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RNDr. Tomáš Neustupa

CHARLES UNIVERSITY, PRAGUE

FACULTY OF MATHEMATICS AND PHYSICS

Department of Numerical Mathematics

Mathematical Modelling of Viscous Incompressible Flow through a Cascade of Profiles

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RNDr. Tomáš Neustupa

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Contents

Introduction

One of the most important subjects in the theory of blade machines (i. e. turbines, compressors, pumps etc.) is the study of flows through blade rows. These flows are rather complex. In general, they are three-dimensional, nonstationary, rotational, turbulent and the domain occupied by the fluid has a complicated geometry. This is the reason that there is a number of various models used for the analysis of the flow though blade rows. Simple one–dimensional models which give only a very rough image of the behaviour of fluid flow in blade rows are now employed rarely. On the other hand, three–dimensional models are so complex that their usage in numerical calculations requires rather long CPU time even on most modern computers. This is why two–dimensional models are still extensively used. We can mention, e. g. the widely used model of plane flow through a cascade of profiles. (For its derivation, see e. g. [8] or [7], Section 3.4.) It has numerous variants taking into account various features, as incompressibility or compressibility of the fluid and variable geometry of casing walls of the through flow part of blade machines leading to the model of flow in a layer of variable thickness. (See, e. g. [2], [8], [29], [33], [44], [52], [56], [57]). The model of plane irrotational incompressible inviscid flow, numerically realized by Martensen's method, was applied in engineering practice for several decades. E. Martensen transformed the problem to a boundary integral equation on a single profile of a cascade in his papers [45], [46] and [47]. Using the Fredholm alternative, he proved the existence and uniqueness of the solution. Martensen's integral equation was later discretized and became the basis for an efficient and accurate numerical method for the solution of flows through cascades of profiles; see ([31], [51]).

The method of E. Martensen can be considered to be the first mathematically founded technique for the simulation of cascade flow problems. The development of modern computers in the last years enables us to consider successively more and more complex models, involving compressibility, viscosity, heat conduction and various other phenomena. A series of approaches to numerical simulation of cascade flow problems is described in ([3], [4], [5], [6], [9], [10], [11], [12], [13], [14], [15], [17], [18], [19], [24], [26], [27], [28]). However, a rigorous mathematical theory is in most of the used models missing.

Motivated by this state, the present thesis deals with the theoretical analysis of an incompressible viscous flow through a plane cascade of profiles. Our model is based on the Navier–Stokes equation (conservation of momentum) and the equation of continuity (conservation of mass). In the first three chapters, we study the stationary problem with suitable boundary conditions. The conditions involve the non–homogeneous Dirichlet boundary condition on the inflow, the no–slip boundary condition on the profiles and the condition of periodicity in one space direction (along the cascade of profiles). The spatially periodic character of the flows enables us to restrict ourselves to a boundary–value problem formulated just on one space period.

The mostly discussed condition is the boundary condition on the outflow. We use a modified "do nothing condition" analogous to the condition used by J. Heywood, R. Rannacher and S. Turek [25] in Chapter I. The name "do nothing" comes from the fact that the condition does not explicitly appear in the weak formulation of the problem and it follows from this formulation if an existing solution is in some sense "smooth". If the convective nonlinear term in the Navier–Stokes equation is written in the "traditional" way as $(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}$, where $\boldsymbol{u}$ is the velocity, then the natural weak formulation (used in [25] and similar to the weak formulation of the boundary value problem with the Dirichlet boundary condition on the whole boundary explained e.g. by R. Temam in [55]) does not exclude eventual backward flows on the output that can bring back to the considered flow field an uncontrollable amount of kinetic energy and consequently, the energy estimate breaks down. The same difficulty was already observed in studies of a flow in a channel by many authors. In order to preserve an energy estimate, S. Kračmar and J. Neustupa in [34] and [37] prescribed an additional boundary condition which restricted the kinetic energy brought back on the outflow and they have therefore described and solved the problem by means of a variational inequality of the Navier–Stokes type. P. Kučera and Z. Skalák solved the problem for "small" data, see [39], [38]. In Chapter I of this thesis, we use a certain nonlinear modification of the boundary condition on the outflow from [25]. The modification was proposed by C. H. Bruneau, F. Fabrie in [1]. It enables us to derive necessary energy estimates and to prove the existence (for a sufficiently small inflow) and uniqueness (for all data sufficiently small).

In the first two sections of Chapter II, we use the Navier–Stokes equation with the nonlinear term in the form $\mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{u}$ (whose two–dimensional analog is $\omega(\boldsymbol{u})\, \boldsymbol{u}^\perp$ – see Section II.1). This is compensated by the presence of pressure in Bernoulli's dynamic form $p + \frac{1}{2}|\boldsymbol{u}|^2$ (which we call the Bernoulli pressure). Since $\omega(\boldsymbol{u})\, \boldsymbol{u}^\perp$ is point–wise perpendicular to $\boldsymbol{u}$, we are able to derive, from a corresponding weak formulation, an energy estimate regardless the backward flows on the outflow. Thus, the "do nothing" boundary condition on the outflow, following from the weak formulation, need not be modified in the sense of C. H. Bruneau, F. Fabrie and can have formally the same form as in [25], however with the Bernoulli pressure $q = p + \frac{1}{2}|\boldsymbol{u}|^2$ instead of the "static" pressure p. The energy estimate further enables us to prove the existence and a uniqueness of the weak solution. We also need the velocity on the inflow to be "small enough" in the proof of existence and all data to be "small enough" in the theorem on uniqueness.

