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Change Point Problem for Censored Data

(Doctoral Thesis)

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Prohlášení

Prohlašuji, že jsem tuto práci vypracovala samostatně a že jsem použila pouze citované zdroje. Souhlasím se zapůjčováním práce.

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Lenka Komárková

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Abstrakt. Práce je zaměřena na problém detekce změny (změn) rozdělení u náhodných veličin, které jsou nezávislé, ale mohou být cenzorovány. Testové statistiky a jim odpovídající odhady jsou sestaveny na základě znalostí pro úplná data. Konkrétně se zde zabýváme pořadovými testovými statistikami maximálního typu vhodnými pro detekování jedné změny a pořadovými MOSUM statistikami založenými na diferencích klouzavých součtů, které se používají v případě, kdy očekáváme více změn. Za platnosti hypotézy, že ke změně rozdělení u cenzorovaných dat nedošlo, je studováno limitní chování uvažovaných testových statistik. Ve speciálním případě, za podmínky shody rozdělení cenzorování, je použit permutační princip. Je ukázáno, že prezentované testy jsou konzistentní. Dále jsou navrženy odhady odvozené od statistiky maximálního typu a jsou vyšetřovány jejich limitní vlastnosti. Teoretické výsledky jsou demonstrovány na simulacích.

Klíčová slova: Analýza přežití; detekce změny; pořadový odhad; pořadový test.

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Abstract. The thesis deals with the problem of detection of a change (changes) in the distribution of variables that are independent but possibly censored. The test statistics and corresponding estimators are derived using the same principle as for uncensored data. We consider max-type rank test statistics applied to one-change problem and MOSUM-type rank statistics suitable for testing multiple changes. The limit behavior for such classes of test statistics under the hypothesis of "no change" in the distribution of censored data is studied. Particularly, under equal censorship, the permutation principle can be used. Moreover, the consistency of our test procedures is shown. Further, rank based estimators of the change point corresponding to the class of the max-type test statistics are proposed and their limit properties are investigated. Theoretical results are demonstrated on simulations.

Keywords: Detection of change; Rank estimator; Rank test; Survival analysis.

Contents

Preface

In the thesis, two important topics from the statistical analysis the change point detection *and* the survival analysis *are brought together, in other words we will show how to detect a change (or changes) in distribution of variables which are not completely observable. The research was initiated by Stute* [**32**]*, because this problem is very important in medical studies and statistical quality control in industry. In spite of this fact, this problem is considered only in a few papers. Contributions of the authors to this area is described in Chapter 1.*

We will study nonparametric methods, particularly, rank-based test procedures and corresponding estimation procedures. We will use the knowledge about the change point problem in the case of completely observable variables, because there has been already much written about it. For a review of the classical change point problems and an extensive reference list, we refer to Csörgő and Horváth [**10**] *and Antoch et al* [**6**].

In Chapter 1, we will introduce the model and formulate the change point problem for randomly censored data. We will summarize the results of other authors and assign the aim of work. In Chapter 2, various rank test statistics as two-sample, max-type and MOSUM-type statistics will be proposed. Estimators of the change point corresponding to max-type test statistics will follow in Chapter 3. The properties of suggested statistics and estimators under the null hypothesis of no change and also under the alternative of one change or multiple changes will be thoroughly studied. For particular situations, we will discuss the assumptions of the presented theorems in Chapter 4. Chapter 5 contains all the calculation and auxiliary results needed in previous chapters. Various simulation studies will be presented in Chapter 6 and finally in Conclusions the summary of open problems and the plan of the future research can be found.

CHAPTER 1

Introduction

1. Formulation of the problem

We introduce the basic notation concerning random censorship models. For more detailed information see e.g. Kalbfleisch and Prentice [**25**]. Typically, $X_1^0, X_2^0, \ldots, X_n^0$ is a sequence of independent nonnegative random variables (*the lifetimes* or *the survival times*), where the index i of X_i^0 corresponds to the chronological order in which the subject of interest (e.g. patient) has entered the study. The patient can be withdrawn from the study due to many reasons, e.g. an accidental death, a migration of human population or limited time of the study. More precisely, the lifetimes can be censored from the right by independent random variables $C_1, C_2, \ldots, C_n$, the so-called *censoring times*. In other words, instead of the survival times $X_1^0, X_2^0, \ldots, X_n^0$ we observe pairs $(X_1, \Delta_1), (X_2, \Delta_2), \ldots, (X_n, \Delta_n)$ only, where

$$X_j = \min(X_j^0, C_j) = \begin{cases} X_j^0, & \text{if } X_j^0 \leq C_j,\ X_j \text{ is uncensored,} \\ C_j, & \text{if } X_j^0 > C_j,\ X_j \text{ is censored,} \end{cases}$$

and

$$\Delta_j = I(X_j^0, C_j) = \begin{cases} 1, & \text{if } X_j \text{ is uncensored,} \\ 0, & \text{if } X_j \text{ is censored,} \end{cases}$$

for $j = 1, 2, \ldots, n$. We assume that the lifetimes and the censoring times are independent variables. Particularly, their distributions need not be the same over the observation period. More precisely, we suppose that for some unknown $\gamma \in (0,1]$ and $\eta \in (0,1]$ (generally, η and γ need not be the same) $X_1^0, X_2^0, \ldots, X_{\lfloor \gamma n \rfloor}^0$ and $X_{\lfloor \gamma n \rfloor + 1}^0, X_{\lfloor \gamma n \rfloor + 2}^0, \ldots, X_n^0$ have absolutely continuous distribution functions F_1 and F_2, respectively, $F_1 \neq F_2$, and $C_1, C_2, \ldots, C_{\lfloor \eta n \rfloor}$ and $C_{\lfloor \eta n \rfloor + 1}, C_{\lfloor \eta n \rfloor + 2} \ldots, C_n$ have absolutely continuous distribution functions G_1 and G_2, respectively, $G_1 \neq G_2$. The point γ (or $\lfloor \gamma n \rfloor$) is called *the change point.* Let f_1, f_2 and g_1, g_2 denote densities corresponding to F_1, F_2 and G_1, G_2, respectively. Notice that the distribution functions F_1, F_2 and G_1, G_2 are unknown.

We wish to test the no-change null hypothesis

$$H_0 : F_1(t) = F_2(t) = F(t) \quad \text{for all } t \in \mathbb{R}, \quad \text{i.e. } \gamma = 1, \tag{1.1}$$

against the one-change alternative hypothesis

$$H_1 : F_1(t) \neq F_2(t) \text{ for some } t \in \mathbb{R}, \quad \text{i.e. } \gamma \in (0,1).$$

This is one of the basic tasks in the change point analysis to decide if there is a change in the model in our case due to medical care. If we reject the no-change null hypothesis, we would like

- to decide if there is just one change or wether there are more changes;
- to locate when the model changed;
- to determine the total number of changes.

Next, we make few notes. Notice that the testing problem does not concern the behavior of the censoring variables even though their distribution has the influence on the distribution of the observed variables $X_j = \min(X_j^0, C_j)$, $j = 1, 2, \dots, n$. By the independence of the lifetimes X_j^0's and the censoring times C_j's, the observed variables $X_1, X_2, \dots, X_n$ have the following distribution function under the null hypothesis H_0 for all $x \in \mathbb{R}$

$$\begin{aligned} H_1(x) &= \mathrm{P}(X_j \le x) = \mathrm{P}\left(\min(X_j^0, C_j) \le x\right) = 1 - (1 - F_1(x))(1 - G_1(x)), \qquad 1 \le j \le \lfloor n\eta \rfloor, \\ H_2(x) &= \mathrm{P}(X_j \le x) = \mathrm{P}\left(\min(X_j^0, C_j) \le x\right) = 1 - (1 - F_1(x))(1 - G_2(x)), \qquad \lfloor n\eta \rfloor < j \le n, \end{aligned}$$

and under the alternative hypothesis H_1 (suppose $\eta \le \gamma$)

$$\begin{aligned} H_1(x) &= \mathrm{P}(X_j \le x) = 1 - (1 - F_1(x))(1 - G_1(x)), && 1 \le j \le \lfloor n\eta \rfloor, \\ H_2(x) &= \mathrm{P}(X_j \le x) = 1 - (1 - F_1(x))(1 - G_2(x)), && \lfloor n\eta \rfloor < j \le \lfloor n\gamma \rfloor, \\ H_3(x) &= \mathrm{P}(X_j \le x) = 1 - (1 - F_2(x))(1 - G_2(x)), && \lfloor n\gamma \rfloor < j \le n. \end{aligned} \tag{1.2}$$

Notice that $(X_1, \Delta_1), (X_2, \Delta_2) \dots, (X_{m_c}, \Delta_{m_c})$ have the common distribution function $L_1(x, d)$ of the following form

$$\begin{aligned} L_1(x, 1) &= \mathrm{P}(X_1 \le x, \Delta_1 = 1) = \mathrm{P}(X_1^0 \le x, X_1^0 \le C_1) \\ &= \iint_{t \le x,\, t \le c} \mathrm{d}F_1(t)\, \mathrm{d}G_1(c) = \int_0^x (1 - G_1(t))\, \mathrm{d}F_1(t), \\ L_1(x, 0) &= \mathrm{P}(X_1 \le x, \Delta_1 = 0) = \mathrm{P}(C_1 \le x, C_1 < X_1^0) \\ &= \iint_{c \le x,\, c < t} \mathrm{d}F_1(t)\, \mathrm{d}G_1(c) = \int_0^x (1 - F_1(c))\, \mathrm{d}G_1(c) \end{aligned} \tag{1.3}$$

and similarly for the distribution function $L_2(x, d)$ corresponding to (X_j, Δ_j), $\lfloor n\eta \rfloor < j \le \lfloor n\gamma \rfloor$, and for $L_3(x, d)$ denoting the distribution function of (X_j, Δ_j), $\lfloor n\gamma \rfloor < j \le n$.

In the following we suppose that the distribution of the censoring times can change ($G_1 \ne G_2$) which can occur more often in practical situations and in this case we will detect the change in the distribution of the survival variables with an appropriate limit test. We will mention the particular situation $G_1 = G_2$ as well. However, we will be able to use the permutation principle and we will obtain in this way an exact test.

Remark 1.1. The time of a change $\lfloor n\eta \rfloor$ in the distribution of the censoring variables is then a nuisance parameter in our testing problem. It is important to realize that in the case of $\gamma = \eta$ it can occur such a situation when the distributions $H_1(x)$ and $H_2(x)$ of the observed variables before and after the change point time $\lfloor n\gamma \rfloor$ are the same, i.e.

$$(1 - F_1(x))(1 - G_1(x)) = (1 - F_2(x))(1 - G_2(x)), \quad \forall\, x \in \mathbb{R}.$$

This equality is valid e.g. when $F_1 = G_2$ and $F_2 = G_1$ or when F_1, F_2 and G_1, G_2, respectively, come from the exponential distribution with the expectations μ_1, μ_2 and μ_1^c, μ_2^c, respectively, and

$$\mu_1 + \mu_1^c = \mu_2 + \mu_2^c.$$

The situation mentioned above is not much probable so we do not take it into account.

Koziol–Green model. The so-called Koziol–Green model (KGM) is a simple model of informative censoring, where the survival function of the censoring times is supposed to be a power of the survival function of the lifetimes, i.e. in our case

$$\forall\, t \ge 0 \qquad 1 - G_i(t) = (1 - F_i(t))^{\beta_i} \quad \text{with } \beta_i > 0,\ i = 1, 2.$$

The parameters β_1 and β_2, respectively, are usually called *the censoring parameters.* This particular model of random censorship (without the change point) was introduced by Koziol and Green [**26**] and it is a single proportional hazard model, because by standard tools we get

$$-\frac{\mathrm{d}\log(1-G_i(t))}{\mathrm{d}t} = -\beta_i \frac{\mathrm{d}\log(1-F_i(t))}{\mathrm{d}t}, \qquad i=1,2,$$
$$\frac{g_i(t)}{1-G_i(t)} = \beta_i \frac{f_i(t)}{1-F_i(t)}, \qquad i=1,2,$$

and

$$\lambda_{G_i}(t) = \beta_i\,\lambda_{F_i}(t), \quad i=1,2,$$

where

$$\lambda_{F_1}(t) = \lim_{\Delta t\to 0^+} \frac{\mathrm{P}(t\le X_j^0 < t+\Delta t \,|\, X_j^0 \ge t)}{\Delta t} = \frac{f_1(t)}{1-F_1(t)}, \qquad 1\le j\le \lfloor n\gamma\rfloor, \tag{1.4}$$
$$\lambda_{F_2}(t) = \lim_{\Delta t\to 0^+} \frac{\mathrm{P}(t\le X_j^0 < t+\Delta t \,|\, X_j^0 \ge t)}{\Delta t} = \frac{f_2(t)}{1-F_2(t)}, \qquad \lfloor n\gamma\rfloor < j\le n, \tag{1.5}$$
$$\lambda_{G_1}(t) = \lim_{\Delta t\to 0^+} \frac{\mathrm{P}(t\le C_j < t+\Delta t \,|\, C_j \ge t)}{\Delta t} = \frac{g_1(t)}{1-G_1(t)}, \qquad 1\le j\le \lfloor n\gamma\rfloor,$$
$$\lambda_{G_2}(t) = \lim_{\Delta t\to 0^+} \frac{\mathrm{P}(t\le C_j < t+\Delta t \,|\, C_j \ge t)}{\Delta t} = \frac{g_2(t)}{1-G_2(t)}, \qquad \lfloor n\gamma\rfloor < j\le n,$$

are the so-called *hazard functions* of the lifetimes and the censoring times before and after the change point $\lfloor n\gamma\rfloor$. The hazard function $\lambda_{F_i}(t)$ specifies the instantaneous rate at which failures occur for items that are surviving at time t. It fully determines the distribution $F_i(t)$. Integrating

$$\lambda_{F_i}(t) = -\frac{\mathrm{d}\log(1-F_i(t))}{\mathrm{d}t}, \quad i=1,2,$$

with respect to t and by $F_i(0)=0$, we get

$$S_i(t) = 1-F_i(t) = \exp\Bigl(-\int_0^t \lambda_{F_i}(u)\,\mathrm{d}u\Bigr) = \exp(-\Lambda_{F_i}(t)), \quad i=1,2,$$

where $S_i(t)$ is *the survivor function* and $\Lambda_{F_i}(t) = \int_0^t \lambda_{F_i}(u)\,\mathrm{d}u$ is called *the cumulative hazard function* corresponding to $F_i(t)$.

Clearly, in this case the expected proportion of uncensored observations is

$$\mathrm{E}\,\Delta_j = \mathrm{P}(X_j^0 \le C_j) = \int_0^\infty (1-F_1(t))^{\beta_1}\,\mathrm{d}F_1(t) = \frac{1}{1+\beta_1}, \qquad 1\le j\le\lfloor n\gamma\rfloor,$$
$$\mathrm{E}\,\Delta_j = \mathrm{P}(X_j^0 \le C_j) = \int_0^\infty (1-F_2(t))^{\beta_2}\,\mathrm{d}F_2(t) = \frac{1}{1+\beta_2}, \qquad \lfloor n\gamma\rfloor < j\le n.$$

REMARK 1.2. In the Koziol–Green model η is equal to γ, therefore under the hypothesis H_0 the censoring variables $C_1, C_2, \ldots, C_n$ are i.i.d. too, and under the alternative H_1 the distributions of the survival times and the censoring times change at the same time point $\lfloor n\gamma\rfloor$ and the distribution functions of the observed variables X_j's are $H_1(x)$ before the change point and $H_3(x)$ after one, see (1.2).

Moreover, if $\beta_1=\beta_2=0$, the survival variables X_j^0's are not censored.

2. Notation

We use the following notation in the rest of the thesis. Let $m = \lfloor n\gamma \rfloor$ denote the change in the distribution of the survival variables $X_1^0, X_2^0, \ldots, X_n^0$ and let $m_c = \lfloor n\eta \rfloor$ denote the change in the distribution of the censoring variables $C_1, C_2, \ldots, C_n$. Write

$$\begin{aligned} 1 - H_{\eta,\gamma}(t) &= \eta(1 - F_1(t))(1 - G_1(t)) + (\gamma - \eta)(1 - F_1(t))(1 - G_2(t)) \\ &\quad + (1 - \gamma)(1 - F_2(t))(1 - G_2(t)), \end{aligned} \tag{1.6}$$

$$1 - H_\gamma(t) = \gamma(1 - F_1(t))(1 - G_1(t)) + (1 - \gamma)(1 - F_2(t))(1 - G_2(t)), \tag{1.7}$$

$$1 - H_\eta(t) = (1 - F(t))(\eta(1 - G_1(t)) + (1 - \eta)(1 - G_2(t))), \tag{1.8}$$

$$1 - H(t) = (1 - F(t))(1 - G(t)) \tag{1.9}$$

for all $t \geq 0$ and

$$\begin{aligned} R_{\eta,\gamma}(t) &= \int_0^t \Big(\eta(1 - G_1(u)) + (\gamma - \eta)(1 - G_2(u))\Big) \,\mathrm{d}F_1(u) \\ &\quad + (1 - \gamma) \int_0^t (1 - G_2(u)) \,\mathrm{d}F_2(u), \end{aligned} \tag{1.10}$$

$$R_\gamma(t) = \gamma \int_0^t (1 - G_1(u)) \,\mathrm{d}F_1(u) + (1 - \gamma) \int_0^t (1 - G_2(u)) \,\mathrm{d}F_2(u), \tag{1.11}$$

$$R_\eta(t) = \int_0^t \Big(\eta(1 - G_1(u)) + (1 - \eta)(1 - G_2(u))\Big) \,\mathrm{d}F(u), \tag{1.12}$$

$$R(t) = \int_0^t (1 - G(u)) \,\mathrm{d}F(u). \tag{1.13}$$

Notice that $1 - H_{\eta,\gamma}(t)$ and $1 - H_\gamma(t)$ are "distribution functions" of $Y(t)/n$ under the alternative H_1 for $m_c < m$ and $m_c = m$, respectively. Similarly, under the hypothesis H_0, $1 - H_\eta(t)$ is a "distribution function" of $Y(t)/n$ in the case of $m_c < n$ and $1 - H(t)$ is a distribution function of $Y(t)/n$ in the case of $m_c = n$. We make use of the notation "distribution function" because there have been used approximations for m_c/n and m/n.

Further, set

$$\begin{aligned} Q_1(t) &= \big(\eta(1 - G_1(t)) + (\gamma - \eta)(1 - G_2(t))\big)\,(1 - F_1(t)), \\ Q_2(t) &= (1 - \gamma)(1 - G_2(t))(1 - F_2(t)). \end{aligned} \tag{1.14}$$

Notice that

$$\begin{aligned} 1 - H_{\eta,\gamma}(t) &= Q_1(t) + Q_2(t), \\ R_{\eta,\gamma}(t) &= \int_0^t (Q_1(u)\lambda_{F_1}(u) + Q_2(u)\lambda_{F_2}(u)) \,\mathrm{d}u, \end{aligned}$$

where $\lambda_{F_1}(t)$ and $\lambda_{F_2}(t)$ are defined in (1.4) and (1.5).

Convention 1.1. In the following text we use simpler notation

$$\lambda_1(t) = \lambda_{F_1}(t) \quad \text{and} \quad \lambda_2(t) = \lambda_{F_2}(t)$$

for the hazard functions $\lambda_{F_1}(t)$ and $\lambda_{F_2}(t)$ corresponding to the lifetime distribution functions $F_1(t)$ and $F_2(t)$.

3. State of arts

Stute [**32**] initiated research in the area of change point analysis for randomly censored data. He suggested estimators for the change point and he studied their properties. He proposed the class of estimators of the form

$$\begin{aligned}\theta_n^* = \hat{\gamma} &= \frac{1}{n} \operatorname*{argmax}_{1\le k<n} \Bigg| \sum_{i=1}^{k} \sum_{j=k+1}^{n} \frac{K(X_i, X_j)\, \Delta_i \Delta_j}{(n-R_i+1)\,(n-R_j+1))} I(\max(X_i, X_j) < \tau_0) \Bigg| \\ &= \operatorname*{argmax}_{1\le k<n} \frac{k(n-k)}{n^2} \left| \int_{-\infty}^{\tau_0}\int_{-\infty}^{\tau_0} \frac{K(x,y)}{(1-\hat{H}_n(x-))\,(1-\hat{H}_n^0(y-))} \,\mathrm{d}\tilde{H}_k(x)\,\mathrm{d}\tilde{H}_k^0(y) \right|,\end{aligned}$$

where R_j is a rank of X_j among $X_1, X_2, \ldots, X_n$ and

$$\hat{H}_k(x) = \frac{1}{k}\sum_{i=1}^{k} I(X_i \le x), \qquad \tilde{H}_k(x) = \frac{1}{k}\sum_{i=1}^{k} I(X_i \le x)\Delta_i, \tag{1.15}$$

$$\hat{H}_k^0(x) = \frac{1}{n-k}\sum_{i=k+1}^{n} I(X_i \le x), \qquad \tilde{H}_k^0(x) = \frac{1}{n-k}\sum_{i=k+1}^{n} I(X_i \le x)\Delta_i \tag{1.16}$$

and $K : \mathbb{R}^2 \to \mathbb{R}$ is a measurable mapping with the antisymmetry property $K(x,y) = -K(y,x)$. We call such mapping K kernel. The value τ_0 is chosen as a positive number fulfilling

$$0 < \tau_0 < \tau := \sup\{x;\ F_i(x) < 1,\ G_i(x) < 1,\ i = 1,2\}.$$

He considered only bounded antisymmetric kernels K which satisfy

$$\int_{-\infty}^{\tau_0}\int_{-\infty}^{\tau_0} \frac{K(x,y)}{(1-F_\gamma(x))\,(1-F_\gamma(y))} \,\mathrm{d}F_1(x)\,\mathrm{d}F_2(y) \ne 0,$$

where $F_\gamma(x) = \gamma F_1(x) + (1-\gamma)F_2(x)$. He proved that under the alternative hypothesis H_1 and the equal censorship $G_1 = G_2$, as $n \to \infty$,

$$|\theta_n^* - \gamma| = O\Big(\frac{\log n}{n}\Big) \quad a.s.$$

His results were extended by **Ferger** [**13**] and **Horváth** [**20**]. They divided the random sample into two groups up to and after the k-th observation and made the comparison, which leads to the estimator

$$\theta_{0,n} = \hat{\gamma} = \frac{1}{n} \operatorname*{argmax}_{1\le k<n} \Big| \frac{1}{k}\sum_{j=1}^{k} X_j - \frac{1}{n-k}\sum_{j=k+1}^{n} X_j \Big|$$

and to their generalization using U-type statistic according to Ferger [**13**]

$$\theta_n = \hat{\gamma} = \frac{1}{n} \operatorname*{argmax}_{1\le k<n} v\Big(\frac{k}{n}\Big) \Big| \sum_{i=k+1}^{n}\sum_{j=1}^{k} K(X_i, X_j) \Big|,$$

where $v : (0,1) \to (0,\infty)$ is a weight-function of the type

$$v(t) = \frac{1}{t^a(1-t)^b}, \quad 0 \le a, b \le 1,$$

and K is an antisymmetric kernel. The former estimator is obtained from the later one letting $K(x,y) = x - y$ and $a = b = 1$. Ferger [**13**] showed that under H_1 and the assumption $G_1 = G_2$, as $n \to \infty$,

$$|\theta_n - \gamma| = O\Big(\frac{1}{n}\Big) \quad a.s.$$

Horváth [**20**] studied the functional

$$Q_n(k) = \frac{k(n-k)}{n^{3/2}}\,(\theta(k) - \theta), \quad 1 \le k < n,$$

with

$$\theta = \int_{-\infty}^{\tau_0}\int_{-\infty}^{\tau_0} \frac{K(x,y)}{(1-F(x))\,(1-F(y))}\,\mathrm{d}F(x)\,\mathrm{d}F(y),$$

$$\theta(k) = \int_{-\infty}^{\tau_0}\int_{-\infty}^{\tau_0} \frac{K(x,y)}{(1-\hat{H}_k(x-))\,(1-\hat{H}_k^0(y-))}\,\mathrm{d}\tilde{H}_k(x)\,\mathrm{d}\tilde{H}_k^0(y),$$

where $\hat{H}_k$, $\tilde{H}_k$ and $\hat{H}_k^0$, $\tilde{H}_k^0$ are defined in (1.15) and (1.16). He considered not only antisymmetric kernels $K(x,y) = -K(y,x)$ but also symmetric ones $K(x,y) = K(y,x)$. Setting $q(t)$ a positive function on (0,1) which is non-decreasing in a neighborhood of 0, non-increasing in a neighborhood of 1 and

$$\inf_{\varepsilon \le t \le 1-\varepsilon} q(t) > 0 \quad \text{for all } 0 < \varepsilon < \frac{1}{2}, \tag{1.17}$$

he proved that under H_0 and $G_1 = G_2$, as $n \to \infty$,

$$\frac{1}{\sigma}\sup_{0<t<1}\frac{|Q_n(nt)|}{q(t)} \xrightarrow{\mathcal{D}} \sup_{0<t<1}\frac{|B(t)|}{q(t)}, \qquad K \text{ antisymmetric},$$

$$\frac{1}{\sigma}\sup_{0<t<1}\frac{|Q_n(nt)|}{q(t)} \xrightarrow{\mathcal{D}} \sup_{0<t<1}\frac{|(1-t)W(t) + t(W(1)-W(t))|}{q(t)}, \qquad K \text{ symmetric},$$

where

$$\sigma^2 = \int_{-\infty}^{\tau_0}\left(\int_{-\infty}^{\tau_0}\frac{K(x,y)}{1-F(y-)}\,\mathrm{d}F(y)\right)^2 \frac{\mathrm{d}L(x,1)}{((1-F(x-))\,(1-G(x-)))^2}$$

with $L(x,1) = L_1(x,1)$ defined in (1.3) and B and W denoting Brownian bridge and Wiener process, respectively, if and only if

$$I_{0,1}(q,c) = \int_0^1 \frac{1}{t(1-t)}\exp\left(-\frac{cq^2(t)}{t(1-t)}\right)\mathrm{d}t < \infty \quad \text{for some } c > 0. \tag{1.18}$$

Moreover, he showed that under H_0 and $G_1 = G_2$, as $n \to \infty$,

$$\mathrm{P}\left(d_1(\log n)\,\frac{1}{\sigma}\max_{1\le k<n}\frac{|Q_n(k)|}{\sqrt{\frac{k}{n}(1-\frac{k}{n})}} \le y + d_2(\log n)\right) \to \exp\left\{-2\mathrm{e}^{-y}\right\}, \quad \forall y \in \mathbb{R},$$

where

$$d_1(t) = \sqrt{2\log t}, \qquad d_2(t) = 2\log t + \tfrac{1}{2}\log\log t - \tfrac{1}{2}\log\pi. \tag{1.19}$$

The last limit property is valid for both classes symmetric and antisymmetric kernels K.

Aly [**2**] applied the idea of combining proposed by Albers and Akritas [**1**] to the change point setup. He treated the uncensored and the censored observations separately. Note that $N_{1,k}$ (resp. $N_{2,k}$) is the number of the uncensored (resp. censored) observations $X_1^1, X_2^1, \ldots, X_{N_{1,k}}^1$ (resp. $X_1^2, X_2^2, \ldots, X_{N_{2,k}}^2$) among $X_1, X_2, \ldots, X_k$ and $\hat{H}_{1k}$ and $\hat{Q}_{1k}$ (resp. $\hat{H}_{2k}$ and $\hat{Q}_{2k}$)

$$\hat{H}_{ik}(x) = \frac{1}{N_{i,k}}\sum_{j=1}^{N_{i,k}} I(X_j^i < x), \qquad i = 1,2,$$

$$\hat{Q}_{ik}(y) = \sup\{x : \hat{H}_{ik}(x) \le y\}, \qquad i = 1,2,$$

are their empirical and quantile processes, respectively. He proposed the test statistics for particular groups of observations (uncensored and censored events) as follows

$$Y_n^i(s,t) = \frac{1}{\sqrt{N_{in}}} \sum_{j=1}^{N_i, \lfloor ns \rfloor} \Psi_t(X_j^i - \hat{Q}_{in}(t)), \quad s,t \in (0,1), \quad i = 1,2,$$

where

$$\begin{aligned} \Psi_t(x) &= -(1-t) && \text{if } x < 0, \\ &= t && \text{if } x \geq 0. \end{aligned}$$

Finally, he mixed both the proposed test statistics $Y_n^i(s,t)$, $i = 1,2$ based on the quantile processes together. He investigated the various modifications of such established test statistics and under H_0 and $G_1(t) = G_2(t)$, as $n \to \infty$, he obtained

$$\begin{aligned}
T_{1,n} &= \max_{i=1,2} \sup_{0 \leq s,t \leq 1} |Y_n^i(s,t)| \xrightarrow{\mathcal{D}} \max_{i=1,2} \sup_{0 \leq s,t \leq 1} |\Gamma_i(s,t)|, \\
T_{2,n} &= \max_{i=1,2} \int_0^1 \int_0^1 (Y_n^i(s,t))^2 \, \mathrm{d}t \, \mathrm{d}s \xrightarrow{\mathcal{D}} \max_{i=1,2} \int_0^1 \int_0^1 (\Gamma_i(s,t))^2 \, \mathrm{d}t \, \mathrm{d}s, \\
T_{3,n}(s_0) &= \frac{1}{\sqrt{s_0(1-s_0)}} \max_{i=1,2} \sup_{0 \leq s,t \leq 1} |Y_n^i(s_0,t)| \xrightarrow{\mathcal{D}} \max_{i=1,2} \sup_{0 \leq t \leq 1} |B_i(t)|, \\
T_{4,n}(t_0) &= \frac{1}{\sqrt{t_0(1-t_0)}} \max_{i=1,2} \sup_{0 \leq s,t \leq 1} |Y_n^i(s,t_0)| \xrightarrow{\mathcal{D}} \max_{i=1,2} \sup_{0 \leq s \leq 1} |B_i(s)|, \\
T_{5,n}(t_0) &= \frac{1}{t_0(1-t_0)} \sum_{i=1}^{2} \int_0^1 (Y_n^i(s,t_0))^2 \, \mathrm{d}s \xrightarrow{\mathcal{D}} \sum_{i=1}^{2} \int_0^1 (B_i(s))^2 \, \mathrm{d}s, \\
T_{6,n} &= 12 \max_{i=1,2} \int_0^1 \int_0^1 Y_n^i(s,t) \, \mathrm{d}t \, \mathrm{d}s \xrightarrow{\mathcal{D}} \max_{i=1,2} Z_i, \\
T_{7,n} &= \frac{12}{\sqrt{2}} \sum_{i=1}^{2} \int_0^1 \int_0^1 Y_n^i(s,t) \, \mathrm{d}t \, \mathrm{d}s \xrightarrow{\mathcal{D}} Z_1, \\
T_{8,n}(t_0) &= \sqrt{\frac{12}{t_0(1-t_0)}} \max_{i=1,2} \int_0^1 Y_n^i(s,t_0) \, \mathrm{d}s \xrightarrow{\mathcal{D}} \max_{i=1,2} Z_i, \\
T_{9,n}(t_0) &= \sqrt{\frac{6}{t_0(1-t_0)}} \sum_{i=1}^{2} \int_0^1 Y_n^i(s,t_0) \, \mathrm{d}s \xrightarrow{\mathcal{D}} Z_1,
\end{aligned}$$

where B_1 and B_2 are two independent Brownian bridges, Z_1 and Z_2 are two independent $\mathrm{N}(0,1)$ random variables and $\Gamma_1(s,t)$ and $\Gamma_2(s,t)$ are two independent mean zero two-parameter Gaussian processes with the same covariance function

$$\mathrm{E}\,\Gamma_i(s,t)\Gamma_i(u,v) = (\min(s,u) - su)(\min(t,v) - tv), \quad i = 1,2.$$

Gombay and Liu [**18**] based their test on a generalization of the Wilcoxon rank statistic

$$\max_{1 \leq k < n} \frac{|\sum_{j=1}^{k} U_j|}{\sqrt{\sum_{j=1}^{n} U_j^2}}$$

with the generalized rank U_j of (X_j, Δ_j)

$$U_j = \sum_{i=1}^{n} \big(I(X_j > X_i, \Delta_i = 1) - I(X_j < X_i, \Delta_j = 1) \big), \quad j = 1, 2, \dots, n.$$

They used the theory of exchangeable variables to investigate its properties. Precisely, they proved that under the no-change hypothesis H_0 and $G_1 = G_2$, as $n \to \infty$,

$$\max_{1 \le k < n} \frac{|\sum_{j=1}^{k} U_j|}{\sqrt{\sum_{j=1}^{n} U_j^2}} \xrightarrow{\mathcal{D}} \sup_{0<t<1} |B(t)|,$$

where B denotes a Brownian bridge. Through this limit distribution which is given by the well-known identity

$$\mathrm{P}(\sup_{0<t<1} |B(t)| > b) = 2 \sum_{i=1}^{\infty} (-1)^{i-1} \exp(-2i^2 b^2), \quad b > 0,$$

we get the approximation of the critical values for our test H_0 versus H_1 under equal censorship. In the case of rejection H_0, they proposed the estimator of the change point $\lfloor n\gamma \rfloor$ as the point k where the test statistic takes its maximum, i.e.

$$\hat{\tau}_n = \widehat{\lfloor n\gamma \rfloor} = \operatorname*{argmax}_{1 \le k < n} |\sum_{j=1}^{k} U_j|$$

and they showed that under alternative H_1 and $G_1 = G_2$ this estimator is consistent with the following rate

$$\Big| \frac{\hat{\tau}_n}{n} - \gamma \Big| = O_{\mathrm{P}}\Big(\frac{1}{n}\Big), \quad n \to \infty.$$

Extensive studies for such procedures were conducted in the doctoral thesis of **Liu** [**28**]. There have been proposed also the weighted-type forms of the previous test statistic

$$\sup_{0<t<1} \frac{|\sum_{j=1}^{\lfloor (n+1)t \rfloor} U_j| / \sqrt{\sum_{j=1}^{n} U_j^2}}{q(t)},$$

where $q(t)$ is a positive function defined on (0,1) with property (1.17), and

$$\max_{1 \le k < n} \frac{|\sum_{j=1}^{k} U_j| / \sqrt{\sum_{j=1}^{n} U_j^2}}{\sqrt{\frac{k}{n}\big(1 - \frac{k-1}{n}\big)}}.$$

Under the no-change hypothesis H_0 and $G_1 = G_2$, as $n \to \infty$,

$$\sup_{0<t<1} \frac{|\sum_{j=1}^{\lfloor (n+1)t \rfloor} U_j| / \sqrt{\sum_{j=1}^{n} U_j^2}}{q(t)} \xrightarrow{\mathcal{D}} \sup_{0<t<1} \frac{|B(t)|}{q(t)}$$

if and only if (1.18) holds, and

$$\mathrm{P}\left(d_1(\log n) \max_{1 \le k < n} \frac{|\sum_{j=1}^{k} U_j| / \sqrt{\sum_{j=1}^{n} U_j^2}}{\sqrt{\frac{k}{n}\big(1 - \frac{k-1}{n}\big)}} \le y + d_2(\log n) \right) \to \exp\big\{ -2\mathrm{e}^{-y} \big\}, \quad \forall y \in \mathbb{R},$$

where $d_1(t)$ and $d_2(t)$ are defined in (1.19).

4. Aim of work

In all the papers listed above the censoring times $C_1, C_2, \ldots, C_n$ are supposed to be i.i.d. variables. **Hušková and Neuhaus** [**23**] developed a test along the lines of the two-sample weighted log-rank tests under the random censoring (see e.g. Neuhaus [**29**] and [**30**]). In contrast to the other mentioned authors, they considered not only the change in the distribution of the survival variables but also the change in the distribution of the censoring variables. We present their point of view in the next chapter and use their results described in Theorem 2.3 and Theorem 2.6 for our research.

The work of Hušková and Neuhaus [**23**] is based on max-type test statistics which are usually applied to one-change point problem. The thesis aims to further study the max-type statistics and develop MOSUM-type tests statistics for one and multiple changes. For investigation we use theory of ranks, mainly the extreme value theorem for the max-type and the MOSUM-type forms of simple linear rank statistics, see Hušková [**21**]. Except asymptotic tests we construct their exact counterparts through the permutation principle which can be used also in change point analysis according to the papers by Antoch, Hušková [**4**] or Hušková [**22**]. We also prove the consistency of the proposed tests using asymptotic representations of the test statistics.

Moreover, we investigate properties of the corresponding max-type estimators under the alternative H_1 and also under the hypothesis H_0. We apply ideas of Gombay and Hušková [**17**], but our investigation is complicated by a nuisance parameter m_c denoting a time of a change in the distribution of the censoring variables.

The useful tools are the limit behavior of empirical processes, theory of counting processes, the Chebyshev inequality and the Kolmogorov-Hájek-Rényi-Chow inequality.

Theoretical results are illustrated by simulations based on the Monte Carlo repetitions or on the resampling methods.

CHAPTER 2

Tests

1. Introduction

We focus on the rank based test procedures for the testing problem (1.1). At first, for simplicity, we attend to the situation, when the parameter γ is known and after that, we concentrate on γ unknown, which is the main problem. In the first case of γ known the problem reduces to *a two-sample problem.* There are a lot of papers offering a solution for such problem, e.g. Neuhaus [**29**] and [**30**]. We briefly summarize commonly used tests. In the second case of γ unknown we construct the rank test statistics for the censored data as for the completely observable data. We present the max-type test statistics which are used when we expect only one change-point and the MOSUM-type test statistics which are used as a diagnostic tool in multiple-change case. We investigate properties of both these classes of test statistics. We also propose other modifications for solving our change point testing problem.

2. Two-sample test statistics (m known)

If the possible change point $m = \lfloor n\gamma \rfloor$ is known, then our problem (1.1), how it was said above, is a common two-sample problem, i.e. we suppose that the lifetimes $X_1^0, X_2^0, \ldots, X_m^0$ and $X_{m+1}^0, X_{m+2}^0, \ldots, X_n^0$ have arbitrary absolutely continuous distribution functions F_1 and F_2, respectively, $F_1 \neq F_2$, and the censoring times $C_1, C_2, \ldots, C_m$ and $C_{m+1}, C_{m+2}, \ldots, C_n$ have absolutely continuous distribution functions G_1 and G_2, respectively, $G_1 \neq G_2$, and our problem is to test the null hypothesis of randomness

$$H_0 : F_1(t) = F_2(t) = F(t) \quad \text{for all } t \in \mathbb{R},$$

where the distribution function $F(t)$ is unknown, against the omnibus alternative hypothesis

$$H_{1m} : F_1(t) \neq F_2(t) \text{ for some } t \in \mathbb{R}$$

with m known. In this case we assume that $\eta = \gamma$ (the distribution G_1 of the censoring variables changed into the distribution G_2 at known time m) or $\eta = 1$ (the censoring variables are i.i.d. under both the hypotheses H_0 and H_{1m} , i.e. the so-called *equal censorship* occurs). We present tests for these particular situations. The rank based test statistic for the two-sample problem has the form

$$S_m(\tau_0) = \sum_{j=1}^{m} a_n(j)$$

with the scores

$$a_n(j) = \int_0^{\tau_0} w_n(t)\, \mathrm{d}N_j(t) - \int_0^{\tau_0} w_n(t)\, \frac{Y_j(t)}{Y(t)}\, \mathrm{d}N(t). \tag{2.1}$$

The process

$$Y(t) = \sum_{j=1}^{n} Y_j(t), \quad Y_j(t) = I(X_j \geq t), \tag{2.2}$$

denotes the number at risk just before time t or the size of the risk set, and

$$N(t) = \sum_{j=1}^{n} N_j(t), \quad N_j(t) = \Delta_j I(X_j \le t), \tag{2.3}$$

counts the observed failures by time t. The value τ_0 denoting the end of medical study is such a positive number for which

$$0 < \tau_0 < \tau := \sup\{x;\, F_i(x) < 1, G_i(x) < 1,\, i = 1, 2\}. \tag{2.4}$$

Since $N(t)$ is a counting process and $w_n(t)$ is the nonnegative function (random or nonrandom) of time, it follows that $\int_0^{\tau_0} w_n(t)\,\mathrm{d}N(t)$ is the Stieltjes integral representation of the sum of the values of w_n at the jump times of N in the interval $[0, \tau_0]$. If we use this notation, we can rewrite the scores in the well-arranged form

$$a_n(j) = I(X_j \le \tau_0)\left(w_n(X_j)\Delta_j - \sum_{i:\, X_i \le X_j} w_n(X_i)\,\frac{\Delta_i}{Y(X_i)} \right), \quad j = 1, 2, \ldots, n.$$

The statistic $S_m(\tau_0)$ is called *the weighted log-rank test statistic* which was studied in a number of papers, e.g. Neuhaus [**29**] and [**30**] or Fleming and Harrington [**14**] or Kalbfleisch and Prentice [**25**]. Now, we focus on the form of the weight function w_n.

The weight function. *An important class of weight functions is*

$$w_n(t) = (\hat{S}_n(t-))^{\rho} \left(\frac{Y(t)}{n}\right)^{\kappa} I(Y(t) > 0), \tag{2.5}$$

where $\rho, \kappa \ge 0$ *and*

$$\hat{S}_n(t-) = \prod_{i: X_i < t} \left(1 - \frac{\Delta_i}{Y(X_i)}\right)$$

is the left-continuous Kaplan–Meier estimate of the survival function. Notice that the weights of the form (2.5) *are bounded* $|w_n(t)| \le 1$ *for all* $t \ge 0$.

Such a class of weighted test statistics includes commonly used test statistics in practice like the log-rank statistic *(*$\rho = 0,\ \kappa = 0$*),* the Prentice–Wilcoxon statistic *(*$\rho = 1,\ \kappa = 0$*) and* the Gehan–Wilcoxon statistic *(*$\rho = 0,\ \kappa = 1$*) which are generalizations of the Savage and the Wilcoxon statistic for uncensored data, for more information see e.g. Hájek et al* [**19**]*. Other discussion will be done in Chapter 4.*

Larger values of $|S_m(\tau_0)|$ indicates that the null hypothesis is violated. The critical region for testing H_0 against H_{1m} has the form as follows

$$|S_m(\tau_0)| > c_n(\alpha),$$

where the critical value $c_n(\alpha)$ is determined in such a way that the test has the prescribed significance level α. The task is to find an appropriate approximation of our critical value. The common way is the approximation through the limit behavior of the statistic $S_m(\tau_0)$, but it requires the large sizes of both samples m and $n - m$.

Under the hypothesis of randomness and some mild conditions the standardized version of this statistic has asymptotically standard normal distribution. This limit property can be obtained through the martingale theory. Let us denote by

$$L_m(\tau_0) = \frac{S_m(\tau_0)}{\sqrt{nV_m(\tau_0)}}$$

and

$$V_m(\tau_0) = \frac{1}{n}\int_0^{\tau_0} w_n^2(t)\, \frac{\sum_{j=1}^m Y_j(t)\, \sum_{j=m+1}^n Y_j(t)}{Y^2(t)}\, \mathrm{d}N(t).$$

THEOREM 2.1. *Suppose that $X_1^0, X_2^0, \dots, X_n^0$, $C_1, C_2, \dots, C_n$ are independent random variables. Let survival variables $X_1^0, X_2^0, \dots, X_n^0$ have an arbitrary absolutely continuous distribution function F. Let censoring variables $C_1, C_2, \dots, C_m$ and $C_{m+1}, C_{m+2}, \dots, C_n$ have absolutely continuous distribution functions G_1 and G_2,respectively, which can be but need not be the same. Let $w_n(s) = v(\hat{S}_n(t-))$ or $w_n(s) = v(\frac{Y(t)}{n})$ for some nonnegative continuous function v with bounded variation on $[0,1]$. Then for all $y \in \mathbb{R}$ we have, as $\min(m, n-m) \to \infty$,*

$$\mathrm{P}(L_m(\tau_0) \le y) \to \Phi(y)$$

with Φ denoting the distribution function of the standard normal distribution $\mathrm{N}(0,1)$.

PROOF. The proof can be found in Fleming and Harrington [**14**], Theorem 7.2.1. □

According to Theorem 2.1 we use quantiles of the standard normal distribution for our decision rule. We reject the hypothesis of randomness $H_0 : F_1 = F_2$ against the alternative $H_{1m} : F_1 \neq F_2$, if

$$|L_m(\tau_0)| = \frac{|S_m(\tau_0)|}{\sqrt{nV_m(\tau_0)}} \ge u_{\frac{\alpha}{2}}$$

with $u_{\frac{\alpha}{2}}$ denoting $100(1-\frac{\alpha}{2})$%-quantile of the standard normal distribution $\mathrm{N}(0,1)$.

Another possibility how to obtain an approximation of the critical value is through one of the resampling methods which gets reasonable approximation even for small sample sizes. But here it is important to mention that this cannot be used in general case as we see below. In the next subsection we describe this method and we call it permutation principle.

Permutation principle. We use the knowledge of *permutation tests* for the two-sample problem. We assume the hypothesis H_0 only under the equal censorship, i.e. we assume the so-called *restricted null hypothesis*

$$\bar{H}_0 :\; F_1(t) = F_2(t),\; G_1(t) = G_2(t), \quad \forall t \in \mathbb{R}.$$

Use in the following the notation

$$\sigma_n^2(\boldsymbol{a}) = \frac{1}{n-1}\sum_{j=1}^n a_n^2(j) \tag{2.6}$$

and denote by

$$(\boldsymbol{X}, \boldsymbol{\Delta})_{(.)} = ((X_{(1)}, \Delta_{[1]}), (X_{(2)}, \Delta_{[2]}), \dots, (X_{(n)}, \Delta_{[n]}))$$

the random sample of observations ordered according to $X_1, X_2, \dots, X_n$, i.e.

$$X_{(1)} \le X_{(2)} \le \dots \le X_{(n)}$$

and $\Delta_{[j]}$'s are corresponding censoring indicators to the variables $X_{(j)}$'s and $\boldsymbol{R} = (R_1, R_2, \dots, R_n)$ the corresponding ranks. Notice that under $\bar{H}_0$ the paired observations

$$(\boldsymbol{X}, \boldsymbol{\Delta}) = ((X_1, \Delta_1), (X_2, \Delta_2), \dots, (X_n, \Delta_n))$$

are i.i.d., $(\boldsymbol{X}, \boldsymbol{\Delta})_{(.)}$ and $\boldsymbol{R}$ are independent and

$$(X_{(R_j)}, \Delta_{[R_j]}) = (X_j, \Delta_j), \quad j = 1, 2, \dots n.$$

Write

$$L_m^{\sigma}(\tau_0) = \sqrt{\frac{n}{m(n-m)}}\,\frac{1}{\sigma_n(\boldsymbol{a})}\sum_{j=1}^{m} a_n(j).$$

Notice that the statistic $L_m^{\sigma}(\tau_0)$ differs from the statistic $L_m(\tau_0)$ only by the standardization.

Then according the principle of permutation tests (see e.g. Lehmann [**27**] or Good [**16**]) the permutation tests related to the test statistic $|L_m^{\sigma}(\tau_0)|$ can be described as the conditional test given $(\boldsymbol{X},\boldsymbol{\Delta})_{(.)}$ and the randomized critical function has the following form

$$\psi(t,(\boldsymbol{X},\boldsymbol{\Delta})_{(.)}) = \begin{cases} 1, & \text{if } |L_m^{\sigma}(\tau_0)| > c_n^*(\alpha,(\boldsymbol{X},\boldsymbol{\Delta})_{(.)}), \\ \nu \in (0,1), & \text{if } |L_m^{\sigma}(\tau_0)| = c_n^*(\alpha,(\boldsymbol{X},\boldsymbol{\Delta})_{(.)}), \\ 0, & \text{if } |L_m^{\sigma}(\tau_0)| < c_n^*(\alpha,(\boldsymbol{X},\boldsymbol{\Delta})_{(.)}), \end{cases} \tag{2.7}$$

where $c_n^*(\alpha,(\boldsymbol{X},\boldsymbol{\Delta})_{(.)})$ stands for the $100(1-\alpha)\%$-quantile corresponding to the conditional distribution of $|L_m^{\sigma}(\tau_0)|$ given $(X_{(1)},\Delta_{[1]}),(X_{(2)},\Delta_{[2]}),\dots,(X_{(n)},\Delta_{[n]})$ and ν is chosen such that under $\bar{H}_0$

$$\mathrm{P}\big(|L_m^{\sigma}(\tau_0)| > c_n^*(\alpha,(\boldsymbol{X},\boldsymbol{\Delta})_{(.)}) \,|\, (\boldsymbol{X},\boldsymbol{\Delta})_{(.)}\big) + \nu\, \mathrm{P}\big(|L_m^{\sigma}(\tau_0)| = c_n^*(\alpha,(\boldsymbol{X},\boldsymbol{\Delta})_{(.)}) \,|\, (\boldsymbol{X},\boldsymbol{\Delta})_{(.)}\big) = \alpha.$$

The conditional distribution $\mathrm{P}\big(|L_m^{\sigma}(\tau_0)| \le x | (\boldsymbol{X},\boldsymbol{\Delta}_{(.)})\big)$ is sometimes called *permutation distribution* and it can be expressed as follows

$$\mathrm{P}\big(|L_m^{\sigma}(\tau_0)| \le x \,|\, (\boldsymbol{X},\boldsymbol{\Delta}_{(.)})\big) = \frac{1}{n!}\,\#\{\boldsymbol{r} \in \mathcal{Q}_n; |L_m^{\sigma}(\tau_0,\boldsymbol{r})| \le x\}, \quad x \in \mathbb{R},$$

where $\mathcal{Q}_n$ is the set of all permutations of $(1,2,\dots,n)$, $\#A$ denotes the cardinality of a set A and $L_m^{\sigma}(\tau_0,\boldsymbol{r})$ is defined as $L_m^{\sigma}(\tau_0)$ with (X_j,Δ_j) replaced by $(X_{(r_j)},\Delta_{[r_j]})$, $j=1,2,\dots,n$.

Notice that the test in (2.7) can be also described as the test with the critical function

$$\psi(t,(\boldsymbol{X},\boldsymbol{\Delta})) = \begin{cases} 1, & \text{if } |L_m^{\sigma}(\tau_0,\boldsymbol{Q})| > c_n^*(\alpha,(\boldsymbol{X},\boldsymbol{\Delta})), \\ \nu \in (0,1), & \text{if } |L_m^{\sigma}(\tau_0,\boldsymbol{Q})| = c_n^*(\alpha,(\boldsymbol{X},\boldsymbol{\Delta})), \\ 0, & \text{if } |L_m^{\sigma}(\tau_0,\boldsymbol{Q})| < c_n^*(\alpha,(\boldsymbol{X},\boldsymbol{\Delta})), \end{cases} \tag{2.8}$$

with $c_n^*(\alpha,(\boldsymbol{X},\boldsymbol{\Delta}))$ denoting the critical value corresponding to the conditional distribution of $|L_m^{\sigma}(\tau_0,\boldsymbol{Q})|$ given $(X_1,\Delta_1),(X_2,\Delta_2),\dots,(X_n,\Delta_n)$ and ν is chosen such that under $\bar{H}_0$

$$\mathrm{P}\big(|L_m^{\sigma}(\tau_0,\boldsymbol{Q})| > c_n^*(\alpha,(\boldsymbol{X},\boldsymbol{\Delta})) \,|\, (\boldsymbol{X},\boldsymbol{\Delta})\big) + \nu\, \mathrm{P}\big(|L_m^{\sigma}(\tau_0,\boldsymbol{Q})| = c_n^*(\alpha,(\boldsymbol{X},\boldsymbol{\Delta})) \,|\, (\boldsymbol{X},\boldsymbol{\Delta})\big) = \alpha.$$

Here $L_m^{\sigma}(\tau_0,\boldsymbol{Q})$ is defined as $L_m^{\sigma}(\tau_0)$ with (X_j,Δ_j) replaced by $(X_{(Q_j)},\Delta_{[Q_j]})$, $j=1,2,\dots,n$, and $\boldsymbol{Q}=(Q_1,Q_2,\dots,Q_n)$ being a random permutation of $(1,2,\dots,n)$, i.e.

$$L_m^{\sigma}(\tau_0,\boldsymbol{Q}) = \sqrt{\frac{n}{m(n-m)}}\,\frac{1}{\sigma_n(\boldsymbol{a})}\sum_{j=1}^{m} a_n(Q_j).$$

The conditional distribution of $|L_m^{\sigma}(\tau_0,\boldsymbol{Q})|$ given $(\boldsymbol{X},\boldsymbol{\Delta})$ has the form

$$\begin{aligned} \mathrm{P}\big(|L_m^{\sigma}(\tau_0,\boldsymbol{Q})| \le x \,|\, (\boldsymbol{X},\boldsymbol{\Delta})\big) &= \frac{1}{n!}\,\#\{\boldsymbol{q} \in \mathcal{Q}_n; |L_m^{\sigma}(\tau_0,\boldsymbol{q})| \le x\}, && x \in \mathbb{R}, \\ &= \frac{1}{\binom{n}{m}}\,\#\{\boldsymbol{q_m} \in \mathcal{Q}_n^m; |L_m^{\sigma}(\tau_0,\boldsymbol{q_m})| \le x\}, && x \in \mathbb{R}, \end{aligned}$$

where $\mathcal{Q}_n$ is the set of all permutations of $(1,2,\dots,n)$ and $\mathcal{Q}_n^m$ is the set of all combinations of $(1,2,\dots,n)$ of size m. Under the restricted null hypothesis $\bar{H}_0$ the distributions of the statistics $L_m^{\sigma}(\tau_0)$ and $L_m^{\sigma}(\tau_0,\boldsymbol{Q})$ are the same and the permutation distribution provides the exact critical values for our testing problem (the level of our testing problem is α). It is clear, that the critical values $c_n^*(\alpha, (\boldsymbol{X}, \boldsymbol{\Delta}))$ depend on the observations $(X_1, \Delta_1), (X_2, \Delta_2), \dots, (X_n, \Delta_n)$.

Now we investigate the behavior of the critical value $c_n^*(\alpha, (\boldsymbol{X}, \boldsymbol{\Delta}))$. Toward this we derive the limit behavior of the permutation distribution of $|L_m^\sigma(\tau_0, \boldsymbol{Q})|$ (resp. $L_m^\sigma(\tau_0, \boldsymbol{Q})$) through the classic theory of ranks and we show that its conditional and unconditional limit distribution coincide.

THEOREM 2.2. *Suppose that $X_1^0, X_2^0, \dots, X_n^0$, $C_1, C_2, \dots, C_n$ are independent random variables. Let $X_1^0, X_2^0, \dots, X_m^0$ and $X_{m+1}^0, X_{m+2}^0, \dots, X_n^0$ have arbitrary absolutely continuous distribution functions F_1 and F_2, respectively (the distribution functions F_1 and F_2 can be the same). Let $C_1, C_2, \dots, C_m$ and $C_{m+1}, C_{m+2}, \dots, C_n$ have absolutely continuous distribution functions G_1 and G_2, respectively (the distribution functions G_1 and G_2 can be the same). Let*

$$\sup_{0 \le t \le \tau_0} |w_n(t) - w(t)| = o_{\mathrm{P}}(1), \tag{2.9}$$

where w is a continuous nonrandom function on $[0, \tau_0]$ and

$$\int_0^{\tau_0} w^2(t)(1 - G_i(t))\,\mathrm{d}F_i(t) > 0, \quad i = 1, 2. \tag{2.10}$$

Then for all $y \in \mathbb{R}$ we have, as $\min(m, n-m) \to \infty$,

$$\mathrm{P}\big(L_m^\sigma(\tau_0, \boldsymbol{Q}) \le y \,|\, (\boldsymbol{X}, \boldsymbol{\Delta})\big) \xrightarrow{\mathrm{P}} \Phi(y).$$

PROOF. Our proof starts with the observation that the variable $S_m(\tau_0, \boldsymbol{Q})$ given $(\boldsymbol{X}, \boldsymbol{\Delta})$ can be viewed as a simple linear rank statistic, where the role of ranks is played by a random permutation $\boldsymbol{Q} = (Q_1, Q_2, \dots, Q_n)$ of (1,2, ...n) and therefore limit theorem on two-sample rank statistics can be applied. By the result of Hájek et al [**19**], the conditional distribution of $L_m(\tau_0, \boldsymbol{Q})$ given $(\boldsymbol{X}, \boldsymbol{\Delta})$ is asymptotically standard normal if the scores satisfy

$$\frac{\max_{1 \le j \le n} (a_n(j) - \bar{a}_n)^2}{\sum_{j=1}^n (a_n(j) - \bar{a}_n)^2} \xrightarrow{\mathrm{P}} 0, \quad n \to \infty. \tag{2.11}$$

By Corollary 5.6 bellow we obtain, as $n \to \infty$,

$$\frac{\max_{1 \le j \le n} (a_n(j) - \bar{a}_n)^2}{\sum_{j=1}^n (a_n(j) - \bar{a}_n)^2} = \frac{\frac{1}{n} \max_{1 \le j \le n} (a_n(j) - \bar{a}_n)^2}{\frac{1}{n} \sum_{j=1}^n (a_n(j) - \bar{a}_n)^2} = \frac{O_{\mathrm{P}}\left(\frac{1}{n}\right)}{c} = o_{\mathrm{P}}(1), \quad c > 0,$$

so the condition (2.11) for the asymptotic normality is fulfilled and the proof is finished. □

Notice that the assumptions of Theorem 2.2 cover both the null restricted hypothesis and alternatives. Moreover, the limit permutation distribution is the same in both cases and does not depend on $(X_1, \Delta_1), (X_2, \Delta_2), \dots, (X_n, \Delta_n)$ and that is why also the unconditional distribution of $L_m^\sigma(\tau_0, \boldsymbol{Q})$ is asymptotically standard normal. Recall that under the null restricted hypothesis $\bar{H}_0$ the distributions of $L_m^\sigma(\tau_0)$ and $L_m^\sigma(\tau_0, \boldsymbol{Q})$ coincide, so we get, as $\min(m, n-m) \to \infty$,

$$\mathrm{P}_{\bar{H}_0}(L_m^\sigma(\tau_0) \le y) \to \Phi(y), \quad \forall\, y \in \mathbb{R}.$$

Thus, the rejection region for testing $\bar{H}_0$ versus H_{1m} based on the limit distribution of $L_m^\sigma(\tau_0)$ is given by

$$|L_m^\sigma(\tau_0)| = \sqrt{\frac{n}{m(n-m)}}\,\frac{1}{\sigma_n(\boldsymbol{a})}\left|\sum_{j=1}^m a_n(j)\right| \ge u_{\frac{\alpha}{2}},$$

where $u_{\frac{\alpha}{2}}$ stands for $100(1 - \frac{\alpha}{2})\%$-quantile of the standard normal distribution $\mathrm{N}(0,1)$.

REMARK 2.1. The test with the critical function (2.8) is called the permutation test and it could be interpreted also as *the bootstrap without replacement.* Checking step by step through the proof one observes that *the bootstrap with replacement* also works here and hence, both variants of the bootstrap provide approximations to the desired critical values.

Finally, we compare the estimators $V_m(\tau_0)$ and $\frac{m(n-m)}{n^2}\,\sigma_n^2(\boldsymbol{a})$ for the variance of the statistic $\frac{1}{\sqrt{n}}\,S_m(\tau_0)$.

LEMMA 2.1. *If* (2.9) *and*

$$\int_0^{\tau_0} w^2(t)(1-G(t))\,\mathrm{d}F(t) > 0. \tag{2.12}$$

hold, then, under $\bar{H}_0$*, as* $n\to\infty$*,*

$$\left|V_m(\tau_0) - \frac{m(n-m)}{n^2}\,\sigma_n^2(\boldsymbol{a})\right| = o_{\mathrm{P}}(1). \tag{2.13}$$

PROOF. By Corollary 5.14 below we have, as $n\to\infty$,

$$\frac{n^2}{m(n-m)}\,V_m(\tau_0) = \int_0^{\tau_0} w^2(t)(1-G(t)\,\mathrm{d}F(t) + o_{\mathrm{P}}(1).$$

Moreover, by Corollary 5.5 below we find that, as $n\to\infty$,

$$\frac{n-1}{n}\,\sigma_n^2(\boldsymbol{a}) = \int_0^{\tau_0} w^2(t)\,\mathrm{d}R(t) + o_{\mathrm{P}}(1) = \int_0^{\tau_0} w^2(t)\,(1-G(t))\,\mathrm{d}F(t) + o_{\mathrm{P}}(1).$$

Thus, the assertion (2.13) holds. □

Consequently, by the Cramer–Slutsky theorem we get that under the restricted null hypothesis $\bar{H}_0$ the test statistics $L_m(\tau_0)$ and $L_m^{\sigma}(\tau_0)$ are asymptotically equivalent, i.e.

$$|L_m(\tau_0) - L_m^{\sigma}(\tau_0)| = o_{\mathrm{P}}(1), \quad \text{as } \min(m,(n-m))\to\infty.$$

3. Max-type test statistics (m unknown, one change case)

Generally, m is unknown, so the change in the distribution of the survival variables can occur at an arbitrary time-point $k=1,2,\ldots,n-1$. Along the lines of a two-sample rank test for randomly censored data, the test procedure is based on

$$L_k(\tau_0) = \frac{S_k(\tau_0)}{\sqrt{nV_k(\tau_0)}}, \quad k=1,2,\ldots,n-1, \tag{2.14}$$

with

$$S_k(\tau_0) = \sum_{j=1}^{k} a_n(j), \quad k=1,2,\ldots,n, \tag{2.15}$$

$$V_k(\tau_0) = \frac{1}{n}\int_0^{\tau_0} w_n^2(t)\,\frac{\sum_{j=1}^{k} Y_j(t)\,\sum_{j=k+1}^{n} Y_j(t)}{Y^2(t)}\,\mathrm{d}N(t) + v_k, \tag{2.16}$$

$$v_k = \frac{k(n-k)}{n^2}\,(I(k\le\log\log n) + I(k\ge n-\log\log n)).$$

The terms v_k, $k=1,2,\ldots,n$, ensure that $V_k(\tau_0)$ are bounded away from 0. The weight functions $w_n = w_n(X_j,\Delta_j; 1\le j\le n)\ge 0$ fulfil, as $n\to\infty$,

$$\sup_{0\le t\le\tau_0} |w_n(t)-w(t)| = o_{\mathrm{P}}\Big((\log\log n)^{-1}\Big), \tag{2.17}$$

where $w(t)$ is a continuous nonrandom function on $[0, \tau_0]$. Under H_0 the condition (2.17) is satisfied for the commonly used weights given by (2.5), see Chapter 4.

We apply *the union-intersection principle*, for more details see e.g. Csörgő and Horváth [**10**]. Since in the one-change point testing problem the alternative is $H_1 = \cup_{k=1}^{n-1} A_k$, $A_k : [\gamma n] = k$, we reject H_0 if at least one of $|L_k(\tau_0)|$, $k = 1, 2, \dots, n-1$, defined in (2.14) is large. This leads to the maximum-type (or max-type) test statistic and the rejection region

$$T_n(\tau_0) = \max_{1 \le k < n} |L_k(\tau_0)| \ge c_n(\alpha), \tag{2.18}$$

where $c_n(\alpha)$ is determined in such a way that the test has the prescribed level α.

To apply this test procedure we need at least an approximation for the critical value $c_n(\alpha)$. We can find this approximation applying the Bonferroni inequality

$$\begin{aligned} \mathrm{P}\Big(\max_{1 \le k < n} |L_k(\tau_0)| \ge c_n(\alpha)\Big) &= \mathrm{P}\Big(\bigcup_{k=1}^{n-1} \{|L_k(\tau_0)| \ge c_n(\alpha)\}\Big) \\ &\le \sum_{k=1}^{n-1} \mathrm{P}\Big(|L_k(\tau_0)| \ge c_n(\alpha)\Big) = n\, \mathrm{P}\Big(|L_k(\tau_0)| \ge c_n(\alpha)\Big). \end{aligned}$$

By Theorem 2.1, the critical value $u_{\frac{\alpha}{2n}}$ of the standard normal distribution can be used as un upper estimate of the critical value $c_n(\alpha)$. The approximate critical values obtained in this way are good enough for small values of n, but they are too conservative for n large. The other way commonly used in the change point analysis, is to get it through the limit distribution of the test statistic $T_n(\tau_0)$ under the no-change null hypothesis H_0.

THEOREM 2.3. *Suppose that $X_1^0, X_2^0, \dots, X_n^0$, $C_1, C_2, \dots, C_n$ are independent random variables. Let $X_1^0, X_2^0, \dots, X_n^0$ have an arbitrary absolutely continuous distribution function F. Let C_1, C_2, $\dots$, $C_{\lfloor n\eta \rfloor}$ and $C_{\lfloor n\eta \rfloor+1}, C_{\lfloor n\eta \rfloor+2}, \dots, C_n$ have absolutely continuous distribution functions G_1 and G_2, respectively, $G_1 \ne G_2$, for some $\eta \in (0, 1]$. Let* (2.17) *be satisfied and let*

$$\int_0^{\tau_0} w^2(t)(1 - G_i(t))\, \mathrm{d}F(t) > 0, \quad i = 1, 2, \tag{2.19}$$

then we have, as $n \to \infty$,

$$\mathrm{P}\left(d_1(\log n)\, T_n(\tau_0) \le y + d_2(\log n)\right) \to \exp\big\{-2\mathrm{e}^{-y}\big\}, \quad \forall y \in \mathbb{R}, \tag{2.20}$$

where d_1 and d_2 are defined in (1.19).

PROOF. The proof can be found in Hušková and Neuhaus [**23**], Theorem 1.1. □

Under H_0 the limit distribution of $T_n(\tau_0)$ belongs to the so-called *extreme value distributions* and the convergence rate is extremely slow (see Csörgő and Horváth [**10**] and Antoch et al [**6**]). Using this limit distribution for an approximation of the critical value $c_n(\alpha)$ in (2.18), we get

$$c_n(\alpha) \doteq \frac{-\log(-\log(\sqrt{1-\alpha})) + d_2(\log n)}{d_1(\log n)}. \tag{2.21}$$

REMARK 2.2. The assertion (2.20) remains true if the distribution of the censoring variables $C_1, C_2, \dots, C_n$ changes more than one time, i.e. there exist $0 = \eta_0 < \eta_1 < \dots < \eta_q < \eta_{q+1} = 1$ with some finite q such that $C_{\lfloor n\eta_i \rfloor+1}, C_{\lfloor n\eta_i \rfloor+2}, \dots, C_{\lfloor n\eta_{i+1} \rfloor}$ have an absolutely continuous distribution function G_{i+1}, $i = 0, 1, \dots, q$.

Permutation principle. In this subsection we show that the permutation principle for the change-point problem can be used similarly to the two-sample problem considered in Section 2. Assume that the parameters γ and η are equal to each other. Thus, under the no-change null hypothesis H_0 the censoring variables $C_1, C_2, \dots, C_n$ are i.i.d. too (so we get the null restricted hypothesis $\bar{H}_0$), and under the alternative H_1 the distributions of the survival and the censoring variables changed at the same time point.

Set

$$T_n^\sigma(\tau_0) = \max_{1\le k<n} |L_k^\sigma(\tau_0)| = \max_{1\le k<n} \sqrt{\frac{n}{k(n-k)}}\,\frac{1}{\sigma_n(\boldsymbol{a})}\left|\sum_{j=1}^k a_n(j)\right|, \tag{2.22}$$

where $\sigma_n^2(\boldsymbol{a})$ is defined in (2.6). The permutation distribution $F_n(\cdot,(\boldsymbol{X},\boldsymbol{\Delta}))$ of the test statistic $T_n^\sigma(\tau_0)$ can be described as the conditional distribution given $(X_1,\Delta_1),\dots,(X_n,\Delta_n)$ of

$$T_n^\sigma(\tau_0,\boldsymbol{Q}) = \max_{1\le k<n} |L_k^\sigma(\tau_0,\boldsymbol{Q})| = \max_{1\le k<n} \sqrt{\frac{n}{k(n-k)}}\frac{1}{\sigma_n(\boldsymbol{a})}\left|\sum_{j=1}^k a_n(Q_j)\right|,$$

where $\boldsymbol{Q} = (Q_1, Q_2, \dots, Q_n)$ is a random permutation of $(1,2,\dots,n)$, precisely it can be expressed as

$$\mathrm{P}\big(T_n^\sigma(\tau_0) \le x \,|\, (\boldsymbol{X},\boldsymbol{\Delta})\big) = \frac{1}{n!}\,\#\{\boldsymbol{q}\in\mathcal{Q}_n; T_n^\sigma(\tau_0,\boldsymbol{q}) \le x\}, \quad x\in\mathbb{R},$$

where $\mathcal{Q}_n$ is the set of all permutations of $(1,2,\dots,n)$. Under the restricted null hypothesis $\bar{H}_0$ the distributions of $T_n^\sigma(\tau_0)$ and $T_n^\sigma(\tau_0,\boldsymbol{Q})$ are the same and the permutation distribution provides the exact critical values for our testing problem. Denoting by $c_n(\alpha,(\boldsymbol{X},\boldsymbol{\Delta}))$ the corresponding $100(1-\alpha)\%$-quantile, the critical function of the exact (permutation) test based on $T_n^\sigma(\tau_0)$ with the level α is given by

$$\psi_2(t,(\boldsymbol{X},\boldsymbol{\Delta})) = \begin{cases} 1, & \text{if } T_n^\sigma(\tau_0) > c_n(\alpha,(\boldsymbol{X},\boldsymbol{\Delta})),\\ \nu_2\in(0,1), & \text{if } T_n^\sigma(\tau_0) = c_n(\alpha,(\boldsymbol{X},\boldsymbol{\Delta})),\\ 0, & \text{if } T_n^\sigma(\tau_0) < c_n(\alpha,(\boldsymbol{X},\boldsymbol{\Delta})),\end{cases}$$

where ν_2 is chosen such that

$$\mathrm{P}_{\bar{H}_0}\left(T_n^\sigma(\tau_0) > c_n(\alpha,(\boldsymbol{X},\boldsymbol{\Delta}))\,|\,(\boldsymbol{X},\boldsymbol{\Delta})\right) + \nu_2\,\mathrm{P}_{\bar{H}_0}\left(T_n^\sigma(\tau_0) = c_n(\alpha,(\boldsymbol{X},\boldsymbol{\Delta}))\,|\,(\boldsymbol{X},\boldsymbol{\Delta})\right) = \alpha.$$

Practically, for large n it is not possible to calculate the value of the statistic $T_n^\sigma(\tau_0,\boldsymbol{q})$ for all $n!$ permutations $\boldsymbol{q}$. So instead, we generate a random sample from all possible permutations of size B large enough and determine the empirical critical value $x_n(\alpha,(\boldsymbol{X},\boldsymbol{\Delta}))$ from this sample. Such calculated critical value $x_n(\alpha,(\boldsymbol{X},\boldsymbol{\Delta}))$ provides a good estimate for the actual value $c_n(\alpha,(\boldsymbol{X},\boldsymbol{\Delta}))$.

Next we derive the limit distribution of the permutation distribution of $T_n^\sigma(\tau_0)$.

THEOREM 2.4. *Suppose that $X_1^0, X_2^0, \dots, X_n^0$, $C_1, C_2, \dots, C_n$ are independent random variables. Let the survival times $X_1^0, X_2^0, \dots, X_{\lfloor n\gamma\rfloor}^0$ have an absolutely continuous distribution function F_1 and $X_{\lfloor n\gamma\rfloor+1}^0, X_{\lfloor n\gamma\rfloor+2}^0, \dots, X_n^0$ have an absolutely continuous distribution function F_2 for some $\gamma\in(0,1]$. Let the censoring times $C_1, C_2, \dots, C_{\lfloor n\gamma\rfloor}$ have an absolutely continuous distribution function G_1 and $C_{\lfloor n\gamma\rfloor+1}, C_{\lfloor n\gamma\rfloor+2}, \dots, C_n$ have an absolutely continuous distribution function G_2. Let (2.9) and (2.10) be satisfied, then we have, as $n\to\infty$,*

$$\mathrm{P}\left(d_1(\log n)\,T_n^\sigma(\tau_0,\boldsymbol{Q}) \le y + d_2(\log n)\,|\,(\boldsymbol{X},\boldsymbol{\Delta})\right) \xrightarrow{\ \mathrm{P}\ } \exp\left\{-2\mathrm{e}^{-y}\right\},\ \forall y\in\mathbb{R}, \tag{2.23}$$

and, moreover, under $\bar{H}_0 : \gamma = 1$, *as* $n \to \infty$,

$$\mathrm{P}\Big(d_1(\log n)\ T_n^{\sigma}(\tau_0) \le y + d_2(\log n)\Big) \to \exp\big\{-2\mathrm{e}^{-y}\big\},\ \forall y \in \mathbb{R}, \tag{2.24}$$

where d_1 *and* d_2 *are defined in* (1.19).

PROOF. We repeat the basic idea of the proof of Theorem 2.2. Realize that the random variables $\sum_{j=1}^{k} a_n(Q_j)$, $k = 1, 2, \dots, n$, given $(\boldsymbol{X}, \boldsymbol{\Delta})$, can be viewed as simple linear rank statistics, where the role of ranks is played by $\boldsymbol{Q} = (Q_1, Q_2, \dots, Q_n)$. Consequently, the statistic $T_n^{\sigma}(\tau_0, \boldsymbol{Q})$ given $(\boldsymbol{X}, \boldsymbol{\Delta})$ can be viewed as a function of a simple linear rank statistic and theorem on rank statistics for change point problem can be used. By Corollary 5.6 below the assumptions (5.47) and (5.48) of Theorem 5.1 below are satisfied for convergence in probability and therefore the assertion (2.23) holds.
Moreover, under $\bar{H}_0$, the random variables $\sum_{j=1}^{k} a_n(j)$, $k = 1, 2, \dots, n$, have the same distribution as $\sum_{j=1}^{k} a_n(Q_j)$. Thus, the distributions of $T_n^{\sigma}(\tau_0)$ and $T_n^{\sigma}(\tau_0, \boldsymbol{Q})$ coincide and the limit distribution does not depend on the condition $(\boldsymbol{X}, \boldsymbol{\Delta})$, we can conclude that (2.24) holds. □

Notice that the assumptions of Theorem 2.4 cover both the restricted null hypothesis $\bar{H}_0 : \gamma = \eta = 1$ and the particular alternative $\bar{H}_1 : \gamma = \eta \in (0, 1)$. Moreover, the limit conditional distribution does not depend on the original observations $(X_1, \Delta_1), (X_2, \Delta_2), \dots, (X_n, \Delta_n)$, so the conditional and unconditional limit distribution of $T_n^{\sigma}(\tau_0, \boldsymbol{Q})$ is the same in both these cases. This means that the critical value for the permutation test provides an approximation for the critical value of the test based on $T_n^{\sigma}(\tau_0)$. The model considered in Theorem 2.4 includes also the Koziol–Green model of the random censorship, see Chapter 1.

Moreover, the assertion (2.23) remains true also under the particular case of multiple-change alternative $\bar{H}_2 : 0 = \gamma_0 = \eta_0 < \gamma_1 = \eta_1 < \dots < \gamma_q = \eta_q < \gamma_{q+1} = \eta_{q+1} = 1$ with finite q (i.e. the distribution of C_j's changes together with X_j's more than ones) as we see below.

COROLLARY 2.1. *Suppose that* $X_1^0, X_2^0, \dots, X_n^0$, $C_1, C_2, \dots, C_n$ *are independent random variables. There exist* $0 = \gamma_0 < \gamma_1 < \dots < \gamma_q < \gamma_{q+1} = 1$ *with some finite* $q \in \mathbb{N}$ *such that variables* $X^0_{\lfloor n\gamma_i \rfloor + 1}, X^0_{\lfloor n\gamma_i \rfloor + 2}, \dots, X^0_{\lfloor n\gamma_{i+1} \rfloor}$ *have an absolutely continuous distribution function* F_{i+1} *and* $C_{\lfloor n\gamma_i \rfloor + 1}, C_{\lfloor n\gamma_1 \rfloor + 2}, \dots, C_{\lfloor n\gamma_{i+1} \rfloor}$ *have an absolutely continuous distribution function* G_{i+1}, $i = 0, 1, \dots, q$. *Let* (2.9) *be satisfied and let*

$$\int_0^{\tau_0} w^2(t)(1 - G_i(t))\,\mathrm{d}F_i(t) > 0, \quad i = 1, 2, \dots, q + 1, \tag{2.25}$$

then for all $y \in \mathbb{R}$ *we have, as* $n \to \infty$,

$$\mathrm{P}\Big(d_1(\log n)\, T_n^{\sigma}(\tau_0, \boldsymbol{Q}) \le y + d_2(\log n) \,|\, (\boldsymbol{X}, \boldsymbol{\Delta})\Big) \xrightarrow{\mathrm{P}} \exp\big\{-2\mathrm{e}^{-y}\big\},$$

where $d_1(t)$ *and* $d_2(t)$ *are defined in* (1.19).

PROOF. We proceed in much the same way as in the proof of Theorem 2.4. We have to check whether the assumptions (5.47) and (5.48) of Theorem 5.1 below are satisfied. Toward this we need a small modification of Lemma 5.4 below. If in the proof of that lemma, the functions $1 - H_{\eta,\gamma}(t)$ and $R_{\eta,\gamma}(t)$ of the forms (1.6) and (1.10) are replaced by

$$\begin{aligned} 1 - H_{\gamma_1,\dots,\gamma_q}(t) = \gamma_1(1 - F_1(t))(1 - G_1(t)) + (\gamma_2 - \gamma_1)(1 - F_2(t))(1 - G_2(t)) \\ + (1 - \gamma_q)(1 - F_{q+1}(t))(1 - G_{q+1}(t)), \end{aligned} \tag{2.26}$$

$$R_{\gamma_1,\ldots,\gamma_q}(t) = \gamma_1 \int_0^t (1 - G_1(u))\,\mathrm{d}F_1(u) + (\gamma_2 - \gamma_1) \int_0^t (1 - G_2(u))\,\mathrm{d}F_2(u) + (1 - \gamma_q) \int_0^t (1 - G_{q+1}(u))\,\mathrm{d}F_{q+1}(u) \tag{2.27}$$

corresponding to the considered multiple-change case, then we get, as $n \to \infty$,

$$\frac{1}{n} \sum_{j=1}^{n} |a_n(j) - \bar{a}_n|^4 = \frac{1}{n} \sum_{j=1}^{n} |a_n(j)|^4 = O_{\mathrm{P}}\Big(\max_{1 \le j \le n} |a_n(j)|^4\Big) = O_{\mathrm{P}}(1),$$

$$\frac{1}{n} \sum_{j=1}^{n} (a_n(j) - \bar{a}_n)^2 = \frac{1}{n} \sum_{j=1}^{n} a_n^2(j) \xrightarrow{\mathrm{P}} \int_0^{\tau_0} w^2(t)\,\mathrm{d}R_{\gamma_1,\ldots,\gamma_q}(t) > 0$$

since $\bar{a}_n = 0$ and (2.25). These relations ensure that the assumptions (5.47) and (5.48) for convergence in probability are fulfilled and the assertion of our corollary follows. □

Now we investigate the convergence relation between $V_k(\tau_0)$ and $\frac{k(n-k)}{n^2}\,\sigma_n^2(\boldsymbol{a})$.

