problems*. Complete classification of their complexity based on the associated structure was obtained. In this thesis we give an alternative proof of this classification theorem in a more algebraic manner. We use modern methods not available to Feder et al. at the time. For every relevant structure, we determine whether a weak near unanimity (WNU) polymorphism of that structure exists. This is sufficient to classify the complexity of the CSP over that structure [1, 2, 4, 5].

Necessary terminology and notation is introduced in Chapter 1. We also lay out theorems about complexity of CSP, these will be used to classify individual reversing problems. In Chapter 2 we prove that there exists a conservative majority polymorphism for reversing problems under some conditions, thus demonstrating polynomial complexity, using the results from Feder and Vardi [6]. Conversely, in Chapter 3 we show NP-completeness of all the remaining reversing problems by applying results from Bulatov et al. [7] and Maróti and McKenzie [4].

1. Preliminaries

In this chapter we introduce all core concepts that will be used to classify the complexity of reversing problems.

First of all, we introduce notation which will be used throughout this thesis. The set of natural numbers will be denoted by $\mathbb{N} = \{0, 1, 2, \ldots\}$. The set of positive integers 1 through n will be denoted by $[n]$ for every positive integer n. We will use function (relation) composition operator $\circ$ defined as follows. Consider functions $f : B \to C$ and $g : A \to B$, then $(f \circ g)(x) = f(g(x))$. Similarly, for relations $F \subset B \times C$ and $G \subset A \times B$, we define $F \circ G = \{(a, c) \mid (\exists b \in B)((a, b) \in G \wedge (b, c) \in F)\}$.

We denote the n-th Cartesian power of set X by X^n. An n-ary *relation* on X is a subset of X^n. An n-ary *operation* on X is a mapping $X^n \to X$. For a function $f : X \to X$ and $k \in \mathbb{N}$, $f^k : X \to X$ signifies that function f is iterated k times. This power operator takes precedence over function evaluation.

$$f^k = \overbrace{f \circ f \cdots f \circ f}^{k}, f^0 = \mathrm{id}_X \quad \text{(Identity on X)}$$

All sets will be considered finite, unless indicated otherwise.

1.1 Constraint Satisfaction Problem (CSP)

In this thesis we opt to formulate CSP as a problem of existence of a homomorphism.

Definition 1 (Signature)**.** *Signature S is a finite ordered collection of*

- *relational symbols, each with its arity,*
- *functional (or operational) symbols, each with its arity,*
- *and constant symbols.*

Definition 2 (Structure)**.** *Let S be a signature. A* structure *$\mathcal{T}$ with signature S consists of a finite set T (called* domain*), one n-ary relation on T for every n-ary relational symbol in S, one operation (or function) $T^n \to T$ for every n-ary functional symbol, and a constant from T for every constant symbol.*

Structure is called relational *if its signature contains no functional and constant symbols. If there are no relational symbols in the signature, the structure is called an* algebra.

The operation in $\mathcal{T}$ corresponding to functional symbol f will be denoted by $f_{\mathcal{T}}$ when the distinction needs to be made among operations of different structures corresponding to the same functional symbol. Analogous notation will be used for relations and constants.

Definition 3 (Homomorphism of structures)**.** *Let $\mathcal{T}, \mathcal{U}$ be two structures over the same signature S with domains T, U. A mapping $h : T \to U$ is called a* homomorphism *from $\mathcal{T}$ to $\mathcal{U}$ if*

- *for every n-ary relational symbol R in S and $a_1, \ldots, a_n \in T$*

$$R_{\mathcal{T}}(a_1, \ldots, a_n) \implies R_{\mathcal{U}}(h(a_1), \ldots, h(a_n)),$$

- *for every n-ary functional symbol f in S and $a_1, \ldots, a_n \in T$*

$$h(f_{\mathcal{T}}(a_1, \ldots, a_n)) = f_{\mathcal{U}}(h(a_1), \ldots, h(a_n)),$$

and

- *for every contant symbol c in S*

$$h(c_{\mathcal{T}}) = c_{\mathcal{U}}.$$

We are now ready to formulate the CSP.

Definition 4 (Constraint satisfaction problem)**.** *We say that a pair $(\mathcal{T},\mathcal{U})$ of two finite structures with the same signature is an instance of the* constraint satisfaction problem. *This instance is* solvable *if there exists a homomorphism h from $\mathcal{T}$ to $\mathcal{U}$. In that case h is said to* satisfy *this instance of CSP or that h is a* solution *to the instance.*

We remark that there are other equivalent formulations of CSP (see for example a survey by Barto et al. [5]), the one chosen is the most suitable in this context.

Definition 5 (CSP over a structure)**.** *Let $\mathcal{T}$ be a structure with signature S. Then the CSP over* template *$\mathcal{T}$ denoted* CSP($\mathcal{T}$) *is the following algorithmic decision problem. Permissible inputs are structures with signature S. The answer for input $\mathcal{V}$ is positive if and only if the instance $(\mathcal{V}, \mathcal{T})$ of the CSP is solvable.*

We can measure the complexity with respect to the size of domain of the input. Then apparently the CSP over any structure belongs to NP, since any homomorphism can be used as a proof of polynomial size.

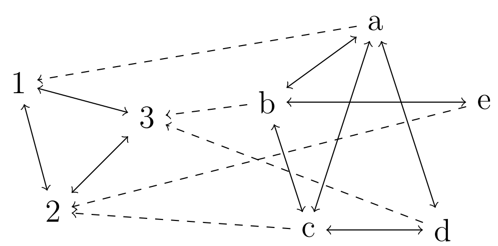
We give two examples of computational problems equivalent to the CSP over a certain structure.

Example (Coloring of graphs). The problem of whether a graph permits a coloring by n colors is equivalent to the CSP over a structure with domain $[n]$. We use the structure with one binary relation $[n]^2 \setminus \mathrm{id}$. The graph to be colored is represented as a structure with one symmetric irreflexive binary relation on the set of vertices which reflects the edges.

If two vertices are connected in the graph, their images must be different in any homomorphism which solves the CSP. If the CSP is solvable, the homomorphism which solves it provides a valid assignment of colors.

On the other hand it can be seen that should we be given a structure with one binary relation, we could transform symmetrization of that relation into a graph, determine if any n-coloring exists and then construct a suitable homomorphism to solve the CSP from that coloring.

The situation is depicted for three-coloring of a graph with vertices a, b, c, d, e. The dashed arrows form a valid homomorphism.



Example (Boolean satisfiability problem). Let A be a set of logical variables. A clause is a logical disjunction of several variables from A or their negations. Logical expression in CNF is then a logical conjunction of several clauses.

Let L be an expression in CNF where every clause contains at most three variables, we call 3-SAT(A, L) the problem to decide whether it is possible to assign logical true and false to variables in A in such a way that L would be true. Every logical expression can be converted to the form accepted by 3-SAT, additionally clauses can be padded by new variables such that all clauses have length three.

3-SAT can be reformulated as CSP over a specific structure $\mathcal{B}$ with domain $B = \{0, 1\}$ and a collection of eight relations $\{R_{\mathcal{B}}^{ijk} : i, j, k \in \{0, 1\}\}$ where $R_{\mathcal{B}}^{ijk} = \{0, 1\}^3 \setminus \{(i, j, k)\}$.

Suppose we want to transform 3-SAT(A, L), where all clauses have length exactly three. In that case we define a structure $\mathcal{A}$ with domain A and a collection of relations $\{R_{\mathcal{A}}^{ijk} : i, j, k \in \{0, 1\}\}$.

Every clause in L then corresponds to one element of a relation in $\mathcal{A}$. We choose the relation based on which literals are negated in the clause. The triple of indices is chosen in such a way that it is the only triple of logical values not satisfying that clause. For example, the clause $(a \vee b \vee \neg c)$ would result in the triple (a, b, c) in the relation $R_{\mathcal{A}}^{0,0,1}$.