The requirement on the smallness of the data in the theorems on uniqueness has the same reason as the similar condition in known theorems on uniqueness of stationary solutions of boundary–value problems for the Navier–Stokes equation with the Dirichlet boundary condition on the whole boundary of the flow field. As examples of non–uniqueness are known in the case of the Dirichlet boundary condition on the whole boundary and large data, it is logical to expect that the non–uniqueness can also take place in the case of our boundary value problem, if the given data are large.

The condition on smallness of the inflow velocity has the same background as the condition on smallness of fluxes between various components of the boundary in known theorems on existence of a weak solution of the boundary–value problem for the Navier–Stokes equation with the Dirichlet boundary condition, see e.g. G. P. Galdi [21] or V. Girault and P.-A. Raviart [23]. The case of a large inflow is solved in Section II.3, however the possibility of the large inflow is again compensated by certain modification of the boundary condition on the outflow.

Chapter III deals with the nonstationary case of the boundary value problem from Chapter I. We present a weak formulation and we prove the existence of a weak solution. Our result extends known theorems on the existence of a weak solution of the Navier–Stokes initial–boundary value problem, proved originally by J. Leray [42] and E. Hopf

[30], and explained in a series of books on the theory of the Navier–Stokes equations, see e.g. O. A. Ladyzhenskaya [41], V. Girault and P.-A. Raviart [23], R. Temam [55] and M. Feistauer [7]. The proof is carried out by the Rothe method, which was for the first time applied to the Navier–Stokes equations by M. Shinbrot in [54]. Due to the possibility of deriving a necessary energy estimate by means of the Gronwall lemma, we do not need the velocity on the inflow to be "small" in Chapter III. On the other hand, the obtained regularity of the weak solution does not enable us to prove its uniqueness even if all the data are "small". It can be expected that, in the case of "small" and regular data (inflow, specific body force, outflow), the existing weak solution is regular and therefore unique, as it is known from the case of the Dirichlet boundary condition on the whole boundary. However, we have not studied this question in the thesis.