LEMMA 2.2. *Let (2.17) be satisfied. Under $\bar{H}_0 : \eta = \gamma = 1$, if (2.12) is fulfilled, then we obtain, as $n \to \infty$,*

$$\max_{(\log n)^{\omega} < k < n - (\log n)^{\omega}} \left| \frac{n^2}{k(n-k)}\, V_k(\tau_0) - \sigma_n^2(\boldsymbol{a}) \right| = o_{\mathrm{P}}\Big((\log\log n)^{-1}\Big), \tag{2.28}$$

where $\omega > 0$ is arbitrary but fixed.

PROOF. By Corollary 5.13 below we get, as $n \to \infty$,

$$\frac{n^2}{k(n-k)}\, V_k(\tau_0) = \left(\int_0^{\tau_0} w^2(t)(1 - G(t))\,\mathrm{d}F(t) + o_{\mathrm{P}}\Big((\log\log n)^{-1}\Big)\right) \left(1 + O_{\mathrm{P}}\left(\sqrt{\frac{\log k}{k}} + \sqrt{\frac{\log(n-k)}{n-k}}\right)\right) \tag{2.29}$$

uniformly in $\log\log n < k < n - \log\log n$ and consequently, as $n \to \infty$,

$$\frac{n^2}{k(n-k)}\, V_k(\tau_0) = \int_0^{\tau_0} w^2(t)\,(1 - G(t))\,\mathrm{d}F(t) + o_{\mathrm{P}}\Big((\log\log n)^{-1}\Big)$$

uniformly in $(\log n)^{\omega} < k < n - (\log n)^{\omega}$.

Further, Corollary 5.5 below says that, as $n \to \infty$,

$$\frac{n-1}{n}\, \sigma_n^2(\boldsymbol{a}) = \int_0^{\tau_0} w^2(t)\,\mathrm{d}R(t) = \int_0^{\tau_0} w^2(t)(1 - G(t))\,\mathrm{d}F(t) + o_{\mathrm{P}}\Big((\log\log n)^{-1}\Big).$$

Thus, the assertion (2.28) holds. □

It can be seen that

$$\max_{1 \le k \le \log\log n} \frac{k(n-k)}{n^2 V_k} \le \frac{k(n-k)}{n^2} \frac{1}{v_k} = 1$$

and together with (2.29) we get, as $n \to \infty$,

$$\max_{1 \le k \le (\log n)^{\omega}} \frac{k(n-k)}{n^2 V_k} = O_{\mathrm{P}}(1).$$

By Corollary 5.15 below we obtain, as $n \to \infty$,

$$\max_{1 \le k \le (\log n)^{\omega}} \sqrt{\frac{n}{k(n-k)}} |S_k(\tau_0)| = o_{\mathrm{P}}(\sqrt{\log\log n}),$$

where $S_k(\tau_0)$ is defined in (2.15). Thus, under the hypothesis $\bar{H}_0 : \eta = \gamma = 1$ and the assumptions (2.12) and (2.17), the test statistics $T_n(\tau_0)$ and $T_n^\sigma(\tau_0)$ are asymptotically equivalent, i.e.

$$|T_n(\tau_0) - T_n^\sigma(\tau_0)| = o_P\Big((\log\log n)^{-1/2}\Big), \quad n \to \infty.$$

Notice that the assumption (2.17) for the weights $w_n(t)$ is stronger than the assumption (2.9) in Theorem 2.4.

Remark 2.3. In the case of equal censorship $G_1(t) = G_2(t)$ for all t we can apply a general principle how to construct tests appropriate for detection of a change in distribution in our setup. Thus, except the max-type test statistic $T_n^\sigma(\tau_0)$ of the form (2.22) can be used e.g. the following max-type test statistics derived by *the maximum likelihood principle*

$$\frac{1}{\sqrt{n}} \max_{1\le k<n} \left(\frac{k}{n}\Big(1-\frac{k}{n}\Big)\right)^{-b_1} \frac{|S_k(\tau_0)|}{\sigma_n(\boldsymbol{a})}, \quad b_1 \in [0, 1/2),$$

and the sum-type test statistics obtained by *the pseudo-Bayes method*

$$\frac{1}{n} \sum_{k=1}^{n} \left(\frac{k}{n}\Big(1-\frac{k}{n}\Big)\right)^{-b_2} \frac{S_k^2(\tau_0)}{\sigma_n^2(\boldsymbol{a})}, \quad b_2 \in [0, 1].$$

For details about various forms of the test (for complete data) and methods of construction see e.g. the work of Antoch et al [**5**]. The max-type form of the test statistic with $b_1 = 0$ and $w_n(t) = Y(t)/n$ was studied by Liu and Gombay [**18**] and extensions for $b_1 \in (0, 1/2]$ have been done by Liu [**28**], see Section 3 in Chapter 1.

4. MOSUM type test statistics (m unknown, multiple change case)

Here we consider the random censorship model with multiple changes. Particularly, we assume that the lifetimes and the censoring times are independent nonnegative variables. There exist $0 = \gamma_0 \le \gamma_1 \le \cdots \le \gamma_q \le \gamma_{q+1} = 1$ with some finite $q \in \mathbb{N}$ such that the lifetimes $X^0_{\lfloor n\gamma_i\rfloor+1}, X^0_{\lfloor n\gamma_i\rfloor+2}, \dots, X^0_{\lfloor n\gamma_{i+1}\rfloor}$ have an absolutely continuous distribution function F_{i+1}, $i = 0, 1, \dots, q$, and there exist also $0 = \eta_0 \le \eta_1 \le \cdots \le \eta_{q_c} \le \gamma_{q_C+1} = 1$ with some finite $q_C \in \mathbb{N}$ such that the censoring times $C_{\lfloor n\eta_j\rfloor+1}, C_{\lfloor n\eta_j\rfloor+2}, \dots, C_{\lfloor n\eta_{j+1}\rfloor}$ have an absolutely continuous distribution function G_{j+1}, $j = 0, 1, \dots, q_C$.

We wish to test the no-change null hypothesis against the multiple-change alternative hypothesis

$$H_0 : \gamma_1 = \gamma_2 = \cdots = \gamma_q = 1 \qquad (F_1(t) = F_2(t) = \cdots = F_{q+1}(t) \quad \text{for all } t \in \mathbb{R}), \tag{2.30}$$

$$H_2 : \gamma_1 < \gamma_2 < \cdots < \gamma_q \in (0,1)$$
$$(F_1(t_1) \ne F_2(t_1),\ F_2(t_2) \ne F_3(t_2), \dots, F_q(t_q) \ne F_{q+1}(t_q) \text{ with some } t_1, t_2, \dots t_q),$$

where the integer $q > 1$ can be known or unknown.

Now, we introduce another class of test statistics for our problem (2.30). They are based on the moving sums (MOSUM) of the statistics $S_k(\tau_0)$ defined in (2.15)

$$\begin{aligned} L_{k,D}(\tau_0) &= S_{k+D}(\tau_0) - 2S_k(\tau_0) + S_{k-D}(\tau_0) \\ &= \sum_{j=k+1}^{k+D} a_n(j) - \sum_{j=k-D+1}^{k} a_n(j), \qquad k = D+1, \dots, n-D-1. \end{aligned} \tag{2.31}$$

The test procedure also depends on D. We assume that $D = D(n)$ satisfies, as $n \to \infty$,

$$\frac{D}{n} \to 0, \qquad \frac{n^{2/(2+u)} \log n}{D} \to 0, \tag{2.32}$$

where u is a positive constant such that

$$\frac{1}{n}\sum_{j=1}^{n}|a_n(j)-\bar{a}_n|^{u+2}=O_{\mathrm{P}}(1),\quad n\to\infty,$$

for the scores defined in (2.1). It is satisfied for every finite $u>0$ because of (5.14) and (5.15) below. The term (2.32) means that D tends to infinity together with n but not too fast.

The MOSUM-type test statistic $L_{k,D}(\tau_0)$ given by (2.31) is convenient to use if we expect more than one change in the distribution of the survival variables, i.e. the alternative H_2. Mainly, it is suitable in the case of equal censorship $G_0(t)=G_1(t)=\cdots=G_{q_c}(t)$ for all t, when the test procedure is simple. This situation will be described in the subsection called Permutation principle. For the sake of completeness, we deal with general event when the distribution of the censoring variables can change. In this case the test procedure becomes rather complicated and we develop it now.

In the following we suppose only at most one change ($q_C\le 1$) in the distribution of the censoring variables given by the parameter $\eta_1=\eta\in(0,1]$. Let us denote the point of the change in the distribution of the censoring times by m_c, i.e. $m_c=\lfloor n\eta\rfloor$. Assume that m_c is known. Consider the random variable

$$T_{n,D}(\tau_0)=\max\left(\max_{D<k<m_c-D}\frac{|L_{k,D}(\tau_0)|}{\sqrt{2D}\sigma_{m_c}(\boldsymbol{a})},\max_{m_c+D<k<n-D}\frac{|L_{k,D}(\tau_0)|}{\sqrt{2D}\sigma^0_{m_c}(\boldsymbol{a})}\right)\tag{2.33}$$

with

$$\sigma^2_{m_c}(\boldsymbol{a})=\frac{1}{m_c-1}\sum_{j=1}^{m_c}(a_n(j)-\bar{a}_{m_c})^2,\qquad \bar{a}_{m_c}=\frac{1}{m_c}\sum_{j=1}^{m_c}a_n(j),\tag{2.34}$$

$$\sigma^{0\,2}_{m_c}(\boldsymbol{a})=\frac{1}{n-m_c-1}\sum_{j=m_c+1}^{n}(a_n(j)-\bar{a}^0_{m_c})^2,\qquad \bar{a}^0_{m_c}=\frac{1}{n-m_c}\sum_{j=m_c+1}^{n}a_n(j),\tag{2.35}$$

where $a_n(j)$ are defined in (2.1).

Theorem 2.5. *Suppose that $X^0_1,X^0_2,\dots,X^0_n$, $C_1,C_2,\dots,C_n$ are independent random variables. Let the lifetimes $X^0_1,X^0_2,\dots,X^0_n$ have an absolutely continuous distribution function F. Let $C_1,C_2,\dots,C_{\lfloor n\eta\rfloor}$ and $C_{\lfloor n\eta\rfloor+1},C_{\lfloor n\eta\rfloor+2},\dots,C_n$ have absolutely continuous distribution functions G_1 and G_2, respectively, $G_1\neq G_2$, for some $\eta\in(0,1]$. Let* (2.9) *and* (2.19) *be satisfied. Then for all $y\in\mathbb{R}$ we have, as $n\to\infty$,*

$$\mathrm{P}\left(d_1\Big(\frac{n}{D}\Big)\,T_{n,D}(\tau_0)\le y+d_2\Big(\frac{n}{D}\Big)+\log\Big(\frac{3}{2}\Big)\right)\to\exp\big\{-2\mathrm{e}^{-y}\big\}\tag{2.36}$$

with d_1 and d_2 defined in (1.19).

Proof. Notice that the random variables $\sum_{j=1}^{k}a_n(j)$, $k=1,2,\dots,n$, have the same distribution as the variables $\sum_{j=1}^{k}a_n(Q_j)$, $k=1,2,\dots,n$, where $\boldsymbol{Q_{m_c}}=(Q_1,Q_2,\dots,Q_{m_c})$ and $\boldsymbol{Q^o_{m_c}}=(Q_{m_c+1},Q_{m_c+2},\dots,Q_n)$ are random permutations of $(1,2,\dots,m_c)$ and $(m_c+1,m_c+2,\dots,n)$, respectively. Moreover, the random variables $\sum_{j=1}^{k}a_n(Q_j)$, $k=1,\dots,n$, given $(\boldsymbol{X},\boldsymbol{\Delta})$, can be viewed as simple linear rank statistics, where the role of ranks is played by $(Q_1,Q_2,\dots,Q_{m_c})$ and $(Q_{m_c+1},Q_{m_c+2},\dots,Q_n)$.

We verify the conditions (5.51), (5.52) and (5.54), (5.55) for convergence in probability to apply Theorem 5.2 below. By Corollary 5.5 below we know that $\max_{1\le j\le n}|a_n(j)|=O_{\mathrm{P}}(1)$, as $n \to \infty$, and by Corollary 5.7 below we obtain $\bar{a}_{m_c} = o_{\mathrm{P}}(1)$, as $n \to \infty$, and consequently $\bar{a}^0_{m_c} = o_{\mathrm{P}}(1)$, as $n \to \infty$, since $\bar{a}_n = 0$. This implies

$$\frac{\sum_{j=1}^{m_c}(a_n(j) - \bar{a}_{m_c})^4}{m_c} = \sum_{i=0}^{4}\left\{\binom{4}{i}\frac{\sum_{j=1}^{m_c}(a_n(j))^{4-i}}{m_c}\left(-\bar{a}_{m_c}\right)^i\right\} = O_{\mathrm{P}}(1),$$
$$\frac{\sum_{j=m_c+1}^{n}(a_n(j) - \bar{a}^0_{m_c})^4}{n-m_c} = \sum_{i=0}^{4}\left\{\binom{4}{i}\frac{\sum_{j=m_c+1}^{n}(a_n(j))^{4-i}}{n-m_c}\left(-\bar{a}^0_{m_c}\right)^i\right\} = O_{\mathrm{P}}(1).$$

Now it is clear that the conditions (5.52) and (5.55) are for convergence in probability fulfilled. Further, by Corollary 5.11 below we have, as $n \to \infty$,

$$\sigma^2_{m_c}(\boldsymbol{a}) \xrightarrow{\mathrm{P}} \int_0^{\tau_0} w^2(t)(1 - G_1(t))\,\mathrm{d}F(t) > 0,$$
$$\sigma^{0\,2}_{m_c}(\boldsymbol{a}) \xrightarrow{\mathrm{P}} \int_0^{\tau_0} w^2(t)(1 - G_2(t))\,\mathrm{d}F(t) > 0,$$

since the assumption (2.19). Thus the assumptions (5.51) and (5.54) are also for convergence in probability satisfied. By Theorem 5.2 below ($m = n$) we obtain, as $n \to \infty$,

$$\mathrm{P}\left(d_1\Big(\frac{m_c}{D}\Big)\max_{D<k<m_c-D}\frac{|L_{k,D}(\tau_0,\boldsymbol{Q_{m_c}})|}{\sqrt{2D}\,\sigma_{m_c}(\boldsymbol{a})} \le y + d_2\Big(\frac{m_c}{D}\Big) + \log\Big(\frac{3}{2}\Big) \mid (\boldsymbol{X},\boldsymbol{\Delta})\right) \xrightarrow{\mathrm{P}} \exp\left\{-2\mathrm{e}^{-y}\right\} \quad (2.37)$$

and

$$\mathrm{P}\left(d_1\Big(\frac{n-m_c}{D}\Big)\max_{m_c+D<k<n-D}\frac{|L_{k,D}(\tau_0,\boldsymbol{Q^o_{m_c}})|}{\sqrt{2D}\,\sigma^0_{m_c}(\boldsymbol{a})} \le y + d_2\Big(\frac{n-m_c}{D}\Big) + \log\Big(\frac{3}{2}\Big) \mid (\boldsymbol{X},\boldsymbol{\Delta})\right) \xrightarrow{\mathrm{P}} \exp\left\{-2\mathrm{e}^{-y}\right\}. \quad (2.38)$$

Moreover, by the Taylor expansion we have, as $n \to \infty$,

$$d_1\Big(\frac{m_c}{D}\Big) - d_1\Big(\frac{n}{D}\Big) = \frac{\log\eta}{\sqrt{2\log(\frac{n}{D})}} + o\left(\left(\log(\frac{n}{D})\right)^{-1}\right), \quad (2.39)$$
$$d_1\Big(\frac{n-m_c}{D}\Big) - d_1\Big(\frac{n}{D}\Big) = \frac{\log(1-\eta)}{\sqrt{2\log(\frac{n}{D})}} + o\left(\left(\log(\frac{n}{D})\right)^{-1}\right), \quad (2.40)$$
$$d_2\Big(\frac{m_c}{D}\Big) - d_2\Big(\frac{n}{D}\Big) = 2\log\eta + \frac{\log\eta}{2\log(\frac{n}{D})} + o\left(\left(\log(\frac{n}{D})\right)^{-1}\right), \quad (2.41)$$
$$d_2\Big(\frac{n-m_c}{D}\Big) - d_2\Big(\frac{n}{D}\Big) = 2\log(1-\eta) + \frac{\log(1-\eta)}{2\log(\frac{n}{D})} + o\left(\left(\log(\frac{n}{D})\right)^{-1}\right). \quad (2.42)$$

By (2.37) we receive, as $n \to \infty$,

$$\mathrm{P}\left(\frac{\max_{D<k<m_c-D}\frac{|L_{k,D}(\tau_0,\boldsymbol{Q_{m_c}})|}{\sqrt{2D}\,\sigma_{m_c}(\boldsymbol{a})}}{d_1\Big(\frac{m_c}{D}\Big)} \le 1 + \frac{y + \frac{1}{2}\log\log\Big(\frac{m_c}{D}\Big) - \frac{1}{2}\log\pi + \log\Big(\frac{3}{2}\Big)}{2\log\Big(\frac{m_c}{D}\Big)} \mid (\boldsymbol{X},\boldsymbol{\Delta})\right) \xrightarrow{\mathrm{P}} \exp\left\{-2\mathrm{e}^{-y}\right\}.$$

Choosing $y = -2\varepsilon \log\Big(\frac{m_c}{D}\Big)$ and then $y = 2\varepsilon \log\Big(\frac{m_c}{D}\Big)$, respectively, for an arbitrary small $\varepsilon > 0$, we get

$$\mathrm{P}\left(\frac{\max_{D<k<m_c-D} \frac{|L_{k,D}(\tau_0,\boldsymbol{Q}_{\boldsymbol{m_c}})|}{\sqrt{2D}\,\sigma_{m_c}(\boldsymbol{a})}}{d_1\Big(\frac{m_c}{D}\Big)} \leq 1-\varepsilon \,\Big|\, (\boldsymbol{X},\boldsymbol{\Delta})\right) \xrightarrow{\mathrm{P}} 0,$$

$$\mathrm{P}\left(\frac{\max_{D<k\ <m_c-D} \frac{|L_{k,D}(\tau_0,\boldsymbol{Q}_{\boldsymbol{m_c}})|}{\sqrt{2D}\,\sigma_{m_c}(\boldsymbol{a})}}{d_1\Big(\frac{m_c}{D}\Big)} \geq 1+\varepsilon \,\Big|\, (\boldsymbol{X},\boldsymbol{\Delta})\right) \xrightarrow{\mathrm{P}} 0, \quad \text{respectively,}$$

since $\log \frac{m_c}{D} \to \infty$. Thus, as $n \to \infty$,

$$\frac{1}{d_1\Big(\frac{m_c}{D}\Big)} \max_{D<k<m_c-D} \frac{|L_{k,D}(\tau_0,\boldsymbol{Q}_{\boldsymbol{m_c}})|}{\sqrt{2D}\,\sigma_{m_c}(\boldsymbol{a})} \xrightarrow{\mathrm{P}} 1 \tag{2.43}$$

and similarly we can proceed for (2.38), i.e. we obtain

$$\frac{1}{d_1\Big(\frac{n-m_c}{D}\Big)} \max_{m_c+D<k<n-D} \frac{|L_{k,D}(\tau_0,\boldsymbol{Q}^{\boldsymbol{o}}_{\boldsymbol{m_c}})|}{\sqrt{2D}\,\sigma^o_{m_c}(\boldsymbol{a})} \xrightarrow{\mathrm{P}} 1. \tag{2.44}$$

Combining (2.39) with (2.43) we get, as $n \to \infty$,

$$\Big(d_1\Big(\frac{m_c}{D}\Big) - d_1\Big(\frac{n}{D}\Big)\Big) \max_{D<k<m_c-D} \frac{|L_{k,D}(\tau_0,\boldsymbol{Q}_{\boldsymbol{m_c}})|}{\sqrt{2D}\,\sigma^o_{m_c}(\boldsymbol{a})} \xrightarrow{\mathrm{P}} \log\eta \tag{2.45}$$

and analogously by (2.40) with (2.44)

$$\Big(d_1\Big(\frac{n-m_c}{D}\Big) - d_1\Big(\frac{n}{D}\Big)\Big) \max_{m_c+D<k<n-D} \frac{|L_{k,D}(\tau_0,\boldsymbol{Q}^{\boldsymbol{o}}_{\boldsymbol{m_c}})|}{\sqrt{2D}\,\sigma^o_{m_c}(\boldsymbol{a})} \xrightarrow{\mathrm{P}} \log(1-\eta). \tag{2.46}$$

By (2.37), (2.38) with (2.41), (2.42) and (2.45), (2.46) we see that, as $n \to \infty$,

$$\mathrm{P}\left(d_1\Big(\frac{n}{D}\Big) \max_{D<k<m_c-D} \frac{|L_{k,D}(\tau_0,\boldsymbol{Q}_{\boldsymbol{m_c}})|}{\sqrt{2D}\,\sigma_{m_c}(\boldsymbol{a})} \leq \log\eta + y + d_2\Big(\frac{n}{D}\Big) + \log\Big(\frac{3}{2}\Big) \,|\, (\boldsymbol{X},\boldsymbol{\Delta})\right)$$
$$\xrightarrow{\mathrm{P}} \exp\left\{-2\mathrm{e}^{-y}\right\} \tag{2.47}$$

and

$$\mathrm{P}\left(d_1\Big(\frac{n}{D}\Big) \max_{m_c+D<k<n-D} \frac{|L_{k,D}(\tau_0,\boldsymbol{Q}^{\boldsymbol{o}}_{\boldsymbol{m_c}})|}{\sqrt{2D}\,\sigma^0_{m_c}(\boldsymbol{a})} \leq \log(1-\eta) + y + d_2\Big(\frac{n}{D}\Big) + \log\Big(\frac{3}{2}\Big) \,|\, (\boldsymbol{X},\boldsymbol{\Delta})\right)$$
$$\xrightarrow{\mathrm{P}} \exp\left\{-2\mathrm{e}^{-y}\right\}. \tag{2.48}$$

Since the random vectors $\boldsymbol{Q}_{\boldsymbol{m_c}} = (Q_1, Q_2, \ldots, Q_{m_c})$ and $\boldsymbol{Q}^{\boldsymbol{o}}_{\boldsymbol{m_c}} = (Q_{m_c+1}, Q_{m_c+2}, \ldots, Q_n)$ are independent, the random variable $T_{n,D}(\tau_0, (\boldsymbol{Q}_{\boldsymbol{m_c}}, \boldsymbol{Q}^{\boldsymbol{o}}_{\boldsymbol{m_c}}))$ is the maximum of the two independent variables. Thus, we get in view of (2.47) and (2.48), as $n \to \infty$,

$$\mathrm{P}\left(d_1\Big(\frac{n}{D}\Big) T_{n,D}(\tau_0, (\boldsymbol{Q}_{\boldsymbol{m_c}}, \boldsymbol{Q}^{\boldsymbol{o}}_{\boldsymbol{m_c}})) \leq y + d_2\Big(\frac{n}{D}\Big) + \log\Big(\frac{3}{2}\Big) \,|\, (\boldsymbol{X},\boldsymbol{\Delta})\right)$$
$$\xrightarrow{\mathrm{P}} \exp\left\{-2\mathrm{e}^{-y+\log\eta}\right\} \exp\left\{-2\mathrm{e}^{-y+\log(1-\eta)}\right\}.$$

Since the limit distribution does not depend on the condition $(\boldsymbol{X},\boldsymbol{\Delta})$ and

$$\exp\left\{-2\mathrm{e}^{-y+\log\eta}\right\} \exp\left\{-2\mathrm{e}^{-y+\log(1-\eta)}\right\} = \exp\left\{-2\mathrm{e}^{-y}\right\},$$

we can conclude that (2.36) holds. □

The rejection criterion of the asymptotic test based on $T_{n,D}(\tau_0)$ has the following form

$$T_{n,D}(\tau_0) > \frac{-\log(-\log(\sqrt{1-\alpha})) + d_2(\frac{n}{D}) + \log(\frac{3}{2})}{d_1(\frac{n}{D})}, \tag{2.49}$$

where d_1 and d_2 are defined in (1.19).

The problem is that the point m_c in the definition (2.33) of $T_{n,D}(\tau_0)$ is usually unknown, so we replace m_c by its consistent estimator

$$\hat{m}_c(\tau_0) = \min\left\{ l : \max_{1\le j<n} \frac{|S_j^c(\tau_0)|}{\sqrt{nV_j^c(\tau_0)}} = \frac{|S_l^c(\tau_0)|}{\sqrt{nV_l^c(\tau_0)}} \right\}, \tag{2.50}$$

where $S_j^c(\tau_0)$ and $V_j^c(\tau_0)$ are defined similarly to $S_j(\tau_0)$ in (2.15) and $V_j(\tau_0)$ in (2.16) but we are focusing on the change in the distribution of the censoring variables, i.e.

$$S_k^c(\tau_0) = \sum_{j=1}^{k} a_n^c(j), \quad k = 1, 2, \dots, n,$$

with the scores

$$a_n^c(j) = \int_0^{\tau_0} w_n^c(t)\, \mathrm{d}N_j^c(t) - \int_0^{\tau_0} w_n^c(t)\, \frac{Y_j(t)}{Y(t)}\, \mathrm{d}N^c(t)$$

and the process

$$N^c(t) = \sum_{j=1}^{n} N_j^c(t), \quad N_j^c(t) = (1-\Delta_j)\, I(X_j \le t),$$

counting the censoring events by time t and

$$V_k^c(\tau_0) = \frac{1}{n}\int_0^{\tau_0} (w_n^c(t))^2\, \frac{\sum_{j=1}^{k} Y_j(t)\, \sum_{j=k+1}^{n} Y_j(t)}{Y^2(t)}\, \mathrm{d}N^c(t) + v_k.$$

The weight functions $w_n^c(X_j, \Delta_j; j = 1, \dots, n) = w_n(X_j, (1-\Delta_j); j = 1, 2, \dots, n) \ge 0$ fulfil, as $n \to \infty$,

$$\sup_{0\le t\le \tau_0} |w_n^c(t) - w^c(t)| = O_{\mathrm{P}}\left(\sqrt{\frac{\log\log n}{n}}\right), \tag{2.51}$$

where $w^c(t)$ is a continuous nonrandom function on $[0, \tau_0]$. The limit behavior of the estimator $\hat{m}_c(\tau_0)$ (or $\hat{m}_2(\tau_0)$ respectively) will be treated in Chapter 3.

Under a bit stronger assumption on the weight function $w_n(t)$ (compare (2.9) with (2.51)) we get the following corollary under the assumption that the distribution functions of the censoring variables G_1 and G_2 depend on n and $G_1 - G_2 = G_{1n} - G_{2n}$ tends to zero in some sense.

Corollary 2.2. *Suppose that $X_{1n}^0, X_{2n}^0, \dots, X_{nn}^0$, $C_{1n}, C_{2n}, \dots, C_{nn}$ are independent random variables. Let the survival variables $X_{1n}^0, X_{2n}^0, \dots, X_{nn}^0$ have an absolutely continuous distribution function F. Let $C_{1n}, C_{2n}, \dots, C_{\lfloor n\eta\rfloor\, n}$ and $C_{\lfloor n\eta\rfloor+1\, n}, C_{\lfloor n\eta\rfloor+2\, n}, \dots, C_{nn}$ have absolutely continuous distribution functions G_{1n} and G_{2n}, respectively, for some $\eta \in (0,1)$, where the distribution functions depend on n. Assume* (2.51) *and the following conditions:*

(C.1): *there exists a hazard function* $\lambda_G(t)$ *such that*

$$\lim_{n\to\infty}\int_0^{\tau_0}|\lambda_{G_{in}}(t)-\lambda_G(t)|\,\mathrm{d}t=0,\quad i=1,2,$$

and, as $n\to\infty$,

$$\frac{n(A_n^c(\tau_0))^2}{\log\log n}\to\infty,$$

where $A_n^c(\tau_0)=\int_0^{\tau_0}w^c(t)(1-F(t))(1-G(t))\,(\lambda_{G_{1n}}(t)-\lambda_{G_{2n}}(t))\,\mathrm{d}t$.

(C.2):

$$J^c(\tau_0)=\int_0^{\tau_0}(w^c(t))^2(1-F(t))\,\mathrm{d}G(t)>0,$$
$$J(\tau_0)=\int_0^{\tau_0}(w(t))^2(1-G(t))\,\mathrm{d}F(t)>0.$$

Then the assertion (2.36) *remains true if* m_c *is replaced by* $\hat{m}_c(\tau_0)$ *of the form* (2.50).

Proof. By Theorem 3.2 below and replacing X_j^0's by C_j's, $j=1,2,\dots,n$, we have, as $n\to\infty$,

$$\frac{\hat{m}_c(\tau_0)-m_c}{n}=o_{\mathrm{P}}\Big((\log\log n)^{-1}\Big).$$

Further, the condition (2.51) for the weights $w_n^c(t)$ ensures the validity of the condition (2.9) for the weights $w_n(t)$. □

Thus, we can use the test based on $T_{n,D}(\tau_0)$ of the form (2.33) with the proposed estimator of m_c only in the particular forms of the "closed" censoring distribution functions specified by the assumption (C.1).

Now, we answer the question what happens when $\eta=1$ and we use the estimator $\hat{m}_c(\tau_0)$ of $\lfloor n\eta\rfloor$.

Corollary 2.3. *Suppose that* $X_1^0,X_2^0,\dots,X_n^0$, $C_1,C_2,\dots,C_n$ *are independent random variables. Let the lifetimes* $X_1^0,X_2^0,\dots,X_n^0$ *have an absolutely continuous distribution function* F. *Let the censoring times* $C_1,C_2,\dots,C_n$ *have an absolutely continuous distribution function* G. *Let* (2.9) *and* (2.12) *be satisfied, then the assertion* (2.36) *remains true if* m_c *is replaced by* $\hat{m}_c(\tau_0)$ *which is defined in* (2.50).

Proof. By Theorem 3.4 below and replacing X_j^0's by C_j's, $j=1,2,\dots n$, we get, as $n\to\infty$,

$$\frac{\hat{m}_c(\tau_0)}{n}\xrightarrow{\mathcal{D}}U,$$

where U is a random variable with distribution $\mathrm{P}(U=0)=\mathrm{P}(U=1)=1/2$. □

The task is what we use instead of the MOSUM-type test. Motivated by the procedure for complete data developed by Vostrikova [**33**], we can base the test on the max-type test statistic $T_n(\tau_0)$ which is used for one-change alternative H_1. If we reject the no-change hypothesis H_0, we estimate m by the point, where the test statistic $T_n(\tau_0)$ takes its maximum, i.e.

$$\hat{m}(\tau_0)=\min\left\{l:\max_{1\le j<n}\frac{|S_j(\tau_0)|}{\sqrt{nV_j(\tau_0)}}=\frac{|S_l(\tau_0)|}{\sqrt{nV_l(\tau_0)}}\right\}.$$

After that we divide the paired observations $(X_1,\Delta_1),(X_2,\Delta_2),\dots,(X_n,\Delta_n)$ into two groups $(X_1,\Delta_1),\dots,(X_{\hat{m}(\tau_0)},\Delta_{\hat{m}(\tau_0)})$ and $(X_{\hat{m}(\tau_0)+1},\Delta_{\hat{m}(\tau_0)+1}),\dots,(X_n,\Delta_n)$ and we apply to each of both samples the test based on $T_{\hat{m}(\tau_0)}(\tau_0)$ or $T_{n-\hat{m}(\tau_0)}(\tau_0)$, respectively. We repeat such procedure as long as in any group the tests do not indicate the change in the distribution of the lifetimes. We obtain critical values from Theorem 2.3. If we choose the significance level α_n such that $\lim_{n\to\infty} \alpha_n = 0$, then this procedure estimates consistently all change points and also the number of changes.

The problem with using the MOSUM-type test disappears, if we suppose e.g. the Koziol–Green model with multiple changes. For such situation we can use permutation principle and simpler form of the test which will be described below.

Permutation principle. We proceed as for the two-sample and the max-type permutation tests in Sections 2 and 3. In the particular situation of the same time-points of changes in the distributions of the survival and the censoring variables, i.e. $\gamma_i = \eta_i$, $i = 1, 2, \dots q$, we consider the following form of the test statistic

$$T^{\sigma}_{n,D}(\tau_0) = \max_{D<k<n-D} \frac{|L_{k,D}(\tau_0)|}{\sqrt{2D}\sigma_n(\boldsymbol{a})}, \tag{2.52}$$

where $L_{k,D}(\tau_0)$ is given by (2.31) and

$$\sigma^2_n(\boldsymbol{a}) = \frac{1}{n-1}\sum_{j=1}^{n}(a_n(j))^2.$$

In the same way as in the previous sections we use common notation with $\boldsymbol{Q} = (Q_1, Q_2, \dots, Q_n)$ being a random permutation of (1,2, …, n), i.e.

$$T^{\sigma}_{n,D}(\tau_0, \boldsymbol{Q}) = \max_{D<k<n-D} \frac{|\sum_{j=k+1}^{k+D} a_n(Q_j) - \sum_{j=k-D+1}^{k} a_n(Q_j|}{\sqrt{2D}\sigma_n(\boldsymbol{a})}$$

and the exact (permutation) test for the restricted null hypothesis $\bar{H}_0$ of i.i.d. survival and i.i.d. censoring variables is given by

$$\psi_3(t, (\boldsymbol{X}, \boldsymbol{\Delta})) = \begin{cases} 1, & \text{if } T^{\sigma}_{n,D}(\tau_0, \boldsymbol{Q}) > c_{n,D}(\alpha, (\boldsymbol{X}, \boldsymbol{\Delta})), \\ \nu_3 \in (0,1), & \text{if } T^{\sigma}_{n,D}(\tau_0, \boldsymbol{Q}) = c_{n,D}(\alpha, (\boldsymbol{X}, \boldsymbol{\Delta})), \\ 0, & \text{if } T^{\sigma}_{n,D}(\tau_0, \boldsymbol{Q}) < c_{n,D}(\alpha, (\boldsymbol{X}, \boldsymbol{\Delta})), \end{cases}$$

where $c_{n,D}(\alpha, (\boldsymbol{X}, \boldsymbol{\Delta}))$ and ν_3 are determined by

$$\mathrm{P}\left(T^{\sigma}_{n,D}(\tau_0, \boldsymbol{Q}) > c_{n,D}(\alpha, (\boldsymbol{X}, \boldsymbol{\Delta})) \,|\, (\boldsymbol{X}, \boldsymbol{\Delta})\right) + \nu_3 \, \mathrm{P}\left(T^{\sigma}_{n,D}(\tau_0, \boldsymbol{Q}) = c_{n,D}(\alpha, (\boldsymbol{X}, \boldsymbol{\Delta})) \,|\, (\boldsymbol{X}, \boldsymbol{\Delta})\right) = \alpha$$

under the restricted null hypothesis $\bar{H}_0$. Thus, $c_{n,D}(\alpha, (\boldsymbol{X}, \boldsymbol{\Delta}))$ stands for $100(1-\alpha)\%$-quantile of the permutation distribution which is expressed as

$$\mathrm{P}(T^{\sigma}_{n,D}(\tau_0) \le x \,|\, (\boldsymbol{X}, \boldsymbol{\Delta})) = \frac{1}{n!}\#\{\boldsymbol{q} \in \mathcal{Q}_n; T^{\sigma}_{n,D}(\tau_0, \boldsymbol{q}) \le x\}, \quad x \in \mathbb{R},$$

where $\mathcal{Q}_n$ is the set of all permutations of $(1, 2, \dots, n)$.

Now, we state the limit distribution of the MOSUM-type test statistic $T_{n,D}(\tau_0, \boldsymbol{Q})$.

Corollary 2.4. *Suppose that $X^0_1, X^0_2, \dots, X^0_n$, $C_1, C_2, \dots, C_n$ are independent random variables. There exist $0 = \gamma_0 < \gamma_1 < \dots < \gamma_q < \gamma_{q+1} = 1$ with some finite $q \in \mathbb{N}$ such that variables $X^0_{\lfloor n\gamma_i \rfloor+1}, X^0_{\lfloor n\gamma_i \rfloor+2}, \dots, X^0_{\lfloor n\gamma_{i+1} \rfloor}$ have an absolutely continuous distribution function F_{i+1} and $C_{\lfloor n\gamma_i \rfloor+1}, C_{\lfloor n\gamma_1 \rfloor+2}, \dots, C_{\lfloor n\gamma_{i+1} \rfloor}$ have an absolutely continuous distribution function G_{i+1} for $i = 0, 1, \dots, q$. Let* (2.9) *and* (2.25) *be satisfied. Then for all $y \in \mathbb{R}$ we have, as $n \to \infty$,*

$$\mathrm{P}\left(d_1\Big(\frac{n}{D}\Big)\, T^{\sigma}_{n,D}(\tau_0, \boldsymbol{Q}) \le y + d_2\Big(\frac{n}{D}\Big) + \log\Big(\frac{3}{2}\Big) \,|\, (\boldsymbol{X}, \boldsymbol{\Delta})\right) \xrightarrow{\mathrm{P}} \exp\left\{-2\mathrm{e}^{-y}\right\}$$

and, moreover under $\bar{H}_0 : \gamma_i = 1$ *for* $i = 1, 2, \dots, q$

$$\mathrm{P}\left(d_1\Big(\frac{n}{D}\Big)\, T^{\sigma}_{n,D}(\tau_0) \le y + d_2\Big(\frac{n}{D}\Big) + \log\Big(\frac{3}{2}\Big)\right) \to \exp\big\{-2\mathrm{e}^{-y}\big\}$$

with d_1 *and* d_2 *defined in* (1.19).

Proof. We use the same steps as in the proof of Corollary 2.1 or Theorem 2.4, respectively. □

Now it can be seen that the asymptotic critical region for testing multiple changes based on $T_{n,D}(\tau_0)$ and $T^{\sigma}_{n,D}(\tau_0)$ is the same and given by (2.49).

5. Limit behavior under alternatives

Here we investigate limit behavior of the considered test statistics. Particularly, we concentrate on the consistency of the tests. Recall the definition.

Definition 2.1. *Let* T_n, $n = 1, 2, \dots$, *be a sequence of test statistics used to test hypothesis* H, *and* R_n, $n = 1, 2, \dots$, *an associated set of level* α *rejection regions. The sequence* T_n *is said to be* consistent *under an alternative hypothesis* H_A *if*

$$\lim_{n\to\infty} \mathrm{P}(T_n \in R_n) = 1$$

when H_A *is true.*

Notice that for the considered test statistics under H_0 we have, as $n \to \infty$,

$$T_n(\tau_0)\,(\log\log n)^{-1/2} = O_{\mathrm{P}}(1),$$
$$T_{n,D}(\tau_0)\,(\log n)^{-1/2} = O_{\mathrm{P}}(1)$$

and under $\bar{H}_0$ we get, as $n \to \infty$,

$$T^{\sigma}_n(\tau_0)\,(\log\log n)^{-1/2} = O_{\mathrm{P}}(1),$$
$$T^{\sigma}_{n,D}(\tau_0)\,(\log n)^{-1/2} = O_{\mathrm{P}}(1).$$

Therefore to show the consistency it suffices to show that under alternatives

$$T_n(\tau_0)\,(\log\log n)^{-1/2} \xrightarrow{\mathrm{P}} \infty, \qquad \text{as } n \to \infty,$$
$$T^{\sigma}_n(\tau_0)\,(\log\log n)^{-1/2} \xrightarrow{\mathrm{P}} \infty, \qquad \text{as } n \to \infty,$$
$$T_{n,D}(\tau_0)\,(\log n)^{-1/2} \xrightarrow{\mathrm{P}} \infty, \qquad \text{as } n \to \infty,$$
$$T^{\sigma}_{n,D}(\tau_0)\,(\log n)^{-1/2} \xrightarrow{\mathrm{P}} \infty, \qquad \text{as } n \to \infty.$$

We prove that it holds.

Max-type test statistic. We present the consistency of the max-type test statistics $T_n(\tau_0)$ or $T^{\sigma}_n(\tau_0)$ defined in (2.18) or (2.22), respectively, used for the test H_0 or $\bar{H}_0$, respectively, against the one-change alternative H_1.

Theorem 2.6. *Suppose that* $X^0_1, X^0_2, \dots, X^0_n$, $C_1, C_2, \dots, C_n$ *are independent random variables. Let the survival variables* $X^0_1, X^0_2, \dots, X^0_{\lfloor n\gamma\rfloor}$ *and* $X^0_{\lfloor n\gamma\rfloor+1}, X^0_{\lfloor n\gamma\rfloor+2}, \dots, X^0_n$ *have absolutely continuous distribution functions* F_1 *and* F_2, *respectively,* $F_1 \ne F_2$, *and let the censoring variables* $C_1, C_2, \dots, C_{\lfloor n\eta\rfloor}$ *and* $C_{\lfloor n\eta\rfloor+1}, C_{\lfloor n\eta\rfloor+2}, \dots, C_n$ *have absolutely continuous distribution* *functions* G_1 *and* G_2*, respectively,* $G_1 \neq G_2$*, for some* $0 < \eta \leq \gamma < 1$*. Let* (2.17) *be satisfied and let*

$$\int_0^{\tau_0} w(t)\,\frac{Q_1(t)\,Q_2(t)}{Q_1(t)+Q_2(t)}\,(\lambda_1(t)-\lambda_2(t))\,\mathrm{d}t \neq 0, \tag{2.53}$$

where $Q_1(t)$ *and* $Q_2(t)$ *are defined in* (1.14) *and* $\lambda_1(t)$ *and* $\lambda_2(t)$ *are the hazard functions corresponding to the distribution functions* $F_1(t)$ *and* $F_2(t)$*, respectively. If*

$$\int_0^{\tau_0} w^2(t)\,\frac{Q_1(t)\,Q_2(t)}{(Q_1(t)+Q_2(t))^2}\,(Q_1(t)\lambda_1(t)+Q_2(t)\lambda_2(t))\,\mathrm{d}t > 0,$$

then, as $n \to \infty$*,*

$$n^{u-\frac{1}{2}}\,T_n(\tau_0) \xrightarrow{\mathrm{P}} \infty$$

for any $u > 0$*.*

Proof. The proof of the theorem can be found in Hušková and Neuhaus [**23**], Theorem 1.2. □

Now, we show the consistency of the test based on $T_n^\sigma(\tau_0)$ which differs from $T_n(\tau_0)$ only by the standardization and which is appropriate only for testing the restricted null hypothesis $\bar{H}_0$.

Theorem 2.7. *Suppose that* $X_1^0, X_2^0, \ldots, X_n^0$*,* $C_1, C_2, \ldots, C_n$ *are independent random variables. Let the survival variables* $X_1^0, X_2^0, \ldots, X_{\lfloor n\gamma\rfloor}^0$ *and* $X_{\lfloor n\gamma\rfloor+1}^0, X_{\lfloor n\gamma\rfloor+2}^0, \ldots, X_n^0$ *have absolutely continuous distribution functions* F_1 *and* F_2*, respectively,* $F_1 \neq F_2$*, and let the censoring variables* $C_1, C_2, \ldots, C_{\lfloor n\eta\rfloor}$ *and* $C_{\lfloor n\eta\rfloor+1}, C_{\lfloor n\eta\rfloor+2}, \ldots, C_n$ *have absolutely continuous distribution functions* G_1 *and* G_2*, respectively,* $G_1 \neq G_2$*, for some* $0 < \eta \leq \gamma < 1$*. Let* (2.9) *and* (2.53) *be satisfied. If*

$$\int_0^{\tau_0} w^2(t)\,\mathrm{d}R_{\eta,\gamma}(t) > 0 \tag{2.54}$$

holds, where $R_{\eta,\gamma}(t)$ *is defined in* (1.10)*. Then, as* $n \to \infty$*,*

$$n^{u-\frac{1}{2}}\,T_n^\sigma(\tau_0) \xrightarrow{\mathrm{P}} \infty \tag{2.55}$$

for any $u > 0$*.*

Proof. By the definition of the test statistic $T_n^\sigma(\tau_0)$ we have

$$T_n^\sigma(\tau_0) = \max_{1\leq k<n} |L_k^\sigma(\tau_0)| \geq |L_m^\sigma(\tau_0)| = \sqrt{\frac{n}{m(n-m)}}\,\frac{|S_m(\tau_0)|}{\sigma_n(\boldsymbol{a})}. \tag{2.56}$$

By Lemma 5.4 below we know that

$$\sigma_n^2(\boldsymbol{a}) \xrightarrow{\mathrm{P}} \int_0^{\tau_0} w^2(t)\,\mathrm{d}R_{\eta,\gamma}(t) > 0, \quad n \to \infty, \tag{2.57}$$

in view of the assumption (2.54).
Corollary 5.7 below gives, as $n \to \infty$,

$$\frac{S_m(\tau_0)}{n} \xrightarrow{\mathrm{P}} \int_0^{\tau_0} w(t)\,\frac{Q_1(t)\,Q_2(t)}{Q_1(t)+Q_2(t)}\,(\lambda_1(t)-\lambda_2(t))\,\mathrm{d}t \neq 0$$

since (2.53). From this and (2.57) it follows that

$$\frac{1}{\sqrt{n}}\,L_m^\sigma(\tau_0) \xrightarrow{\mathrm{P}} \sqrt{\frac{1}{\gamma(1-\gamma)}}\,\frac{\int_0^{\tau_0} w(t)\,\frac{Q_1(t)\,Q_2(t)}{Q_1(t)+Q_2(t)}\,(\lambda_1(t)-\lambda_2(t))\,\mathrm{d}t}{\sqrt{\int_0^{\tau_0} w^2(t)\,\mathrm{d}R_{\eta,\gamma}(t)}} \neq 0, \quad n \to \infty.$$

This together with (2.56) ensure that the assertion (2.55) of the theorem holds. □

Notice that the test statistic $T_n^\sigma(\tau_0)$ is used for testing $\bar{H}_0 : \gamma = \eta = 1$ as we said above, but the alternative H_1 can be supposed general, i.e. $\gamma \in (0,1)$, $\eta \in (0,1]$ need not be the same. We can investigate also the consistency of $T_n^\sigma(\tau_0)$ under special alternatives.

Corollary 2.5. *Suppose that $X_1^0, X_2^0, \ldots, X_n^0$, $C_1, C_2, \ldots, C_n$ are independent random variables. Let the survival variables $X_1^0, X_2^0, \ldots, X_{\lfloor n\gamma \rfloor}^0$ and $X_{\lfloor n\gamma \rfloor+1}^0, X_{\lfloor n\gamma \rfloor+2}^0, \ldots, X_n^0$ have absolutely continuous distribution functions F_1 and F_2, respectively, $F_1 \neq F_2$, and let the censoring variables $C_1, C_2, \ldots, C_{\lfloor n\gamma \rfloor}$ and $C_{\lfloor n\gamma \rfloor+1}, C_{\lfloor n\gamma \rfloor+2}, \ldots, C_n$ have absolutely continuous distribution functions G_1 and G_2, respectively, $G_1 \neq G_2$, for some $\gamma \in (0,1)$. Let (2.9) be satisfied. If (2.10) and*

$$\int_0^{\tau_0} w(t)\, \frac{(1-F_1(t))(1-F_2(t))(1-G_1(t))(1-G_2(t))}{1-H_\gamma(t)}\, (\lambda_1(t)-\lambda_2(t))\, \mathrm{d}t \neq 0 \tag{2.58}$$

hold. Then we have, as $n \to \infty$,

$$n^{u-\frac{1}{2}}\, T_n^\sigma(\tau_0) \xrightarrow{\mathrm{P}} \infty$$

for any $u > 0$.

Proof. We use the same steps as in the proof of Theorem 2.7, but the assertion of Lemma 5.4 is replaced by the assertion of Corollary 5.5 and further, we apply instead of the first part of Corollary 5.7 the second one. □

Corollary 2.6. *Suppose that $X_1^0, X_2^0, \ldots, X_n^0$, $C_1, C_2, \ldots, C_n$ are independent random variables. Let $X_1^0, X_2^0, \ldots, X_{\lfloor n\gamma \rfloor}^0$ and $X_{\lfloor n\gamma \rfloor+1}^0, X_{\lfloor n\gamma \rfloor+2}^0, \ldots, X_n^0$ have absolutely continuous distribution functions F_1 and F_2, respectively, $F_1 \neq F_2$, for some $\gamma \in (0,1)$. Let $C_1, C_2, \ldots, C_n$ have an absolutely continuous distribution function G . Let (2.9) be satisfied. If*

$$\int_0^{\tau_0} w(t)\, \frac{(1-F_1(t))(1-F_2(t))(1-G(t))}{\gamma(1-F_1(t))+(1-\gamma)(1-F_2(t))}\, (\lambda_1(t)-\lambda_2(t))\, \mathrm{d}t \neq 0, \tag{2.59}$$

$$\int_0^{\tau_0} w^2(t)\,(1-G(t)\,\mathrm{d}F_i(t) > 0, \quad i = 1,2,$$

hold. Then we have, as $n \to \infty$,

$$n^{u-\frac{1}{2}}\, T_n^\sigma(\tau_0) \xrightarrow{\mathrm{P}} \infty$$

for any $u > 0$.

Proof. Putting $G_1(t) = G_2(t) = G(t)$ for all t we see that the assertion follows directly from Corollary 2.5. □

The assumptions of Theorems 2.6 and 2.7 will be discussed in Chapter 4.

MOSUM-type test statistic. In the following theorems we will derive the consistency of the MOSUM-type test based on $T_{n,D}(\tau_0)$ or $T_{n,D}^\sigma(\tau_0)$ given by (2.33) or (2.52), respectively. Recall that $T_{n,D}(\tau_0)$ can be used more frequently, but we have to estimate the time of a change in the distribution of the censoring variables.

First, supposing that m_c is known, we will treat the limit behavior of $T_{n,D}(\tau_0)$ under the one-change alternative H_1 for the lifetime distributions and fixed as well local alternatives for the censoring distribution functions G_1, G_2 and G_{1n}, G_{2n}, respectively. Unfortunately, the results are non-transparent and they are presented only for the sake of completeness. Additionally, we will not deal with the situation with the estimated m_c.

After that, we will study the consistency of $T^{\sigma}_{n,D}(\tau_0)$ also under the general alternative H_1, but mainly under the particular multiple-change alternatives, where the time-points for changes in the distribution of the survival and the censoring variables coincide ($\gamma_i = \eta_i \in (0,1)$, $i = 1,2,\dots,q$ with $q \geq 1$), or the changes occur only in the distribution of the lifetimes ($\gamma_i \in (0,1)$, $i = 1,2,\dots,q$) and the censoring variables are i.i.d.

THEOREM 2.8. *Suppose that $X^0_1, X^0_2, \dots, X^0_n$, $C_1, C_2, \dots, C_n$ are independent random variables. Let the survival variables $X^0_1, X^0_2, \dots, X^0_{\lfloor n\gamma\rfloor}$ and $X^0_{\lfloor n\gamma\rfloor+1}, X^0_{\lfloor n\gamma\rfloor+2}, \dots, X^0_n$ have absolutely continuous distribution functions F_1 and F_2, respectively, $F_1 \neq F_2$, and let the censoring variables $C_1, C_2, \dots, C_{\lfloor n\eta\rfloor}$ and $C_{\lfloor n\eta\rfloor+1}, C_{\lfloor n\eta\rfloor+2}, \dots, C_n$ have absolutely continuous distribution functions G_1 and G_2, respectively, $G_1 \neq G_2$, for some $0 < \eta < \gamma < 1$. Let (2.9) and*

$$\int_0^{\tau_0} w(t)\frac{(1-F_1(t))(\eta(1-G_1(t)) + (1-\eta)(1-G_2(t)))\,Q_2(t)}{1-H_{\eta,\gamma}(t)}\,(\lambda_2(t)-\lambda_1(t))\,\mathrm{d}t \neq 0 \tag{2.60}$$

be satisfied. If

$$\begin{aligned}\int_0^{\tau_0} w^2(t)(1-G_1(t))\,\mathrm{d}F_1(t) > 2\iint_{t_1\le t_2} &\frac{w(t_1)w(t_2)(1-F_1(t_2))(1-G_1(t_2))Q_2(t_2)}{(1-H_{\eta,\gamma}(t_1))(1-H_{\eta,\gamma}(t_2))}\\ &(Q_1(t_1)\lambda_1(t_1)+Q_2(t_1)\lambda_2(t_1))\,(\lambda_1(t_2)-\lambda_2(t_2))\,\mathrm{d}t_1\,\mathrm{d}t_2\\ +2\iint_{t_1\le t_2} &\frac{w(t_1)w(t_2)(1-F_1(t_1))(1-G_1(t_1))(1-F_1(t_2))(1-G_1(t_2))Q_2(t_1)Q_2(t_2)}{(1-H_{\eta,\gamma}(t_1))(1-H_{\eta,\gamma}(t_2))}\\ &(\lambda_1(t_1)-\lambda_2(t_1))(\lambda_1(t_2)-\lambda_2(t_2))\,\mathrm{d}t_1\,\mathrm{d}t_2\end{aligned} \tag{2.61}$$

and

$$\begin{aligned}&\frac{\gamma-\eta}{1-\eta}\int_0^{\tau_0} w^2(t)(1-G_2(t))\,\mathrm{d}F_1(t) + \frac{1-\gamma}{1-\eta}\int_0^{\tau_0} w^2(t)(1-G_2(t))\,\mathrm{d}F_2(t)\\ &\quad+2\frac{\eta}{1-\eta}\iint_{t_1\le t_2}\frac{w(t_1)w(t_2)(1-F_1(t_2))(1-G_1(t_2))Q_2(t_2)}{(1-H_{\eta,\gamma}(t_1))(1-H_{\eta,\gamma}(t_2))}\\ &\quad(Q_1(t_1)\lambda_1(t_1)+Q_2(t_1)\lambda_2(t_1))\,(\lambda_1(t_2)-\lambda_2(t_2))\,\mathrm{d}t_1\,\mathrm{d}t_2\\ &>2\Big(\frac{\eta}{1-\eta}\Big)^2\iint_{t_1\le t_2}\frac{w(t_1)w(t_2)(1-F_1(t_1))(1-G_1(t_1))(1-F_1(t_2))(1-G_1(t_2))Q_2(t_1)Q_2(t_2)}{(1-H_{\eta,\gamma}(t_1))(1-H_{\eta,\gamma}(t_2))}\\ &\quad(\lambda_1(t_1)-\lambda_2(t_1))(\lambda_1(t_2)-\lambda_2(t_2))\,\mathrm{d}t_1\,\mathrm{d}t_2,\end{aligned} \tag{2.62}$$

where $Q_1(t)$ and $Q_2(t)$ are defined in (1.14) and $\lambda_1(t)$ and $\lambda_2(t)$ are the hazard functions corresponding to the distribution functions $F_1(t)$ and $F_2(t)$, respectively. Then, as $n \to \infty$,

$$D^{u-\frac{1}{2}}\,T_{n,D}(\tau_0) \xrightarrow{\ \mathrm{P}\ } \infty$$

for any $u > 0$.

PROOF. It can be assumed w.l.g. that $m \in (m_c + D, n - D)$. By the definition of $T_{n,D}(\tau_0)$ we have

$$\begin{aligned}T_{n,D}(\tau_0) &\geq \frac{|L_{m,D}(\tau_0)|}{\sqrt{2D}\,\sigma^0_{m_c}(\boldsymbol{a})} = \frac{|S_{m+D}(\tau_0) - S_m(\tau_0) + S_{m-D}(\tau_0)|}{\sqrt{2D}\,\sigma^0_{m_c}(\boldsymbol{a})}\\ &= \frac{|\sum_{j=m+1}^{m+D} a_n(j) - \sum_{j=m-D+1}^{m} a_n(j)|}{\sqrt{2D}\,\sigma^0_{m_c}(\boldsymbol{a})}.\end{aligned}$$

In view of Lemma 5.6 below the conditions (2.61) and (2.62) ensure that limits of $\sigma^2_{m_c}(\boldsymbol{a})$ and $\sigma^{0\,2}_{m_c}(\boldsymbol{a})$ tend to a positive constant. By Corollary 5.8 below we have, as $n \to \infty$,

$$\frac{1}{D}\sum_{j=m+1}^{m+D} a_n(j) = \int_0^{\tau_0} w(t)\frac{(1-F_1(t))(1-F_2(t))(1-G_2(t))(\eta(1-G_1(t))+(\gamma-\eta)(1-G_2(t)))}{1-H_{\eta,\gamma}(t)}$$
$$\left(\frac{\mathrm{d}F_2(t)}{1-F_2(t)} - \frac{\mathrm{d}F_1(t)}{1-F_1(t)}\right) + o_{\mathrm{P}}(1)$$
$$\frac{1}{D}\sum_{j=m-D+1}^{m} a_n(j) = (1-\gamma)\int_0^{\tau_0} w(t)\frac{(1-F_1(t))(1-F_2(t))(1-G_2(t))^2}{1-H_{\eta,\gamma}(t)}\left(\frac{\mathrm{d}F_1(t)}{1-F_1(t)} - \frac{\mathrm{d}F_2(t)}{1-F_2(t)}\right)$$
$$+ o_{\mathrm{P}}(1)$$

and consequently

$$\frac{1}{D}\left(\sum_{j=m+1}^{m+D} a_n(j) - \sum_{j=m-D+1}^{m} a_n(j)\right)$$
$$= \int_0^{\tau_0} w(t)\frac{(1-F_1(t))(1-F_2(t))(1-G_2(t))(\eta(1-G_1(t))+(1-\eta)(1-G_2(t)))}{1-H_{\eta,\gamma}(t)}$$
$$\left(\frac{\mathrm{d}F_2(t)}{1-F_2(t)} - \frac{\mathrm{d}F_1(t)}{1-F_1(t)}\right) + o_{\mathrm{P}}(1)$$
$$= \frac{1}{1-\gamma}\int_0^{\tau_0} w(t)\frac{(1-F_1(t))(\eta(1-G_1(t))+(1-\eta)(1-G_2(t)))\,Q_2(t)}{1-H_{\eta,\gamma}(t)}\,(\lambda_2(t)-\lambda_1(t))\;\mathrm{d}t$$
$$+ o_{\mathrm{P}}(1).$$

This implies, as $n \to \infty$,

$$\frac{1}{D}\,L_{m,D}(\tau_0) = \frac{1}{1-\gamma}\int_0^{\tau_0} w(t)\frac{(1-F_1(t))(\eta(1-G_1(t))+(1-\eta)(1-G_2(t)))\,Q_2(t)}{1-H_{\eta,\gamma}(t)}$$
$$(\lambda_2(t)-\lambda_1(t))\;\mathrm{d}t + o_{\mathrm{P}}(1)$$

which completes the proof. □

Recall that the MOSUM-test statistic $T_{n,D}(\tau_0)$ is convenient to use for the situation of the different but very "closed" censoring distribution functions $G_{1n}(t)$ and $G_{2n}(t)$ and therefore we take into account also this situation.

Corollary 2.7. *Suppose that $X^0_{1n}, X^0_{2n}, \dots, X^0_{nn}$, $C_{1n}, C_{2n}, \dots, C_{nn}$ are independent random variables. Let the survival variables $X^0_{1n}, X^0_{2n}, \dots, X^0_{\lfloor n\gamma\rfloor\, n}$ and $X^0_{\lfloor n\gamma\rfloor+1\, n}, X^0_{\lfloor n\gamma\rfloor+2\, n}, \dots, X^0_{nn}$ have absolutely continuous distribution functions F_1 and F_2, respectively, $F_1 \neq F_2$, and let the censoring variables $C_{1n}, C_{2n}, \dots, C_{\lfloor n\eta\rfloor\, n}$ and $C_{\lfloor n\eta\rfloor+1\, n}, C_{\lfloor n\eta\rfloor+2\, n}, \dots, C_{nn}$ have absolutely continuous distribution functions G_{1n} and G_{2n}, respectively, for some $0 < \eta < \gamma < 1$. The distribution functions G_{1n} and G_{2n} fulfil the condition (A.2) in Chapter 3. Let* (2.9) *and* (2.59) *be* *satisfied. If*

$$\int_0^{\tau_0} w^2(t)(1-G(t))\,\mathrm{d}F_1(t) > 2(1-\gamma)\iint_{t_1\le t_2} \frac{w(t_1)\,(\gamma f_1(t_1)+(1-\gamma)f_2(t_1))}{\gamma(1-F_1(t_1))+(1-\gamma)(1-F_2(t_1))}$$
$$\frac{w(t_2)(1-F_1(t_2))(1-F_2(t_2))(1-G(t_2))}{\gamma(1-F_1(t_2))+(1-\gamma)(1-F_2(t_2))}(\lambda_1(t_2)-\lambda_2(t_2))\,\mathrm{d}t_1\,\mathrm{d}t_2$$
$$+\,2(1-\gamma)^2\iint_{t_1\le t_2} \frac{w(t_1)(\,1-F_1(t_1))(1-F_2(t_1))\,(1-G(t_1))}{\gamma(1-F_1(t_1))+(1-\gamma)(1-F_2(t_1))}$$
$$\frac{w(t_2)\,(1-F_1(t_2))\,(1-F_2(t_2))\,(1-G(t_2))}{\gamma(1-F_1(t_2))+(1-\gamma)(1-F_2(t_2))}\,(\lambda_1(t_1)-\lambda_2(t_1))\,(\lambda_1(t_2)-\lambda_2(t_2))\,\mathrm{d}t_1\,\mathrm{d}t_2$$

and

$$\frac{\gamma-\eta}{1-\eta}\int_0^{\tau_0} w^2(t)(1-G(t))\,\mathrm{d}F_1(t)+\frac{1-\gamma}{1-\eta}\int_0^{\tau_0} w^2(t)(1-G(t))\,\mathrm{d}F_2(t)$$
$$+\,2\frac{(1-\gamma)\eta}{1-\eta}\iint_{t_1\le t_2} \frac{w(t_1)\,(\gamma f_1(t_1)+(1-\gamma)f_2(t_1))}{\gamma(1-F_1(t_1))+(1-\gamma)(1-F_2(t_1))}$$
$$\frac{w(t_2)(1-F_1(t_2))(1-F_2(t_2))(1-G(t_2))}{\gamma(1-F_1(t_2))+(1-\gamma)(1-F_2(t_2))}(\lambda_1(t_2)-\lambda_2(t_2))\,\mathrm{d}t_1\,\mathrm{d}t_2$$
$$>2\Big(\frac{\eta\,(1-\gamma)}{1-\eta}\Big)^2\iint_{t_1\le t_2} \frac{w(t_1)(1-F_1(t_1))(1-F_2(t_1))(1-G(t_1))(1-F_1(t_2))}{\gamma(1-F_1(t_1))+(1-\gamma)(1-F_2(t_1))}$$
$$\frac{w(t_2)(1-F_2(t_2))(1-G(t_2))}{(\gamma(1-F_1(t_2))+(1-\gamma)(1-F_2(t_2)))}(\lambda_1(t_1)-\lambda_2(t_1))(\lambda_1(t_2)-\lambda_2(t_2))\,\mathrm{d}t_1\,\mathrm{d}t_2,$$

where $\lambda_1(t)$ and $\lambda_2(t)$ are the hazard functions corresponding to the distribution functions $F_1(t)$ and $F_2(t)$, respectively. Then, as $n\to\infty$,

$$D^{u-\frac{1}{2}}\,T_{n,D}(\tau_0)\xrightarrow{\mathrm{P}}\infty$$

for any $u>0$.

Proof. The corollary follows directly from Theorem 2.8 if we use the assumption (A.2) for the convergence of G_{in}, $i=1,2$. □

Now, under the general one-change alternative H_1, we prove the consistency for the test based on $T^{\sigma}_{n,D}(\tau_0)$ which is appropriate for testing $\bar{H}_0$.

Theorem 2.9. *Suppose that $X_1^0, X_2^0, \dots, X_n^0$, $C_1, C_2, \dots, C_n$ are independent random variables. Let the survival variables $X_1^0, X_2^0, \dots, X_{\lfloor n\gamma\rfloor}^0$ and $X_{\lfloor n\gamma\rfloor+1}^0, X_{\lfloor n\gamma\rfloor+2}^0, \dots, X_n^0$ have absolutely continuous distribution functions F_1 and F_2, respectively, $F_1\ne F_2$, and let the censoring variables $C_1, C_2, \dots, C_{\lfloor n\eta\rfloor}$ and $C_{\lfloor n\eta\rfloor+1}, C_{\lfloor n\eta\rfloor+2}, \dots, C_n$ have absolutely continuous distribution functions G_1 and G_2, respectively, $G_1\ne G_2$, for some $0<\eta<\gamma<1$. Let* (2.9) *and* (2.60) *be satisfied. If* (2.54) *holds, then, as $n\to\infty$,*

$$D^{u-\frac{1}{2}}\,T^{\sigma}_{n,D}(\tau_0)\xrightarrow{\mathrm{P}}\infty$$

for any $u>0$.

Proof. The proof is nearly the same as the proof of Theorem 2.8 only instead of investigation of the limit behavior of $\sigma^2_{m_c}(\boldsymbol{a})$ or $\sigma^{0\,2}_{m_c}(\boldsymbol{a})$ given by (2.34) and (2.35) under the considered alternative we investigate the behavior of $\sigma_n^2(\boldsymbol{a})$. Lemma 5.4 below yields

$$\sigma_n^2(\boldsymbol{a}) \xrightarrow{\mathrm{P}} \int_0^{\tau_0} w^2(t)\,\mathrm{d}R_{\eta,\gamma}(t), \quad n \to \infty,$$

and by the assumption (2.54) we obtain that $\sigma_n^2(\boldsymbol{a})$ is asymptotically bounded away from 0. □

We show also the consistency of the test based on $T^{\sigma}_{n,D}(\tau_0)$ under the special alternatives which are more frequent in this case.

THEOREM 2.10. *Suppose that* $X_1^0, X_2^0, \dots, X_n^0$, $C_1, C_2, \dots, C_n$ *are independent random variables. There exist* $0 = \gamma_0 < \gamma_1 < \cdots < \gamma_q < \gamma_{q+1} = 1$ *with some finite* $q \in \mathbb{N}$ *such that variables* $X^0_{\lfloor n\gamma_i\rfloor+1}, X^0_{\lfloor n\gamma_i\rfloor+2}, \dots, X^0_{\lfloor n\gamma_{i+1}\rfloor}$ *have an absolutely continuous distribution function* F_{i+1} *and* $C_{\lfloor n\gamma_i\rfloor+1}, C_{\lfloor n\gamma_1\rfloor+2}, \dots, C_{\lfloor n\gamma_{i+1}\rfloor}$ *have an absolutely continuous distribution function* G_{i+1} *for* $i = 0, 1, \dots, q$. *Further, suppose that* $F_{i+1} \neq F_i$ *and* $G_{i+1} \neq G_i$, $i = 1, 2, \dots, q$. *Let* (2.9) *and* (2.25) *be satisfied. If*

$$\max_{i=1,2,\dots,q}\Bigg|\sum_{j=1}^{q+1}\Bigg\{(\gamma_j-\gamma_{j-1})\int_0^{\tau_0} w(t)\,\frac{(1-F_j(t))(1-G_j(t))}{1-H_{\gamma_1,\dots,\gamma_q}(t)}$$
$$((1-F_{i+1}(t))(1-G_{i+1}(t))(\lambda_{i+1}(t)-\lambda_j(t)) - (1-F_i(t))(1-G_i(t))(\lambda_i(t)-\lambda_j(t)))\,\mathrm{d}t\Bigg\}\Bigg| > 0 \tag{2.63}$$

holds, where $H_{\gamma_1,\dots,\gamma_q}(t)$ *is defined in* (2.26). *Then we have, as* $n \to \infty$,

$$D^{u-\frac{1}{2}}\, T^{\sigma}_{n,D}(\tau_0) \xrightarrow{\mathrm{P}} \infty$$

for any $u > 0$.

PROOF. It can be assumed w.l.g. that

$$\lfloor n\gamma_{i-1}\rfloor \le \lfloor n\gamma_i\rfloor - D < \lfloor n\gamma_i\rfloor + D \le \lfloor n\gamma_{i+1}\rfloor, \quad i = 1, 2, \dots, q.$$

By the definition of the test statistic $T^{\sigma}_{n,D}(\tau_0)$ we have

$$T^{\sigma}_{n,D}(\tau_0) \ge \max_{i=1,2,\dots,q} \frac{\left|\sum_{j=\lfloor n\gamma_i\rfloor+1}^{\lfloor n\gamma_i\rfloor+D} a_n(j) - \sum_{j=\lfloor n\gamma_i\rfloor-D+1}^{\lfloor n\gamma_i\rfloor} a_n(j)\right|}{\sqrt{2D}\,\sigma_n(\boldsymbol{a})}.$$

By a small modification of Corollary 5.5 below we receive, as $n \to \infty$,

$$\begin{aligned}\sigma_n^2(\boldsymbol{a}) &= \int_0^{\tau_0} w^2(t)\,\mathrm{d}R_{\gamma_1,\dots,\gamma_q}(t) + o_{\mathrm{P}}(1)\\ &= \gamma_1\int_0^{\tau_0} w^2(t)\,(1-G_1(t))\,\mathrm{d}F_1(t) + (\gamma_2-\gamma_1)\int_0^{\tau_0} w^2(t)\,(1-G_2(t))\,\mathrm{d}F_2(t) + \cdots\\ &\quad + (1-\gamma_q)\int_0^{\tau_0} w^2(t)\,(1-G_{q+1}(t))\,\mathrm{d}F_{q+1}(t) + o_{\mathrm{P}}(1)\end{aligned}$$

since the definition (2.27) of $R_{\gamma_1,\dots,\gamma_q}(t)$. By the assumption (2.25) of our corollary we get that $\sigma_n^2(\boldsymbol{a})$ is asymptotically bounded away from 0.

Further, we use the same steps as in the proof of Corollary 5.8 below. Thus we obtain, as $n \to \infty$,

$$\frac{1}{D}\sum_{j=\lfloor n\gamma_i\rfloor+1}^{\lfloor n\gamma_i\rfloor+D} a_n(j) = \int_0^{\tau_0} w(t)\,\mathrm{d}\,\mathrm{E}\,N_{\lfloor n\gamma_{i+1}\rfloor}(t) - \int_0^{\tau_0} w(t)\,\frac{\mathrm{E}\,Y_{\lfloor n\gamma_{i+1}\rfloor}(t)}{1-H_{\gamma_1,\dots,\gamma_q}(t)}\,\mathrm{d}R_{\gamma_1,\dots,\gamma_q}(t) + o_{\mathrm{P}}(1)$$

$$
\begin{aligned}
&= \int_0^{\tau_0} w(t)\,(1-G_{i+1}(t))\,\mathrm{d}F_{i+1}(t) - \int_0^{\tau_0} w(t)\,\frac{(1-F_{i+1}(t))(1-G_{i+1}(t))}{1-H_{\gamma_1,\dots,\gamma_q}(t)}\,\mathrm{d}R_{\gamma_1,\dots,\gamma_q}(t) + o_{\mathrm{P}}(1)\\
&= \gamma_1 \int_0^{\tau_0} w(t)\,\frac{(1-F_1(t))(1-F_{i+1}(t))(1-G_1(t))(1-G_{i+1}(t))}{1-H_{\gamma_1,\dots,\gamma_q}(t)}\left(\frac{\mathrm{d}F_{i+1}(t)}{1-F_{i+1}(t)} - \frac{\mathrm{d}F_1(t)}{1-F_1(t)}\right)\\
&+ (\gamma_2-\gamma_1)\int_0^{\tau_0} w(t)\,\frac{(1-F_2(t))(1-F_{i+1}(t))(1-G_2(t))(1-G_{i+1}(t))}{1-H_{\gamma_1,\dots,\gamma_q}(t)}\left(\frac{\mathrm{d}F_{i+1}(t)}{1-F_{i+1}(t)} - \frac{\mathrm{d}F_2(t)}{1-F_2(t)}\right)\\
&+ \cdots\\
&+ (1-\gamma_q)\int_0^{\tau_0} w(t)\,\frac{(1-F_{q+1}(t))(1-F_{i+1}(t))(1-G_{q+1}(t))(1-G_{i+1}(t))}{1-H_{\gamma_1,\dots,\gamma_q}(t)}\left(\frac{\mathrm{d}F_{i+1}(t)}{1-F_{i+1}(t)} - \frac{\mathrm{d}F_{q+1}(t)}{1-F_{q+1}(t)}\right)\\
&+ o_{\mathrm{P}}(1)
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{1}{D}\sum_{j=\lfloor n\gamma_i\rfloor-D+1}^{\lfloor n\gamma_i\rfloor} a_n(j) = \int_0^{\tau_0} w(t)\,\mathrm{d}\,\mathrm{E}\,N_{\lfloor n\gamma_i\rfloor}(t) - \int_0^{\tau_0} w(t)\,\frac{\mathrm{E}\,Y_{\lfloor n\gamma_i\rfloor}(t)}{1-H_{\gamma_1,\dots,\gamma_q}(t)}\,\mathrm{d}R_{\gamma_1,\dots,\gamma_q}(t) + o_{\mathrm{P}}(1)\\
&= \int_0^{\tau_0} w(t)\,(1-G_i(t))\,\mathrm{d}F_i(t) - \int_0^{\tau_0} w(t)\,\frac{(1-F_i(t))(1-G_i(t))}{1-H_{\gamma_1,\dots,\gamma_q}(t)}\,\mathrm{d}R_{\gamma_1,\dots,\gamma_q}(t) + o_{\mathrm{P}}(1)\\
&= \gamma_1 \int_0^{\tau_0} w(t)\,\frac{(1-F_1(t))(1-F_i(t))(1-G_1(t))(1-G_i(t))}{1-H_{\gamma_1,\dots,\gamma_q}(t)}\left(\frac{\mathrm{d}F_i(t)}{1-F_i(t)} - \frac{\mathrm{d}F_1(t)}{1-F_1(t)}\right)\\
&+ (\gamma_2-\gamma_1)\int_0^{\tau_0} w(t)\,\frac{(1-F_2(t))(1-F_i(t))(1-G_2(t))(1-G_i(t))}{1-H_{\gamma_1,\dots,\gamma_q}(t)}\left(\frac{\mathrm{d}F_i(t)}{1-F_i(t)} - \frac{\mathrm{d}F_2(t)}{1-F_2(t)}\right)\\
&+ \cdots\\
&+ (1-\gamma_q)\int_0^{\tau_0} w(t)\,\frac{(1-F_{q+1}(t))(1-F_i(t))(1-G_{q+1}(t))(1-G_i(t))}{1-H_{\gamma_1,\dots,\gamma_q}(t)}\left(\frac{\mathrm{d}F_i(t)}{1-F_i(t)} - \frac{\mathrm{d}F_{q+1}(t)}{1-F_{q+1}(t)}\right)\\
&+ o_{\mathrm{P}}(1).
\end{aligned}
$$

This and the assumption (2.63) imply, as $n \to \infty$,

$$
\max_{i=1,2,\dots,q} \frac{\left|\sum_{j=\lfloor n\gamma_i\rfloor+1}^{\lfloor n\gamma_i\rfloor+D} a_n(j) - \sum_{j=\lfloor n\gamma_i\rfloor-D+1}^{\lfloor n\gamma_i\rfloor} a_n(j)\right|}{D} \xrightarrow{\mathrm{P}} \mathit{const} > 0.
$$

□

Corollary 2.8. *Suppose that $X_1^0, X_2^0, \dots, X_n^0$, $C_1, C_2, \dots, C_n$ are independent random variables. There exist $0 = \gamma_0 < \gamma_1 < \cdots < \gamma_q < \gamma_{q+1} = 1$ with some finite $q \in \mathbb{N}$ such that variables $X^0_{\lfloor n\gamma_i\rfloor+1}, X^0_{\lfloor n\gamma_i\rfloor+2}, \dots, X^0_{\lfloor n\gamma_{i+1}\rfloor}$ have an absolutely continuous distribution function F_{i+1} for $i = 0, 1, \dots q$ and $F_{i+1} \neq F_i$ for $i = 1, 2, \dots, q$. Let $C_1, C_2, \dots, C_n$ be i.i.d. variables with an absolutely continuous distribution function G . Let (2.9) be satisfied. If*

$$
\max_{i=1,2,\dots,q}\left|\sum_{j=1}^{q+1}\left\{(\gamma_j-\gamma_{j-1})\int_0^{\tau_0} w(t)\,\frac{(1-F_j(t))\,(1-G(t))}{\gamma_1(1-F_1(t))+\cdots+(1-\gamma_q)(1-F_{q+1}(t))}\right.\right.
$$
$$
\left.\left.((1-F_{i+1}(t))(\lambda_{i+1}(t)-\lambda_j(t)) - (1-F_i(t))(\lambda_i(t)-\lambda_j(t)))\,\mathrm{d}t\right\}\right| > 0
$$

and

$$\int_0^{\tau_0} w^2(t)\,(1-G(t))\,\mathrm{d}F_i(t) > 0, \quad i = 1, 2, \dots, q+1,$$

hold. Then we have, as $n \to \infty$,

$$D^{u-\frac{1}{2}}\, T^{\sigma}_{n,D}(\tau_0) \xrightarrow{\mathrm{P}} \infty$$

for any $u > 0$.

PROOF. Putting $G_1(t) = G_2(t) = \dots = G_{q+1}(t) = G(t)$ for all t we conclude that the assertion follows from Theorem 2.10. □

COROLLARY 2.9. *Suppose that $X_1^0, X_2^0, \dots, X_n^0$, $C_1, C_2, \dots, C_n$ are independent random variables. Let $X_1^0, X_2^0, \dots, X^0_{\lfloor n\gamma \rfloor}$ and $X^0_{\lfloor n\gamma \rfloor+1}, X^0_{\lfloor n\gamma \rfloor+2}, \dots, X_n^0$ have absolutely continuous distribution function F_1 and F_2, respectively, and let $C_1, C_2, \dots, C_{\lfloor n\gamma \rfloor}$ and $C_{\lfloor n\gamma \rfloor+1}, C_{\lfloor n\gamma \rfloor+2}, \dots, C_n$ have absolutely continuous distribution functions G_1 and G_2, respectively, for some $\gamma \in (0,1)$. Let* (2.9) *be satisfied. If* (2.10) *and* (2.58) *hold. Then we have, as $n \to \infty$,*

$$D^{u-\frac{1}{2}}\, T^{\sigma}_{n,D}(\tau_0) \xrightarrow{\mathrm{P}} \infty$$

for any $u > 0$.