We claim that 3-SAT(A, L) is satisfiable if and only if there is a homomorphism from $\mathcal{A}$ to $\mathcal{B}$.

If the 3-SAT has a solution, then there exists a homomorphism from $\mathcal{A}$ to $\mathcal{B}$. Indeed, we can fix an assignment f of logical values to A satisfying L and f is in fact the desired homomorphism. For every element of a relation in $\mathcal{A}$. If we take any triple (a, b, c) in $A_{i,j,k}$, we know that the corresponding clause is satisfied by f. Therefore $f(a, b, c)$ cannot be (i, j, k) and we get $R_{\mathcal{B}}^{i,j,k}(a, b, c)$.

Conversely, let the CSP be satisfied by homomorphism f. In that case f is an assignment satisfying 3-SAT(A, L). Every clause corresponds to an element of a relation in $\mathcal{A}$. Therefore the one combination of logical values not satisfying that clause is not assigned to variables in it.

Feder and Vardi [6] in their influential paper conjectured that the CSP over any (finite) relational structure is either solvable in polynomial time, or NP-complete. Recently, their conjecture was confirmed independently by Zhuk [1] and Bulatov [2].

Theorem 1 (CSP Dichotomy Theorem)**.** *For every relational structure $\mathcal{T}$, $\mathrm{CSP}(\mathcal{T})$ is either solvable in polynomial time or NP-complete.*

The complexity of CSPs over general finite structures (not necessarily relational) is currently open and so is even the CSP over algebras. Feder et al. [3] expand an earlier result [6] and show that even CSPs over algebras with two unary operations cover, in the following sense, all relational CSPs.

Theorem 2. *Consider a relational structure $\mathcal{T}$. There exists an algebra $\mathcal{A}$ with two operations of arity 2, such that $\mathrm{CSP}(\mathcal{T})$ is polynomial-time Turing-equivalent to $\mathrm{CSP}(\mathcal{A})$.*

On the other hand, unary operations in the signature can be replaced by their graphs by the following theorem.

Theorem 3. *Let $\mathcal{T}$ be an algebra with signature S of only unary and constant symbols. Signature S' is obtained from S by replacing operational symbol by a binary relational and every constant symbol by a unary relational symbol. Relational structure $\mathcal{T}'$ with signature S' is obtained from $\mathcal{T}$ by interpreting every operation as a binary relation and every constant as a unary relation (with exactly one element). Then $\mathrm{CSP}(\mathcal{T})$ is (polynomial-time many-one) equivalent to $\mathrm{CSP}(\mathcal{T}')$.*

1.2 Algebraic Approach

We now continue by introducing the algebraic theory useful for classification of the complexity of CSPs.

Definition 6 (Compatibility)**.** *For any domain X and a k-ary operation $\tau : X^k \to X$ we say that τ is* compatible *with an n-ary relation R on X (or that τ* preserves *R) if the following holds for all $x_i^j \in X$, $i \in [n], j \in [k]$:*

$$(\forall j \in [k] : R(x_1^j, x_2^j, \dots, x_n^j)) \implies$$
$$R(\tau(x_1^1, x_1^2, \dots, x_1^k), \tau(x_2^1, x_2^2, \dots, x_2^k), \dots, \tau(x_n^1, x_n^2, \dots, x_n^k))$$

We define compatibility with an n-ary operation (function) μ on X similarly:
If for all $j \in [k]$ we denote $\mu(x_1^j, x_2^j, \dots, x_n^j) = y_j$, then the following must hold

$$\mu(\tau(x_1^1, x_1^2, \dots, x_1^k), \tau(x_2^1, x_2^1, \dots, x_2^k), \dots, \tau(x_n^1, x_n^2, \dots, x_n^k)) = \tau(y_1, y_2, \dots, y_k)$$

Compatibility with a constant c is then given by the condition $\tau(c, \dots, c) = c$.

An operation is called a polymorphism *of a structure if it is compatible with every operation, relation and constant of the structure.*

An unary operation can also be naturally interpreted as a binary relation. One can see that compatibility is defined consistently for both interpretations.

Fortunately, we can limit ourselves to low arities of operations in this thesis. On the other hand, we can also apply a unary operation $o : X \to X$ to more than one argument. This is just a shorthand notation for applying the operation to each argument separately, e.g. $o(x_1, x_2, \dots, x_n) := (o(x_1), o(x_2), \dots, o(x_n))$.

Observation. For compatibility with a unary operation $\iota : X \to X$, it can be seen that it is sufficient that an operation $p : X^n \to X$ satisfies $\iota(p(x_1, \dots, x_n)) = p(\iota(x_1, \dots, x_n))$ for all n-tuples $x_1, \dots, x_n \in X$. Essentially this means that the two operations commute, i.e., $p \circ \iota = \iota \circ p$. This may be referred to as *commutativity condition.*

The following observation states that compatibility is preserved under composition and inverse and will be very important later on.

Observation. Let $\tau : X^n \to X$ be compatible with binary relations P, R on X. Under this assumption, τ is compatible also with the composition $P \circ R$ and the inverse relation $P^{-1} := \{(x, y) \mid P(y, x)\}$.

Next, we define two special kinds of operations which will help us classify the complexity.

Definition 7 (Near unanimity & majority)**.** *An operation $\alpha : A^k \to A$ is said to be a* near unanimity *operation if the following holds for all $a_j \in A, j \in [k]$:*

$$(\forall a \in A)\Big(\Big|\{i \in [k] \mid a_i \neq a\}\Big| \leq 1 \implies \alpha(a_1, a_2 \dots, a_k) = a\Big)$$

Near unanimity operation for $k = 3$ is called a majority*. We say further that majority operation $\beta : A^3 \to A$ is* conservative *if $\forall a, b, c \in A : \beta(a, b, c) \in \{a, b, c\}$.*

Definition 8 (Weak near unanimity – WNU)**.** *An operation $\tau : A^k \to A$ is said to be a* weak near unanimity *operation if the following set is singleton for all $a \in A^k$ and $b \in A$:*

$$\left\{ \tau(\overbrace{a, \dots, a}^{j-1}, b, \overbrace{a, \dots, a}^{k-j}) \mid j \in [k] \right\}$$

There are several theorems using (weak) near-unanimities. which determine the complexity of some CSP. The following theorem will be used prove NP-completeness. It is obtained by combining the results of Bulatov et al. [7], and Maróti and McKenzie [4] (see also [5]).

Theorem 4. *Let $\mathcal{T}$ be a relational structure. If there is no arity $n \geq 3$ such that an n-ary weak near unanimity polymorphism on $\mathcal{T}$ would exist,* $\mathrm{CSP}(\mathcal{T})$ *is NP-complete.*

In the sense of Theorem 3, unary operations may be equivalently represented by binary relations and constants can be represented by unary relations. Therefore the following corollary can be formulated:

Corollary. Theorem 4 also holds for algebras with unary operations and constants.

Conversely, it was shown by Bulatov and Zhuk that the $\mathrm{CSP}(\mathcal{T})$ is solvable in polynomial time otherwise. To show that CSP is solvable in polynomial time, we will use an approach suggested by the following theorem [6]:

Theorem 5. *Let $\mathcal{T}$ be a relational structure. If there exists a near unanimity polymorphism of arity $n \geq 3$, then* $\mathrm{CSP}(\mathcal{T})$ *is solvable in polynomial time.*

Just like above, we can use this theorem even for structures with unary operations and constants by Theorem 3.

Corollary. Theorem 5 also holds for algebras with unary operations and constants.

We remark that CSPs over structures with a near unanimity polymorphism have a property called *bounded strict width* [6], which permits the use of a special greedy algorithm for solving them.

1.3 Reversing problems

In the next chapters we will focus on some very specific CSPs which will be defined in this section. First of all we want to establish some basic understanding of graphs of unary operations.

By the *graph* of a unary operation ζ on X we mean the directed graph (with loops allowed) where X forms the set of vertices, denoted by $V(\zeta)$, and the set of edges contains all pairs (a, b) where $\zeta(a) = b$.