Geometry of the problem

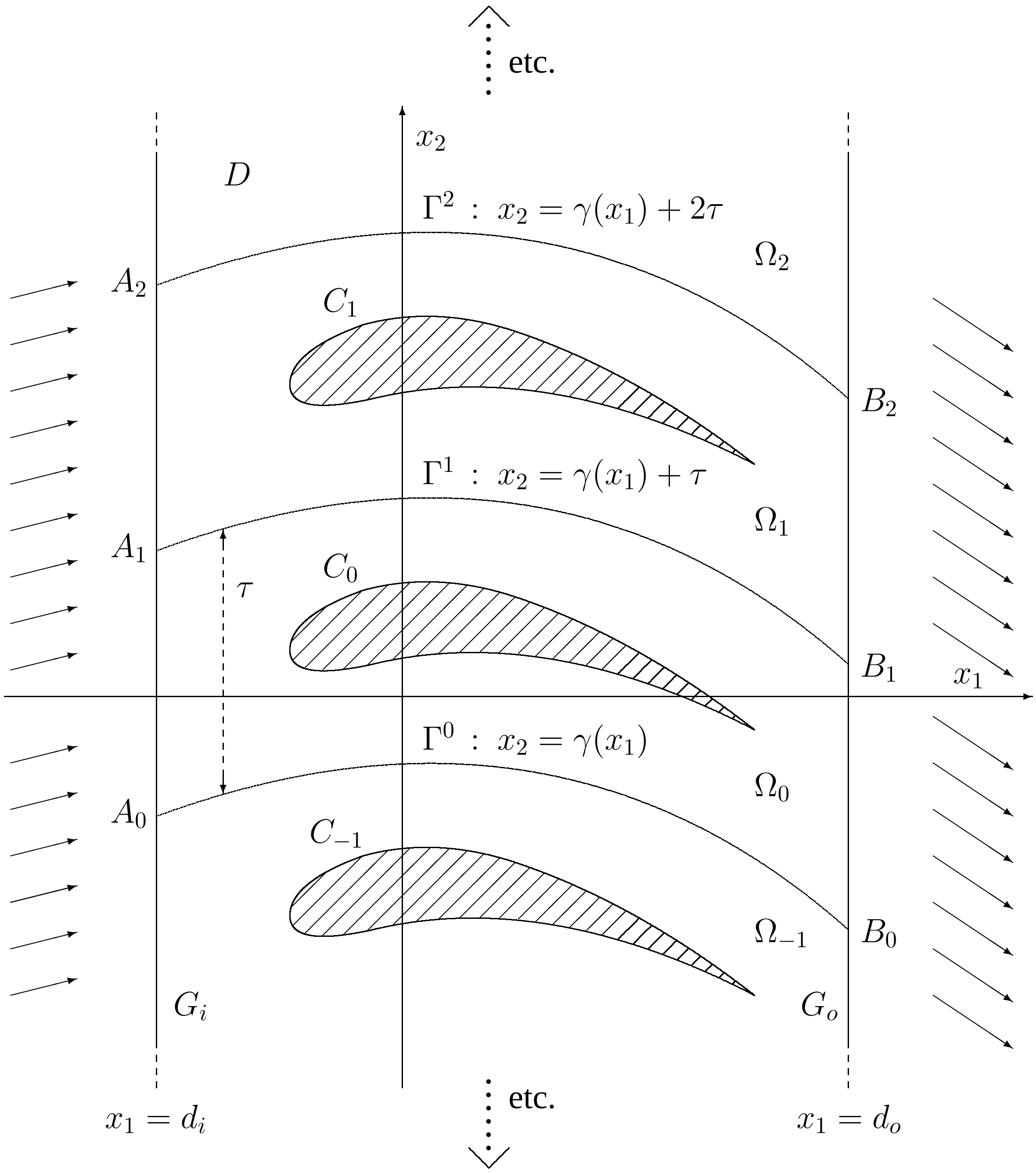

Fig. 1

Let us suppose that C_0 is a simple closed curve in $\mathbb{R}^2$ which is piecewise of the class C^2 and whose interior and exterior are domains with a Lipschitz–continuous boundary. For $k \in \mathbb{Z}$ we put

$$C_k \, = \, \{ \, (x_1, x_2 + k\tau); \; (x_1, x_2) \in C_0 \},$$

where τ is a positive constant. We assume that τ is so large that the curves C_k are mutually disjoint. By $\operatorname{Int} C_k$ we denote the interior of the curve C_k. The set

$$M = \bigcup_{k=-\infty}^{+\infty} \overline{\operatorname{Int} C_k}$$

is called a *cascade of profiles* and each of its components $\overline{\operatorname{Int} C_k}$ is called a *profile*. Number τ is called the *period* of the cascade.

Let d_i and d_o be two real numbers which are chosen so that $d_i < d_o$ and all the profiles are lying inside the strip $\{[x_1, x_2] \in \mathbb{R}^2;\ d_i < x_1 < d_o\}$. We shall study the motion of a viscous incompressible fluid in the unbounded domain

$$D = \{[x_1, x_2] \in \mathbb{R}^2;\ d_i < x_1 < d_o\} \setminus M.$$

We suppose that the fluid enters the domain D through the straight line G_i, given by the equation $x_1 = d_i$, and it leaves the domain D through the straight line G_o, given by the equation $x_1 = d_o$. Now it is logical to call the straight line G_i the *inlet* (or the *inflow*) and the straight line G_o the *outlet* (or the *outflow*). In accordance with the notation G_i and G_o, we further denote by G_w the union of all the curves C_k: $G_w = \cup_{k=-\infty}^{+\infty} C_k$. The set G_w represents an infinite family of fixed walls inside the flow field; this is why we use the subscript w.

Let us assume the existence of a function $x_2 = \gamma(x_1)$, defined in the interval $[d_i, d_o]$, which belongs to $C^1([d_i, d_o])$ and whose graph separates the profiles $\overline{\operatorname{Int} C_0}$ and $\overline{\operatorname{Int} C_{-1}}$. It means that the curve C_0 is lying above the graph of the function γ and the curve C_{-1} is lying below the graph of γ. (See Fig. 1.) We denote by $A_0 = (d_i, \gamma(d_i))$ and $B_0 = (d_o, \gamma(d_o))$ the end points of the graph of γ.

Let us denote by Γ^k (for $k \in \mathbb{Z}$) the curve which is identical with the graph of the function $x_2 = \gamma(x_1) + k\tau$ for $d_i \le x_1 \le d_o$.

Now each of these curves separates the profiles $\overline{\operatorname{Int} C_{k-1}}$ and $\overline{\operatorname{Int} C_k}$. We denote by A_k, respectively B_k the end points of the curve Γ^k. It means that $A_k = (d_i, \gamma(d_i) + k\tau)$ and $B_k = (d_o, \gamma(d_o) + k\tau)$.

Let us denote by Ω_k (for $k \in \mathbb{Z}$) a subdomain of D, whose boundary consists of the line segments $A_k A_{k+1}$, $B_k B_{k+1}$ and the curves Γ^k, Γ^{k+1} and C_k.

Our goal is to study flow in the whole domain D. This domain is unbounded, however its shape repeats periodically with the period τ in the x_2–direction. Thus, it is reasonable to deal with flow which is τ–periodic in the x_2–direction. This assumption enables us to restrict our considerations the flow in one spatial period of the domain D only, e.g. the flow in the domain Ω_0, naturally with appropriate conditions of periodicity on the curves $x_2 = \gamma(x_1)$ and $x_2 = \gamma(x_1) + \tau$, $d_i \le x_1 \le d_o$.

Let us further, for simplicity, write only Ω instead of Ω_0 and let us also use this notation:

- Γ_i – the line segment $A_0 A_1$,
- Γ_o – the line segment $B_0 B_1$,
- Γ_- – the graph of the function $x_2 = \gamma(x_1)$, $d_i \le x_1 \le d_o$,
- Γ_+ – the graph of the function $x_2 = \gamma(x_1) + \tau$, $d_i \le x_1 \le d_o$,

Γ_w – coincides with the curve C_0.