PROOF. The assertion is a direct consequence of Theorem 2.10 by instituting of $q = 1$ into the assumptions (2.25) and (2.63) in view of

$$\begin{aligned}
&\sum_{j=1}^{2} \Bigg\{ (\gamma_j - \gamma_{j-1}) \int_0^{\tau_0} w(t)\, \frac{(1-F_j(t))(1-G_j(t))}{1-H_\gamma(t)} \\
&\qquad ((1-F_2(t))(1-G_2(t))(\lambda_2(t)-\lambda_j(t)) - (1-F_1(t))(1-G_1(t))(\lambda_1(t)-\lambda_j(t)))\,\mathrm{d}t \Bigg\} \\
&= \gamma \int_0^{\tau_0} w(t)\, \frac{(1-F_1(t))(1-G_1(t))}{1-H_\gamma(t)}\, (1-F_2(t))(1-G_2(t))(\lambda_2(t)-\lambda_1(t))\,\mathrm{d}t \\
&\quad - (1-\gamma) \int_0^{\tau_0} w(t)\, \frac{(1-F_2(t))(1-G_2(t))}{1-H_\gamma(t)}\, (1-F_1(t))(1-G_1(t))(\lambda_1(t)-\lambda_2(t))\,\mathrm{d}t \\
&= \int_0^{\tau_0} w(t)\, \frac{(1-F_1(t))(1-F_2(t))(1-G_1(t))(1-G_2(t))}{1-H_\gamma(t)}\, (\lambda_2(t)-\lambda_1(t))\,\mathrm{d}t \neq 0
\end{aligned}$$

or instead this calculation we can use directly Corollary 5.9 below. □

CHAPTER 3

Estimators

1. Introduction

The previous chapter deals with tests on the stability of censorship models. The null hypothesis H_0 claims that the distribution F of the survival times remains the same during the whole observation period and the alternative H_1 claims that at unknown time point $m = \lfloor n\gamma \rfloor$ the distribution F changes. Contrary to noncensored data, the situation is complicated by the parameter $m_c = \lfloor n\eta \rfloor$, the time of a change in the distribution G of the censoring variables.

Recall that $X^0_{1n}, X^0_{2n}, \dots, X^0_{mn}$ and $X^0_{m+1\,n}, X^0_{m+2\,n}, \dots, X^0_{nn}$ are independent with the absolutely continuous distribution functions F_{1n} and F_{2n}, respectively, $F_{1n} \neq F_{2n}$. The lifetimes are censored from the right by the independent random variables $C_{1n}, C_{2n}, \dots, C_{m_c n}$ and $C_{m_c+1\,n}, C_{m_c+2\,n}, \dots, C_{nn}$ which have the absolutely continuous distribution functions G_{1n} and G_{2n}, respectively, $G_{1n} \neq G_{2n}$, and are independent of the lifetimes.

We will distinguish the situations, when the differences $F_{1n} - F_{2n}$ either do not depend on n or tend to 0 in a certain way. The former case is called *the fixed alternatives* and the latter one *the local alternatives*.

Now, there are various versions for the assumption on the distribution functions F_{1n}, F_{2n} and G_{1n}, G_{2n}.

(S.1): the differences $F_{1n} - F_{2n}$ and $G_{1n} - G_{2n}$ do not depend on n
(the fixed alternatives for both types of the distribution functions);

(S.2): the differences $F_{1n} - F_{2n}$ go to 0 in some sense and $G_{1n} - G_{2n}$ do not depend on n
(the local alternatives for F_{1n}, F_{2n} and the fixed alternatives for G_{1n}, G_{2n});

(S.3): the differences $F_{1n} - F_{2n}$ and $G_{1n} - G_{2n}$ go to 0 in some sense
(the local alternatives for both types of the distribution functions).

We will treat the third situation describing the local alternatives. The remaining two situations (S.1) and (S.2) will be discussed in the next chapter.

We assume the following assumption:

(A.1): there exists $0 < \gamma < 1$ such that $m = \lfloor n\gamma \rfloor$.

Convention 3.1. In order to simplify the notation, we shall suppress the dependence of the random variables on n, i.e. we shall drop the index n and write $X^0_1, X^0_2, \dots, X^0_n$ and $C_1, C_2, \dots, C_n$, and F_i, G_i, $i = 1, 2$, whenever it does not cause a problem.

In the case of no censoring there have been published a number of papers and books interested in estimating the change-point m, e.g. Darkhovsky [**11**], Csörgő and Horváth [**8**], Dümbgen [**12**] considered nonparametric setup, Antoch et al [**7**] studied estimators for the change-point based on partial sums and Gombay and Hušková [**17**] investigated the limit behavior of rank based estimators. Antoch and Hušková [**3**] focused on the most often used estimators as the least square time estimators, M-estimators and rank estimators and they described their properties. Detection of a change point and related problems can be also found in Csörgő and Horváth [**10**].

Motivated by this, we introduce estimators of the change-point m based on the class of the rank statistics $S_k(\tau_0)$ given by (2.15). We define two classes of corresponding estimators of the form

$$\hat{m}_1(\tau_0) = \min\left\{k : \max_{1\le j<n} |S_j(\tau_0)| = |S_k(\tau_0)|\right\} = \operatorname*{argmax}_{1\le k<n} |S_k(\tau_0)|, \tag{3.1}$$

$$\hat{m}_2(\tau_0) = \min\left\{k : \max_{1\le j<n} \frac{|S_j(\tau_0)|}{\sqrt{nV_j(\tau_0)}} = \frac{|S_k(\tau_0)|}{\sqrt{nV_k(\tau_0)}}\right\} = \operatorname*{argmax}_{1\le k<n} \frac{|S_k(\tau_0)|}{\sqrt{nV_k(\tau_0)}}. \tag{3.2}$$

We shall investigate limit properties of such classes of estimators, namely the rate of consistency and the limit distribution under the no-change hypothesis. Recall (1.4) and (1.5). In view of Theorems 2.6 and 2.7, we realize that the term

$$\int_0^{\tau_0} w(t)\,\frac{\sum_{j=1}^m \frac{Y_j(t)}{m}\sum_{j=m+1}^n \frac{Y_j(t)}{n-m}}{\frac{Y(t)}{n}}\left(\frac{f_{1n}(t)}{1-F_{1n}(t)} - \frac{f_{2n}(t)}{1-F_{2n}(t)}\right)\mathrm{d}t$$
$$= \int_0^{\tau_0} w(t)\,\frac{\sum_{j=1}^m \frac{Y_j(t)}{m}\sum_{j=m+1}^n \frac{Y_j(t)}{n-m}}{\frac{Y(t)}{n}}\left(\lambda_{F_{1n}}(t) - \lambda_{F_{2n}}(t)\right)\mathrm{d}t$$

plays a central role in the consistency of the max-type test statistics and it is clear that it will be important also in the limit behavior of the proposed estimators. Regarding the proofs of the theorems mentioned above, we concentrate on the representations of $S_{m_c}(\tau_0)$ and $S_m(\tau_0)$ in the former case or $L_{m_c}(\tau_0)$ and $L_m(\tau_0)$ in the later case, where $L_k(\tau_0) = \frac{S_k(\tau_0)}{\sqrt{nV_k(\tau_0)}}$.

Further, we assume the following:

(A.2): there exists a distribution function $G(t)$ such that

$$\lim_{n\to\infty} \sup_{0\le t\le\tau_0} |G_{in}(t) - G(t)| = 0, \quad i = 1, 2,$$

(A.3): there exists a hazard function $\lambda_F(t)$ such that

$$\lim_{n\to\infty} \int_0^{\tau_0} |\lambda_{F_{in}}(t) - \lambda_F(t)|\,\mathrm{d}t = 0, \quad i = 1, 2, \tag{3.3}$$

and if $n \to \infty$, then

$$n\,A_n^2(\tau_0) \to \infty \qquad \text{for } \hat{m}_1(\tau_0), \tag{3.4}$$

$$\frac{n\,A_n^2(\tau_0)}{\log\log n} \to \infty \qquad \text{for } \hat{m}_2(\tau_0), \tag{3.5}$$

where

$$A_n(\tau_0) = \int_0^{\tau_0} w(t)(1-F(t))(1-G(t))\left(\lambda_{F_{1n}}(t) - \lambda_{F_{2n}}(t)\right)\mathrm{d}t. \tag{3.6}$$

(A.4): $J(\tau_0) > 0$, where

$$J(\tau_0) = \int_0^{\tau_0} w^2(t)(1-G(t))\,\mathrm{d}F(t). \tag{3.7}$$

We assume the validity of the alternative hypothesis of the change point problem which is described in the assumption (A.1). The assumption (A.4) is a technical condition ensuring that $\frac{n^2}{k(n-k)}V_k(\tau_0)$ are asymptotically bounded away from zero. The assumption (A.2) expresses "closeness" of G_{1n} and G_{2n} and the assumption (A.3) expresses "closeness" of F_{1n} and F_{2n}, i.e. they ensure the situation (S.3). More precisely, the term $\lambda_{F_{1n}}(t) - \lambda_{F_{2n}}(t)$, which is the difference of the hazard functions for the lifetimes before and after the change-point $m = m(n)$, reflects the discrepancy between the distribution functions F_{1n} and F_{2n}. The assumption (3.3) entails that, as $n \to \infty$,

$$|A_n(\tau_0)| \to 0$$

and moreover, as $n \to \infty$,

$$\sup_{0 \le t \le \tau_0} |\Lambda_{in}(t) - \Lambda(t)| \le \sup_{0 \le t \le \tau_0} \int_0^{\tau_0} |\lambda_{in}(t) - \lambda(t)| \, \mathrm{d}t \to 0,$$

where

$$\Lambda(t) = \int_0^t \lambda(u) \, \mathrm{d}u = \int_0^t \frac{f(u)}{1 - F(u)} \, \mathrm{d}u = -\log(1 - F(t)), \quad 0 < t \le \tau_0,$$

is the cumulative hazard function. Thus,

$$\lim_{n \to \infty} \sup_{0 \le t \le \tau_0} |F_{in}(t) - F(t)| = 0, \quad i = 1, 2, \tag{3.8}$$

in view of the continuity of the logarithmic function.

Hence, the considered alternative in the first case ($|A_n(\tau_0)| \approx \frac{1}{\sqrt{n}}$) is local and contiguous and in the second case ($|A_n(\tau_0)| \approx \sqrt{\frac{\log \log n}{n}}$) is local but not contiguous according to (3.4) and (3.5), respectively.

Remark 3.1. How it was said above, except the local alternatives for the distributions F_{1n} and F_{2n} of the lifetime variables, we suppose that the distributions G_{1n} and G_{2n} of the censoring variables fulfil also the local alternatives. For the fixed censoring distribution functions G_1 and G_2, the investigation of $\hat{m}_i(\tau_0)$, $i = 1, 2$, is much more elaborate because of the jumps expressing by $S_{m_c}(\tau_0)/m_c$ and $(S_m(\tau_0) - S_{m_c}(\tau_0))/(m - m_c)$ are not in this case asymptotically the same, see Lemma 5.5 below or Section 4 in Chapter 4.

We suppose for the weights $w_n(t)$, as $n \to \infty$,

$$\sup_{0 \le t \le \tau_0} |w_n(t) - w(t)| = \begin{cases} O_{\mathrm{P}}\left(\frac{1}{\sqrt{n}}\right) & \text{for } \hat{m}_1(\tau_0), \\ O_{\mathrm{P}}\left(\sqrt{\frac{\log \log n}{n}}\right) & \text{for } \hat{m}_2(\tau_0), \end{cases} \tag{3.9}$$

where $w(t)$ is a continuous nonrandom function on $[0, \tau_0]$. The property poses the class of commonly used weights of the form (2.5), see Chapter 4, Section 2. Combining (3.9) with (3.4) and (3.5), respectively, we obtain for both cases

$$\sup_{0 \le t \le \tau_0} |w_n(t) - w(t)| = o_{\mathrm{P}}(|A_n(\tau_0)|), \quad \text{as } n \to \infty. \tag{3.10}$$

Convention 3.2. In the following we omit τ_0 whenever it is possible.

It can be assumed w.l.g. that $0 < m_c < m < n$. The main idea of the following proofs is

$$S_k = \sum_{j=1}^{k} a_n(j) \overset{\mathcal{D}}{=} \sum_{j=1}^{k} a_n(Q_j) = S_k(\boldsymbol{Q_{m_c}}), \qquad 1 \le k \le m_c,$$

$$S_k = \sum_{j=1}^{k} a_n(j) \overset{\mathcal{D}}{=} \sum_{j=1}^{k} a_n(Q_j) = S_k(\boldsymbol{Q_{m-m_c}}), \qquad m_c + 1 \le k \le m,$$

$$S_k = \sum_{j=1}^{k} a_n(j) \overset{\mathcal{D}}{=} \sum_{j=1}^{k} a_n(Q_j) = S_k(\boldsymbol{Q_{n-m}}), \qquad m + 1 \le k \le n,$$

where $\boldsymbol{Q_{m_c}} = (Q_1, Q_2, \dots Q_{m_c})$ and $\boldsymbol{Q_{m-m_c}} = (Q_{m_c+1}, Q_{m_c+2}, \dots Q_m)$ are random permutations of $(1, 2, \dots, m_c)$, $(m_c+1, m_c+2, \dots, m)$, respectively, and $\boldsymbol{Q_{n-m}} = (Q_{m+1}, Q_{m+2}, \dots, Q_n)$ is a random permutation of $(m+1, m+2, \dots, n)$. Therefore given $(\boldsymbol{X}, \boldsymbol{\Delta})$ the random variables $S_k(\boldsymbol{Q_{m_c}})$ for $k = 1, 2, \dots, m_c$, $S_k(\boldsymbol{Q_{m-m_c}})$ for $k = m_c+1, m_c+2, \dots, m$ and $S_k(\boldsymbol{Q_{n-m}})$ for $k = m+1, m+2, \dots, n$ can be viewed as rank statistics, where the role of ranks is played by $\boldsymbol{Q_{m_c}}$, $\boldsymbol{Q_{m-m_c}}$ or $\boldsymbol{Q_{n-m}}$, respectively . Moreover, we can decompose the partial sums S_k, $k = 1, 2, \dots, n$, as follows

$$S_k = \left(S_k - \frac{k}{m_c}\, S_{m_c}\right) + \frac{k}{m_c} S_{m_c}, \qquad 1 \le k \le m_c,$$

$$S_k = \left((S_k - S_{m_c}) - \frac{k - m_c}{m - m_c}\,(S_m - S_{m_c})\right) + S_{m_c} + \frac{k - m_c}{m - m_c}\,(S_m - S_{m_c}), \quad m_c < k \le m,$$

$$S_k = \left((S_k - S_m) - \frac{k - m}{n - m}\,(S_n - S_m)\right) + S_m + \frac{k - m}{n - m}\,(S_n - S_m), \qquad m < k \le n.$$

From the above, it can be seen that

$$S_k - \frac{k}{m_c} S_{m_c} = \sum_{j=1}^{k} (a_n(j) - \bar{a}_{m_c}) \stackrel{\mathcal{D}}{=} \sum_{j=1}^{k} (a_n(Q_j) - \bar{a}_{m_c}), \quad 1 \le k \le m_c,$$

$$(S_k - S_{m_c}) - \frac{k - m_c}{m - m_c}\,(S_m - S_{m_c}) \stackrel{\mathcal{D}}{=} \sum_{j=m_c+1}^{k} (a_n(Q_j) - \bar{a}_{m-m_c}), \quad m_c < k \le m,$$

and

$$(S_k - S_m) - \frac{k - m}{n - m}\,(S_n - S_m) \stackrel{\mathcal{D}}{=} \sum_{j=m+1}^{k} (a_n(Q_j) - \bar{a}_{n-m}), \quad m < k \le n,$$

with

$$\bar{a}_{m_c} = \frac{1}{m_c} \sum_{j=1}^{m_c} a_n(j), \qquad \bar{a}_{m-m_c} = \frac{1}{m - m_c} \sum_{j=m_c+1}^{m} a_n(j), \qquad \bar{a}_{n-m} = \frac{1}{n - m} \sum_{j=m+1}^{n} a_n(j).$$

Hence, given $(\boldsymbol{X}, \boldsymbol{\Delta})$ asymptotic results of the behavior of simple linear rank statistics under the hypothesis of randomness can be used. Additionally, we need the limit behavior of $\frac{S_{m_c}}{m_c}$ and $\frac{S_m - S_{m_c}}{m - m_c}$. Since $S_n = 0$, the behavior of $\frac{S_n - S_m}{n - m} = -\frac{S_m}{n - m}$ follows from the previous ones.

2. Rate of consistency

In next two theorems we develop the rate of consistency of the considered classes of estimators $\hat{m}_i(\tau_0)$, $i = 1, 2$.

THEOREM 3.1. *Let* (3.9) *and the assumptions (A.1), (A.2), (A.3) and (A.4) be satisfied. Then, as* $n \to \infty$, *we have*

$$\hat{m}_1(\tau_0) - m = O_{\mathrm{P}}(1/A_n^2(\tau_0)),$$

where $A_n(\tau_0)$ *is defined in* (3.6).

PROOF. First, we examine the limit behavior of S_{m_c} and S_m. Corollary 5.10 below asserts that, as $n \to \infty$,

$$\frac{S_{m_c}}{m_c} = \frac{n - m}{n}\, A_n\,(1 + o_{\mathrm{P}}(1)), \tag{3.11}$$

$$\frac{S_m - S_{m_c}}{m - m_c} = \frac{n - m}{n}\, A_n\,(1 + o_{\mathrm{P}}(1)) \tag{3.12}$$

and consequently, as $n \to \infty$,

$$\frac{S_m}{m} = \frac{n-m}{n} A_n \left(1 + o_{\mathrm{P}}(1)\right). \tag{3.13}$$

Now we investigate the behavior of $|S_k|$.

$$\left| \max_{1\le k\le m_c} |S_k| - \max_{1\le k\le m_c} \frac{k}{m_c} |S_{m_c}| \right| \le \max_{1\le k<m_c} \left| S_k - \frac{k}{m_c} S_{m_c} \right|$$

and similarly

$$\begin{aligned} &\left| \max_{m_c+1\le k\le m-nh} |S_k| - \max_{m_c+1\le k\le m-nh} \left| S_{m_c} + \frac{k-m_c}{m-m_c} (S_m - S_{m_c}) \right| \right| \\ &\qquad\qquad \le \max_{m_c+1\le k<m} \left| (S_k - S_{m_c}) - \frac{k-m_c}{m-m_c} (S_m - S_{m_c}) \right|, \end{aligned}$$

where $h \in (0, \gamma-\eta)$ is an arbitrary fixed constant. By Lemma 5.10 below we obtain, as $n \to \infty$,

$$\begin{aligned} \max_{1\le k\le m_c} |S_k| &= \max_{1\le k\le m_c} \frac{k}{m_c} |S_{m_c}| + O_{\mathrm{P}}\left(\sqrt{m_c}\right) = |S_{m_c}| + O_{\mathrm{P}}\left(\sqrt{m_c}\right) \\ &= \frac{n-m}{n} m_c |A_n| \left(1 + o_{\mathrm{P}}(1)\right) + O_{\mathrm{P}}\left(\sqrt{m_c}\right) \\ &= \frac{n-m}{n} m_c |A_n| \left(1 + o_{\mathrm{P}}(1) + O_{\mathrm{P}}\left(\frac{1}{\sqrt{m_c}\,|A_n|}\right)\right) \\ &= \frac{n-m}{n} m_c |A_n| \left(1 + o_{\mathrm{P}}(1)\right) \end{aligned} \tag{3.14}$$

since (3.11) and regarding (3.4). Moreover, using (3.12) we get, as $n \to \infty$,

$$\begin{aligned} &\max_{m_c+1\le k\le m-nh} |S_k| = \max_{m_c+1\le k\le m-nh} \left| S_{m_c} + \frac{k-m_c}{m-m_c} (S_m - S_{m_c}) \right| + O_{\mathrm{P}}\left(\sqrt{m}\right) \\ &= \max_{m_c+1\le k\le m-nh} \left| m_c \frac{n-m}{n} A_n \left(1 + o_{\mathrm{P}}(1)\right) + k \frac{n-m}{n} A_n \left(1 + o_{\mathrm{P}}(1)\right) \right| + O_{\mathrm{P}}\left(\sqrt{m}\right) \\ &= \left| m_c \frac{n-m}{n} A_n \left(1 + o_{\mathrm{P}}(1)\right) + (m - nh - m_c) \frac{n-m}{n} A_n \left(1 + o_{\mathrm{P}}(1)\right) \right| + O_{\mathrm{P}}\left(\sqrt{m}\right) \\ &= \left| (m-nh) \frac{n-m}{n} A_n + \frac{n-m}{n} A_n \left(m_c\, o_{\mathrm{P}}(1) + (m-nh)\, o_{\mathrm{P}}(1) \right) \right| + O_{\mathrm{P}}\left(\sqrt{m}\right) \\ &= \left| (m-nh) \frac{n-m}{n} A_n + (m-nh) \frac{n-m}{n} A_n\, o_{\mathrm{P}}(1) \right| + O_{\mathrm{P}}\left(\sqrt{m}\right) \\ &= (m-nh) \frac{n-m}{n} |A_n| \left(1 + o_{\mathrm{P}}(1)\right) + O_{\mathrm{P}}\left(\sqrt{m}\right) \\ &= (m-nh) \frac{n-m}{n} |A_n| \left(1 + o_{\mathrm{P}}(1)\right). \end{aligned} \tag{3.15}$$

This together with (3.14) imply, as $n \to \infty$,

$$\frac{\max_{1\le k\le m_c} |S_k|}{\max_{m_c+1\le k\le m-nh} |S_k|} = \frac{m_c \frac{n-m}{n} |A_n| \left(1 + o_{\mathrm{P}}(1)\right)}{(m-nh) \frac{n-m}{n} |A_n| \left(1 + o_{\mathrm{P}}(1)\right)} = \frac{\eta}{\gamma - h} + o_{\mathrm{P}}(1) \tag{3.16}$$

and comparing (3.13) with (3.15) we receive, as $n \to \infty$,

$$\max_{m_c+1\le k\le m-nh} \frac{|S_k|}{|S_m|} = \frac{(m-nh) \frac{n-m}{n} |A_n| \left(1 + o_{\mathrm{P}}(1)\right)}{m \frac{n-m}{n} |A_n| \left(1 + o_{\mathrm{P}}(1)\right)} = \frac{\gamma - h}{\gamma} + o_{\mathrm{P}}(1). \tag{3.17}$$

From (3.16) and (3.17), we can conclude, as $n \to \infty$,

$$\begin{aligned}
&\mathrm{P}(\hat{m}_1 \leq m - nh) = \mathrm{P}(\max_{1 \leq k \leq m-nh} |S_k| \geq \max_{m-nh<k\leq m} |S_k|) \\
&\leq \mathrm{P}(\max(\max_{1\leq k\leq m_c} |S_k|, \max_{m_c+1\leq k\leq m-nh} |S_k|) \geq |S_m|) \\
&= \mathrm{P}(\max_{m_c+1\leq k \leq m-nh} |S_k| \geq |S_m|) + o(1) = o(1), \qquad (3.18)
\end{aligned}$$

since $\frac{\eta}{\gamma-h} < 1$ and $\frac{\gamma-h}{\gamma} < 1$. Now we see that $\hat{m}_1/n$ is a consistent estimator of the change point γ.

Further, we investigate the rate of the consistency. We treat the behavior of the estimator $\hat{m}_1$ in the open neighborhood $(m-nh, m+nh)$ of the change point m, where $h \in (0, \min(\gamma-\eta, 1-\gamma))$ is arbitrary but fixed. It is sufficient to investigate the behavior of $\max_{m-nh<k\leq m}(S_k^2 - S_m^2)$. The behavior of $\max_{m+1\leq k<m+nh}(S_k^2 - S_m^2)$ would be obtained analogously and hence it is omitted. Obviously,

$$\begin{aligned}
&\max_{m-nh<k\leq m}(S_k^2 - S_m^2) \\
&= \max_{m-nh<k\leq m} \left\{\left((S_k - S_{m_c}) - \frac{k-m_c}{m-m_c}(S_m - S_{m_c}) + \left(S_{m_c} + \frac{k-m_c}{m-m_c}(S_m - S_{m_c})\right)\right)^2 - S_m^2\right\}.
\end{aligned}$$

We use the decomposition

$$\begin{aligned}
&\left((S_k - S_{m_c}) - \frac{k-m_c}{m-m_c}(S_m - S_{m_c}) + \left(S_{m_c} + \frac{k-m_c}{m-m_c}(S_m - S_{m_c})\right)\right)^2 - S_m^2 \\
&\qquad = U_{k1,1} + U_{k2,1} + U_{k3,1}, \qquad (3.19)
\end{aligned}$$

where

$$\begin{aligned}
U_{k1,1} &= \left((S_k - S_{m_c}) - \frac{k-m_c}{m-m_c}(S_m - S_{m_c})\right)^2, \\
U_{k2,1} &= 2\left((S_k - S_{m_c}) - \frac{k-m_c}{m-m_c}(S_m - S_{m_c})\right)\left(S_{m_c} + \frac{k-m_c}{m-m_c}(S_m - S_{m_c})\right), \\
U_{k3,1} &= \left(S_{m_c} + \frac{k-m_c}{m-m_c}(S_m - S_{m_c})\right)^2 - S_m^2.
\end{aligned}$$

By (3.18) and since $\max_{m-C/A_n^2<k\leq m}(S_k^2 - S_m^2) \geq S_m^2 - S_m^2 = 0$, where $C > 0$, we obtain, as $n \to \infty$,

$$\begin{aligned}
&\mathrm{P}\left(\hat{m}_1 \leq m - \frac{C}{A_n^2}\right) = \mathrm{P}\left(m - nh < \hat{m}_1 \leq m - \frac{C}{A_n^2}\right) + \mathrm{P}\left(\hat{m}_1 \leq m - nh\right) \\
&= \mathrm{P}\left(\max_{m_c<k\leq m-C/A_n^2}(S_k^2 - S_m^2) \geq \max_{m-C/A_n^2<k\leq m}(S_k^2 - S_m^2)\right) + o(1) \\
&\leq \mathrm{P}\left(\max_{m_c<k\leq m-C/A_n^2}(S_k^2 - S_m^2) \geq 0\right) + o(1) \\
&= \mathrm{P}\left(\max_{m_c<k\leq m-C/A_n^2} U_{k3,1}\left(1 + \frac{U_{k1,1}}{U_{k3,1}} + \frac{U_{k2,1}}{U_{k3,1}}\right) \geq 0\right) + o(1)
\end{aligned}$$

and by Lemma 5.13 below we get

$$\lim_{C\to\infty} \limsup_{n\to\infty} \mathrm{P}\left(\hat{m}_1 \leq m - \frac{C}{A_n^2}\right) = 0.$$

Similar arguments yield

$$\lim_{C\to\infty}\limsup_{n\to\infty} \mathrm{P}\left(\hat{m}_1 \geq m + \frac{C}{A_n^2}\right) = 0.$$

□

THEOREM 3.2. *Let* (3.9) *and the assumptions (A.1), (A.2), (A.3) and (A.4) be satisfied. Then, as* $n \to \infty$, *we have*

$$\hat{m}_2(\tau_0) - m = O_{\mathrm{P}}(1/A_n^2(\tau_0)),$$

where $A_n(\tau_0)$ *is defined in* (3.6).

PROOF. We use the steps similar to those in the proof of the previous theorem but we take into account the standardization. Let start with the term V_k. By the definition V_k in (2.16) we have

$$V_k \geq v_k = \frac{k(n-k)}{n^2}, \quad k \leq \log\log n \text{ or } k \geq n - \log\log n,$$

and consequently

$$|L_k| = \frac{|S_k|}{\sqrt{n\,V_k}} \leq \sqrt{\frac{n}{k(n-k)}}\,|S_k| \tag{3.20}$$

for $k \leq \log\log n$ or $k \geq n - \log\log n$. By Lemma 5.8 below we obtain, as $n \to \infty$,

$$V_k = \frac{k(n-k)}{n^2}\,J(\tau_0)\,(1 + o_{\mathrm{P}}(1))$$

uniformly in $\log\log n < k < n - \log\log n$ and therefore

$$|L_k| = \frac{|S_k|}{\sqrt{n\,V_k}} = \sqrt{\frac{n}{k(n-k)}}\,\frac{|S_k|}{\sqrt{J(\tau_0)\,(1+o_{\mathrm{P}}(1))}} \tag{3.21}$$

uniformly in $\log\log n < k < n - \log\log n$. Now we treat the behavior of $\sqrt{\frac{n}{k(n-k)}}\,|S_k|$. First,

$$\left|\max_{1\leq k\leq m_c}\sqrt{\frac{n}{k(n-k)}}\,|S_k| - \max_{1\leq k\leq m_c}\sqrt{\frac{n}{k(n-k)}}\,\frac{k}{m_c}|S_{m_c}|\right| \leq \max_{1\leq k<m_c}\sqrt{\frac{n}{k(n-k)}}\left|S_k - \frac{k}{m_c}S_{m_c}\right|$$

and by Lemma 5.10 below we obtain, as $n \to \infty$,

$$\begin{aligned}
\max_{1\leq k\leq m_c}\sqrt{\frac{n}{k(n-k)}}\,|S_k| &= \max_{1\leq k\leq m_c}\sqrt{\frac{n}{k(n-k)}}\,\frac{k}{m_c}\,|S_{m_c}| + O_{\mathrm{P}}\left(\sqrt{\log\log m_c}\right)\\
&= \max_{1\leq k\leq m_c}\sqrt{\frac{k\,n}{n-k}}\,\frac{|S_{m_c}|}{m_c} + O_{\mathrm{P}}\left(\sqrt{\log\log m_c}\right)\\
&= \sqrt{\frac{m_c\,n}{n-m_c}}\,\frac{|S_{m_c}|}{m_c} + O_{\mathrm{P}}\left(\sqrt{\log\log m_c}\right)\\
&= \sqrt{\frac{m_c\,n}{n-m_c}}\,\frac{n-m}{n}\,|A_n|\,(1+o_{\mathrm{P}}(1)) + O_{\mathrm{P}}\left(\sqrt{\log\log m_c}\right)\\
&= \sqrt{\frac{m_c\,n}{n-m_c}}\,\frac{n-m}{n}\,|A_n|\left(1 + o_{\mathrm{P}}(1) + O_{\mathrm{P}}\left(\frac{\sqrt{\log\log m_c}}{\sqrt{n}\,|A_n|}\right)\right)\\
&= \sqrt{\frac{m_c\,n}{n-m_c}}\,\frac{n-m}{n}\,|A_n|\,(1+o_{\mathrm{P}}(1))
\end{aligned} \tag{3.22}$$

since (3.11) and regarding (3.5). Analogously

$$\Bigg|\max_{m_c+1\le k\le m-nh}\sqrt{\frac{n}{k(n-k)}}\,|S_k| - \max_{m_c+1\le k\le m-nh}\sqrt{\frac{n}{k(n-k)}}\left|S_{m_c}+\frac{k-m_c}{m-m_c}\,(S_m-S_{m_c})\right|\Bigg|$$
$$\le \max_{m_c+1\le k<m}\sqrt{\frac{n}{k(n-k)}}\left|(S_k-S_{m_c})-\frac{k-m_c}{m-m_c}(S_m-S_{m_c})\right|,$$

where $h\in(0,\gamma-\eta)$ is an arbitrary fixed constant. This with the assertion of Lemma 5.10 below imply, as $n\to\infty$,

$$\max_{m_c+1\le k\le m-nh}\sqrt{\frac{n}{k(n-k)}}\,|S_k| = \max_{m_c+1\le k\le m-nh}\sqrt{\frac{n}{k(n-k)}}\left|S_{m_c}+\frac{k-m_c}{m-m_c}\,(S_m-S_{m_c})\right| + O_{\mathrm{P}}\left(\sqrt{\log\log m}\right) \quad (3.23)$$

and using (3.11) and (3.12) we obtain, as $n\to\infty$,

$$\begin{aligned}
&\max_{m_c+1\le k\le m-nh}\sqrt{\frac{n}{k(n-k)}}\left|S_{m_c}+\frac{k-m_c}{m-m_c}\,(S_m-S_{m_c})\right|\\
&=\max_{m_c+1\le k\le m-nh}\sqrt{\frac{n}{k(n-k)}}\left|m_c\,\frac{n-m}{n}\,A_n\,(1+o_{\mathrm{P}}(1))+(k-m_c)\,\frac{n-m}{n}\,A_n\,(1+o_{\mathrm{P}}(1))\right|\\
&=\max_{m_c+1\le k\le m-nh}\sqrt{\frac{n}{k(n-k)}}\left|k\,\frac{n-m}{n}\,A_n+\frac{n-m}{n}\,A_n\left(m_c\,o_{\mathrm{P}}(1)+k\,o_{\mathrm{P}}(1)\right)\right|\\
&=\max_{m_c+1\le k\le m-nh}\sqrt{\frac{n}{k(n-k)}}\left|k\,\frac{n-m}{n}\,A_n+k\,\frac{n-m}{n}\,A_n\,o_{\mathrm{P}}(1)\right|\\
&=\max_{m_c+1\le k\le m-nh}\sqrt{\frac{kn}{n-k}}\,\frac{n-m}{n}\,|A_n|\,(1+o_{\mathrm{P}}(1))\\
&=\sqrt{\frac{(m-nh)\,n}{n-m+nh}}\,\frac{n-m}{n}\,|A_n|\,(1+o_{\mathrm{P}}(1)). &(3.24)
\end{aligned}$$

Combining (3.23) with (3.24) we see

$$\begin{aligned}
\max_{m_c+1\le k\le m-nh}\sqrt{\frac{n}{k(n-k)}}|S_k| &= \sqrt{\frac{(m-nh)\,n}{n-m+nh}}\,\frac{n-m}{n}\,|A_n|\,(1+o_{\mathrm{P}}(1))+O_{\mathrm{P}}\left(\sqrt{\log\log m}\right)\\
&=\sqrt{\frac{(m-nh)\,n}{n-m+nh}}\,\frac{n-m}{n}\,|A_n|\,(1+o_{\mathrm{P}}(1)) &(3.25)
\end{aligned}$$

in view of (3.5). From (3.22) and (3.25) it can be seen that, as $n\to\infty$,

$$\begin{aligned}
\frac{\max_{1\le k\le m_c}\sqrt{\frac{n}{k(n-k)}}\,|S_k|}{\max_{m_c+1\le k\le m-nh}\sqrt{\frac{n}{k(n-k)}}\,|S_k|} &= \frac{\sqrt{\frac{m_c\,n}{n-m_c}}\,\frac{n-m}{n}\,|A_n|\,(1+o_{\mathrm{P}}(1))}{\sqrt{\frac{(m-nh)\,n}{n-m+nh}}\,\frac{n-m}{n}\,|A_n|\,(1+o_{\mathrm{P}}(1))}\\
&=\sqrt{\frac{\eta\,(1-\gamma+h)}{(\gamma-h)\,(1-\eta)}}\,(1+o_{\mathrm{P}}(1)) &(3.26)
\end{aligned}$$

and analogously

$$\frac{\max_{m_c+1\le k\le m-nh}\sqrt{\frac{n}{k(n-k)}}\,|S_k|}{\sqrt{\frac{n}{m(n-m)}}\,|S_m|} = \frac{\sqrt{\frac{(m-nh)\,n}{n-m+nh}}\,\frac{n-m}{n}\,|A_n|\,(1+o_{\mathrm{P}}(1))}{\sqrt{\frac{m\,n}{n-m}}\,\frac{n-m}{n}\,|A_n|\,(1+o_{\mathrm{P}}(1))}$$

$$= \sqrt{\frac{(\gamma-h)\,(1-\gamma)}{\gamma\,(1-\gamma+h)}}\,(1+o_{\mathrm{P}}(1)). \tag{3.27}$$

From (3.26) and (3.27) together with (3.20) and (3.21), we can conclude, as $n\to\infty$,

$$\begin{aligned}\mathrm{P}(\hat m_2 \le m-nh) &= \mathrm{P}(\max_{1\le k\le m-nh}|L_k| \ge \max_{m-nh<k\le m}|L_k|)\\ &= \mathrm{P}(\max_{1\le k\le m-nh}|L_k| \ge |L_m|)\\ &= \mathrm{P}(\max(\max_{1\le k\le m_c}|L_k|, \max_{m_c+1\le k\le m-nh}|L_k|) \ge |L_m|)\\ &= \mathrm{P}(\max_{m_c+1\le k\le m-nh}|L_k| \ge |L_m|) + o(1) = o(1) \end{aligned} \tag{3.28}$$

since $\sqrt{\frac{\eta\,(1-\gamma+h)}{(\gamma-h)\,(1-\eta)}} < 1$ and $\sqrt{\frac{(\gamma-h)\,(1-\gamma)}{\gamma\,(1-\gamma+h)}} < 1$. Therefore it is clear that

$$\frac{\hat m_2}{n} \xrightarrow{\mathrm{P}} \gamma, \quad n\to\infty.$$

Now, we investigate the rate of the consistency. It suffices to treat the behavior of $\hat m_2$ in the neighborhood $(m-nh, m+nh)$ of the change point m, where $h\in(0,\min(\gamma-\eta,1-\gamma))$ is an arbitrary fixed constant. We study the behavior of $\max_{m-nh<k\le m}(L_k^2-L_m^2)$. The behavior of $\max_{m+1\le k<m+nh}(L_k^2-L_m^2)$ can be treated analogously and hence it is omitted. Clearly,

$$\begin{aligned}&\max_{m-nh<k\le m}(L_k^2-L_m^2)\\ &= \max_{m-nh<k\le m}\left\{\frac{\Big((S_k-S_{m_c}) - \frac{k-m_c}{m-m_c}(S_m-S_{m_c}) + \Big(S_{m_c}+\frac{k-m_c}{m-m_c}(S_m-S_{m_c})\Big)\Big)^2}{n\,V_k} - \frac{S_m^2}{n\,V_m}\right\}.\end{aligned}$$

We use the decomposition

$$\left\{\frac{\Big((S_k-S_{m_c}) - \frac{k-m_c}{m-m_c}(S_m-S_{m_c}) + \Big(S_{m_c}+\frac{k-m_c}{m-m_c}(S_m-S_{m_c})\Big)\Big)^2}{n\,V_k} - \frac{S_m^2}{n\,V_m}\right\} = U_{k1,2}+U_{k2,2}+U_{k3,2}, \tag{3.29}$$

where

$$\begin{aligned}U_{k1,2} &= \frac{\Big((S_k-S_{m_c}) - \frac{k-m_c}{m-m_c}(S_m-S_{m_c})\Big)^2}{n\,V_k},\\ U_{k2,2} &= 2\,\frac{(S_k-S_{m_c}) - \frac{k-m_c}{m-m_c}(S_m-S_{m_c})}{n\,V_k}\Big(S_{m_c}+\frac{k-m_c}{m-m_c}(S_m-S_{m_c})\Big),\\ U_{k3,2} &= \frac{\Big(S_{m_c}+\frac{k-m_c}{m-m_c}(S_m-S_{m_c})\Big)^2}{n\,V_k} - \frac{S_m^2}{nV_k}\frac{V_k}{V_m}.\end{aligned}$$

By (3.28) and since $\max_{m-C/A_n^2<k\le m}(L_k^2-L_m^2) \ge L_m^2-L_m^2 = 0$ with $C>0$, we obtain, as $n\to\infty$,

$$\mathrm{P}\left(\hat m_2 \le m-\frac{C}{A_n^2}\right) = \mathrm{P}\left(m-nh<\hat m_2\le m-\frac{C}{A_n^2}\right) + \mathrm{P}\left(\hat m_2\le m-nh\right)$$

$$= \mathrm{P}\left(\max_{m_c<k\le m-C/A_n^2}(L_k^2-L_m^2)\ge \max_{m-C/A_n^2<k\le m}(L_k^2-L_m^2)\right)+o(1)$$
$$\le \mathrm{P}\left(\max_{m_c<k\le m-C/A_n^2}(L_k^2-L_m^2)\ge 0\right)+o(1)$$
$$= \mathrm{P}\left(\max_{m_c<k\le m-C/A_n^2}U_{k3,2}\left(1+\frac{U_{k1,2}}{U_{k3,2}}+\frac{U_{k2,2}}{U_{k3,2}}\right)\ge 0\right)+o(1)$$

and by Lemma 5.14 below we get

$$\lim_{C\to\infty}\limsup_{n\to\infty}\mathrm{P}\left(\hat{m}_2\le m-\frac{C}{A_n^2}\right)=0.$$

Similar arguments give that also

$$\lim_{C\to\infty}\limsup_{n\to\infty}\mathrm{P}\left(\hat{m}_2\ge m+\frac{C}{A_n^2}\right)=0.$$

□

3. Behavior under no-change hypothesis

The limit behavior of the considered estimators $\hat{m}_i(\tau_0)$, $i=1,2$, of m are quite different in the case of no change in the lifetime distribution as shown in the next two theorems. We investigate the limit behavior of $\hat{m}_1(\tau_0)$ under the restricted null hypothesis $\bar{H}_0$ and the limit behavior of $\hat{m}_2(\tau_0)$ under the null hypothesis H_0.

THEOREM 3.3. *Suppose that $X_1^0, X_2^0, \dots, X_n^0$, $C_1, C_2, \dots, C_n$ are independent random variables. Let $X_1^0, X_2^0, \dots, X_n^0$ have an arbitrary absolutely continuous distribution function F and let C_1, C_2, …, C_n have an absolutely continuous distribution function G. Let* (2.9) *and* (2.12) *be satisfied, then we have, as $n\to\infty$,*

$$\frac{\hat{m}_1(\tau_0)}{n}\xrightarrow{\mathcal{D}}\min\left\{t\in[0,1];|B(t)|=\max_{0\le v\le 1}B(v)|\right\}=\operatorname*{argmax}_{0\le t\le 1}|B(t)|$$

with $\{B(t);t\in[0,1]\}$ being a standard Brownian bridge.

PROOF. The random variables S_k, $k=1,\dots,n$, have the same distribution as $S_k(\boldsymbol{Q})$, where $\boldsymbol{Q}=(Q_1,Q_2,\dots,Q_n)$ is a random permutation of $(1,2,\dots,n)$. Moreover, the random variables $S_k(\boldsymbol{Q})$, $k=1,\dots,n$, given $(\boldsymbol{X},\boldsymbol{\Delta})$, can be viewed as simple linear rank statistics, where the role of ranks is played by $\boldsymbol{Q}$. Notice that the estimator $\hat{m}_1(\tau_0,\boldsymbol{Q})=\operatorname{argmax}_{1\le k<n}|S_k(\boldsymbol{Q})|$ can be viewed as a rank estimator. To apply Theorem 1.2 in Gombay, Hušková [**17**], we have to verify that the scores $a_n(j)$ defined in (2.1) fulfil for convergence in probability (5.47) and (5.48). Since Corollary 5.6 below ensures the conditions (5.47) and (5.48) for convergence in probability, Theorem 1.2 in Gombay, Hušková [**17**] can be used, so the assertion of the theorem holds. □

THEOREM 3.4. *Suppose that $X_1^0, X_2^0, \dots, X_n^0$, $C_1, C_2, \dots, C_n$ are independent random variables. Let $X_1^0, X_2^0, \dots, X_n^0$ be i.i.d. variables with an arbitrary absolutely continuous distribution function F. Let C_1, C_2, …, $C_{\lfloor n\eta\rfloor}$ and $C_{\lfloor n\eta\rfloor+1}, C_{\lfloor n\eta\rfloor+2}, \dots, C_n$ have absolutely continuous distribution functions G_1 and G_2, respectively, $G_1\ne G_2$, for some $\eta\in(0,1]$. Let* (2.17) *and* (2.19) *be satisfied, then we have for an arbitrary $\varepsilon\in(0,1/2)$, as $n\to\infty$,*

$$\mathrm{P}(\hat{m}_2(\tau_0)<n\varepsilon)\to\frac{1}{2}\quad \textit{and}\quad \mathrm{P}(\hat{m}_2(\tau_0)>n(1-\varepsilon))\to\frac{1}{2}.$$

PROOF. Hušková and Neuhaus [**23**] showed in the proof of Theorem 1.1 (Theorem 2.3 in the thesis) that

$$\mathrm{P}\left(\max_{1\le k<n}\frac{|S_k|}{\sqrt{nV_k}}=\max\Bigl(\max_{k\in\cup_{i=1}^{6}I_{in}}\frac{|S_k|}{\sqrt{nV_k}}=\max\Bigl(\max_{k\in I_{2n}}\frac{|S_k|}{\sqrt{nV_k}},\max_{k\in I_{5n}}\frac{|S_k|}{\sqrt{nV_k}}\Bigr)\right)\to 1,\quad n\to\infty,$$

where

$$\begin{aligned}
I_{1n}&=\left\{1,\dots,\lfloor(\log n)^A\rfloor\right\},\\
I_{2n}&=\left\{\lfloor(\log n)^A\rfloor+1,\dots,\lfloor n(\log\log n)^{-B}\rfloor\right\},\\
I_{3n}&=\left\{\lfloor n(\log\log n)^{-B}\rfloor+1,\dots,m_c\right\},\\
I_{4n}&=\left\{m_c+1,\dots,n-\lfloor n(\log\log n)^{-B}\rfloor\right\},\\
I_{5n}&=\left\{n-\lfloor n(\log\log n)^{-B}\rfloor+1,\dots,n-\lfloor(\log n)^A\rfloor\right\},\\
I_{6n}&=\left\{n-\lfloor(\log n)^A\rfloor+1,\dots,n-1\right\}
\end{aligned}$$

with arbitrary $A>0$ and $B\in(1,2)$. Since that, we get, as $n\to\infty$,

$$\mathrm{P}\left(\max_{1\le k<n\,\varepsilon_n}\frac{|S_k|}{\sqrt{nV_k}}\ge\max_{n\,\varepsilon_n\le k<n}\frac{|S_k|}{\sqrt{nV_k}}\right)\to\frac{1}{2},$$

$$\mathrm{P}\left(\max_{n(1-\varepsilon_n)<k<n}\frac{|S_k|}{\sqrt{nV_k}}\ge\max_{1\le k\le n(1-\varepsilon_n)}\frac{|S_k|}{\sqrt{nV_k}}\right)\to\frac{1}{2}$$

for $\varepsilon_n\in[(\log\log n)^{-B},1/2)$. Letting $\varepsilon_n\to\varepsilon$, we get the desired result. $\square$

CHAPTER 4

Discussion on assumptions

1. Introduction

In this chapter we will verify and discuss the conditions of theorems from the previous two chapters for particular cases of weights and various underlying lifetime distribution functions.

Recall that the lifetimes $X_1^0, X_2^0, \dots, X_{\lfloor n\gamma \rfloor}^0$ and $X_{\lfloor n\gamma \rfloor+1}^0, X_{\lfloor n\gamma \rfloor+2}^0, \dots, X_n^0$ are independent with the absolutely continuous distribution functions F_1 and F_2, respectively, $F_1 \neq F_2$. The censoring variables $C_1, \dots, C_{\lfloor n\eta \rfloor}$ and $C_{\lfloor n\eta \rfloor+1}, \dots, C_n$ are independent with the absolutely continuous distribution functions G_1 and G_2, respectively, $G_1 \neq G_2$, and they are independent of the lifetimes. We suppose that parameters $\gamma \in (0,1]$ and $\eta \in (0,1]$ are unknown and need not be the same. Notice that we take into account both the no-change hypothesis H_0 and also the one-change alternative H_1.

We focus on the general condition for the limit behavior of weights which we need in all the assertions

$$\sup_{0\le t\le \tau_0} |w_n(t) - w(t)| = o_{\mathrm{P}}\Big(\frac{1}{b_n}\Big), \tag{4.1}$$

where b_n is a sequence with the form

$$b_n = \begin{cases} 1/|A_n(\tau_0)|, & \text{(the estimators } m_1(\tau_0) \text{ and } m_2(\tau_0)\text{)}, \\ \log\log n, & \text{(the max-type test statistic } T_n(\tau_0)\text{)}, \\ 1, & \text{(otherwise)}, \end{cases}$$

where $A_n(\tau_0)$ is given by (3.6), the estimators $m_1(\tau_0)$ and $m_2(\tau_0)$ are defined in (3.1) and (3.2) and $T_n(\tau_0) = \max_{1\le k<n} |L_k(\tau_0)|$ with $L_k(\tau_0)$ of the form (2.14).

We also concentrate on the other technical conditions as follows

$$\int_0^{\tau_0} w^2(t)\ (1 - G_i(t))\,\mathrm{d}F_j(t) > 0, \quad i,j = 1,2, \tag{4.2}$$

$$\int_0^{\tau_0} w(t)\, \frac{Q_1(t)\,Q_2(t)}{Q_1(t) + Q_2(t)}\,(\lambda_1(t) - \lambda_2(t))\ \mathrm{d}t \neq 0, \tag{4.3}$$

$$\int_0^{\tau_0} w(t)\, \frac{(1 - F_1(t))\,(\eta(1 - G_1(t)) + (1-\eta)(1 - G_2(t))\,Q_2(t)}{Q_1(t) + Q_2(t)}\,(\lambda_2(t) - \lambda_1(t))\ \mathrm{d}t \neq 0, \tag{4.4}$$

where $Q_1(t)$ and $Q_2(t)$ are given by (1.14). Notice that (4.2) in Theorems 2.2–2.5 ensures that the limit of $\frac{n^2}{k(n-k)}V_k(\tau_0)$ or $\sigma_n^2(\boldsymbol{a})$, respectively, is bounded away from 0. The condition (4.3) plays a central role in the consistency of the max-type test statistics, see Theorem 2.6 or 2.7 and (4.4) in the consistency of the MOSUM-type tests, see Theorems 2.8 and 2.9.

Further, we discus the conditions (A.2) – (A.4) for behavior of the developed estimators under the local alternatives described in Theorems 3.1 and 3.2 for the particular types of distribution.

Finally, we address our attention on the situations (S.1) and (S.2) describing various versions (fixed, local) of the alternative hypothesis. Recall that the situation (S.3) was supposed in Chapter 3.

2. The weights

In the following, we focus on the weights defined in (2.5), i.e. it is adequate to investigate only these three types of weights

$$\begin{aligned} w_{1,n}(t) &= 1, \\ w_{2,n}(t) &= \left(\frac{Y(t)}{n}\right)^{\kappa}, && \kappa > 0, \\ w_{3,n}(t) &= (\hat{S}_n(t-))^{\rho}, && \rho > 0. \end{aligned}$$

Notice that $w_{1,n}(t)$ fulfils (4.1) elementary.

By Lemma 5.2 below we obtain, as $n \to \infty$,

$$\sup_{0 \le t \le \tau_0} \left| \frac{Y(t)}{n} - (1 - H_{\eta,\gamma}(t)) \right| = O_{\mathrm{P}}\left(\frac{1}{\sqrt{n}}\right) \tag{4.5}$$

and consequently

$$w_2(t) = \begin{cases} ((1-F(t))(1-G(t)))^{\kappa}, & \text{under } \bar{H}_0 : \gamma = \eta = 1, \\ (1 - H_{\eta}(t))^{\kappa}, & \text{under } H_0 : \gamma = 1, \\ (1 - H_{\gamma}(t))^{\kappa}, & \text{under } \bar{H}_1 : \gamma = \eta \in (0,1), \\ (1 - H_{\eta,\gamma}(t))^{\kappa}, & \text{under } H_1 : \gamma \in (0,1), \end{cases}$$

where $H_\eta(t)$, $H_\gamma(t)$ and $H_{\eta,\gamma}(t)$ are defined in (1.8), (1.7) and (1.6), respectively.

For the Kaplan–Meier estimate we use the Taylor expansion

$$\begin{aligned} \log(\hat{S}_n(t)) &= \sum_{i: X_i \le t} \log\Big(1 - \frac{\Delta_i}{Y(X_i)}\Big) = \sum_{i: X_i \le t} \Big(-\frac{\Delta_i}{Y(X_i)} - \frac{\Delta_i^2}{2Y^2(X_i)} - \frac{\Delta_i^3}{3Y^3(X_i)} - \cdots \Big) \\ &= \sum_{i: X_i \le t} -\frac{\Delta_i}{Y(X_i)} \Big(1 + \frac{\Delta_i}{2Y(X_i)} + \frac{\Delta_i^2}{3Y^2(X_i)} + \cdots \Big) \quad \text{for } 0 \le t \le \tau_0 \end{aligned}$$

and, as $\frac{\Delta_i}{Y(X_i)} \xrightarrow{\mathrm{P}} 0$,

$$\begin{aligned} 1 + \frac{\Delta_i}{2Y(X_i)} + \frac{\Delta_i^2}{3Y^2(X_i)} + \cdots &= 1 + \frac{\Delta_i}{Y(X_i)} O_{\mathrm{P}}\left(\sum_{j=0}^{\infty} \left(\frac{\Delta_i}{Y(X_i)} \right)^j \right) \\ = 1 + O_{\mathrm{P}}\left(\frac{\frac{\Delta_i}{Y(X_i)}}{1 - \frac{\Delta_i}{Y(X_i)}} \right) &= 1 + O_{\mathrm{P}}\Big(\frac{1}{n}\Big) = 1 + o_{\mathrm{P}}\Big(\frac{1}{\sqrt{n}}\Big). \end{aligned}$$

Thus

$$\log(\hat{S}_n(t)) = -\int_0^t \left(1 + o_{\mathrm{P}}\Big(\frac{1}{\sqrt{n}}\Big)\right) \frac{\mathrm{d}N(u)}{Y(u)} \tag{4.6}$$

and by (4.5) we have, as $n \to \infty$,

$$-\int_0^t \frac{\mathrm{d}N(u)}{Y(u)} = -\int_0^t \frac{1}{1 - H_{\eta,\gamma}(u)} \mathrm{d}\frac{N(u)}{n} \left(1 + O_{\mathrm{P}}\Big(\frac{1}{\sqrt{n}}\Big)\right). \tag{4.7}$$

Choosing $v(t) = \frac{1}{1-H_{\eta,\gamma}(u)}$ in Lemma 5.3 below we get, as $n \to \infty$,

$$\sup_{0\le t\le\tau_0}\left|\int_0^t \frac{1}{1-H_{\eta,\gamma}(u)}\,\mathrm{d}\frac{N(u)}{n} - \int_0^t \frac{1}{1-H_{\eta,\gamma}(u)}\,\mathrm{d}R_{\eta,\gamma}(u)\right| \le \int_0^{\tau_0} \frac{1}{1-H_{\eta,\gamma}(u)}\left|\mathrm{d}\frac{N(u)}{n} - \mathrm{d}R_{\eta,\gamma}(u)\right| = O_{\mathrm{P}}\Big(\frac{1}{\sqrt{n}}\Big). \tag{4.8}$$

Combining (4.6) and (4.7) with (4.8) we obtain, as $n \to \infty$,

$$\sup_{0\le t\le\tau_0}\left|\hat{S}_n(t) - \exp\Big(-\int_0^t \frac{1}{1-H_{\eta,\gamma}(u)}\,\mathrm{d}R_{\eta,\gamma}(u)\Big)\right| = O_{\mathrm{P}}\Big(\frac{1}{\sqrt{n}}\Big), \tag{4.9}$$

where $H_{\eta,\gamma}(t)$ and $R_{\eta,\gamma}(t)$ are given by (1.6) and (1.10). Thus,

$$w_3(t) = \begin{cases} (1-F(t))^\rho, & \text{under } \bar{H}_0,\\ \exp\Big(-\rho\int_0^t \frac{1}{1-H_\eta(u)}\,\mathrm{d}R_\eta(u)\Big), & \text{under } H_0,\\ \exp\Big(-\rho\int_0^t \frac{1}{1-H_\gamma(u)}\,\mathrm{d}R_\gamma(u)\Big), & \text{under } \bar{H}_1,\\ \exp\Big(-\rho\int_0^t \frac{1}{1-H_{\eta,\gamma}(u)}\,\mathrm{d}R_{\eta,\gamma}(u)\Big), & \text{under } H_1, \end{cases}$$

in view of

$$\exp\Big(-\rho\int_0^t \frac{\mathrm{d}F(u)}{1-F(u)}\Big) = \exp\Big(-\rho\int_0^t \lambda(u)\,\mathrm{d}(u)\Big) = (\exp(-\Lambda(t)))^\rho = (1-F(t))^\rho,$$

where $\lambda(t)$ is a hazard function and $\Lambda(t)$ is a corresponding cumulative hazard function.

By (4.5) and (4.9), under both the restricted and the unrestricted hypotheses and also the alternatives we get, as $n \to \infty$,

$$\sup_{0\le t\le\tau_0}|w_{in}(t) - w_i(t)| = O_{\mathrm{P}}\Big(\frac{1}{\sqrt{n}}\Big), \quad i = 1,2,3. \tag{4.10}$$

It is clear that the requirements on the rate $o_{\mathrm{P}}(1)$ and $o_{\mathrm{P}}\Big((\log\log n)^{-1}\Big)$ are in all cases fulfilled. Since (3.9) implies (3.10), we see that (4.10) ensures also the rate $o_{\mathrm{P}}(|A_n|)$.

Convention 4.1. In the following text of this section we suppose w.l.g. that $f_i(t) > 0$, $g_i(t) > 0$ for $t > 0$, i.e. consequently, the distribution functions $F_1(t)$, $F_2(t)$ and $G_1(t)$, $G_2(t)$ are strictly increasing on $[0,\infty)$.

Second, we discuss the condition (4.2). Notice that the obtained $w(t)$ are continuous and strictly decreasing functions on $[0,\tau_0]$ or $w(t) = 1$, i.e. we have

$$\int_0^{\tau_0} w^2(t)(1-G_i(t))\,\mathrm{d}F_j(t) \ge w^2(\tau_0)(1-G_i(\tau_0))\int_0^{\tau_0}\mathrm{d}F_j(t) = w^2(\tau_0)(1-G_i(\tau_0))F_j(\tau_0) > 0 \tag{4.11}$$

since the property (2.4) of τ_0 and in view of $w(\tau_0) > 0$ for the considered weights.

Third, we treat the condition (4.3) which is adequate for the consistency of the max-type tests. Defining $0/0 := 0$ we can rewrite our condition as follows

$$\int_0^\infty w(t)\,\frac{Q_1(t)\,Q_2(t)}{1-H_{\eta,\gamma}(t)}\,(\lambda_1(t)-\lambda_2(t))\,\mathrm{d}t \ne 0.$$

Recall $w(t)\,\frac{Q_1(t)\,Q_2(t)}{1-H_{\eta,\gamma}(t)}$ is continuous and strictly decreasing for $t \geq 0$. By integration by parts, we obtain

$$(\Lambda_1(t)-\Lambda_2(t))\,w(t)\,\frac{Q_1(t)\,Q_2(t)}{1-H_{\eta,\gamma}(t)}\bigg|_0^\infty - \int_0^\infty (\Lambda_1(t)-\Lambda_2(t))\,\mathrm{d}\Big(w(t)\,\frac{Q_1(t)\,Q_2(t)}{1-H_{\eta,\gamma}(t)}\Big) \neq 0.$$

Now it is clear that (4.3) is equivalent to

$$\int_0^\infty (\Lambda_1(t)-\Lambda_2(t))\,\mathrm{d}\Big(w(t)\,\frac{Q_1(t)\,Q_2(t)}{1-H_{\eta,\gamma}(t)}\Big) \neq 0.$$

If there exists $t_0 \in [0,\tau_0]$ such that $F_1(t_0) \neq F_2(t_0)$ which is equivalent to $\Lambda_1(t_0) \neq \Lambda_2(t_0)$ then due to continuity there exists an open neighborhood of t_0, i.e. (t_0-h, t_0+h) for $h>0$, and our statistics $T_n(\infty)$ and $T_n^\sigma(\infty)$, respectively, are consistent. We proceed similarly for (4.4) replacing $Q_1(t)$ in nominator by $(1-F_1(t))(\eta(1-G_1(t))+(1-\eta))(1-G_2(t)))$, thus the MOSUM-type test statistics $T_{n,D}(\infty)$ and $T_{n,D}^\sigma(\infty)$ are also consistent.

Notice that it is more difficult to choose appropriate weights. One usually uses also the knowledge of the lifetime distribution of the sample at hand. It is known that the logrank weight function is appropriate for the extreme value distribution and the Prentice-Wilcoxon weight function for the logistic distribution described below, for more information see e.g. Fleming and Harrington [**14**], Section 7.4.

3. Lifetime distributions

Here we focus on the assumptions (A.2) – (A.4) needed in Theorems 3.1 and 3.2.

We consider such types of the lifetime distribution

(1) the exponential distribution E(δ), i.e. $F_A(t) = 1-\exp(-\delta t)$, $\delta>0$, $t \geq 0$;
(2) the extreme value distribution EV(δ), i.e. $F_B(t) = 1-\exp(-\mathrm{e}^{\delta t})$, $\delta>0$, $t \geq 0$;
(3) the logistic distribution L(δ), i.e $F_C(t) = (1+\mathrm{e}^{-\delta t})^{-1}$, $\delta>0$, $t \geq 0$;
(4) the log-normal distribution LN(δ), i.e. $F_D(t) = \Phi(\log(\delta t))$, $\delta>0$, $t \geq 0$;
(5) the Weibull distribution W(δ), i.e. $F_E(t) = 1-\exp(-(\delta t)^4)$, $\delta>0$, $t \geq 0$.

Notice that all above mentioned distribution functions are strictly increasing for $t \geq 0$ and $F(0)=0$.

We compute their hazard function by the formula $\lambda(t) = f(t)/(1-F(t))$, so we get

(1) $f_A(t) = \delta\exp(-\delta t)$ and consequently $\lambda_A(t) = \delta$;
(2) $f_B(t) = \delta\exp(-\mathrm{e}^{\delta t})\mathrm{e}^{\delta t}$ and consequently $\lambda_B(t) = \delta\exp\delta t$;
(3) $f_C(t) = \delta(1+\mathrm{e}^{-\delta t})^{-2}\,\mathrm{e}^{-\delta t}$ and $\lambda_C(t) = \delta/(1+\mathrm{e}^{-\delta t})$;
(4) $f_D(t) = \phi(\log(\delta t))/t$ and $\lambda_D(t) = \phi(\log(\delta t))/\left(t(1-\Phi(\log(\delta t)))\right)$;
(5) $f_E(t) = 4\delta^4 t^3\,\exp(-(\delta t)^4)$ and $\lambda_E(t) = 4\delta^4 t^3$.

Notice that the hazard function $\lambda(t)$ is a constant function in t for the exponential distribution. Further, it is evident that the hazard functions $\lambda(t)$ for the extreme value, logistic and the Weibull distribution are strictly increasing and continuous in t. Finally, the hazard function for the log-normal distribution is continuous and has value 0 at $t=0$, increases to a maximum (which is approximately at $t=1/\delta$) and then decreases, approaching to zero with $t\to\infty$. The detailed investigation of behavior of the hazard function $\lambda_D(t)$ will be omitted. Thus, we cannot easily verify for the log-normal distribution the following considered conditions, but we will do at least simulations for behavior of change-point estimators in case of this lifetime distribution. Moreover, according to Kalbfleisch, Prentice [**25**] the log-logistic distribution provides a good approximation to the log-normal one.

Convention 4.2. In the following, we assume (S.3) and we suppose that $F_{in}(t)$ has the parameter $\delta_{in} > 0$ and $G_{in}(t)$ is chosen also from the distributions presented above with parameters $\delta_{i\,C,n} > 0$, $i = 1, 2$.

Notice that

$$\int_0^{\tau_0} |\lambda_{F_{in}}(t) - \lambda_F(t)| \leq \sup_{0\leq t\leq \tau_0} |\lambda_{F_{in}}(t) - \lambda_F(t)| = \tau_0\, |\lambda_{F_{in}}(\tau_0) - \lambda_F(\tau_0)|$$
$$= |\delta_{in} - \delta|\, O(1)$$

for the considered lifetime distributions except the log-normal one. Clearly, if exist $\delta, \delta_C > 0$ such that

$$\lim_{n\to\infty} |\delta_{in} - \delta| = 0, \qquad i = 1, 2, \tag{4.12}$$
$$\lim_{n\to\infty} |\delta_{i\,C,n} - \delta_C| = 0, \qquad i = 1, 2,$$

the conditions (A.2) and (3.3) in (A.3) are satisfied. Further, (4.12) implies that

$$\lim_{n\to\infty} |\delta_{1n} - \delta_{2n}| = 0.$$

If moreover,

$$\sqrt{n}\, |\delta_{1n} - \delta_{2n}| \to \infty, \quad \text{as } n \to \infty,$$

or

$$\frac{\sqrt{n}\, |\delta_{1n} - \delta_{2n}|}{\sqrt{\log\log n}} \to \infty, \quad \text{as } n \to \infty,$$

then the condition (3.4) or (3.5), respectively, in (A.3) is also fulfilled in view of

$$|A_n(\tau_0)| = |\int_0^{\tau_0} w(t)(1 - F(t))(1 - G(t))\, (\lambda_{F_{1n}}(t) - \lambda_{F_{2n}}(t))\, \mathrm{d}t|$$
$$\geq \inf_{0\leq t\leq \tau_0} |\lambda_{F_{1n}}(t) - \lambda_{F_{2n}}(t)|\, \tau_0\, w(\tau_0)\, (1 - F(\tau_0))(1 - G(\tau_0))$$
$$= |\lambda_{F_{1n}}(0) - \lambda_{F_{2n}}(0)|\, \tau_0\, w(\tau_0)\, (1 - F(\tau_0))(1 - G(\tau_0))$$

and

$$|\lambda_{1n}(0) - \lambda_{2n}(0)| = \begin{cases} |\delta_{1n} - \delta_{2n}| & \text{for the exponential, extreme value, logistic distribution,} \\ |\delta_{1n}^4 - \delta_{2n}^4| & \text{for the Weibull distribution.} \end{cases}$$

The condition (A.4) is satisfied trivially for all the lifetime distributions, see (4.11) in the previous section.

4. Local and fixed alternatives

We analyze what happens under the situations (S.1) and (S.2), respectively, i.e. at least the distributions G_{1n} and G_{2n} of the censoring variables do not depend on n. We treat the limit behavior of S_{m_c} and S_m or $S_{m_c}/\sqrt{nV_{m_c}}$ and $S_m/\sqrt{nV_m}$, respectively, which influence the limit behavior of the max-type estimators of the change point $\hat{m}_1(\tau_0)$ or $\hat{m}_2(\tau_0)$, respectively.

By Lemma 5.5 and Corollary 5.7 below we get, as $n \to \infty$,

$$\frac{S_{m_c}}{n} = \frac{n-m}{n}\left(\eta\, I_1(\tau_0) + o_{\mathrm{P}}\Big(\frac{1}{b_n}\Big)\right),$$
$$\frac{S_m}{n} = \frac{n-m}{n}\left(\eta\, I_1(\tau_0) + (\gamma - \eta)\, I_2(\tau_0) + o_{\mathrm{P}}\Big(\frac{1}{b_n}\Big)\right),$$

where $I_1(\tau_0)$, $I_2(\tau_0)$ are defined in (5.20), (5.21), respectively, and b_n is a prescribed rate for convergence of weights $w_n(t)$. By Corollary 5.14 below we obtain, as $n \to \infty$,

$$\frac{S_{m_c}}{\sqrt{nV_{m_c}}} = \sqrt{\frac{n\,(n-m_c)}{m_c}}\;\frac{n-m}{n-m_c}\;\frac{\eta\, I_1(\tau_0) + o_{\mathrm{P}}\left(\frac{1}{b_n}\right)}{\sqrt{\frac{\gamma-\eta}{1-\eta}\, J_{1112}(\tau_0) + \frac{1-\gamma}{1-\eta}\, J_{1212}(\tau_0) + o_{\mathrm{P}}\left(\frac{1}{b_n}\right)}},$$

$$\frac{S_m}{\sqrt{nV_m}} = \sqrt{\frac{n\,(n-m)}{m}}\;\frac{\eta\, I_1(\tau_0) + (\gamma-\eta)\, I_2(\tau_0) + o_{\mathrm{P}}\left(\frac{1}{b_n}\right)}{\sqrt{\frac{\eta}{\gamma}\, J_{1212}(\tau_0) + \frac{\gamma-\eta}{\gamma}\, J_{1222}(\tau_0) + o_{\mathrm{P}}\left(\frac{1}{b_n}\right)}}$$

with $J_{ijkl}(\tau_0)$ of the form (5.36).

If we suppose the so-called *ordered hazards alternative*, i.e. $\lambda_1(t) \geq \lambda_2(t)$ or $\lambda_1(t) \leq \lambda_2(t)$, respectively, for all t, which implies the alternative of stochastic ordering $F_1(t) \geq F_2(t)$ or $F_1(t) \leq F_2(t)$, respectively, then the terms $I_1(\tau_0)$ and $I_2(\tau_0)$ has the same sign $+$ or $-$. If we use the distributions introduced in Section 3 (except the log-normal case) and the distribution functions $F_1(t)$, $F_2(t)$ differs in the parameter δ only, then it is clear that the ordered hazard alternative is fulfilled, even with sharp inequality $>$ or $<$, respectively. Thus, $|S_{m_c}|/n$ is asymptotically stochastically smaller than $|S_m|/n$ for the situations (S.1) and (S.2).

The problem is the standardization. Under the situation (S.2) of the local alternatives for the lifetime distributions and the fixed alternatives for the censoring distributions, we get, as $n \to \infty$,

$$\begin{aligned}
\frac{S_{m_c}}{\sqrt{nV_{m_c}}} &= \sqrt{\frac{m_c\,(n-m_c)}{n}}\;\frac{n-m}{n-m_c} \\
&\quad\cdot \frac{\int_0^{\tau_0} w(t)\, \frac{(1-F(t))(1-G_1(t))(1-G_2(t))}{\eta(1-G_1(t))+(1-\eta)(1-G_2(t))} (\lambda_{1n}(t)-\lambda_{2n}(t))\, \mathrm{d}t\, (1+o(1)) + o_{\mathrm{P}}\left(\frac{1}{b_n}\right)}{\sqrt{\int_0^{\tau_0} w(t)\, \frac{(1-G_1(t))(1-G_2(t))}{(\eta(1-G_1(t))+(1-\eta)(1-G_2(t)))^2}\, \mathrm{d}R_\eta(t) + o_{\mathrm{P}}(1)}} \\
&= \sqrt{\frac{m_c\,(n-m_c)}{n}}\;\frac{n-m}{n-m_c} \\
&\quad\cdot \frac{\int_0^{\tau_0} w(t)\, \frac{(1-F(t))(1-G_1(t))(1-G_2(t))}{\eta(1-G_1(t))+(1-\eta)(1-G_2(t))} (\lambda_{1n}(t)-\lambda_{2n}(t))\, \mathrm{d}t\, (1+o(1)) + o_{\mathrm{P}}\left(\frac{1}{b_n}\right)}{\sqrt{\int_0^{\tau_0} w(t)\, \frac{(1-G_1(t))(1-G_2(t))}{\eta(1-G_1(t))+(1-\eta)(1-G_2(t))}\, \mathrm{d}F(t) + o_{\mathrm{P}}(1)}}
\end{aligned}$$

and

$$\begin{aligned}
\frac{S_m}{\sqrt{nV_m}} &= \sqrt{\frac{m\,(n-m)}{n}} \\
&\quad\cdot \frac{\int_0^{\tau_0} w(t)\, \frac{(1-F(t))(\eta(1-G_1(t))+(\gamma-\eta)(1-G_2(t)))(1-G_2(t))}{\eta(1-G_1(t))+(1-\eta)(1-G_2(t))} (\lambda_{1n}(t)-\lambda_{2n}(t))\, \mathrm{d}t\, (1+o(1)) + o_{\mathrm{P}}\left(\frac{1}{b_n}\right)}{\sqrt{\int_0^{\tau_0} w(t)\, \frac{(\eta(1-G_1(t))+(\gamma-\eta)(1-G_2(t)))(1-G_2(t))}{(\eta(1-G_1(t))+(1-\eta)(1-G_2(t)))^2}\, \mathrm{d}R_\eta(t) + o_{\mathrm{P}}(1)}} \\
&= \sqrt{\frac{m\,(n-m)}{n}} \\
&\quad\cdot \frac{\int_0^{\tau_0} w(t)\, \frac{(1-F(t))(\eta(1-G_1(t))+(\gamma-\eta)(1-G_2(t)))(1-G_2(t))}{\eta(1-G_1(t))+(1-\eta)(1-G_2(t))} (\lambda_{1n}(t)-\lambda_{2n}(t))\, \mathrm{d}t\, (1+o(1)) + o_{\mathrm{P}}\left(\frac{1}{b_n}\right)}{\sqrt{\int_0^{\tau_0} w(t)\, \frac{(\eta(1-G_1(t))+(\gamma-\eta)(1-G_2(t)))(1-G_2(t))}{\eta(1-G_1(t))+(1-\eta)(1-G_2(t))}\, \mathrm{d}F(t) + o_{\mathrm{P}}(1)}},
\end{aligned}$$

where $F(t)$ is the limit distribution function of $F_{in}(t)$, $i = 1, 2$, $0 \leq t \leq \tau_0$, (recall (3.8)).

Summarizing the above comments, it seems that $\hat{m}_1(\tau_0)/n$ in contrast to $\hat{m}_2(\tau_0)/n$ is a consistent estimator of γ not only under (S.3) but even under (S.2) but the proof becomes more complicated than the proof of Theorem 3.1, because we have to work not only with one term $A_n(\tau_0)$ but with two terms given by

$$A_{1n}(\tau_0) = \int_0^{\tau_0} w(t) \frac{(1-F(t))(1-G_1(t))(1-G_2(t))}{\eta(1-G_1(t))+(1-\eta)(1-G_2(t))} (\lambda_{1n}(t) - \lambda_{2n}(t))\,\mathrm{d}t,$$

$$A_{2n}(\tau_0) = \int_0^{\tau_0} w(t) \frac{(1-F(t))(1-G_2(t))^2}{\eta(1-G_1(t))+(1-\eta)(1-G_2(t))} (\lambda_{1n}(t) - \lambda_{2n}(t))\,\mathrm{d}t.$$

CHAPTER 5

Technical Calculation

1. Introduction

Recall that the survival variables $X_1^0, X_2^0, \ldots, X_{\lfloor n\gamma \rfloor}^0$ and $X_{\lfloor n\gamma \rfloor+1}^0, X_{\lfloor n\gamma \rfloor+2}^0, \ldots, X_n^0$ are independent with the absolutely continuous distribution functions F_1 and F_2, respectively, $F_1 \neq F_2$. The lifetimes are censored from the right by the independent random variables $C_1, C_2, \ldots, C_{\lfloor n\eta \rfloor}$ and $C_{\lfloor n\eta \rfloor+1}, C_{\lfloor n\eta \rfloor+2}, \ldots, C_n$ which have the absolutely continuous distribution functions G_1 and G_2, respectively, $G_1 \neq G_2$. The censoring times are independent of the lifetimes. We consider in most cases of this chapter that the distribution functions F_1, F_2 and G_1, G_2 are fixed. The results can be easily modified for the situation, when the distribution functions depend on n. It is mainly needed in Chapter 3. Important results will be rewritten also for this case with the notation "(local alternatives)". Notice that parameters γ and η are unknown constants from $(0,1]$ and $m = \lfloor n\gamma \rfloor$, $m_c = \lfloor n\eta \rfloor$. Further, recall

$$\sup_{0 \le t \le \tau_0} |w_n(t) - w(t)| = o_{\mathrm{P}}\Big(\frac{1}{b_n}\Big), \tag{5.1}$$

where b_n is a sequence with the form

$$b_n = \begin{cases} 1/|A_n(\tau_0)|, & \text{(estimators)}, \\ \log\log n, & \text{(the test statistic } T_n(\tau_0)), \\ 1, & \text{(otherwise)}, \end{cases} \tag{5.2}$$

with $A_n(\tau_0)$ defined in (3.6).

In the following sections we prove a number of technical lemmas which were used in Chapters 2 and 3 to investigate properties of the suggested statistics and the estimators. Some references to those can be found also in Chapter 4. Further, for brevity, we omit τ_0 mainly in the terms $S_k(\tau_0)$, $V_k(\tau_0)$ and $A_n(\tau_0)$.

2. Approximation of the processes $Y(t)$ and $N(t)$

In this section we present a useful representation of the processes $Y(t)$ denoting the size of risk set and $N(t)$ counting the observed failures by time t, see their definitions in (2.2) and (2.3).