Similarly, the graph of a binary relation R on X has $X = V(R)$ as vertices and edges correspond to pairs in R.

Observation (Structure of graphs of unary operations). For any unary operation $\eta : U \to U$, over a finite domain U, the following holds for its graph:

- For every vertex $v \in V(\eta)$, the out-degree of v is 1.
- Every weakly-connected component contains exactly one oriented cycle (or loop).

Definition 9 (Reversed operation)**.** *Let $\tau : X \to X$ be a unary operation with only one cycle in its graph. Then operation $\theta : X \to X$ is said to be the* reverse *of τ, if it has the same graph with the only difference that the cycle is oriented in the opposite direction.*

Observation. If θ is the reverse of τ, also τ is the reverse of θ.

Observation. Reverse of an operation with one cycle of length at most two is the same operation.

Definition 10 (Reversing problems)**.** *The* CSP *over a structure with two unary functions in the signature which are reverses of one another is called a* reversing problem.

Definition 11 (Pendant chain)**.** *For a graph of any unary operation η over a finite domain U, we will say that the directed path $v_n, v_{n-1}, \dots, v_1$ of vertices not belonging to a cycle is a* pendant chain *on v if v_n has in-degree* 0 *and $v = \eta(v_1)$ is part of a cycle.*

We will also say that v_i is in distance i from v. We denote by $v^{[j]}$ the set of all elements on pendant chains on v in distance j.

When we use this notation the relevant operation must be apparent from the context, as the notation does not indicate it. We can see however that an operation and its reverse have the same pendant chains.

Note that one vertex may belong to many different pendant chains.

At the end of this chapter we present the theorems which will be proved in the next chapters.

Theorem 6. *The* CSP *over structure $\mathcal{T}$ with signature of only constant symbols and unary function symbols is solvable in polynomial time given at least one of the following conditions holds:*

1. *The signature of $\mathcal{T}$ contains at most one unary symbol.*
2. *Graph of every operation in $\mathcal{T}$ is a disjoint union of directed cycles.*
3. *There are exactly two unary operations π and σ, the graph of π contains exactly one cycle, it has length four, and σ is the reverse of π. In addition, one of the following holds:*
   - (a) *All vertices have distance at most one from the oriented cycle (in graph of both π and σ).*
   - (b) *All vertices have distance at most two from the oriented cycle and two non-neighboring vertices of the cycle do not have any pendant chains (in graph of both π and σ).*

It can be shown that in all of the above cases, there exists a conservative majority polymorphism. Furthermore, addition of unary relations does not increase the complexity. The proof of this theorem is the content of Chapter 2.

Theorem 7. *A* CSP *over structure $\mathcal{T}$ with two unary operations σ, π with only one cycle that are reverses of one another in the signature is NP-hard given at least one of the following holds:*

1. *Length of the cycle is odd and greater than one and there is at least one pendant chain.*
2. *Length of the cycle is even and greater than four and there is at least one pendant chain.*
3. *Length of the cycle is four and there is a pendant chain of length three or more.*

4. *Length of the cycle is four and there are two adjacent elements with pendant chains in the cycle of which at least one has length two or more.*

The proof of this theorem is the content of Chapter 3, we show that weak near unanimity polymorphisms cannot exist in these cases.

The two theorems combined classify the complexity of every reversing problem.

Corollary. Every reversing problem is either solvable in polynomial time, or NP-complete.

2. Polynomial Cases

We will start off by looking at two elementary cases – when there is only one unary operation in the signature and when all operations are bijective (i.e., permutations).

In the following lemmata, we will say that ternary operation τ picks position i from (a_1, a_2, a_3) if $\tau(a_1, a_2, a_3) = a_i$.

Lemma 8. *For any domain A and unary operation $f : A \to A$ there exists a conservative majority operation $\tau : A^3 \to A$ compatible with f.*

Proof. We define the result of a ternary operation τ on a triple $(a, b, c) \in A^3$ as follows.

(D1) If $\forall k \in \mathbb{N} : \left|\{f^k(a), f^k(b), f^k(c)\}\right| = 3$, we set $\tau(a, b, c) := a$.

(D2) Otherwise there exists a smallest $k \in \mathbb{N}$ such that $f^k(a, b, c)$ has a majority element. In such case we define τ to pick the left-most position that contains the majority element in $f^k(a, b, c)$.

It remains to verify that τ really possesses the claimed properties. The operation τ is apparently well-defined and satisfies the conservative majority condition. We now proceed to show that it is compatible with f. It suffices to show that $f(\tau(a, b, c)) = \tau(f(a, b, c))$ for all $a, b, c \in A$.

If $\tau(a, b, c)$ was defined by (D1), also $\tau(f(a, b, c))$ was defined by (D1) and $f(\tau(a, b, c)) = f(a) = \tau(f(a, b, c))$.

Else if $\tau(a, b, c)$ was defined by (D2) with $k \geq 1$, also $\tau(f(a, b, c))$ was defined by (D2) with k one less. Therefore the positions chosen to define τ are the same in both cases. The desired equality follows.

Otherwise k was 0. In that case since (a, b, c) has a majority element, also $f(a, b, c)$ has a majority element. Moreover it is the image of the majority element in (a, b, c). This completes the proof. □

Lemma 9. *Suppose a domain A and a finite set of unary operations $\{f_i \mid i \in [n]\}$, $f_i : A \to A$. If the graph of every f_i is a union of disjoint oriented cycles, there exists a conservative majority operation $\tau : A^3 \to A$ compatible with every operation f_i.*

Proof. We define a suitable ternary conservative majority operation τ on triple $(a, b, c) \in A^3$:

(D1) If (a, b, c) has a majority element, we set to $\tau(a, b, c)$ to that majority element,

(D2) otherwise $\tau(a, b, c) := a$.

Every f_i is bijective, this means that (a, b, c) has a majority if and only if $f_i(a, b, c)$ has a majority.

Now we show that τ is compatible with every operation f_i. If $\tau(a, b, c)$ is defined by (D1), we have $f_i(\tau(a, b, c)) = f_i(a) = \tau(f_i(a, b, c))$. Similarly, if (a, b, c) has a majority element, $f_i(a, b, c)$ has a majority element, in that case both $f_i(\tau(a, b, c))$ and $\tau(f_i(a, b, c))$ are equal to the f_i-image of that majority element. □

The following lemmata address two special cases of the reversing problem which are solvable in polynomial time. Consider the set of all structures with two unary operations σ, π which are reverses of one another and where the graph of each is an oriented cycle of

$$
\begin{tikzcd}
0 \arrow[r] \arrow[dr, phantom, "\sigma"] & 1 \arrow[d] & & 0 \arrow[d] \arrow[dr, phantom, "\pi"] & 1 \arrow[l] \\
3 \arrow[u] & 2 \arrow[l] & & 3 \arrow[r] & 2 \arrow[u]
\end{tikzcd}
$$

Figure 2.1: Cycles of σ and π

length four, possibly with some pendant chains. Let us denote this set by $\mathbf{H}$ and elements on the cycles as in Figure 2.1 for every $H \in \mathbf{H}$.

Let us define the parity of an element outside the cycle in σ inductively to be the opposite of its image in σ.

Lemma 10. *If all pendant chains of $H \in \mathbf{H}$ have length at most one, then there exists a conservative majority operation $\tau : H^3 \to H$ compatible with σ and π.*

Proof. **(D1)** We begin by setting value of τ for all triples with a majority element to that majority element. (This is forced by the definition of majority.)

(D2) For (a, b, c) with elements of both parities, we set τ to pick the left-most position with an element of the majority parity.

(D3) For all remaining triples (a, b, c) both $\sigma(a, b, c)$ and $\pi(a, b, c)$ have a majority element (they are three elements of the inner cycle of the same parity). Moreover, there exists a position which is part of the majority in both of those images. We set τ to pick the left-most such position.