The symbols $(\Gamma_i)^\circ$, $(\Gamma_o)^\circ$, $(\Gamma_-)^\circ$, $(\Gamma_+)^\circ$ denote the curves Γ_i, Γ_o, Γ_-, Γ_+ without their end points:

$(\Gamma_i)^\circ$ – the open line segment A_0A_1,

$(\Gamma_o)^\circ$ – the open line segment B_0B_1,

$(\Gamma_-)^\circ$ – the graph of the function $x_2 = \gamma(x_1),\ d_i < x_1 < d_o$,

$(\Gamma_+)^\circ$ – the graph of the function $x_2 = \gamma(x_1) + \tau,\ d_i < x_1 < d_o$.

Using this notation, we have

$$\partial\Omega = \Gamma_i \cup \Gamma_o \cup \Gamma_- \cup \Gamma_+ \cup \Gamma_w .$$

Fig. 2 shows the domain Ω with all the points and curves which form its boundary.

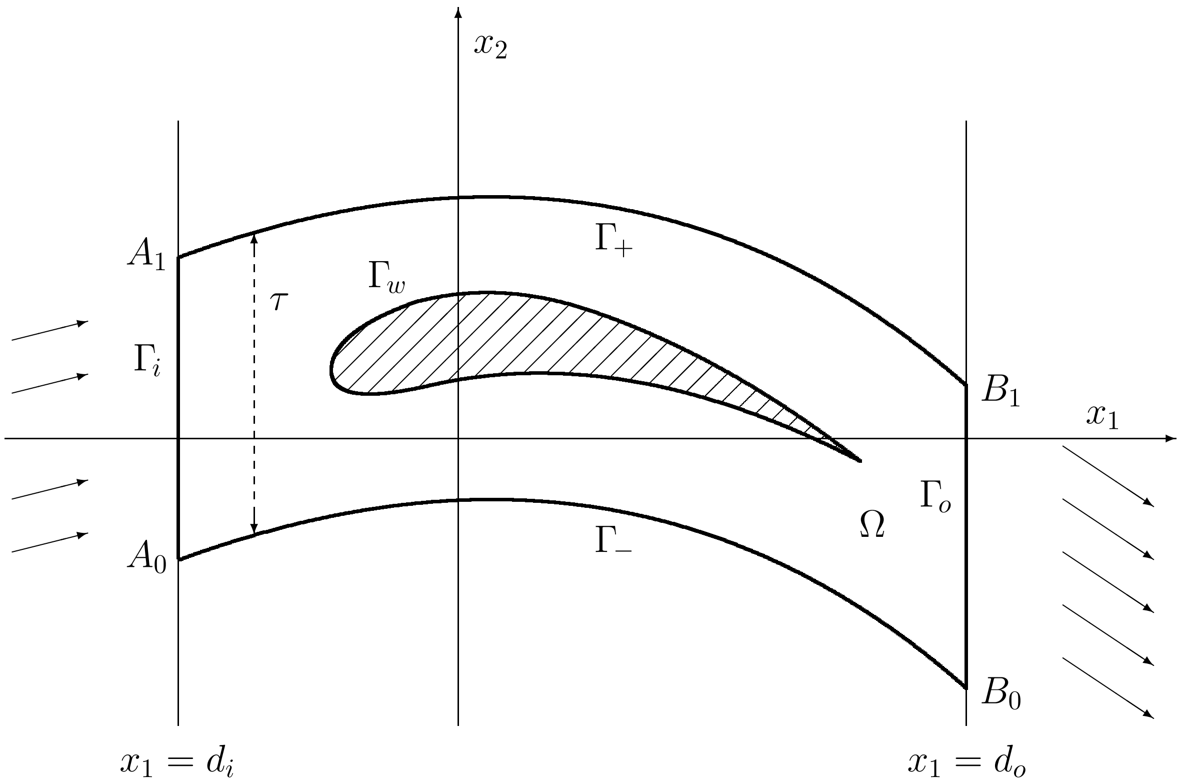

Fig. 2

Chapter I

Stationary problem with a nonlinear mixed boundary condition on the outflow

I.1 Classical formulation

I.1.1 Equations of motion. We study the flow of a viscous incompressible Newtonian fluid. For simplicity, we assume that the constant density of the fluid is equal to one. Furthermore, in Chapter I, we consider a stationary plane flow. It means that all quantities are independent of time and the space variable x_3 and the component of velocity in the direction of the x_3–axis equals zero.

We denote by $\boldsymbol{u}$ $(= (u_1, u_2))$ the **velocity** and by p the **pressure** in the fluid, $\boldsymbol{f}$ $(= (f_1, f_2))$ is the density of the volume force and constant $\nu > 0$ is the **kinematic viscosity**.

The condition of incompressibility is expressed by the equation

$$\operatorname{div} \boldsymbol{u} \,=\, 0. \tag{1}$$

Due to the constant density, this equation also expresses the conservation of mass in the fluid.