LEMMA 5.1. *Assume $0 < m_c \le m \le n$. For any $A > 0$ there exist $C_A, D_A > 0$ such that*

$$\mathrm{P}\left(\sup_{0 \le t \le \tau_0} \left| \sum_{j=1}^{k} \frac{Y_j(t) - \mathrm{E}\, Y_j(t)}{k} \right| > C_A \sqrt{\frac{\log k}{k}} \right) \le \frac{D_A}{k^A}, \qquad 1 \le k \le n,$$

$$\mathrm{P}\left(\sup_{0 \le t \le \tau_0} \left| \sum_{j=k+1}^{n} \frac{Y_j(t) - \mathrm{E}\, Y_j(t)}{n-k} \right| > C_A \sqrt{\frac{\log (n-k)}{n-k}} \right) \le \frac{D_A}{(n-k)^A}, \qquad 1 \le k < n.$$

PROOF. Notice that for all $t \ge 0$

$$\mathrm{E}\, Y_j(t) = \mathrm{P}(X_j \ge t) = 1 - H_1(t) = (1 - F_1(t))(1 - G_1(t)), \qquad 1 \le j \le m_c,$$

$$\begin{aligned}
\mathrm{E}\,Y_j(t) &= \mathrm{P}(X_j \geq t) = 1 - H_2(t) = (1-F_1(t))(1-G_2(t)), && m_c < j \leq m,\\
\mathrm{E}\,Y_j(t) &= \mathrm{P}(X_j \geq t) = 1 - H_3(t) = (1-F_2(t))(1-G_2(t)), && m < j \leq n,
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{E}\,Y_j(t) - Y_j(t) &= I(X_j < t) - H_1(t) = I(H_1(X_j) < H_1(t)) - H_1(t), && 1 \leq j \leq m_c,\\
\mathrm{E}\,Y_j(t) - Y_j(t) &= I(X_j < t) - H_2(t) = I(H_2(X_j) < H_2(t)) - H_2(t), && m_c < j \leq m,\\
\mathrm{E}\,Y_j(t) - Y_j(t) &= I(X_j < t) - H_3(t) = I(H_3(X_j) < H_3(t)) - H_3(t), && m < j \leq n. \qquad (5.3)
\end{aligned}$$

It can be seen that $U_1 = H_1(X_1), \ldots, U_{m_c} = H_1(X_{m_c})$, $U_{m_c+1} = H_2(X_{m_c}), \ldots, U_m = H_2(X_m)$, $U_{m+1} = H_3(X_{m+1}), \ldots, U_n = H_3(X_n)$ are independent variables coming from the uniform distribution $\mathrm{U}(0,1)$ and therefore Lemma 5.12 below can be applied. By a small modification for $\sum_{j=1}^{k}(I(U_i \leq t) - t)$, $k = 1, 2, \ldots, n$, and $\sum_{j=k+1}^{n}(I(U_i \leq t) - t)$, $k = 1, 2, \ldots, n-1$, respectively, we get

$$\begin{aligned}
&\mathrm{P}\left(\sqrt{k}\sup_{0\leq t\leq 1}\left|\frac{1}{k}\sum_{j=1}^{k}(Y_j(t) - \mathrm{E}\,Y_j(t))\right| > \varepsilon_1\right) \leq 2\exp\{-2\varepsilon_1^2\}, && \forall\,\varepsilon_1 > 0,\\
&\mathrm{P}\left(\sqrt{n-k}\sup_{0\leq t\leq 1}\left|\frac{1}{n-k}\sum_{j=k+1}^{n}(Y_j(t) - \mathrm{E}\,Y_j(t))\right| > \varepsilon_2\right) \leq 2\exp\{-2\varepsilon_2^2\}, && \forall\,\varepsilon_2 > 0,
\end{aligned}$$

and by the choices of $\varepsilon_1 = C_A\sqrt{\log k}$ and $\varepsilon_2 = C_A\sqrt{\log(n-k)}$ we obtain our assertion. □

COROLLARY 5.1. *(local alternatives) Assume* $0 < m_c \leq m \leq n$. *Let (A.2) and* (3.8) *be satisfied. Then, as* $n \to \infty$,

$$\begin{aligned}
\sup_{0\leq t\leq \tau_0}\sum_{j=1}^{k}\frac{|Y_j(t) - (1-F(t))(1-G(t))|}{k} &= o_{\mathrm{P}}(1) && \textit{uniformly in } \log\log n < k \leq n,\\
\sup_{0\leq t\leq \tau_0}\sum_{j=k+1}^{n}\frac{|Y_j(t) - (1-F(t))(1-G(t))|}{n-k} &= o_{\mathrm{P}}(1) && \textit{uniformly in } 1 \leq k < n - \log\log n.
\end{aligned} \qquad (5.4)$$

PROOF. The assumptions (A.2) and (3.8) imply, as $n \to \infty$,

$$\sup_{0\leq t\leq \tau_0}|\,\mathrm{E}\,Y_j(t) - (1-F(t))(1-G(t))| = o(1), \quad 1 \leq j \leq n,$$

and by Lemma 5.1 we get, as $n \to \infty$,

$$\begin{aligned}
\sup_{0\leq t\leq \tau_0}\sum_{j=1}^{k}\frac{|Y_j(t) - \mathrm{E}\,Y_j(t)|}{k} &= O_{\mathrm{P}}\left(\sqrt{\frac{\log k}{k}}\right) && \text{uniformly in } \log\log n < k \leq n,\\
\sup_{0\leq t\leq \tau_0}\sum_{j=k+1}^{n}\frac{|Y_j(t) - \mathrm{E}\,Y_j(t)|}{n-k} &= O_{\mathrm{P}}\left(\sqrt{\frac{\log(n-k)}{n-k}}\right) && \text{uniformly in } 1 \leq k < n - \log\log n.
\end{aligned}$$

Now it is clear that, as $n \to \infty$,

$$\begin{aligned}
&\sup_{0\le t\le\tau_0} \sum_{j=1}^{k} \frac{|Y_j(t) - (1-F(t))(1-G(t))|}{k} \\
&\le \sup_{0\le t\le\tau_0} \sum_{j=1}^{k} \frac{|Y_j(t) - \mathrm{E}\,Y_j(t)|}{k} + \sup_{0\le t\le\tau_0} \sum_{j=1}^{k} \frac{|\,\mathrm{E}\,Y_j(t) - (1-F(t))\,(1-G(t))|}{k} \\
&= O_{\mathrm{P}}\left(\sqrt{\frac{\log k}{k}}\right) + o(1) = o_{\mathrm{P}}(1)
\end{aligned}$$

uniformly in $\log\log n < k \le n$. In the same way we obtain the approximation (5.4) and hence the proof is finished. □

Lemma 5.2. *Assume* $0 < m_c < m < n$. *We have, as* $n \to \infty$,

$$\begin{aligned}
&\sup_{0\le t\le\tau_0} \frac{1}{m_c} \left|\sum_{j=1}^{m_c}(Y_j(t) - \mathrm{E}\,Y_1(t))\right| + \sup_{0\le t\le\tau_0} \frac{1}{m-m_c} \left|\sum_{j=m_c+1}^{m} (Y_j(t) - \mathrm{E}\,Y_m(t))\right| \\
&+ \sup_{0\le t\le\tau_0} \frac{1}{n-m} \left|\sum_{j=m+1}^{n} (Y_j(t) - \mathrm{E}\,Y_n(t))\right| = O_{\mathrm{P}}\left(\frac{1}{\sqrt{n}}\right), \qquad (5.5)
\end{aligned}$$

$$\begin{aligned}
&\sup_{0\le t\le\tau_0} \frac{1}{m_c} \left|\sum_{j=1}^{m_c}(N_j(t) - \mathrm{E}\,N_1(t))\right| + \sup_{0\le t\le\tau_0} \frac{1}{m-m_c} \left|\sum_{j=m_c+1}^{m} (N_j(t) - \mathrm{E}\,N_m(t))\right| \\
&+ \sup_{0\le t\le\tau_0} \frac{1}{n-m} \left|\sum_{j=m+1}^{n} (N_j(t) - \mathrm{E}\,N_n(t))\right| = O_{\mathrm{P}}\left(\frac{1}{\sqrt{n}}\right) \qquad (5.6)
\end{aligned}$$

and also, as $n \to \infty$,

$$\sup_{0\le t\le\tau_0} \left|\frac{Y(t)}{n} - (1 - H_{\eta,\gamma}(t))\right| = O_{\mathrm{P}}\left(\frac{1}{\sqrt{n}}\right), \qquad \sup_{0\le t\le\tau_0} \left|\frac{N(t)}{n} - R_{\eta,\gamma}(t)\right| = O_{\mathrm{P}}\left(\frac{1}{\sqrt{n}}\right),$$

where $H_{\eta,\gamma}(t)$ *and* $R_{\eta,\gamma}(t)$ *are defined in* (1.6) *and* (1.10).

Proof. Notice that for all $t \ge 0$

$$\begin{aligned}
\mathrm{E}\,N_j(t) &= \mathrm{P}(X_j \le t, \Delta_j = 1) = L_1(t,1) = \int_0^t (1 - G_1(u))\,\mathrm{d}F_1(u), && 1 \le j \le m_c, \\
\mathrm{E}\,N_j(t) &= \mathrm{P}(X_j \le t, \Delta_j = 1) = L_2(t,1) = \int_0^t (1 - G_2(u))\,\mathrm{d}F_1(u), && m_c < j \le m, \\
\mathrm{E}\,N_j(t) &= \mathrm{P}(X_j \le t, \Delta_j = 1) = L_3(t,1) = \int_0^t (1 - G_2(u))\,\mathrm{d}F_2(u), && m < j \le n,
\end{aligned}$$

and

$$\begin{aligned}
N_j(t) - \mathrm{E}\,N_j(t) &= I(X_j \le t)\,\Delta_j - L_1(t,1) = I(L_1(X_j,1) \le L_1(t,1)) - L_1(t,1), && 1 \le j \le m_c, \\
N_j(t) - \mathrm{E}\,N_j(t) &= I(X_j \le t)\,\Delta_j - L_2(t,1) = I(L_2(X_j,1) \le L_2(t,1)) - L_2(t,1), && m_c < j \le m, \\
N_j(t) - \mathrm{E}\,N_j(t) &= I(X_j \le t)\,\Delta_j - L_3(t,1) = I(L_3(X_j,1) \le L_3(t,1)) - L_3(t,1), && m < j \le n,
\end{aligned}$$

and recall (5.3). Using the same idea as in the proof of Lemma 5.1 we see that the assertions (5.5) and (5.6) are direct consequences of Lemma 5.12 below. Further,

$$\begin{aligned}
&\sup_{0\le t\le\tau_0}\Big|\frac{Y(t)}{n}-(1-H_{\eta,\gamma}(t))\Big| \le \sup_{0\le t\le\tau_0}\Big|\frac{m_c}{n}\Big(\frac{1}{m_c}\sum_{j=1}^{m_c}(Y_j(t)-(1-F_1(t))(1-G_1(t)))\Big)\Big|\\
&\quad+\Big(\frac{m_c}{n}-\eta\Big)\sup_{0\le t\le\tau_0}|(1-F_1(t))(1-G_1(t))|\\
&\quad+\sup_{0\le t\le\tau_0}\Big|\frac{m-m_c}{n}\Big(\frac{1}{m-m_c}\sum_{j=m_c+1}^{m}(Y_j(t)-(1-F_1(t))(1-G_2(t)))\Big)\Big|\\
&\quad+\Big(\frac{m-m_c}{n}-(\gamma-\eta)\Big)\sup_{0\le t\le\tau_0}|(1-F_1(t))(1-G_2(t))|\\
&\quad+\sup_{0\le t<\tau_0}\Big|\frac{n-m}{n}\Big(\frac{1}{n-m}\sum_{j=m+1}^{n}(Y_j(t)-(1-F_2(t))(1-G_2(t)))\Big)\Big|\\
&\quad+\Big(\frac{n-m}{n}-(1-\gamma)\Big)\sup_{0\le t\le\tau_0}|(1-F_2(t))(1-G_2(t))|\\
&\le\frac{m_c}{n}\sup_{0\le t\le\tau_0}\Big|\frac{1}{m_c}\sum_{j=1}^{m_c}(Y_j(t)-(1-F_1(t))(1-G_1(t)))\Big|\\
&\quad+\frac{m-m_c}{n}\sup_{0\le t\le\tau_0}\Big|\frac{1}{m-m_c}\sum_{j=m_c+1}^{m}(Y_j(t)-(1-F_1(t))(1-G_2(t)))\Big|\\
&\quad+\frac{n-m}{n}\sup_{0\le t\le\tau_0}\Big|\frac{1}{n-m}\sum_{j=m+1}^{n}(Y_j(t)-(1-F_2(t))(1-G_2(t)))\Big|\\
&\quad+\Big(\frac{m_c}{n}-\eta\Big)+\Big(\frac{m-m_c}{n}-(\gamma-\eta)\Big)+\Big(\frac{n-m}{n}-(1-\gamma)\Big)\\
&=O_{\mathrm P}\left(\frac{1}{\sqrt{n}}\right),\quad n\to\infty,
\end{aligned}$$

and similarly we proceed for $N(t)$ and $R_{\eta,\gamma}(t)$. □

COROLLARY 5.2. *Assume $0<m_c=m<n$. Then we have, as $n\to\infty$,*

$$\sup_{0\le t\le\tau_0}\frac{1}{m}\left|\sum_{j=1}^{m}(Y_j(t)-\mathrm{E}\,Y_1(t))\right|+\sup_{0\le t\le\tau_0}\frac{1}{n-m}\left|\sum_{j=m+1}^{n}(Y_j(t)-\mathrm{E}\,Y_n(t))\right|=O_{\mathrm P}\left(\frac{1}{\sqrt{n}}\right),$$

$$\sup_{0\le t\le\tau_0}\frac{1}{m}\left|\sum_{j=1}^{m}(N_j(t)-\mathrm{E}\,N_1(t))\right|+\sup_{0\le t\le\tau_0}\frac{1}{n-m}\left|\sum_{j=m+1}^{n}(N_j(t)-\mathrm{E}\,N_n(t))\right|=O_{\mathrm P}\left(\frac{1}{\sqrt{n}}\right)$$

and also, as $n\to\infty$,

$$\sup_{0\le t\le\tau_0}\left|\frac{Y(t)}{n}-(1-H_\gamma(t))\right|=O_{\mathrm P}\left(\frac{1}{\sqrt{n}}\right),\quad \sup_{0\le t\le\tau_0}\left|\frac{N(t)}{n}-R_\gamma(t)\right|=O_{\mathrm P}\left(\frac{1}{\sqrt{n}}\right),$$

where $H_\gamma(t)$ and $R_\gamma(t)$ are defined in (1.7) *and* (1.11).
Assume $0 < m_c < m = n$. Then we have, as $n \to \infty$,

$$\sup_{0\le t\le\tau_0} \frac{1}{m_c}\left|\sum_{j=1}^{m_c}(Y_j(t) - \mathrm{E}\,Y_1(t))\right| + \sup_{0\le t\le\tau_0}\frac{1}{n-m_c}\left|\sum_{j=m_c+1}^{n}(Y_j(t) - \mathrm{E}\,Y_n(t))\right| = O_{\mathrm{P}}\left(\frac{1}{\sqrt{n}}\right),$$

$$\sup_{0\le t\le\tau_0} \frac{1}{m_c}\left|\sum_{j=1}^{m_c}(N_j(t) - \mathrm{E}\,N_1(t))\right| + \sup_{0\le t\le\tau_0}\frac{1}{n-m_c}\left|\sum_{j=m_c+1}^{n}(N_j(t) - \mathrm{E}\,N_n(t))\right| = O_{\mathrm{P}}\left(\frac{1}{\sqrt{n}}\right)$$

and also, as $n \to \infty$,

$$\sup_{0\le t\le\tau_0}\left|\frac{Y(t)}{n} - (1 - H_\eta(t))\right| = O_{\mathrm{P}}\left(\frac{1}{\sqrt{n}}\right), \quad \sup_{0\le t\le\tau_0}\left|\frac{N(t)}{n} - R_\eta(t)\right| = O_{\mathrm{P}}\left(\frac{1}{\sqrt{n}}\right),$$

where $H_\eta(t)$ and $R_\eta(t)$ are defined in (1.8) *and* (1.12).
Assume $0 < m_c = m = n$. Then we have, as $n \to \infty$,

$$\sup_{0\le t\le\tau_0}\left|\frac{Y(t)}{n} - (1 - H(t))\right| = O_{\mathrm{P}}\left(\frac{1}{\sqrt{n}}\right), \quad \sup_{0\le t\le\tau_0}\left|\frac{N(t)}{n} - R(t)\right| = O_{\mathrm{P}}\left(\frac{1}{\sqrt{n}}\right),$$

where $H(t)$ and $R(t)$ are defined in (1.9) *and* (1.13).

Proof. The assertions follow directly from Lemma 5.2. □

Lemma 5.3. *Assume $0 < m_c < m < n$. Let $v(t)$ is a continuous function on $[0,\tau_0]$, then we have, as $n \to \infty$,*

$$\left|\int_0^{\tau_0} v(t)\left(\mathrm{d}\Big(\frac{\sum_{j=1}^{m_c} N_j(t)}{m_c}\Big) - \mathrm{d}\,\mathrm{E}\,N_1(t)\right)\right| = O_{\mathrm{P}}\left(\frac{1}{\sqrt{m_c}}\right), \tag{5.7}$$

$$\left|\int_0^{\tau_0} v(t)\left(\mathrm{d}\Big(\frac{\sum_{j=m_c+1}^{m} N_j(t)}{m-m_c}\Big) - \mathrm{d}\,\mathrm{E}\,N_m(t)\right)\right| = O_{\mathrm{P}}\left(\frac{1}{\sqrt{m}}\right), \tag{5.8}$$

$$\left|\int_0^{\tau_0} v(t)\left(\mathrm{d}\Big(\frac{\sum_{j=m+1}^{n} N_j(t)}{n-m}\Big) - \mathrm{d}\,\mathrm{E}\,N_n(t)\right)\right| = O_{\mathrm{P}}\left(\frac{1}{\sqrt{n-m}}\right) \tag{5.9}$$

and also, as $n \to \infty$,

$$\int_0^{\tau_0} v(t)\,\mathrm{d}\Big(\frac{N(t)}{n}\Big) = \int_0^{\tau_0} v(t)\,\mathrm{d}R_{\eta,\gamma}(t) + O_{\mathrm{P}}\left(\frac{1}{\sqrt{n}}\right), \tag{5.10}$$

where $R_{\eta,\gamma}(t)$ is defined in (1.10).

Proof. Since the function $v(t)$ is continuous on $[0,\tau_0]$, it is also bounded. Further, we have by (1.3) and direct calculations

$$\begin{aligned}
&\left|\int_0^{\tau_0} v(t)\left(\mathrm{d}\Big(\frac{\sum_{j=1}^{m_c} N_j(t)}{m_c}\Big) - \mathrm{d}\,\mathrm{E}\,N_1(t)\right)\right| \\
&\qquad = \left|\frac{1}{m_c}\sum_{j=1}^{m_c} I(X_j \le \tau_0)\,v(X_j)\,\Delta_j - \int_0^{\tau_0} v(t)(1 - G_1(t))\,\mathrm{d}F_1(t)\right| \\
&\qquad\qquad = \left|\frac{1}{m_c}\sum_{j=1}^{m_c} I(X_j \le \tau_0)\,v(X_j)\,\Delta_j - \mathrm{E}\left(v(X_1)\,\Delta_1\,I(X_1 \le \tau_0)\right)\right|.
\end{aligned}$$

Since the function $v(t)$ is bounded on $[0,\tau_0]$, we see that $v(X_1), v(X_2), \dots, v(X_n)$ are bounded i.i.d. variables hence $\operatorname{var} v(X_1) < \infty$. Then by the Chebyshev inequality we get for every $\varepsilon > 0$

$$\mathrm{P}\left(\left|\frac{1}{m_c}\sum_{j=1}^{m_c}\Big(I(X_j \le \tau_0)\, v(X_j)\,\Delta_j - \mathrm{E}\,(v(X_1)\,\Delta_1\, I(X_1 \le \tau_0))\Big)\right| \ge \frac{\varepsilon}{\sqrt{m_c}}\right) \le \frac{\operatorname{var} v(X_1)}{\varepsilon^2}$$

and from this it follows that, as $n \to \infty$,

$$\left|\frac{1}{m_c}\sum_{j=1}^{m_c} I(X_j \le \tau_0)\, v(X_j)\,\Delta_j - \mathrm{E}\,(v(X_1)\,\Delta_1\, I(X_1 \le \tau_0))\right| = O_{\mathrm{P}}\Big(\frac{1}{\sqrt{m_c}}\Big),$$

so the assertion (5.7) holds. We use the similar steps for (5.8) and (5.9). The assertion (5.10) follows directly from the results (5.7) – (5.9). ☐

Corollary 5.3. *Let $v(t)$ is a continuous function on $[0,\tau_0]$.*
Assume $0 < m_c = m < n$, then we have, as $n \to \infty$,

$$\int_0^{\tau_0} v(t)\,\mathrm{d}\Big(\frac{N(t)}{n}\Big) = \int_0^{\tau_0} v(t)\,\mathrm{d}R_\gamma(t) + O_{\mathrm{P}}\left(\frac{1}{\sqrt{n}}\right),$$

where $R_\gamma(t)$ is defined in (1.11).
Assume $0 < m_c < m = n$, then we have, as $n \to \infty$,

$$\int_0^{\tau_0} v(t)\,\mathrm{d}\Big(\frac{N(t)}{n}\Big) = \int_0^{\tau_0} v(t)\,\mathrm{d}R_\eta(t) + O_{\mathrm{P}}\left(\frac{1}{\sqrt{n}}\right),$$

where $R_\eta(t)$ is defined in (1.12).
Assume $0 < m_c = m = n$, then we have, as $n \to \infty$,

$$\int_0^{\tau_0} v(t)\,\mathrm{d}\Big(\frac{N(t)}{n}\Big) = \int_0^{\tau_0} v(t)\,\mathrm{d}R(t) + O_{\mathrm{P}}\left(\frac{1}{\sqrt{n}}\right),$$

where $R(t)$ is defined in (1.13).

Proof. The assertions are easy consequences of Lemma 5.3. ☐

Corollary 5.4. *(local alternatives) Assume $0 < m_c < m < n$. Let (A.2) and* (3.3) *be satisfied and $v(t)$ is a continuous function on $[0,\tau_0]$, then we have, as $n \to \infty$,*

$$\int_0^{\tau_0} v(t)\,\mathrm{d}\Big(\frac{N(t)}{n}\Big) = \int_0^{\tau_0} v(t)\,(1-G(t))\,\mathrm{dF(t)} + o_{\mathrm{P}}(1).$$

Proof. It can be seen that

$$\mathrm{E}\left(\int_0^{\tau_0} v(t)\,\mathrm{d}N_1(t)\right) = \mathrm{E}\,(v(X_1)\,\Delta_1\, I(X_1 \le \tau_0)) = \int_0^{\tau_0} v(t)\,(1-F_{1n}(t))(1-G_{1n}(t))\,\lambda_{1n}(t)\,\mathrm{d}t,$$

where $\lambda_{1n}(t)$ is a hazard function corresponding to the distribution function $F_{1n}(t)$, and

$$\begin{aligned}
&\left|\int_0^{\tau_0} v(t)(1-F_{1n}(t))(1-G_{1n}(t))\ \lambda_{1n}(t)\,\mathrm{d}t - \int_0^{\tau_0} v(t)(1-F(t))(1-G(t))\,\lambda(t)\,\mathrm{d}t\right| \\
&\le \left|\int_0^{\tau_0} v(t)(F(t)-F_{1n}(t))(G(t)-G_{1n}(t))\,\lambda_{1n}(t)\,\mathrm{d}t\right| \\
&\quad + \left|\int_0^{\tau_0} v(t)(1-F(t))(1-G(t))\,(\lambda_{1n}(t)-\lambda(t))\,\mathrm{d}t\right|.
\end{aligned}$$

Direct calculations yield

$$\begin{aligned}
&\left|\int_0^{\tau_0} v(t)(F(t)-F_{1n}(t))(G(t)-G_{1n}(t))\,\lambda_{1n}(t)\,\mathrm{d}t\right| \\
&\leq \sup_{0\leq t\leq\tau_0}|v(t)| \sup_{0\leq t\leq\tau_0}|F_{1n}(t)-F(t)| \sup_{0\leq t\leq\tau_0}|G_{1n}(t)-G(t)|\,|\Lambda_{1n}(\tau_0)| \\
&\leq \sup_{0\leq t\leq\tau_0}|v(t)| \sup_{0\leq t\leq\tau_0}|F_{1n}(t)-F(t)| \sup_{0\leq t\leq\tau_0}|G_{1n}(t)-G(t)|
\end{aligned}$$

with $\Lambda_{1n}(t)$ denoting a cumulative hazard function corresponding to the distribution function $F_{1n}(t)$, and

$$\left|\int_0^{\tau_0} v(t)(1-F(t))(1-G(t))\,(\lambda_{1n}-\lambda(t))\,\mathrm{d}t\right| \leq \sup_{0\leq t\leq\tau_0}|v(t)| \int_0^{\tau_0} |\lambda_{1n}-\lambda(t)|\,\mathrm{d}t.$$

By (A.2) and (3.3) and regarding that (3.3) implies (3.8), we get, as $n\to\infty$,

$$\begin{aligned}
\mathrm{E}\left(v(X_1)\,\Delta_1\, I(X_1\leq\tau_0)\right) &= \int_0^{\tau_0} v(t)\,(1-F(t))(1-G(t))\,\lambda(t)\,\mathrm{d}(t)+o(1) \\
&= \int_0^{\tau_0} v(t)\,(1-G(t))\,\mathrm{d}F(t)+o(1). \qquad (5.11)
\end{aligned}$$

Analogously, we receive, as $n\to\infty$,

$$\mathrm{E}\left(\int_0^{\tau_0} v(t)\,\mathrm{d}N_m(t)\right) = \mathrm{E}\left(v(X_m)\,\Delta_m\, I(X_m\leq\tau_0)\right) = \int_0^{\tau_0} v(t)\,(1-G(t))\,\mathrm{d}F(t)+o(1), \quad (5.12)$$

$$\mathrm{E}\left(\int_0^{\tau_0} v(t)\,\mathrm{d}N_n(t)\right) = \mathrm{E}\left(v(X_n)\,\Delta_n\, I(X_n\leq\tau_0)\right) = \int_0^{\tau_0} v(t)\,(1-G(t))\,\mathrm{d}F(t)+o(1). \quad (5.13)$$

Using the steps in the proof of Lemma 5.3 together with the results (5.11) – (5.13) we see that our assertion holds. □

3. Properties of scores $a_n(j)$

In this section we investigate the properties of the scores $a_n(j)$ for $1\leq j\leq n$ using the representations of $Y(t)$ and $N(t)$ developed in the previous section.

LEMMA 5.4. *Let the condition* (5.1) *for the weights with* b_n *of the form* (5.2) *be satisfied. The scores* $a_n(j)$ *defined in* (2.1) *have the following properties*

$$\bar{a}_n = \frac{1}{n}\sum_{j=1}^n a_n(j) = 0. \qquad (5.14)$$

Further, assume $0 < m_c < m < n$. *Then we have, as* $n\to\infty$,

$$\max_{1\leq j\leq n}|a_n(j)| = O_{\mathrm{P}}(1), \qquad (5.15)$$

$$\frac{1}{n}\sum_{j=1}^n a_n^2(j) = \int_0^{\tau_0} w^2(t)\,\mathrm{d}R_{\eta,\gamma}(t) + o_{\mathrm{P}}\Big(\frac{1}{b_n}\Big). \qquad (5.16)$$

PROOF. First, it can be seen that

$$\sum_{j=1}^n a_n(j) = \int_0^{\tau_0} w_n(t)\,\mathrm{d}\sum_{j=1}^n N_j(t) - \int_0^{\tau_0} w_n(t)\,\frac{\sum_{j=1}^n Y_j(t)}{Y(t)}\,\mathrm{d}N(t) = 0.$$

Second, by Lemma 5.2 we obtain, as $n \to \infty$,

$$
\begin{aligned}
|a_n(j)| &\le \left| \int_0^{\tau_0} w_n(t) \mathrm{d}N_j(t) \right| + \left| \int_0^{\tau_0} \frac{w_n(t) Y_j(t)}{Y(t)} \,\mathrm{d}N(t) \right| \\
&\le \sup_{0 \le t \le \tau_0} |w_n(t) - w(t)| + \sup_{0 \le t \le \tau_0} |w(t)| + \left| \int_0^{\tau_0} \frac{w_n(t)}{1 - H_{\eta,\gamma}(t)} \,\mathrm{d}\frac{N(t)}{n} \left(1 + O_{\mathrm{P}}\Big(\frac{1}{\sqrt{n}}\Big)\right) \right| \\
&\le \left(\sup_{0 \le t \le \tau_0} |w_n(t) - w(t)| + \sup_{0 \le t \le \tau_0} |w(t)| \right) \left(1 + \frac{N(\tau_0)}{n(1 - H_{\eta,\gamma}(\tau_0))} \left(1 + O_{\mathrm{P}}\Big(\frac{1}{\sqrt{n}}\Big)\right) \right) \\
&= O_{\mathrm{P}}(1)
\end{aligned}
$$

uniformly in $1 \le j \le n$.

Third, we get by a standard computation, as $n \to \infty$,

$$
\begin{aligned}
&\frac{1}{n} \sum_{j=1}^n a_n^2(j) = \frac{1}{n} \sum_{j=1}^n \left(\int_0^{\tau_0} w_n(t) \,\mathrm{d}N_j(t) - \int_0^{\tau_0} \frac{w_n(t) Y_j(t)}{Y(t)} \,\mathrm{d}N(t) \right)^2 \\
&= \frac{1}{n} \sum_{j=1}^n \Bigg\{ \left(\int_0^{\tau_0} w_n(t) \,\mathrm{d}N_j(t) \right)^2 - 2 \int_0^{\tau_0} \int_0^{\tau_0} \frac{w_n(t_1)\, w_n(t_2)}{Y(t_1)} \, Y_j(t_1) \,\mathrm{d}N(t_1) \,\mathrm{d}N_j(t_2) \\
&\quad + \int_0^{\tau_0} \int_0^{\tau_0} \frac{w_n(t_1)\, w_n(t_2)}{Y(t_1) Y(t_2)} \, Y_j(t_1)\, Y_j(t_2) \,\mathrm{d}N(t_1) \,\mathrm{d}N(t_2) \Bigg\} \\
&= \frac{1}{n} \sum_{j=1}^n \Bigg\{ w_n^2(X_j)\, \Delta_j \, I(X_j \le \tau_0) - 2 \int_0^{\tau_0} \frac{w_n(t_1) w_n(X_j)}{Y(t_1)} \, I(t_1 \le X_j \le \tau_0)\, \Delta_j \,\mathrm{d}N(t_1) \\
&\quad + \int_0^{\tau_0} \int_0^{\tau_0} \frac{w_n(t_1)\, w_n(t_2)}{Y(t_1)\, Y(t_2)} \, I(X_j \ge \max(t_1, t_2)) \,\mathrm{d}N(t_1) \,\mathrm{d}N(t_2) \Bigg\} \\
&= \frac{1}{n} \sum_{j=1}^n \Bigg\{ \int_0^{\tau_0} w_n^2(t) \,\mathrm{d}N_j(t) - 2 \iint_{t_1 \le t_2} \frac{w_n(t_1) w_n(t_2)}{Y(t_1)} \,\mathrm{d}N(t_1) \,\mathrm{d}N_j(t_2) \\
&\quad + 2 \iint_{t_1 \le t_2} \frac{w_n(t_1)\, w_n(t_2)\, Y_j(t_2)}{Y(t_1)\, Y(t_2)} \,\mathrm{d}N(t_1) \,\mathrm{d}N(t_2) - \int_0^{\tau_0} \frac{w_n^2(t)}{Y^2(t)} \, Y_j(t) \,\mathrm{d}N(t) \Bigg\} \\
&= \frac{1}{n} \int_0^{\tau_0} w_n^2(t) \,\mathrm{d}\Big(\sum_{j=1}^n N_j(t) \Big) - \frac{2}{n} \iint_{t_1 \le t_2} \frac{w_n(t_1)\, w_n(t_2)}{Y(t_1)} \,\mathrm{d}N(t_1) \,\mathrm{d}\Big(\sum_{j=1}^n N_j(t_2) \Big) \\
&\quad + \frac{2}{n} \iint_{t_1 \le t_2} \frac{w_n(t_1)\, w_n(t_2) \sum_{j=1}^n Y_j(t_2)}{Y(t_1)\, Y(t_2)} \,\mathrm{d}N(t_1) \,\mathrm{d}N(t_2) - \frac{1}{n} \int_0^{\tau_0} \frac{w_n^2(t) \, \sum_{j=1}^n Y_j(t)}{Y^2(t)} \,\mathrm{d}N(t) \\
&= \frac{1}{n} \int_0^{\tau_0} w_n^2(t) \,\mathrm{d}N(t) - \frac{2}{n} \iint_{t_1 \le t_2} \frac{w_n(t_1)\, w_n(t_2)}{Y(t_1)} \,\mathrm{d}N(t_1) \,\mathrm{d}N(t_2) \\
&\quad + \frac{2}{n} \iint_{t_1 \le t_2} \frac{w_n(t_1)\, w_n(t_2)}{Y(t_1)} \,\mathrm{d}N(t_1) \,\mathrm{d}N(t_2) - \frac{1}{n} \int_0^{\tau_0} \frac{w_n^2(t)}{Y(t)} \,\mathrm{d}N(t) \\
&= \frac{1}{n} \int_0^{\tau_0} w_n^2(t) \frac{Y(t) - 1}{Y(t)} \,\mathrm{d}N(t) = \frac{1}{n} \int_0^{\tau_0} w_n^2(t) \,\mathrm{d}N(t) \, \left(1 + O_{\mathrm{P}}\Big(\frac{1}{n}\Big) \right)
\end{aligned}
\tag{5.17}
$$

and

$$\begin{aligned}
&\Big|\int_0^{\tau_0} w_n^2(t)\,\mathrm{d}\Big(\frac{N(t)}{n}\Big) - \int_0^{\tau_0} w^2(t)\,\mathrm{d}R_{\eta,\gamma}(t)\Big| \\
&\le \Big(\frac{N(\tau_0)}{n}\Big) \sup_{0\le t\le\tau_0} |w_n^2(t) - w^2(t)| + \Big|\int_0^{\tau_0} w^2(t)\,\Big(\mathrm{d}\Big(\frac{N(t)}{n}\Big) - \mathrm{d}R_{\eta,\gamma}(t)\Big)\Big| \\
&\le \Big(\frac{N(\tau_0)}{n}\Big) \sup_{0\le t\le\tau_0} |w_n(t) - w(t)| \Big(\sup_{0\le t\le\tau_0} |w_n(t) - w(t)| + 2 \sup_{0\le t\le\tau_0} |w(t)|\Big) \\
&\quad + \Big|\int_0^{\tau_0} w^2(t)\,\Big(\mathrm{d}\Big(\frac{N(t)}{n}\Big) - \mathrm{d}R_{\eta,\gamma}(t)\Big)\Big| \\
&= o_{\mathrm{P}}\Big(\frac{1}{b_n}\Big) + \Big|\int_0^{\tau_0} w^2(t)\,\Big(\mathrm{d}\Big(\frac{N(t)}{n}\Big) - \mathrm{d}R_{\eta,\gamma}(t)\Big)\Big| \qquad (5.18)
\end{aligned}$$

in view of the assumption (5.1) for the weights $w_n(t)$. Particularly, since the function $w(t)$ is continuous on $t \in [0, \tau_0]$, the function $v(t) = w^2(t)$ is also continuous on $t \in [0, \tau_0]$. By (5.10) we have, as $n \to \infty$,

$$\Big|\int_0^{\tau_0} w^2(t)\,\mathrm{d}\Big(\frac{N(t)}{n}\Big) - \int_0^{\tau_0} w^2(t)\,\mathrm{d}R_{\eta,\gamma}(t)\Big| = O_{\mathrm{P}}\Big(\frac{1}{\sqrt{n}}\Big). \qquad (5.19)$$

Combining (5.17) together with (5.18) and (5.19), we obtain the desired result (5.16). □

Corollary 5.5. *Let the condition* (5.1) *with* b_n *of the form* (5.2) *be satisfied. Assume* $0 < m_c = m < n$. *Then we have, as* $n \to \infty$,

$$\max_{1\le j\le n} |a_n(j)| = O_{\mathrm{P}}(1), \qquad \frac{1}{n}\sum_{j=1}^n a_n^2(j) = \int_0^{\tau_0} w^2(t)\,\mathrm{d}R_\gamma(t) + o_{\mathrm{P}}\Big(\frac{1}{b_n}\Big),$$

where $R_\gamma(t)$ *is defined in* (1.11).
Assume $0 < m_c < m = n$. *Then we have, as* $n \to \infty$,

$$\max_{1\le j\le n} |a_n(j)| = O_{\mathrm{P}}(1), \qquad \frac{1}{n}\sum_{j=1}^n a_n^2(j) = \int_0^{\tau_0} w^2(t)\,\mathrm{d}R_\eta(t) + o_{\mathrm{P}}\Big(\frac{1}{b_n}\Big),$$

where $R_\eta(t)$ *is defined in* (1.12).
Assume $0 < m_c = m = n$. *Then we have, as* $n \to \infty$,

$$\max_{1\le j\le n} |a_n(j)| = O_{\mathrm{P}}(1), \qquad \frac{1}{n}\sum_{j=1}^n a_n^2(j) = \int_0^{\tau_0} w^2(t)\,\mathrm{d}R(t) + o_{\mathrm{P}}\Big(\frac{1}{b_n}\Big),$$

where $R(t)$ *is defined in* (1.13).

Proof. The assertions follow directly from Lemma 5.4. □

Corollary 5.6. *Let the condition* (5.1) *with* $b_n = 1$ *be satisfied. Assume* $0 < m_c \le m \le n$. *If*

$$\int_0^{\tau_0} w^2(t)\,(1 - G_1(t))\,\mathrm{d}F_1(t) > 0,$$
$$\int_0^{\tau_0} w^2(t)\,(1 - G_2(t))\,\mathrm{d}F_1(t) > 0,$$
$$\int_0^{\tau_0} w^2(t)\,(1 - G_2(t))\,\mathrm{d}F_2(t) > 0,$$

then we have, as $n \to \infty$,

$$\frac{1}{n}\max_{1\le j\le n}(a_n(j)-\bar{a}_n)^2 = O_{\mathrm{P}}\Big(\frac{1}{n}\Big),$$
$$\frac{1}{n}\sum_{j=1}^{n}|a_n(j)-\bar{a}_n|^4 = O_{\mathrm{P}}(1),$$
$$\frac{1}{n}\sum_{j=1}^{n}(a_n(j)-\bar{a}_n)^2 \xrightarrow{\mathrm{P}} const > 0.$$

PROOF. Since $\bar{a}_n = 0$, the assertions follow directly from Lemma 5.4 or Corollary 5.5, respectively. □

4. Behavior of partial sums S_{m_c} and S_m

In this section we give representations for S_{m_c} and S_m which are important for the investigation of limit behavior of the statistics under the alternatives. We use the following notation

$$I_1(\tau_0) = \int_0^{\tau_0} w(t)\,\frac{(1-F_1(t))(1-F_2(t))(1-G_1(t))(1-G_2(t))}{1-H_{\eta,\gamma}(t)}\left(\frac{\mathrm{d}F_1(t)}{1-F_1(t)} - \frac{\mathrm{d}F_2(t)}{1-F_2(t)}\right), \tag{5.20}$$
$$I_2(\tau_0) = \int_0^{\tau_0} w(t)\,\frac{(1-F_1(t))(1-F_2(t))(1-G_2(t))^2}{1-H_{\eta,\gamma}(t)}\left(\frac{\mathrm{d}F_1(t)}{1-F_1(t)} - \frac{\mathrm{d}F_2(t)}{1-F_2(t)}\right) \tag{5.21}$$

with $H_{\eta,\gamma}(t)$ given by (1.6).

LEMMA 5.5. *Let the condition for the weights* (5.1) *with* b_n *of the form* (5.2) *be satisfied. Assume* $0 < m_c < m < n$, *we have, as* $n \to \infty$,

$$\frac{S_{m_c}}{m_c} = \frac{n-m}{n}\,I_1(\tau_0) + o_{\mathrm{P}}\Big(\frac{1}{b_n}\Big), \tag{5.22}$$
$$\frac{S_m - S_{m_c}}{m-m_c} = \frac{n-m}{n}\,I_2(\tau_0) + o_{\mathrm{P}}\Big(\frac{1}{b_n}\Big), \tag{5.23}$$

where $I_1(\tau_0)$ *and* $I_2(\tau_0)$ *are defined in* (5.20) *and* (5.21), *respectively.*

PROOF. Notice that

$$\begin{aligned}
\frac{n}{m_c(n-m)}\,S_{m_c} &= \frac{1}{m_c}\sum_{j=1}^{m_c}\left(\int_0^{\tau_0} w_n(t)\,\frac{\frac{1}{n-m}\sum_{j=m_c+1}^{m} Y_j(t)}{\frac{Y(t)}{n}}\,\mathrm{d}N_j(t)\right)\\
&- \frac{1}{n-m}\sum_{j=m_c+1}^{m}\left(\int_0^{\tau_0} w_n(t)\,\frac{\frac{1}{m_c}\sum_{j=1}^{m_c} Y_j(t)}{\frac{Y(t)}{n}}\,\mathrm{d}N_j(t)\right)\\
&+ \frac{1}{m_c}\sum_{j=1}^{m_c}\left(\int_0^{\tau_0} w_n(t)\,\frac{\frac{1}{n-m}\sum_{j=m+1}^{n} Y_j(t)}{\frac{Y(t)}{n}}\,\mathrm{d}N_j(t)\right)\\
&- \frac{1}{n-m}\sum_{j=m+1}^{n}\left(\int_0^{\tau_0} w_n(t)\,\frac{\frac{1}{m_c}\sum_{j=1}^{m_c} Y_j(t)}{\frac{Y(t)}{n}}\,\mathrm{d}N_j(t)\right).
\end{aligned}$$

Hence by Lemma 5.2, as $n \to \infty$,

$$\begin{aligned}
\frac{n}{m_c(n-m)} S_{m_c} &= \frac{1}{m_c} \sum_{j=1}^{m_c} \left(\int_0^{\tau_0} w_n(t) \, \frac{\frac{m-m_c}{n-m} \operatorname{E} Y_m(t)}{1 - H_{\eta,\gamma}(t)} \, \mathrm{d}N_j(t) \right) \left(1 + O_{\mathrm{P}}\Big(\frac{1}{\sqrt{n}}\Big) \right) \\
&- \frac{1}{n-m} \sum_{j=m_c+1}^{m} \left(\int_0^{\tau_0} w_n(t) \, \frac{\operatorname{E} Y_1(t)}{1 - H_{\eta,\gamma}(t)} \, \mathrm{d}N_j(t) \right) \left(1 + O_{\mathrm{P}}\Big(\frac{1}{\sqrt{n}}\Big) \right) \\
&+ \frac{1}{m_c} \sum_{j=1}^{m_c} \left(\int_0^{\tau_0} w_n(t) \, \frac{\operatorname{E} Y_n(t)}{1 - H_{\eta,\gamma}(t)} \, \mathrm{d}N_j(t) \right) \left(1 + O_{\mathrm{P}}\Big(\frac{1}{\sqrt{n}}\Big) \right) \\
&- \frac{1}{n-m} \sum_{j=m+1}^{n} \left(\int_0^{\tau_0} w_n(t) \, \frac{\operatorname{E} Y_1(t)}{1 - H_{\eta,\gamma}(t)} \, \mathrm{d}N_j(t) \right) \left(1 + O_{\mathrm{P}}\Big(\frac{1}{\sqrt{n}}\Big) \right)
\end{aligned}$$

and

$$\begin{aligned}
&\left| \int_0^{\tau_0} w_n(t) \, \frac{\operatorname{E} Y_m(t)}{1 - H_{\eta,\gamma}(t)} \, \mathrm{d}\Big(\sum_{j=1}^{m_c} \frac{N_j(t)}{m_c} \Big) - \int_0^{\tau_0} w(t) \, \frac{\operatorname{E} Y_m(t)}{1 - H_{\eta,\gamma}(t)} \, \mathrm{d} \operatorname{E} N_1(t) \right| \\
&\le \left(\frac{\sum_{j=1}^{m_c} N_j(\tau_0)}{m_c \, (1 - H_{\eta,\gamma}(\tau_0))} \right) \sup_{0 \le t \le \tau_0} |w_n(t) - w(t)| \\
&\quad + \left| \int_0^{\tau_0} w(t) \, \frac{\operatorname{E} Y_m(t)}{1 - H_{\eta,\gamma}(t)} \left(\mathrm{d}\Big(\sum_{j=1}^{m_c} \frac{N_j(t)}{m_c} \Big) - \mathrm{d} \operatorname{E} N_1(t) \right) \right| \\
&= \left| \int_0^{\tau_0} w(t) \frac{\operatorname{E} Y_m(t)}{1 - H_{\eta,\gamma}(t)} \left(\mathrm{d}\Big(\sum_{j=1}^{m_c} \frac{N_j(t)}{m_c} \Big) - \mathrm{d} \operatorname{E} N_1(t) \right) \right| + o_{\mathrm{P}}\Big(\frac{1}{b_n}\Big). \qquad (5.24)
\end{aligned}$$

Since the functions

$$\begin{aligned}
v_1(t) &= \frac{w(t) \operatorname{E} Y_1(t)}{1 - H_{\eta,\gamma}(t)} = \frac{w(t)(1 - F_1(t))(1 - G_1(t))}{1 - H_{\eta,\gamma}(t)}, \\
v_2(t) &= \frac{w(t) \operatorname{E} Y_m(t)}{1 - H_{\eta,\gamma}(t)} = \frac{w(t)(1 - F_1(t))(1 - G_2(t))}{1 - H_{\eta,\gamma}(t)}, \\
v_3(t) &= \frac{w(t) \operatorname{E} Y_n(t)}{1 - H_{\eta,\gamma}(t)} = \frac{w(t)(1 - F_2(t))(1 - G_2(t))}{1 - H_{\eta,\gamma}(t)}
\end{aligned}$$

are continuous on $[0, \tau_0]$, we get by Lemma 5.3 and (5.24), as $n \to \infty$,

$$\left| \int_0^{\tau_0} w_n(t) \, \frac{\operatorname{E} Y_m(t)}{1 - H_{\eta,\gamma}(t)} \, \mathrm{d}\Big(\sum_{j=1}^{m_c} \frac{N_j(t)}{m_c} \Big) - \int_0^{\tau_0} w(t) \, \frac{\operatorname{E} Y_m(t)}{1 - H_{\eta,\gamma}(t)} \, \mathrm{d} \operatorname{E} N_1(t) \right| = o_{\mathrm{P}}\Big(\frac{1}{b_n}\Big),$$

$$\left| \int_0^{\tau_0} w_n(t) \, \frac{\operatorname{E} Y_1(t)}{1 - H_{\eta,\gamma}(t)} \, \mathrm{d}\Big(\sum_{j=m_c+1}^{m} \frac{N_j(t)}{m - m_c} \Big) - \int_0^{\tau_0} w(t) \, \frac{\operatorname{E} Y_1(t)}{1 - H_{\eta,\gamma}(t)} \, \mathrm{d} \operatorname{E} N_m(t) \right| = o_{\mathrm{P}}\Big(\frac{1}{b_n}\Big),$$

$$\left| \int_0^{\tau_0} w_n(t) \, \frac{\operatorname{E} Y_n(t)}{1 - H_{\eta,\gamma}(t)} \, \mathrm{d}\Big(\sum_{j=1}^{m_c} \frac{N_j(t)}{m_c} \Big) - \int_0^{\tau_0} w(t) \, \frac{\operatorname{E} Y_n(t)}{1 - H_{\eta,\gamma}(t)} \, \mathrm{d} \operatorname{E} N_1(t) \right| = o_{\mathrm{P}}\Big(\frac{1}{b_n}\Big),$$

$$\left| \int_0^{\tau_0} w_n(t) \, \frac{\mathrm{E}\, Y_1(t)}{1 - H_{\eta,\gamma}(t)} \, \mathrm{d}\Big(\sum_{j=m+1}^{n} \frac{N_j(t)}{n-m} \Big) - \int_0^{\tau_0} w(t) \, \frac{\mathrm{E}\, Y_1(t)}{1 - H_{\eta,\gamma}(t)} \, \mathrm{d}\, \mathrm{E}\, N_n(t) \right| = o_{\mathrm{P}}\Big(\frac{1}{b_n}\Big).$$

This gives

$$\frac{n}{m_c(n-m)} \, S_{m_c} = \frac{m - m_c}{n-m} \int_0^{\tau_0} \frac{w(t)}{1 - H_{\eta,\gamma}(t)} \, (\mathrm{E}\, Y_m(t) \, \mathrm{d}\, \mathrm{E}\, N_1(t) - \mathrm{E}\, Y_1(t) \, \mathrm{d}\, \mathrm{E}\, N_m(t))$$
$$+ \int_0^{\tau_0} \frac{w(t)}{1 - H_{\eta,\gamma}(t)} \, (\mathrm{E}\, Y_n(t) \, \mathrm{d}\, \mathrm{E}\, N_1(t) - \mathrm{E}\, Y_1(t) \, \mathrm{d}\, \mathrm{E}\, N_n(t)) + o_{\mathrm{P}}\Big(\frac{1}{b_n}\Big).$$

Since

$$\int_0^{\tau_0} \frac{w(t)}{1 - H_{\eta,\gamma}(t)} \, (\mathrm{E}\, Y_m(t) \, \mathrm{d}\, \mathrm{E}\, N_1(t) - \mathrm{E}\, Y_1(t) \, \mathrm{d}\, \mathrm{E}\, N_m(t)) = 0$$

and

$$\int_0^{\tau_0} \frac{w(t)}{1 - H_{\eta,\gamma}(t)} \, (\mathrm{E}\, Y_n(t) \, \mathrm{d}\, \mathrm{E}\, N_1(t) - \mathrm{E}\, Y_1(t) \, \mathrm{d}\, \mathrm{E}\, N_n(t))$$
$$= \int_0^{\tau_0} w(t) \, \frac{\mathrm{E}\, Y_n(t) \, (1 - G_1(t))}{1 - H_{\eta,\gamma}(t)} \, \mathrm{d}F_1(t) - \int_0^{\tau_0} w(t) \, \frac{\mathrm{E}\, Y_1(t)(1 - G_2(t))}{1 - H_{\eta,\gamma}(t)} \, \mathrm{d}F_2(t)$$
$$= \int_0^{\tau_0} w(t) \, \frac{\mathrm{E}\, Y_1(t) \; \mathrm{E}\, Y_n(t)}{1 - H_{\eta,\gamma}(t)} \left(\frac{\mathrm{d}F_1(t)}{1 - F_1(t)} - \frac{\mathrm{d}F_2(t)}{1 - F_2(t)} \right),$$

we obtain the desired result (5.22).

Proceeding similarly to S_{m_c} we obtain, as $n \to \infty$,

$$\frac{n}{n-m} \, \frac{S_m - S_{m_c}}{m - m_c} = \frac{1}{m - m_c} \sum_{j=m_c+1}^{m} \left(\int_0^{\tau_0} w_n(t) \, \frac{\frac{1}{n-m} \sum_{j=m+1}^{n} Y_j(t)}{\frac{Y(t)}{n}} \, \mathrm{d}N_j(t) \right)$$
$$- \frac{1}{n-m} \sum_{j=m+1}^{n} \left(\int_0^{\tau_0} w_n(t) \, \frac{\frac{1}{m-m_c} \sum_{j=m_c+1}^{m} Y_j(t)}{\frac{Y(t)}{n}} \, \mathrm{d}N_j(t) \right)$$
$$+ \frac{1}{m - m_c} \sum_{j=m_c+1}^{m} \left(\int_0^{\tau_0} w_n(t) \, \frac{\frac{1}{n-m} \sum_{j=1}^{m_c} Y_j(t)}{\frac{Y(t)}{n}} \, \mathrm{d}N_j(t) \right)$$
$$- \frac{1}{n-m} \sum_{j=1}^{m_c} \left(\int_0^{\tau_0} w_n(t) \, \frac{\frac{1}{m-m_c} \sum_{j=m_c+1}^{m} Y_j(t)}{\frac{Y(t)}{n}} \, \mathrm{d}N_j(t) \right)$$
$$= \int_0^{\tau_0} \frac{w(t)}{1 - H_{\eta,\gamma}(t)} \, (\mathrm{E}\, Y_n(t) \, \mathrm{d}\, \mathrm{E}\, N_m(t) - \mathrm{E}\, Y_m(t) \, \mathrm{d}\, \mathrm{E}\, N_n(t))$$
$$+ \frac{m_c}{n-m} \int_0^{\tau_0} \frac{w(t)}{1 - H_{\eta,\gamma}(t)} \, (\mathrm{E}\, Y_1(t) \, \mathrm{d}\, \mathrm{E}\, N_m(t) - \mathrm{E}\, Y_m(t) \, \mathrm{d}\, \mathrm{E}\, N_1(t)) + o_{\mathrm{P}}\Big(\frac{1}{b_n}\Big).$$

Since

$$\int_0^{\tau_0} \frac{w(t)}{1 - H_{\eta,\gamma}(t)} \, (\mathrm{E}\, Y_n(t) \, \mathrm{d}\, \mathrm{E}\, N_m(t) - \mathrm{E}\, Y_m(t) \, \mathrm{d}\, \mathrm{E}\, N_n(t))$$
$$= \int_0^{\tau_0} w(t) \, \frac{\mathrm{E}\, Y_n(t)(1 - G_2(t))}{1 - H_{\eta,\gamma}(t)} \, \mathrm{d}F_1(t) - \int_0^{\tau_0} w(t) \, \frac{\mathrm{E}\, Y_m(t)(1 - G_2(t))}{1 - H_{\eta,\gamma}(t)} \, \mathrm{d}F_2(t)$$
$$= \int_0^{\tau_0} w(t) \, \frac{\mathrm{E}\, Y_m(t) \; \mathrm{E}\, Y_n(t)}{1 - H_{\eta,\gamma}(t)} \left(\frac{\mathrm{d}F_1(t)}{1 - F_1(t)} - \frac{\mathrm{d}F_2(t)}{1 - F_2(t)} \right)$$

and

$$\int_0^{\tau_0} \frac{w(t)}{1 - H_{\eta,\gamma}(t)} \left(\mathrm{E}\, Y_1(t) \,\mathrm{d}\, \mathrm{E}\, N_m(t) - \mathrm{E}\, Y_m(t) \,\mathrm{d}\, \mathrm{E}\, N_1(t)\right) = 0,$$

we get the result (5.23). □

COROLLARY 5.7. *Let the condition* (5.1) *with* b_n *of the form* (5.2) *be satisfied.*
Assume $0 < m_c < m < n$. *Then we have, as* $n \to \infty$,

$$\frac{S_m}{n} = \frac{n-m}{n} \left(\eta\, I_1(\tau_0) + (\gamma - \eta)\, I_2(\tau_0)\right) + o_{\mathrm{P}}\Big(\frac{1}{b_n}\Big),$$

where $I_1(\tau_0)$, $I_2(\tau_0)$ *are defined in* (5.20) *and* (5.21).
Assume $0 < m_c = m < n$. *Then we have, as* $n \to \infty$,

$$\frac{S_m}{m} = \frac{n-m}{n} \int_0^{\tau_0} w(t) \frac{(1 - F_1(t))(1 - F_2(t))(1 - G_1(t))(1 - G_2(t))}{1 - H_\gamma(t)} \left(\frac{\mathrm{d}F_1(t)}{1 - F_1(t)} - \frac{\mathrm{d}F_2(t)}{1 - F_2(t)}\right) + o_{\mathrm{P}}\Big(\frac{1}{b_n}\Big),$$

where $H_\gamma(t)$ *is defined in* (1.7).
Assume $0 < m_c < m = n$. *The we have, as* $n \to \infty$,

$$\frac{S_{m_c}}{m_c} = o_{\mathrm{P}}\Big(\frac{1}{b_n}\Big).$$

PROOF. The assertions are direct consequences of Lemma 5.5. □

COROLLARY 5.8. *Let the condition* (5.1) *with* $b_n = 1$ *be satisfied.*
Assume $0 < m_c < m < n$. *Then we have, as* $n \to \infty$,

$$-\frac{1}{D} \sum_{j=m+1}^{m+D} a_n(j) = \eta\, I_1(\tau_0) + (\gamma - \eta)\, I_2(\tau_0) + o_{\mathrm{P}}(1),$$

$$\frac{1}{D} \sum_{j=m-D+1}^{m} a_n(j) = (1 - \gamma)\, I_2(\tau_0) + o_{\mathrm{P}}(1),$$

where D *is defined in* (2.32) *and* $I_1(\tau_0)$, $I_2(\tau_0)$ *are given by* (5.20), (5.21), *respectively.*

PROOF. Notice that

$$\frac{1}{D} \sum_{j=m+1}^{m+D} a_n(j) = \frac{1}{D} \left(\int_0^{\tau_0} w_n(t) \,\mathrm{d} \sum_{j=m+1}^{m+D} N_j(t) - \int_0^{\tau_0} w_n(t) \frac{\sum_{j=m+1}^{m+D} Y_j(t)}{Y(t)} \,\mathrm{d}N(t) \right).$$

Hence by a small modification of Lemma 5.2 we have, as $n \to \infty$,

$$\frac{1}{D} \sum_{j=m+1}^{m+D} a_n(j) = \int_0^{\tau_0} w_n(t) \,\mathrm{d}\Big(\sum_{j=m+1}^{m+D} \frac{N_j(t)}{D} \Big) - \int_0^{\tau_0} w_n(t) \frac{\mathrm{E}\, Y_n(t)}{1 - H_{\eta,\gamma}(t)} \,\mathrm{d}\frac{N(t)}{n} \left(1 + O_{\mathrm{P}}\Big(\frac{1}{\sqrt{D}}\Big)\right)$$

and

$$\begin{aligned}
&\left| \int_0^{\tau_0} w_n(t) \frac{\mathrm{E}\, Y_n(t)}{1 - H_{\eta,\gamma}(t)} \,\mathrm{d}\frac{N(t)}{n} - \int_0^{\tau_0} w(t) \frac{\mathrm{E}\, Y_n(t)}{1 - H_{\eta,\gamma}(t)} \,\mathrm{d}R_{\eta,\gamma}(t) \right| \\
&\le \left(\frac{N(\tau_0)}{n\,(1 - H_{\eta,\gamma}(\tau_0))} \right) \sup_{0 \le t \le \tau_0} |w_n(t) - w(t)| + \left| \int_0^{\tau_0} w(t) \frac{\mathrm{E}\, Y_n(t)}{1 - H_{\eta,\gamma}(t)} \left(\mathrm{d}\frac{N(t)}{n} - \mathrm{d}R_{\eta,\gamma}(t) \right) \right| \\
&= \left| \int_0^{\tau_0} w(t) \frac{\mathrm{E}\, Y_n(t)}{1 - H_{\eta,\gamma}(t)} \left(\mathrm{d}\Big(\frac{N(t)}{n} - \mathrm{d}R_{\eta,\gamma}(t)\Big) \right| + o_{\mathrm{P}}(1). \qquad (5.25)
\end{aligned}$$

Since the functions w(t) and

$$v(t) = \frac{w(t)\,\mathrm{E}\,Y_n(t)}{1-H_{\eta,\gamma}(t)} = \frac{w(t)(1-F_2(t))(1-G_2(t))}{1-H_{\eta,\gamma}(t)}$$

are continuous on $[0,\tau_0]$, we get by a small modification of Lemma 5.3 together with (5.25), as $n \to \infty$,

$$\left| \int_0^{\tau_0} w_n(t)\,\mathrm{d}\Big(\sum_{j=m+1}^{m+D} \frac{N_j(t)}{D}\Big) - \int_0^{\tau_0} w(t)\,\mathrm{d}\,\mathrm{E}\,N_n(t) \right| = O_{\mathrm{P}}\Big(\frac{1}{\sqrt{D}}\Big) + o_{\mathrm{P}}(1) = o_{\mathrm{P}}(1),$$

$$\left| \int_0^{\tau_0} w_n(t)\,\frac{\mathrm{E}\,Y_n(t)}{1-H_{\eta,\gamma}(t)}\,\mathrm{d}\frac{N(t)}{n} - \int_0^{\tau_0} w(t)\,\frac{\mathrm{E}\,Y_n(t)}{1-H_{\eta,\gamma}(t)}\,\mathrm{d}R_{\eta,\gamma}(t) \right| = O_{\mathrm{P}}\Big(\frac{1}{\sqrt{D}}\Big) + o_{\mathrm{P}}(1) = o_{\mathrm{P}}(1).$$

This gives

$$\begin{aligned}
\frac{1}{D}\sum_{j=m+1}^{m+D} a_n(j) &= \int_0^{\tau_0} w(t)\,\mathrm{d}\,\mathrm{E}\,N_n(t) - \int_0^{\tau_0} w(t)\,\frac{\mathrm{E}\,Y_n(t)}{1-H_{\eta,\gamma}(t)}\,\mathrm{d}R_{\eta,\gamma}(t) + o_{\mathrm{P}}(1)\\
&= \int_0^{\tau_0} w(t)\,(1-G_2(t))\,\mathrm{d}F_2(t) - \int_0^{\tau_0} w(t)\,\frac{(1-F_2(t))\,(1-G_2(t))}{1-H_{\eta,\gamma}(t)}\,\mathrm{d}R_{\eta,\gamma}(t) + o_{\mathrm{P}}(1)\\
&= \eta \int_0^{\tau_0} w(t)\,\frac{(1-F_1(t))\,(1-F_2(t))\,(1-G_1(t))\,(1-G_2(t))}{1-H_{\eta,\gamma}(t)} \left(\frac{\mathrm{d}F_2(t)}{1-F_2(t)} - \frac{\mathrm{d}F_1(t)}{1-F_1(t)}\right)\\
&\quad + (\gamma-\eta) \int_0^{\tau_0} w(t)\,\frac{(1-F_1(t))\,(1-F_2(t))\,(1-G_2(t))^2}{1-H_{\eta,\gamma}(t)} \left(\frac{\mathrm{d}F_2(t)}{1-F_2(t)} - \frac{\mathrm{d}F_1(t)}{1-F_1(t)}\right)\\
&\quad + o_{\mathrm{P}}(1).
\end{aligned}$$

For the term $\frac{1}{D}\sum_{j=m-D+1}^{m} a_n(j)$ we proceed analogously

$$\begin{aligned}
\frac{1}{D}\sum_{j=m-D+1}^{m} a_n(j) &= \frac{1}{D}\left(\int_0^{\tau_0} w_n(t)\,\mathrm{d}\sum_{j=m-D+1}^{m} N_j(t) - \int_0^{\tau_0} w_n(t)\,\frac{\sum_{j=m-D+1}^{m} Y_j(t)}{Y(t)}\,\mathrm{d}N(t) \right)\\
&= \int_0^{\tau_0} w_n(t)\,\mathrm{d}\Big(\sum_{j=m-D+1}^{m} \frac{N_j(t)}{D}\Big) - \int_0^{\tau_0} w_n(t)\,\frac{\mathrm{E}\,Y_m(t)}{1-H_{\eta,\gamma}(t)}\,\mathrm{d}\frac{N(t)}{n}\left(1 + O_{\mathrm{P}}\Big(\frac{1}{\sqrt{D}}\Big)\right)\\
&= \int_0^{\tau_0} w(t)\,\mathrm{d}\,\mathrm{E}\,N_m(t) - \int_0^{\tau_0} w(t)\,\frac{\mathrm{E}\,Y_m(t)}{1-H_{\eta,\gamma}(t)}\,\mathrm{d}R_{\eta,\gamma}(t) + o_{\mathrm{P}}(1)\\
&= \int_0^{\tau_0} w(t)\,(1-G_2(t))\,\mathrm{d}F_1(t) - \int_0^{\tau_0} w(t)\,\frac{(1-F_1(t))(1-G_2(t))}{1-H_{\eta,\gamma}(t)}\,\mathrm{d}R_{\eta,\gamma}(t) + o_{\mathrm{P}}(1)\\
&= (1-\gamma) \int_0^{\tau_0} w(t)\,\frac{(1-F_1(t))(1-F_2(t))(1-G_2(t))^2}{1-H_{\eta,\gamma}(t)} \left(\frac{\mathrm{d}F_1(t)}{1-F_1(t)} - \frac{\mathrm{d}F_2(t)}{1-F_2(t)}\right)\\
&\quad + o_{\mathrm{P}}(1).
\end{aligned}$$

By the definitions (5.20) and (5.21) of $I_1(\tau_0)$ and $I_2(\tau_0)$ the proof is finished. □

COROLLARY 5.9. *Let the condition* (5.1) *with* $b_n = 1$ *be satisfied. Assume* $0 < m_c = m < n$. *Then we have, as* $n \to \infty$,

$$-\frac{1}{D}\sum_{j=m+1}^{m+D} a_n(j) = \gamma \int_0^{\tau_0} w(t)\,\frac{(1-F_1(t))(1-F_2(t))(1-G_1(t))(1-G_2(t))}{1-H_\gamma(t)}$$
$$\left(\frac{\mathrm{d}F_1(t)}{1-F_1(t)} - \frac{\mathrm{d}F_2(t)}{1-F_2(t)}\right) + o_{\mathrm{P}}(1),$$
$$\frac{1}{D}\sum_{j=m-D+1}^{m} a_n(j) = (1-\gamma) \int_0^{\tau_0} w(t)\,\frac{(1-F_1(t))(1-F_2(t))(1-G_1(t))(1-G_2(t))}{1-H_\gamma(t)}$$
$$\left(\frac{\mathrm{d}F_1(t)}{1-F_1(t)} - \frac{\mathrm{d}F_2(t)}{1-F_2(t)}\right) + o_{\mathrm{P}}(1),$$

where D *is defined in* (2.32) *and* $H_\gamma(t)$ *is given by* (1.7).

PROOF. We use the same steps as in the proof of Corollary 5.8. Thus we get, as $n \to \infty$,

$$\frac{1}{D}\sum_{j=m+1}^{m+D} a_n(j) = \int_0^{\tau_0} w(t)\,\mathrm{d}\,\mathrm{E}\,N_n(t) - \int_0^{\tau_0} w(t)\frac{\mathrm{E}\,Y_n(t)}{1-H_\gamma(t)}\,\mathrm{d}R_\gamma(t) + o_{\mathrm{P}}(1)$$
$$= \int_0^{\tau_0} w(t)\,(1-G_2(t))\,\mathrm{d}F_2(t) - \int_0^{\tau_0} w(t)\,\frac{(1-F_2(t))\,(1-G_2(t))}{1-H_\gamma(t)}\,\mathrm{d}R_\gamma(t) + o_{\mathrm{P}}(1)$$
$$= \gamma \int_0^{\tau_0} w(t)\,\frac{(1-F_1(t))\,(1-F_2(t))\,(1-G_1(t))\,(1-G_2(t))}{1-H_\gamma(t)}\left(\frac{\mathrm{d}F_2(t)}{1-F_2(t)} - \frac{\mathrm{d}F_1(t)}{1-F_1(t)}\right)$$
$$+ o_{\mathrm{P}}(1)$$

and

$$\frac{1}{D}\sum_{j=m-D+1}^{m} a_n(j) = \int_0^{\tau_0} w(t)\,\mathrm{d}\,\mathrm{E}\,N_m(t) - \int_0^{\tau_0} w(t)\,\frac{\mathrm{E}\,Y_m(t)}{1-H_\gamma(t)}\,\mathrm{d}R_\gamma(t) + o_{\mathrm{P}}(1)$$
$$= \int_0^{\tau_0} w(t)\,(1-G_1(t))\,\mathrm{d}F_1(t) - \int_0^{\tau_0} w(t)\,\frac{(1-F_1(t))\,(1-G_1(t))}{1-H_\gamma(t)}\,\mathrm{d}R_\gamma(t) + o_{\mathrm{P}}(1)$$
$$= (1-\gamma) \int_0^{\tau_0} w(t)\,\frac{(1-F_1(t))\,(1-F_2(t))\,(1-G_1(t))\,(1-G_2(t))}{1-H_\gamma(t)}\left(\frac{\mathrm{d}F_1(t)}{1-F_1(t)} - \frac{\mathrm{d}F_2(t)}{1-F_2(t)}\right)$$
$$+ o_{\mathrm{P}}(1),$$

where $R_\gamma(t)$ is given by (1.11). Thus, the proof is finished. □

Notice that $I_1(\tau_0)$ and $I_2(\tau_0)$ can depend on n.

COROLLARY 5.10. *(local alternatives) Let the condition* (5.1) *with* $b_n = 1/|A_n|$ *be satisfied. Assume* $0 < m_c < m < n$. *Then, under the assumptions (A.2) and (A.3), we have, as* $n \to \infty$,

$$\frac{S_{m_c}}{m_c} = \frac{n-m}{n}\,A_n\,(1 + o_{\mathrm{P}}(1)),$$
$$\frac{S_m - S_{m_c}}{m - m_c} = \frac{n-m}{n}\,A_n\,(1 + o_{\mathrm{P}}(1)),$$

where A_n *is defined in* (3.6).

Proof. By Lemma 5.5 and the choice $b_n = 1/|A_n|$, we know that

$$\frac{S_{m_c}}{m_c} = \frac{n-m}{n}\int_0^{\tau_0} w(t)\frac{(1-F_{1n}(t))(1-F_{2n}(t))(1-G_{1n}(t))(1-G_{2n}(t))}{1-H_{\eta,\gamma}(t)}\,(\lambda_{1n}(t)-\lambda_{2n}(t))\,\mathrm{d}t + o_{\mathrm{P}}(|A_n|)$$

which gives together with (A.2) and (A.3)

$$\frac{S_{m_c}}{m_c} = \frac{n-m}{n}\int_0^{\tau_0} w(t)\frac{(1-F(t))^2(1-G(t))^2}{1-H(t)}\,(\lambda_{1n}(t)-\lambda_{2n}(t))\,\mathrm{d}t\,(1+o(1)) + o_{\mathrm{P}}(|A_n|),$$

where $1-H(t) = (1-F(t))\,(1-G(t))$. By the definition (3.6) of A_n, we get, as $n\to\infty$,

$$\frac{S_{m_c}}{m_c} = \frac{n-m}{n}\,A_n\,(1+o(1)) + o_{\mathrm{P}}(|A_n|) = \frac{n-m}{n}\,A_n\,(1+o_{\mathrm{P}}(1)).$$

In the same way as for $\frac{S_{m_c}}{m_c}$ we proceed for $\frac{S_m - S_{m_c}}{m-m_c}$. □

5. Behavior of $\sigma^2_{m_c}$, $\sigma^2_{m-m_c}$ and $\sigma^{0\,2}_{m_c}$

In this section we develop the representations for $\sigma^2_{m_c}(\boldsymbol{a})$, $\sigma^2_{m-m_c}(\boldsymbol{a})$ and $\sigma^{0\,2}_{m_c}(\boldsymbol{a})$ which are given by

$$\sigma^2_{m_c}(\boldsymbol{a}) = \frac{1}{m_c-1}\sum_{j=1}^{m_c}(a_n(j)-\bar a_{m_c})^2, \qquad \bar a_{m_c} = \frac{1}{m_c}\sum_{j=1}^{m_c} a_n(j),$$

$$\sigma^2_{m-m_c}(\boldsymbol{a}) = \frac{1}{m-m_c-1}\sum_{j=m_c+1}^{m}(a_n(j)-\bar a_{m-m_c})^2, \qquad \bar a_{m-m_c} = \frac{1}{m-m_c}\sum_{j=m_c+1}^{m} a_n(j),$$

$$\sigma^{0\,2}_{m_c}(\boldsymbol{a}) = \frac{1}{n-m_c-1}\sum_{j=m_c+1}^{n}(a_n(j)-\bar a^0_{m_c})^2, \qquad \bar a^0_{m_c} = \frac{1}{n-m_c}\sum_{j=m_c+1}^{n} a_n(j),$$

where $a_n(j)$ are defined in (2.1).