The image of a majority element in a triple is a majority in image of that triple, this implies the commutativity condition for τ of triples defined by (D1).

We see that image of any element in σ and π has the opposite parity than the pre-image. That means that positions of dominant parity in pre-image are the same as in image.

Images of triples defined by (D2) in σ or π may not have a majority and τ for them is then defined by (D2) as projections to the same position. Alternatively, these images do have a majority, in which case it is the image of both of the elements of the same parity. Again this implies the commutativity condition.

For any triple (a, b, c) with τ defined in (D3) we see that the images have a majority and τ of that image is equal to the majority element. And $\tau(a, b, c)$ is a pre-image of that element.

We have shown the properties of compatibility for τ by demonstrating commutativity condition for all triples, which completes the proof.

□

Lemma 11. *If all pendant chains in $H \in \mathbf{H}$ end in 0 or 2 and have length at most two, then there exists a conservative majority operation $\tau : H^3 \to H$ compatible with π and σ.*

Proof. Once again, we construct such an operation τ in steps.

(D1) For (a, b, c) with elements of both parities, we set τ to pick the left-most position with an element of the majority parity.

(D2) For all other (a, b, c) there is at least one position that contains the majority element for all images of (a, b, c) in any composition of operations π and σ with a majority. We choose $\tau(a, b, c)$ to pick the left-most such a position from (a, b, c).

Just like in the previous lemma, from parity arguments triples from (D1) are not (pre-)images of triples from (D2). We can therefore analyze τ for these triples separately. The situation for (D1) is analogous to the previous lemma and need not be discussed further.

Denote by C the set of operations containing σ, π, and any composition of them (in particular π^2, σ^2, $\pi \circ \sigma$, and $\sigma \circ \pi$).

We now show that when we look at $\alpha(a, b, c)$ with $\tau(a, b, c)$ defined in (D2) for any $\alpha \in C$, τ either picks the same position from $\alpha(a, b, c)$ as from (a, b, c), or some position to the left of it, in which case there is the same element in both of those positions in $\alpha(a, b, c)$.

This claim is a bit complicated to prove. It can be seen that it is sufficient to show that the claim holds for π and σ, since the property is maintained by composition.

Suppose for contradiction that we have an image of a triple (a, b, c) in π (without loss of generality) which has a different position picked by τ than (a, b, c) and elements in the two positions are different. We get a contradiction looking at what elements are in the two positions in $\pi(a, b, c)$. Both of them holding an element from the cycle already implies that those elements would be equal, otherwise there is no way that the two positions would be part of every majority.

On the other hand, if both of them were parts of pendant chains, $\pi^2(a, b, c)$ would contain three elements from the cycle of the same parity – thus a majority with the two positions holding majority elements. Thus both the positions in (a, b, c) would need to contain elements in distance two from the cycle on pendant chains of the same element. In that case though, the positions with majority in images of $\sigma(a, b, c)$ would need to be the same as for $\pi(a, b, c)$. Since clearly neither $\sigma(a, b, c)$, nor $\pi(a, b, c)$ can have a majority, both of those positions must be part of majority in image in every $\alpha \in C$. So τ must pick the one more to the left even from (a, b, c).

The last possibility is hence that one of the two positions in $\pi(a, b, c)$ contains an element of the cycle while the other does not. Both $\pi^2(a, b, c)$ and $(\sigma \circ \pi)(a, b, c)$ would both need to have a majority with the two positions holding the majority element. Elements on the two positions may not be equal to each other in both of those images however. This is in contradiction with the positions being picked in definition (D2). So the claim holds.

Should we be able to choose the position in (D2) for all triples, compatibility would be guaranteed by the claim that has just been established. Now it suffices to show that the choice of position in (D2) is possible for all relevant triples.

If τ is defined for a triple by (D2) and there is a majority element in the triple, elements on the same positions form a majority in image of this triple in any operation α from C, thus a position is picked according to (D2).

Otherwise, if we have (a, b, c) such that a, b, c are all on a pendant chain of the same element in the same distance from the cycle, there is a suitable position. If the distance from the cycle is one, it suffices to pick the left-most position. If the distance is two and there is a majority in the π-image, τ picks the left-most majority position from the π-image. If there is not a majority in the π image, the left-most position is suitable.

Otherwise, when we have a triple (a, b, c) where without loss of generality both $a, b \in 0^{[\mathbf{1}]}$, or $0^{[\mathbf{2}]}$, or $2^{[\mathbf{1}]}$, or $2^{[\mathbf{2}]}$, then all images of (a, b, c) have a majority with the first two positions in it. Therefore the first position is picked in (D2). (Similarly for triples that differ only by permutation.)

Otherwise, a triple with τ defined in (D2) does not have two elements from any of $0^{[\mathbf{1}]}$, $0^{[\mathbf{2}]}$, $2^{[\mathbf{1}]}$, or $2^{[\mathbf{2}]}$. It must therefore be in $(0^{[\mathbf{2}]} \cup 2^{[\mathbf{2}]} \cup \{0,2\})^3$ or $(0^{[\mathbf{1}]} \cup 2^{[\mathbf{1}]} \cup \{1,3\})^3$ and all the elements are different.

We cannot get $\alpha(0) = \alpha(2)$, nor $\alpha(1) = \alpha(3)$ for any $\alpha \in C$. (Because α restricted on the cycle must be bijective.) Similarly for $a \in 0^{[\mathbf{1}]},\ b \in 2^{[\mathbf{1}]}$ always $\alpha(a) \neq \alpha(b)$. And for $c \in 0^{[\mathbf{2}]},\ d \in 2^{[\mathbf{2}]}$ always $\alpha(c) \neq \alpha(d)$.

Therefore when we have any remaining triple defined by (D2) it contains two elements whose positions may never be part of the same majority. The two remaining pairs of positions may of course hold majority elements. Clearly, in (D2) we pick a position that is in both of those pairs though.

We have constructed τ and proven the lemma.

□

With the lemmata we have established, we can now prove Theorem 6.

Proof. [Of Theorem 6] In all cases there exists a conservative majority compatible with π and σ. In case 1 by Lemma 8, in case 2 by Lemma 9, in case 3(a) by Lemma 10, and in case 3(b) by Lemma 11. Any conservative majority is compatible with all constants. We can now conclude that the theorem holds invoking Theorem 5 and its Corollary.

□

Additionally, it can be seen that any conservative majority is by definition compatible with all unary relations. In all of the lemmata, we constructed conservative majority polymorphism. Thus Theorem 6 still holds if unary relations are also permitted in the signature.

3. NP-hard Cases

In this chapter we will show that all the remaining reversing problems are NP-hard. We will again work with unary operations σ and π on X which are reverses of one another and the graph of both has exactly one oriented cycle γ of length at least 3. Elements of this cycle will be denoted by elements of the finite cyclic group $Z_n = \{0, 1, \dots, n-1\}$ with addition modulo n, where n is the length of the cycle. We denote the elements so that $\sigma(x) = x + 1$ for all elements x of the cycle.

It will also be useful to look at the relation $\omega = \pi \circ \sigma^{-1}$. (Compatibility with π and σ implies compatibility with ω.) In particular, we are interested in the structure of the subgraph of graph of ω which is induced by vertices of γ. The situation is different for even and odd length of the cycle. For even cycles there are two oriented cycles, one with even elements, the other with odd. Additionally, there may be loops. A loop at x is present if and only if $x^{[\mathbf{1}]}$ is non-empty. For $y \in x^{[\mathbf{1}]}$ clearly $\sigma(y) = x$ and $\pi(y) = x$. For odd cycles, there is one oriented cycle. Again loop at x is present if and only if $x^{[\mathbf{1}]}$ is non-empty. Both situations are depicted in Figure 3.1 and 3.2.

Observation. Let D be a domain with a unary operation $\alpha : D \to D$ and let $t : D^r \to D$ be compatible with α. If the graph of α only has one cycle κ of length l, then for every $x_1, x_2, \dots, x_r \in V(\kappa)$ it holds that $t(x_1, x_2, \dots, x_r) \in V(\kappa)$.