The conservation of momentum is expressed by the Navier–Stokes equations

$$\sum_{j=1}^{2} u_j \, \frac{\partial u_i}{\partial x_j} \,=\, f_i \,-\, \frac{\partial p}{\partial x_i} \,+\, \nu \, \Delta u_i \qquad (i = 1, 2). \tag{2}$$

System (2) can be written in the form of one vector equation

$$(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} \,=\, \boldsymbol{f} \,-\, \nabla p \,+\, \nu \, \Delta \boldsymbol{u}. \tag{3}$$

I.1.2 Boundary conditions on ∂D. We assume that the known velocity profile $\boldsymbol{g}$ is prescribed on G_i. This leads to the Dirichlet boundary condition

$$\boldsymbol{u} \,=\, \boldsymbol{g} \qquad \text{on } \; G_i\,. \tag{4}$$

Due to the viscosity, it is reasonable to prescribe the no–slip boundary condition on the union G_w of the surfaces of all profiles:

$$\boldsymbol{u} \,=\, \boldsymbol{0} \qquad \text{on } \; G_w\,. \tag{5}$$

As it was already mentioned in Introduction, there are more possibilities how to choose a boundary condition on the part G_o of the boundary of D (the supposed outflow). In this section, and in Sections I.2–I.4, we shall use the boundary condition

$$-\nu \, \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}} \,+\, p \cdot \boldsymbol{n} \,-\, \frac{1}{2} \, (\boldsymbol{u} \cdot \boldsymbol{n})^{-} \, \boldsymbol{u} \,=\, \boldsymbol{h} \qquad \text{on } \; G_o \tag{6}$$

where $\boldsymbol{n}$ is the outer normal vector and $\boldsymbol{h}$ is a given function. For $a \in \mathbb{R}$ we set $a^+ = (|a| + a)/2$ and $a^- = (|a| - a)/2$.

I.1.3 Classical formulation of the problem in the domain D. Given continuous functions $\boldsymbol{f} = (f_1, f_2)$ (in D), $\boldsymbol{g} = (g_1, g_2)$ (on G_i) and $\boldsymbol{h} = (h_1, h_2)$ (on G_o), periodic with period τ in variable the x_2, we look for functions $\boldsymbol{u} = (u_1, u_2)$ and p, periodic in the domain D with period τ in the variable x_2, which have these properties:

a) $\boldsymbol{u} \in C^2(\overline{D})$

b) $p \in C^1(\overline{D})$

c) The pair of functions $(\boldsymbol{u}, p)$ is a classical solution of equations (1), (3) in the domain D and the functions $\boldsymbol{u}$, p satisfy the boundary conditions (4) (on G_i), (5) (on G_w) and (6) (on G_o).

The solution $(\boldsymbol{u}, p)$ of this problem will be called the **classical solution in domain** D.

The domain D is unbounded. Due to technical reasons, it is easier to study the existence and further properties of a solution in a bounded domain. Moreover, it is impossible to solve numerically the problem in the whole unbounded domain. Therefore we restrict our considerations to the domain Ω, which is one period of the D.

We consider a classical solution in D, which is τ–periodic in variable x_2, restricted to the domain Ω, representing one space period of the domain D, to be a classical solution in Ω. From this, we naturally obtain the following classical formulation of the problem in Ω.

I.1.4 Classical formulation of the problem in the domain Ω. Given continuous functions $\boldsymbol{f} = (f_1, f_2)$ (in $\overline{\Omega}$), $\boldsymbol{g} = (g_1, g_2)$ (on Γ_i) and $\boldsymbol{h} = (h_1, h_2)$ (on Γ_o), satisfying the conditions of periodicity

$$\boldsymbol{f}(x_1, x_2) = \boldsymbol{f}(x_1, x_2 + \tau) \qquad \text{for } (x_1, x_2) \in \Gamma_- \,, \tag{7}$$

$$\boldsymbol{g}(A_0) = \boldsymbol{g}(A_1), \tag{8}$$

$$\boldsymbol{h}(B_0) = \boldsymbol{h}(B_1). \tag{9}$$

We look for functions $\boldsymbol{u} = (u_1, u_2)$ and p such that

a) $\boldsymbol{u} \in C^2(\overline{\Omega})$

b) $p \in C^1(\overline{\Omega})$

c) the pair $(\boldsymbol{u}, p)$ satisfies equations (1), (3) in the domain Ω,

d) the functions $\boldsymbol{u}$, p satisfy the boundary conditions (4), (5), (6), restricted to the boundary of Ω:

$$\boldsymbol{u} = \boldsymbol{g} \qquad \text{on } \Gamma_i \,, \tag{10}$$

$$\boldsymbol{u} = \boldsymbol{0} \qquad \text{on } \Gamma_w \,, \tag{11}$$

$$-\nu \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}} + p\boldsymbol{n} - \frac{1}{2}(\boldsymbol{u} \cdot \boldsymbol{n})^- \boldsymbol{u} = \boldsymbol{h} \qquad \text{on } (\Gamma_o)^\circ \tag{12}$$

and the periodicity conditions in the variable x_2:

$$\boldsymbol{u}(x_1, x_2+\tau) = \boldsymbol{u}(x_1, x_2) \qquad \text{for } (x_1, x_2) \in \Gamma_- \,, \tag{13}$$

$$\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}}(x_1, x_2+\tau) = -\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}}(x_1, x_2) \qquad \text{for } (x_1, x_2) \in (\Gamma_-)^\circ \,, \tag{14}$$

$$p(x_1, x_2+\tau) = p(x_1, x_2) \qquad \text{for } (x_1, x_2) \in \Gamma_- \,. \tag{15}$$

We shall call the couple $(\boldsymbol{u}, p)$ with the above properties the **classical solution in the domain** Ω.