Lemma 5.6. *Let the condition for the weights* (5.1) *with b_n of the form* (5.2) *be satisfied. Assume $0 < m_c < m < n$. Then we have, as $n\to\infty$,*

$$\begin{aligned}\sigma^2_{m_c}(\boldsymbol{a}) = &\int_0^{\tau_0} w^2(t)(1-G_1(t))\,\mathrm{d}F_1(t)\\ &-2\iint_{t_1\le t_2} w(t_1)w(t_2)\,\frac{(1-F_1(t_2))(1-G_1(t_2))}{(1-H_{\eta,\gamma}(t_1))(1-H_{\eta,\gamma}(t_2))}\\ &\quad(Q_1(t_1)\lambda_1(t_1)+Q_2(t_1)\lambda_2(t_1))\,Q_2(t_2)\,(\lambda_1(t_2)-\lambda_2(t_2))\,\mathrm{d}t_1\,\mathrm{d}t_2\\ &-2\iint_{t_1\le t_2} w(t_1)w(t_2)\,\frac{(1-F_1(t_1))(1-G_1(t_1))(1-F_1(t_2))(1-G_1(t_2))}{(1-H_{\eta,\gamma}(t_1))(1-H_{\eta,\gamma}(t_2))}\\ &\quad Q_2(t_1)Q_2(t_2)\,(\lambda_1(t_1)-\lambda_2(t_1))(\lambda_1(t_2)-\lambda_2(t_2))\,\mathrm{d}t_1\,\mathrm{d}t_2 + o_{\mathrm{P}}\Big(\frac{1}{b_n}\Big),\end{aligned}$$

$$\begin{aligned}
\sigma^2_{m-m_c}(\boldsymbol{a}) = &\int_0^{\tau_0} w^2(t)(1-G_2(t))\,\mathrm{d}F_1(t)\\
&-2\iint_{t_1\le t_2} w(t_1)w(t_2)\,\frac{(1-F_1(t_2))(1-G_2(t_2))}{(1-H_{\eta,\gamma}(t_1))\,(1-H_{\eta,\gamma}(t_2))}\\
&(Q_1(t_1)\lambda_1(t_1)+Q_2(t_1)\lambda_2(t_1))\,Q_2(t_2)\,(\lambda_1(t_2)-\lambda_2(t_2))\,\mathrm{d}t_1\,\mathrm{d}t_2\\
&-2\iint_{t_1\le t_2} w(t_1)w(t_2)\,\frac{(1-F_1(t_1))(1-G_2(t_1))(1-F_1(t_2))(1-G_2(t_2))}{(1-H_{\eta,\gamma}(t_1))(1-H_{\eta,\gamma}(t_2))}\\
&Q_2(t_1)Q_2(t_2)\,(\lambda_1(t_1)-\lambda_2(t_1))(\lambda_1(t_2)-\lambda_2(t_2))\,\mathrm{d}t_1\,\mathrm{d}t_2+o_{\mathrm{P}}\Big(\frac{1}{b_n}\Big),\\
\sigma^{0\,2}_{m_c}(\boldsymbol{a}) = &\frac{\gamma-\eta}{1-\eta}\int_0^{\tau_0} w^2(t)(1-G_2(t))\,\mathrm{d}F_1(t)+\frac{1-\gamma}{1-\eta}\int_0^{\tau_0} w^2(t)(1-G_2(t))\,\mathrm{d}F_2(t)\\
&+2\,\frac{\eta}{1-\eta}\iint_{t_1\le t_2} w(t_1)w(t_2)\,\frac{(1-F_1(t_2))(1-G_1(t_2))}{(1-H_{\eta,\gamma}(t_1))(1-H_{\eta,\gamma}(t_2))}\\
&(Q_1(t_1)\lambda_1(t_1)+Q_2(t_1)\lambda_2(t_1))\,Q_2(t_2)\,(\lambda_1(t_2)-\lambda_2(t_2))\,\mathrm{d}t_1\,\mathrm{d}t_2\\
&-2\Big(\frac{\eta}{1-\eta}\Big)^2\iint_{t_1\le t_2} w(t_1)w(t_2)\,\frac{(1-F_1(t_1))(1-G_1(t_1))(1-F_1(t_2))(1-G_1(t_2))}{(1-H_{\eta,\gamma}(t_1))(1-H_{\eta,\gamma}(t_2))}\\
&Q_2(t_1)Q_2(t_2)\,(\lambda_1(t_1)-\lambda_2(t_1))(\lambda_1(t_2)-\lambda_2(t_2))\,\mathrm{d}t_1\,\mathrm{d}t_2+o_{\mathrm{P}}\Big(\frac{1}{b_n}\Big),
\end{aligned}$$

where $H_{\eta,\gamma}(t)$ is defined in (1.6), *$Q_1(t)$ and $Q_2(t)$ are given by* (1.14) *and $\lambda_1(t)$ and $\lambda_2(t)$ are the hazard functions corresponding to the distribution functions $F_1(t)$ and $F_2(t)$, respectively.*

Proof. First, we treat the limit behavior of $\frac{1}{m_c}\sum_{j=1}^{m_c} a_n^2(j)$ using

$$\begin{aligned}
&\frac{1}{m_c}\sum_{j=1}^{m_c} a_n^2(j) = \frac{1}{m_c}\sum_{j=1}^{m_c}\Big(\int_0^{\tau_0} w_n(t)\,\mathrm{d}N_j(t)-\int_0^{\tau_0}\frac{w_n(t)Y_j(t)}{Y(t)}\,\mathrm{d}N(t)\Big)^2\\
&=\frac{1}{m_c}\sum_{j=1}^{m_c}\Bigg\{\Big(\int_0^{\tau_0} w_n(t)\,\mathrm{d}N_j(t)\Big)^2-2\int_0^{\tau_0}\!\!\int_0^{\tau_0}\frac{w_n(t_1)\,w_n(t_2)}{Y(t_1)}\,Y_j(t_1)\,\mathrm{d}N(t_1)\,\mathrm{d}N_j(t_2)\\
&\quad+\int_0^{\tau_0}\!\!\int_0^{\tau_0}\frac{w_n(t_1)\,w_n(t_2)}{Y(t_1)Y(t_2)}\,Y_j(t_1)\,Y_j(t_2)\,\mathrm{d}N(t_1)\,\mathrm{d}N(t_2)\Bigg\}\\
&=\frac{1}{m_c}\sum_{j=1}^{m_c}\Bigg\{w_n^2(X_j)\,\Delta_j\,I(X_j\le\tau_0)-2\int_0^{\tau_0}\frac{w_n(t_1)w_n(X_j)}{Y(t_1)}\,I(t_1\le X_j\le\tau_0)\,\Delta_j\,\mathrm{d}N(t_1)\\
&\quad+\int_0^{\tau_0}\!\!\int_0^{\tau_0}\frac{w_n(t_1)\,w_n(t_2)}{Y(t_1)\,Y(t_2)}\,I(X_j\ge\max(t_1,t_2))\,\mathrm{d}N(t_1)\,\mathrm{d}N(t_2)\Bigg\}\\
&=\frac{1}{m_c}\sum_{j=1}^{m_c}\Bigg\{\int_0^{\tau_0} w_n^2(t)\,\mathrm{d}N_j(t)-2\iint_{t_1\le t_2}\frac{w_n(t_1)w_n(t_2)}{Y(t_1)}\,\mathrm{d}N(t_1)\,\mathrm{d}N_j(t_2)\\
&\quad+2\iint_{t_1\le t_2}\frac{w_n(t_1)\,w_n(t_2)\,Y_j(t_2)}{Y(t_1)\,Y(t_2)}\,\mathrm{d}N(t_1)\,\mathrm{d}N(t_2)-\int_0^{\tau_0}\frac{w_n^2(t)}{Y^2(t)}\,Y_j(t)\,\mathrm{d}N(t)\Bigg\}
\end{aligned}$$

$$= \frac{1}{m_c} \int_0^{\tau_0} w_n^2(t)\,\mathrm{d}\Big(\sum_{j=1}^{m_c} N_j(t)\Big) - \frac{2}{m_c} \iint_{t_1 \le t_2} \frac{w_n(t_1)\, w_n(t_2)}{Y(t_1)}\,\mathrm{d}N(t_1)\,\mathrm{d}\Big(\sum_{j=1}^{m_c} N_j(t_2)\Big)$$
$$+ \frac{2}{m_c} \iint_{t_1 \le t_2} \frac{w_n(t_1)\, w_n(t_2) \sum_{j=1}^{m_c} Y_j(t_2)}{Y(t_1)\, Y(t_2)}\,\mathrm{d}N(t_1)\,\mathrm{d}N(t_2) - \frac{1}{m_c} \int_0^{\tau_0} \frac{w_n^2(t)\, \sum_{j=1}^{m_c} Y_j(t)}{Y^2(t)}\,\mathrm{d}N(t).$$

By Lemma 5.2 we get, as $n \to \infty$,

$$\begin{aligned}
\frac{1}{m_c} \sum_{j=1}^{m_c} a_n^2(j) &= \frac{1}{m_c} \int_0^{\tau_0} w_n^2(t)\,\mathrm{d}\Big(\sum_{j=1}^{m_c} N_j(t)\Big) \\
&\quad - \frac{2}{m_c} \iint_{t_1 \le t_2} \frac{w_n(t_1)\, w_n(t_2)}{1 - H_{\eta,\gamma}(t_1)}\,\mathrm{d}\Big(\frac{N(t_1)}{n}\Big)\mathrm{d}\Big(\sum_{j=1}^{m_c} N_j(t_2)\Big)\,\Big(1 + O_{\mathrm{P}}\Big(\frac{1}{\sqrt{n}}\Big)\Big) \\
&\quad + \frac{2}{m_c} \iint_{t_1 \le t_2} \frac{w_n(t_1)\, w_n(t_2) \sum_{j=1}^{m_c} \mathrm{E}\, Y_j(t_2)}{(1 - H_{\eta,\gamma}(t_1))\,(1 - H_{\eta,\gamma}(t_2))}\,\mathrm{d}\Big(\frac{N(t_1)}{n}\Big)\,\mathrm{d}\Big(\frac{N(t_2)}{n}\Big)\,\Big(1 + O_{\mathrm{P}}\Big(\frac{1}{\sqrt{n}}\Big)\Big) \\
&\quad - \frac{1}{m_c\, n} \int_0^{\tau_0} \frac{w_n^2(t)\, \sum_{j=1}^{m_c} \mathrm{E}\, Y_j(t)}{(1 - H_{\eta,\gamma}(t))^2}\,\mathrm{d}\Big(\frac{N(t)}{n}\Big)\,\Big(1 + O_{\mathrm{P}}\Big(\frac{1}{\sqrt{n}}\Big)\Big) \\
&= \int_0^{\tau_0} w_n^2(t)\,\mathrm{d}\Big(\sum_{j=1}^{m_c} \frac{N_j(t)}{m_c}\Big) \\
&\quad - 2 \iint_{t_1 \le t_2} \frac{w_n(t_1)\, w_n(t_2)}{1 - H_{\eta,\gamma}(t_1)}\,\mathrm{d}\Big(\frac{N(t_1)}{n}\Big)\,\mathrm{d}\Big(\sum_{j=1}^{m_c} \frac{N_j(t_2)}{m_c}\Big)\,\Big(1 + O_{\mathrm{P}}\Big(\frac{1}{\sqrt{n}}\Big)\Big) \\
&\quad + 2 \iint_{t_1 \le t_2} \frac{w_n(t_1)\, w_n(t_2)\, \mathrm{E}\, Y_1(t_2)}{(1 - H_{\eta,\gamma}(t_1))\,(1 - H_{\eta,\gamma}(t_2))}\,\mathrm{d}\Big(\frac{N(t_1)}{n}\Big)\,\mathrm{d}\Big(\frac{N(t_2)}{n}\Big)\,\Big(1 + O_{\mathrm{P}}\Big(\frac{1}{\sqrt{n}}\Big)\Big) \\
&\quad - \frac{1}{n} \int_0^{\tau_0} \frac{w_n^2(t)\, \mathrm{E}\, Y_1(t)}{(1 - H_{\eta,\gamma}(t))^2}\,\mathrm{d}\Big(\frac{N(t)}{n}\Big)\,\Big(1 + O_{\mathrm{P}}\Big(\frac{1}{\sqrt{n}}\Big)\Big) \qquad (5.26)
\end{aligned}$$

and for the first integral

$$\begin{aligned}
&\Big| \int_0^{\tau_0} w_n^2(t)\,\mathrm{d}\Big(\frac{\sum_{j=1}^{m_c} N_j(t)}{m_c}\Big) - \int_0^{\tau_0} w^2(t)\,\mathrm{d}EN_1(t) \Big| \\
&\le \Big(\frac{\sum_{j=1}^{m_c} N_j(t)}{m_c}\Big) \sup_{0 \le t \le \tau_0} |w_n^2(t) - w^2(t)| + \Big| \int_0^{\tau_0} w^2(t)\,\Big(\mathrm{d}\Big(\frac{\sum_{j=1}^{m_c} N_j(t)}{m_c}\Big) - \mathrm{d}\,\mathrm{E}\, N_1(t)\Big) \Big| \\
&\le \Big(\frac{\sum_{j=1}^{m_c} N_j(t)}{m_c}\Big) \sup_{0 \le t \le \tau_0} |w_n(t) - w(t)|\,\Big(\sup_{0 \le t \le \tau_0} |w_n(t) - w(t)| + 2 \sup_{0 \le t \le \tau_0} |w(t)|\Big) \\
&\quad + \Big| \int_0^{\tau_0} w^2(t)\,\Big(\mathrm{d}\Big(\frac{\sum_{j=1}^{m_c} N_j(t)}{m_c}\Big) - \mathrm{d}\,\mathrm{E}\, N_1(t)\Big) \Big| \\
&= o_{\mathrm{P}}\Big(\frac{1}{b_n}\Big) + \Big| \int_0^{\tau_0} w^2(t)\,\Big(\mathrm{d}\Big(\frac{\sum_{j=1}^{m_c} N_j(t)}{m_c}\Big) - \mathrm{d}EN_1(t)\Big) \Big|
\end{aligned}$$

in view of the assumption for the weights w_n. Particularly, since the function $w(t)$ is continuous on $t \in [0, \tau_0]$, the function $v(t) = w^2(t)$ is also continuous on $t \in [0, \tau_0]$. By Lemma 5.3 we have, as $n \to \infty$,

$$\left| \int_0^{\tau_0} w_n^2(t)\,\mathrm{d}\Big(\frac{\sum_{j=1}^{m_c} N_j(t)}{m_c}\Big) - \int_0^{\tau_0} w^2(t)\,\mathrm{d}\,\mathrm{E}\,N_1(t) \right| = O_{\mathrm{P}}\Big(\frac{1}{\sqrt{n}}\Big) + o_{\mathrm{P}}\Big(\frac{1}{b_n}\Big) = o_{\mathrm{P}}\Big(\frac{1}{b_n}\Big).$$

Similarly we can proceed for each integral since

$$\begin{aligned} v_1(t) &= \frac{w(t)}{1-H_{\eta,\gamma}(t)}, \\ v_2(t) &= \frac{w(t)\,\mathrm{E}\,Y_1(t)}{1-H_{\eta,\gamma}(t)} = \frac{w(t)(1-F_1(t))(1-G_1(t))}{1-H_{\eta,\gamma}(t)}, \\ v_3(t) &= \frac{w^2(t)\,\mathrm{E}\,Y_1(t)}{(1-H_{\eta,\gamma}(t))^2} = \frac{w^2(t)(1-F_1(t))(1-G_1(t))}{(1-H_{\eta,\gamma}(t))^2} \end{aligned}$$

are continuous functions on $[0,\tau_0]$, so for the remaining integrals can be Lemma 5.3 also applied. Thus, the term (5.26) can be rewritten as follows, as $n \to \infty$,

$$\begin{aligned} \frac{1}{m_c}\sum_{j=1}^{m_c} a_n^2(j) &= \int_0^{\tau_0} w^2(t)\,\mathrm{d}\,\mathrm{E}\,N_1(t) - 2\iint_{t_1\le t_2} \frac{w(t_1)\,w(t_2)}{1-H_{\eta,\gamma}(t_1)}\,\mathrm{d}R_{\eta,\gamma}(t_1)\,\mathrm{d}\,\mathrm{E}\,N_1(t_2) \\ &\quad + 2\iint_{t_1\le t_2} \frac{w(t_1)\,w(t_2)\,\mathrm{E}\,Y_1(t_2)}{(1-H_{\eta,\gamma}(t_1))\,(1-H_{\eta,\gamma}(t_2))}\,\mathrm{d}R_{\eta,\gamma}(t_1)\,\mathrm{d}R_{\eta,\gamma}(t_2) - \frac{1}{n}\int_0^{\tau_0} \frac{w^2(t)\;\mathrm{E}\,Y_1(t)}{(1-H_{\eta,\gamma}(t))^2}\,\mathrm{d}R_{\eta,\gamma}(t) \\ &\quad + o_{\mathrm{P}}\Big(\frac{1}{b_n}\Big) \\ &= \int_0^{\tau_0} w^2(t)(1-G_1(t))\,\mathrm{d}F_1(t) - 2\iint_{t_1\le t_2} \frac{w(t_1)\,w(t_2)}{1-H_{\eta,\gamma}(t_1)}\,(1-G_1(t_2))\,\mathrm{d}R_{\eta,\gamma}(t_1)\,\mathrm{d}F_1(t_2) \\ &\quad + 2\iint_{t_1\le t_2} \frac{w(t_1)\,w(t_2)}{(1-H_{\eta,\gamma}(t_1))\,(1-H_{\eta,\gamma}(t_2))}\,(1-F_1(t_2))(1-G_1(t_2))\,\mathrm{d}R_{\eta,\gamma}(t_1)\,\mathrm{d}R_{\eta,\gamma}(t_2) \\ &\quad + o_{\mathrm{P}}\Big(\frac{1}{b_n}\Big) \end{aligned} \tag{5.27}$$

and it can be seen that

$$\begin{aligned} &\iint_{t_1\le t_2} \frac{w(t_1)\,w(t_2)}{1-H_{\eta,\gamma}(t_1)}\,(1-G_1(t_2))\,\mathrm{d}R_{\eta,\gamma}(t_1)\,\mathrm{d}F_1(t_2) \\ &= \iint_{t_1\le t_2} \frac{w(t_1)\,w(t_2)}{(1-H_{\eta,\gamma}(t_1))\,(1-H_{\eta,\gamma}(t_2))}\,(1-G_1(t_2))(Q_1(t_2)+Q_2(t_2))\,\mathrm{d}R_{\eta,\gamma}(t_1)\,\mathrm{d}F_1(t_2) \\ &= \iint_{t_1\le t_2} \frac{w(t_1)\,w(t_2)}{(1-H_{\eta,\gamma}(t_1))\,(1-H_{\eta,\gamma}(t_2))}\,(1-G_1(t_2))Q_1(t_2)\,\mathrm{d}R_{\eta,\gamma}(t_1)\,\mathrm{d}F_1(t_2) \\ &\quad + \iint_{t_1\le t_2} \frac{w(t_1)\,w(t_2)}{(1-H_{\eta,\gamma}(t_1))\,(1-H_{\eta,\gamma}(t_2))}\,(1-F_1(t_2)(1-G_1(t_2))Q_2(t_2)\lambda_1(t_2)\,\mathrm{d}R_{\eta,\gamma}(t_1)\,\mathrm{d}t_2 \end{aligned} \tag{5.28}$$

and

$$\iint_{t_1\le t_2} \frac{w(t_1)\,w(t_2)}{(1-H_{\eta,\gamma}(t_1))\,(1-H_{\eta,\gamma}(t_2))}\,(1-F_1(t_2))(1-G_1(t_2))\,\mathrm{d}R_{\eta,\gamma}(t_1)\,\mathrm{d}R_{\eta,\gamma}(t_2)$$

$$= \iint_{t_1 \le t_2} \frac{w(t_1)\, w(t_2)}{(1 - H_{\eta,\gamma}(t_1))\,(1 - H_{\eta,\gamma}(t_2))}\,(1 - F_1(t_2))(1 - G_1(t_2))\,\mathrm{d}R_{\eta,\gamma}(t_1)$$
$$\left(\frac{Q_1(t_2)}{1 - F_1(t_2)}\,\mathrm{d}F_1(t_2) + \frac{Q_2(t_2)}{1 - F_2(t_2)}\,\mathrm{d}F_2(t_2)\right)$$
$$= \iint_{t_1 \le t_2} \frac{w(t_1)\, w(t_2)}{(1 - H_{\eta,\gamma}(t_1))\,(1 - H_{\eta,\gamma}(t_2))}\,(1 - G_1(t_2))Q_1(t_2)\,\mathrm{d}R_{\eta,\gamma}(t_1)\,\mathrm{d}F_1(t_2)$$
$$+ \iint_{t_1 \le t_2} \frac{w(t_1)\, w(t_2)}{(1 - H_{\eta,\gamma}(t_1))\,(1 - H_{\eta,\gamma}(t_2))}\,(1 - F_1(t_2))(1 - G_1(t_2))Q_2(t_2)\lambda_2(t_2)\,\mathrm{d}R_{\eta,\gamma}(t_1)\,\mathrm{d}t_2. \tag{5.29}$$

Comparing (5.28) and (5.29) we get, as $n \to \infty$,

$$\iint_{t_1 \le t_2} \frac{w(t_1)\, w(t_2)}{1 - H_{\eta,\gamma}(t_1)}\,(1 - G_1(t_2))\,\mathrm{d}R_{\eta,\gamma}(t_1)\,\mathrm{d}F_1(t_2)$$
$$- \iint_{t_1 \le t_2} \frac{w(t_1)\, w(t_2)}{(1 - H_{\eta,\gamma}(t_1))\,(1 - H_{\eta,\gamma}(t_2))}\,(1 - F_1(t_2))(1 - G_1(t_2))\,\mathrm{d}R_{\eta,\gamma}(t_1)\,\mathrm{d}R_{\eta,\gamma}(t_2)$$
$$= \iint_{t_1 \le t_2} \frac{w(t_1)\, w(t_2)}{(1 - H_{\eta,\gamma}(t_1))\,(1 - H_{\eta,\gamma}(t_2))}\,(1 - F_1(t_2))(1 - G_1(t_2))Q_2(t_2)\,(\lambda_1(t_2) - \lambda_2(t_2))$$
$$\mathrm{d}R_{\eta,\gamma}(t_1)\,\mathrm{d}t_2$$
$$= \iint_{t_1 \le t_2} \frac{w(t_1)\, w(t_2)}{(1 - H_{\eta,\gamma}(t_1))\,(1 - H_{\eta,\gamma}(t_2))}\,(1 - F_1(t_2))(1 - G_1(t_2))Q_2(t_2)\,(\lambda_1(t_2) - \lambda_2(t_2))$$
$$(Q_1(t_1)\lambda_1(t_1) + Q_2(t_1)\lambda_2(t_1))\,\mathrm{d}t_1\,\mathrm{d}t_2,$$

which implies together with (5.27)

$$\frac{1}{m_c}\sum_{j=1}^{m_c} a_n^2(j) = \int_0^{\tau_0} w^2(t)(1 - G_1(t))\,\mathrm{d}F_1(t)$$
$$- 2\iint_{t_1 \le t_2} \frac{w(t_1)\, w(t_2)}{(1 - H_{\eta,\gamma}(t_1))\,(1 - H_{\eta,\gamma}(t_2))}\,(1 - F_1(t_2))(1 - G_1(t_2))Q_2(t_2)\,(\lambda_1(t_2) - \lambda_2(t_2))$$
$$(Q_1(t_1)\lambda_1(t_1) + Q_2(t_1)\lambda_2(t_1))\,\mathrm{d}t_1\,\mathrm{d}t_2 + o_{\mathrm{P}}\Big(\frac{1}{b_n}\Big). \tag{5.30}$$

Repeating the steps for $j = m_c+1, m_c+2, \ldots, m$, we get instead of (5.30) the following formula

$$\frac{1}{m - m_c}\sum_{j=m_c+1}^{m} a_n^2(j) = \int_0^{\tau_0} w^2(t)(1 - G_2(t))\,\mathrm{d}F_1(t)$$
$$- 2\iint_{t_1 \le t_2} \frac{w(t_1)\, w(t_2)}{(1 - H_{\eta,\gamma}(t_1))\,(1 - H_{\eta,\gamma}(t_2))}\,(1 - F_1(t_2))(1 - G_2(t_2))Q_2(t_2)\,(\lambda_1(t_2) - \lambda_2(t_2))$$
$$(Q_1(t_1)\lambda_1(t_1) + Q_2(t_1)\lambda_2(t_1))\,\mathrm{d}t_1\,\mathrm{d}t_2 + o_{\mathrm{P}}\Big(\frac{1}{b_n}\Big). \tag{5.31}$$

By Lemma 5.4 we know that

$$\frac{1}{n}\sum_{j=1}^{n} a_n^2(j) = \eta\int_0^{\tau_0} w^2(t)(1-G_1(t))\,\mathrm{d}F_1(t) + (\gamma-\eta)\int_0^{\tau_0} w^2(t)(1-G_2(t))\,\mathrm{d}F_1(t) + (1-\gamma)\int_0^{\tau_0} w^2(t)(1-G_2(t))\,\mathrm{d}F_2(t) + o_{\mathrm{P}}\Big(\frac{1}{b_n}\Big)$$

and by (5.30) we obtain, as $n\to\infty$,

$$\frac{1}{n-m_c}\sum_{j=m_c+1}^{n} a_n^2(j) = \frac{\gamma-\eta}{1-\eta}\int_0^{\tau_0} w^2(t)(1-G_2(t))\,\mathrm{d}F_1(t) + \frac{1-\gamma}{1-\eta}\int_0^{\tau_0} w^2(t)(1-G_2(t))\,\mathrm{d}F_2(t)$$
$$+ 2\frac{\eta}{1-\eta}\iint_{t_1\le t_2} \frac{w(t_1)\,w(t_2)}{(1-H_{\eta,\gamma}(t_1))\,(1-H_{\eta,\gamma}(t_2))}(1-F_1(t_2))(1-G_1(t_2))Q_2(t_2)\,(\lambda_1(t_2)-\lambda_2(t_2))$$
$$(Q_1(t_1)\lambda_1(t_1)+Q_2(t_1)\lambda_2(t_1))\,\mathrm{d}t_1\,\mathrm{d}t_2 + o_{\mathrm{P}}\Big(\frac{1}{b_n}\Big). \tag{5.32}$$

Next, we turn to $\bar{a}^2_{m_c}$ and $\bar{a}^2_{m-m_c}$. By Lemma 5.5 we have, as $n\to\infty$,

$$\bar{a}^2_{m_c} = \Big(\int_0^{\tau_0} \frac{w(t)}{1-H_{\eta,\gamma}(t)}(1-F_1(t))(1-G_1(t))Q_2(t)(\lambda_1(t)-\lambda_2(t))\,\mathrm{d}t + o_{\mathrm{P}}\Big(\frac{1}{b_n}\Big)\Big)^2$$
$$= 2\iint_{t_1\le t_2} \frac{w(t_1)\,w(t_2)}{(1-H_{\eta,\gamma}(t_1))\,(1-H_{\eta,\gamma}(t_1))}(1-F_1(t_1))(1-G_1(t_1))(1-F_1(t_2))(1-G_1(t_2))$$
$$Q_2(t_1)Q_2(t_2)(\lambda_1(t_1)-\lambda_2(t_1))(\lambda_1(t_2)-\lambda_2(t_2))\,\mathrm{d}t_1\,\mathrm{d}t_2 + o_{\mathrm{P}}\Big(\frac{1}{b_n}\Big) \tag{5.33}$$

and

$$\bar{a}^2_{m-m_c} = \Big(\int_0^{\tau_0} \frac{w(t)}{1-H_{\eta,\gamma}(t)}(1-F_1(t))(1-G_2(t))Q_2(t)(\lambda_1(t)-\lambda_2(t))\,\mathrm{d}t + o_{\mathrm{P}}\Big(\frac{1}{b_n}\Big)\Big)^2$$
$$= 2\iint_{t_1\le t_2} \frac{w(t_1)\,w(t_2)}{(1-H_{\eta,\gamma}(t_1))\,(1-H_{\eta,\gamma}(t_1))}(1-F_1(t_1))(1-G_2(t_1))(1-F_1(t_2))(1-G_2(t_2))$$
$$Q_2(t_1)Q_2(t_2)(\lambda_1(t_1)-\lambda_2(t_1))(\lambda_1(t_2)-\lambda_2(t_2))\,\mathrm{d}t_1\,\mathrm{d}t_2 + o_{\mathrm{P}}\Big(\frac{1}{b_n}\Big). \tag{5.34}$$

By Lemma 5.4 we know that $\bar{a}_n = 0$ which gives together with (5.33)

$$\bar{a}^{0\,2}_{m_c} = 2\Big(\frac{\eta}{1-\eta}\Big)^2\iint_{t_1\le t_2} \frac{w(t_1)\,w(t_2)}{(1-H_{\eta,\gamma}(t_1))\,(1-H_{\eta,\gamma}(t_1))}(1-F_1(t_1))(1-G_1(t_1))$$
$$(1-F_1(t_2))(1-G_1(t_2))Q_2(t_1)Q_2(t_2)(\lambda_1(t_1)-\lambda_2(t_1))(\lambda_1(t_2)-\lambda_2(t_2))\,\mathrm{d}t_1\,\mathrm{d}t_2$$
$$+ o_{\mathrm{P}}\Big(\frac{1}{b_n}\Big). \tag{5.35}$$

Combining (5.30) with (5.33) we obtain the assertion of the theorem for $\sigma^2_{m_c}(\boldsymbol{a})$ and combination of (5.31) and (5.34) gives the assertion for $\sigma^2_{m-m_c}(\boldsymbol{a})$. Finally, by (5.32) and (5.35) we get the assertion for $\sigma^{0\,2}_{m_c}(\boldsymbol{a})$. □

COROLLARY 5.11. *Let the condition* (5.1) *with* b_n *of the form* (5.2) *be satisfied. Assume* $0 < m_c < m = n$. *Then we have, as* $n\to\infty$,

$$\sigma^2_{m_c}(\boldsymbol{a}) = \int_0^{\tau_0} w^2(t)\,(1-G_1(t))\,\mathrm{d}F(t) + o_{\mathrm{P}}\Big(\frac{1}{b_n}\Big),$$

$$\sigma^2_{m-m_c}(\boldsymbol{a}) = \int_0^{\tau_0} w^2(t)\,(1-G_2(t))\,\mathrm{d}F(t) + o_{\mathrm{P}}\Big(\frac{1}{b_n}\Big),$$
$$\sigma^{0\,2}_{m_c}(\boldsymbol{a}) = \int_0^{\tau_0} w^2(t)\,(1-G_2(t))\,\mathrm{d}F(t) + o_{\mathrm{P}}\Big(\frac{1}{b_n}\Big).$$

Proof. Since $F_1(t) = F_2(t) = F(t)$ for all t, the assertion follows from Lemma 5.6. □

Corollary 5.12. *(local alternatives) Let the condition* (5.1) *with* $b_n = 1/|A_n|$ *be satisfied. Assume* $0 < m_c < m < n$. *Then, under the assumptions (A.2) and (A.3), we have, as* $n \to \infty$,

$$\sigma^2_{m_c}(\boldsymbol{a}) = \int_0^{\tau_0} w^2(t)\,(1-G(t))\,\mathrm{d}F(t) + o_{\mathrm{P}}(1),$$
$$\sigma^2_{m-m_c}(\boldsymbol{a}) = \int_0^{\tau_0} w^2(t)\,(1-G(t))\,\mathrm{d}F(t) + o_{\mathrm{P}}(1),$$
$$\sigma^{0\,2}_{m_c}(\boldsymbol{a}) = \int_0^{\tau_0} w^2(t)\,(1-G(t))\,\mathrm{d}F(t) + o_{\mathrm{P}}(1).$$

Proof. Lemma 5.6 for the choice of $b_n = 1/|A_n|$ together with the assumptions (A.2) and (A.3) imply the assertions of the corollary. □

6. Behavior of standardization V_k

In this section we focus on representation for V_k defined in (2.16). Let us denote by

$$J_{ijkl}(\tau_0) = \int_0^{\tau_0} w^2(t)\,\frac{(1-F_i(t))(1-F_j(t))(1-G_k(t))(1-G_l(t))}{(1-H_{\eta,\gamma}(t))^2}\,\mathrm{d}R_{\eta,\gamma}(t) \tag{5.36}$$

for $i,j,k,l = 1,2$ and with $H_{\eta,\gamma}(t)$ and $R_{\eta,\gamma}(t)$ given by (1.6) and (1.10).

Lemma 5.7. *Let the condition for the weights* (5.1) *with* b_n *of the form* (5.2) *be satisfied. Assume* $0 < m_c < m < n$. *Then we have, as* $n \to \infty$,

$$\frac{n^2}{k(n-k)}\,V_k = \Big(\frac{m_c-k}{n-k}\,J_{1111}(\tau_0) + \frac{m-m_c}{n-k}\,J_{1112}(\tau_0) + \frac{n-m}{n-k}\,J_{1212}(\tau_0) + o_{\mathrm{P}}\Big(\frac{1}{b_n}\Big)\Big)$$
$$\Big(1 + O_{\mathrm{P}}\Big(\sqrt{\frac{\log k}{k}} + \sqrt{\frac{\log(n-k)}{n-k}}\Big)\Big) \tag{5.37}$$

uniformly in $\log\log n < k \le m_c$,

$$\frac{n^2}{k(n-k)}\,V_k = \Big(\frac{m_c(m-k)}{k(n-k)}\,J_{1112}(\tau_0) + \frac{m_c(n-m)}{k(n-k)}\,J_{1212}(\tau_0) + \frac{(k-m_c)(m-k)}{k(n-k)}J_{1122}(\tau_0)$$
$$+ \frac{(k-m_c)(n-m)}{k(n-k)}J_{1222}(\tau_0) + o_{\mathrm{P}}\Big(\frac{1}{b_n}\Big)\Big)\Big(1 + O_{\mathrm{P}}\Big(\sqrt{\frac{\log k}{k}} + \sqrt{\frac{\log(n-k)}{n-k}}\Big)\Big) \tag{5.38}$$

uniformly in $m_c \le k \le m$,

$$\frac{n^2}{k(n-k)}\,V_k = \Big(\frac{m_c}{k}\,J_{1212}(\tau_0) + \frac{m-m_c}{k}\,J_{1222}(\tau_0) + \frac{k-m}{k}\,J_{2222}(\tau_0) + o_{\mathrm{P}}\Big(\frac{1}{b_n}\Big)\Big)$$
$$\Big(1 + O_{\mathrm{P}}\Big(\sqrt{\frac{\log k}{k}} + \sqrt{\frac{\log(n-k)}{n-k}}\Big)\Big) \tag{5.39}$$

uniformly in $m \le k < n - \log\log n$ *and where* $J_{ijkl}(\tau_0)$ *given by* (5.36).

PROOF. Using Lemmas 5.1 and 5.2 we approximate the term V_k as follows

$$\frac{n^2}{k(n-k)}\,V_k = \int_0^{\tau_0} w_n^2(t)\,\frac{\frac{1}{k}\left(\sum_{j=1}^k Y_j(t)\right)\frac{1}{n-k}\left(\sum_{j=k+1}^n Y_j(t)\right)}{\left(\frac{Y(t)}{n}\right)^2}\,\mathrm{d}\frac{N(t)}{n}$$

$$= \int_0^{\tau_0} w_n^2(t)\,\frac{\frac{1}{k}\left(\sum_{j=1}^k \mathrm{E}\,Y_j(t)\right)\frac{1}{n-k}\left(\sum_{j=k+1}^n \mathrm{E}\,Y_j(t)\right)}{\left(\frac{1}{n}\sum_{j=1}^n \mathrm{E}\,Y_j(t)\right)^2}\,\mathrm{d}\frac{N(t)}{n}$$

$$\left(1+O_{\mathrm{P}}\left(\sqrt{\frac{\log k}{k}}+\sqrt{\frac{\log(n-k)}{n-k}}\right)\right)$$

$$= \int_0^{\tau_0} w_n^2(t)\,\frac{\mathrm{E}\,Y_1(t)\left\{\frac{m_c-k}{n-k}\,\mathrm{E}\,Y_1(t)+\frac{m-m_c}{n-k}\,\mathrm{E}\,Y_m(t)+\frac{n-m}{n-k}\,\mathrm{E}\,Y_n(t)\right\}}{(1-H_{\eta,\gamma}(t))^2}\,\mathrm{d}\frac{N(t)}{n}$$

$$\left(1+O_{\mathrm{P}}\left(\sqrt{\frac{\log k}{k}}+\sqrt{\frac{\log(n-k)}{n-k}}\right)\right)$$

$$= \left(\frac{m_c-k}{n-k}\int_0^{\tau_0} w_n^2(t)\,\frac{\mathrm{E}\,Y_1^2(t)}{(1-H_{\eta,\gamma}(t))^2}\,\mathrm{d}\frac{N(t)}{n} + \frac{m-m_c}{n-k}\int_0^{\tau_0} w_n^2(t)\,\frac{\mathrm{E}\,Y_1(t)\,\mathrm{E}\,Y_m(t)}{(1-H_{\eta,\gamma}(t))^2}\,\mathrm{d}\frac{N(t)}{n}\right.$$

$$\left. + \frac{n-m}{n-k}\int_0^{\tau_0} w_n^2(t)\,\frac{\mathrm{E}\,Y_1(t)\,\mathrm{E}\,Y_n(t)}{(1-H_{\eta,\gamma}(t))^2}\,\mathrm{d}\frac{N(t)}{n}\right)\left(1+O_{\mathrm{P}}\left(\sqrt{\frac{\log k}{k}}+\sqrt{\frac{\log(n-k)}{n-k}}\right)\right)$$

uniformly in $\log\log n < k \le m_c$. Using the steps similar to (5.18) – (5.19) in the proof of Lemma 5.4 only instead of $v(t) = w^2(t)$ we have

$$v_1(t) = w^2(t)\,\frac{(1-F_1(t))^2(1-G_1(t))^2}{(1-H_{\eta,\gamma}(t))^2},$$

$$v_2(t) = w^2(t)\,\frac{(1-F_1(t))^2(1-G_1(t)(1-G_2(t))}{(1-H_{\eta,\gamma}(t))^2},$$

$$v_3(t) = w^2(t)\,\frac{(1-F_1(t))(1-F_2(t))(1-G_1(t))(1-G_2(t))}{(1-H_{\eta,\gamma}(t))^2}.$$

Thus, we get the result (5.37).

Analogously,

$$\frac{n^2}{k(n-k)}\,V_k = \int_0^{\tau_0} w_n^2(t)\,\frac{\frac{1}{k}\left(\sum_{j=1}^k \mathrm{E}\,Y_j(t)\right)\frac{1}{n-k}\left(\sum_{j=k+1}^n \mathrm{E}\,Y_j(t)\right)}{\left(\frac{1}{n}\sum_{j=1}^n \mathrm{E}\,Y_j(t)\right)^2}\,\mathrm{d}\frac{N(t)}{n}$$

$$\left(1+O_{\mathrm{P}}\left(\sqrt{\frac{\log k}{k}}+\sqrt{\frac{\log(n-k)}{n-k}}\right)\right)$$

$$= \int_0^{\tau_0} w_n^2(t)\,\frac{\left\{\frac{m_c}{k}\,\mathrm{E}\,Y_1(t)+\frac{k-m_c}{k}\,\mathrm{E}\,Y_m(t)\right\}\left\{\frac{m-k}{n-k}\,\mathrm{E}\,Y_m(t)+\frac{n-m}{n-k}\,\mathrm{E}\,Y_n(t)\right\}}{(1-H_{\eta,\gamma}(t))^2}\,\mathrm{d}\frac{N(t)}{n}$$

$$\left(1+O_{\mathrm{P}}\left(\sqrt{\frac{\log k}{k}}+\sqrt{\frac{\log(n-k)}{n-k}}\right)\right)$$

$$
\begin{aligned}
&= \Bigg(\frac{m_c(m-k)}{k(n-k)} \int_0^{\tau_0} w_n^2(t)\, \frac{\mathrm{E}\, Y_1(t)\ \mathrm{E}\, Y_m(t)}{(1-H_{\eta,\gamma}(t))^2}\, \mathrm{d}\frac{N(t)}{n} \\
&\quad + \frac{m_c(n-m)}{k(n-k)} \int_0^{\tau_0} w_n^2(t)\, \frac{\mathrm{E}\, Y_1(t)\ \mathrm{E}\, Y_n(t)}{(1-H_{\eta,\gamma}(t))^2}\, \mathrm{d}\frac{N(t)}{n} \\
&\quad + \frac{(k-m_c)(m-k)}{k(n-k)} \int_0^{\tau_0} w_n^2(t)\, \frac{\mathrm{E}\, Y_m^2(t)}{(1-H_{\eta,\gamma}(t))^2}\, \mathrm{d}\frac{N(t)}{n} \\
&\quad + \frac{(k-m_c)(n-m)}{k(n-k)} \int_0^{\tau_0} w_n^2(t)\, \frac{\mathrm{E}\, Y_m(t)\ \mathrm{E}\, Y_n(t)}{(1-H_{\eta,\gamma}(t))^2}\, \mathrm{d}\frac{N(t)}{n} \Bigg) \\
&\quad \left(1 + O_{\mathrm{P}}\left(\sqrt{\frac{\log k}{k}} + \sqrt{\frac{\log(n-k)}{n-k}}\right)\right)
\end{aligned}
$$

uniformly in $m_c \le k \le m$ and

$$
\begin{aligned}
\frac{n^2}{k(n-k)}\, V_k &= \int_0^{\tau_0} w_n^2(t)\, \frac{\frac{1}{k}\left(\sum_{j=1}^{k} \mathrm{E}\, Y_j(t)\right) \frac{1}{n-k}\left(\sum_{j=k+1}^{n} \mathrm{E}\, Y_j(t)\right)}{\left(\frac{1}{n}\sum_{j=1}^{n} \mathrm{E}\, Y_j(t)\right)^2}\, \mathrm{d}\frac{N(t)}{n} \\
&\quad \left(1 + O_{\mathrm{P}}\left(\sqrt{\frac{\log k}{k}} + \sqrt{\frac{\log(n-k)}{n-k}}\right)\right) \\
&= \int_0^{\tau_0} w_n^2(t)\, \frac{\left\{\frac{m_c}{k}\ \mathrm{E}\, Y_1(t) + \frac{m-m_c}{k}\ \mathrm{E}\, Y_{m_c}(t) + \frac{k-m}{k}\, \mathrm{E}\, Y_n(t)\right\}\ \mathrm{E}\, Y_n(t)}{(1-H_{\eta,\gamma}(t))^2}\, \mathrm{d}\frac{N(t)}{n} \\
&\quad \left(1 + O_{\mathrm{P}}\left(\sqrt{\frac{\log k}{k}} + \sqrt{\frac{\log(n-k)}{n-k}}\right)\right) \\
&= \Bigg(\frac{m_c}{k} \int_0^{\tau_0} w_n^2(t)\, \frac{\mathrm{E}\, Y_1(t)\ \mathrm{E}\, Y_n(t)}{(1-H_{\eta,\gamma}(t))^2}\, \mathrm{d}\frac{N(t)}{n} \\
&\quad + \frac{m-m_c}{k} \int_0^{\tau_0} w_n^2(t)\, \frac{\mathrm{E}\, Y_m(t)\ \mathrm{E}\, Y_n(t)}{(1-H_{\eta,\gamma}(t))^2}\, \mathrm{d}\frac{N(t)}{n} \\
&\quad + \frac{k-m}{k} \int_0^{\tau_0} w_n^2(t)\, \frac{\mathrm{E}\, Y_n^2(t)}{(1-H_{\eta,\gamma}(t))^2}\, \mathrm{d}\frac{N(t)}{n} \Bigg) \left(1 + O_{\mathrm{P}}\left(\sqrt{\frac{\log k}{k}} + \sqrt{\frac{\log(n-k)}{n-k}}\right)\right)
\end{aligned}
$$

uniformly in $m \le k < n - \log\log n$. Repeating the steps as (5.18) – (5.19) in the proof of Lemma 5.4, we obtain the results (5.38) and (5.39). □

Corollary 5.13. *Let the condition* (5.1) *with* b_n *of the form* (5.2) *be satisfied. Assume* $0 < m_c = m = n$. *Then we have, as* $n \to \infty$,

$$
\frac{n^2}{k(n-k)}\, V_k = \left(\int_0^{\tau_0} w^2(t)\, \mathrm{d}R(t) + o_{\mathrm{P}}\left(\frac{1}{b_n}\right)\right) \left(1 + O_{\mathrm{P}}\left(\sqrt{\frac{\log k}{k}} + \sqrt{\frac{\log(n-k)}{n-k}}\right)\right)
$$

uniformly in $\log\log n < k < n - \log\log n$ *with* $R(t)$ *given by* (1.13).

Proof. Using Lemmas 5.1 and 5.2 we approximate the term V_k as follows

$$
\frac{n^2}{k(n-k)}\, V_k = \int_0^{\tau_0} w_n^2(t) \frac{\frac{1}{k}\left(\sum_{j=1}^{k} Y_j(t)\right) \frac{1}{n-k}\left(\sum_{j=k+1}^{n} Y_j(t)\right)}{\left(\frac{Y(t)}{n}\right)^2}\, \mathrm{d}\frac{N(t)}{n}
$$

$$= \int_0^{\tau_0} w_n^2(t) \frac{\frac{1}{k}\left(\sum_{j=1}^{k} \mathrm{E}\, Y_j(t)\right) \frac{1}{n-k}\left(\sum_{j=k+1}^{n} \mathrm{E}\, Y_j(t)\right)}{\left(\frac{1}{n} \sum_{j=1}^{n} \mathrm{E}\, Y_j(t)\right)^2} \,\mathrm{d}\frac{N(t)}{n}$$
$$\left(1 + O_{\mathrm{P}}\left(\sqrt{\frac{\log k}{k}} + \sqrt{\frac{\log(n-k)}{n-k}}\right)\right)$$
$$= \int_0^{\tau_0} w_n^2(t) \frac{(\mathrm{E}\, Y_1(t))^2}{(1-H(t))^2} \,\mathrm{d}\frac{N(t)}{n} \left(1 + O_{\mathrm{P}}\left(\sqrt{\frac{\log k}{k}} + \sqrt{\frac{\log(n-k)}{n-k}}\right)\right)$$
$$= \int_0^{\tau_0} w_n^2(t) \,\mathrm{d}\frac{N(t)}{n} \left(1 + O_{\mathrm{P}}\left(\sqrt{\frac{\log k}{k}} + \sqrt{\frac{\log(n-k)}{n-k}}\right)\right)$$

uniformly in $\log\log n < k < n - \log\log n$ and. Further, we proceed analogously to (5.18) – (5.19) in the proof of Lemma 5.4. Thus, the proof is finished. □

COROLLARY 5.14. *Let the condition* (5.1) *with* b_n *of the form* (5.2) *be satisfied. Assume* $0 < m_c < m < n$*. Then we have, as* $n \to \infty$*,*

$$\frac{n^2}{m_c(n-m_c)} V_{m_c} = \frac{m-m_c}{n-m_c} J_{1112}(\tau_0) + \frac{n-m}{n-m_c} J_{1212}(\tau_0) + o_{\mathrm{P}}\Big(\frac{1}{b_n}\Big), \tag{5.40}$$

and

$$\frac{n^2}{m(n-m)} V_m = \frac{m_c}{m} J_{1212}(\tau_0) + \frac{m-m_c}{m} J_{1222}(\tau_0) + o_{\mathrm{P}}\Big(\frac{1}{b_n}\Big), \tag{5.41}$$

where $J_{ijkl}(\tau_0)$ *is defined in* (5.36)*.*
Assume $0 < m_c = m < n$*. Then we have, as* $n \to \infty$*,*

$$\frac{n^2}{m(n-m)} V_m = \int_0^{\tau_0} w^2(t) \frac{(1-F_1(t))(1-F_2(t))(1-G_1(t))(1-G_2(t))}{(1-H_\gamma(t))^2} \,\mathrm{d}R_\gamma(t) + o_{\mathrm{P}}\Big(\frac{1}{b_n}\Big), \tag{5.42}$$

where $H_\gamma(t)$ *and* $R_\gamma(t)$ *are defined in* (1.7) *and* (1.11)*.*
Assume $0 < m_c = m = n$*. Then we have, as* $n \to \infty$*,*

$$\frac{n^2}{m(n-m)} V_m = \int_0^{\tau_0} w^2(t) \,\mathrm{d}R(t) + o_{\mathrm{P}}\Big(\frac{1}{b_n}\Big), \tag{5.43}$$

where $R(t)$ *is defined in* (1.13)*.*

PROOF. To obtain the terms (5.40) and (5.41), we apply the same steps as in the proof of Lemma 5.7, but we use for approximation only Lemma 5.2.
The assertions (5.42) and (5.43) follow directly from (5.41). □

LEMMA 5.8. *(local alternatives) Let the assumptions (A.2) – (A.4) and the condition* (5.1) *with* $b_n = 1/|A_n|$ *be satisfied. Assume* $0 < m_c < m < n$*. Then we have, as* $n \to \infty$*,*

$$\frac{n^2}{k(n-k)} V_k = J(\tau_0)\,(1 + o_{\mathrm{P}}(1))$$

uniformly in $\log\log n < k < n - \log\log n$*, where* $J(\tau_0)$ *is given by* (3.7)*.*

PROOF. The proof is quite close to that of Lemma 5.7. By Corollary 5.1 we obtain, as $n \to \infty$,

$$\begin{aligned}
\frac{n^2}{k(n-k)} V_k &= \int_0^{\tau_0} w_n^2(t) \frac{\frac{1}{k}\Big(\sum_{j=1}^k Y_j(t)\Big) \frac{1}{n-k}\Big(\sum_{j=k+1}^n Y_j(t)\Big)}{\Big(\frac{1}{n}\sum_{j=1}^n Y_j(t)\Big)^2} \,\mathrm{d}\frac{N(t)}{n} \\
&= \int_0^{\tau_0} w_n^2(t) \frac{\frac{1}{k}\Big(\sum_{j=1}^k (1-H(t))\Big) \frac{1}{n-k}\Big(\sum_{j=k+1}^n (1-H(t))\Big)}{\Big(\frac{1}{n}\sum_{j=1}^n (1-H(t))\Big)^2} \,\mathrm{d}\frac{N(t)}{n}\,(1+o_{\mathrm{P}}(1)) \\
&= \int_0^{\tau_0} w_n^2(t)\,\mathrm{d}\frac{N(t)}{n}\,(1+o_{\mathrm{P}}(1)), \qquad (5.44)
\end{aligned}$$

where $1-H(t) = (1-F(t))(1-G(t))$, and replacing $\int_0^{\tau_0} w^2(t)\,\mathrm{d}R_{\eta,\gamma}(t)$ by $J(\tau_0)$ in (5.18), we obtain, as $n \to \infty$,

$$\begin{aligned}
\left|\int_0^{\tau_0} w_n^2(t)\,\mathrm{d}\Big(\frac{N(t)}{n}\Big) - J(\tau_0)\right| &= \left|\int_0^{\tau_0} w_n^2(t)\,\mathrm{d}\Big(\frac{N(t)}{n}\Big) - \int_0^{\tau_0} w^2(t)\,(1-G(t))\,\mathrm{d}F(t)\right| \\
&= \left|\int_0^{\tau_0} w^2(t)\left(\mathrm{d}\Big(\frac{N(t)}{n}\Big) - (1-G(t))\,\mathrm{d}F(t)\right)\right| + o_{\mathrm{P}}(|A_n|) \quad (5.45)
\end{aligned}$$

since $b_n = 1/|A_n|$. By Corollary 5.4 with $v(t) = w^2(t)$ we have, as $n \to \infty$,

$$\left|\int_0^{\tau_0} w^2(t)\,\mathrm{d}\Big(\frac{N(t)}{n}\Big) - \int_0^{\tau_0} w^2(t)\,(1-G(t))\,\mathrm{d}F(t)\right| = o_{\mathrm{P}}(1). \qquad (5.46)$$

The results (5.44) – (5.46) imply our assertion. □

7. Simple Linear Rank Statistics of S_k

Consider the simple linear rank statistics

$$T_{0,k}(\boldsymbol{b}) = \sum_{j=1}^k (b_n(Q_j) - \bar{b}_n), \quad k = 1, 2, \ldots, n,$$

where $\boldsymbol{Q} = (Q_1, Q_2, \ldots, Q_n)$ denotes a random permutation of $(1, 2, \ldots, n)$.
Further, $b_n(1), b_n(2), \ldots, b_n(n)$ are scores and

$$\bar{b}_n = \frac{1}{n}\sum_{j=1}^n b_n(j), \qquad \sigma_n^2(\boldsymbol{b}) = \frac{1}{n-1}\sum_{j=1}^n (b_n(j) - \bar{b}_n)^2.$$

THEOREM 5.1. *Let $(Q_1, Q_2, \ldots, Q_n)$ be a random permutation of $(1, 2, \ldots, n)$. Let the scores $b_n(1), b_n(2), \ldots, b_n(n)$ satisfy*

$$\liminf_{n\to\infty} \frac{1}{n}\sum_{j=1}^n (b_n(j) - \bar{b}_n)^2 \geq D_1, \qquad (5.47)$$

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{j=1}^n |b_n(j) - \bar{b}_n|^{2+u} \leq D_2 \qquad (5.48)$$

for some positive constants D_1, D_2 and u. Then, as $n \to \infty$, for all $y \in \mathbb{R}$ we have

$$\mathrm{P}\left(d_1(\log n) \max_{1\leq k<n} \sqrt{\frac{n}{k(n-k)}}\frac{|T_{0,k}(\boldsymbol{b})|}{\sigma_n(\boldsymbol{b})} \leq y + d_2(\log n)\right) \to \mathrm{e}^{-2\mathrm{e}^{-y}},$$

where $d_1(t)$ *and* $d_2(t)$ *are defined in* (1.19)*.*

PROOF. The assertion is a direct consequence of Theorem 2 in Hušková [**21**]. □

Further, consider the simple linear rank statistics

$$T_{1,k}(\boldsymbol{b}) = \sum_{j=1}^{k}(b_{m_c}(Q_j) - \bar{b}_{m_c}), \quad k = 1, 2, \dots, m_c, \tag{5.49}$$

and

$$T_{2,k}(\boldsymbol{b}) = \sum_{j=m_c+1}^{k}(b_{m-m_c}(Q_j) - \bar{b}_{m-m_c}), \quad k = m_c+1, m_c+2, \dots, m, \tag{5.50}$$

where $\boldsymbol{Q_{m_c}} = (Q_1, Q_2, \dots, Q_{m_c})$ and $\boldsymbol{Q_{m-m_c}} = (Q_{m_c+1}, Q_{m_c+2}, \dots Q_m)$ are random permutations of $(1, 2, \dots, m_c)$ and $(m_c+1, m_c+2, \dots, m)$, respectively.
Further, $b_{m_c}(1), b_{m_c}(2), \dots, b_{m_c}(m_c)$ and $b_{m-m_c}(m_c+1), b_{m-m_c}(m_c+2), \dots, b_{m-m_c}(m)$ are scores and

$$\bar{b}_{m_c} = \frac{1}{m_c}\sum_{j=1}^{m_c} b_{m_c}(j), \qquad \bar{b}_{m-m_c} = \frac{1}{m-m_c}\sum_{j=m_c+1}^{m} b_{m-m_c}(j).$$

Put

$$\sigma^2_{m_c}(\boldsymbol{b}) = \frac{1}{m_c-1}\sum_{j=1}^{m_c}(b_{m_c}(j) - \bar{b}_{m_c})^2,$$

$$\sigma^2_{m-m_c}(\boldsymbol{b}) = \frac{1}{m-m_c-1}\sum_{j=m_c+1}^{m}(b_{m-m_c}(j) - \bar{b}_{m-m_c})^2.$$

THEOREM 5.2. *Assume* $0 < m_c < m \le n$*.*

(1) *Let* $(Q_1, Q_2, \dots, Q_{m_c})$ *be a random permutation of* $(1, 2, \dots, m_c)$ *and let the scores* $b_{m_c}(1), b_{m_c}(2), \dots, b_{m_c}(m_c)$ *satisfy*

$$\liminf_{n\to\infty} \frac{1}{n}\sum_{j=1}^{m_c}(b_{m_c}(j) - \bar{b}_{m_c})^2 \ge D_1^*, \tag{5.51}$$

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{j=1}^{m_c}|b_{m_c}(j) - \bar{b}_{m_c}|^{2+u^*} \le D_2^* \tag{5.52}$$

for some positive constants D_1^**,* D_2^* *and* u^**. Then, as* $n \to \infty$*, for all* $y \in \mathbb{R}$ *we have*

$$\mathrm{P}\Big(d_1(\log m_c) \max_{1\le k<m_c} \sqrt{\frac{m_c}{k(m_c-k)}}\frac{|T_{1,k}(\boldsymbol{b})|}{\sigma_{m_c}(\boldsymbol{b})} \le y + d_2(\log m_c)\Big) \to \mathrm{e}^{-2\mathrm{e}^{-y}}$$

with $d_1(t)$ *and* $d_2(t)$ *given by* (1.19) *and*

$$\mathrm{P}\Big(\max_{1\le k<m_c v} \frac{|T_{1,k}(\boldsymbol{b})|}{\sqrt{m_c}\,\sigma_{m_c}(\boldsymbol{b})} \le y\Big) \to \mathrm{P}\Big(\max_{0\le t<v}|B(t)| \le y\Big), \quad v \in (0, 1],$$

where $\{B(t), t \in [0,1]\}$ *denotes a Brownian bridge. If moreover, as* $n \to \infty$

$$\frac{D}{n} \to 0, \qquad \frac{n^{2/(2+u^*)}\log n}{D} \to 0, \tag{5.53}$$

then for all $y \in \mathbb{R}$ we have, as $n \to \infty$,

$$\mathrm{P}\left(d_1\Big(\frac{m_c}{D}\Big) \max_{D<k<m_c-D} \frac{|T_{1,k+D}(\boldsymbol{b}) - 2T_{1,k}(\boldsymbol{b}) + T_{1,k-D}(\boldsymbol{b})|}{\sqrt{2D}\,\sigma_{m_c}(\boldsymbol{b})} \le y + d_2\Big(\frac{m_c}{D}\Big) + \log\Big(\frac{3}{2}\Big)\right) \to \mathrm{e}^{-2\mathrm{e}^{-y}}.$$

(2) *Let $(Q_{m_c+1}, Q_{m_c+2}, \dots, Q_m)$ be a random permutation of $(m_c+1, m_c+2, \dots, m)$ and let the scores $b_{m-m_c}(m_c+1), b_{m-m_c}(m_c+2), \dots, b_{m-m_c}(m)$ satisfy*

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{j=m_c+1}^{m} (b_{m-m_c}(j) - \bar{b}_{m-m_c})^2 \ge D_1^*, \tag{5.54}$$

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{j=m_c+1}^{m} |b_{m-m_c}(j) - \bar{b}_{m-m_c}|^{2+u^*} \le D_2^* \tag{5.55}$$

for some positive constants D_1^*, D_2^* and u^*. Then, as $n \to \infty$, for all $y \in \mathbb{R}$ we have

$$\mathrm{P}\left(d_1(\log(m-m_c)) \max_{m_c+1\le k<m} \sqrt{\frac{m-m_c}{(k-m_c)(m-k)}} \frac{|T_{2,k}(\boldsymbol{b})|}{\sigma_{m-m_c}(\boldsymbol{b})} \le y + d_2(\log(m-m_c))\right) \to \mathrm{e}^{-2\mathrm{e}^{-y}}$$

and

$$\mathrm{P}\left(\max_{m_c+1\le k<mv} \frac{|T_{2,k}(\boldsymbol{b})|}{\sqrt{m-m_c}\,\sigma_{m-m_c}(\boldsymbol{b})} \le y\right) \to \mathrm{P}\left(\max_{0\le t<v} |B(t)| \le y\right), \quad v \in (0,1].$$

If moreover (5.53)*, as $n \to \infty$, is fulfilled, then for all $y \in \mathbb{R}$ we have, as $n \to \infty$,*

$$\mathrm{P}\left(d_1\Big(\frac{m-m_c}{D}\Big) \max_{m_c+D<k<m-D} \frac{|T_{2,k+D}(\boldsymbol{b}) - 2T_{2,k}(\boldsymbol{b}) + T_{2,k-D}(\boldsymbol{b})|}{\sqrt{2D}\sigma_{m-m_c}(\boldsymbol{b})} \le y + d_2\Big(\frac{m-m_c}{D}\Big) + \log\Big(\frac{3}{2}\Big)\right) \to \mathrm{e}^{-2\mathrm{e}^{-y}}.$$

PROOF. The assertions follow from the Theorem 1 and Theorem 2 in Hušková [**21**]. □

LEMMA 5.9. *Let us denote by $\sigma\{Q_1, Q_2, \dots, Q_{k-1}\}$, $\sigma\{Q_{m_c+1}, Q_{m_c+2}, \dots, Q_{m_c+k-1}\}$ σ-fields generated by $Q_1, Q_2, \dots, Q_{k-1}$ and $Q_{m_c+1}, Q_{m_c+2}, \dots, Q_{m_c+k-1}$, respectively. The sequences*

$$\left\{\frac{T_{1,k}(\boldsymbol{b})}{m_c-k}, \sigma\{Q_1, Q_2, \dots, Q_{k-1}\};\ k = 1, \dots, m_c-1\}\right\},$$

and

$$\left\{\frac{T_{2,k}(\boldsymbol{b})}{m-k}, \sigma\{Q_{m_c+1}, Q_{m_c+2}, \dots, Q_{k-1}\};\ k = m_c+1, m_c+2, \dots, m-1\}\right\}$$

form the martingales, where $T_{1,k}(\boldsymbol{b})$ and $T_{2,k}(\boldsymbol{b})$ are given by (5.49) *and* (5.50)*, respectively.*

PROOF. Direct calculation yields

$$\begin{aligned}
&\mathrm{E}\left(\frac{1}{m_c-k} \sum_{i=1}^{k}\Big(b_{m_c}(Q_i) - \bar{b}_{m_c}\Big) \,\Big|\, Q_1, Q_2, \dots, Q_{k-1}\right) \\
&= \frac{1}{m_c-k} \sum_{i=1}^{k-1}\Big(b_{m_c}(Q_i) - \bar{b}_{m_c}\Big) + \frac{1}{m_c-k}\mathrm{E}\left(\Big(b_{m_c}(Q_i) - \bar{b}_{m_c}\Big) \,\Big|\, Q_1, Q_2, \dots, Q_{k-1}\right)
\end{aligned}$$

$$= \frac{1}{m_c - k} \sum_{i=1}^{k-1} \Big(b_{m_c}(Q_i) - \bar{b}_{m_c} \Big) + \frac{1}{m_c - k} \sum_{i=k}^{m_c} \Big(b_{m_c}(Q_i) - \bar{b}_{m_c} \Big) \frac{1}{m_c - k + 1}$$

with probability 1. Since

$$\sum_{i=1}^{m_c} \Big(b_{m_c}(Q_i) - \bar{b}_{m_c} \Big) = 0,$$

we conclude that

$$\sum_{i=k}^{m_c} \Big(b_{m_c}(Q_i) - \bar{b}_{m_c} \Big) = -\sum_{i=1}^{k-1} \Big(b_{m_c}(Q_i) - \bar{b}_{m_c} \Big)$$

and consequently

$$\begin{aligned} &\mathrm{E}\left(\frac{1}{m_c - k} \sum_{i=1}^{k} \Big(b_{m_c}(Q_i) - \bar{b}_{m_c} \Big) \,\Big|\, Q_1, Q_2, \ldots, Q_{k-1} \right) \\ &= \frac{1}{m_c - k} \sum_{i=1}^{k-1} \Big(b_{m_c}(Q_i) - \bar{b}_{m_c} \Big) - \frac{1}{(m_c - k)\,(m_c - k + 1)} \sum_{i=1}^{k-1} \Big(b_{m_c}(Q_i) - \bar{b}_{m_c} \Big) \\ &= \left(\frac{1}{m_c - k} - \frac{1}{(m_c - k)\,(m_c - k + 1)} \right) \sum_{i=1}^{k-1} \Big(b_{m_c}(Q_i) - \bar{b}_{m_c} \Big) \\ &= \frac{1}{m_c - (k-1)} \sum_{i=1}^{k-1} \Big(b_{m_c}(Q_i) - \bar{b}_{m_c} \Big) \end{aligned}$$

with probability 1. This means that $T_{1,k}(\boldsymbol{b})/(m_c - k)$, $k = 1, 2, \ldots, m_c - 1$ forms a martingale. The assertion on $T_{2,k}(\boldsymbol{b})/(m - k)$, $k = m_c + 1, m_c + 2, \ldots, m - 1$, can be shown in the same way therefore it is omitted. □

Recall that

$$S_k - \frac{k}{m_c} S_{m_c} \overset{\mathcal{D}}{=} \sum_{j=1}^{k} (a_n(Q_j) - \bar{a}_{m_c}), \quad 1 \le k \le m_c,$$

and

$$(S_k - S_{m_c}) - \frac{k - m_c}{m - m_c} (S_m - S_{m_c}) \overset{\mathcal{D}}{=} \sum_{j=m_c+1}^{k} (a_n(Q_j) - \bar{a}_{m-m_c}), \quad m_c < k \le m,$$

where $(Q_1, Q_2, \ldots Q_{m_c})$, $(Q_{m_c+1}, Q_{m_c+2}, \ldots Q_m)$ are random permutations of $(1, 2, \ldots, m_c)$ and $(m_c + 1, m_c + 2, \ldots, m)$. Moreover, given $(\boldsymbol{X}, \boldsymbol{\Delta}) = ((X_1, \Delta_1), (X_2, \Delta_2), \ldots, (X_n, \Delta_n))$, the random variables $\sum_{j=1}^{k} (a_n(Q_j) - \bar{a}_{m_c})$, $k = 1, 2, \ldots, m_c$, and $\sum_{j=m_c+1}^{k} (a_n(Q_j) - \bar{a}_{m-m_c})$, $k = m_c + 1, m_c + 2, \ldots, m$, can be viewed as simple linear rank statistics $T_{1,k}(\boldsymbol{b})$, $k = 1, 2, \ldots, m_c$, and $T_{2,k}(\boldsymbol{b})$, $k = m_c + 1, m_c + 2, \ldots, m$, respectively, where $b_{m_c}(j) = a_n(j)$ for $1 \le j \le m_c$ and $b_{m-m_c}(j) = a_n(j)$ for $m_c + 1 \le j \le m$. Thus, in the following, we apply given $(\boldsymbol{X}, \boldsymbol{\Delta})$ the results for linear rank statistics presented in Theorem 5.2 to S_k of the form (2.15).

LEMMA 5.10. *(local alternatives) Let the condition* (5.1) *for* $b_n = 1/|A_n|$ *be satisfied. Assume* $0 < m_c < m < n$. *Then, under the assumptions (A.2)–(A.4), we have, as* $n \to \infty$,

$$\max_{1 \le k < m_c} \left| S_k - \frac{k}{m_c} S_{m_c} \right| = O_{\mathrm{P}}\Big(\sqrt{m_c} \Big),$$

$$\max_{m_c + 1 \le k < m} \left| S_k - S_{m_c} - \frac{k - m_c}{m - m_c} (S_m - S_{m_c}) \right| = O_{\mathrm{P}}\Big(\sqrt{m} \Big)$$

and

$$\max_{1\le k<m_c}\sqrt{\frac{n}{k(n-k)}}\left|S_k-\frac{k}{m_c}\,S_{m_c}\right|=O_{\mathrm{P}}\left(\sqrt{\log\log m_c}\right),$$

$$\max_{m_c+1\le k<m}\sqrt{\frac{n}{k(n-k)}}\left|(S_k-S_{m_c})-\frac{k-m_c}{m-m_c}\,(S_m-S_{m_c})\right|=O_{\mathrm{P}}\left(\sqrt{\log\log m}\right),$$

where S_k is defined in (2.15).

PROOF. We have to verify the conditions (5.51), (5.52) or (5.54), (5.55), respectively, to use both parts of Theorem 5.2. We do that for convergence in probability. Corollary 5.12 asserts that, as $n\to\infty$,

$$\sigma^2_{m_c}(\boldsymbol{a})=\int_0^{\tau_0}w^2(t)\,(1-G(t))\,\mathrm{d}F(t)+o_{\mathrm{P}}(1)=J(\tau_0)+o_{\mathrm{P}}(1),\tag{5.56}$$

$$\sigma^2_{m-m_c}(\boldsymbol{a})=\int_0^{\tau_0}w^2(t)\,(1-G(t))\,\mathrm{d}F(t)+o_{\mathrm{P}}(1)=J(\tau_0)+o_{\mathrm{P}}(1).\tag{5.57}$$

Further, Lemma 5.4 gives that $\max_{1\le j\le n}|a_n(j)|=O_{\mathrm{P}}(1)$, as $n\to\infty$. Thus, from this and Corollary 5.10 we can infer, as $n\to\infty$,

$$\frac{1}{m_c}\sum_{j=1}^{m_c}\left|a_n(j)-\frac{S_{m_c}}{m_c}\right|^4=\sum_{i=0}^{4}\left\{\binom{4}{i}\frac{\sum_{j=1}^{m_c}(a_n(j))^{4-i}}{m_c}\left(-\frac{S_{m_c}}{m_c}\right)^i\right\}=O_{\mathrm{P}}(1),$$

$$\begin{aligned}\frac{1}{m-m_c}\sum_{j=m_c+1}^{m}\left|a_n(j)-\frac{S_m-S_{m_c}}{m-m_c}\right|^4&=\sum_{i=0}^{4}\left\{\binom{4}{i}\frac{\sum_{j=m_c+1}^{m}(a_n(j))^{4-i}}{m-m_c}\left(-\frac{S_m-S_{m_c}}{m-m_c}\right)^i\right\}\\&=O_{\mathrm{P}}(1).\end{aligned}$$

Since that and regarding that the term $J(\tau_0)$ in (5.56) and (5.57) is positive, we can apply Theorem 5.2.

By Theorem 5.2 we have, as $n\to\infty$,

$$\mathrm{P}\left(\max_{1\le k<m_c}\sqrt{\frac{m_c}{k(m_c-k)}}\left|\sum_{j=1}^{k}(a_n(Q_j)-\bar a_{m_c})\right|\le D_3\sqrt{\log\log m_c}\;\middle|\;(\boldsymbol{X},\boldsymbol{\Delta})\right)\xrightarrow{\mathrm{P}}1,$$

$$\mathrm{P}\left(\max_{1\le k<m_c}\left|\sum_{j=1}^{k}(a_n(Q_j)-\bar a_{m_c})\right|\le D_3^*\sqrt{m_c}\;\middle|\;(\boldsymbol{X},\boldsymbol{\Delta})\right)\xrightarrow{\mathrm{P}}1$$

for some $D_3,D_3^*>0$. Hence, the convergence holds also unconditionally, i.e.

$$\max_{1\le k<m_c}\sqrt{\frac{m_c}{k(m_c-k)}}\left|S_k-\frac{k}{m_c}\,S_{m_c}\right|=O_{\mathrm{P}}\left(\sqrt{\log\log m_c}\right),\tag{5.58}$$

$$\max_{1\le k<m_c}\left|S_k-\frac{k}{m_c}\,S_{m_c}\right|=O_{\mathrm{P}}\left(\sqrt{m_c}\right).$$

Analogously, we obtain, as $n\to\infty$,

$$\max_{m_c+1\le k<m}\sqrt{\frac{m-m_c}{(k-m_c)(m-k)}}\left|(S_k-S_{m_c})-\frac{k-m_c}{m-m_c}\,(S_m-S_{m_c})\right|=O_{\mathrm{P}}\left(\sqrt{\log\log m}\right)\tag{5.59}$$

and

$$\max_{m_c+1\le k<m}\left|(S_k-S_{m_c})-\frac{k-m_c}{m-m_c}\,(S_m-S_{m_c})\right|=O_{\mathrm{P}}\left(\sqrt{m}\right).$$

Further, by (5.58), as $n \to \infty$,

$$\max_{1\le k<m_c} \sqrt{\frac{n}{k(n-k)}} \left|S_k - \frac{k}{m_c} S_{m_c}\right| = \max_{1\le k<m_c} \sqrt{\frac{m_c}{k(m_c-k)}} \left|S_k - \frac{k}{m_c}\, S_{m_c}\right| \sqrt{\frac{n(m_c-k)}{m_c(n-k)}}$$
$$= \sqrt{\frac{m_c}{k(m_c-k)}} \left|S_k - \frac{k}{m_c}\, S_{m_c}\right| O(1) = O_{\mathrm{P}}\left(\sqrt{\log\log m_c}\right)$$

and by (5.59), as $n \to \infty$,

$$\max_{m_c+1\le k<m} \sqrt{\frac{n}{k(n-k)}} \left|(S_k - S_{m_c}) - \frac{k-m_c}{m-m_c}\,(S_m - S_{m_c})\right|$$
$$= \max_{m_c+1\le k<m} \sqrt{\frac{m-m_c}{(k-m_c)(m-k)}} \left|(S_k - S_{m_c}) - \frac{k-m_c}{m-m_c}\,(S_m - S_{m_c})\right|$$
$$\cdot \sqrt{\frac{n(k-m_c)(m-k)}{k(n-k)(m-m_c)}}$$
$$= \max_{m_c+1\le k<m} \sqrt{\frac{m-m_c}{(k-m_c)(m-k)}} \left|(S_k - S_{m_c}) - \frac{k-m_c}{m-m_c}\,(S_m - S_{m_c})\right| O(1)$$
$$= O_{\mathrm{P}}\left(\sqrt{\log\log m}\right).$$

□

Corollary 5.15. *Assume* $0 < m_c = m = n$. *Let* (5.1) *for* $b_n = 1$ *be satisfied. If* (2.12) *be fulfilled, then we have, as* $n \to \infty$,

$$\max_{1\le k\le(\log n)^\omega} \sqrt{\frac{n}{k(n-k)}}\, |S_k| = o_{\mathrm{P}}(\sqrt{\log\log n}),$$

where $\omega > 0$ *is arbitrary but fixed and* S_k *is defined in* (2.15).

Proof. We proceed in much the same way as in the proof of Lemma 5.10. We can apply Theorem 5.1, since Corollary 5.6 ensure the assumptions (5.47) and (5.48) for convergence in probability. Thus, repeating the idea of the previous proof when we derived (5.58) among other things, we receive in our case

$$\max_{1\le k\le(\log n)^\omega} \sqrt{\frac{n}{k(n-k)}}\, |S_k| = O_{\mathrm{P}}\left(\sqrt{\log\log(\log n)^\omega}\right) = o_{\mathrm{P}}(\sqrt{\log\log n}), \quad n \to \infty.$$

The proof is complete. □

8. Auxiliary Results

Here we summarize other important assertions which are needed in Chapter 3 of this thesis.

Lemma 5.11. *Let* $\{X_n, \mathcal{F}_n; n \ge 1\}$ *be a submartingale and* $\{c_n; n \ge 1\}$ *be a nonincreasing sequence of nonnegative numbers. Then for every* $\varepsilon > 0$,

$$\mathrm{P}\left(\max_{1\le k\le n} c_k\, X_k \ge \varepsilon\right) \le \frac{c_n^2 \operatorname{var} X_n + \sum_{i=1}^{n-1} (c_i - c_{i+1})^2 \operatorname{var} X_i}{\varepsilon^2}. \tag{5.60}$$

In particular, (5.60) *yields*

$$\mathrm{P}\left(\max_{1\le k\le n} X_k \ge \varepsilon\right) \le \frac{\operatorname{var} X_n}{\varepsilon^2}. \tag{5.61}$$

Proof. The assertion is the so-called Kolmogorov-Hájek-Rényi-Chow Inequality for submartingales, see e.g. Sen [**31**], p. 13. □

Lemma 5.12. *Denote by* $U_1, U_2, \ldots, U_n$ *i.i.d. random variables with the uniform distribution* $U(0,1)$. *There is a constant* C *such that*

$$\mathrm{P}\left(\sqrt{n}\sup_{0\le t\le 1}\left|\frac{1}{n}\sum_{j=1}^{n}(I\{U_j\le t\}-t)\right|>\varepsilon\right)\le C\exp\{-2\varepsilon^2\} \tag{5.62}$$

for all $\varepsilon>0$. *We can choose* $C=2$ *in* (5.62).

Proof. The proof can be found in Csörgő, Horváth [**9**], Chapter 3, Section 3.1, Lemma 1.4. □

First, recall the decomposition (3.19) for $k=m_c+1, m_c+2, \ldots, m$

$$\Big((S_k-S_{m_c})-\frac{k-m_c}{m-m_c}(S_m-S_{m_c})+(S_{m_c}+\frac{k-m_c}{m-m_c}(S_m-S_{m_c}))\Big)^2-S_m^2=U_{k1,1}+U_{k2,1}+U_{k3,1},$$

where

$$U_{k1,1}=\Big((S_k-S_{m_c})-\frac{k-m_c}{m-m_c}(S_m-S_{m_c})\Big)^2 \tag{5.63}$$

$$U_{k2,1}=2\Big((S_k-S_{m_c})-\frac{k-m_c}{m-m_c}(S_m-S_{m_c})\Big)\Big(S_{m_c}+\frac{k-m_c}{m-m_c}(S_m-S_{m_c})\Big), \tag{5.64}$$

$$U_{k3,1}=\left(S_{m_c}+\frac{k-m_c}{m-m_c}(S_m-S_{m_c})\right)^2-S_m^2. \tag{5.65}$$

It will be shown that $U_{k3,1}$ is the dominating term in (3.19) which is needed in the proof of Theorem 3.1.

Lemma 5.13. *Under the assumptions of Theorem 3.1 we have*

$$\lim_{C\to\infty}\limsup_{n\to\infty}\mathrm{P}\left(\max_{m-nh<k\le m-C/A_n^2}\frac{|U_{k1,1}|}{|U_{k3,1}|}\ge\varepsilon\right)=0,\quad\forall\,\varepsilon>0, \tag{5.66}$$

$$\lim_{C\to\infty}\limsup_{n\to\infty}\mathrm{P}\left(\max_{m-nh<k\le m-C/A_n^2}\frac{|U_{k2,1}|}{|U_{k3,1}|}\ge\varepsilon\right)=0,\quad\forall\,\varepsilon>0, \tag{5.67}$$

$$\max_{m-nh<k\le m-C/A_n^2}U_{k3,1}\xrightarrow{\mathrm{P}}-\infty,\quad\forall\,C>0,\quad \text{as } n\to\infty, \tag{5.68}$$

with $h\in(0,\gamma-\eta)$ *denoting an arbitrary fixed constant.*

Proof. By Corollary 5.10, we obtain, as $n\to\infty$,

$$\frac{k-m_c}{m-m_c}(S_m-S_{m_c})=(k-m_c)\,\frac{n-m}{n}A_n(1+o_{\mathrm{P}}(1)),\quad m_c+1\le k\le m$$

and further, as $n\to\infty$,

$$S_{m_c}=m_c\,\frac{n-m}{n}\,A_n\,(1+o_{\mathrm{P}}(1)),$$
$$S_m=m\,\frac{n-m}{n}\,A_n\,(1+o_{\mathrm{P}}(1))$$

and consequently, as $n\to\infty$,

$$U_{k2,1}=2\Big((S_k-S_{m_c})-\frac{k-m_c}{m-m_c}(S_m-S_{m_c})\Big)\Big(S_{m_c}+\frac{k-m_c}{m-m_c}(S_m-S_{m_c})\Big)$$

$$
\begin{aligned}
&= 2\Big((S_k - S_{m_c}) - \frac{k - m_c}{m - m_c}(S_m - S_{m_c})\Big) \\
&\quad \left(m_c \, \frac{n-m}{n}\, A_n \,(1 + o_{\mathrm{P}}(1)) + (k - m_c)\, \frac{n-m}{n}\, A_n \,(1 + o_{\mathrm{P}}(1))\right) \\
&= 2\Big((S_k - S_{m_c}) - \frac{k - m_c}{m - m_c}(S_m - S_{m_c})\Big) \left(k \, \frac{n-m}{n}\, A_n + \frac{n-m}{n}\, A_n \left(m_c \, o_{\mathrm{P}}(1) + k \, o_{\mathrm{P}}(1)\right)\right) \\
&= 2\Big((S_k - S_{m_c}) - \frac{k - m_c}{m - m_c}(S_m - S_{m_c})\Big)\, k \, \frac{n-m}{n}\, A_n (1 + o_{\mathrm{P}}(1)) \qquad (5.69)
\end{aligned}
$$

uniformly in $m - nh < k \le m - C/A_n^2$ with $C > 0$. By direct calculation

$$
\begin{aligned}
U_{k3,1} &= \left(S_{m_c} + \frac{k - m_c}{m - m_c}\,(S_m - S_{m_c})\right)^2 - S_m^2 \\
&= \left(S_{m_c} + \frac{k - m_c}{m - m_c}\,(S_m - S_{m_c}) - S_m\right)\left(S_{m_c} + \frac{k - m_c}{m - m_c}\,(S_m - S_{m_c}) + S_m\right) \\
&= \left(S_{m_c}\, \frac{m - m_c - k + m_c}{m - m_c} + S_m \, \frac{k - m_c - m + m_c}{m - m_c}\right) \\
&\quad \left(S_{m_c}\, \frac{m - m_c - k + m_c}{m - m_c} + S_m \, \frac{k - m_c + m - m_c}{m - m_c}\right) \\
&= \left(S_{m_c}\, \frac{m-k}{m - m_c} + S_m \, \frac{k-m}{m - m_c}\right)\left(S_{m_c}\, \frac{m-k}{m - m_c} + S_m \, \frac{k + m - 2m_c}{m - m_c}\right) \\
&= (S_{m_c} - S_m)\, \frac{m-k}{m-m_c}\left((S_{m_c} - S_m)\, \frac{m-k}{m-m_c} + 2\, S_m\right) \\
&= -(m-k)\, \frac{n-m}{n}\, A_n (1 + o_{\mathrm{P}}(1)) \\
&\quad \left(-(m-k)\, \frac{n-m}{n}\, A_n \,(1 + o_{\mathrm{P}}(1)) + 2m \, \frac{n-m}{n}\, A_n \,(1 + o_{\mathrm{P}}(1))\right) \\
&= (m-k)^2 \left(\frac{n-m}{n}\right)^2 A_n^2 \,(1 + o_{\mathrm{P}}(1)) - 2m(m-k)\left(\frac{n-m}{n}\right)^2 A_n^2 \,(1 + o_{\mathrm{P}}(1)) \\
&= (m-k)(m-k-2m)\left(\frac{n-m}{n}\right)^2 A_n^2 \\
&\quad + (m-k)\left(\frac{n-m}{n}\right)^2 A_n^2 \,((m-k)\, o_{\mathrm{P}}(1) + m o_{\mathrm{P}}(1)) \\
&= -(m-k)(m+k)\left(\frac{n-m}{n}\right)^2 A_n^2 + m(m-k)\left(\frac{n-m}{n}\right)^2 A_n^2 \, o_{\mathrm{P}}(1) \qquad (5.70)
\end{aligned}
$$

uniformly in $m - nh + 1 \le k \le m - C/A_n^2$ and it can be seen

$$
\begin{aligned}
&-(m-k)(2m - C/A_n^2) \le -(m-k)(m+k) \le -(m-k)(2m - nh + 1) \\
&-2\gamma\,(m-k)n\,(1 + o(1)) \le -(m-k)(m+k) \le -(2\gamma - h)\,(m-k)n\,(1 + o(1)) \qquad (5.71)
\end{aligned}
$$

for $m - nh + 1 \le k \le m - C/A_n^2$. By (5.70) and (5.71), we obtain, as $n \to \infty$,

$$
\begin{aligned}
&-2\gamma\,(m-k)n \left(\frac{n-m}{n}\right)^2 A_n^2 \,(1 + o_{\mathrm{P}}(1)) + \gamma(m-k)n\left(\frac{n-m}{n}\right)^2 A_n^2 \, o_{\mathrm{P}}(1) \le U_{k3,1} \\
&\qquad \le -(2\gamma - h)\,(m-k)n \left(\frac{n-m}{n}\right)^2 A_n^2 \,(1 + o_{\mathrm{P}}(1)) + \gamma(m-k)n\left(\frac{n-m}{n}\right)^2 A_n^2 \, o_{\mathrm{P}}(1)
\end{aligned}
$$

and comparing the terms by $o_{\mathrm{P}}(1)$ we get, as $n \to \infty$,

$$\begin{aligned} -2\gamma(m-k)n\Big(\frac{n-m}{n}\Big)^2 A_n^2\,(1+o_{\mathrm{P}}(1)) &\le U_{k3,1} \le -(2\gamma-h)(m-k)n\Big(\frac{n-m}{n}\Big)^2 A_n^2\,(1+o_{\mathrm{P}}(1)) \\ -2\gamma(1-\gamma)^2\,(m-k)n\,A_n^2\,(1+o_{\mathrm{P}}(1)) &\le U_{k3,1} \le -(2\gamma-h)(1-\gamma)^2\,(m-k)n\,A_n^2\,(1+o_{\mathrm{P}}(1)) \end{aligned} \tag{5.72}$$

for $m-nh+1 \le k \le m-C/A_n^2$. Since the term $-(m-k)$ in (5.72) is increasing in k, we get, as $n \to \infty$,

$$\max_{m-nh<k\le m-C/A_n^2} U_{k3,1} \;\ge\; -2\gamma(1-\gamma)^2\,\frac{C}{A_n^2}\,n\,A_n^2\,(1+o_{\mathrm{P}}(1)) \;=\; -2\gamma(1-\gamma)^2\,Cn\,(1+o_{\mathrm{P}}(1)).$$

This gives (5.68).

By (5.63) and Lemma 5.10 we obtain, as $n \to \infty$,

$$\begin{aligned} &\max_{m-nh<k\le m-C/A_n^2} \frac{|U_{k1,1}|}{n(m-k)A_n^2} \\ &\qquad = \frac{1}{(C/A_n^2)\,A_n^2}\,O_{\mathrm{P}}\left(\max_{m_c+1\le k<m}\left(\frac{(S_k-S_{m_c})-\frac{k-m_c}{m-m_c}\,(S_m-S_{m_c})}{\sqrt{n}}\right)^2\right) = \frac{O_{\mathrm{P}}(1)}{C} \end{aligned}$$

and hence by (5.72), we have, as $n \to \infty$

$$\max_{m-nh<k\le m-C/A_n^2} \frac{|U_{k1,1}|}{|U_{k3,1}|} = \frac{O_{\mathrm{P}}(1)}{C}, \quad C>0,$$

which concludes the assertion (5.66).

Further, by (5.69) we have, as $n \to \infty$,

$$\begin{aligned} &\max_{m-nh<k\le m-C/A_n^2} \frac{|U_{k2,1}|}{n(m-k)A_n^2} \\ &\qquad = 2\,\frac{m(n-m)}{n^2}\,O_{\mathrm{P}}\left(\max_{m_c+1\le k\le m-C/A_n^2} \frac{|(S_k-S_{m_c})-\frac{k-m_c}{m-m_c}(S_m-S_{m_c})|}{(m-k)|A_n|}\right). \end{aligned} \tag{5.73}$$

Since $(X_{m_c+1},\Delta_{m_c+1}),(X_{m_c+2},\Delta_{m_c+2}),\dots,(X_m,\Delta_m)$ are i.i.d. pairs of random variables,

$$(S_k-S_{m_c})-\frac{k-m_c}{m-m_c}\,(S_m-S_{m_c}) \stackrel{\mathcal{D}}{=} \sum_{j=m_c+1}^{k}(a_n(Q_j)-\bar a_{m-m_c}), \quad m_c+1\le k\le m,$$

where $(Q_{m_c+1},Q_{m_c+2},\dots Q_m)$ is a random permutation of $(m_c+1,m_c+2,\dots,m)$. Hence, given $(\boldsymbol{X},\boldsymbol{\Delta})$, properties of simple linear rank statistics can be used because of substituting the ranks by a random permutation. By Lemma 5.9 given $(\boldsymbol{X},\boldsymbol{\Delta})$ the sequence

$$\left\{\frac{\sum_{i=m_c+1}^{k}\Big(a_n(Q_i)-\bar a_{m-m_c}\Big)}{m-k}, \sigma\{Q_{m_c+1},Q_{m_c+2},\dots,Q_{k-1}\}; k=m_c+1,m_c+2,\dots,m-1\right\}$$

forms a martingale. Using the Kolmogorov inequality (5.61), we receive $\forall \varepsilon > 0$

$$\mathrm{P}\left(\max_{m_c+1 \le k \le m - C/A_n^2} \frac{|(S_k - S_{m_c}) - \frac{k-m_c}{m-m_c}(S_m - S_{m_c})|}{(m-k)|A_n|} \ge \varepsilon | (\boldsymbol{X}, \boldsymbol{\Delta}) \right)$$

$$\le \frac{1}{A_n^2 \varepsilon^2} \frac{\mathrm{var}(S_{m-C/A_n^2} - S_{m_c} - \frac{m-C/A_n^2-m_c}{m-m_c}(S_m - S_{m_c}))}{(C/A_n^2)^2}$$

$$= \frac{1}{A_n^2\varepsilon^2} \frac{(m-m_c-C/A_n^2)\, C/A_n^2}{(m-m_c)\, C^2/A_n^4}\, \sigma^2_{m-m_c} = \frac{(m-m_c-C/A_n^2)}{(m-m_c)C\varepsilon^2}\, \sigma^2_{m-m_c}(\boldsymbol{a}), \quad (5.74)$$

where

$$\sigma^2_{m-m_c}(\boldsymbol{a}) = \frac{1}{m-m_c-1} \sum_{i=m_c+1}^{m} (a_n(i) - \bar{a}_{m-m_c})^2 .$$

Putting (5.74) and the assertion of Corollary 5.12 for $\sigma^2_{m-m_c}(\boldsymbol{a})$ together, we obtain for convergence in probability

$$\lim_{C\to\infty} \limsup_{n\to\infty} \mathrm{P}\left(\max_{m_c+1 \le k \le m-C/A_n^2} \frac{|(S_k - S_{m_c}) - \frac{k-m_c}{m-m_c}(S_m - S_{m_c})|}{(m-k)|A_n|} \ge \varepsilon \,\Big|\, (\boldsymbol{X}, \boldsymbol{\Delta}) \right) = 0 \quad (5.75)$$

for all $\varepsilon > 0$. Now, it can be seen from (5.72) and (5.73) that

$$\lim_{C\to\infty} \limsup_{n\to\infty} \mathrm{P}\left(\max_{m-nh<k\le m-C/A_n^2} \frac{|U_{k2,1}|}{|U_{k3,1}|} \ge \varepsilon \,\Big|\, (\boldsymbol{X}, \boldsymbol{\Delta}) \right) = 0, \quad \forall\, \varepsilon > 0,$$

for convergence in probability. It is clear that the convergence holds also unconditionally, i.e. we prove the assertion (5.67). Thus, the whole proof is finished. □

Second, recall the decomposition (3.29) for $k = m_c+1, m_c+2, \dots, m$

$$\frac{\left((S_k - S_{m_c}) - \frac{k-m_c}{m-m_c}(S_m - S_{m_c}) + (S_{m_c} + \frac{k-m_c}{m-m_c}(S_m - S_{m_c}))\right)^2}{n\,V_k} - \frac{S_m^2}{n\,V_m} = U_{k1,2} + U_{k2,2} + U_{k3,2},$$

where

$$U_{k1,2} = \frac{U_{k1,1}}{n\,V_k}, \qquad U_{k2,2} = \frac{U_{k2,1}}{n\,V_k}, \qquad U_{k3,2} = \frac{U_{k3,1}}{n\,V_k} + \frac{S_m^2}{nV_k}\left(1 - \frac{V_k}{V_m}\right), \quad (5.76)$$

where $U_{k1,1}$, $U_{k2,1}$ and $U_{k3,1}$ are given by (5.63) – (5.65). It will be shown that $U_{k3,2}$ is the dominating term in (3.29) which is needed in the proof of Theorem 3.2.