Proof. Clearly $\alpha^l(x_j) = x_j$ for all j. Therefore we must also have $\alpha^l(t(x_1, x_2, \dots, x_r)) = t(\alpha^l(x_1), \alpha^l(x_2), \dots, \alpha^l(x_r)) = t(x_1, x_2, \dots, x_r)$. This is only possible if $t(x_1, x_2, \dots, x_r)$ is part of the cycle.

□

To prove that some weak near unanimity cannot exist, it will be useful to know that existence of some weak near unanimity implies existence of another with the same arity which maps vectors that only contain copies of one element to that element.

Lemma 12. *Let $f : X^k \to X$ be compatible with both π and σ. Then there exists an operation $F : X \to X$ compatible with σ and π with $F(f(x, \dots, x)) = x$ for every $x \in V(\gamma)$.*

Proof. Let $x \in V(\gamma)$. We have already made an observation that the value $f(x, \dots, x) = y$ must also lie in the cycle.

For two different $x_1, x_2 \in V(\gamma)$ it must hold that $f(x_1, \dots, x_1) \neq f(x_2, \dots, x_2)$. Suppose the opposite is true for some x_1, x_2. Then $x_1 = \pi^r(x_2)$ for some minimum r. Since f is compatible with π, we also have $\pi^r(f(x_1, \dots, x_1)) = f(x_2, \dots, x_2) = f(x_1, \dots, x_1)$. This implies the existence of a cycle of length r in graph of π, which is not the case since the only cycle in π is γ and it has length greater than r.

Now it is apparent that $h : x \mapsto f(x, x, \dots, x)$ is a bijection on $V(\gamma)$.

We define F for elements of $V(\gamma)$ as $F(x) = h^{-1}(x)$. (So F is essentially a shift along the cycle for all elements of $V(\gamma)$.) We then choose arbitrarily one of the longest pendant chains for every element of $V(\gamma)$, denote the elements in the chosen elements by $z(x, i) \in x^{[\boldsymbol{i}]}$ for every x. To complete the definition of F, for $y \in x^{[\boldsymbol{i}]}$ for some $x \in V(\gamma)$ we define $F(y) = z(F(x), i)$.

If this definition is correct, compatibility with π and σ can be easily seen.

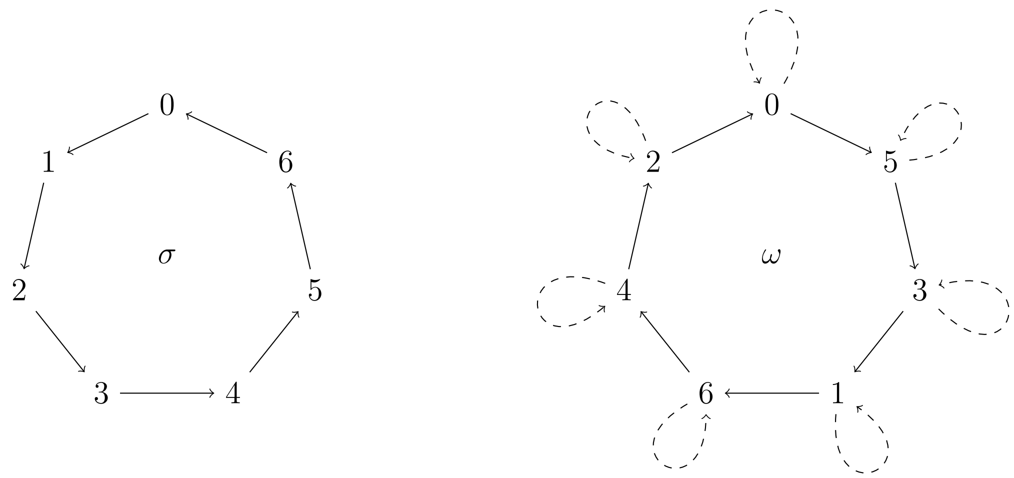


Figure 3.1: Graph induced by the cycle in σ and ω for odd cycle

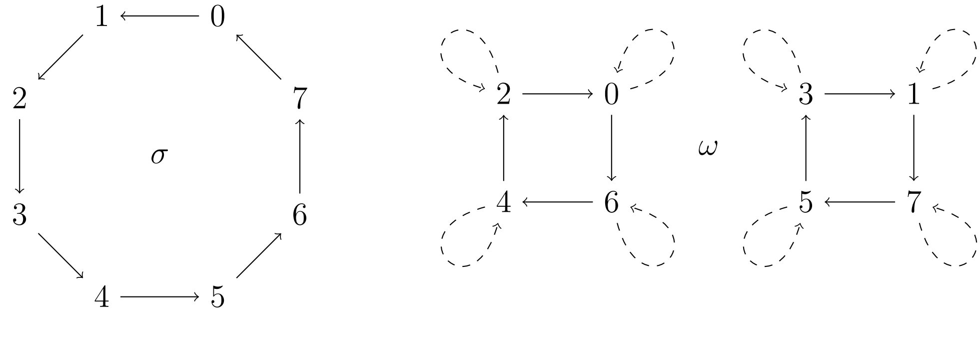


Figure 3.2: Graph induced by the cycle in σ and ω for even cycle

It remains to show that this definition is correct. In particular, we need to prove for every $x \in V(\gamma)$ and for every pendant chain of x that $F(x)$ has a chain at least as long. This will be obvious from the claim that for $y \in x^{[i]}$ we have $f(y,\ldots,y) \in (f(x,\ldots,x))^{[i]}$. In other words, f must map tuples of an element from a pendant chain ending in x to an element of a pendant chain ending in $f(x,\ldots,x)$ in the same distance from the end of the chain. This claim can be proved by induction on the distance of the element from the end of the chain.

To establish the base case, consider any $y \in x^{[\mathbf{1}]}$. We have $\pi(y,\ldots,y) = \sigma(y,\ldots,y) = (x,\ldots,x)$ and therefore also $\pi(f(y,\ldots,y)) = \sigma(f(y,\ldots,y)) = f(x,\ldots,x)$ from compatibility. Only elements of pendant chains have the same image in π and σ. Additionally, $(y,\ldots,y)$ is a pre-image of $(x,\ldots,x)$ in π. Therefore $f(y,\ldots,y)$ is on a pendant chain ending in $f(x,\ldots,x)$ in distance 1.

Suppose for the induction case that the proposition holds for all elements of distance at most i from the end of a pendant chain. For any $y \in x^{[\boldsymbol{i+1}]}$, let $\pi(y) = z \in x^{[\boldsymbol{i}]}$. From the induction hypothesis $f(z,\ldots,z) \in (f(x,\ldots,x))^{[\boldsymbol{i}]}$. Then $f(y,\ldots,y)$ is a pre-image of $f(z,\ldots,z)$ in π, thus a member of $(f(x,\ldots,x))^{[\boldsymbol{i+1}]}$.

It can now be seen that F is well-defined and compatible with π and σ and it has the desired properties.

$\square$

Observation. Under the conditions of the previous lemma $F \circ f$ is an operation compatible with π and σ.

Lemma 13. *Suppose that the length of γ is odd. For every positive integer k, if there exists a weak near unanimity operation $f : X^k \to X$ compatible with σ and π, then there also exists an operation $g : X^k \to X$ still compatible with σ and π, which is near unanimity when restricted to $V(\gamma)$.*

Proof. If there are no pendant chains in the graph of π (respectively σ), we can extend the definition from Lemma 9 to any arity, thus obtaining the desired near unanimity. Now without loss of generality $0^{[\mathbf{1}]}$ is non-empty. (Recall the notation we introduced.)