It is obvious from the formulation of problems I.1.3 and I.1.4 that the classical solution in the domain D, restricted to $\overline{\Omega}$, is a classical solution in the domain Ω., on the other hand the next theorem shows that, if we extend a classical solution in Ω periodically (with period τ) in the x_2–direction, we obtain a classical solution in D.

I.1.5 Theorem 1. Let $\boldsymbol{u}$, p be a solution of problem I.1.4. If we extend $\boldsymbol{u}$, p from $\overline{\Omega}$ onto $\overline{D}$ as τ–periodic functions in the x_2–direction, we obtain a solution of problem I.1.3.

Proof. One can deduce from the formulation of problem I.1.4 that the extended functions $\boldsymbol{u}$ and p have these properties:

i) The function $\boldsymbol{u}$ is continuous in $\overline{D}$, its 1st order partial derivatives are continuous in $D \cup G_o$, it satisfies the boundary condition (4) (on G_i) and the boundary condition (5) (on G_w) and $\boldsymbol{u}$ satisfies the equation of continuity (1) in D.

ii) The function p is continuous in $D \cup G_o$ and $\boldsymbol{u}$, p satisfy the boundary condition (9) (on G_o).

iii) The function $\boldsymbol{u}$ (respectively p) is of the class C^2 (respectively C^1) in each of the domains Ω_i ($i \in \mathbb{Z}$).

iv) Functions $\boldsymbol{u}$, p satisfy the Navier–Stokes equations (3) in each of the domains Ω_i ($i \in \mathbb{Z}$).

v) The 2nd order partial derivatives of $\boldsymbol{u}$ are continuously extendable both from Ω_0 ($\equiv \Omega$) and from Ω_{-1} onto the curve $x_2 = \gamma(x_1)$; $d_i < x_1 < d_o$ ($\equiv (\Gamma_-)^\circ$). Let us denote by $u_{i,jk}^{(0)}$ the continuous extension of the partial derivative $\partial^2 u_i / \partial x_j \, \partial x_k$ from Ω_0 and by $u_{i,jk}^{(-1)}$ the continuous extension of $\partial^2 u_i / \partial x_j \, \partial x_k$ from Ω_{-1}.

vi) The 1st order partial derivatives of p are continuously extendable both from Ω_0 and from Ω_{-1} onto the curve $(\Gamma_-)^\circ$. Let us denote by $p_{,i}^{(0)}$ the continuous extension of the partial derivative $\partial p / \partial x_i$ from Ω_0 and by $p_{,i}^{(-1)}$ the continuous extension of $\partial p / \partial x_i$ from Ω_{-1}.

In order to show that the extended functions $\boldsymbol{u}$, p represent a classical solution in D, it is sufficient to verify that

1) $\boldsymbol{u}$ (respectively p) has continuous 2nd order partial derivatives (respectively the 1st order partial derivatives) in D and

2) $\boldsymbol{u}$ and p satisfy the Navier–Stokes equations (3) everywhere in D.

To prove the statement in item 1), it remains to show the continuity of the 2nd order partial derivatives of the function $\boldsymbol{u}$ and the continuity of the 1st order partial derivatives of the function p at all points of the curve $(\Gamma_-)^\circ$. It means that we need to show that the identities

$$u^{(0)}_{i,jk}(x_1,x_2) = u^{(-1)}_{i,jk}(x_1,x_2) \qquad \text{for } (x_1,x_2)\in(\Gamma_-)^\circ \text{ and } i,j,k=1,2, \tag{16}$$

$$p^{(0)}_{,i}(x_1,x_2) = p^{(-1)}_{,i}(x_1,x_2) \qquad \text{for } (x_1,x_2)\in(\Gamma_-)^\circ \text{ and } i=1,2 \tag{17}$$

hold for all $(x_1,x_2)\in(\Gamma_-)^\circ$. Let us therefore assume that $\boldsymbol{x}=(x_1,x_2)$ is an arbitrarily chosen point on $(\Gamma_-)^\circ$. We can assume without loss of generality that the tangent line to curve $(\Gamma_-)^\circ$ at point $\boldsymbol{x}$ is parallel with the x_1–axis. (Otherwise we can turn the coordinate axes and work in a new Cartesian system x_1', x_2' in which the tangent to curve $(\Gamma_-)^\circ$ at the point $\boldsymbol{x}$ is parallel with the x_1'–axis.)

Partial derivatives

$$\frac{\partial u_1}{\partial x_1}, \quad \frac{\partial u_1}{\partial x_2}, \quad \frac{\partial u_2}{\partial x_1}, \quad \frac{\partial u_2}{\partial x_2} \tag{18}$$

are continuous at the point $\boldsymbol{x}$. The assumption that the tangent to curve $(\Gamma_-)^\circ$ at the point $\boldsymbol{x}$ is parallel with the x_1–axis and the fact that the 2nd order partial derivatives of the function $\boldsymbol{u}$ have a continuous extension both from Ω_0 and from Ω_1 onto $(\Gamma_-)^\circ$ imply that all the partial derivatives listed in (18) have a continuous derivative at $\boldsymbol{x}$ in the tangential direction to $(\Gamma_-)^\circ$, i.e. the derivative with respect to x_1. Hence

$$u^{(0)}_{i,jk}(x_1,x_2) = u^{(-1)}_{i,jk}(x_1,x_2) = \frac{\partial^2 u_i}{\partial x_j\,\partial x_k}(x_1,x_2) \tag{19}$$

for $i,j,k=1,2$,$\{j,k\}\neq\{2,2\}$. By analogy, we derive that

$$p^{(0)}_{,1}(x_1,x_2) = p^{(-1)}_{,1}(x_1,x_2) = \frac{\partial p}{\partial x_1}(x_1,x_2). \tag{20}$$