Lemma 5.14. *Under the assumptions of Theorem 3.2 we have*

$$\lim_{n\to\infty} \mathrm{P}\left(\max_{m-nh<k\le m-C/A_n^2} \frac{|U_{k1,2}|}{|U_{k3,2}|} \ge \varepsilon \right) = 0, \quad \forall\, \varepsilon > 0, \quad (5.77)$$

$$\lim_{C\to\infty} \limsup_{n\to\infty} \mathrm{P}\left(\max_{m-nh<k\le m-C/A_n^2} \frac{|U_{k2,2}|}{|U_{k3,2}|} \ge \varepsilon \right) = 0, \quad \forall\, \varepsilon > 0, \quad (5.78)$$

$$\lim_{C\to\infty} \limsup_{n\to\infty} \max_{m-nh<k\le m-C/A_n^2} U_{k3,2} = -\infty \quad \textit{for convergence in probability}, \quad (5.79)$$

where $h \in (0, \gamma - \eta)$ is an arbitrary fixed constant.

Proof. We compute

$$V_m - V_k = \frac{1}{n}\int_0^{\tau_0} w_n^2(t)\,\frac{\sum_{j=1}^m Y_j(t)\sum_{j=m+1}^n Y_j(t) - \sum_{j=1}^k Y_j(t)\sum_{j=k+1}^n Y_j(t)}{Y^2(t)}\,\mathrm{d}N(t)$$
$$= \frac{1}{n}\int_0^{\tau_0} \frac{w_n^2(t)}{Y^2(t)}\left\{\sum_{j=1}^m Y_j(t)\Big(\sum_{j=m+1}^n Y_j(t) - \sum_{j=k+1}^n Y_j(t)\Big) + \sum_{j=k+1}^n Y_j(t)\Big(\sum_{j=1}^m Y_j(t) - \sum_{j=1}^k Y_j(t)\Big)\right\}\mathrm{d}N(t)$$
$$= \frac{1}{n}\int_0^{\tau_0} \frac{w_n^2(t)}{Y^2(t)}\left\{-\sum_{j=1}^m Y_j(t)\sum_{j=k+1}^m Y_j(t) + \sum_{j=k+1}^n Y_j(t)\sum_{j=k+1}^m Y_j(t)\right\}\mathrm{d}N(t)$$
$$= \frac{1}{n}\int_0^{\tau_0} \frac{w_n^2(t)}{Y^2(t)}\left\{\sum_{j=k+1}^m Y_j(t)\Big(\sum_{j=k+1}^n Y_j(t) - \sum_{j=1}^m Y_j(t)\Big)\right\}\mathrm{d}N(t)$$
$$= \frac{1}{n}\int_0^{\tau_0} w_n^2(t)\,\frac{\sum_{j=k+1}^m Y_j(t)\Big(\sum_{j=m+1}^n Y_j(t) - \sum_{j=1}^k Y_j(t)\Big)}{Y^2(t)}\,\mathrm{d}N(t)$$
$$= \int_0^{\tau_0} w_n^2(t)\,\frac{\sum_{j=k+1}^m Y_j(t)\,\sum_{j=m+1}^n Y_j(t)}{Y^2(t)}\,\mathrm{d}\Big(\frac{N(t)}{n}\Big) - \int_0^{\tau_0} w_n^2(t)\,\frac{\sum_{j=k+1}^m Y_j(t)\,\sum_{j=1}^k Y_j(t)}{Y^2(t)}\,\mathrm{d}\Big(\frac{N(t)}{n}\Big)$$

for $m - nh < k \le m - C/A_n^2$ with $C > 0$ and further we proceed in the same way as in the proof of Lemma 5.8 and that is why we omit individual steps and we present only the final form, i.e. we get, as $n \to \infty$,

$$\begin{aligned}
V_m - V_k &= \frac{(m-k)(n-m)}{n^2}\,J(\tau_0)(1 + o_{\mathrm{P}}(1)) - \frac{(m-k)k}{n^2}\,J(\tau_0)(1 + o_{\mathrm{P}}(1)) \\
&= \frac{(m-k)(n-m-k)}{n^2}\,J(\tau_0) + \frac{m-k}{n^2}\,J(\tau_0)\,((n-m)\,o_{\mathrm{P}}(1) + k\,o_{\mathrm{P}}(1)) \\
&= \frac{(m-k)(n-m-k)}{n^2}\,J(\tau_0) + \frac{(m-k)(n-m)}{n^2}\,J(\tau_0)\,o_{\mathrm{P}}(1) \\
&= \frac{m-k}{n^2}\,J(\tau_0)\,((n-m-k) + (n-m)\,o_{\mathrm{P}}(1))
\end{aligned} \tag{5.80}$$

uniformly in $m - nh < k \le m - C/A_n^2$, where $J(\tau_0)$ is defined in (3.7). By Corollary 5.10 and Lemma 5.8 we obtain, as $n \to \infty$,

$$\frac{S_m^2}{V_m} = \frac{m^2\left(\frac{n-m}{n}\right)^2 A_n^2\,(1 + o_{\mathrm{P}}(1))}{\frac{m(n-m)}{n^2}\,J(\tau_0)\,(1 + o_{\mathrm{P}}(1))} = m(n-m)\,\frac{A_n^2}{J(\tau_0)}\,(1 + o_{\mathrm{P}}(1)). \tag{5.81}$$

Thus, combination of (5.80) and (5.81) implies, as $n \to \infty$,

$$\begin{aligned}
S_m^2\left(1 - \frac{V_k}{V_m}\right) &= \frac{S_m^2}{V_m}\,(V_m - V_k) \\
&= \frac{m(n-m)}{n^2}\,(m-k)\,A_n^2\,(1 + o_{\mathrm{P}}(1))\,((n-m-k) + (n-m)\,o_{\mathrm{P}}(1)) \\
&= \frac{m(n-m)}{n^2}\,(m-k)(m-m-k)\,A_n^2 + m(m-k)\left(\frac{n-m}{n}\right)^2 A_n^2\,o_{\mathrm{P}}(1)
\end{aligned}$$

uniformly in $m - nh < k \le m - C/A_n^2$ and by (5.70) we get

$$\begin{aligned} nV_k\, U_{k3,2} &= U_{k3,1} + S_m^2 \left(1 - \frac{V_k}{V_m}\right) \\ &= -(m^2-k^2)\left(\frac{n-m}{n}\right)^2 A_n^2 + m(n-m)\frac{(m-k)(n-m-k)}{n^2}\, A_n^2 \\ &\quad + m(m-k)\left(\frac{n-m}{n}\right)^2 A_n^2\, o_{\mathrm{P}}(1) \\ &= -(m-k)\,\frac{n-m}{n^2} A_n^2\, ((m+k)(n-m) - m(n-m-k)) + m(m-k)\,\frac{n-m}{n} A_n^2\, o_{\mathrm{P}}(1) \\ &= -k(m-k)\,\frac{n-m}{n} A_n^2 + \frac{m}{k}\, k(m-k)\,\frac{n-m}{n} A_n^2\, o_{\mathrm{P}}(1) \\ &= -k(m-k)\,\frac{n-m}{n} A_n^2\, (1 + o_{\mathrm{P}}(1)) \end{aligned} \tag{5.82}$$

uniformly in $m - nh < k \le m - C/A_n^2$ since

$$1 + o(1) = \frac{m}{m - C/A_n^2} \le \frac{m}{k} \le \frac{m}{m - nh + 1} = \frac{\gamma}{\gamma - h} + o(1).$$

By Lemma 5.10 and (5.63) with (5.76), we get, as $n \to \infty$,

$$\begin{aligned} &\max_{m-nh<k\le m-C/A_n^2} \frac{nV_k\, |U_{k1,2}|}{k(m-k)A_n^2} = \max_{m-nh<k\le m-C/A_n^2} \frac{|U_{k1,1}|}{k(m-k)A_n^2} \\ &= \frac{1}{m\, A_n^2} \max_{m-nh<k\le m-C/A_n^2} \frac{m}{k(m-k)} \left((S_k - S_{m_c}) - \frac{k-m_c}{m-m_c}(S_m - S_{m_c})\right)^2 \\ &= O_{\mathrm{P}}\left(\frac{1}{m\, A_n^2} \max_{m_c+1\le k<m} \frac{m-m_c}{(k-m_c)(m-m_c-k)} \left((S_k - S_{m_c}) - \frac{k-m_c}{m-m_c}(S_m - S_{m_c})\right)^2\right) \\ &= O_{\mathrm{P}}\left(\frac{\log\log m}{m\, A_n^2}\right) \end{aligned}$$

and hence by (5.82), we have, as $n \to \infty$,

$$\max_{m-nh<k\le m-C/A_n^2} \frac{|U_{k1,2}|}{|U_{k3,2}|} = O_{\mathrm{P}}\left(\frac{\log\log m}{m\, A_n^2}\right),$$

which concludes the assertion (5.77) in view of (3.5).

The approximation (5.69) gives, as $n \to \infty$,

$$nV_k\, U_{k2,2} = U_{k2,1} = 2\left((S_k - S_{m_c}) - \frac{k-m_c}{m-m_c}(S_m - S_{m_c})\right) k\, \frac{n-m}{n}\, A_n (1 + o_{\mathrm{P}}(1))$$

uniformly in $m - nh < k \le m - C/A_n^2$ and it can be seen that

$$\begin{aligned} \max_{m-nh<k\le m-C/A_n^2} \frac{nV_k\, |U_{k2,2}|}{k(m-k)A_n^2} &= \max_{m-nh<k\le m-C/A_n^2} \frac{|U_{k2,1}|}{k(m-k)A_n^2} \\ &= 2\,\frac{n-m}{n}\, O_{\mathrm{P}}\left(\max_{m_c+1\le k\le m-C/A_n^2} \frac{|(S_k - S_{m_c}) - \frac{k-m_c}{m-m_c}(S_m - S_{m_c})|}{(m-k)|A_n|}\right) \end{aligned}$$

which implies together with (5.75) and (5.82) that (5.78) holds.

By Lemma 5.8 and (5.82) we have, as $n \to \infty$,

$$U_{k3,2} = -\frac{k(m-k)\,\frac{n-m}{n} A_n^2\, (1 + o_{\mathrm{P}}(1))}{n\, V_k} = -\frac{k(m-k)\,\frac{n-m}{n} A_n^2\, (1 + o_{\mathrm{P}}(1))}{\frac{k(n-k)}{n}\, J(\tau_0)\, (1 + o_{\mathrm{P}}(1))}$$

$$= -\frac{m-k}{n-k}\,(n-m)\,\frac{A_n^2}{J(\tau_0)}\,(1+o_{\mathrm{P}}(1)) \tag{5.83}$$

uniformly in $m - nh < k \leq m - C/A_n^2$. Since the term $-\frac{m-k}{n-k}$ in (5.83) is increasing in k, we get, as $n \to \infty$,

$$\max_{m-nh<k\leq m-C/A_n^2} U_{k3,2} = -\frac{C}{J(\tau_0)}\,\frac{n-m}{n-m+C/A_n^2}\,(1+o_{\mathrm{P}}(1)) = -\frac{C}{J(\tau_0)}\,(1+o_{\mathrm{P}}(1)), \quad n \to \infty,$$

i.e. the assertion (5.79) is proved. Thus, the proof is finished. □

CHAPTER 6

Simulations

1. Introduction

To illustrate the proposed tests and estimators we prepare a simulation study. First, we present results of simulated and asymptotic critical values for the considered max-type and MOSUM-type tests described in Chapter 2. Second, we simulate the power for the max-type procedure to check a finite sample behavior. Third, we simulate the distribution of the corresponding max-type estimators under the one-change alternative H_1 and also under the no-change hypothesis H_0. The limit properties of estimators were studied in Chapter 3.

We use the three types of weights (2.5) for

(1) the *log-rank*-type test (LR) with $\rho = 0$, $\kappa = 0$;
(2) the *Gehan–Wilcoxon*-type test (GW) with $\rho = 0$, $\kappa = 1$;
(3) the *Prentice–Wilcoxon*-type test (PW) with $\rho = 1$, $\kappa = 0$;

and we consider three types of distributions

- the exponential distribution $\mathrm{E}(\delta_n)$, i.e. $F(x) = 1 - \exp(-\delta_n x)$, $\delta_n > 0$, $x > 0$;
- the log-normal distribution $\mathrm{L}(\delta_n)$, i.e. $F(x) = \Phi(\log(\delta_n x))$, $\delta_n > 0$, $x > 0$;
- the Weibull distribution $\mathrm{W}(\delta_n)$, i.e. $F(x) = 1 - \exp(-(\delta_n x)^4)$, $\delta_n > 0$, $x > 0$.

The choice of the distributions follows Neuhaus [**29**].

Further, for simplicity, we assume that $\tau_0 = \infty$. For the simulation we use the statistical software *R v.1.5.1* made by The R Development Core Team.

2. Critical values

2.1. Critical values for the statistic T_n. We simulate the distribution of $T_n(\tau_0)$ through the Monte Carlo repetitions. We perform 10000 simulations for each case. From such simulated distribution we determine critical values for the test based on the max-type statistic $T_n(\tau_0)$ with the rejection region (2.18).

Suppose the classical model of the random censorship (RCM) only with a change in the distribution of the censoring variables given by the parameter $\eta \in (0,1)$, i.e. our model fulfils the no-change hypothesis in the distribution of the survival variables H_0. We proceed with $n = 100; 200$ as follows:

(1) The survival times $X_1^0, X_2^0, \ldots, X_n^0$ are simulated from the chosen distribution $F = \mathrm{E}(1)$ or $\mathrm{L}(1)$, respectively.
(2) The censoring times $C_1, C_2, \ldots, C_n$ are simulated using the chosen combination of parameters

$$C_1, C_2, \ldots, C^0_{\lfloor n\eta \rfloor} \sim G_1, \qquad G_1 = \mathrm{E}(1) \text{ (or } \mathrm{L}(1))$$
$$C_{\lfloor n\eta \rfloor+1}, C^0_{\lfloor n\eta \rfloor+2}, \ldots, C^0_n \sim G_2, \qquad G_2 = \mathrm{E}(\delta_{C,n}) \text{ (or } \mathrm{L}(\delta_{C,n}))$$

(we use $\eta = 0.25; 0.5; 0.75$, $\delta_{C,n} = 1; 1.5; 2; 3$).
(3) The pairs $(X_1, \Delta_1), (X_2, \Delta_2), \ldots, (X_n, \Delta_n)$ are computed.
(4) The value of the statistic $T_n(\tau_0)$ is calculated and its value stored.

(5) The steps (1) – (4) are repeated 10^4 times.
(6) The 10%, 5%, 2.5% and 1% empirical critical values related to the empirical distribution function of $T_n(\tau_0)$ are computed and used as an estimator of the actual critical values.

In Tables 1 and 2 the results of the simulation for various sample sizes n are summarized and in Table 3 the results of the simulation for the particular situation where the survival variables $X_1^0, X_2^0, \ldots, X_n^0$ are not censored can be found. For comparing, in Table 4 the asymptotic critical values according to the formula (2.21) are determined.

				exponential				log-normal			
n	η	$\delta_{C,n}$	w_n	10%	5%	2.5%	1%	10%	5%	2.5%	1%
100	0.25	1.0	LR	3.462	4.188	4.937	6.230	3.479	4.127	4.769	5.876
100	0.25	1.0	GW	3.293	3.929	4.625	5.643	3.315	3.909	4.658	5.773
100	0.25	1.0	PW	3.386	4.047	4.745	5.845	3.368	4.038	4.750	5.813
100	0.25	1.5	LR	3.445	4.126	4.823	5.791	3.467	4.176	4.977	6.151
100	0.25	1.5	GW	3.357	4.002	4.743	5.734	3.312	4.021	4.834	5.950
100	0.25	1.5	PW	3.396	4.014	4.792	5.972	3.406	4.077	4.744	5.787
100	0.25	2.0	LR	3.492	4.190	4.923	6.197	3.539	4.160	4.923	5.950
100	0.25	2.0	GW	3.290	3.889	4.694	5.811	3.353	3.991	4.691	5.733
100	0.25	2.0	PW	3.363	3.984	4.783	6.140	3.426	4.106	4.909	6.160
100	0.25	3.0	LR	3.453	4.119	4.889	6.061	3.524	4.253	5.112	6.455
100	0.25	3.0	GW	3.338	4.004	4.782	5.943	3.367	4.101	4.936	6.332
100	0.25	3.0	PW	3.381	4.053	4.837	6.081	3.510	4.351	5.183	6.428
100	0.50	1.5	LR	3.476	4.140	4.977	6.017	3.559	4.202	4.906	6.109
100	0.50	1.5	GW	3.339	4.010	4.624	5.696	3.353	4.044	4.754	5.883
100	0.50	1.5	PW	3.361	4.027	4.724	5.765	3.408	4.095	4.825	5.950
100	0.50	2.0	LR	3.472	4.148	4.986	6.230	3.532	4.191	4.945	6.229
100	0.50	2.0	GW	3.299	3.944	4.719	6.028	3.346	4.009	4.756	5.998
100	0.50	2.0	PW	3.411	4.087	4.863	5.958	3.431	4.147	4.980	6.417
100	0.50	3.0	LR	3.492	4.150	4.835	5.974	3.612	4.302	5.152	6.595
100	0.50	3.0	GW	3.380	4.047	4.927	6.254	3.436	4.146	4.985	6.284
100	0.50	3.0	PW	3.383	4.090	4.855	6.200	3.444	4.157	4.951	6.326
100	0.75	1.5	LR	3.503	4.211	4.969	6.201	3.465	4.155	4.811	5.926
100	0.75	1.5	GW	3.349	4.025	4.784	5.758	3.361	4.023	4.819	6.025
100	0.75	1.5	PW	3.398	4.108	4.871	5.969	3.410	4.068	4.772	5.874
100	0.75	2.0	LR	3.464	4.120	4.823	5.944	3.540	4.180	4.954	6.197
100	0.75	2.0	GW	3.329	4.003	4.811	5.873	3.400	4.017	4.738	5.963
100	0.75	2.0	PW	3.401	4.088	4.842	6.091	3.446	4.110	4.846	6.107
100	0.75	3.0	LR	3.495	4.179	4.840	6.008	3.582	4.252	5.172	6.525
100	0.75	3.0	GW	3.354	4.008	4.772	6.066	3.412	4.135	4.905	6.108
100	0.75	3.0	PW	3.445	4.098	4.822	6.062	3.476	4.259	5.191	6.544

TABLE 1. Empirical critical values for $T_{100}(\tau_0)$.

The simulated critical values are very stable when the value of $\delta_{C,n}$ increases from $\delta_{C,n} = 1$ corresponding to no change in the distribution of the censoring variables C_i's to $\delta_{C,n} = 3$ corresponding to an evident change in the censoring distribution. There is also no visible effect on the critical values by the choice of the time of such change given by $\lfloor n\eta \rfloor$. Moreover, the empirical critical values are only slightly influenced by the choice of the weights and the underlying distribution. Comparing the results in Tables 1, 2 with Table 3, we can see that the empirical critical values in the case of censored variables X_i^0's are similar to their counterparts in the case of uncensored X_i^0's. Surprisingly, the simulated critical values are substantially larger than the corresponding asymptotic ones (see Table 4), which is probably influenced by large variability of $V_k(\tau_0)$ defined in (2.16) and it needs another extended investigation.

n	η	$\delta_{C,n}$	w_n	exponential 10%	5%	2.5%	1%	log-normal 10%	5%	2.5%	1%
200	0.25	1.0	LR	3.519	4.200	4.864	6.155	3.540	4.159	4.976	6.054
200	0.25	1.0	GW	3.434	4.044	4.845	6.032	3.377	3.967	4.680	6.004
200	0.25	1.0	PW	3.442	4.056	4.912	6.102	3.383	4.022	4.851	6.141
200	0.25	1.5	LR	3.523	4.173	4.872	6.170	3.546	4.252	5.016	6.365
200	0.25	1.5	GW	3.417	4.008	4.775	5.896	3.424	4.098	4.938	6.435
200	0.25	1.5	PW	3.417	4.092	4.872	6.260	3.455	4.133	4.885	6.375
200	0.25	2.0	LR	3.607	4.326	5.144	6.418	3.593	4.315	5.148	6.228
200	0.25	2.0	GW	3.393	4.028	4.825	6.409	3.434	4.050	4.830	6.099
200	0.25	2.0	PW	3.468	4.101	4.847	6.155	3.508	4.157	4.991	6.176
200	0.25	3.0	LR	3.586	4.270	5.076	6.523	3.663	4.462	5.373	6.741
200	0.25	3.0	GW	3.461	4.129	4.979	6.268	3.444	4.122	4.937	6.302
200	0.25	3.0	PW	3.483	4.193	4.967	6.198	3.555	4.302	5.233	6.589
200	0.50	1.5	LR	3.537	4.181	4.910	6.077	3.601	4.233	5.070	6.256
200	0.50	1.5	GW	3.479	4.130	4.866	6.138	3.444	4.116	4.907	6.041
200	0.50	1.5	PW	3.494	4.137	4.939	6.234	3.500	4.127	4.938	6.148
200	0.50	2.0	LR	3.569	4.209	5.013	6.299	3.625	4.341	5.100	6.399
200	0.50	2.0	GW	3.479	4.173	5.029	6.197	3.393	4.085	4.896	6.133
200	0.50	2.0	PW	3.472	4.119	4.888	6.151	3.466	4.165	4.942	6.322
200	0.50	3.0	LR	3.566	4.273	5.091	6.328	3.635	4.432	5.355	6.869
200	0.50	3.0	GW	3.422	4.082	4.922	5.958	3.522	4.218	5.018	6.291
200	0.50	3.0	PW	3.479	4.140	4.874	6.078	3.581	4.329	5.216	6.598
200	0.75	1.5	LR	3.528	4.143	4.952	6.148	3.564	4.178	4.883	6.049
200	0.75	1.5	GW	3.411	3.980	4.776	5.928	3.416	4.063	4.895	6.145
200	0.75	1.5	PW	3.448	4.088	4.942	6.102	3.472	4.157	4.975	6.313
200	0.75	2.0	LR	3.567	4.231	5.020	6.202	3.607	4.326	5.043	6.500
200	0.75	2.0	GW	3.433	4.126	4.917	6.101	3.489	4.132	4.993	6.320
200	0.75	2.0	PW	3.493	4.175	4.935	5.986	3.605	4.347	5.323	6.942
200	0.75	3.0	LR	3.624	4.379	5.239	6.516	3.676	4.361	5.409	6.769
200	0.75	3.0	GW	3.440	4.115	4.861	6.339	3.512	4.237	5.224	6.534
200	0.75	3.0	PW	3.462	4.174	5.064	6.486	3.642	4.384	5.234	6.540

TABLE 2. Empirical critical values for $T_{200}(\tau_0)$.

n	w_n	exponential 10%	5%	2.5%	1%	log-normal 10%	5%	2.5%	1%
100	LR	3.440	4.033	4.694	5.898	3.482	4.116	4.924	5.992
100	GW	3.264	3.892	4.610	5.874	3.341	3.958	4.771	5.899
100	PW	3.311	3.918	4.564	5.560	3.344	3.977	4.692	5.883
200	LR	3.520	4.126	4.771	5.961	3.540	4.153	4.889	6.097
200	GW	3.395	4.002	4.731	5.837	3.417	4.027	4.794	6.039
200	PW	3.395	4.017	4.741	5.808	3.432	4.060	4.739	5.867

TABLE 3. Empirical critical values for $T_n(\tau_0)$ - no censoring, no change.

n	10%	5%	2.5%	1%
50	3.181	3.617	4.045	4.604
100	3.226	3.637	4.041	4.570
150	3.249	3.650	4.043	4.558
200	3.264	3.659	4.045	4.551

TABLE 4. Asymptotic critical values for $T_n(\tau_0)$ and $T_n^{\sigma}(\tau_0)$.

2.2. Critical values for the statistic T_n^σ. We present critical values for the permutation test based on the statistic $T_n^\sigma(\tau_0)$ defined in (2.22). In this case in contrast to the previous we conduct only one sample of observations and from it we generate randomly B permutations (where B is large enough, in our case B is chosen 10000). We compute the critical value $c_n^*(\alpha, (\boldsymbol{X}, \boldsymbol{\Delta}))$ from this simulated permutation distribution of $T_n^\sigma(\tau_0, \boldsymbol{Q})$. Recall that we can use such test only in the special case of $\eta = \gamma$, so we suppose the Koziol-Green model (KGM) of random censorship (see Chapter 1), which satisfies this condition.

We proceed with the sample sizes $n = 100; 200$ as follows:

(1) $X_1^0, X_2^0, \dots, X_n^0$ are simulated using the chosen combination of parameters

$$X_1^0, X_2^0, \dots, X_{\lfloor n\gamma \rfloor}^0 \sim F_1, \qquad F_1 = \mathrm{E}(1) \text{ (or } \mathrm{L}(1))$$

$$X_{\lfloor n\gamma \rfloor+1}^0, X_{\lfloor n\gamma \rfloor+2}^0, \dots, X_n^0 \sim F_2, \qquad F_2 = \mathrm{E}(\delta_n) \text{ (or } \mathrm{L}(\delta_n))$$

(we use $\gamma = 0.25; 0.5; 0.75,\ \delta_n = 1; 1.5; 2; 3$).

(2) $C_1, C_2, \dots, C_n$ fulfilling KGM are simulated
(we use the censoring parameter $\beta = 0; 0.5; 1$).

(3) Pairs $(X_1, \Delta_1), (X_2, \Delta_2), \dots, (X_n, \Delta_n)$ are computed.

(4) A random permutation $\boldsymbol{q} = (q_1, q_2, \dots, q_n)$ of $(1, 2, \dots, n)$ is generated.

(5) $T_n^\sigma(\tau_0, \boldsymbol{Q})$ with $\boldsymbol{Q} = \boldsymbol{q}$ is calculated and its value stored.

(6) The steps $(4)-(5)$ are repeated 10^4 times.

(7) The 10%, 5%, 2.5% and 1% empirical critical values related to the empirical distribution function of $T_n^\sigma(\tau_0, \boldsymbol{Q})$ are computed and used as an estimator of the actual ones.

The empirical critical values for $T_n^\sigma(\tau_0, \boldsymbol{Q})$ obtained through the permutation principle do reasonable approximation of the critical values for $T_n^\sigma(\tau_0)$, see Theorem 2.4. In Tables 5, 6 (for $n = 100$) and in Tables 7, 8 (for $n = 200$) the results of the simulation are shown. The corresponding asymptotic critical values can be found in Table 4. They are the same as for the test based on the max-type rank statistic $T_n(\tau_0)$.

The simulated values are almost not influenced by the change both in the model given by δ_n and the underlying distribution F and there is no influence of the location of the "true" change-point $m = \lfloor n\gamma \rfloor$. The critical values obtained through the permutation principle are in nearly all cases substantially smaller than the corresponding asymptotic ones. In the case of no censoring ($\beta = 0$) the empirical critical values for the log-rank-type test are much larger than for other two tests, even sometimes larger than the asymptotic counterparts (and they are very similar to the asymptotic critical values), but in other cases we can see similar results for the considered weights. It seems that the expected proportion of censoring $\beta/(\beta + 1)$ for $\beta > 0$ does not play an important role at least in our setting, trade off the results for $\beta = 0.5$ (33% censoring) and $\beta = 1$ (50% censoring). Comparing the obtained critical values with the results in Antoch, Hušková [**4**] for completely observable data, we see similar patterns. The difference between the empirical critical values for $T_n(\tau_0)$ (see Tables 1, 2, 3) and $T_n^\sigma(\tau_0)$ (see the parts of Tables 5 and 7 with $\delta_n = 1$ which means no change in the distribution of the lifetimes) is caused by the choice of standardization of the statistic $S_k(\tau_0)$ which behaves different at tails of the observation period, see Lemma 2.2.

n	γ	δ_n	β	w_n	exponential 10%	5%	2.5%	1%	log-normal 10%	5%	2.5%	1%
100	0.25	1.0	0	LR	3.036	3.455	4.282	4.300	3.084	3.434	4.129	4.300
100	0.25	1.0	0	GW	2.695	2.926	3.137	3.387	2.709	2.939	3.148	3.378
100	0.25	1.0	0	PW	2.704	2.949	3.125	3.367	2.706	2.944	3.155	3.416
100	0.25	1.0	0.5	LR	2.922	3.248	3.561	3.699	2.840	3.220	3.445	3.638
100	0.25	1.0	0.5	GW	2.732	2.960	3.178	3.441	2.763	2.998	3.227	3.514
100	0.25	1.0	0.5	PW	2.708	2.946	3.160	3.415	2.707	2.949	3.168	3.425
100	0.25	1.0	1	LR	2.898	3.006	3.204	3.457	2.829	3.201	3.363	3.574
100	0.25	1.0	1	GW	2.784	3.034	3.273	3.543	2.781	3.025	3.275	3.608
100	0.25	1.0	1	PW	2.738	2.969	3.169	3.415	2.733	2.969	3.193	3.391
100	0.25	1.5	0	LR	3.027	3.393	4.087	4.300	3.031	3.35	3.958	4.300
100	0.25	1.5	0	GW	2.709	2.945	3.163	3.410	2.712	2.955	3.164	3.384
100	0.25	1.5	0	PW	2.716	2.954	3.159	3.417	2.720	2.978	3.225	3.479
100	0.25	1.5	0.5	LR	2.839	3.231	3.587	3.729	2.898	3.293	3.894	4.181
100	0.25	1.5	0.5	GW	2.736	2.978	3.188	3.442	2.722	2.946	3.229	3.500
100	0.25	1.5	0.5	PW	2.724	2.963	3.171	3.458	2.721	2.959	3.173	3.438
100	0.25	1.5	1	LR	2.776	3.072	3.293	3.558	2.761	3.001	3.205	3.481
100	0.25	1.5	1	GW	2.797	3.043	3.254	3.537	2.851	3.177	3.476	3.777
100	0.25	1.5	1	PW	2.734	2.956	3.162	3.411	2.751	3.003	3.203	3.454
100	0.25	2.0	0	LR	3.045	3.392	4.148	4.300	3.065	3.455	4.122	4.300
100	0.25	2.0	0	GW	2.694	2.922	3.126	3.365	2.721	2.952	3.160	3.387
100	0.25	2.0	0	PW	2.709	2.925	3.152	3.395	2.728	2.950	3.134	3.372
100	0.25	2.0	0.5	LR	2.942	3.487	3.487	3.711	2.775	3.024	3.274	3.532
100	0.25	2.0	0.5	GW	2.744	2.978	3.193	3.419	2.721	2.985	3.205	3.451
100	0.25	2.0	0.5	PW	2.723	2.962	3.171	3.410	2.707	2.921	3.142	3.390
100	0.25	2.0	1	LR	2.800	3.019	3.198	3.489	2.742	2.988	3.207	3.455
100	0.25	2.0	1	GW	2.800	3.075	3.304	3.551	2.798	3.065	3.268	3.536
100	0.25	2.0	1	PW	2.739	2.980	3.206	3.440	2.738	2.989	3.187	3.452
100	0.25	3.0	0	LR	3.075	3.392	3.981	4.300	3.073	3.435	4.300	4.300
100	0.25	3.0	0	GW	2.695	2.921	3.124	3.400	2.714	2.930	3.149	3.391
100	0.25	3.0	0	PW	2.698	2.938	3.148	3.419	2.711	2.950	3.150	3.403
100	0.25	3.0	0.5	LR	2.875	3.270	3.930	3.930	2.853	3.218	3.512	3.671
100	0.25	3.0	0.5	GW	2.722	2.971	3.183	3.444	2.727	2.965	3.205	3.483
100	0.25	3.0	0.5	PW	2.731	2.966	3.147	3.404	2.700	2.912	3.107	3.393
100	0.25	3.0	1	LR	3.018	3.100	3.350	3.687	2.848	3.215	3.311	3.590
100	0.25	3.0	1	GW	2.803	3.065	3.283	3.532	2.748	2.990	3.215	3.505
100	0.25	3.0	1	PW	2.738	2.978	3.208	3.485	2.747	2.989	3.202	3.454

TABLE 5. Empirical critical values for $T^{\sigma}_{100}(\tau_0, \boldsymbol{Q})$ with $\gamma = 0.25$.

					exponential				log-normal			
n	γ	δ_n	β	w_n	10%	5%	2.5%	1%	10%	5%	2.5%	1%
100	0.50	1.5	0	LR	3.093	3.481	4.300	4.300	3.077	3.433	4.129	4.300
100	0.50	1.5	0	GW	2.684	2.922	3.139	3.377	2.722	2.971	3.178	3.418
100	0.50	1.5	0	PW	2.708	2.949	3.161	3.407	2.697	2.933	3.160	3.432
100	0.50	1.5	0.5	LR	2.932	3.264	3.553	3.713	2.871	3.226	3.964	3.990
100	0.50	1.5	0.5	GW	2.760	3.003	3.212	3.484	2.740	2.974	3.180	3.445
100	0.50	1.5	0.5	PW	2.730	2.972	3.197	3.469	2.717	2.934	3.161	3.413
100	0.50	1.5	1	LR	2.812	3.066	3.244	3.562	2.786	2.997	3.191	3.440
100	0.50	1.5	1	GW	2.749	3.017	3.233	3.523	2.780	3.028	3.251	3.529
100	0.50	1.5	1	PW	2.727	2.973	3.199	3.446	2.732	2.982	3.198	3.437
100	0.50	2.0	0	LR	3.050	3.392	4.045	4.300	3.086	3.483	4.300	4.300
100	0.50	2.0	0	GW	2.700	2.930	3.170	3.403	2.704	2.950	3.154	3.366
100	0.50	2.0	0	PW	2.688	2.938	3.127	3.436	2.715	2.940	3.152	3.405
100	0.50	2.0	0.5	LR	2.969	3.265	3.626	3.740	2.952	3.267	3.621	3.807
100	0.50	2.0	0.5	GW	2.749	2.991	3.195	3.449	2.722	2.960	3.141	3.415
100	0.50	2.0	0.5	PW	2.732	2.961	3.200	3.447	2.722	2.954	3.172	3.404
100	0.50	2.0	1	LR	2.735	2.958	3.176	3.423	2.894	3.059	3.264	3.515
100	0.50	2.0	1	GW	2.779	3.032	3.271	3.536	2.833	3.105	3.311	3.589
100	0.50	2.0	1	PW	2.716	2.954	3.160	3.426	2.737	2.972	3.207	3.467
100	0.50	3.0	0	LR	3.086	3.441	4.288	4.300	3.009	3.318	4.129	4.300
100	0.50	3.0	0	GW	2.711	2.955	3.176	3.389	2.704	2.939	3.162	3.430
100	0.50	3.0	0	PW	2.718	2.973	3.171	3.412	2.724	2.954	3.174	3.444
100	0.50	3.0	0.5	LR	3.037	3.400	4.175	4.544	2.969	3.129	3.382	3.686
100	0.50	3.0	0.5	GW	2.777	3.009	3.263	3.512	2.734	2.984	3.203	3.449
100	0.50	3.0	0.5	PW	2.735	2.981	3.193	3.449	2.731	2.949	3.187	3.403
100	0.50	3.0	1	LR	2.735	2.963	3.156	3.445	2.859	3.185	3.572	3.572
100	0.50	3.0	1	GW	2.800	3.023	3.246	3.533	2.767	3.015	3.236	3.511
100	0.50	3.0	1	PW	2.725	2.977	3.173	3.429	2.717	2.963	3.167	3.373
100	0.75	1.5	0	LR	3.045	3.431	4.267	4.300	3.050	3.434	4.031	4.300
100	0.75	1.5	0	GW	2.698	2.912	3.103	3.394	2.728	2.983	3.194	3.432
100	0.75	1.5	0	PW	2.736	2.956	3.141	3.462	2.718	2.925	3.140	3.403
100	0.75	1.5	0.5	LR	2.887	3.285	3.666	3.742	2.946	3.269	3.565	3.747
100	0.75	1.5	0.5	GW	2.728	2.968	3.190	3.501	2.756	2.986	3.185	3.432
100	0.75	1.5	0.5	PW	2.717	2.938	3.151	3.333	2.700	2.907	3.124	3.373
100	0.75	1.5	1	LR	2.776	3.036	3.312	3.633	2.805	3.131	3.387	3.551
100	0.75	1.5	1	GW	2.796	3.088	3.346	3.634	2.814	3.112	3.358	3.647
100	0.75	1.5	1	PW	2.725	2.966	3.189	3.455	2.731	2.966	3.196	3.468
100	0.75	2.0	0	LR	3.018	3.381	4.110	4.300	3.060	3.443	4.222	4.300
100	0.75	2.0	0	GW	2.703	2.934	3.128	3.381	2.729	2.960	3.180	3.451
100	0.75	2.0	0	PW	2.721	2.953	3.146	3.416	2.715	2.954	3.172	3.410
100	0.75	2.0	0.5	LR	2.868	3.251	3.404	3.736	2.826	3.191	3.408	3.580
100	0.75	2.0	0.5	GW	2.724	2.968	3.193	3.439	2.754	2.997	3.220	3.508
100	0.75	2.0	0.5	PW	2.751	2.973	3.173	3.453	2.709	2.945	3.141	3.390
100	0.75	2.0	1	LR	2.802	3.124	3.257	3.513	2.867	3.267	3.961	4.029
100	0.75	2.0	1	GW	2.766	2.994	3.195	3.483	2.761	2.995	3.222	3.463
100	0.75	2.0	1	PW	2.763	3.021	3.230	3.495	2.724	2.973	3.190	3.417
100	0.75	3.0	0	LR	3.026	3.409	4.070	4.300	3.023	3.392	4.048	4.300
100	0.75	3.0	0	GW	2.728	2.978	3.179	3.423	2.727	2.944	3.169	3.407
100	0.75	3.0	0	PW	2.698	2.926	3.153	3.379	2.716	2.949	3.166	3.408
100	0.75	3.0	0.5	LR	2.883	3.259	3.384	3.724	2.898	3.268	3.798	3.820
100	0.75	3.0	0.5	GW	2.729	2.962	3.168	3.417	2.739	2.958	3.185	3.482
100	0.75	3.0	0.5	PW	2.723	2.941	3.190	3.414	2.721	2.959	3.15	3.420
100	0.75	3.0	1	LR	2.812	3.137	3.208	3.485	2.891	3.035	3.264	3.543
100	0.75	3.0	1	GW	2.755	3.012	3.219	3.476	2.774	3.028	3.241	3.535
100	0.75	3.0	1	PW	2.752	2.985	3.189	3.496	2.736	2.986	3.213	3.526

TABLE 6. Empirical critical values for $T^{\sigma}_{100}(\tau_0, \boldsymbol{Q})$ with $\gamma = 0.50; 0.75$.

					exponential				log-normal			
n	γ	δ_n	β	w_n	10%	5%	2.5%	1%	10%	5%	2.5%	1%
200	0.25	1.0	0	LR	3.148	3.617	3.956	4.951	3.119	3.578	3.936	4.951
200	0.25	1.0	0	GW	2.800	3.035	3.246	3.502	2.799	3.036	3.244	3.500
200	0.25	1.0	0	PW	2.790	3.020	3.259	3.513	2.804	3.040	3.261	3.552
200	0.25	1.0	0.5	LR	3.079	3.320	3.714	4.450	3.049	3.427	3.710	4.264
200	0.25	1.0	0.5	GW	2.847	3.080	3.318	3.579	2.829	3.068	3.303	3.570
200	0.25	1.0	0.5	PW	2.827	3.062	3.287	3.566	2.818	3.049	3.281	3.551
200	0.25	1.0	1	LR	2.930	3.212	3.580	4.327	2.928	3.266	3.397	3.727
200	0.25	1.0	1	GW	2.916	3.180	3.394	3.677	2.911	3.172	3.418	3.691
200	0.25	1.0	1	PW	2.812	3.060	3.293	3.513	2.825	3.061	3.296	3.527
200	0.25	1.5	0	LR	3.114	3.501	3.936	4.951	3.144	3.572	3.936	4.951
200	0.25	1.5	0	GW	2.819	3.053	3.268	3.515	2.786	3.008	3.211	3.474
200	0.25	1.5	0	PW	2.789	3.027	3.240	3.520	2.805	3.037	3.255	3.542
200	0.25	1.5	0.5	LR	3.062	3.461	4.287	4.287	3.023	3.373	3.645	3.890
200	0.25	1.5	0.5	GW	2.836	3.087	3.318	3.587	2.819	3.074	3.288	3.535
200	0.25	1.5	0.5	PW	2.804	3.038	3.263	3.529	2.812	3.046	3.278	3.550
200	0.25	1.5	1	LR	2.926	3.227	3.586	4.157	2.912	3.200	3.507	3.663
200	0.25	1.5	1	GW	2.941	3.219	3.422	3.755	2.872	3.099	3.350	3.646
200	0.25	1.5	1	PW	2.822	3.045	3.284	3.534	2.820	3.054	3.268	3.549
200	0.25	2.0	0	LR	3.105	3.539	3.936	4.951	3.122	3.550	3.936	4.951
200	0.25	2.0	0	GW	2.802	3.044	3.217	3.473	2.832	3.066	3.298	3.567
200	0.25	2.0	0	PW	2.787	3.020	3.249	3.523	2.827	3.079	3.287	3.566
200	0.25	2.0	0.5	LR	2.972	3.372	3.627	4.048	3.038	3.372	3.469	3.856
200	0.25	2.0	0.5	GW	2.839	3.070	3.272	3.549	2.843	3.100	3.316	3.543
200	0.25	2.0	0.5	PW	2.812	3.076	3.294	3.538	2.819	3.064	3.289	3.528
200	0.25	2.0	1	LR	2.964	3.171	3.453	3.786	3.035	3.145	3.382	3.649
200	0.25	2.0	1	GW	2.860	3.123	3.336	3.624	2.880	3.136	3.375	3.654
200	0.25	2.0	1	PW	2.836	3.067	3.311	3.564	2.827	3.072	3.274	3.505
200	0.25	3.0	0	LR	3.127	3.587	3.978	4.951	3.136	3.577	3.944	4.951
200	0.25	3.0	0	GW	2.809	3.046	3.256	3.515	2.821	3.053	3.273	3.530
200	0.25	3.0	0	PW	2.792	3.042	3.243	3.464	2.811	3.041	3.240	3.488
200	0.25	3.0	0.5	LR	3.012	3.284	3.666	4.350	2.949	3.247	3.615	4.121
200	0.25	3.0	0.5	GW	2.827	3.069	3.299	3.559	2.840	3.089	3.299	3.558
200	0.25	3.0	0.5	PW	2.806	3.031	3.247	3.463	2.815	3.040	3.252	3.509
200	0.25	3.0	1	LR	2.892	3.201	3.545	3.875	2.939	3.291	3.465	3.735
200	0.25	3.0	1	GW	2.904	3.173	3.422	3.697	2.898	3.171	3.435	3.768
200	0.25	3.0	1	PW	2.836	3.073	3.312	3.569	2.824	3.042	3.286	3.534

TABLE 7. Empirical critical values for $T^{\sigma}_{200}(\tau_0, \boldsymbol{Q})$ with $\gamma = 0.25$.

					exponential				log-normal			
n	γ	δ_n	β	w_n	10%	5%	2.5%	1%	10%	5%	2.5%	1%
200	0.50	1.5	0	LR	3.110	3.513	3.965	4.951	3.130	3.534	3.936	4.951
200	0.50	1.5	0	GW	2.803	3.036	3.267	3.529	2.822	3.063	3.268	3.551
200	0.50	1.5	0	PW	2.800	3.038	3.256	3.525	2.805	3.048	3.271	3.517
200	0.50	1.5	0.5	LR	3.057	3.411	3.745	4.659	3.011	3.300	3.657	4.386
200	0.50	1.5	0.5	GW	2.844	3.089	3.326	3.606	2.830	3.061	3.287	3.566
200	0.50	1.5	0.5	PW	2.827	3.047	3.290	3.559	2.820	3.046	3.263	3.508
200	0.50	1.5	1	LR	2.882	3.170	3.492	3.657	2.948	3.228	3.351	3.623
200	0.50	1.5	1	GW	2.905	3.175	3.421	3.772	2.880	3.167	3.381	3.645
200	0.50	1.5	1	PW	2.832	3.075	3.277	3.578	2.841	3.064	3.273	3.555
200	0.50	2.0	0	LR	3.116	3.554	3.936	4.951	3.147	3.568	3.942	4.951
200	0.50	2.0	0	GW	2.819	3.069	3.276	3.519	2.811	3.050	3.235	3.493
200	0.50	2.0	0	PW	2.814	3.060	3.271	3.530	2.796	3.050	3.279	3.529
200	0.50	2.0	0.5	LR	2.878	3.137	3.397	3.687	2.965	3.252	3.608	4.305
200	0.50	2.0	0.5	GW	2.839	3.087	3.302	3.572	2.826	3.087	3.321	3.560
200	0.50	2.0	0.5	PW	2.834	3.067	3.271	3.536	2.817	3.055	3.279	3.553
200	0.50	2.0	1	LR	2.831	3.092	3.334	3.590	2.912	3.105	3.346	3.628
200	0.50	2.0	1	GW	2.911	3.201	3.432	3.699	2.867	3.117	3.324	3.603
200	0.50	2.0	1	PW	2.828	3.067	3.286	3.542	2.822	3.045	3.265	3.515
200	0.50	3.0	0	LR	3.119	3.516	3.936	4.951	3.132	3.549	3.936	4.951
200	0.50	3.0	0	GW	2.805	3.031	3.230	3.496	2.808	3.037	3.257	3.543
200	0.50	3.0	0	PW	2.817	3.055	3.276	3.518	2.806	3.040	3.256	3.514
200	0.50	3.0	0.5	LR	2.979	3.360	4.071	4.071	2.916	3.263	3.523	4.040
200	0.50	3.0	0.5	GW	2.812	3.050	3.276	3.537	2.847	3.092	3.274	3.539
200	0.50	3.0	0.5	PW	2.815	3.048	3.292	3.541	2.819	3.055	3.279	3.525
200	0.50	3.0	1	LR	2.977	3.423	3.664	3.834	2.887	3.149	3.433	3.585
200	0.50	3.0	1	GW	2.903	3.156	3.397	3.661	2.894	3.162	3.391	3.692
200	0.50	3.0	1	PW	2.832	3.055	3.286	3.562	2.811	3.069	3.306	3.580
200	0.75	1.5	0	LR	3.091	3.534	3.936	4.951	3.050	3.434	4.031	4.300
200	0.75	1.5	0	GW	2.820	3.059	3.283	3.550	2.728	2.983	3.194	3.432
200	0.75	1.5	0	PW	2.799	3.038	3.247	3.513	2.718	2.925	3.140	3.403
200	0.75	1.5	0.5	LR	2.952	3.315	3.704	3.837	2.946	3.269	3.565	3.747
200	0.75	1.5	0.5	GW	2.852	3.096	3.313	3.591	2.756	2.986	3.185	3.432
200	0.75	1.5	0.5	PW	2.817	3.058	3.274	3.518	2.700	2.907	3.124	3.373
200	0.75	1.5	1	LR	2.994	3.352	3.842	3.842	2.805	3.131	3.387	3.551
200	0.75	1.5	1	GW	2.937	3.221	3.440	3.735	2.814	3.112	3.358	3.647
200	0.75	1.5	1	PW	2.827	3.070	3.280	3.585	2.731	2.966	3.196	3.468
200	0.75	2.0	0	LR	3.116	3.574	3.936	4.951	3.060	3.443	4.222	4.300
200	0.75	2.0	0	GW	2.817	3.056	3.272	3.485	2.729	2.960	3.180	3.451
200	0.75	2.0	0	PW	2.804	3.058	3.264	3.528	2.715	2.954	3.172	3.410
200	0.75	2.0	0.5	LR	2.995	3.377	3.975	3.975	2.826	3.191	3.408	3.580
200	0.75	2.0	0.5	GW	2.816	3.044	3.266	3.540	2.754	2.997	3.220	3.508
200	0.75	2.0	0.5	PW	2.826	3.036	3.248	3.497	2.709	2.945	3.141	3.390
200	0.75	2.0	1	LR	2.856	3.161	3.491	3.839	2.867	3.267	3.961	4.029
200	0.75	2.0	1	GW	2.862	3.120	3.363	3.656	2.761	2.995	3.222	3.463
200	0.75	2.0	1	PW	2.831	3.067	3.288	3.532	2.724	2.973	3.190	3.417
200	0.75	3.0	0	LR	3.103	3.467	3.936	4.951	3.023	3.392	4.048	4.300
200	0.75	3.0	0	GW	2.793	3.029	3.252	3.490	2.727	2.944	3.169	3.407
200	0.75	3.0	0	PW	2.793	3.030	3.232	3.505	2.716	2.949	3.166	3.408
200	0.75	3.0	0.5	LR	3.038	3.338	3.619	4.081	2.898	3.268	3.798	3.820
200	0.75	3.0	0.5	GW	2.843	3.078	3.309	3.557	2.739	2.958	3.185	3.482
200	0.75	3.0	0.5	PW	2.806	3.064	3.295	3.558	2.721	2.959	3.150	3.420
200	0.75	3.0	1	LR	2.875	3.151	3.387	3.689	2.891	3.035	3.264	3.543
200	0.75	3.0	1	GW	2.880	3.138	3.378	3.657	2.774	3.028	3.241	3.535
200	0.75	3.0	1	PW	2.836	3.078	3.316	3.568	2.736	2.986	3.213	3.526

TABLE 8. Empirical critical values for $T^{\sigma}_{200}(\tau_0, \boldsymbol{Q})$ with $\gamma = 0.50; 0.75$.

2.3. Critical values for the statistic $T_{n,D}$. Suppose the classical model of random censorship, where the distribution G_1 of the censoring variables can change in the distribution G_2 at time $m_c = \lfloor n\eta \rfloor$. We simulate critical values for the test based on the MOSUM-type statistic $T_{n,D}(\tau_0)$ defined in (2.33), where m_c is estimated. We conduct the Monte Carlo simulations analogously to $T_n(\tau_0)$, see Subsection 2.1. We choose $D = 0.05n$ or $0.1n$ or $D = \lfloor \sqrt{n} \rfloor$ which are commonly used.

We proceed with $n = 100; 200$ as follows:

(1) The survival times $X_1^0, X_2^0, \ldots, X_n^0$ are simulated from the chosen distribution $F = \mathrm{E}(1)$ or $\mathrm{L}(1)$, respectively.

(2) The censoring times $C_1, C_2, \ldots, C_n$ are simulated using the chosen combination of parameters

$$C_1, C_2, \ldots, C^0_{\lfloor n\eta \rfloor} \sim G_1, \qquad G_1 = \mathrm{E}(1) \text{ (or } \mathrm{L}(1))$$
$$C_{\lfloor n\eta \rfloor+1}, C^0_{\lfloor n\eta \rfloor+2}, \ldots, C^0_n \sim G_2, \qquad G_2 = \mathrm{E}(\delta_{C,n}) \text{ (or } \mathrm{L}(\delta_{C,n}))$$

(we use $\eta = 0.25; 0.5; 0.75$, $\delta_{C,n} = 1; 1.5; 2; 3$).

(3) The pairs $(X_1, \Delta_1), (X_2, \Delta_2), \ldots, (X_n, \Delta_n)$ are computed.

(4) The estimator $\hat{m}_c(\tau_0)$ is calculated and its value is used for the calculation of the test statistic.

(5) The value of the statistic $T_{n,D}(\tau_0)$ is determined and its value stored.

(6) The steps (1) − (5) are repeated 10^4 times.

(7) Empirical critical values related to the empirical distribution function of $T_{n,D}(\tau_0)$ are computed and used as an estimator of the actual critical values.

The sample critical values relevant to $\alpha = 10\%$, 5%, 2.5% and 1% can be found in Table 9 (for $n = 100$) and in Tables 10, 11 (for $n = 200$). In Table 12 the corresponding asymptotic critical values determined according to (2.49) are summarized.

We can observe that the critical values are almost not influenced by the location and the size of a change in the distribution of the censoring variables given by η and $\delta_{C,n}$. Recall that $\delta_{C,n} = 1$ means that $C_1, C_2, \ldots, C_n$ are i.i.d. variables. There is also no visible effect on the sample critical values if we use different underlying distribution functions. The values are only slightly influenced by the choice of the weights. Comparing the results in Tables 9 and 10, we see that the simulated values are a bit higher for $n = 200$ in spite of the fact that n/D is still the same. Further, the critical values in Tables 9 − 11 obtained through the Monte Carlo simulations is substantially smaller than their asymptotic counterparts presented in Table 12.

					exponential				log-normal			
n	D	η	$\delta_{C,n}$	w_n	10%	5%	2.5%	1%	10%	5%	2.5%	1%
100	5	0.25	1.0	LR	3.085	3.279	3.420	3.632	3.067	3.261	3.441	3.664
100	5	0.25	1.0	GW	3.063	3.269	3.448	3.672	3.051	3.243	3.425	3.651
100	5	0.25	1.0	PW	3.042	3.209	3.367	3.555	3.028	3.220	3.374	3.576
100	5	0.25	1.5	LR	3.064	3.255	3.412	3.612	3.084	3.285	3.454	3.664
100	5	0.25	1.5	GW	3.085	3.282	3.461	3.663	3.088	3.292	3.463	3.676
100	5	0.25	1.5	PW	3.051	3.220	3.368	3.562	3.056	3.248	3.391	3.598
100	5	0.25	2.0	LR	3.058	3.234	3.404	3.625	3.074	3.273	3.453	3.685
100	5	0.25	2.0	GW	3.080	3.288	3.489	3.726	3.105	3.309	3.500	3.758
100	5	0.25	2.0	PW	3.038	3.226	3.398	3.611	3.057	3.248	3.428	3.638
100	5	0.25	3.0	LR	3.066	3.271	3.450	3.707	3.121	3.319	3.504	3.759
100	5	0.25	3.0	GW	3.112	3.321	3.523	3.750	3.143	3.376	3.583	3.842
100	5	0.25	3.0	PW	3.055	3.257	3.453	3.672	3.082	3.275	3.458	3.673
100	5	0.50	1.5	LR	3.051	3.229	3.400	3.598	3.063	3.269	3.460	3.671
100	5	0.50	1.5	GW	3.059	3.228	3.406	3.616	3.072	3.264	3.430	3.645
100	5	0.50	1.5	PW	3.049	3.219	3.379	3.586	3.044	3.231	3.382	3.599
100	5	0.50	2.0	LR	3.060	3.240	3.397	3.585	3.070	3.240	3.408	3.639
100	5	0.50	2.0	GW	3.072	3.264	3.436	3.651	3.070	3.276	3.438	3.628
100	5	0.50	2.0	PW	3.034	3.217	3.372	3.565	3.057	3.235	3.404	3.601
100	5	0.50	3.0	LR	3.049	3.236	3.417	3.627	3.068	3.243	3.423	3.639
100	5	0.50	3.0	GW	3.048	3.253	3.441	3.659	3.089	3.281	3.477	3.706
100	5	0.50	3.0	PW	3.050	3.235	3.406	3.625	3.060	3.249	3.420	3.624
100	5	0.75	1.5	LR	3.069	3.246	3.410	3.611	3.080	3.259	3.434	3.643
100	5	0.75	1.5	GW	3.047	3.247	3.417	3.609	3.055	3.229	3.403	3.604
100	5	0.75	1.5	PW	3.033	3.224	3.378	3.575	3.034	3.198	3.363	3.539
100	5	0.75	2.0	LR	3.065	3.263	3.451	3.671	3.075	3.263	3.445	3.615
100	5	0.75	2.0	GW	3.045	3.231	3.401	3.622	3.063	3.231	3.408	3.613
100	5	0.75	2.0	PW	3.049	3.227	3.396	3.595	3.028	3.202	3.348	3.564
100	5	0.75	3.0	LR	3.065	3.249	3.425	3.631	3.067	3.268	3.452	3.643
100	5	0.75	3.0	GW	3.048	3.246	3.426	3.643	3.067	3.251	3.422	3.666
100	5	0.75	3.0	PW	3.031	3.203	3.369	3.570	3.032	3.217	3.370	3.563
100	10	0.25	1.0	LR	2.936	3.150	3.337	3.574	2.941	3.153	3.351	3.603
100	10	0.25	1.0	GW	2.957	3.177	3.356	3.594	2.944	3.140	3.347	3.574
100	10	0.25	1.0	PW	2.936	3.154	3.343	3.557	2.944	3.133	3.342	3.562
100	10	0.25	1.5	LR	2.938	3.138	3.331	3.562	2.938	3.143	3.326	3.538
100	10	0.25	1.5	GW	2.902	3.115	3.329	3.582	2.928	3.145	3.336	3.550
100	10	0.25	1.5	PW	2.929	3.153	3.334	3.576	2.943	3.157	3.340	3.592
100	10	0.25	2.0	LR	2.932	3.148	3.326	3.574	2.935	3.151	3.360	3.601
100	10	0.25	2.0	GW	2.925	3.159	3.355	3.612	2.893	3.107	3.316	3.572
100	10	0.25	2.0	PW	2.921	3.134	3.322	3.571	2.919	3.138	3.317	3.517
100	10	0.25	3.0	LR	2.942	3.154	3.337	3.592	2.939	3.148	3.343	3.542
100	10	0.25	3.0	GW	2.887	3.099	3.298	3.568	2.854	3.089	3.302	3.540
100	10	0.25	3.0	PW	2.910	3.126	3.340	3.582	2.917	3.122	3.309	3.542
100	10	0.50	1.5	LR	2.936	3.150	3.335	3.571	2.930	3.141	3.322	3.537
100	10	0.50	1.5	GW	2.919	3.134	3.328	3.604	2.904	3.119	3.306	3.522
100	10	0.50	1.5	PW	2.935	3.141	3.332	3.558	2.938	3.138	3.308	3.553
100	10	0.50	2.0	LR	2.922	3.122	3.308	3.560	2.900	3.118	3.299	3.540
100	10	0.50	2.0	GW	2.887	3.113	3.302	3.530	2.893	3.106	3.279	3.498
100	10	0.50	2.0	PW	2.908	3.121	3.288	3.499	2.902	3.098	3.278	3.478
100	10	0.50	3.0	LR	2.915	3.131	3.308	3.528	2.906	3.111	3.282	3.508
100	10	0.50	3.0	GW	2.856	3.052	3.243	3.467	2.852	3.050	3.237	3.479
100	10	0.50	3.0	PW	2.888	3.093	3.295	3.520	2.895	3.095	3.277	3.478
100	10	0.75	1.5	LR	2.927	3.136	3.327	3.561	2.932	3.153	3.360	3.598
100	10	0.75	1.5	GW	2.954	3.179	3.377	3.569	2.915	3.142	3.331	3.568
100	10	0.75	1.5	PW	2.927	3.131	3.297	3.551	2.924	3.135	3.316	3.527
100	10	0.75	2.0	LR	2.927	3.134	3.322	3.530	2.939	3.140	3.324	3.577
100	10	0.75	2.0	GW	2.918	3.127	3.326	3.580	2.934	3.160	3.339	3.543
100	10	0.75	2.0	PW	2.941	3.146	3.327	3.573	2.929	3.144	3.331	3.567
100	10	0.75	3.0	LR	2.913	3.131	3.349	3.598	2.924	3.137	3.339	3.574
100	10	0.75	3.0	GW	2.903	3.128	3.317	3.553	2.892	3.120	3.301	3.538
100	10	0.75	3.0	PW	2.912	3.135	3.326	3.551	2.917	3.138	3.326	3.568

TABLE 9. Critical values for $T_{100,D}(\tau_0)$.

n	D	η	$\delta_{C,n}$	w_n	exponential 10%	exponential 5%	exponential 2.5%	exponential 1%	log-normal 10%	log-normal 5%	log-normal 2.5%	log-normal 1%
200	10	0.25	1.0	LR	3.250	3.436	3.628	3.847	3.250	3.467	3.625	3.841
200	10	0.25	1.0	GW	3.245	3.443	3.610	3.822	3.232	3.427	3.597	3.790
200	10	0.25	1.0	PW	3.221	3.418	3.578	3.827	3.225	3.426	3.599	3.853
200	10	0.25	1.5	LR	3.231	3.433	3.634	3.872	3.230	3.437	3.635	3.856
200	10	0.25	1.5	GW	3.232	3.423	3.589	3.828	3.239	3.449	3.615	3.876
200	10	0.25	1.5	PW	3.226	3.421	3.594	3.797	3.224	3.418	3.614	3.842
200	10	0.25	2.0	LR	3.247	3.432	3.622	3.856	3.231	3.447	3.638	3.874
200	10	0.25	2.0	GW	3.235	3.442	3.638	3.873	3.228	3.452	3.651	3.907
200	10	0.25	2.0	PW	3.198	3.408	3.600	3.847	3.215	3.407	3.600	3.802
200	10	0.25	3.0	LR	3.240	3.437	3.614	3.838	3.251	3.484	3.680	3.930
200	10	0.25	3.0	GW	3.231	3.452	3.649	3.906	3.251	3.493	3.713	3.962
200	10	0.25	3.0	PW	3.234	3.448	3.647	3.862	3.248	3.478	3.672	3.917
200	10	0.50	1.5	LR	3.239	3.442	3.627	3.872	3.224	3.425	3.598	3.801
200	10	0.50	1.5	GW	3.242	3.433	3.617	3.828	3.223	3.429	3.630	3.851
200	10	0.50	1.5	PW	3.222	3.409	3.592	3.826	3.209	3.404	3.590	3.810
200	10	0.50	2.0	LR	3.230	3.420	3.606	3.829	3.231	3.438	3.619	3.819
200	10	0.50	2.0	GW	3.209	3.420	3.598	3.823	3.215	3.398	3.610	3.850
200	10	0.50	2.0	PW	3.212	3.396	3.575	3.823	3.214	3.423	3.596	3.802
200	10	0.50	3.0	LR	3.233	3.445	3.653	3.862	3.252	3.459	3.651	3.878
200	10	0.50	3.0	GW	3.214	3.428	3.618	3.862	3.228	3.435	3.636	3.891
200	10	0.50	3.0	PW	3.192	3.382	3.575	3.789	3.234	3.426	3.611	3.803
200	10	0.75	1.5	LR	3.240	3.429	3.600	3.790	3.251	3.445	3.624	3.848
200	10	0.75	1.5	GW	3.230	3.419	3.611	3.852	3.218	3.403	3.603	3.840
200	10	0.75	1.5	PW	3.237	3.430	3.593	3.828	3.230	3.415	3.585	3.790
200	10	0.75	2.0	LR	3.236	3.438	3.604	3.859	3.246	3.438	3.618	3.803
200	10	0.75	2.0	GW	3.208	3.416	3.610	3.816	3.219	3.410	3.602	3.838
200	10	0.75	2.0	PW	3.215	3.404	3.568	3.757	3.222	3.418	3.593	3.812
200	10	0.75	3.0	LR	3.237	3.453	3.617	3.829	3.239	3.435	3.610	3.840
200	10	0.75	3.0	GW	3.213	3.416	3.605	3.830	3.192	3.407	3.585	3.832
200	10	0.75	3.0	PW	3.210	3.412	3.567	3.814	3.201	3.387	3.555	3.770
200	20	0.25	1.0	LR	3.066	3.281	3.466	3.660	3.079	3.292	3.487	3.703
200	20	0.25	1.0	GW	3.048	3.285	3.476	3.722	3.058	3.278	3.484	3.728
200	20	0.25	1.0	PW	3.070	3.296	3.498	3.743	3.087	3.303	3.499	3.762
200	20	0.25	1.5	LR	3.035	3.260	3.445	3.677	3.038	3.277	3.475	3.691
200	20	0.25	1.5	GW	3.060	3.262	3.454	3.689	3.012	3.230	3.445	3.650
200	20	0.25	1.5	PW	3.063	3.272	3.449	3.703	3.034	3.243	3.451	3.680
200	20	0.25	2.0	LR	3.040	3.243	3.447	3.713	3.048	3.275	3.465	3.725
200	20	0.25	2.0	GW	3.018	3.235	3.456	3.705	3.016	3.249	3.444	3.684
200	20	0.25	2.0	PW	3.026	3.237	3.426	3.634	3.032	3.248	3.437	3.686
200	20	0.25	3.0	LR	3.014	3.236	3.427	3.654	3.006	3.243	3.442	3.681
200	20	0.25	3.0	GW	2.988	3.212	3.444	3.671	2.990	3.210	3.434	3.668
200	20	0.25	3.0	PW	3.037	3.250	3.442	3.684	3.024	3.223	3.433	3.666
200	20	0.50	1.5	LR	3.066	3.288	3.471	3.711	3.041	3.268	3.477	3.707
200	20	0.50	1.5	GW	3.024	3.245	3.450	3.681	3.037	3.266	3.478	3.710
200	20	0.50	1.5	PW	3.051	3.265	3.488	3.754	3.023	3.257	3.434	3.663
200	20	0.50	2.0	LR	3.043	3.257	3.434	3.667	3.044	3.263	3.462	3.670
200	20	0.50	2.0	GW	3.030	3.256	3.438	3.697	3.009	3.227	3.399	3.651
200	20	0.50	2.0	PW	3.007	3.224	3.428	3.655	3.024	3.253	3.476	3.705
200	20	0.50	3.0	LR	3.026	3.248	3.444	3.668	3.029	3.247	3.457	3.690
200	20	0.50	3.0	GW	2.994	3.199	3.396	3.623	3.006	3.217	3.411	3.631
200	20	0.50	3.0	PW	3.021	3.246	3.461	3.679	3.022	3.245	3.442	3.654
200	20	0.75	1.5	LR	3.057	3.278	3.453	3.657	3.070	3.293	3.498	3.716
200	20	0.75	1.5	GW	3.049	3.267	3.452	3.703	3.069	3.281	3.463	3.695
200	20	0.75	1.5	PW	3.059	3.293	3.482	3.727	3.067	3.283	3.468	3.692
200	20	0.75	2.0	LR	3.038	3.265	3.449	3.678	3.052	3.290	3.494	3.708
200	20	0.75	2.0	GW	3.040	3.267	3.463	3.715	3.035	3.242	3.445	3.699
200	20	0.75	2.0	PW	3.035	3.247	3.445	3.701	3.062	3.295	3.511	3.746
200	20	0.75	3.0	LR	3.040	3.266	3.464	3.673	3.035	3.247	3.442	3.680
200	20	0.75	3.0	GW	3.014	3.259	3.439	3.657	3.013	3.223	3.437	3.673
200	20	0.75	3.0	PW	3.048	3.256	3.469	3.725	3.032	3.253	3.425	3.674

TABLE 10. Critical values for $T_{200,D}(\tau_0)$.

					exponential				log-normal			
n	D	η	$\delta_{C,n}$	w_n	10%	5%	2.5%	1%	10%	5%	2.5%	1%
200	14	0.25	1.0	LR	3.152	3.352	3.549	3.826	3.162	3.372	3.548	3.774
200	14	0.25	1.0	GW	3.153	3.366	3.545	3.758	3.164	3.366	3.552	3.792
200	14	0.25	1.0	PW	3.154	3.377	3.567	3.791	3.167	3.400	3.574	3.785
200	14	0.25	1.5	LR	3.156	3.373	3.549	3.770	3.166	3.379	3.565	3.798
200	14	0.25	1.5	GW	3.146	3.356	3.539	3.771	3.141	3.357	3.549	3.807
200	14	0.25	1.5	PW	3.165	3.377	3.575	3.779	3.140	3.363	3.550	3.790
200	14	0.25	2.0	LR	3.156	3.378	3.568	3.765	3.138	3.360	3.537	3.748
200	14	0.25	2.0	GW	3.147	3.359	3.572	3.881	3.138	3.360	3.538	3.790
200	14	0.25	2.0	PW	3.144	3.350	3.543	3.783	3.159	3.367	3.545	3.785
200	14	0.25	3.0	LR	3.151	3.365	3.534	3.749	3.164	3.394	3.599	3.829
200	14	0.25	3.0	GW	3.144	3.360	3.560	3.851	3.124	3.352	3.585	3.828
200	14	0.25	3.0	PW	3.133	3.336	3.528	3.757	3.156	3.363	3.554	3.751
200	14	0.50	1.5	LR	3.158	3.371	3.573	3.830	3.150	3.371	3.556	3.769
200	14	0.50	1.5	GW	3.142	3.353	3.552	3.755	3.145	3.369	3.570	3.847
200	14	0.50	1.5	PW	3.146	3.342	3.550	3.776	3.150	3.355	3.527	3.756
200	14	0.50	2.0	LR	3.150	3.340	3.517	3.739	3.142	3.360	3.543	3.794
200	14	0.50	2.0	GW	3.124	3.335	3.509	3.728	3.132	3.333	3.517	3.749
200	14	0.50	2.0	PW	3.167	3.357	3.535	3.763	3.147	3.339	3.519	3.766
200	14	0.50	3.0	LR	3.147	3.354	3.550	3.814	3.157	3.363	3.555	3.748
200	14	0.50	3.0	GW	3.105	3.326	3.505	3.726	3.116	3.332	3.511	3.750
200	14	0.50	3.0	PW	3.132	3.335	3.517	3.752	3.142	3.345	3.541	3.718
200	14	0.75	1.5	LR	3.156	3.357	3.549	3.779	3.154	3.362	3.571	3.790
200	14	0.75	1.5	GW	3.161	3.375	3.565	3.781	3.142	3.347	3.538	3.765
200	14	0.75	1.5	PW	3.157	3.344	3.545	3.791	3.174	3.393	3.573	3.794
200	14	0.75	2.0	LR	3.163	3.388	3.568	3.798	3.158	3.359	3.567	3.805
200	14	0.75	2.0	GW	3.132	3.346	3.544	3.768	3.144	3.353	3.533	3.768
200	14	0.75	2.0	PW	3.150	3.369	3.562	3.761	3.133	3.345	3.502	3.734
200	14	0.75	3.0	LR	3.156	3.360	3.548	3.784	3.163	3.372	3.578	3.800
200	14	0.75	3.0	GW	3.122	3.335	3.517	3.762	3.145	3.344	3.529	3.737
200	14	0.75	3.0	PW	3.151	3.357	3.555	3.768	3.128	3.323	3.513	3.754

TABLE 11. Critical values for $T_{200,D}(\tau_0)$ with $D = \lfloor\sqrt{n}\rfloor$.

n	D	10%	5%	2.5%	1%
100	5	3.806	4.100	4.389	4.766
100	10	3.634	3.970	4.299	4.729
200	10	3.806	4.100	4.389	4.766
200	14	3.722	4.034	4.341	4.741
200	20	3.634	3.970	4.299	4.729

TABLE 12. Asymptotic critical values for $T_{n,D}(\tau_0)$ and $T^{\sigma}_{n,D}(\tau_0)$.

2.4. Critical values for the statistic $T^{\sigma}_{n,D}$. Recall that the test based on the MOSUM-type statistic $T^{\sigma}_{n,D}(\tau_0)$ of the form (2.52) can be used in the particular cases of $\eta = \gamma$ or under the assumption that $C_1, C_2, \dots, C_n$ are i.i.d. and the change can occur in the lifetime distribution only. We suppose KGM with a change point (see Chapter 1), which satisfies the condition $\gamma = \eta$. We use permutation principle to obtain critical values as for $T^{\sigma}_{n}(\tau_0)$ in Subsection 2.2.

We proceed with the sample sizes $n = 100; 200$ and $D = 0.05n$ or $0.1n$ or $D = \lfloor\sqrt{n}\rfloor$ as follows:

(1) $X^0_1, X^0_2, \dots, X^0_n$ are simulated using the chosen combination of parameters

$$X^0_1, X^0_2, \dots, X^0_{\lfloor n\gamma\rfloor} \sim F_1, \qquad F_1 = \mathrm{E}(1) \text{ (or L(1))}$$

$$X^0_{\lfloor n\gamma\rfloor+1}, X^0_{\lfloor n\gamma\rfloor+2}, \dots, X^0_n \sim F_2, \qquad F_2 = \mathrm{E}(\delta_n) \text{ (or L}(\delta_n))$$

(we use $\gamma = 0.25; 0.5$, $\delta_n = 1; 2; 3$).

(2) $C_1, C_2, \dots, C_n$ fulfilling KGM are simulated (we use the censoring parameter $\beta = 0; 0.5; 1$).

(3) Pairs $(X_1, \Delta_1), (X_2, \Delta_2), \dots, (X_n, \Delta_n)$ are computed.

(4) A random permutation $\boldsymbol{q} = (q_1, q_2, \dots, q_n)$ of $(1, 2, \dots, n)$ is generated.

(5) $T^{\sigma}_{n}(\tau_0, \boldsymbol{Q})$ with $\boldsymbol{Q} = \boldsymbol{q}$ is calculated and its value stored.

(6) The steps (4) – (5) are repeated 10^4 times.

(7) Empirical critical values related to the empirical distribution function of $T^{\sigma}_{n,D}(\tau_0, \boldsymbol{Q})$ are computed and used as an estimator of the actual ones.

In Tables 13 – 17 the simulated critical values for various choices of parameters n, D, γ, β and δ_n are reported. Recall that $\beta = 0$ means no censoring and $\delta_n = 1$ means no change-point in our censorship model.