We focus on the graph of ω. In fact, we will only be interested in the subgraph induced by $V(\gamma)$. Let $x \in V(\gamma), x \neq 0$ and look at the sequence W:

$$\begin{array}{ccccccccc}
(& 0, & 0, & \ldots & 0, & x, & 0, & \ldots\ 0 &), \\
 & & & & \downarrow \omega & & & & \\
(& 0, & 0, & \ldots & 0, & x-2, & 0, & \ldots\ 0 &), \\
 & & & & \vdots & & & & \\
(& 0, & 0, & \ldots & 0, & 2, & 0, & \ldots\ 0 &), \\
 & & & & \downarrow \omega & & & & \\
(& -2, & 0, & \ldots & 0, & 0, & 0, & \ldots\ 0 &), \\
 & & & & \vdots & & & & \\
(& x, & 0, & \ldots & 0, & 0, & 0, & \ldots\ 0 &)
\end{array}$$

There are $|V(\gamma)|$ tuples in the sequence W and, at each coordinate, consecutive pairs of elements are in relation ω.

We see that the first and last tuples in the sequence W have the same value of f from weak unanimity. Values of f of tuples in the sequence W must form a walk in the graph of ω. There is only one possible walk in the graph of ω starting and ending in the same element of $V(\gamma)$ of length $|V(\gamma)|-1$ (length is the number of edges on the walk), a walk staying at the said element. From this we can conclude that the value of f for all tuples in the sequence is the same, let us denote it by s_0.

As a consequence, we get that the value $f(0,\ldots,0,y,0,\ldots,0) = s_0$ for any position of any $y \in V(\gamma)$ in the vector, if $y \neq 0$.

It only remains to show that $f(0,\ldots,0) = s_0$. Since $f(0,\ldots,0)$ must be in the pre-image of both $f(0,\ldots,0,-2) = s_0$ in ω and $f(0,\ldots,0,2) = s_0$ in $\sigma \circ \pi^{-1}$ (from compatibility of f with those relations). We already argued that for compatibility reasons operations must assign elements of the cycle to tuples composed solely of elements of the cycle. Therefore $s_0 \in V(\gamma)$. Pre-image of s_0 in ω can be only s_0 or $s_0 - 2$. Similarly, pre-image of s_0 in $\sigma \circ \pi^{-1}$ can be only s_0 or $s_0 + 2$. The only possible value is thus s_0 again.

We have shown that all k-tuples with 0 on at least $k-1$ positions have the same value of f, i.e., s_0. Shifting these tuples along γ by σ we get that all k-tuples with $m \in V(\gamma)$ on at least $k-1$ positions have the same value of f, $s_0 + m$.

Now composing f with the homomorphism guaranteed from the previous lemma, we get the desired operation which is near unanimity when restricted to $V(\gamma)$.

$\square$

We are now ready to rule out existence of weak near-unanimities for cycles of length other than four.

Lemma 14. *Suppose the cycle γ is of odd length $2k+1 = l \geq 3$ and the cycle has at least one pendant chain. Then no operation compatible with σ and π which is a weak near unanimity exists (of any arity ≥ 3).*

Before presenting the proof, let us introduce some additional notation. When working with vectors of some fixed size, we will use arrows to represent that some number of consecutive coordinates have the same value. For example $(\leftarrow a, b, c \rightarrow)$ denotes all vectors whose coordinates begin with zero or more as, these are followed by single b, and the rest of the coordinates are cs.

Proof. There is a pendant chain ending in the element 0, without loss of generality. We will prove this lemma by supposing that such an operation exists and reaching a contradiction. As we have shown in Lemma 13, we can assume that there exists an operation $t : X^k \to X$ compatible with σ and π which is a near unanimity when restricted to tuples from $V(\gamma)$.

We want to show that $t(\leftarrow 2, 0 \rightarrow) = 0$ for all such vectors. We prove this by induction on the number of twos in the vector. We get the base case $t(0,\ldots,0) = 0$, as well as $t(\leftarrow 0, 2, 0 \rightarrow) = 0$, from near unanimity properties of t.

From compatibility, the sequence $\{t(\leftarrow 2-2i, 0, -2i \rightarrow)\}_{i\in\{0,1,\ldots,l\}}$ must be a closed walk in the graph of ω. The structure of ω permits two options for every i, either $t(\leftarrow -2i, 0, -2i-2 \rightarrow) = t(\leftarrow 2-2i, 0, -2i \rightarrow)$, or $t(\leftarrow -2i, 0, -2i-2 \rightarrow) = t(\leftarrow 2-2i, 0, -2i \rightarrow) - 2$ must hold. In conjunction with the requirement of a closed walk, either $\{t(\leftarrow 2-2i, 0, -2i \rightarrow) \mid i \in \mathbb{Z}_l\}$ is a singleton set, or $t(\leftarrow -2i, 0, -2i-2 \rightarrow)$ $= t(\leftarrow 2-2i, 0, -2i \rightarrow) - 2$ for all i.

Let us return to the induction case. Suppose $t(\overbrace{\leftarrow 2}^{i}, 0 \rightarrow) = 0$, then $t(\overbrace{\leftarrow 0}^{i}, 0, 2k-1 \rightarrow)$ must be either $-2 = 2k-1$ or 0. The latter case leads to a contradiction as follows.

As stated, it would hold that $t(\overbrace{\leftarrow 4}^{i}, 0, 2 \rightarrow) = 0$, from the derived properties of t on the sequence. Note that $0 \neq 4 \neq 2 \neq 0$. Apparently, $t(\leftarrow 0, -4, 0 \rightarrow) = 0$ for all positions of -4 from near unanimity properties. The vectors $t(\leftarrow 0, -4, 0 \rightarrow)$ and $(\leftarrow 2, -2, 0 \rightarrow)$ are related by $\sigma \circ \pi^{-1}$. From this observation and compatibility of t with $\sigma \circ \pi^{-1}$, $t(\leftarrow 2, -2, 0 \rightarrow) \in \{0, 2\}$. Directly applying σ twice yields $t(\leftarrow 4, 0, 2 \rightarrow) \in \{2, 4\}$, the contradiction we wanted.

So $t(\overbrace{\leftarrow 0}^{i}, 0, -2 \rightarrow) = -2$. Therefore $t(\sigma^2(\overbrace{\leftarrow 0}^{i+1}, -2 \rightarrow)) = t(\overbrace{\leftarrow 2}^{i+1}, 0 \rightarrow) = \sigma^2(-2) = 0$. Having completed the induction, we notice that $2 = t(2, \ldots, 2) = 0$ is a contradiction.

No t with the desired properties can exist.

□

Lemma 15. *Let γ be of even length $l > 4$. If there is a pendant chain in γ, then for any arity $r \geq 3$ there does not exist any weak near unanimity $t : X^r \rightarrow X$ compatible with σ and π.*

Proof. We prove the lemma by contradiction. Without loss of generality, there is an element A on a pendant chain in distance one from 0. Let the arity r be any fixed integer not less than 3 and suppose that there exists a weak near unanimity operation $t : X^r \rightarrow X$ compatible with σ and π.

First we show that all values $t(\leftarrow 0, l-2 \rightarrow) \in \{c, c-2\}$ for $c = t(0 \rightarrow) \in V(\gamma)$. We can deduce $(\leftarrow A, l-1 \rightarrow) \in c^{[\mathbf{1}]} \cup \{c-1\}$ by looking at the pre-image of $t(0 \rightarrow)$ in σ and then directly $t(\leftarrow 0, l-2 \rightarrow) \in \{c, c-2\}$ from applying π. Actually, this holds for any permutation of coordinates.

Furthermore, the case $t(l-2, 0 \rightarrow) = c-2$ leads to a contradiction as described below. The reasoning is similar to the one used to prove the previous lemma. Elements of the form $(2n, 0 \rightarrow)$ for $n \in \mathbb{Z}_{l/2}$ represent an oriented cycle in the graph of ω. Length of the cycle is $l/2$. Value of t for consecutive elements must therefore differ either always by 0, or 2. (Otherwise compatibility is violated somewhere.) From $t(l-2, 0 \rightarrow) = c-2$ and $t(0 \rightarrow) = c$, we get $t(2n, 0 \rightarrow) = c+2n$ for all $n \in \mathbb{Z}_{l/2}$. It follows that $t(2, 0 \rightarrow) = c+2$ and $t(0, l-2 \rightarrow) = c$ by compatibility with π^2. From the conditions of weak near unanimity we get $t(\leftarrow l-2, 0) = c$.