Due to the continuity equation, the identities

$$\frac{\partial u_2}{\partial x_2} = -\frac{\partial u_1}{\partial x_1}, \qquad \frac{\partial^2 u_2}{\partial x_2^2} = -\frac{\partial^2 u_1}{\partial x_1\,\partial x_2}$$

hold both in Ω_0 and in Ω_{-1}. Using the second identity in (19) (with $i=2$, $j=1$, $k=2$), we obtain

$$u^{(0)}_{2,22}(x_1,x_2) = -u^{(0)}_{1,12}(x_1,x_2) = -u^{(-1)}_{1,12}(x_1,x_2) = u^{(-1)}_{2,22}(x_1,x_2).$$

Hence,

$$\frac{\partial^2 u_2}{\partial x_2^2}(x_1,x_2) = -\frac{\partial^2 u_1}{\partial x_1\,\partial x_2}(x_1,x_2). \tag{21}$$

All the equalities (19)–(21), the possibilities of a continuous extension of the 2nd order partial derivatives of $\boldsymbol{u}$ and 1st order partial derivatives of p both from Ω_0 and Ω_{-1} to the point $\boldsymbol{x}$ and the validity of the Navier–Stokes equations (2) in $\Omega_0 \cup \Omega_{-1}$ imply that the equations

$$\begin{aligned}\sum_{j=1}^{2} u_j \frac{\partial u_1}{\partial x_j} &= f_1 - \frac{\partial p}{\partial x_1} + \nu \left(\frac{\partial^2 u_1}{\partial x_1^2} + u_{1,22}^{(0)} \right) \\ &= f_1 - \frac{\partial p}{\partial x_1} + \nu \left(\frac{\partial^2 u_1}{\partial x_1^2} + u_{1,22}^{(-1)} \right), \end{aligned} \tag{22}$$

$$\begin{aligned}\sum_{j=1}^{2} u_j \frac{\partial u_2}{\partial x_j} &= f_2 - p_{,2}^{(0)} + \nu \left(\frac{\partial^2 u_2}{\partial x_1^2} + \frac{\partial^2 u_2}{\partial x_2^2} \right) \\ &= f_1 - p_{,2}^{(-1)} + \nu \left(\frac{\partial^2 u_2}{\partial x_1^2} + \frac{\partial^2 u_2}{\partial x_2^2} \right) \end{aligned} \tag{23}$$

hold at the point $\boldsymbol{x}$. From (22) we obtain

$$u_{1,22}^{(0)}(x_1, x_2) \;=\; u_{1,22}^{(-1)}(x_1, x_2) \;=\; \frac{\partial^2 u_1}{\partial x_2^2}(x_1, x_2) \tag{24}$$

and from (23) we obtain

$$p_{,2}^{(0)}(x_1, x_2) \;=\; p_{,2}^{(-1)}(x_1, x_2) \;=\; \frac{\partial p}{\partial x_2}(x_1, x_2). \tag{25}$$

Equalities (16) and (17) at the chosen point $\boldsymbol{x}$ can now be derived from (19)–(25).

The validity of the Navier–Stokes equations (2) at the point $\boldsymbol{x}$ follows from (22) and (23). The point $\boldsymbol{x}$ was chosen arbitrarily on $(\Gamma_-)^\circ$. Hence, equations (2) hold at all points of $(\Gamma_-)^\circ$. The periodicity of the functions $\boldsymbol{u}$ and p implies that equations (2) are satisfied on all curves $x_2 = \gamma(x_1) + k\tau$; $d_i < x_1 < d_o$, $k \in \mathbb{Z}$. Thus, using also statement iii), we can conclude that the Navier–Stokes equations (2) hold in the whole domain D. This completes the proof. □

I.2 Weak formulation

I.2.1 Definition of function spaces. By analogy with the definitions of the classical solution of the problem in the domain D (subsection I.1.3) and the classical solution of the problem in the domain Ω (subsection I.1.4), we will define a weak solution in the domain Ω. We shall need function spaces whose definitions are presented in this subsection.

- $H^1(\Omega)$ is the usual Sobolev space of scalar–valued functions with the scalar product

$$(u, v)_{H^1(\Omega)} \;:=\; \int_\Omega \big(uv + \nabla u \cdot \nabla v \big) \, \mathrm{d}\boldsymbol{x}.$$

- $H^1(\Omega)^2 := \big[H^1(\Omega) \big]^2$; the scalar product in this space is defined by

$$(\boldsymbol{u}, \boldsymbol{v})_{H^1(\Omega)^2} \;:=\; \sum_{i=1}^{2} (u_i, v_i)_{H^1(\Omega)}$$

where $\boldsymbol{u} = (u_1, u_2)$ and $\boldsymbol{v} = (v_1, u_2)$.

