

Review of the Doctoral Thesis
“Coloring Problems in Geometric Context”
submitted by Aleš Přivětivý

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Summary

The thesis consists of three independent parts.

1. The first part concerns the combinatorial discrepancy for the set system \mathcal{S}_n^3 of sums of three arithmetic progressions in $\{0, 1, \dots, n-1\}$. This is a very natural extension of the question for single arithmetic progressions, which was for a long time an important open problem in discrepancy theory and is now well-known to be $\Theta(n^{1/4})$ (Roth (1964), Beck (1981), and Matoušek and Spencer (1996)).

Improving on earlier bounds by Hebbinghaus (2004), the candidate shows that the discrepancy of \mathcal{S}_n^3 is at least $\Omega(n^{1/2})$, which is almost tight, since a standard probabilistic argument shows that any set system of size polynomial in n has discrepancy at most $O(\sqrt{n \log n})$. The proof uses the eigenvalue bound method. In the main technical lemma, for every prime n and integer $0 < j < n$, a set $C_j \subset \{0, 1, \dots, n-1\}$ is constructed such that (i) C_j is the sum of three arithmetic progressions, (ii) $jk \pmod n \in \{0, \dots, n/6\} \cup \{5n/6, \dots, n-1\}$ for all $k \in C_j$ and (iii) $|C_j| \geq \Omega(n)$. Shifted copies of the sets C_j form a so-called *wrapped* system of n^2 sets, whose discrepancy asymptotically lower bounds that of \mathcal{S}_n^3 . Moreover, the Laplacian of the wrapped system is a circulant matrix, whose eigenvalues are easy to analyze; specifically, Properties (ii) and (iii) ensure that the smallest eigenvalue of the Laplacian is $\Omega(n^2)$. For a set system of size n^2 , this implies that the discrepancy is at least $\Omega(\sqrt{n})$.

These results were very recently further strengthened by Hebbinghaus (2006), who proved that the same lower bound of $\Omega(\sqrt{n})$ also holds for sums of two arithmetic progressions. The thesis also contains an alternative proof of this result.

2. The second part of the thesis concerns graph (vertex) colorings with small monochromatic connected components. Specifically, it is shown that for

every two-coloring of a triangulated d -dimensional grid of sidelength n , there exists a monochromatic component of size at least n^{d-1}/\sqrt{d} , which is tight up to a constant factor, as shown by the diagonal layer coloring; for a grid with all diagonals added, the lower is improved to $n^{d-1} - d^2n^{d-2}$, which is tight up to the lower-order term.

The first step of the proof uses topological arguments. The main lemma states that for a sufficiently connected simple graph G on n vertices (namely, if G can be extended to a simply connected simplicial complex by gluing in higher-dimensional faces (triangles suffice)), there is a monochromatic connected *separator*, i.e., a subset of vertices upon whose removal all remaining connected components have size at most $n/2$.

Grid graphs with added diagonals satisfy the assumptions of the topological lemma. The second step uses edge and vertex isoperimetric inequalities in the grid to deduce lower bounds for the size of a monochromatic connected separator in each of the two cases.

3. The third part of the thesis concerns the problem of reconstructing a set of points in the plane from finitely many orthogonal projections (“discrete X-rays”). It is shown that for every integer $k > 0$, any generic set of k projection directions is “good” in the sense that allows to uniquely reconstruct any set of $2^{Ck/\log k}$ or fewer points, for some constant C . Here, “generic” means that for each k , there is a finite list of $2k$ -variate polynomials such that the coordinates of each bad k -tuple of projection directions lie in the zero set of one of the polynomials. This is complemented by upper bounds that show that for any k projection directions, one can construct two sets of 1.81712^k points with the same projections (for multisets, the base is further improved to 1.79964).

For the proof of the lower bound, the first step is again linear-algebraic. From a potential counterexample of two sets of n points having the same projections in directions u_1, \dots, u_k , a bipartite “interchange” graph is constructed whose edges are partitioned (“colored”) into k perfect matchings. For any subgraph of this interchange graph with the right number $(2n-2)$ of edges, the incidence matrix is in a suitable enriched to a square matrix with $2k$ indeterminates, and it is shown that the determinant of the resulting matrices vanishes if the coordinates of the u_i ’s are substituted for the indeterminates.

In the second step, it is shown that under certain combinatorial conditions on the subgraph, this determinant is actually nonzero as a polynomial, and by a quite intricate probabilistic argument it is shown that for $n \leq 2^{Ck/\log k}$, there is a suitable subgraph that meets these conditions.

Comments

There are a couple of small inaccuracies in the presentation: First, in the definition of a circulant matrix (pp. 12–13) the rows should be shifted to the

right, not to the left, in order to guarantee the simple form of eigenvectors and λ -values. Secondly, the specification of the parameter d_2 in proof of Lemma 2.8 (p. 18) is not quite correct, it should be $d_2 = -d_1 \lfloor n/c_1 \rfloor \pmod n$. Neither of these affects the validity of the proofs. Beyond this, I only found a few typos and minor mistakes too trivial to deserve mentioning.

Evaluation

This is a very nice thesis that contains a number of interesting new results. Moreover, it is also well-written; the presentation is well-structured, and concise, and apart from the two minor comments above, very precise and clear.

Personally, my favorite result is the auxiliary topological lemma on monochromatic connected separators, which I think is of independent interest and has a significant potential for further extensions and applications.

I am particularly impressed by the variety of methods that are used in the proofs: algebraic techniques as well as topological arguments are combined with clever observations and intricate combinatorial and probabilistic arguments. The candidate's work shows a well-rounded general mathematical background as well as technical strength and versatility.

Summarizing, I think this thesis meets high standards and should be accepted as a doctoral dissertation.



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To whom it may concern:

Please find enclosed my evaluation of the dissertation "Coloring Problems in Geometric Context" submitted by Aleš Přívětivý.

Summarizing, I find the thesis to be of high quality and recommend that it be accepted as a doctoral dissertation.

Yours sincerely

Uli Wagner

Enclosures

- Detailed evaluation