Now we realize that there would need to be at least one j such that $t(\overbrace{\leftarrow l-2}^{j}, 0 \rightarrow) =$ $c-2$ and $t(\overbrace{\leftarrow l-2}^{j+1}, 0 \rightarrow) = c$. We would also get $t(\overbrace{\leftarrow 0}^{j+1}, 2 \rightarrow) = c+2$ from compatibility with σ^2. There must be an edge from $t(\overbrace{\leftarrow 0}^{j+1}, 2 \rightarrow)$ to $t(\overbrace{\leftarrow l-2}^{j}, 0 \rightarrow)$ in the graph of ω from compatibility. However, there is no edge from $c+2$ to $c-2$ in ω. We have reached a contradiction and got $t(l-2, 0 \rightarrow) = c$.

Let $k \geq 2$ be the smallest integer such that there exists $w \in \{0, l-2\}^r$ with k positions equal to $l-2$ and $t(w) = c-2$. Without loss of generality $w = (\overbrace{\leftarrow l-2}^{k}, 0 \rightarrow)$.

$$\begin{tikzcd}
B \arrow[r] & A \arrow[r] & 0 \arrow[r] & 1 \arrow[d] \\
C \arrow[u] & \sigma & 3 \arrow[u] & 2 \arrow[l]
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[r] & A \arrow[r] & 0 \arrow[d] & 1 \arrow[l] \\
C \arrow[u] & \pi & 3 \arrow[r] & 2 \arrow[u]
\end{tikzcd}$$

Figure 3.3: Situation with a chain of length 3 on the four-cycle

Now look at $(\overbrace{\leftarrow 0}^{k-1}, 4, 0 \rightarrow)$, $(\overbrace{\leftarrow l-4}^{k-1}, 0 \rightarrow)$, and $(\overbrace{\leftarrow l-2}^{k-1}, 2, 0 \rightarrow)$. We get $t(\overbrace{\leftarrow 0}^{k-1}, 4, 0 \rightarrow) = c$, $t(\overbrace{\leftarrow l-4}^{k-1}, 0 \rightarrow) = c$. This follows again from being part of a cycle of length $l/2$ in the graph ω where the value of t for all elements is c. Combination of the two yields $(\overbrace{\leftarrow l-2}^{k-1}, 2, 0 \rightarrow) = c$. By applying π twice, we get $t(\overbrace{\leftarrow l-4}^{k-1}, 0, l-2 \rightarrow) = c-2$.

Seeing that $t(\overbrace{\leftarrow l-2}^{k-1}, 0, 0 \rightarrow) = c$, values of t on the cycle with $(\overbrace{\leftarrow 2n-2}^{k-1}, 0, 2n \rightarrow)$ must be $c + 2n$. Notably $t(\overbrace{\leftarrow 0}^{k}, 2 \rightarrow) = c + 2$, which implies $t(\overbrace{\leftarrow l-2}^{k}, 0 \rightarrow) = c$, this is a contradiction with the choice of k.

The remaining option is $t(2 \rightarrow) = c$, which is inconsistent with $t(0 \rightarrow) = c$. Thus no t can exist.

$\square$

We have shown so far that weak near-unanimities do not exist for lengths of γ three and five or more. In the following two lemmata we prove that for the cases where γ has length four that were not covered by the previous chapter there are no weak near-unanimities.

Lemma 16. *If γ has length four and it has an element with a pendant-chain of length at least three, then no operation of arity at least three compatible with σ and π which would be a weak near unanimity can exist. (Other pendant chains are also permitted.)*

Proof. We denote the elements on the pendant chains by A, B, C as in Figure 3.3. Let $t : X^r \to X$ be a weak near unanimity compatible with π and σ for some $r \geq 3$.

Denote $x = t(0, 2 \rightarrow)$. (The value of x must be inside the cycle γ.) By compatibility with π, we have two possible choices for $t(A, 1 \rightarrow)$, either some member of $x^{[\mathbf{1}]}$ or $x - 1$.

If $t(A, 1 \rightarrow) \in x^{[\mathbf{1}]}$, we get $t(B, 0 \rightarrow) \in x^{[\mathbf{2}]}$, using compatibility of t with σ. Similarly, $t(C, 1, A \rightarrow) \in x^{[\mathbf{3}]}$, using compatibility with π. Compatibility with σ forces $t(B, 2, 0 \rightarrow) \in x^{[\mathbf{2}]}$, compatibility with σ again gives $t(A, 3, 1 \rightarrow) \in x^{[\mathbf{1}]}$ and finally using π we obtain $t(0, 2, 0 \rightarrow) = x = t(2, 0 \rightarrow)$ from WNU conditions. Applying σ twice: $t(0, 2 \rightarrow) = x + 2$, a contradiction.

So $t(A, 1 \rightarrow)$ must be $x - 1$, which yields $t(0 \rightarrow) = x - 2$ by applying π. We now get a similar choice for $t(B, 0 \rightarrow)$, which is a pre-image of $t(A, 1 \rightarrow)$ in σ. It can be either $x - 2$ or member of $(x-1)^{[\mathbf{1}]}$.

Suppose $t(B, 0 \rightarrow) \in (x-1)^{[\mathbf{1}]}$. We immediately get $t(C, A, 1 \rightarrow) \in (x-1)^{[\mathbf{2}]}$ from compatibility with π. Then $t(B, 0, 2 \rightarrow) \in (x-1)^{[\mathbf{1}]}$, $t(A, 1, 3 \rightarrow) = (x-1)$, and $t(2, 0 \rightarrow) = t(0, 2, 0 \rightarrow) = x$, all by compatibility with σ. Now applying σ twice: $x + 2 = t(0, 2 \rightarrow)$, which is a contradiction.

So $t(B, 0 \rightarrow)$ must be $x-2$ and, by applying σ, clearly also $t(A, 1 \rightarrow) = x-1$. Now we want to show by induction that $t(\leftarrow 3, A, 1 \rightarrow)$ must be $x-1$ for any number of 3s at the beginning. We already know that the base case holds.

The induction step involves showing $t(\overbrace{\leftarrow 3}^{i}, A, 1 \rightarrow) = x-1$ implies $t(\overbrace{\leftarrow 3}^{i+1}, A, 1 \rightarrow) = x-1$. Applying σ triple times to $(\overbrace{\leftarrow 3}^{i}, A, 1 \rightarrow)$ and realizing compatibility yields $t(\overbrace{\leftarrow 2}^{i+1}, 0 \rightarrow) = x+2$. Therefore $t(\overbrace{\leftarrow 3}^{i+1}, A, 1 \rightarrow) \in \{x-1\} \cup (x+2)^{[\mathbf{1}]}$ from compatibility with π. No value from $(x+2)^{[\mathbf{1}]}$ is permissible: We would get $t(\overbrace{\leftarrow 2}^{i+1}, B, 0 \rightarrow) \in (x+2)^{[\mathbf{2}]}$ by compatibility with σ, directly forcing $t(\overbrace{\leftarrow 1}^{i+1}, C, A \rightarrow) \in (x+2)^{[\mathbf{3}]}$ again by σ. Now, applying π three times, we would get $t(\overbrace{\leftarrow 2}^{i+1}, 0, 2 \rightarrow) = x+2$. This is in conflict with the definition of x and WNU properties of t, a contradiction.