It is evident that the asymptotic critical values in Table 12 are conservative with respect to the permutation procedure (see Tables 13 – 17), in other words we do not reject the hypothesis in more cases using the limit decision rule. The size of the differences between the empirical and the asymptotic counterparts increases with size $1 - \alpha$. Comparing the obtained sample critical values in below parts of Tables 13 and 14 with $D = 10$ with the simulations for uncensored data made by Hušková and Slabý [**24**], we see the similar results. Further, we observe that the simulated critical values are almost not influenced by the amount of the change expressed by δ_n, the location of a change $\lfloor\gamma n\rfloor$ and the underlying (exponential or log-normal) distribution function. There is also no visible effect of the expected proportion of censoring $\beta/(1 + \beta)$. The critical values obtained through the permutation principle are only slightly influenced by the choice of the weights $w_n(t)$. Notice that the critical values for $T^{\sigma}_{n,D}(\tau_0)$, i.e. critical values for $T^{\sigma}_{n,D}(\tau_0, \boldsymbol{Q})$ obtained under the restricted null hypothesis $\bar{H}_0$, can be found in Tables 13, 15 or 17 in the top frame (the rows with $\delta_n = 1$).

						exponential				log-normal			
n	D	γ	δ_n	β	w_n	10%	5%	2.5%	1%	10%	5%	2.5%	1%
100	5	0.25	1	0	LR	3.258	3.495	3.718	3.975	3.283	3.543	3.762	4.012
100	5	0.25	1	0	GW	3.063	3.237	3.390	3.564	3.052	3.226	3.368	3.543
100	5	0.25	1	0	PW	3.052	3.226	3.379	3.575	3.052	3.237	3.390	3.553
100	5	0.25	1	0.5	LR	3.233	3.441	3.642	3.878	3.205	3.421	3.605	3.814
100	5	0.25	1	0.5	GW	3.067	3.236	3.398	3.593	3.066	3.245	3.398	3.551
100	5	0.25	1	0.5	PW	3.065	3.228	3.387	3.590	3.064	3.233	3.388	3.565
100	5	0.25	1	1	LR	3.115	3.298	3.452	3.647	3.152	3.365	3.545	3.794
100	5	0.25	1	1	GW	3.111	3.294	3.485	3.653	3.107	3.286	3.434	3.652
100	5	0.25	1	1	PW	3.085	3.252	3.402	3.599	3.087	3.274	3.410	3.599
100	5	0.25	2	0	LR	3.258	3.506	3.714	3.973	3.241	3.471	3.678	3.936
100	5	0.25	2	0	GW	3.063	3.237	3.379	3.575	3.052	3.226	3.368	3.575
100	5	0.25	2	0	PW	3.063	3.237	3.412	3.608	3.052	3.216	3.368	3.553
100	5	0.25	2	0.5	LR	3.204	3.420	3.620	3.822	3.171	3.383	3.571	3.798
100	5	0.25	2	0.5	GW	3.077	3.248	3.411	3.601	3.069	3.238	3.406	3.588
100	5	0.25	2	0.5	PW	3.083	3.243	3.390	3.556	3.052	3.226	3.378	3.532
100	5	0.25	2	1	LR	3.122	3.300	3.493	3.690	3.104	3.298	3.456	3.660
100	5	0.25	2	1	GW	3.080	3.269	3.410	3.612	3.079	3.265	3.423	3.629
100	5	0.25	2	1	PW	3.088	3.267	3.451	3.651	3.074	3.255	3.399	3.595
100	5	0.25	3	0	LR	3.243	3.471	3.653	3.900	3.253	3.474	3.705	3.960
100	5	0.25	3	0	GW	3.063	3.226	3.379	3.543	3.052	3.226	3.368	3.543
100	5	0.25	3	0	PW	3.041	3.216	3.368	3.532	3.052	3.226	3.379	3.586
100	5	0.25	3	0.5	LR	3.154	3.365	3.555	3.788	3.169	3.374	3.553	3.786
100	5	0.25	3	0.5	GW	3.062	3.235	3.403	3.589	3.066	3.245	3.391	3.538
100	5	0.25	3	0.5	PW	3.073	3.246	3.420	3.600	3.073	3.234	3.398	3.558
100	5	0.25	3	1	LR	3.103	3.301	3.458	3.695	3.170	3.370	3.538	3.777
100	5	0.25	3	1	GW	3.109	3.325	3.486	3.718	3.097	3.277	3.450	3.644
100	5	0.25	3	1	PW	3.088	3.269	3.438	3.647	3.084	3.260	3.431	3.630
100	10	0.25	1	0	LR	3.021	3.241	3.445	3.737	3.038	3.273	3.489	3.701
100	10	0.25	1	0	GW	2.983	3.199	3.384	3.607	3.014	3.206	3.407	3.638
100	10	0.25	1	0	PW	2.991	3.191	3.384	3.569	3.014	3.214	3.376	3.592
100	10	0.25	1	0.5	LR	3.011	3.223	3.415	3.662	3.009	3.223	3.416	3.664
100	10	0.25	1	0.5	GW	3.006	3.219	3.412	3.625	3.012	3.216	3.406	3.601
100	10	0.25	1	0.5	PW	3.005	3.222	3.396	3.643	3.006	3.203	3.371	3.586
100	10	0.25	1	1	LR	3.000	3.231	3.423	3.674	3.019	3.226	3.431	3.678
100	10	0.25	1	1	GW	2.990	3.205	3.382	3.591	3.012	3.211	3.415	3.629
100	10	0.25	1	1	PW	3.005	3.227	3.402	3.654	3.003	3.210	3.400	3.606
100	10	0.25	2	0	LR	3.009	3.232	3.438	3.697	3.021	3.258	3.476	3.733
100	10	0.25	2	0	GW	3.006	3.222	3.414	3.615	3.021	3.222	3.399	3.615
100	10	0.25	2	0	PW	2.998	3.191	3.368	3.584	3.006	3.199	3.376	3.592
100	10	0.25	2	0.5	LR	3.011	3.233	3.451	3.664	3.024	3.251	3.451	3.679
100	10	0.25	2	0.5	GW	3.023	3.219	3.414	3.619	3.009	3.212	3.397	3.590
100	10	0.25	2	0.5	PW	3.003	3.217	3.421	3.644	3.005	3.209	3.398	3.630
100	10	0.25	2	1	LR	3.024	3.238	3.429	3.658	3.007	3.211	3.401	3.591
100	10	0.25	2	1	GW	2.993	3.225	3.421	3.641	3.016	3.222	3.406	3.673
100	10	0.25	2	1	PW	2.988	3.198	3.395	3.613	3.013	3.217	3.415	3.645
100	10	0.25	3	0	LR	3.019	3.243	3.463	3.702	3.011	3.236	3.429	3.666
100	10	0.25	3	0	GW	3.021	3.222	3.407	3.584	3.006	3.222	3.399	3.607
100	10	0.25	3	0	PW	3.014	3.214	3.407	3.646	2.998	3.199	3.391	3.592
100	10	0.25	3	0.5	LR	3.040	3.257	3.445	3.672	3.039	3.254	3.448	3.680
100	10	0.25	3	0.5	GW	3.005	3.198	3.391	3.634	3.005	3.194	3.379	3.618
100	10	0.25	3	0.5	PW	3.009	3.216	3.404	3.603	3.008	3.218	3.418	3.609
100	10	0.25	3	1	LR	3.009	3.217	3.409	3.611	3.011	3.219	3.413	3.630
100	10	0.25	3	1	GW	3.004	3.215	3.410	3.637	3.015	3.217	3.390	3.626
100	10	0.25	3	1	PW	3.005	3.201	3.396	3.604	3.009	3.204	3.379	3.582

TABLE 13. Critical values for $T^{\sigma}_{100,D}(\tau_0, \boldsymbol{Q})$ with $\gamma = 0.25$.

						exponential				log-normal			
n	D	γ	δ_n	β	w_n	10%	5%	2.5%	1%	10%	5%	2.5%	1%
100	5	0.5	2	0	LR	3.248	3.473	3.684	3.965	3.265	3.497	3.686	3.955
100	5	0.5	2	0	GW	3.063	3.226	3.368	3.553	3.052	3.237	3.401	3.575
100	5	0.5	2	0	PW	3.063	3.226	3.379	3.553	3.052	3.237	3.379	3.532
100	5	0.5	2	0.5	LR	3.205	3.421	3.632	3.899	3.168	3.367	3.558	3.766
100	5	0.5	2	0.5	GW	3.078	3.265	3.417	3.597	3.071	3.262	3.396	3.574
100	5	0.5	2	0.5	PW	3.079	3.252	3.421	3.617	3.061	3.239	3.397	3.574
100	5	0.5	2	1	LR	3.145	3.358	3.534	3.770	3.090	3.279	3.473	3.673
100	5	0.5	2	1	GW	3.113	3.288	3.448	3.646	3.119	3.297	3.459	3.646
100	5	0.5	2	1	PW	3.082	3.259	3.417	3.610	3.096	3.284	3.446	3.636
100	5	0.5	3	0	LR	3.285	3.523	3.735	3.992	3.255	3.481	3.691	3.954
100	5	0.5	3	0	GW	3.063	3.237	3.379	3.564	3.063	3.226	3.379	3.553
100	5	0.5	3	0	PW	3.074	3.226	3.390	3.553	3.052	3.216	3.368	3.521
100	5	0.5	3	0.5	LR	3.159	3.367	3.555	3.758	3.161	3.375	3.555	3.772
100	5	0.5	3	0.5	GW	3.098	3.275	3.417	3.602	3.085	3.272	3.431	3.618
100	5	0.5	3	0.5	PW	3.084	3.248	3.386	3.569	3.057	3.234	3.385	3.574
100	5	0.5	3	1	LR	3.176	3.366	3.549	3.732	3.066	3.257	3.411	3.583
100	5	0.5	3	1	GW	3.073	3.253	3.406	3.592	3.113	3.302	3.466	3.687
100	5	0.5	3	1	PW	3.085	3.273	3.431	3.630	3.086	3.288	3.478	3.682
100	10	0.5	2	0	LR	3.020	3.248	3.477	3.734	3.007	3.230	3.431	3.645
100	10	0.5	2	0	GW	3.014	3.214	3.384	3.607	3.014	3.207	3.399	3.630
100	10	0.5	2	0	PW	2.998	3.214	3.376	3.615	2.998	3.206	3.391	3.653
100	10	0.5	2	0.5	LR	3.006	3.199	3.386	3.605	3.018	3.217	3.424	3.683
100	10	0.5	2	0.5	GW	3.013	3.218	3.402	3.596	3.008	3.203	3.401	3.614
100	10	0.5	2	0.5	PW	3.004	3.228	3.431	3.667	3.015	3.226	3.431	3.634
100	10	0.5	2	1	LR	2.994	3.200	3.408	3.655	3.006	3.230	3.424	3.663
100	10	0.5	2	1	GW	3.011	3.213	3.395	3.633	2.994	3.197	3.394	3.661
100	10	0.5	2	1	PW	3.002	3.218	3.406	3.646	3.000	3.204	3.395	3.637
100	10	0.5	3	0	LR	2.999	3.226	3.444	3.707	3.025	3.227	3.425	3.678
100	10	0.5	3	0	GW	3.006	3.229	3.414	3.607	3.014	3.222	3.384	3.592
100	10	0.5	3	0	PW	3.006	3.229	3.399	3.676	2.983	3.199	3.376	3.584
100	10	0.5	3	0.5	LR	3.037	3.251	3.442	3.632	3.011	3.221	3.402	3.624
100	10	0.5	3	0.5	GW	2.989	3.203	3.406	3.605	2.996	3.198	3.379	3.576
100	10	0.5	3	0.5	PW	2.989	3.179	3.361	3.589	3.011	3.221	3.415	3.669
100	10	0.5	3	1	LR	3.004	3.211	3.417	3.629	3.008	3.228	3.405	3.634
100	10	0.5	3	1	GW	3.006	3.228	3.419	3.625	3.005	3.224	3.418	3.631
100	10	0.5	3	1	PW	3.000	3.198	3.367	3.629	3.015	3.208	3.390	3.599

TABLE 14. Critical values for $T^{\sigma}_{100,D}(\tau_0, \boldsymbol{Q})$ with $\gamma = 0.5$.

n	D	γ	δ_n	β	w_n	exponential 10%	exponential 5%	exponential 2.5%	exponential 1%	log-normal 10%	log-normal 5%	log-normal 2.5%	log-normal 1%
200	10	0.25	1	0	LR	3.351	3.585	3.818	4.096	3.348	3.569	3.779	4.077
200	10	0.25	1	0	GW	3.249	3.438	3.616	3.794	3.230	3.412	3.574	3.771
200	10	0.25	1	0	PW	3.257	3.439	3.628	3.856	3.234	3.427	3.604	3.794
200	10	0.25	1	0.5	LR	3.298	3.504	3.685	3.956	3.324	3.538	3.736	3.990
200	10	0.25	1	0.5	GW	3.262	3.455	3.639	3.827	3.242	3.440	3.616	3.807
200	10	0.25	1	0.5	PW	3.250	3.443	3.619	3.858	3.250	3.436	3.605	3.841
200	10	0.25	1	1	LR	3.295	3.505	3.698	3.960	3.276	3.499	3.693	3.919
200	10	0.25	1	1	GW	3.255	3.452	3.638	3.875	3.261	3.454	3.629	3.845
200	10	0.25	1	1	PW	3.251	3.451	3.631	3.842	3.265	3.455	3.619	3.873
200	10	0.25	2	0	LR	3.351	3.589	3.810	4.072	3.359	3.592	3.809	4.065
200	10	0.25	2	0	GW	3.276	3.458	3.635	3.840	3.245	3.423	3.601	3.802
200	10	0.25	2	0	PW	3.257	3.450	3.620	3.825	3.241	3.423	3.597	3.809
200	10	0.25	2	0.5	LR	3.314	3.544	3.752	3.979	3.337	3.557	3.741	3.968
200	10	0.25	2	0.5	GW	3.260	3.457	3.631	3.862	3.266	3.455	3.637	3.845
200	10	0.25	2	0.5	PW	3.254	3.446	3.611	3.841	3.257	3.457	3.626	3.846
200	10	0.25	2	1	LR	3.296	3.500	3.687	3.935	3.295	3.494	3.670	3.890
200	10	0.25	2	1	GW	3.266	3.458	3.623	3.840	3.263	3.455	3.638	3.870
200	10	0.25	2	1	PW	3.264	3.461	3.633	3.840	3.259	3.451	3.637	3.837
200	10	0.25	3	0	LR	3.345	3.574	3.772	4.067	3.345	3.571	3.760	4.006
200	10	0.25	3	0	GW	3.257	3.446	3.616	3.840	3.241	3.438	3.604	3.809
200	10	0.25	3	0	PW	3.237	3.419	3.601	3.786	3.245	3.435	3.612	3.829
200	10	0.25	3	0.5	LR	3.324	3.542	3.743	4.001	3.318	3.544	3.750	4.014
200	10	0.25	3	0.5	GW	3.246	3.432	3.607	3.827	3.252	3.441	3.604	3.805
200	10	0.25	3	0.5	PW	3.243	3.425	3.594	3.787	3.246	3.439	3.613	3.809
200	10	0.25	3	1	LR	3.284	3.484	3.674	3.898	3.304	3.504	3.686	3.917
200	10	0.25	3	1	GW	3.264	3.465	3.635	3.854	3.269	3.472	3.667	3.883
200	10	0.25	3	1	PW	3.253	3.456	3.635	3.840	3.255	3.451	3.629	3.867
200	20	0.25	1	0	LR	3.120	3.356	3.577	3.834	3.096	3.330	3.558	3.800
200	20	0.25	1	0	GW	3.112	3.316	3.527	3.734	3.106	3.327	3.527	3.748
200	20	0.25	1	0	PW	3.098	3.325	3.543	3.797	3.133	3.355	3.535	3.775
200	20	0.25	1	0.5	LR	3.119	3.327	3.534	3.783	3.130	3.363	3.558	3.801
200	20	0.25	1	0.5	GW	3.113	3.337	3.529	3.745	3.123	3.345	3.549	3.759
200	20	0.25	1	0.5	PW	3.125	3.316	3.509	3.723	3.119	3.337	3.552	3.793
200	20	0.25	1	1	LR	3.127	3.344	3.521	3.769	3.119	3.348	3.558	3.833
200	20	0.25	1	1	GW	3.111	3.317	3.526	3.757	3.113	3.342	3.549	3.761
200	20	0.25	1	1	PW	3.130	3.343	3.546	3.794	3.116	3.339	3.530	3.801
200	20	0.25	2	0	LR	3.116	3.342	3.555	3.811	3.142	3.372	3.590	3.841
200	20	0.25	2	0	GW	3.120	3.327	3.532	3.764	3.133	3.355	3.546	3.808
200	20	0.25	2	0	PW	3.114	3.325	3.502	3.737	3.120	3.336	3.508	3.726
200	20	0.25	2	0.5	LR	3.121	3.349	3.549	3.813	3.116	3.343	3.563	3.833
200	20	0.25	2	0.5	GW	3.125	3.349	3.552	3.795	3.106	3.328	3.545	3.768
200	20	0.25	2	0.5	PW	3.123	3.338	3.550	3.802	3.134	3.365	3.564	3.821
200	20	0.25	2	1	LR	3.117	3.354	3.549	3.780	3.122	3.324	3.518	3.775
200	20	0.25	2	1	GW	3.123	3.327	3.520	3.756	3.116	3.328	3.512	3.748
200	20	0.25	2	1	PW	3.118	3.339	3.529	3.800	3.121	3.328	3.522	3.783
200	20	0.25	3	0	LR	3.115	3.343	3.549	3.798	3.125	3.340	3.541	3.808
200	20	0.25	3	0	GW	3.120	3.344	3.543	3.764	3.144	3.371	3.571	3.764
200	20	0.25	3	0	PW	3.114	3.330	3.532	3.748	3.117	3.300	3.483	3.734
200	20	0.25	3	0.5	LR	3.131	3.351	3.553	3.792	3.114	3.344	3.567	3.777
200	20	0.25	3	0.5	GW	3.104	3.340	3.530	3.775	3.118	3.346	3.518	3.754
200	20	0.25	3	0.5	PW	3.134	3.343	3.538	3.779	3.104	3.323	3.507	3.737
200	20	0.25	3	1	LR	3.104	3.336	3.531	3.768	3.127	3.342	3.581	3.815
200	20	0.25	3	1	GW	3.105	3.313	3.544	3.763	3.113	3.338	3.514	3.766
200	20	0.25	3	1	PW	3.122	3.324	3.544	3.767	3.112	3.336	3.503	3.758

TABLE 15. Critical values for $T^{\sigma}_{200,D}(\tau_0, \boldsymbol{Q})$ with $\gamma = 0.25$.

						exponential				log-normal			
n	D	γ	δ_n	β	w_n	10%	5%	2.5%	1%	10%	5%	2.5%	1%
200	10	0.5	2	0	LR	3.341	3.573	3.795	4.073	3.350	3.602	3.830	4.084
200	10	0.5	2	0	GW	3.245	3.442	3.628	3.894	3.249	3.423	3.593	3.825
200	10	0.5	2	0	PW	3.253	3.438	3.612	3.813	3.253	3.442	3.612	3.786
200	10	0.5	2	0.5	LR	3.340	3.553	3.729	3.992	3.295	3.506	3.683	3.914
200	10	0.5	2	0.5	GW	3.249	3.432	3.601	3.842	3.242	3.438	3.604	3.845
200	10	0.5	2	0.5	PW	3.253	3.444	3.604	3.801	3.248	3.438	3.602	3.815
200	10	0.5	2	1	LR	3.312	3.500	3.697	3.935	3.278	3.490	3.670	3.916
200	10	0.5	2	1	GW	3.284	3.483	3.647	3.862	3.251	3.441	3.604	3.831
200	10	0.5	2	1	PW	3.243	3.431	3.593	3.817	3.265	3.458	3.648	3.878
200	10	0.5	3	0	LR	3.343	3.579	3.803	4.056	3.349	3.569	3.792	4.056
200	10	0.5	3	0	GW	3.261	3.465	3.643	3.859	3.237	3.427	3.601	3.809
200	10	0.5	3	0	PW	3.234	3.427	3.604	3.805	3.253	3.442	3.608	3.840
200	10	0.5	3	0.5	LR	3.272	3.486	3.683	3.896	3.328	3.550	3.768	4.022
200	10	0.5	3	0.5	GW	3.256	3.440	3.601	3.810	3.240	3.442	3.621	3.840
200	10	0.5	3	0.5	PW	3.245	3.443	3.636	3.865	3.250	3.457	3.628	3.823
200	10	0.5	3	1	LR	3.314	3.516	3.707	3.936	3.329	3.560	3.756	4.007
200	10	0.5	3	1	GW	3.274	3.472	3.646	3.835	3.258	3.448	3.638	3.848
200	10	0.5	3	1	PW	3.243	3.436	3.597	3.823	3.262	3.454	3.619	3.798
200	20	0.5	2	0	LR	3.117	3.336	3.561	3.833	3.096	3.340	3.539	3.772
200	20	0.5	2	0	GW	3.139	3.374	3.565	3.808	3.131	3.360	3.554	3.767
200	20	0.5	2	0	PW	3.136	3.357	3.576	3.778	3.122	3.352	3.524	3.740
200	20	0.5	2	0.5	LR	3.110	3.315	3.517	3.738	3.122	3.345	3.564	3.839
200	20	0.5	2	0.5	GW	3.117	3.331	3.528	3.764	3.120	3.330	3.536	3.731
200	20	0.5	2	0.5	PW	3.132	3.358	3.576	3.792	3.127	3.358	3.549	3.781
200	20	0.5	2	1	LR	3.128	3.345	3.556	3.811	3.135	3.352	3.555	3.818
200	20	0.5	2	1	GW	3.116	3.323	3.526	3.766	3.124	3.333	3.538	3.791
200	20	0.5	2	1	PW	3.125	3.344	3.528	3.778	3.105	3.318	3.516	3.746
200	20	0.5	3	0	LR	3.117	3.350	3.548	3.790	3.112	3.335	3.545	3.823
200	20	0.5	3	0	GW	3.117	3.355	3.557	3.764	3.131	3.360	3.549	3.803
200	20	0.5	3	0	PW	3.122	3.349	3.535	3.773	3.136	3.341	3.551	3.775
200	20	0.5	3	0.5	LR	3.108	3.334	3.548	3.802	3.118	3.331	3.531	3.773
200	20	0.5	3	0.5	GW	3.120	3.331	3.549	3.760	3.108	3.350	3.540	3.784
200	20	0.5	3	0.5	PW	3.130	3.351	3.553	3.829	3.127	3.350	3.542	3.778
200	20	0.5	3	1	LR	3.112	3.327	3.529	3.751	3.117	3.331	3.526	3.748
200	20	0.5	3	1	GW	3.110	3.339	3.541	3.774	3.103	3.313	3.503	3.731
200	20	0.5	3	1	PW	3.120	3.329	3.536	3.752	3.127	3.348	3.546	3.774

TABLE 16. Critical values for $T^{\sigma}_{200,D}(\tau_0, \boldsymbol{Q})$ with $\gamma = 0.5$.

n	D	γ	δ_n	β	w_n	exponential 10%	5%	2.5%	1%	log-normal 10%	5%	2.5%	1%
200	14	0.25	1	0	LR	3.249	3.462	3.657	3.903	3.241	3.464	3.672	3.905
200	14	0.25	1	0	GW	3.203	3.412	3.608	3.791	3.203	3.406	3.592	3.837
200	14	0.25	1	0	PW	3.193	3.392	3.575	3.820	3.210	3.406	3.585	3.850
200	14	0.25	1	0.5	LR	3.220	3.434	3.616	3.849	3.236	3.459	3.671	3.909
200	14	0.25	1	0.5	GW	3.189	3.397	3.583	3.821	3.195	3.401	3.592	3.821
200	14	0.25	1	0.5	PW	3.204	3.408	3.593	3.825	3.191	3.394	3.587	3.815
200	14	0.25	1	1	LR	3.230	3.438	3.623	3.825	3.217	3.432	3.652	3.887
200	14	0.25	1	1	GW	3.207	3.412	3.603	3.840	3.201	3.402	3.587	3.814
200	14	0.25	1	1	PW	3.191	3.409	3.584	3.772	3.202	3.407	3.579	3.765
200	14	0.25	2	0	LR	3.235	3.451	3.657	3.903	3.228	3.454	3.654	3.901
200	14	0.25	2	0	GW	3.203	3.406	3.588	3.810	3.193	3.412	3.582	3.781
200	14	0.25	2	0	PW	3.207	3.415	3.618	3.840	3.197	3.399	3.575	3.778
200	14	0.25	2	0.5	LR	3.232	3.436	3.630	3.865	3.232	3.450	3.676	3.892
200	14	0.25	2	0.5	GW	3.194	3.411	3.618	3.887	3.193	3.398	3.585	3.790
200	14	0.25	2	0.5	PW	3.209	3.420	3.608	3.836	3.213	3.424	3.619	3.877
200	14	0.25	2	1	LR	3.214	3.424	3.627	3.872	3.216	3.438	3.626	3.854
200	14	0.25	2	1	GW	3.188	3.398	3.596	3.817	3.204	3.428	3.622	3.800
200	14	0.25	2	1	PW	3.211	3.396	3.582	3.803	3.207	3.413	3.608	3.838
200	14	0.25	3	0	LR	3.258	3.504	3.710	3.967	3.227	3.447	3.672	3.932
200	14	0.25	3	0	GW	3.197	3.402	3.569	3.774	3.197	3.386	3.569	3.774
200	14	0.25	3	0	PW	3.200	3.412	3.602	3.824	3.187	3.402	3.585	3.827
200	14	0.25	3	0.5	LR	3.222	3.436	3.630	3.891	3.221	3.438	3.631	3.878
200	14	0.25	3	0.5	GW	3.194	3.391	3.568	3.780	3.180	3.392	3.574	3.811
200	14	0.25	3	0.5	PW	3.187	3.381	3.584	3.777	3.203	3.409	3.589	3.799
200	14	0.25	3	1	LR	3.211	3.421	3.623	3.855	3.222	3.431	3.628	3.864
200	14	0.25	3	1	GW	3.203	3.428	3.619	3.866	3.205	3.405	3.600	3.825
200	14	0.25	3	1	PW	3.220	3.412	3.592	3.810	3.208	3.407	3.591	3.810
200	14	0.50	2	0	LR	3.246	3.491	3.694	3.969	3.217	3.449	3.666	3.942
200	14	0.50	2	0	GW	3.216	3.438	3.605	3.837	3.216	3.415	3.582	3.814
200	14	0.50	2	0	PW	3.193	3.409	3.588	3.794	3.200	3.412	3.595	3.807
200	14	0.50	2	0.5	LR	3.244	3.453	3.653	3.909	3.204	3.419	3.605	3.862
200	14	0.50	2	0.5	GW	3.217	3.407	3.592	3.828	3.215	3.407	3.583	3.807
200	14	0.50	2	0.5	PW	3.206	3.424	3.602	3.818	3.201	3.400	3.595	3.815
200	14	0.50	2	1	LR	3.231	3.449	3.631	3.880	3.230	3.440	3.636	3.859
200	14	0.50	2	1	GW	3.197	3.390	3.571	3.786	3.203	3.426	3.612	3.853
200	14	0.50	2	1	PW	3.201	3.415	3.602	3.834	3.209	3.417	3.593	3.843
200	14	0.50	3	0	LR	3.230	3.459	3.684	3.951	3.245	3.475	3.681	3.936
200	14	0.50	3	0	GW	3.200	3.409	3.588	3.817	3.210	3.412	3.601	3.814
200	14	0.50	3	0	PW	3.213	3.422	3.605	3.846	3.206	3.412	3.614	3.837
200	14	0.50	3	0.5	LR	3.235	3.468	3.682	3.943	3.204	3.412	3.615	3.871
200	14	0.50	3	0.5	GW	3.205	3.413	3.605	3.819	3.196	3.401	3.589	3.784
200	14	0.50	3	0.5	PW	3.209	3.418	3.610	3.824	3.210	3.403	3.576	3.784
200	14	0.50	3	1	LR	3.212	3.420	3.626	3.871	3.211	3.431	3.606	3.816
200	14	0.50	3	1	GW	3.205	3.428	3.625	3.866	3.198	3.397	3.608	3.809
200	14	0.50	3	1	PW	3.196	3.404	3.594	3.801	3.199	3.405	3.593	3.818

TABLE 17. Critical values for $T^{\sigma}_{200,D}(\tau_0, \boldsymbol{Q})$ with $D = \lfloor\sqrt{n}\rfloor$.

3. Power

3.1. Power of the statistic T_n. Now, we focus on the behavior of the test statistic $T_n(\tau_0)$ under the alternative H_1 that the change in the distribution of the survival times occurs, because beside observing the first type error of the test it is necessary to mention the second type error. In other words we address our attention on the power of the test. Recall that our situation is complicated by the change in the distribution of the censoring variables.

We proceed with $n = 100; 200$ as follows:

(1) The survival times $X_1^0, X_2^0, \dots, X_n^0$ are generated from the model

$$X_1^0, X_2^0, \dots, X_{\lfloor n\gamma \rfloor}^0 \sim F_1, \qquad F_1 = \mathrm{E}(1) \text{ (or } \mathrm{L}(1); \mathrm{W}(1))$$
$$X_{\lfloor n\gamma \rfloor+1}^0, X_{\lfloor n\gamma \rfloor+2}^0, \dots, X_n^0 \sim F_2, \qquad F_2 = \mathrm{E}(\delta_n) \text{ (or } \mathrm{L}(\delta_n); \mathrm{W}(\delta_n))$$

(we use $\gamma = 0.25; 0.5$, $\delta_n = 1.5; 2; 3; 4$ for the exponential and the log-normal distribution and $\delta_n = 1.2; 1.4; 1.6; 1.8$ for the Weibull distribution).

(2) The censoring times $C_1, C_2, \dots, C_n$ are generated from the model

$$C_1, C_2, \dots, C_{\lfloor n\eta \rfloor} \sim G_1, \qquad G_1 = \mathrm{E}(1) \text{ (or } \mathrm{L}(1); \mathrm{W}(1))$$
$$C_{\lfloor n\eta \rfloor+1}, C_{\lfloor n\eta \rfloor+2}, \dots, C_n \sim G_2, \qquad G_2 = \mathrm{E}(\delta_{C,n}) \text{ (or } \mathrm{L}(\delta_{C,n}); \mathrm{W}(\delta_{C,n}))$$

(we use $\eta = 0.25; 0.5; 0.75$, $\delta_{C,n} = 2; 3$ for the exponential and the log-normal distribution and $\delta_{C,n} = 1.4; 1.6$ for the Weibull distribution).

(3) The pairs of observations $(X_1, \Delta_1), (X_2, \Delta_2), \dots, (X_n, \Delta_n)$ are computed.

(4) The value of the statistic $T_n(\tau_0)$ is calculated and compared with the asymptotic critical value.

(5) The steps $(1) - (4)$ are repeated 10^4 times.

(6) The relative frequency of the rejected cases is determined.

In Tables $18 - 23$ the simulated power for the max-type test procedure based on the test statistic $T_n(\tau_0)$ for the sample sizes $n = 100$ and 200 and three chosen underlying distributions is presented. The power is determined by 10000 repetitions for the levels $\alpha = 10\%$, 5%, 2.5% and 1% which correspond to columns in the main part of the tables.

Here it is reasonable to recall that in the lifetime model we suppose both a change in the mean and a change in the variance:

(1) the exponential distribution:

$$\mathrm{E}\, X_i^0 = 1, \qquad \operatorname{var} X_i^0 = 1, \qquad 1 \le i \le \lfloor n\gamma \rfloor,$$
$$\mathrm{E}\, X_i^0 = 1/\delta_n, \qquad \operatorname{var} X_i^0 = 1/\delta_n^2, \qquad \lfloor n\gamma \rfloor < i \le n,$$

(2) the log-normal distribution:

$$\mathrm{E} \log(X_i^0) = 0, \qquad \operatorname{var} \log(X_i^0) = 1, \qquad 1 \le i \le \lfloor n\gamma \rfloor,$$
$$\mathrm{E} \log(X_i^0) = -\log(\delta_n), \qquad \operatorname{var} \log(X_i^0) = 1, \qquad \lfloor n\gamma \rfloor < i \le n,$$

and

$$\mathrm{E}\, X_i^0 = \mathrm{e}^{1/2}, \qquad \operatorname{var} X_i^0 = \mathrm{e}(\mathrm{e} - 1), \qquad 1 \le i \le \lfloor n\gamma \rfloor,$$
$$\mathrm{E}\, X_i^0 = \mathrm{e}^{1/2}/\delta_n, \qquad \operatorname{var} X_i^0 = \mathrm{e}(\mathrm{e} - 1)/\delta_{\mathrm{n}}^2, \qquad \lfloor n\gamma \rfloor < i \le n,$$

(3) the Weibull distribution:

$$\mathrm{E}\,X_i^0 = \Gamma\Big(\frac{5}{4}\Big), \qquad \operatorname{var} X_i^0 = \left(\Gamma\Big(\frac{3}{2}\Big) - \Gamma^2\Big(\frac{5}{4}\Big)\right), \qquad 1 \le i \le \lfloor n\gamma \rfloor,$$

$$\mathrm{E}\,X_i^0 = \Gamma\Big(\frac{5}{4}\Big)/\delta_n, \qquad \operatorname{var} X_i^0 = \left(\Gamma\Big(\frac{3}{2}\Big) - \Gamma^2\Big(\frac{5}{4}\Big)\right)/\delta_n^2, \qquad \lfloor n\gamma \rfloor < i \le n,$$

where $\Gamma(s)$ denotes Gamma function.

In Tables 18 − 23, we see that the power is almost not influenced by the choice of η which expresses the location of a change in the distribution of the censoring variables. The results are better for $\gamma = 0.5$ than for $\gamma = 0.25$ which describe the change-points in the distribution of the lifetime variables. Further, the power increases rapidly with the growing change amount δ_n in the survival times, but on the contrary, the power decreases slightly with growing $\delta_{C,n}$ for the censoring times. If we focus on the weights, we observe that in most cases the log-rank type test gives a little higher power regardless of using the exponential (Tables 18 and 21) or the Weibull (Tables 20 and 23) distribution. In the log-normal case (Tables 19 and 22) the Prentice-Wilcoxon test is sometimes better. Comparing Tables 18, 19, 20 with 21, 22, 23, it is evident that the simulated power goes up also with the larger sample size n.

					$\gamma = 0.25$				$\gamma = 0.50$			
n	δ_n	η	$\delta_{C,n}$	w_n	10%	5%	2.5%	1%	10%	5%	2.5%	1%
100	2	0.25	2	LR	0.286	0.157	0.083	0.038	0.418	0.270	0.156	0.073
100	2	0.25	2	GW	0.184	0.093	0.047	0.025	0.311	0.184	0.104	0.047
100	2	0.25	2	PW	0.228	0.115	0.060	0.028	0.363	0.222	0.125	0.060
100	2	0.25	3	LR	0.268	0.143	0.072	0.034	0.365	0.227	0.132	0.065
100	2	0.25	3	GW	0.162	0.081	0.045	0.025	0.268	0.156	0.092	0.045
100	2	0.25	3	PW	0.216	0.110	0.057	0.031	0.320	0.193	0.108	0.053
100	2	0.50	2	LR	0.295	0.163	0.085	0.038	0.425	0.265	0.157	0.072
100	2	0.50	2	GW	0.187	0.094	0.051	0.026	0.334	0.196	0.109	0.051
100	2	0.50	2	PW	0.232	0.114	0.057	0.025	0.379	0.229	0.134	0.057
100	2	0.50	3	LR	0.285	0.155	0.079	0.034	0.396	0.253	0.149	0.067
100	2	0.50	3	GW	0.175	0.085	0.048	0.027	0.299	0.176	0.100	0.050
100	2	0.50	3	PW	0.221	0.113	0.057	0.026	0.347	0.213	0.125	0.058
100	2	0.75	2	LR	0.300	0.162	0.084	0.032	0.468	0.306	0.183	0.086
100	2	0.75	2	GW	0.200	0.100	0.053	0.028	0.351	0.213	0.118	0.054
100	2	0.75	2	PW	0.238	0.123	0.061	0.029	0.402	0.248	0.141	0.060
100	2	0.75	3	LR	0.299	0.167	0.085	0.037	0.447	0.295	0.177	0.083
100	2	0.75	3	GW	0.193	0.095	0.052	0.028	0.336	0.199	0.113	0.051
100	2	0.75	3	PW	0.235	0.122	0.059	0.030	0.393	0.241	0.138	0.061
100	3	0.25	2	LR	0.682	0.482	0.290	0.117	0.838	0.709	0.541	0.325
100	3	0.25	2	GW	0.455	0.248	0.113	0.037	0.707	0.530	0.355	0.175
100	3	0.25	2	PW	0.569	0.359	0.186	0.060	0.780	0.623	0.444	0.241
100	3	0.25	3	LR	0.611	0.403	0.237	0.093	0.764	0.605	0.432	0.242
100	3	0.25	3	GW	0.389	0.200	0.089	0.031	0.626	0.444	0.284	0.127
100	3	0.25	3	PW	0.513	0.306	0.149	0.046	0.707	0.532	0.361	0.185
100	3	0.50	2	LR	0.703	0.512	0.321	0.135	0.864	0.735	0.578	0.359
100	3	0.50	2	GW	0.478	0.271	0.128	0.040	0.742	0.576	0.396	0.201
100	3	0.50	2	PW	0.564	0.352	0.185	0.060	0.800	0.653	0.480	0.271
100	3	0.50	3	LR	0.673	0.483	0.286	0.114	0.804	0.656	0.493	0.289
100	3	0.50	3	GW	0.428	0.228	0.104	0.037	0.679	0.507	0.336	0.167
100	3	0.50	3	PW	0.555	0.335	0.171	0.055	0.748	0.588	0.411	0.219
100	3	0.75	2	LR	0.721	0.530	0.334	0.142	0.897	0.799	0.645	0.415
100	3	0.75	2	GW	0.502	0.290	0.138	0.042	0.778	0.616	0.433	0.229
100	3	0.75	2	PW	0.597	0.379	0.192	0.062	0.832	0.686	0.518	0.298
100	3	0.75	3	LR	0.714	0.523	0.331	0.144	0.879	0.763	0.598	0.383
100	3	0.75	3	GW	0.486	0.278	0.128	0.041	0.743	0.578	0.403	0.207
100	3	0.75	3	PW	0.581	0.363	0.186	0.061	0.820	0.671	0.497	0.286
100	4	0.25	2	LR	0.912	0.795	0.622	0.349	0.979	0.937	0.861	0.691
100	4	0.25	2	GW	0.722	0.507	0.286	0.090	0.922	0.831	0.686	0.455
100	4	0.25	2	PW	0.821	0.636	0.415	0.170	0.955	0.893	0.786	0.571
100	4	0.25	3	LR	0.870	0.731	0.540	0.277	0.951	0.881	0.771	0.568
100	4	0.25	3	GW	0.661	0.436	0.223	0.066	0.865	0.743	0.572	0.333
100	4	0.25	3	PW	0.772	0.569	0.354	0.129	0.921	0.825	0.688	0.459
100	4	0.50	2	LR	0.922	0.814	0.658	0.398	0.984	0.953	0.891	0.746
100	4	0.50	2	GW	0.748	0.540	0.314	0.104	0.938	0.855	0.727	0.504
100	4	0.50	2	PW	0.835	0.655	0.432	0.175	0.964	0.907	0.810	0.616
100	4	0.50	3	LR	0.904	0.786	0.616	0.347	0.966	0.911	0.815	0.630
100	4	0.50	3	GW	0.710	0.488	0.266	0.087	0.905	0.798	0.643	0.414
100	4	0.50	3	PW	0.808	0.620	0.404	0.158	0.941	0.866	0.743	0.525
100	4	0.75	2	LR	0.936	0.839	0.684	0.428	0.988	0.966	0.916	0.787
100	4	0.75	2	GW	0.768	0.562	0.339	0.117	0.952	0.886	0.771	0.552
100	4	0.75	2	PW	0.838	0.672	0.457	0.188	0.971	0.926	0.834	0.645
100	4	0.75	3	LR	0.932	0.832	0.682	0.423	0.984	0.955	0.893	0.744
100	4	0.75	3	GW	0.758	0.556	0.331	0.110	0.938	0.858	0.732	0.512
100	4	0.75	3	PW	0.832	0.661	0.438	0.181	0.967	0.913	0.818	0.623

TABLE 18. Simulated power for $T_{100}(\tau_0)$ and the exponential distribution.

					$\gamma = 0.25$				$\gamma = 0.50$			
n	δ_n	η	$\delta_{C,n}$	w_n	10%	5%	2.5%	1%	10%	5%	2.5%	1%
100	2	0.25	2	LR	0.362	0.211	0.110	0.044	0.555	0.388	0.251	0.128
100	2	0.25	2	GW	0.300	0.149	0.071	0.032	0.537	0.355	0.214	0.098
100	2	0.25	2	PW	0.333	0.175	0.087	0.035	0.552	0.381	0.238	0.112
100	2	0.25	3	LR	0.353	0.199	0.107	0.046	0.509	0.351	0.228	0.110
100	2	0.25	3	GW	0.275	0.134	0.067	0.033	0.479	0.309	0.183	0.083
100	2	0.25	3	PW	0.325	0.169	0.083	0.035	0.520	0.348	0.217	0.108
100	2	0.50	2	LR	0.366	0.208	0.106	0.043	0.577	0.413	0.277	0.146
100	2	0.50	2	GW	0.306	0.148	0.064	0.030	0.572	0.401	0.249	0.114
100	2	0.50	2	PW	0.349	0.183	0.089	0.037	0.589	0.420	0.267	0.131
100	2	0.50	3	LR	0.377	0.213	0.114	0.046	0.558	0.403	0.268	0.144
100	2	0.50	3	GW	0.300	0.146	0.070	0.034	0.535	0.363	0.226	0.108
100	2	0.50	3	PW	0.354	0.186	0.090	0.037	0.553	0.386	0.250	0.121
100	2	0.75	2	LR	0.362	0.203	0.105	0.043	0.607	0.434	0.288	0.145
100	2	0.75	2	GW	0.317	0.155	0.073	0.032	0.598	0.410	0.258	0.124
100	2	0.75	2	PW	0.361	0.191	0.091	0.037	0.617	0.438	0.284	0.136
100	2	0.75	3	LR	0.378	0.215	0.115	0.049	0.609	0.439	0.296	0.149
100	2	0.75	3	GW	0.320	0.159	0.075	0.034	0.585	0.413	0.257	0.123
100	2	0.75	3	PW	0.353	0.183	0.089	0.034	0.606	0.442	0.290	0.144
100	3	0.25	2	LR	0.809	0.637	0.429	0.206	0.942	0.867	0.748	0.555
100	3	0.25	2	GW	0.771	0.539	0.288	0.082	0.950	0.877	0.752	0.521
100	3	0.25	2	PW	0.806	0.612	0.372	0.133	0.955	0.888	0.779	0.567
100	3	0.25	3	LR	0.783	0.611	0.413	0.188	0.916	0.829	0.695	0.486
100	3	0.25	3	GW	0.717	0.468	0.234	0.067	0.909	0.804	0.642	0.393
100	3	0.25	3	PW	0.780	0.567	0.338	0.120	0.925	0.835	0.698	0.477
100	3	0.50	2	LR	0.817	0.647	0.446	0.214	0.950	0.888	0.786	0.595
100	3	0.50	2	GW	0.789	0.569	0.322	0.093	0.958	0.895	0.786	0.563
100	3	0.50	2	PW	0.823	0.621	0.383	0.137	0.961	0.904	0.806	0.609
100	3	0.50	3	LR	0.806	0.638	0.436	0.213	0.934	0.862	0.754	0.554
100	3	0.50	3	GW	0.755	0.525	0.286	0.081	0.939	0.863	0.731	0.497
100	3	0.50	3	PW	0.800	0.604	0.373	0.137	0.942	0.869	0.750	0.545
100	3	0.75	2	LR	0.827	0.661	0.464	0.223	0.962	0.903	0.814	0.637
100	3	0.75	2	GW	0.804	0.592	0.345	0.111	0.967	0.911	0.818	0.607
100	3	0.75	2	PW	0.829	0.636	0.407	0.155	0.974	0.923	0.834	0.648
100	3	0.75	3	LR	0.821	0.654	0.454	0.222	0.952	0.893	0.796	0.613
100	3	0.75	3	GW	0.798	0.577	0.331	0.100	0.961	0.902	0.791	0.579
100	3	0.75	3	PW	0.830	0.640	0.403	0.149	0.963	0.911	0.815	0.618
100	4	0.25	2	LR	0.969	0.911	0.788	0.544	0.997	0.987	0.964	0.892
100	4	0.25	2	GW	0.962	0.870	0.672	0.304	0.998	0.991	0.970	0.884
100	4	0.25	2	PW	0.974	0.905	0.756	0.436	0.998	0.993	0.974	0.910
100	4	0.25	3	LR	0.958	0.888	0.752	0.490	0.992	0.975	0.938	0.840
100	4	0.25	3	GW	0.941	0.817	0.585	0.225	0.993	0.974	0.933	0.796
100	4	0.25	3	PW	0.966	0.880	0.705	0.375	0.996	0.982	0.948	0.841
100	4	0.50	2	LR	0.970	0.914	0.806	0.566	0.997	0.990	0.973	0.915
100	4	0.50	2	GW	0.970	0.889	0.708	0.350	0.998	0.992	0.976	0.916
100	4	0.50	2	PW	0.975	0.913	0.771	0.463	0.999	0.994	0.979	0.924
100	4	0.50	3	LR	0.969	0.909	0.786	0.537	0.995	0.984	0.957	0.881
100	4	0.50	3	GW	0.958	0.855	0.649	0.292	0.996	0.986	0.959	0.864
100	4	0.50	3	PW	0.969	0.897	0.743	0.424	0.998	0.989	0.966	0.889
100	4	0.75	2	LR	0.972	0.915	0.801	0.568	0.998	0.993	0.979	0.931
100	4	0.75	2	GW	0.973	0.902	0.731	0.379	0.999	0.995	0.982	0.929
100	4	0.75	2	PW	0.980	0.926	0.794	0.481	0.999	0.996	0.986	0.945
100	4	0.75	3	LR	0.973	0.916	0.804	0.567	0.998	0.992	0.976	0.918
100	4	0.75	3	GW	0.968	0.892	0.722	0.370	0.999	0.994	0.979	0.916
100	4	0.75	3	PW	0.976	0.918	0.779	0.468	0.999	0.995	0.983	0.926

TABLE 19. Simulated power for $T_{100}(\tau_0)$ and the log-normal distribution.

n	δ_n	η	$\delta_{C,n}$	w_n	$\gamma = 0.25$ 10%	5%	2.5%	1%	$\gamma = 0.50$ 10%	5%	2.5%	1%
100	1.4	0.25	1.4	LR	0.804	0.632	0.428	0.198	0.899	0.798	0.649	0.424
100	1.4	0.25	1.4	GW	0.564	0.337	0.154	0.046	0.790	0.631	0.448	0.235
100	1.4	0.25	1.4	PW	0.705	0.485	0.277	0.099	0.860	0.740	0.577	0.339
100	1.4	0.25	1.6	LR	0.668	0.466	0.278	0.111	0.796	0.642	0.470	0.268
100	1.4	0.25	1.6	GW	0.428	0.220	0.097	0.035	0.646	0.463	0.296	0.133
100	1.4	0.25	1.6	PW	0.587	0.368	0.194	0.062	0.746	0.588	0.414	0.224
100	1.4	0.50	1.4	LR	0.868	0.720	0.529	0.271	0.926	0.840	0.716	0.507
100	1.4	0.50	1.4	GW	0.638	0.408	0.209	0.059	0.844	0.705	0.541	0.314
100	1.4	0.50	1.4	PW	0.760	0.565	0.347	0.125	0.898	0.791	0.646	0.415
100	1.4	0.50	1.6	LR	0.833	0.678	0.478	0.230	0.840	0.717	0.558	0.345
100	1.4	0.50	1.6	GW	0.564	0.336	0.159	0.046	0.744	0.578	0.397	0.202
100	1.4	0.50	1.6	PW	0.721	0.506	0.300	0.108	0.802	0.656	0.491	0.282
100	1.4	0.75	1.4	LR	0.900	0.779	0.608	0.343	0.975	0.933	0.855	0.685
100	1.4	0.75	1.4	GW	0.711	0.485	0.276	0.087	0.913	0.814	0.665	0.430
100	1.4	0.75	1.4	PW	0.793	0.606	0.386	0.148	0.955	0.885	0.768	0.545
100	1.4	0.75	1.6	LR	0.899	0.778	0.604	0.338	0.963	0.912	0.822	0.631
100	1.4	0.75	1.6	GW	0.691	0.460	0.251	0.078	0.889	0.780	0.616	0.377
100	1.4	0.75	1.6	PW	0.790	0.599	0.381	0.146	0.935	0.852	0.721	0.505
100	1.6	0.25	1.4	LR	0.994	0.975	0.924	0.763	0.999	0.996	0.986	0.946
100	1.6	0.25	1.4	GW	0.942	0.834	0.638	0.306	0.992	0.975	0.928	0.792
100	1.6	0.25	1.4	PW	0.976	0.913	0.781	0.492	0.998	0.991	0.967	0.890
100	1.6	0.25	1.6	LR	0.975	0.920	0.813	0.563	0.991	0.974	0.930	0.818
100	1.6	0.25	1.6	GW	0.865	0.684	0.435	0.148	0.961	0.906	0.797	0.576
100	1.6	0.25	1.6	PW	0.945	0.839	0.660	0.345	0.987	0.956	0.890	0.732
100	1.6	0.50	1.4	LR	0.998	0.990	0.961	0.852	0.999	0.998	0.992	0.963
100	1.6	0.50	1.4	GW	0.961	0.878	0.714	0.397	0.995	0.985	0.958	0.869
100	1.6	0.50	1.4	PW	0.982	0.938	0.834	0.582	0.999	0.994	0.981	0.925
100	1.6	0.50	1.6	LR	0.995	0.980	0.935	0.788	0.996	0.986	0.960	0.880
100	1.6	0.50	1.6	GW	0.935	0.822	0.614	0.281	0.985	0.956	0.891	0.729
100	1.6	0.50	1.6	PW	0.975	0.916	0.792	0.504	0.993	0.975	0.937	0.825
100	1.6	0.75	1.4	LR	0.999	0.995	0.978	0.900	1.000	0.999	0.999	0.993
100	1.6	0.75	1.4	GW	0.977	0.923	0.793	0.503	0.999	0.995	0.984	0.935
100	1.6	0.75	1.4	PW	0.988	0.957	0.875	0.645	1.000	0.999	0.994	0.970
100	1.6	0.75	1.6	LR	0.998	0.991	0.972	0.893	1.000	0.999	0.997	0.986
100	1.6	0.75	1.6	GW	0.972	0.902	0.761	0.456	0.998	0.991	0.971	0.906
100	1.6	0.75	1.6	PW	0.985	0.951	0.868	0.624	0.999	0.997	0.989	0.951
100	1.8	0.25	1.4	LR	1.000	1.000	0.998	0.985	1.000	1.000	1.000	0.999
100	1.8	0.25	1.4	GW	0.997	0.984	0.941	0.767	1.000	1.000	0.999	0.989
100	1.8	0.25	1.4	PW	1.000	0.997	0.981	0.892	1.000	1.000	1.000	0.998
100	1.8	0.25	1.6	LR	1.000	0.998	0.991	0.944	1.000	1.000	0.999	0.993
100	1.8	0.25	1.6	GW	0.990	0.956	0.864	0.573	0.999	0.997	0.989	0.950
100	1.8	0.25	1.6	PW	0.998	0.989	0.954	0.799	1.000	1.000	0.997	0.981
100	1.8	0.50	1.4	LR	1.000	1.000	0.999	0.992	1.000	1.000	1.000	1.000
100	1.8	0.50	1.4	GW	0.998	0.991	0.963	0.835	1.000	1.000	0.999	0.996
100	1.8	0.50	1.4	PW	0.999	0.996	0.984	0.917	1.000	1.000	1.000	0.998
100	1.8	0.50	1.6	LR	1.000	1.000	0.998	0.988	1.000	1.000	0.999	0.997
100	1.8	0.50	1.6	GW	0.997	0.983	0.931	0.743	1.000	0.999	0.996	0.979
100	1.8	0.50	1.6	PW	0.999	0.995	0.977	0.880	1.000	1.000	0.999	0.992
100	1.8	0.75	1.4	LR	1.000	1.000	1.000	0.997	1.000	1.000	1.000	1.000
100	1.8	0.75	1.4	GW	0.999	0.995	0.977	0.884	1.000	1.000	1.000	1.000
100	1.8	0.75	1.4	PW	1.000	0.999	0.991	0.940	1.000	1.000	1.000	1.000
100	1.8	0.75	1.6	LR	1.000	1.000	0.999	0.996	1.000	1.000	1.000	1.000
100	1.8	0.75	1.6	GW	0.999	0.995	0.971	0.857	1.000	1.000	1.000	0.997
100	1.8	0.75	1.6	PW	1.000	0.998	0.988	0.933	1.000	1.000	1.000	1.000

TABLE 20. Simulated power for $T_{100}(\tau_0)$ and the Weibull distribution.

					$\gamma = 0.25$				$\gamma = 0.50$			
n	δ_n	η	$\delta_{C,n}$	w_n	10%	5%	2.5%	1%	10%	5%	2.5%	1%
200	1.5	0.25	2	LR	0.223	0.124	0.071	0.036	0.302	0.176	0.101	0.052
200	1.5	0.25	2	GW	0.157	0.088	0.051	0.029	0.236	0.138	0.077	0.041
200	1.5	0.25	2	PW	0.188	0.101	0.061	0.032	0.275	0.164	0.092	0.048
200	1.5	0.25	3	LR	0.211	0.121	0.070	0.038	0.274	0.173	0.099	0.050
200	1.5	0.25	3	GW	0.155	0.088	0.052	0.029	0.210	0.121	0.071	0.040
200	1.5	0.25	3	PW	0.193	0.109	0.061	0.035	0.243	0.145	0.085	0.045
200	1.5	0.50	2	LR	0.243	0.133	0.075	0.037	0.321	0.202	0.120	0.058
200	1.5	0.50	2	GW	0.159	0.086	0.050	0.027	0.257	0.152	0.086	0.044
200	1.5	0.50	2	PW	0.200	0.106	0.060	0.033	0.281	0.169	0.097	0.051
200	1.5	0.50	3	LR	0.229	0.130	0.077	0.039	0.296	0.180	0.105	0.054
200	1.5	0.50	3	GW	0.157	0.086	0.052	0.031	0.226	0.131	0.078	0.040
200	1.5	0.50	3	PW	0.190	0.103	0.057	0.028	0.266	0.163	0.096	0.050
200	1.5	0.75	2	LR	0.245	0.134	0.075	0.039	0.340	0.212	0.127	0.060
200	1.5	0.75	2	GW	0.171	0.094	0.053	0.029	0.258	0.150	0.084	0.045
200	1.5	0.75	2	PW	0.201	0.105	0.059	0.033	0.302	0.180	0.102	0.049
200	1.5	0.75	3	LR	0.240	0.139	0.079	0.041	0.341	0.210	0.123	0.060
200	1.5	0.75	3	GW	0.169	0.091	0.053	0.030	0.252	0.149	0.082	0.044
200	1.5	0.75	3	PW	0.202	0.112	0.061	0.032	0.286	0.171	0.101	0.048
200	2.0	0.25	2	LR	0.556	0.369	0.222	0.097	0.714	0.550	0.384	0.211
200	2.0	0.25	2	GW	0.373	0.205	0.097	0.037	0.553	0.389	0.249	0.119
200	2.0	0.25	2	PW	0.463	0.284	0.149	0.059	0.645	0.478	0.318	0.161
200	2.0	0.25	3	LR	0.493	0.319	0.184	0.079	0.623	0.456	0.298	0.153
200	2.0	0.25	3	GW	0.317	0.171	0.086	0.036	0.479	0.313	0.187	0.089
200	2.0	0.25	3	PW	0.430	0.246	0.127	0.053	0.567	0.396	0.254	0.126
200	2.0	0.50	2	LR	0.589	0.400	0.248	0.108	0.738	0.585	0.424	0.238
200	2.0	0.50	2	GW	0.391	0.218	0.108	0.042	0.605	0.432	0.284	0.139
200	2.0	0.50	2	PW	0.490	0.310	0.167	0.064	0.670	0.504	0.340	0.175
200	2.0	0.50	3	LR	0.564	0.382	0.225	0.100	0.665	0.498	0.343	0.181
200	2.0	0.50	3	GW	0.359	0.197	0.100	0.039	0.525	0.364	0.224	0.106
200	2.0	0.50	3	PW	0.463	0.284	0.149	0.058	0.614	0.446	0.292	0.150
200	2.0	0.75	2	LR	0.614	0.427	0.261	0.115	0.796	0.646	0.480	0.280
200	2.0	0.75	2	GW	0.424	0.242	0.121	0.045	0.644	0.469	0.310	0.155
200	2.0	0.75	2	PW	0.502	0.312	0.173	0.065	0.723	0.563	0.394	0.207
200	2.0	0.75	3	LR	0.605	0.414	0.252	0.115	0.762	0.608	0.454	0.261
200	2.0	0.75	3	GW	0.414	0.236	0.118	0.045	0.609	0.441	0.290	0.145
200	2.0	0.75	3	PW	0.504	0.315	0.170	0.065	0.687	0.524	0.361	0.194
200	3.0	0.25	2	LR	0.970	0.923	0.830	0.628	0.993	0.979	0.946	0.854
200	3.0	0.25	2	GW	0.861	0.716	0.527	0.272	0.968	0.921	0.833	0.652
200	3.0	0.25	2	PW	0.925	0.829	0.676	0.425	0.985	0.959	0.907	0.776
200	3.0	0.25	3	LR	0.943	0.867	0.746	0.530	0.980	0.946	0.882	0.738
200	3.0	0.25	3	GW	0.800	0.626	0.424	0.190	0.922	0.840	0.718	0.501
200	3.0	0.25	3	PW	0.897	0.777	0.603	0.350	0.964	0.916	0.830	0.653
200	3.0	0.50	2	LR	0.976	0.937	0.852	0.669	0.995	0.985	0.960	0.886
200	3.0	0.50	2	GW	0.882	0.748	0.560	0.297	0.976	0.936	0.860	0.704
200	3.0	0.50	2	PW	0.931	0.837	0.699	0.454	0.988	0.967	0.922	0.807
200	3.0	0.50	3	LR	0.966	0.915	0.816	0.612	0.986	0.962	0.911	0.798
200	3.0	0.50	3	GW	0.842	0.689	0.494	0.250	0.953	0.893	0.788	0.595
200	3.0	0.50	3	PW	0.917	0.812	0.660	0.415	0.978	0.939	0.870	0.714
200	3.0	0.75	2	LR	0.983	0.942	0.867	0.704	0.998	0.993	0.980	0.931
200	3.0	0.75	2	GW	0.893	0.775	0.595	0.335	0.984	0.954	0.898	0.759
200	3.0	0.75	2	PW	0.944	0.863	0.729	0.478	0.994	0.979	0.947	0.852
200	3.0	0.75	3	LR	0.978	0.941	0.868	0.699	0.998	0.990	0.970	0.904
200	3.0	0.75	3	GW	0.885	0.756	0.567	0.313	0.980	0.946	0.879	0.723
200	3.0	0.75	3	PW	0.938	0.857	0.716	0.468	0.991	0.972	0.932	0.824

TABLE 21. Simulated power for $T_{200}(\tau_0)$ and the exponential distribution.

					$\gamma = 0.25$				$\gamma = 0.50$			
n	δ_n	η	$\delta_{C,n}$	w_n	10%	5%	2.5%	1%	10%	5%	2.5%	1%
200	1.5	0.25	2	LR	0.284	0.156	0.085	0.043	0.392	0.253	0.153	0.077
200	1.5	0.25	2	GW	0.232	0.123	0.069	0.032	0.392	0.249	0.146	0.067
200	1.5	0.25	2	PW	0.266	0.143	0.078	0.037	0.399	0.258	0.160	0.085
200	1.5	0.25	3	LR	0.268	0.154	0.087	0.041	0.380	0.244	0.156	0.082
200	1.5	0.25	3	GW	0.232	0.124	0.069	0.037	0.350	0.227	0.135	0.063
200	1.5	0.25	3	PW	0.267	0.153	0.084	0.043	0.376	0.242	0.147	0.078
200	1.5	0.50	2	LR	0.279	0.157	0.089	0.042	0.428	0.276	0.171	0.085
200	1.5	0.50	2	GW	0.248	0.130	0.068	0.030	0.420	0.265	0.159	0.075
200	1.5	0.50	2	PW	0.264	0.140	0.073	0.035	0.431	0.279	0.169	0.080
200	1.5	0.50	3	LR	0.293	0.166	0.093	0.045	0.411	0.272	0.174	0.088
200	1.5	0.50	3	GW	0.239	0.128	0.068	0.032	0.399	0.259	0.161	0.077
200	1.5	0.50	3	PW	0.274	0.155	0.083	0.043	0.406	0.270	0.167	0.084
200	1.5	0.75	2	LR	0.275	0.156	0.082	0.038	0.437	0.292	0.182	0.094
200	1.5	0.75	2	GW	0.251	0.134	0.071	0.036	0.436	0.281	0.171	0.080
200	1.5	0.75	2	PW	0.270	0.143	0.076	0.036	0.451	0.301	0.181	0.085
200	1.5	0.75	3	LR	0.288	0.168	0.098	0.046	0.439	0.290	0.179	0.094
200	1.5	0.75	3	GW	0.259	0.140	0.072	0.035	0.433	0.285	0.169	0.083
200	1.5	0.75	3	PW	0.278	0.160	0.085	0.041	0.435	0.283	0.176	0.084
200	2.0	0.25	2	LR	0.688	0.513	0.339	0.161	0.845	0.727	0.580	0.375
200	2.0	0.25	2	GW	0.674	0.467	0.264	0.103	0.873	0.749	0.594	0.376
200	2.0	0.25	2	PW	0.712	0.519	0.320	0.139	0.879	0.766	0.624	0.410
200	2.0	0.25	3	LR	0.666	0.486	0.312	0.147	0.798	0.671	0.521	0.327
200	2.0	0.25	3	GW	0.615	0.406	0.226	0.082	0.812	0.680	0.516	0.311
200	2.0	0.25	3	PW	0.678	0.477	0.291	0.122	0.825	0.696	0.543	0.340
200	2.0	0.50	2	LR	0.701	0.523	0.347	0.161	0.873	0.762	0.619	0.416
200	2.0	0.50	2	GW	0.685	0.486	0.284	0.108	0.894	0.786	0.646	0.423
200	2.0	0.50	2	PW	0.729	0.529	0.334	0.143	0.902	0.794	0.663	0.450
200	2.0	0.50	3	LR	0.704	0.517	0.339	0.161	0.837	0.718	0.574	0.379
200	2.0	0.50	3	GW	0.669	0.457	0.269	0.101	0.855	0.740	0.583	0.374
200	2.0	0.50	3	PW	0.714	0.521	0.327	0.145	0.863	0.748	0.606	0.400
200	2.0	0.75	2	LR	0.711	0.534	0.351	0.164	0.892	0.793	0.657	0.454
200	2.0	0.75	2	GW	0.709	0.509	0.308	0.122	0.908	0.812	0.680	0.468
200	2.0	0.75	2	PW	0.743	0.553	0.358	0.153	0.915	0.829	0.700	0.494
200	2.0	0.75	3	LR	0.707	0.525	0.345	0.165	0.888	0.783	0.648	0.443
200	2.0	0.75	3	GW	0.700	0.499	0.297	0.120	0.897	0.801	0.657	0.449
200	2.0	0.75	3	PW	0.736	0.544	0.350	0.153	0.906	0.808	0.675	0.470
200	3.0	0.25	2	LR	0.994	0.977	0.935	0.812	0.999	0.997	0.991	0.968
200	3.0	0.25	2	GW	0.995	0.976	0.921	0.738	1.000	0.999	0.995	0.977
200	3.0	0.25	2	PW	0.997	0.986	0.945	0.818	1.000	0.999	0.995	0.979
200	3.0	0.25	3	LR	0.990	0.968	0.913	0.766	0.998	0.993	0.980	0.934
200	3.0	0.25	3	GW	0.987	0.955	0.875	0.646	0.999	0.996	0.985	0.948
200	3.0	0.25	3	PW	0.994	0.976	0.924	0.767	0.999	0.998	0.991	0.960
200	3.0	0.50	2	LR	0.994	0.980	0.943	0.824	1.000	0.999	0.995	0.979
200	3.0	0.50	2	GW	0.995	0.977	0.932	0.777	1.000	1.000	0.997	0.983
200	3.0	0.50	2	PW	0.995	0.984	0.949	0.831	1.000	0.999	0.997	0.988
200	3.0	0.50	3	LR	0.992	0.975	0.934	0.814	0.999	0.997	0.989	0.96
200	3.0	0.50	3	GW	0.993	0.973	0.912	0.731	0.999	0.998	0.992	0.968
200	3.0	0.50	3	PW	0.996	0.982	0.944	0.811	1.000	0.999	0.994	0.973
200	3.0	0.75	2	LR	0.994	0.982	0.948	0.837	1.000	0.999	0.998	0.984
200	3.0	0.75	2	GW	0.995	0.984	0.942	0.803	1.000	1.000	0.998	0.991
200	3.0	0.75	2	PW	0.998	0.989	0.957	0.846	1.000	1.000	0.998	0.992
200	3.0	0.75	3	LR	0.995	0.980	0.939	0.825	1.000	0.999	0.995	0.980
200	3.0	0.75	3	GW	0.996	0.980	0.938	0.790	1.000	0.999	0.997	0.985
200	3.0	0.75	3	PW	0.996	0.987	0.952	0.837	1.000	0.999	0.998	0.987

TABLE 22. Simulated power for $T_{200}(\tau_0)$ and the log-normal distribution.

					$\gamma = 0.25$				$\gamma = 0.50$			
n	δ_n	η	$\delta_{C,n}$	w_n	10%	5%	2.5%	1%	10%	5%	2.5%	1%
200	1.2	0.25	1.4	LR	0.505	0.317	0.180	0.076	0.615	0.442	0.294	0.157
200	1.2	0.25	1.4	GW	0.321	0.172	0.085	0.037	0.470	0.307	0.185	0.085
200	1.2	0.25	1.4	PW	0.435	0.260	0.137	0.055	0.561	0.399	0.254	0.125
200	1.2	0.25	1.6	LR	0.398	0.246	0.138	0.058	0.476	0.321	0.199	0.103
200	1.2	0.25	1.6	GW	0.245	0.132	0.069	0.035	0.370	0.232	0.136	0.065
200	1.2	0.25	1.6	PW	0.346	0.188	0.102	0.047	0.446	0.298	0.188	0.096
200	1.2	0.50	1.4	LR	0.601	0.417	0.250	0.110	0.662	0.504	0.351	0.190
200	1.2	0.50	1.4	GW	0.381	0.214	0.110	0.045	0.540	0.369	0.233	0.113
200	1.2	0.50	1.4	PW	0.496	0.307	0.165	0.066	0.616	0.449	0.306	0.155
200	1.2	0.50	1.6	LR	0.577	0.390	0.235	0.099	0.536	0.374	0.245	0.129
200	1.2	0.50	1.6	GW	0.331	0.183	0.092	0.040	0.429	0.276	0.167	0.084
200	1.2	0.50	1.6	PW	0.479	0.295	0.158	0.064	0.495	0.341	0.216	0.112
200	1.2	0.75	1.4	LR	0.644	0.459	0.293	0.132	0.805	0.661	0.496	0.295
200	1.2	0.75	1.4	GW	0.441	0.263	0.134	0.051	0.642	0.477	0.321	0.164
200	1.2	0.75	1.4	PW	0.544	0.351	0.206	0.082	0.734	0.580	0.408	0.225
200	1.2	0.75	1.6	LR	0.654	0.466	0.298	0.139	0.771	0.618	0.466	0.270
200	1.2	0.75	1.6	GW	0.431	0.247	0.127	0.049	0.607	0.435	0.281	0.137
200	1.2	0.75	1.6	PW	0.535	0.353	0.201	0.079	0.698	0.536	0.372	0.204
200	1.4	0.25	1.4	LR	0.994	0.978	0.937	0.817	0.998	0.992	0.978	0.933
200	1.4	0.25	1.4	GW	0.940	0.845	0.693	0.424	0.985	0.962	0.909	0.786
200	1.4	0.25	1.4	PW	0.981	0.940	0.853	0.654	0.996	0.988	0.966	0.894
200	1.4	0.25	1.6	LR	0.966	0.910	0.813	0.608	0.983	0.952	0.895	0.761
200	1.4	0.25	1.6	GW	0.852	0.701	0.499	0.238	0.938	0.867	0.754	0.545
200	1.4	0.25	1.6	PW	0.945	0.863	0.718	0.462	0.973	0.936	0.865	0.709
200	1.4	0.50	1.4	LR	0.997	0.990	0.971	0.907	0.999	0.996	0.990	0.961
200	1.4	0.50	1.4	GW	0.965	0.903	0.793	0.559	0.994	0.978	0.948	0.857
200	1.4	0.50	1.4	PW	0.989	0.966	0.905	0.749	0.997	0.992	0.978	0.928
200	1.4	0.50	1.6	LR	0.996	0.986	0.958	0.867	0.991	0.973	0.940	0.851
200	1.4	0.50	1.6	GW	0.943	0.860	0.713	0.446	0.969	0.931	0.858	0.694
200	1.4	0.50	1.6	PW	0.985	0.952	0.884	0.693	0.985	0.962	0.913	0.792
200	1.4	0.75	1.4	LR	0.999	0.996	0.987	0.949	1.000	1.000	0.999	0.994
200	1.4	0.75	1.4	GW	0.983	0.946	0.864	0.675	0.999	0.997	0.987	0.952
200	1.4	0.75	1.4	PW	0.992	0.976	0.933	0.804	1.000	0.998	0.995	0.981
200	1.4	0.75	1.6	LR	0.999	0.994	0.981	0.940	1.000	0.999	0.999	0.990
200	1.4	0.75	1.6	GW	0.978	0.934	0.843	0.634	0.998	0.992	0.974	0.921
200	1.4	0.75	1.6	PW	0.993	0.974	0.933	0.803	0.999	0.997	0.991	0.967
200	1.6	0.25	1.4	LR	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
200	1.6	0.25	1.4	GW	1.000	0.999	0.996	0.972	1.000	1.000	1.000	0.999
200	1.6	0.25	1.4	PW	1.000	1.000	0.999	0.994	1.000	1.000	1.000	1.000
200	1.6	0.25	1.6	LR	1.000	1.000	0.999	0.994	1.000	1.000	1.000	0.999
200	1.6	0.25	1.6	GW	0.999	0.992	0.971	0.881	1.000	1.000	0.997	0.986
200	1.6	0.25	1.6	PW	1.000	0.999	0.995	0.973	1.000	1.000	1.000	0.998
200	1.6	0.50	1.4	LR	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
200	1.6	0.50	1.4	GW	1.000	1.000	0.998	0.987	1.000	1.000	1.000	1.000
200	1.6	0.50	1.4	PW	1.000	1.000	1.000	0.997	1.000	1.000	1.000	1.000
200	1.6	0.50	1.6	LR	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
200	1.6	0.50	1.6	GW	1.000	0.999	0.994	0.970	1.000	1.000	0.999	0.997
200	1.6	0.50	1.6	PW	1.000	1.000	0.999	0.995	1.000	1.000	1.000	0.999
200	1.6	0.75	1.4	LR	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
200	1.6	0.75	1.4	GW	1.000	1.000	0.999	0.993	1.000	1.000	1.000	1.000
200	1.6	0.75	1.4	PW	1.000	1.000	1.000	0.999	1.000	1.000	1.000	1.000
200	1.6	0.75	1.6	LR	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
200	1.6	0.75	1.6	GW	1.000	1.000	0.999	0.991	1.000	1.000	1.000	1.000
200	1.6	0.75	1.6	PW	1.000	1.000	1.000	0.999	1.000	1.000	1.000	1.000

TABLE 23. Simulated power for $T_{200}(\tau_0)$ and the Weibull distribution.

3.2. Power of the statistic T_n^σ. We focus on the power of the test based on the rank test statistic $T_n^\sigma(\tau_0)$ which is appropriate for testing the restricted null hypothesis $\bar{H}_0$ which can be considered e.g. in the Koziol–Green model with a change-point. We suppose that the alternative H_1 in such model occurs, i.e. the lifetime distribution and the censoring distribution change at the same time-point $1 < \lfloor n\gamma \rfloor = \lfloor n\eta \rfloor < n$, but the proportion of censoring before and after the change-point is still the same.

We proceed with $n = 100; 200$ as follows:

(1) $X_1^0, X_2^0, \dots, X_n^0$ are simulated using the chosen combination of parameters

$$X_1^0, X_2^0, \dots, X_{\lfloor n\gamma \rfloor}^0 \sim F_1, \qquad F_1 = \mathrm{E}(1) \text{ (or } \mathrm{L}(1))$$

$$X_{\lfloor n\gamma \rfloor+1}^0, X_{\lfloor n\gamma \rfloor+2}^0, \dots, X_n^0 \sim F_2, \qquad F_2 = \mathrm{E}(\delta_n) \text{ (or } \mathrm{L}(\delta_n))$$

(we use $\gamma = 0.25; 0.5,\ \delta_n = 1.5; 2; 3; 4$).

(2) $C_1, C_2, \dots, C_n$ fulfilling KGM are simulated
(we use the censoring parameter $\beta = 0; 0.5; 1$).

(3) The pairs of observations $(X_1, \Delta_1), (X_2, \Delta_2), \dots, (X_n, \Delta_n)$ are computed.

(4) The value of the statistic $T_n(\tau_0)$ is calculated and compared with the asymptotic critical value.

(5) The steps (1) − (4) are repeated 10^4 times.

(6) The relative frequency of the rejected cases is determined.

In Tables 24 and 25 the results of the simulated power for the max-type test procedure based on the test statistic $T_n^\sigma(\tau_0)$ and nearly all the combinations of parameters γ, δ_n and β are summarized. We use the exponential and the log-normal distribution.

We can observe similarly to Tables 18, 19 and 21, 22 that the power grows with increasing δ_n and also with the increasing sample size n. Further, the output is influenced by the choice of the parameter γ such that in most cases we get worse results for $\gamma = 0.25$, i.e. for the time of a change in the distribution of the survival variables which is near to the tail of observation period. If we focus on the expected proportion of censoring $\beta/(1+\beta)$ ($\beta = 0$ means no censoring), the power decreases with the greater ratio of censoring. Comparing the weights, we get in most cases worse results for the Gehan–Wilcoxon test than for other two types of the test. Finally, we obtain slightly better results for the log-normal distribution than in case of the exponential one. But recall that for the same δ_n mean and variance for such distributions are different, only the relative change of these characteristics after the change-point is the same for both underlying distributions.

					exponential				log-normal			
n	γ	δ_n	β	w_n	10%	5%	2.5%	1%	10%	5%	2.5%	1%
100	0.25	2	0	LR	0.655	0.487	0.333	0.143	0.562	0.390	0.245	0.094
100	0.25	2	0	GW	0.392	0.222	0.102	0.029	0.493	0.296	0.149	0.042
100	0.25	2	0	PW	0.388	0.218	0.106	0.026	0.488	0.296	0.150	0.045
100	0.25	2	0.5	LR	0.437	0.269	0.139	0.042	0.487	0.312	0.163	0.053
100	0.25	2	0.5	GW	0.218	0.092	0.028	0.004	0.378	0.188	0.072	0.012
100	0.25	2	0.5	PW	0.281	0.135	0.053	0.011	0.424	0.232	0.100	0.026
100	0.25	2	1	LR	0.309	0.160	0.067	0.015	0.404	0.233	0.113	0.028
100	0.25	2	1	GW	0.148	0.051	0.013	0.001	0.299	0.129	0.038	0.005
100	0.25	2	1	PW	0.212	0.090	0.028	0.004	0.361	0.182	0.075	0.015
100	0.25	3	0	LR	0.966	0.916	0.835	0.644	0.945	0.880	0.771	0.553
100	0.25	3	0	GW	0.828	0.683	0.501	0.263	0.934	0.844	0.691	0.414
100	0.25	3	0	PW	0.824	0.677	0.494	0.257	0.937	0.842	0.683	0.413
100	0.25	3	0.5	LR	0.841	0.702	0.515	0.274	0.906	0.796	0.636	0.380
100	0.25	3	0.5	GW	0.589	0.379	0.204	0.060	0.870	0.704	0.481	0.196
100	0.25	3	0.5	PW	0.700	0.500	0.308	0.116	0.895	0.756	0.559	0.285
100	0.25	3	1	LR	0.688	0.497	0.305	0.112	0.850	0.696	0.500	0.244
100	0.25	3	1	GW	0.413	0.209	0.084	0.014	0.779	0.558	0.315	0.082
100	0.25	3	1	PW	0.575	0.361	0.183	0.048	0.840	0.661	0.442	0.181
100	0.25	4	0	LR	0.997	0.990	0.971	0.904	0.996	0.987	0.963	0.881
100	0.25	4	0	GW	0.956	0.895	0.789	0.569	0.995	0.982	0.944	0.814
100	0.25	4	0	PW	0.958	0.898	0.786	0.571	0.996	0.982	0.948	0.814
100	0.25	4	0.5	LR	0.962	0.902	0.796	0.562	0.987	0.961	0.899	0.725
100	0.25	4	0.5	GW	0.823	0.656	0.441	0.186	0.985	0.936	0.818	0.530
100	0.25	4	0.5	PW	0.892	0.763	0.582	0.315	0.988	0.960	0.880	0.660
100	0.25	4	1	LR	0.871	0.731	0.535	0.270	0.969	0.909	0.786	0.521
100	0.25	4	1	GW	0.645	0.416	0.213	0.051	0.947	0.841	0.623	0.272
100	0.25	4	1	PW	0.798	0.615	0.394	0.151	0.970	0.907	0.775	0.478
100	0.50	2	0	LR	0.640	0.416	0.226	0.060	0.593	0.378	0.201	0.053
100	0.50	2	0	GW	0.531	0.341	0.185	0.061	0.673	0.477	0.294	0.119
100	0.50	2	0	PW	0.530	0.335	0.181	0.059	0.670	0.476	0.284	0.113
100	0.50	2	0.5	LR	0.403	0.214	0.088	0.020	0.473	0.268	0.126	0.029
100	0.50	2	0.5	GW	0.326	0.169	0.070	0.017	0.550	0.355	0.193	0.066
100	0.50	2	0.5	PW	0.366	0.194	0.088	0.022	0.557	0.355	0.192	0.057
100	0.50	2	1	LR	0.270	0.121	0.043	0.006	0.389	0.203	0.085	0.021
100	0.50	2	1	GW	0.247	0.113	0.042	0.008	0.482	0.280	0.139	0.040
100	0.50	2	1	PW	0.269	0.127	0.050	0.010	0.464	0.264	0.128	0.035
100	0.50	3	0	LR	0.980	0.930	0.805	0.490	0.967	0.901	0.770	0.478
100	0.50	3	0	GW	0.947	0.870	0.736	0.496	0.988	0.962	0.898	0.726
100	0.50	3	0	PW	0.942	0.858	0.723	0.482	0.988	0.961	0.899	0.736
100	0.50	3	0.5	LR	0.807	0.616	0.381	0.133	0.895	0.756	0.545	0.255
100	0.50	3	0.5	GW	0.767	0.587	0.397	0.183	0.955	0.884	0.753	0.500
100	0.50	3	0.5	PW	0.814	0.649	0.449	0.209	0.954	0.878	0.743	0.484
100	0.50	3	1	LR	0.620	0.389	0.199	0.049	0.810	0.618	0.389	0.142
100	0.50	3	1	GW	0.607	0.409	0.232	0.075	0.904	0.787	0.598	0.323
100	0.50	3	1	PW	0.644	0.436	0.246	0.090	0.893	0.764	0.573	0.300
100	0.50	4	0	LR	1.000	0.996	0.977	0.864	0.999	0.994	0.973	0.872
100	0.50	4	0	GW	0.996	0.985	0.950	0.844	1.000	0.999	0.995	0.972
100	0.50	4	0	PW	0.996	0.985	0.952	0.843	1.000	0.999	0.994	0.972
100	0.50	4	0.5	LR	0.948	0.849	0.655	0.327	0.987	0.944	0.837	0.555
100	0.50	4	0.5	GW	0.938	0.849	0.701	0.439	0.998	0.989	0.964	0.860
100	0.50	4	0.5	PW	0.956	0.879	0.744	0.489	0.997	0.986	0.951	0.832
100	0.50	4	1	LR	0.809	0.610	0.377	0.133	0.939	0.824	0.630	0.330
100	0.50	4	1	GW	0.813	0.644	0.453	0.207	0.986	0.955	0.872	0.649
100	0.50	4	1	PW	0.849	0.691	0.484	0.233	0.980	0.938	0.845	0.606

TABLE 24. Simulated power for $T^{\sigma}_{100}(\tau_0)$.

					exponential				log-normal			
n	γ	δ_n	β	w_n	10%	5%	2.5%	1%	10%	5%	2.5%	1%
200	0.25	1.5	0	LR	0.495	0.346	0.217	0.110	0.410	0.255	0.144	0.069
200	0.25	1.5	0	GW	0.281	0.146	0.068	0.020	0.352	0.198	0.094	0.025
200	0.25	1.5	0	PW	0.275	0.144	0.066	0.016	0.343	0.192	0.094	0.027
200	0.25	1.5	0.5	LR	0.335	0.199	0.102	0.035	0.334	0.198	0.106	0.035
200	0.25	1.5	0.5	GW	0.162	0.069	0.026	0.005	0.277	0.133	0.051	0.013
200	0.25	1.5	0.5	PW	0.205	0.092	0.038	0.010	0.304	0.159	0.069	0.019
200	0.25	1.5	1	LR	0.240	0.124	0.055	0.014	0.295	0.161	0.074	0.022
200	0.25	1.5	1	GW	0.120	0.042	0.013	0.003	0.221	0.099	0.035	0.007
200	0.25	1.5	1	PW	0.157	0.070	0.025	0.004	0.262	0.129	0.050	0.012
200	0.25	2.0	0	LR	0.927	0.857	0.751	0.573	0.878	0.780	0.645	0.442
200	0.25	2.0	0	GW	0.768	0.608	0.437	0.233	0.883	0.764	0.604	0.359
200	0.25	2.0	0	PW	0.757	0.598	0.430	0.230	0.877	0.758	0.599	0.368
200	0.25	2.0	0.5	LR	0.788	0.648	0.486	0.280	0.828	0.696	0.537	0.324
200	0.25	2.0	0.5	GW	0.520	0.338	0.191	0.070	0.793	0.630	0.437	0.211
200	0.25	2.0	0.5	PW	0.621	0.441	0.279	0.117	0.821	0.680	0.503	0.268
200	0.25	2.0	1	LR	0.625	0.449	0.290	0.130	0.772	0.616	0.449	0.242
200	0.25	2.0	1	GW	0.376	0.205	0.093	0.023	0.716	0.517	0.320	0.128
200	0.25	2.0	1	PW	0.500	0.315	0.172	0.059	0.770	0.600	0.413	0.198
200	0.25	3.0	0	LR	0.999	0.998	0.996	0.987	1.000	0.999	0.995	0.980
200	0.25	3.0	0	GW	0.993	0.978	0.945	0.854	1.000	0.999	0.994	0.974
200	0.25	3.0	0	PW	0.992	0.977	0.947	0.858	1.000	0.999	0.994	0.973
200	0.25	3.0	0.5	LR	0.996	0.985	0.957	0.869	0.999	0.994	0.983	0.935
200	0.25	3.0	0.5	GW	0.944	0.866	0.743	0.517	0.998	0.992	0.972	0.903
200	0.25	3.0	0.5	PW	0.969	0.925	0.842	0.668	0.998	0.996	0.983	0.936
200	0.25	3.0	1	LR	0.963	0.914	0.827	0.638	0.995	0.984	0.956	0.868
200	0.25	3.0	1	GW	0.837	0.683	0.498	0.254	0.994	0.978	0.929	0.778
200	0.25	3.0	1	PW	0.927	0.837	0.701	0.471	0.997	0.989	0.965	0.877
200	0.50	1.5	0	LR	0.502	0.317	0.170	0.065	0.442	0.273	0.140	0.049
200	0.50	1.5	0	GW	0.372	0.220	0.109	0.036	0.488	0.317	0.182	0.067
200	0.50	1.5	0	PW	0.377	0.221	0.112	0.034	0.494	0.317	0.176	0.062
200	0.50	1.5	0.5	LR	0.320	0.168	0.078	0.018	0.362	0.205	0.101	0.032
200	0.50	1.5	0.5	GW	0.239	0.118	0.048	0.013	0.405	0.233	0.119	0.040
200	0.50	1.5	0.5	PW	0.270	0.140	0.063	0.015	0.406	0.240	0.120	0.039
200	0.50	1.5	1	LR	0.229	0.113	0.045	0.010	0.305	0.158	0.069	0.016
200	0.50	1.5	1	GW	0.176	0.077	0.032	0.007	0.344	0.190	0.091	0.026
200	0.50	1.5	1	PW	0.196	0.096	0.038	0.008	0.342	0.188	0.087	0.026
200	0.50	2.0	0	LR	0.961	0.900	0.782	0.543	0.926	0.838	0.696	0.460
200	0.50	2.0	0	GW	0.894	0.785	0.641	0.412	0.968	0.914	0.820	0.627
200	0.50	2.0	0	PW	0.897	0.793	0.642	0.413	0.968	0.915	0.824	0.633
200	0.50	2.0	0.5	LR	0.796	0.632	0.436	0.208	0.860	0.721	0.544	0.293
200	0.50	2.0	0.5	GW	0.702	0.529	0.356	0.171	0.917	0.826	0.676	0.453
200	0.50	2.0	0.5	PW	0.746	0.577	0.405	0.203	0.915	0.821	0.676	0.452
200	0.50	2.0	1	LR	0.627	0.428	0.254	0.092	0.783	0.615	0.425	0.208
200	0.50	2.0	1	GW	0.541	0.361	0.209	0.083	0.860	0.734	0.557	0.327
200	0.50	2.0	1	PW	0.590	0.407	0.248	0.101	0.856	0.725	0.557	0.325
200	0.50	3.0	0	LR	1.000	1.000	0.999	0.996	1.000	0.999	0.998	0.990
200	0.50	3.0	0	GW	1.000	0.999	0.997	0.982	1.000	1.000	1.000	0.999
200	0.50	3.0	0	PW	1.000	0.999	0.994	0.977	1.000	1.000	1.000	0.998
200	0.50	3.0	0.5	LR	0.997	0.985	0.948	0.814	0.999	0.995	0.984	0.930
200	0.50	3.0	0.5	GW	0.989	0.963	0.910	0.780	1.000	0.999	0.997	0.987
200	0.50	3.0	0.5	PW	0.992	0.980	0.943	0.830	1.000	0.999	0.996	0.983
200	0.50	3.0	1	LR	0.959	0.890	0.764	0.516	0.995	0.980	0.941	0.809
200	0.50	3.0	1	GW	0.938	0.863	0.738	0.517	0.999	0.996	0.986	0.943
200	0.50	3.0	1	PW	0.961	0.908	0.807	0.589	0.999	0.996	0.985	0.936

TABLE 25. Simulated power for $T^{\sigma}_{200}(\tau_0)$.