We may construct the following sequence and conclude that $t(2 \rightarrow) = x+2$.

$$
\begin{array}{ccccccc|c}
 & & & v & & & & t(v) \\
\hline
(& 0, & 0, & 0, & \dots & 0 &) & x+2 \\
 & & & \uparrow \pi & & & & \\
(& A, & 1, & 1, & \dots & 1 &) & x-1 \\
 & & & \downarrow \sigma & & & & \\
(& 0, & 2, & 2, & \dots & 2 &) & x \\
 & & & \downarrow \sigma & & & & \\
(& 1, & 3, & 3, & \dots & 3 &) & x+1 \\
 & & & \downarrow \sigma & & & & \\
(& 2, & 0, & 0, & \dots & 0 &) & x+2 \\
 & & & \uparrow \pi & & & & \\
(& 3, & A, & 1, & \dots & 1 &) & x-1 \\
 & & & \downarrow \sigma & & & & \\
(& 0, & 0, & 2, & \dots & 2 &) & x \\
 & & & \downarrow \sigma & & & & \\
(& 1, & 1, & 3, & \dots & 3 &) & x+1 \\
 & & & \downarrow \sigma & & & & \\
(& 2, & 2, & 0, & \dots & 0 &) & x+2 \\
 & & & \vdots & & & & \\
(& 2, & 2, & 2, & \dots & 2 &) & x+2
\end{array}
$$

From $\sigma^2(0 \rightarrow) = (2 \rightarrow)$ and compatibility with σ, it must hold $\sigma^2(t(0 \rightarrow)) =$

$$\begin{tikzcd}
A \arrow[r] & 0 \arrow[r] & 1 \arrow[d] \arrow[dl, phantom, "\sigma"] & C \arrow[l] \\
B \arrow[u] & 3 \arrow[u] & 2 \arrow[l] &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r] & 0 \arrow[d] & 1 \arrow[l] \arrow[dl, phantom, "\pi"] & C \arrow[l] \\
B \arrow[u] & 3 \arrow[r] & 2 \arrow[u] &
\end{tikzcd}$$

Figure 3.4: Situation with chains of lengths 1 and 2 on adjacent vertices of the four-cycle

$\sigma^2(x-1) \neq x-1 = t(2 \to)$. So every choice eventually lead to a contradiction and no t can exist.

$\square$

Lemma 17. *Let γ have length four. If there are two adjacent elements in γ where one has a pendant-chain of length at least 2 and the other of length at least 1, then no operation with arity three or more compatible with σ and π which would be a weak near unanimity can exist. (Other pendant chains are also permitted.)*

Proof. We denote the elements on the pendant chains by A, B, C as in Figure 3.4.

To prove the lemma, we suppose there is a weak near unanimity operation $t\colon X^r \to X$ compatible with σ, π and reach a contradiction. We will follow Figure 3.5. All vectors are now assumed to have r coordinates.

Let us denote by x the value of $t(0,2 \to)$. As before $t(A,1 \to) \in \{x-1\} \cup x^{[\mathbf{1}]}$.

Now suppose $t(A,1 \to) \in x^{[\mathbf{1}]}$, then $t(B,0,C \to) = t(0,B,C \to) \in x^{[\mathbf{2}]}$ and $t(3,A,1 \to) \in x^{[\mathbf{1}]}$, $t(2,0 \to) = x$, then $t(1,3 \to) = x-1$, then $t(0,2 \to) = x-2$. A contradiction. Therefore $t(A,1 \to) = x-1$.

Having established the base case, we now proceed to show by induction that $t(\leftarrow 3, A, 1 \to) = x-1$ for any number of 3s at the beginning.

From $t(\overbrace{3,\dots,3}^{i},A,1 \to) = x-1$, it follows that $t(\overbrace{0,\dots,0}^{i+1},2 \to) = x$, $t(\overbrace{1,\dots,1}^{i+1},3 \to) = x+1$, $t(\overbrace{2,\dots,2}^{i+1},0 \to) = x+2$ and $t(\overbrace{3,\dots,3}^{i+1},A,1 \to) \in \{x+3\} \cup (x+2)^{[\mathbf{1}]}$.

Also from properties of WNU $t(\overbrace{1,\dots,1}^{i+1},A,1 \to) = x-1$, it follows that $t(\overbrace{0,\dots,0}^{i+1},B,C \to) \in \{x+2\} \cup (x-1)^{[\mathbf{1}]}$, which forces $t(\overbrace{3,\dots,3}^{i+1},A,1 \to) \in \{x+1, x-1\}$.

Combining the two requirements: $t(\overbrace{3,\dots,3}^{i+1},A,1 \to) = x-1 = x+3$.

With the induction now complete, we get $\pi \circ t(\leftarrow 3, A) = t(\leftarrow 2, 0) = x+2$. This is inconsistent with $t(0,2 \to) = x$ due to WNU properties of t.

$\square$

Theorem 7 can now be finally proved.

Proof. [Of Theorem 7] WNU polymorphisms cannot exist in any of the four cases. In case 1 by Lemma 14, in case 2 by Lemma 15, in case 3 by Lemma 16, and in case 4 by Lemma 17.

$$
\begin{tikzcd}
 & (0,2\rightarrow) \arrow[r, "\sigma"] & (1,3\rightarrow) \arrow[r, "\sigma"] & (2,0\rightarrow) & \\
(A,1\rightarrow) \arrow[ur, "\sigma"] \arrow[r, equal, "\mathrm{WNU}"] \arrow[dr, equal, "\mathrm{WNU}"'] & (1,A,1\rightarrow) & (0,B,C\rightarrow) \arrow[l, "\sigma"'] \arrow[r, "\pi"] & (3,A,1\rightarrow) \arrow[u, "\pi"'] & \\
 & (\underbrace{1,\dots,1}_{i+1},A,1\rightarrow) & (\underbrace{0,\dots,0}_{i+1},B,C\rightarrow) \arrow[l, "\sigma"'] \arrow[dr, "\pi"] & & \\
(\overbrace{3,\dots,3}^{i},A,1\rightarrow) \arrow[dr, "\sigma"] & & & (\overbrace{3,\dots,3}^{i+1},A,1\rightarrow) \arrow[d, "\pi"] & \\
 & (\underbrace{0,\dots,0}_{i+1},2\rightarrow) \arrow[r, "\sigma"] & (\underbrace{1,\dots,1}_{i+1},3\rightarrow) \arrow[r, "\sigma"] & (\underbrace{2,\dots,2}_{i+1},0\rightarrow) &
\end{tikzcd}
$$

Figure 3.5: Relations among tuples from proof of Lemma 17

By the Corollary of Theorem 4, the CSPs are NP-complete.

$\square$

Bibliography

[1] Dmitriy Zhuk. A proof of the CSP dichotomy conjecture. *Journal of the ACM*, 67(5), August 2020.

[2] Andrei A. Bulatov. A dichotomy theorem for nonuniform csps. In *2017 IEEE 58th Annual Symposium on Foundations of Computer Science (FOCS)*, pages 319–330, 2017.

[3] Tomás Feder, Florent Madelaine, and Iain A. Stewart. Dichotomies for classes of homomorphism problems involving unary functions. *Theoretical Computer Science*, 314(1–2):1–43, February 2004.

[4] Miklós Maróti and Ralph McKenzie. Existence theorems for weakly symmetric operations. *Algebra universalis*, 59(3–4):463–489, Nov 2008.

[5] Libor Barto, Andrei Krokhin, and Ross Willard. Polymorphisms, and How to Use Them. In Andrei Krokhin and Stanislav Zivny, editors, *The Constraint Satisfaction Problem: Complexity and Approximability*, volume 7 of *Dagstuhl Follow-Ups*, pages 1–44. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, Dagstuhl, Germany, 2017.

[6] Tomás Feder and Moshe Y. Vardi. The computational structure of monotone monadic snp and constraint satisfaction: A study through datalog and group theory. *SIAM Journal on Computing*, 28(1):57–104, Jan 1998.

[7] Andrei Bulatov, Peter Jeavons, and Andrei Krokhin. Classifying the complexity of constraints using finite algebras. *SIAM Journal on Computing*, 34(3):720–742, Jan 2005.