4. Estimators

4.1. Behavior of $\hat{m}_1$. We prepare the Monte Carlo simulation experiment to illustrate the properties of the proposed estimator $\hat{m}_1(\tau_0)$ defined in (3.1).

Suppose RCM with $m = \lfloor n\gamma \rfloor$. We proceed with $n = 100; 200; 300$ as follows:

(1) The survival times $X_1^0, X_2^0, \dots, X_n^0$ are simulated using the chosen combination of parameters

$$X_1^0, X_2^0, \dots, X_{\lfloor n\gamma \rfloor}^0 \sim F_1, \qquad F_1 = \mathrm{E}(1) \text{ (or L(1))}$$
$$X_{\lfloor n\gamma \rfloor+1}^0, X_{\lfloor n\gamma \rfloor+2}^0, \dots, X_n^0 \sim F_2, \qquad F_2 = \mathrm{E}(\delta_n) \text{ (or } \mathrm{L}(\delta_n))$$

(we use $\gamma = 0.25; 0.5; 0.75; 1,\ \delta_n = 2; 3; 4$).

(2) The censoring times $C_1, C_2, \dots, C_n$ are simulated using the chosen combination of parameters

$$C_1, C_2, \dots, C_{\lfloor n\eta \rfloor}^0 \sim G_1, \qquad G_1 = \mathrm{E}(1) \text{ (or L(1))}$$
$$C_{\lfloor n\eta \rfloor+1}, C_{\lfloor n\eta \rfloor+2}^0, \dots, C_n^0 \sim G_2, \qquad G_2 = \mathrm{E}(\delta_{C,n}) \text{ (or } \mathrm{L}(\delta_{C,n}))$$

(we use $\eta = 0.25; 0.5; 0.75; 1,\ \delta_{C,n} = 2; 3$).

(3) The pairs of observations $(X_1, \Delta_1), \dots, (X_n, \Delta_n)$ are computed.

(4) The estimator $\hat{m}_1(\tau_0)$ is calculated and its value stored.

(5) The steps $(1)-(4)$ are repeated 10^4 times.

(6) The histogram with relative frequency of $\hat{m}_1(\tau_0)$ is drawn.

Some graphical output can be found in Figures $1-8$. We consider situations corresponding to the alternative H_1 ($\gamma \in (0, 1)$) and also corresponding to the hypothesis H_0 ($\gamma = 1$). We present histograms of $\hat{m}_1(\tau_0)$ for the exponential and the log-normal lifetime distribution. Recall that in Chapter 4 we were not able to verify precisely the assumptions (A.2) and (A.3) in case of the log-normal distribution.

In Figure 1 we see the results for the exponential distribution and various choices of η. There are nearly the same histograms which are highly positively skewed. It means that the shape of graphs is not influenced by the location of a change in the distribution of the censoring variables. Notice that $\eta = 1$ means no change in the distribution of the censoring variables. Further, the peak of the graphs is located on the "right" place $m = 25$. We observe that the peak is a bit higher for the log-rank type weights with respect to other two common weights. For the log-normal distribution we see histograms very similar to histograms of the log-rank type estimators in the exponential case, see Figure 2 and the first row in Figure 1. In other words, the change in the lifetime distribution expressed by $\delta_n = 2$ is slightly more evident for the log-normal distribution than for the exponential distribution.

Figures 3 and 4 demonstrate that the peak of the histograms increases with larger value of the sample size n. This is mainly visible between the histograms for $n = 200$ and $n = 300$ for the exponential underlying distribution. Further, we observe that the distribution of $\hat{m}_1(\tau_0)$ is more or less symmetrical and the main peak of the graphs is in the neighborhood of $\lfloor 0.5n \rfloor$.

Now, we focus on the behavior of the estimator for various choices of $\gamma \in (0, 1]$ which is shown in Figures 5 and 6. In the case of $\gamma = 0.25$ the graphs are positively skewed and in the case of $\gamma = 0.75$ they are negatively skewed. The hight of peak is nearly the same for both choices of γ and the top is lower than for the change-point $\gamma = 0.5$ which occurs in the middle of the observation period, see the situations with $n = 100$ in Figures 3 or 4, respectively. Notice that in Theorem 3.3 we develop the limit distribution of $\hat{m}_1(\tau_0)$ only under the restricted null hypothesis $\bar{H}_0 : \gamma = \eta = 1$, but it seems from the last column ($\gamma = 1$) in Figures 5 and 6 that we get the same distribution also under the more general hypothesis $H_0 : \gamma = 1$ with $\eta \in (0, 1]$.

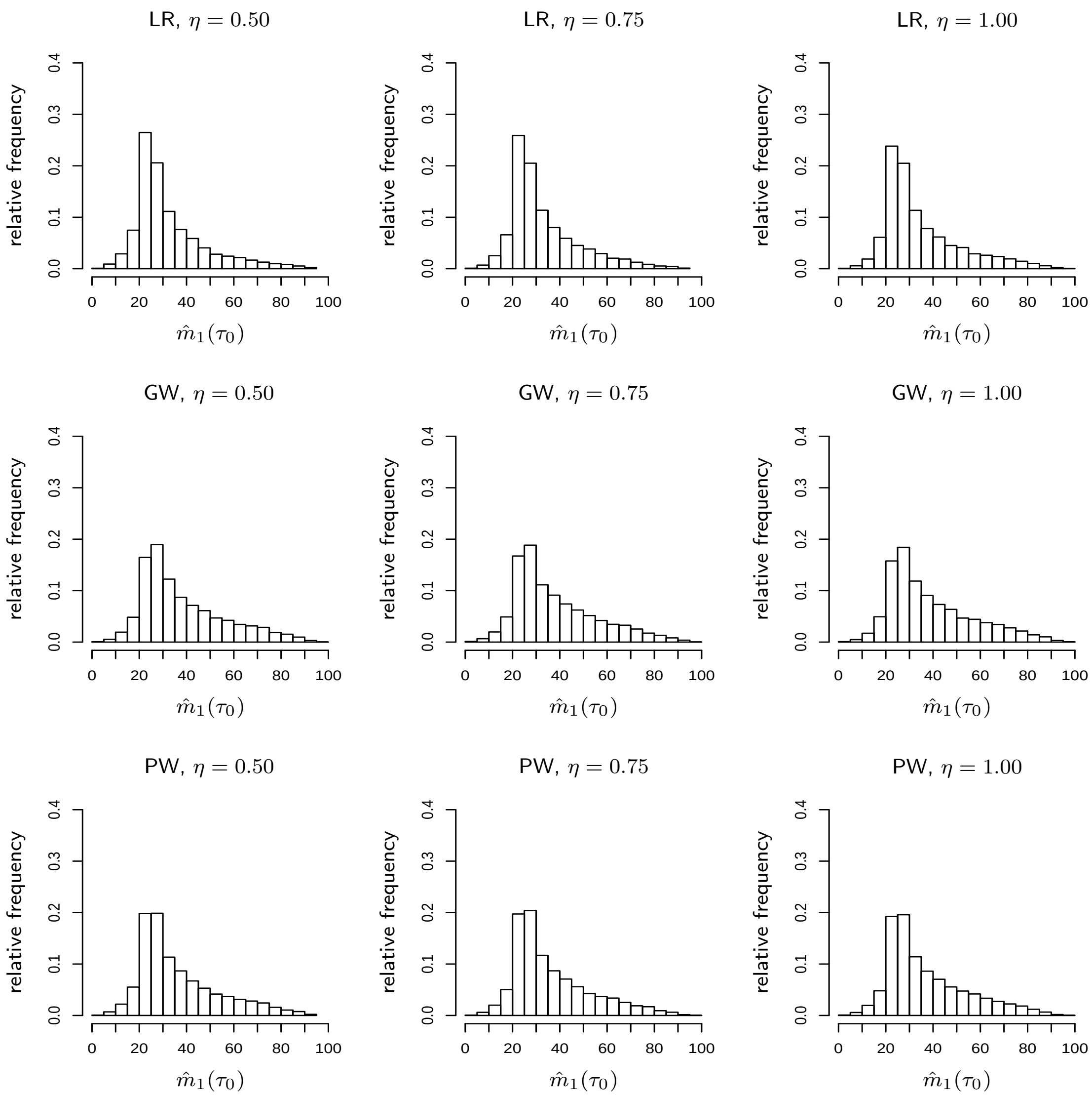

FIGURE 1. Histograms of $\hat{m}_1(\tau_0)$ for $\gamma = 0.25$, $n = 100$, $\delta_n = \delta_{C,n} = 2$ and the exponential distribution.

In Figures 7 and 8 we observe that the peak of histograms is more evident with larger δ_n and we get slightly better output for the log-normal lifetime distribution than for the exponential one. Notice that we choose $\delta_{C,n} = 3$ in contrast to $\delta_{C,n} = 2$ in other figures, so we can compare the graphs in the first column of Figures 3 and 7 or Figures 4 and 8 with the same $\gamma = 0.5$, $\eta = 0.25$, $\delta_n = 2$ and $n = 100$. The graphs are nearly optically similar to each other, but it can be observed a very small difference in the hight of peak which decreases a little with $\delta_{C,n} = 3$ with respect to $\delta_{C,n} = 2$.

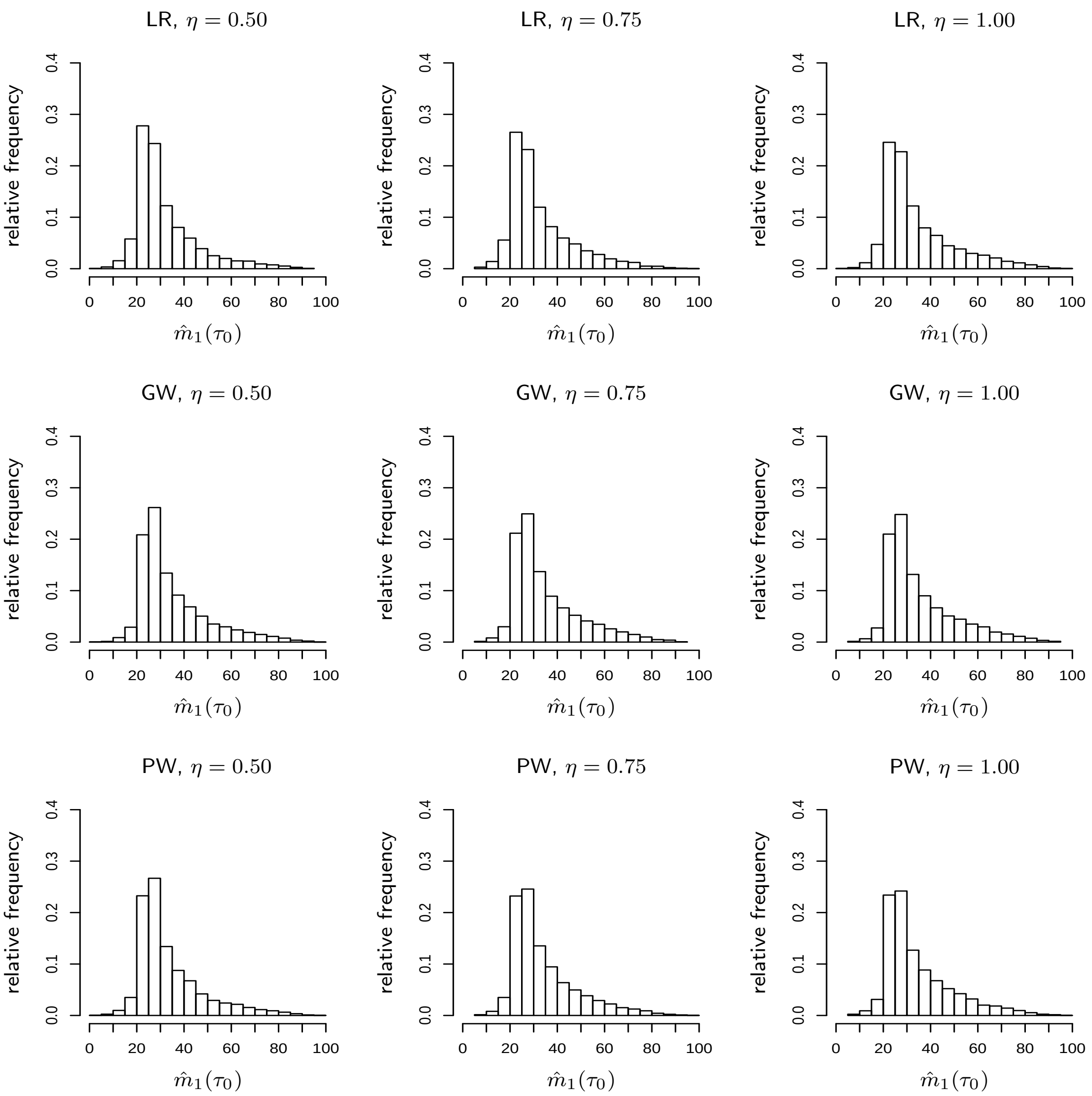

FIGURE 2. Histograms of $\hat{m}_1(\tau_0)$ for $\gamma = 0.25$, $n = 100$, $\delta_n = \delta_{C,n} = 2$ and the log-normal distribution.

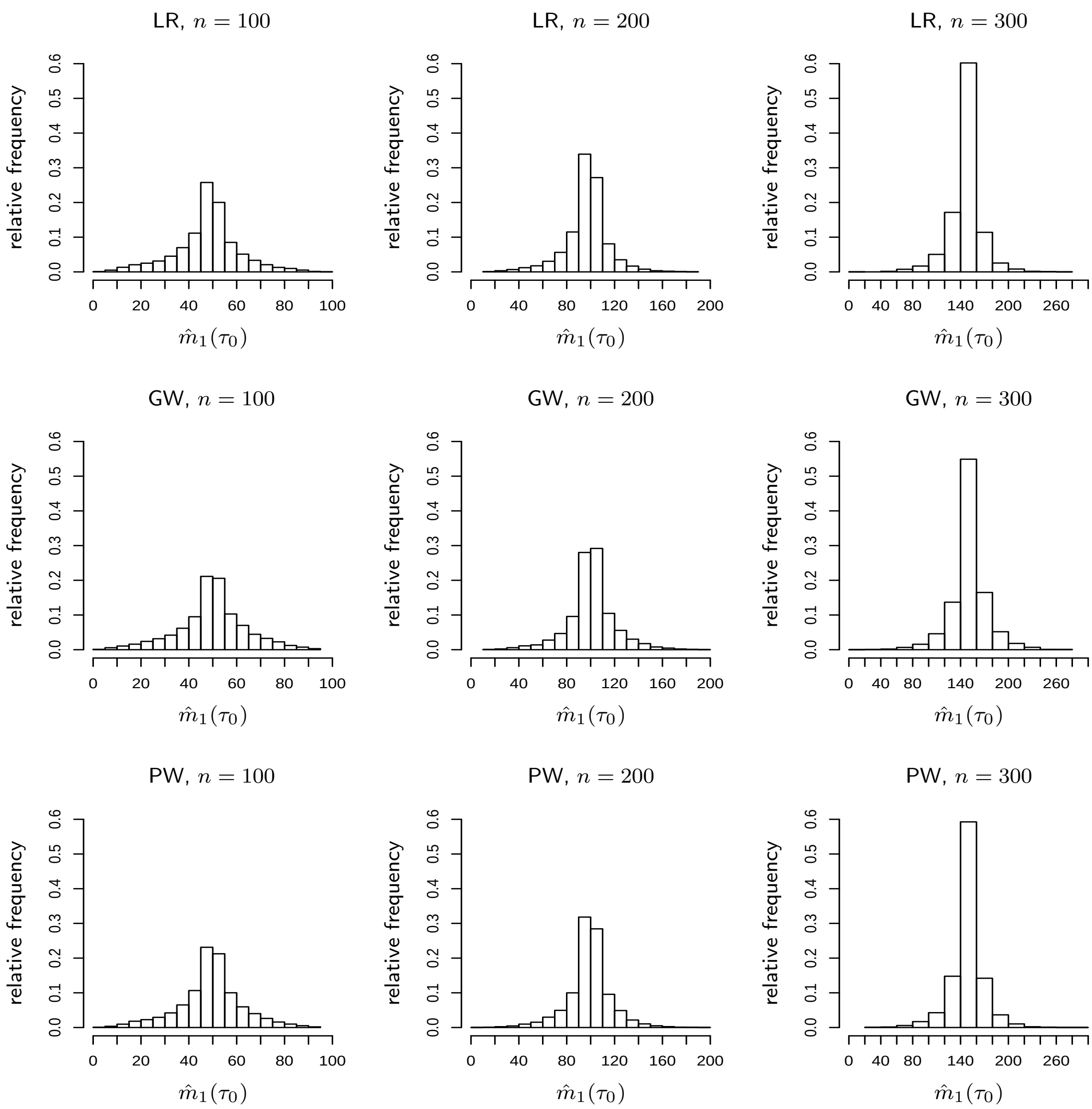

FIGURE 3. Histograms of $\hat{m}_1(\tau_0)$ for $\gamma = 0.5$, $\eta = 0.25$, $\delta_n = \delta_{C,n} = 2$ and the exponential distribution.

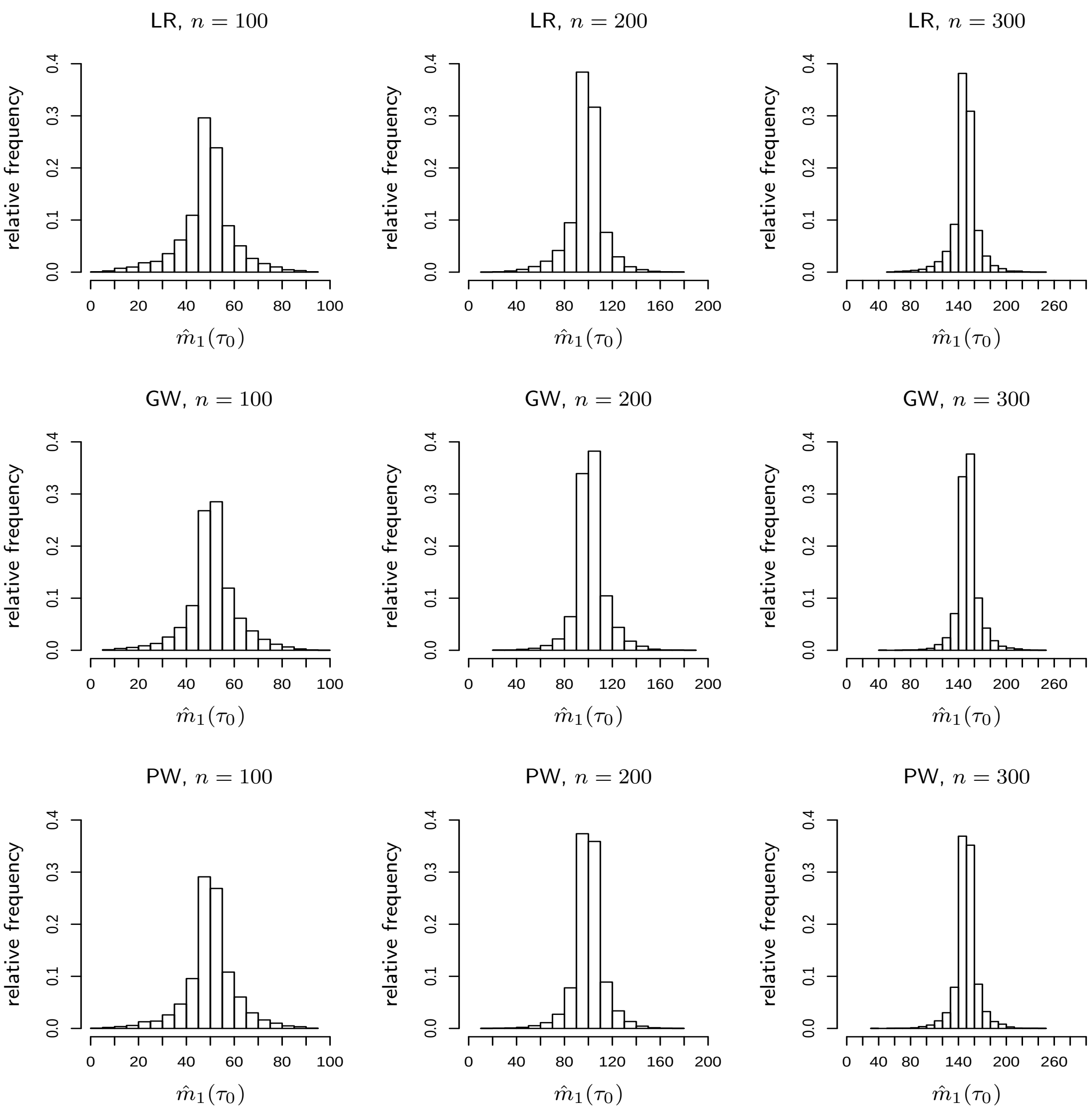

FIGURE 4. Histograms of $\hat{m}_1(\tau_0)$ for $\gamma = 0.5$, $\eta = 0.25$, $\delta_n = \delta_{C,n} = 2$ and the log-normal distribution.

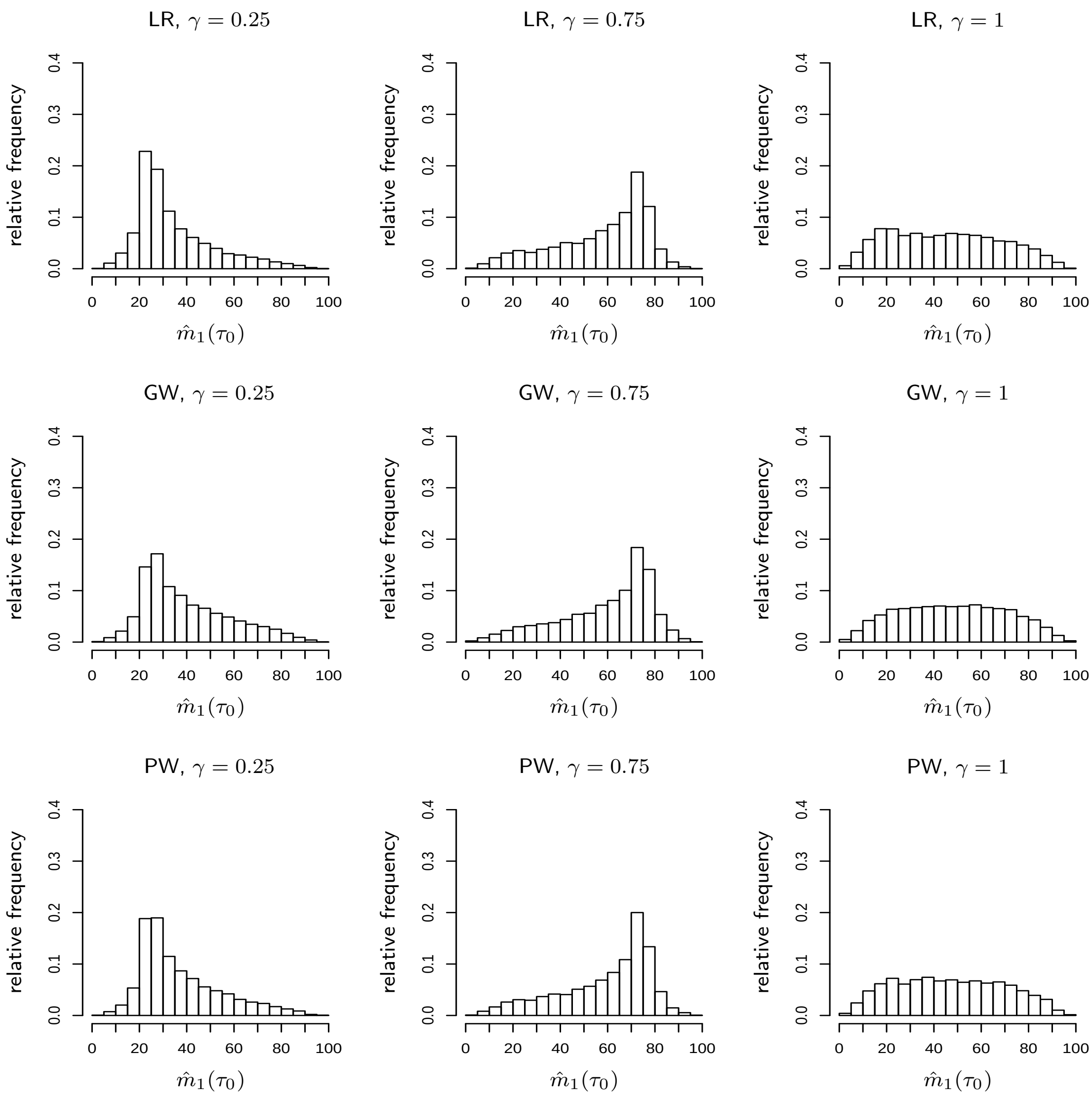

FIGURE 5. Histograms of $\hat{m}_1(\tau_0)$ for $\eta = 0.25$, $n = 100$, $\delta_n = \delta_{C,n} = 2$ and the exponential distribution.

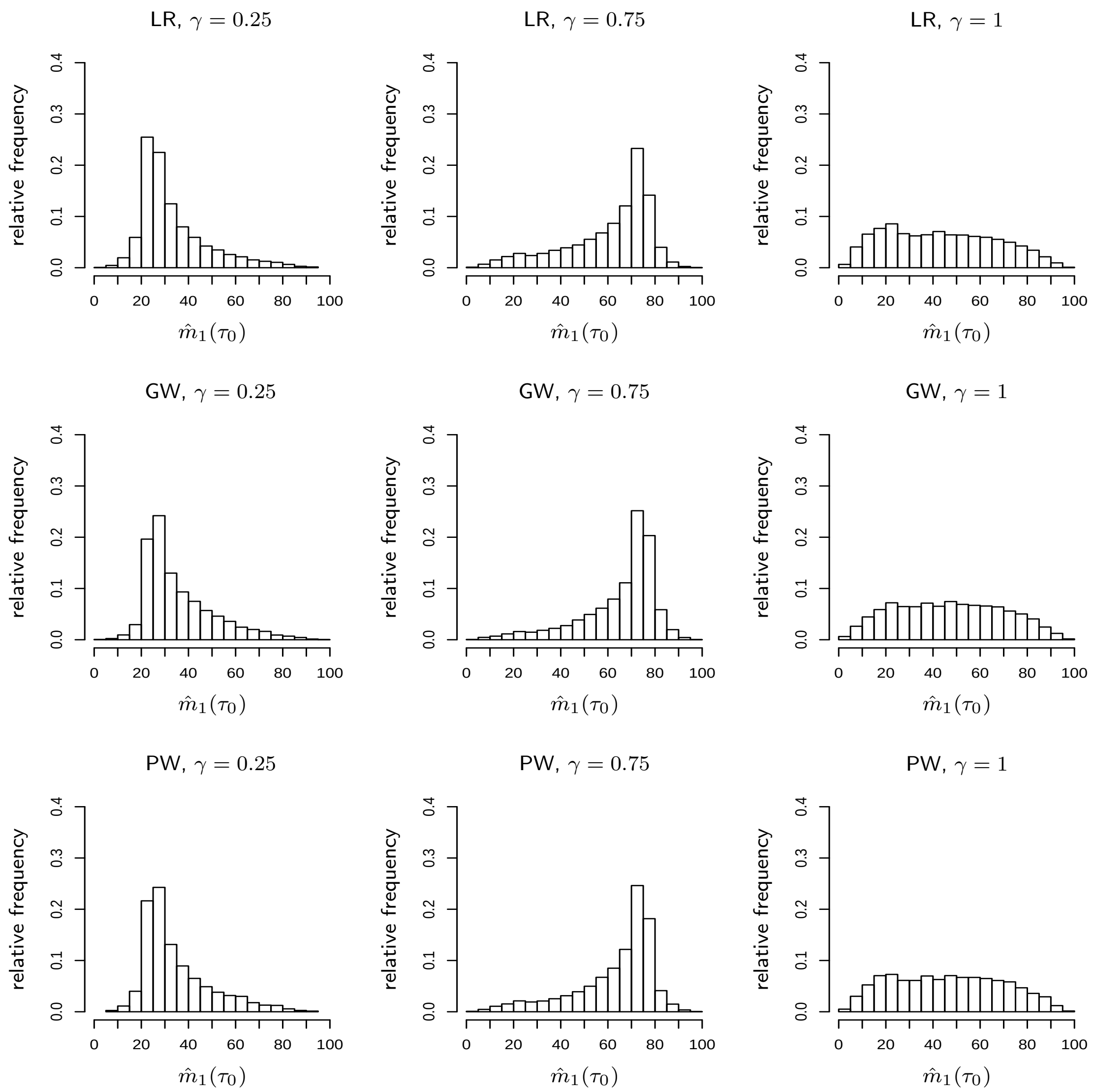

FIGURE 6. Histograms of $\hat{m}_1(\tau_0)$ for $\eta = 0.25$, $n = 100$, $\delta_n = \delta_{C,n} = 2$ and the log-normal distribution.

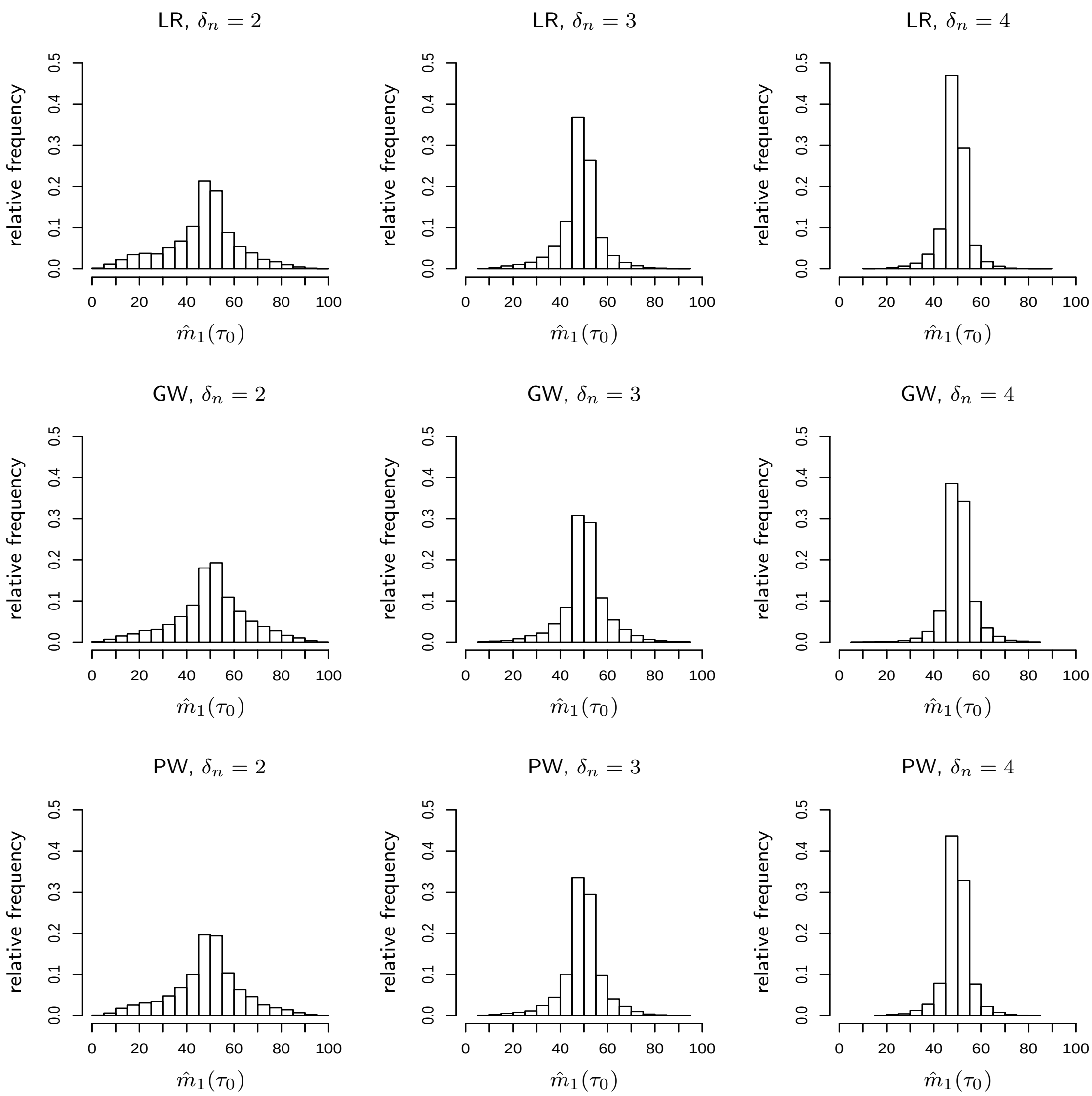

FIGURE 7. Histograms of $\hat{m}_1(\tau_0)$ for $\gamma = 0.5$, $\eta = 0.25$, $n = 100$, $\delta_{C,n} = 3$ and the exponential distribution.

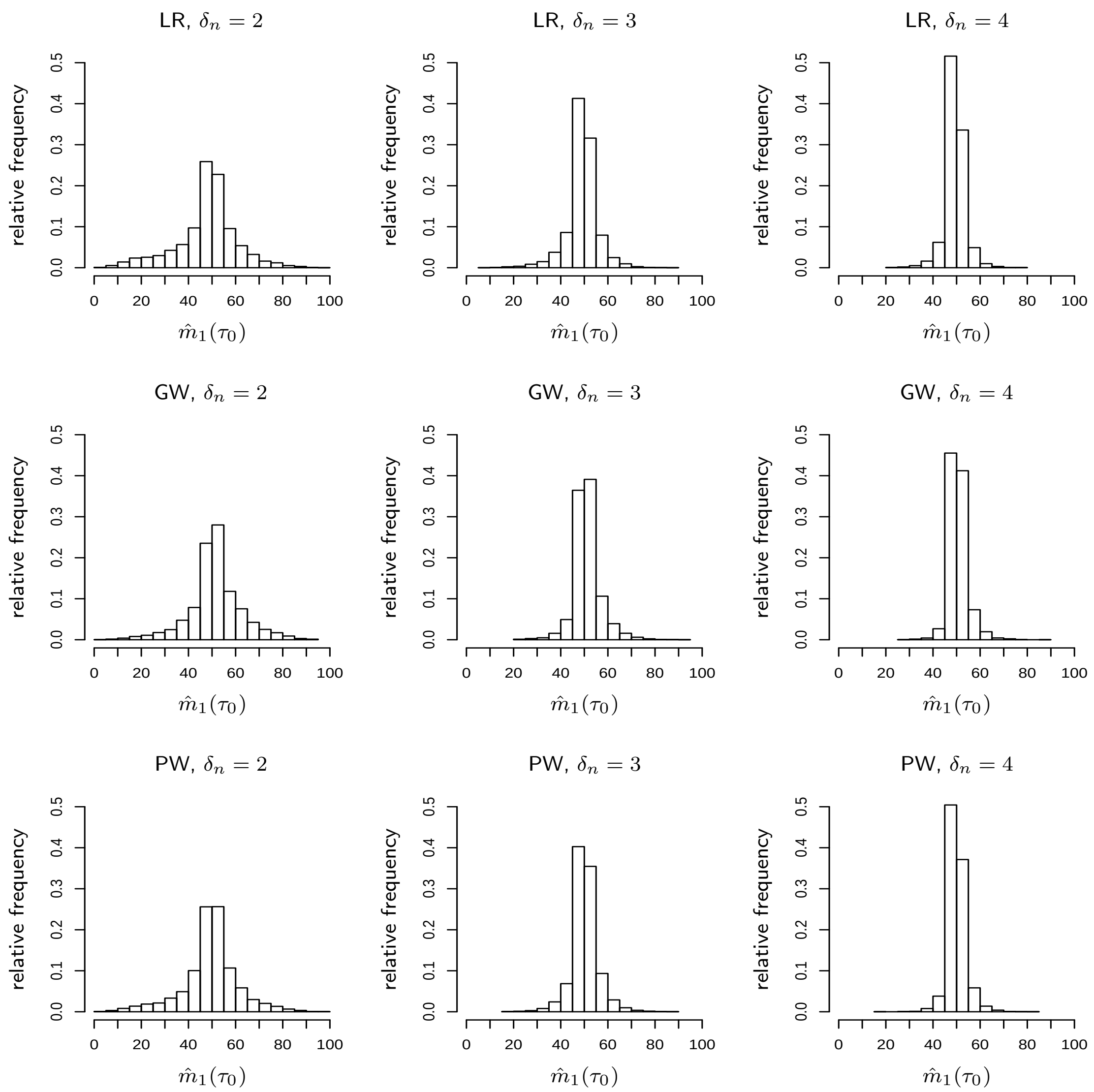

FIGURE 8. Histograms of $\hat{m}_1(\tau_0)$ for $\gamma = 0.5$, $\eta = 0.25$, $n = 100$, $\delta_{C,n} = 3$ and the log-normal distribution.

4.2. Behavior of $\hat{m}_2$. By the Monte Carlo simulations, we check the performance of the proposed estimator $\hat{m}_2(\tau_0)$ defined in (3.2) for finite sample situation.

Suppose RCM with $m = \lfloor n\gamma \rfloor$. We proceed with $n = 100; 200; 300$ as follows:

(1) The survival times $X_1^0, X_2^0, \ldots, X_n^0$ are simulated using the chosen combination of parameters

$$X_1^0, X_2^0, \ldots, X_{\lfloor n\gamma \rfloor}^0 \sim F_1, \qquad F_1 = \mathrm{E}(1) \text{ (or } \mathrm{L}(1))$$
$$X_{\lfloor n\gamma \rfloor+1}^0, X_{\lfloor n\gamma \rfloor+2}^0, \ldots, X_n^0 \sim F_2, \qquad F_2 = \mathrm{E}(\delta_n) \text{ (or } \mathrm{L}(\delta_n))$$

(we use $\gamma = 0.25; 0.5; 0.75; 1,\ \delta_n = 2; 3; 4$).

(2) The censoring times $C_1, C_2, \ldots, C_n$ are simulated using the chosen combination of parameters

$$C_1, C_2, \ldots, C_{\lfloor n\eta \rfloor}^0 \sim G_1, \qquad G_1 = \mathrm{E}(1) \text{ (or } \mathrm{L}(1))$$
$$C_{\lfloor n\eta \rfloor+1}, C_{\lfloor n\eta \rfloor+2}^0, \ldots, C_n^0 \sim G_2, \qquad G_2 = \mathrm{E}(\delta_{C,n}) \text{ (or } \mathrm{L}(\delta_{C,n}))$$

(we use $\eta = 0.25; 0.5; 0.75; 1,\ \delta_{C,n} = 2; 3$).

(3) The pairs of observations $(X_1, \Delta_1), \ldots, (X_n, \Delta_n)$ are computed.
(4) The estimator $\hat{m}_2(\tau_0)$ is calculated and its value stored.
(5) The steps (1) – (4) are repeated 10^4 times.
(6) The histogram with relative frequency of $\hat{m}_2(\tau_0)$ is drawn.

In Figures 9 – 16 the results of our simulations for the same choices of parameters as in Figures 1 – 8 are presented, so we will be able to compare differences in behavior of the estimators $\hat{m}_1(\tau_0)$ and $\hat{m}_2(\tau_0)$ which differ from each other by the standardization $V_k(\tau_0)$ of the form (2.16).

In Figures 9 and 10 the histograms of $\hat{m}_2(\tau_0)$ for $\gamma = 0.25$ and various choices of η and type of weights $w_n(t)$ can be found. We see again that the shape of the histograms is not influenced by location of the point of a change $\lfloor n\eta \rfloor$ in the distribution of the censoring variables (no change in the censoring variables is expressed by $\eta = 1$). We see that the main peak of the histograms of $\hat{m}_2(\tau_0)$ is in the neighborhood of $m = 25$ and is slightly more evident for the log-normal distribution which corresponds with the results of simulated power for the max-type test statistic $T_n(\tau_0)$ in Table 18 and 19, the parts with $\gamma = 0.25$. Comparing the graphs in Figures 9, 10 with Figures 1, 2, we see different shapes of the graphs at the tails of observation period. The estimator $\hat{m}_2(\tau_0)$ supposed to improve the situations when the change-point $\lfloor n\gamma \rfloor$ occurs on the tails, but for $\gamma = 0.25$ this improvement is not yet visible.

We see in Figures 11 and 12 the histograms of $\hat{m}_2(\tau_0)$ for $\gamma = 0.5$ and various choices of the sample size n and the most common types of weights. As in Figures 3 and 4, we see that the hight of peak of the histogram increases with growing n. Further, we observe bigger difference in the hight of the top between $n = 200$ and $n = 300$ than between $n = 100$ and $n = 200$ for both considered underlying distributions.

In Figures 13 and 14 the histograms of $\hat{m}_2(\tau_0)$ for various choices of the change-point γ and type of weights and the underlying distribution can be found. Notice that for $\gamma = 1$ there is no change in the distribution of the survival variables and in this case the histogram has two main peaks at the tails of our observation period which corresponds to the assertion of Theorem 3.4. Further, we observe that for $\gamma = 0.75$ there is the problem with the peak in the left-neighborhood of n, but the main peak is still located in the neighborhood of $\lfloor n\gamma \rfloor$ which corresponds to the assertion of Theorem 3.2. The problem with the "rival" peak nearly vanishes with $\gamma = 0.25$ (in the same figures) or $\gamma = 0.5$ (see Figures 11, 12).

In Figures 15 and 16 the histograms of $\hat{m}_2(\tau_0)$ for the change-point at the half of the observation period and the different sizes of the change amount δ_n in the distribution of the lifetimes

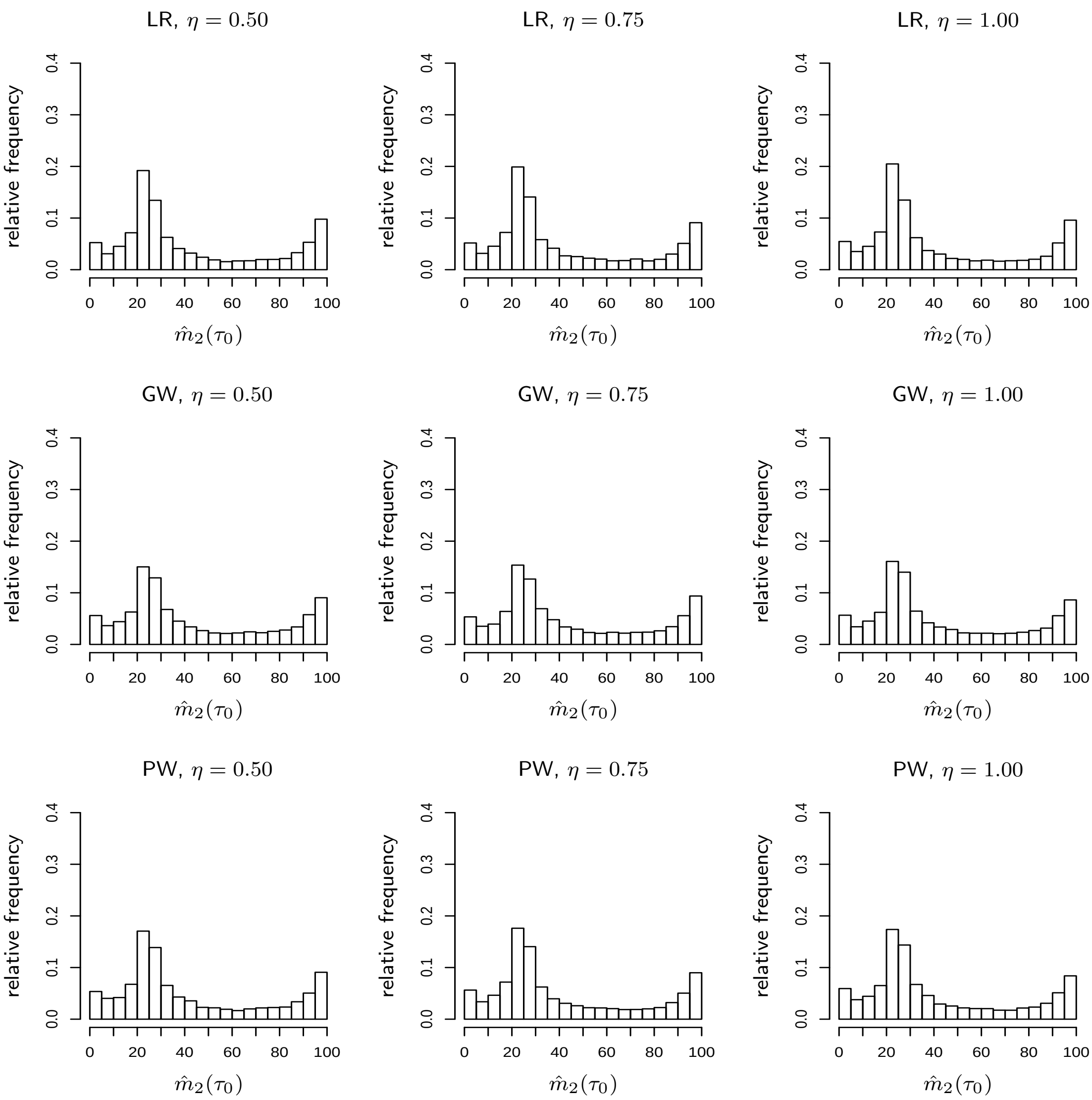

FIGURE 9. Histograms of $\hat{m}_2(\tau_0)$ for $\gamma = 0.25$, $n = 100$, $\delta_n = \delta_{C,n} = 2$ and the exponential distribution.

and different types of weights are shown. It can be seen that the peak of the graphs of relative frequency of the estimator is more evident with larger δ_n. Analogous results were obtained for the simulated power, see the parts of Tables 18 and 19 with $\gamma = 0.5$. If we compare the graphs with their counterparts in Figures 7 and 8, we detect that the peak for $\gamma = 0.5$ is higher for $\hat{m}_1(\tau_0)$ than for $\hat{m}_2(\tau_0)$ and moreover, the interquartile range is narrower in the case of $\hat{m}_1(\tau_0)$.

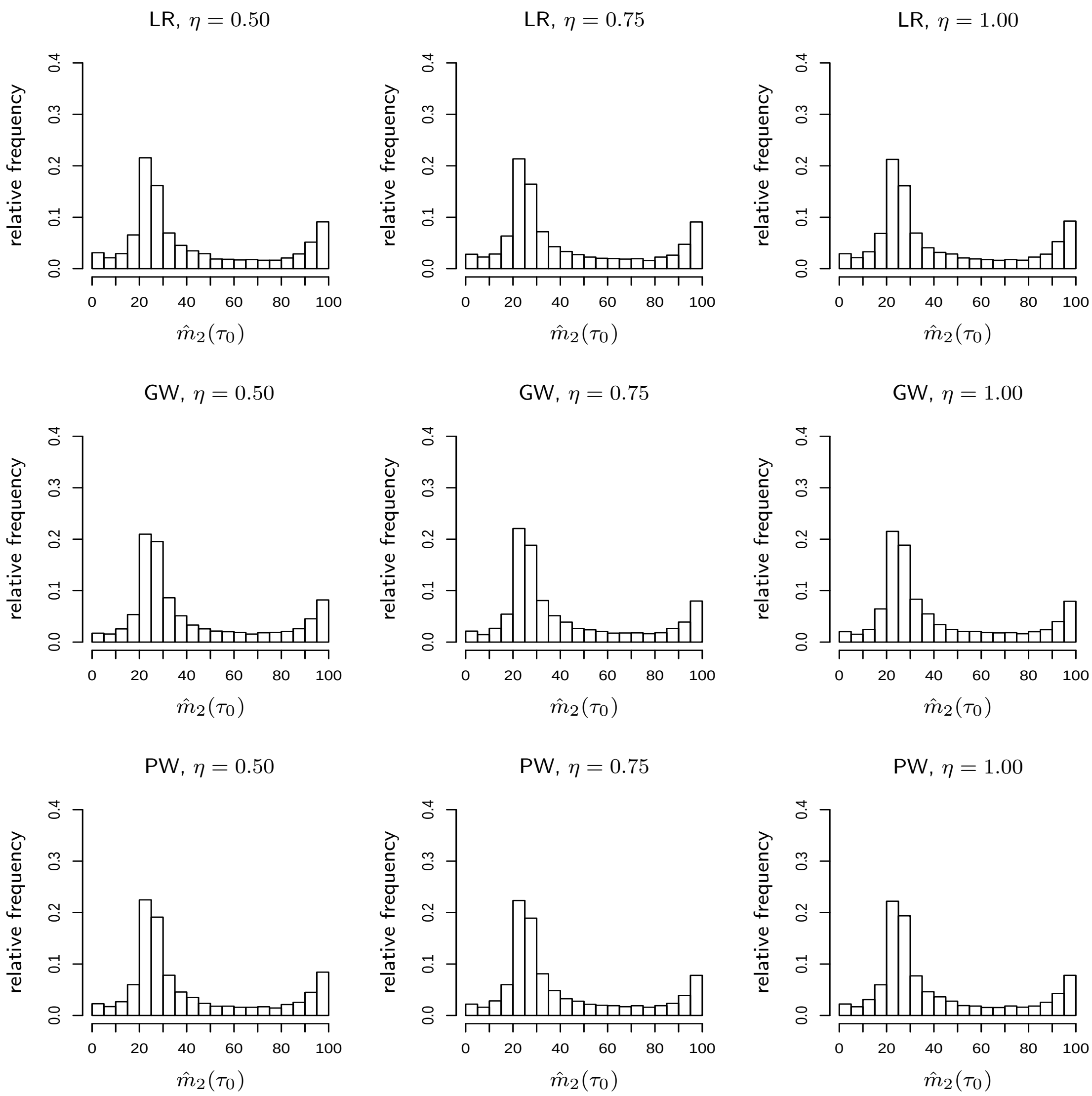

FIGURE 10. Histograms of $\hat{m}_2(\tau_0)$ for $\gamma = 0.25$, $n = 100$, $\delta_n = \delta_{C,n} = 2$ and the log-normal distribution.

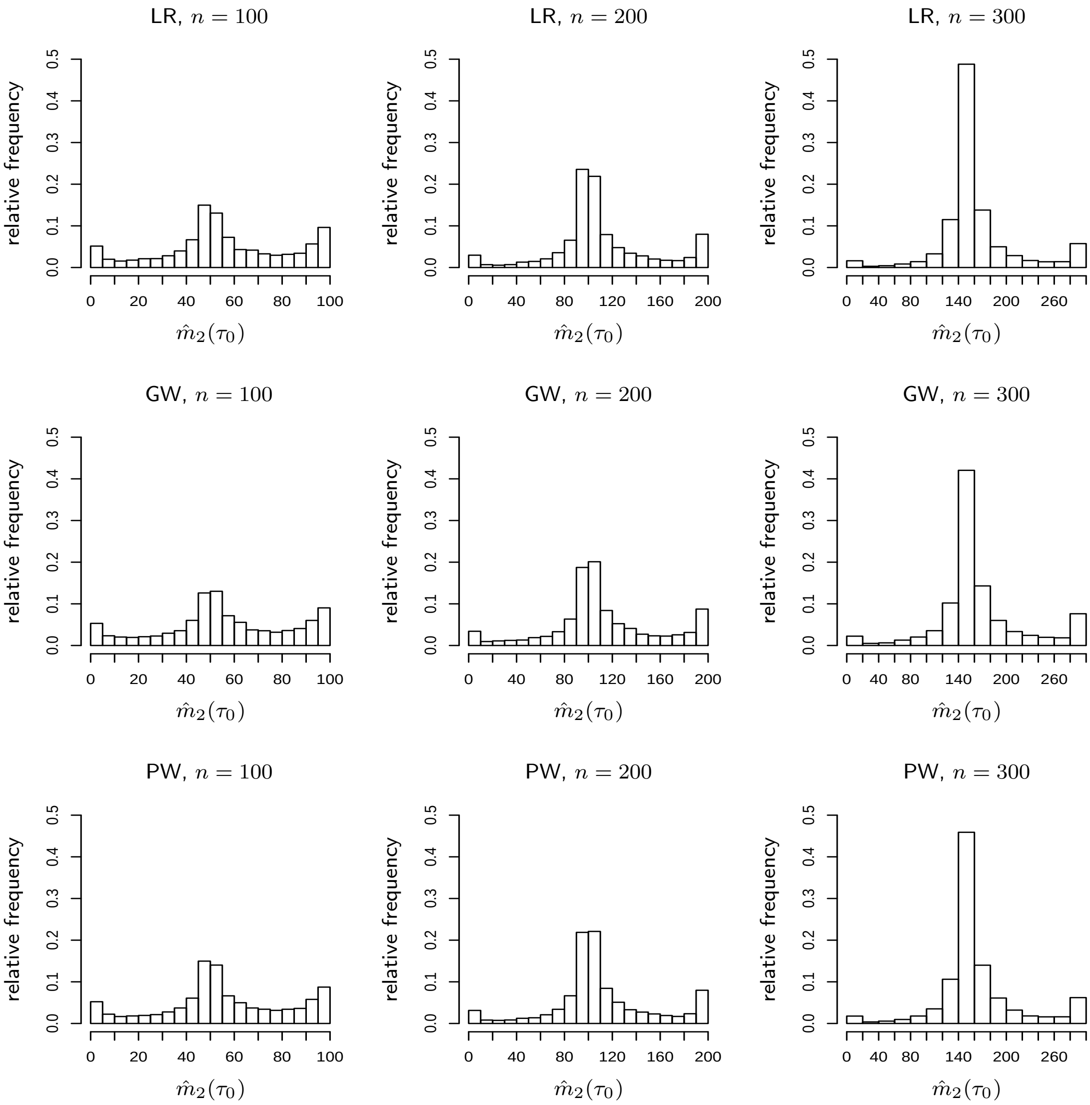

FIGURE 11. Histograms of $\hat{m}_2(\tau_0)$ for $\gamma = 0.5$, $\eta = 0.25$, $\delta_n = \delta_{C,n} = 2$ and the exponential distribution.

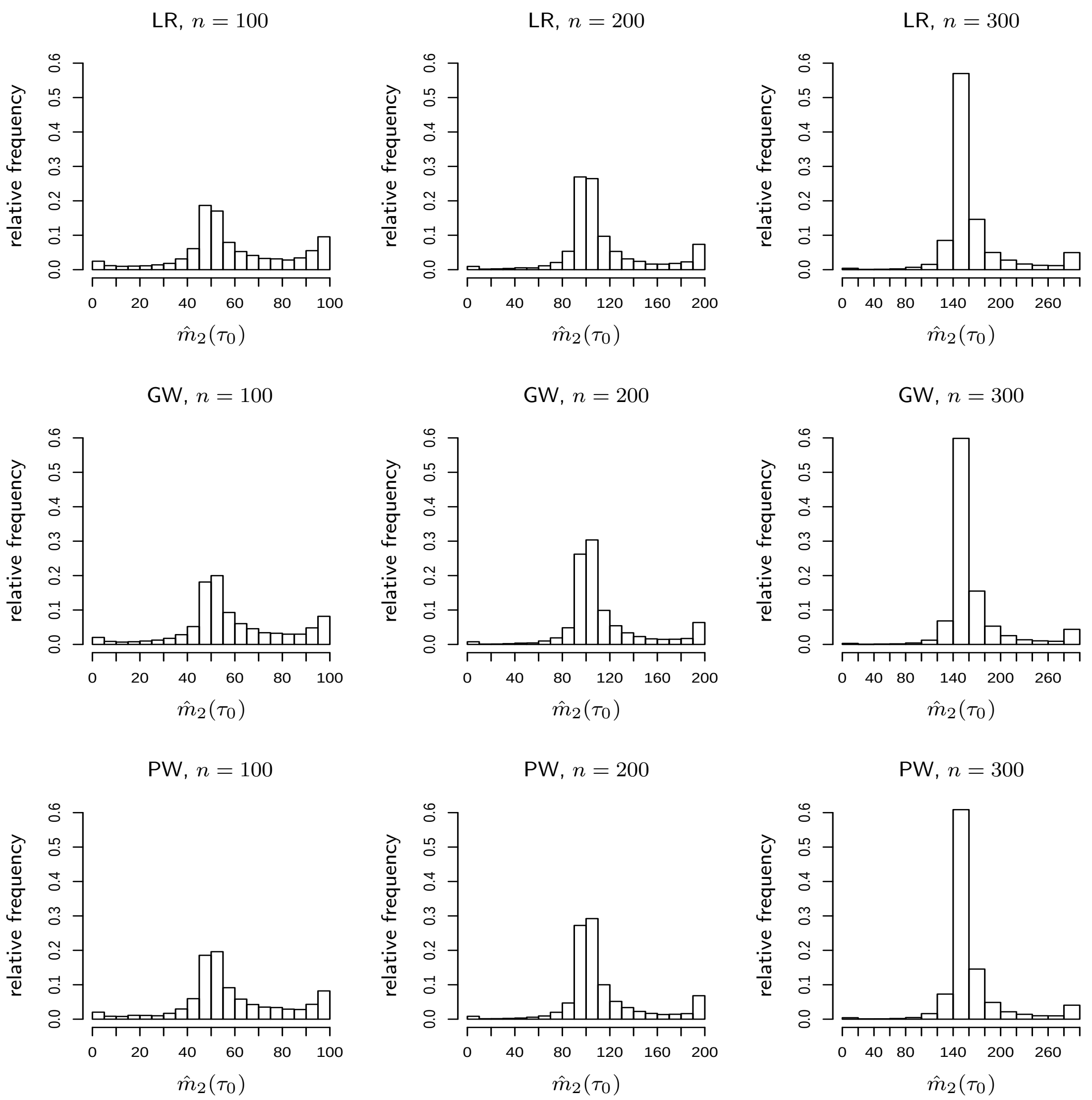

FIGURE 12. Histograms of $\hat{m}_2(\tau_0)$ for $\gamma = 0.5$, $\eta = 0.25$, $\delta_n = \delta_{C,n} = 2$ and the log-normal distribution.

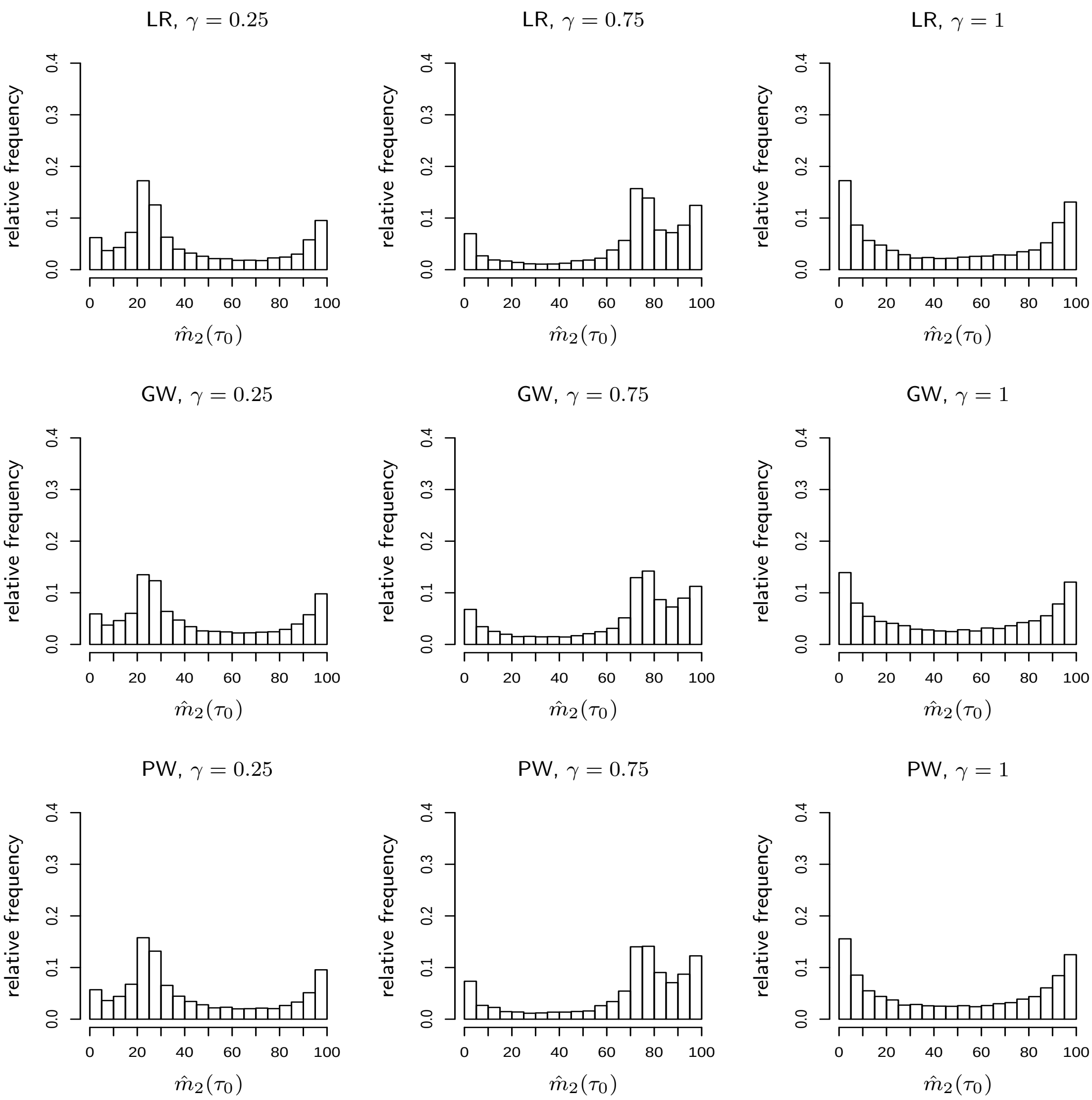

FIGURE 13. Histograms of $\hat{m}_2(\tau_0)$ for $\eta = 0.25$, $n = 100$, $\delta_n = \delta_{C,n} = 2$ and the exponential distribution.

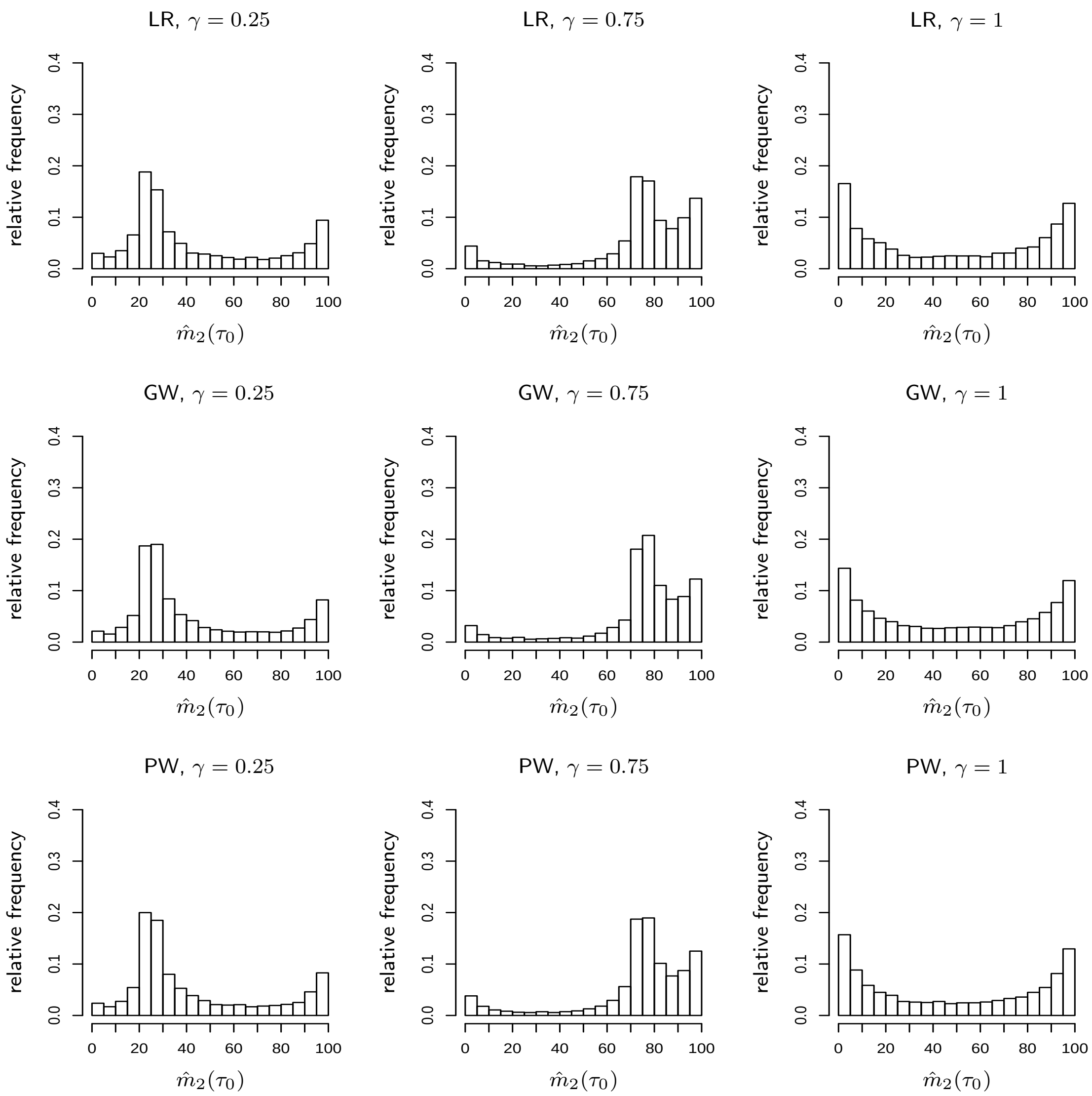

FIGURE 14. Histograms of $\hat{m}_2(\tau_0)$ for $\eta = 0.25$, $n = 100$, $\delta_n = \delta_{C,n} = 2$ and the log-normal distribution.

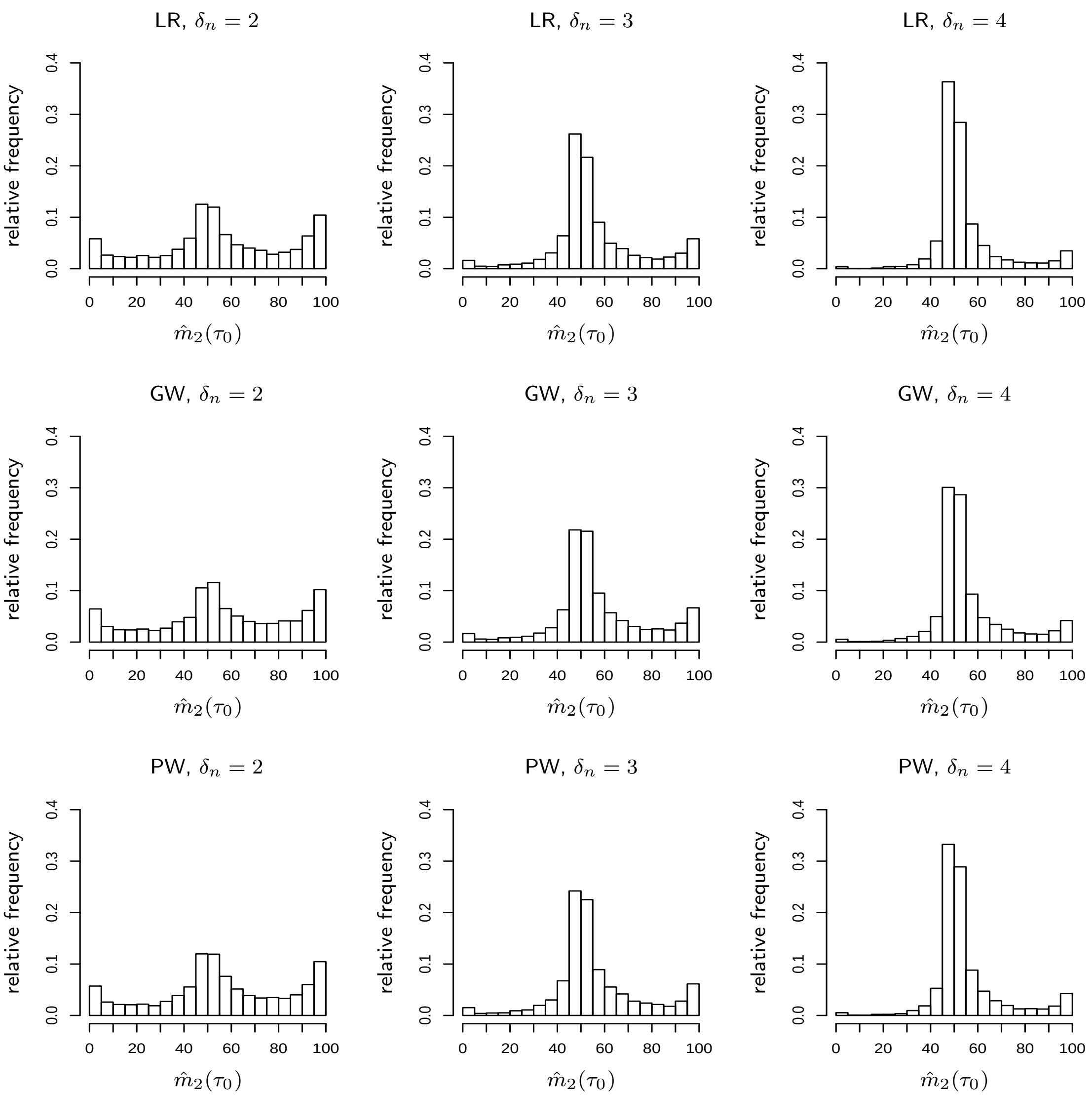

FIGURE 15. Histograms of $\hat{m}_2(\tau_0)$ for $\gamma = 0.5$, $\eta = 0.25$, $n = 100$, $\delta_{C,n} = 3$ and the exponential distribution.

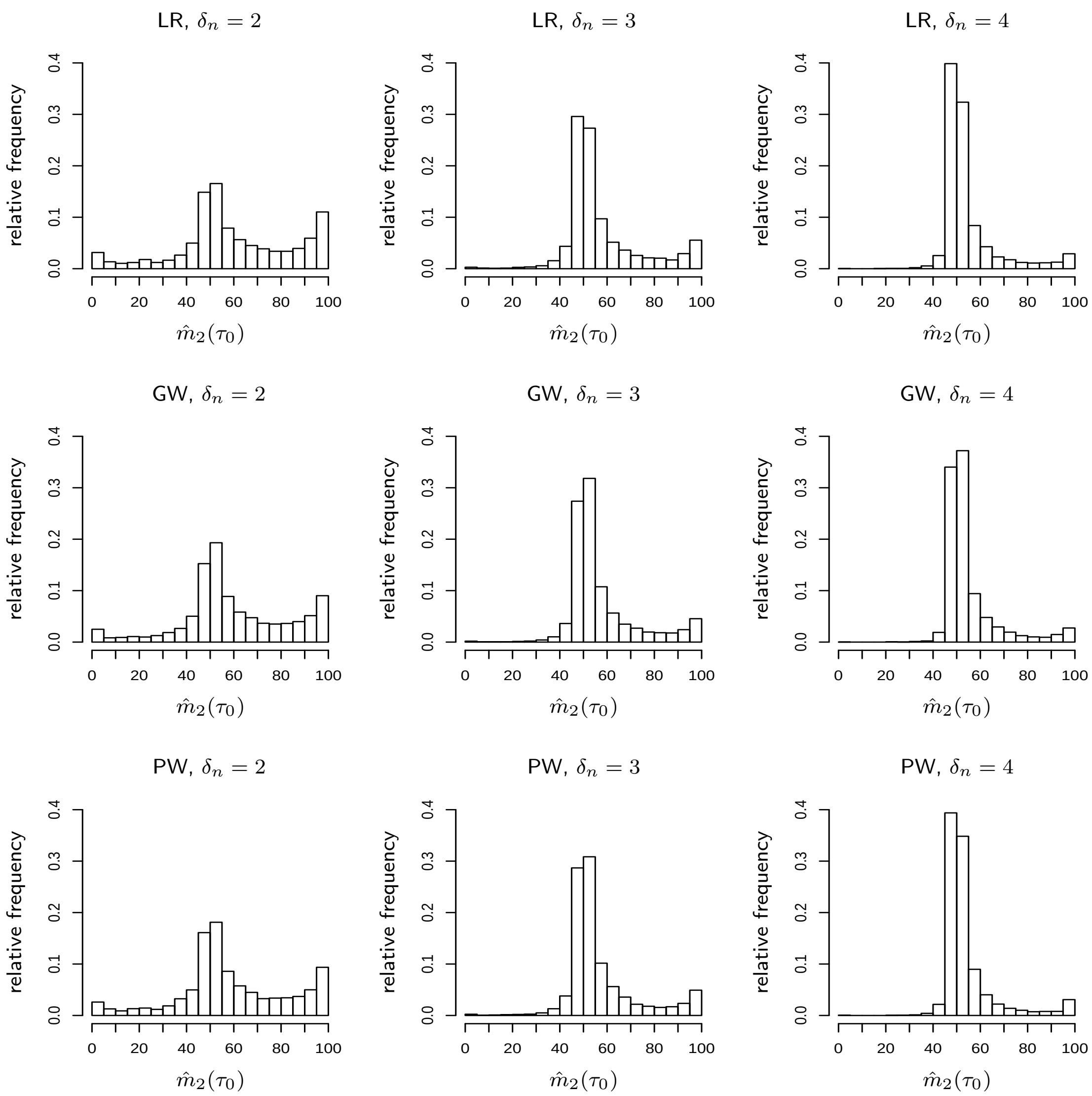

FIGURE 16. Histograms of $\hat{m}_2(\tau_0)$ for $\gamma = 0.5$, $\eta = 0.25$, $n = 100$, $\delta_{C,n} = 3$ and the log-normal distribution.

Conclusions

The aim of this thesis was to extend the results recently given by Hušková and Neuhaus [**23**]. *For testing, we considered not only one-change point form of the alternative but also the multiple-change case. Additionally, we dealt with the estimators of the change point. Assuming that also the distribution of the censoring variables can change at an unknown point we faced quite some difficulties. We have succeeded to derive a number of new results but still many problems remain open.*

We mention some of them:

(1) *We have studied performance of the tests under the null hypothesis and proved consistency. However, it would be worthwhile to study properties of the test under the local alternatives.*

(2) *Concerning the estimators, only the rate of consistency has been studied. The open question is their limit distribution. Further, it should be solved how to avoid the assumption of the local alternatives for the distribution of the censoring times.*

(3) *Another type of tests and estimators can be introduced and studied, e.g. the Kolmogorov–Smirnov type test statistic as a generalization of Neuhaus* [**29**].

(4) *We have considered absolutely continuous distributions only. The open question is to study the case of ties, a motivation can be found in two-sample censorship model treated e.g. by Neuhaus* [**29**].

(5) *We have presented a change point problem for the lifetimes which are censored from the right by the censoring times. It is also worth to study the situation, when the lifetimes are doubly censored (current status data), i.e. they can be censored either from the right or from the left. In this case tests and estimators for the change point problem could be derived along the lines of Gehan* [**15**] *who developed a basic two-sample test under such censoring.*

(6) *A possible future research can also be oriented to a general problem of detection a change point in models used in survival analysis, for extensive list of failure time models (without a change point) see e.g. Kalbfleisch and Prentice* [**25**].

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