-
$$X \;=\; \big\{\boldsymbol{v} \in H^1(\Omega)^2;\; \boldsymbol{v} = \boldsymbol{0} \text{ a.e. in } \Gamma_i \cup \Gamma_w,\; \boldsymbol{v}(x_1, x_2 + \tau) = \boldsymbol{v}(x_1, x_2) \text{ for a.a. } (x_1, x_2) \in \Gamma_-\big\}. \tag{26}$$

(The conditions on the curves Γ_i, Γ_w and Γ_- are interpreted in the sense of traces.)

-
$$V \;=\; \Big\{\boldsymbol{v} \in X;\; \operatorname{div} \boldsymbol{v} = 0 \;\text{ a.e. in } \Omega\Big\}.$$

I.2.2 Theorem 2 (on equivalent norms in X).

The formula

$$|||\boldsymbol{v}||| \;=\; \left(\int_\Omega \sum_{i,j=1}^{2} \left(\frac{\partial v_i}{\partial x_j}\right)^2 \mathrm{d}\boldsymbol{x}\right)^{1/2} \tag{27}$$

defines a norm in X, equivalent with the norm $\|\,.\,\|_{H^1(\Omega)^2}$: there exist constants $c_1 > 0$ and $c_2 > 0$ such that the inequalities

$$c_1\, \|\boldsymbol{v}\|_{H^1(\Omega)^2} \;\le\; |||\boldsymbol{v}||| \;\le\; c_2\, \|\boldsymbol{v}\|_{H^1(\Omega)^2} \tag{28}$$

hold for every function $\boldsymbol{v} \in X$.

Proof: The second inequality in (28), i.e. $|||\boldsymbol{v}||| \le c_2\, \|\boldsymbol{v}\|_{H^1(\Omega)^2}$, is obvious. We shall further prove the first inequality in (28). The procedure is standard: From Remark 5.11.4 in the book [40], p. 299, it follows that if Ω^* is a non–empty open subset of Ω, then there exists $c_3 > 0$ such that

$$\|\boldsymbol{v}\|_{H^1(\Omega)^2} \;\le\; c_3\, |||\boldsymbol{v}||| \;+\; c_3 \left(\int_{\Omega^*} |\boldsymbol{v}(\boldsymbol{x})|^2 \,\mathrm{d}\boldsymbol{x}\right)^{1/2} \tag{29}$$

for all functions $\boldsymbol{v} \in X$. We define the domain Ω^* as an open rectangle $(a, b) \times (c, d)$ whose side $\{a\} \times (c, d)$ is a subset of Γ_i and the whole rectangle is in Ω. Suppose at first that $\boldsymbol{v} \in \mathcal{X}$. Then

$$\boldsymbol{v}(\boldsymbol{x}) \;=\; \boldsymbol{v}(x_1, x_2) \;=\; \boldsymbol{v}(a, x_2) \;+\; \int_a^{x_1} \frac{\partial \boldsymbol{v}}{\partial x_1}(t, x_2)\,\mathrm{d}t$$

for $\boldsymbol{x} = (x_1, x_2) \in \Omega^*$. If we take into account that the point (a, x_2) lies on Γ_i, we have $\boldsymbol{v}(a, x_2) = \boldsymbol{0}$ and get

$$\begin{aligned}
|\boldsymbol{v}(\boldsymbol{x})|^2 \;&=\; \left(\int_a^{x_1} \frac{\partial \boldsymbol{v}}{\partial x_1}(t, x_2)\,\mathrm{d}t\right)^2 \;\le\; \int_a^{x_1} \left|\frac{\partial \boldsymbol{v}}{\partial x_1}(t, x_2)\right|^2 \mathrm{d}t\; (x_1 - a) \\
&\le\; (b - a) \int_a^{x_1} \left|\frac{\partial \boldsymbol{v}}{\partial x_1}(t, x_2)\right|^2 \mathrm{d}t, \\
\int_{\Omega^*} |\boldsymbol{v}(\boldsymbol{x})|^2 \,\mathrm{d}\boldsymbol{x} \;&=\; \int_a^b \int_c^d |\boldsymbol{v}(x_1, x_2)|^2 \,\mathrm{d}x_2\, \mathrm{d}x_1
\end{aligned}$$

$$
\begin{aligned}
&\le \int_a^b \int_c^d \left((b-a) \int_a^{x_1} \left| \frac{\partial \boldsymbol{v}}{\partial x_1}(t, x_2) \right|^2 \mathrm{d}t \right) \mathrm{d}x_2 \, \mathrm{d}x_1 \\
&\le (b-a) \int_a^b \int_c^d \int_a^b \left| \frac{\partial \boldsymbol{v}}{\partial x_1}(t, x_2) \right|^2 \mathrm{d}t \, \mathrm{d}x_2 \, \mathrm{d}x_1 \\
&\le (b-a)^2 \int_c^d \int_a^b \left| \frac{\partial \boldsymbol{v}}{\partial x_1}(x_1, x_2) \right|^2 \mathrm{d}x_1 \, \mathrm{d}x_2 \\
&= (b-a)^2 \int_{\Omega^*} \left| \frac{\partial \boldsymbol{v}}{\partial x_1}(\boldsymbol{x}) \right|^2 \mathrm{d}\boldsymbol{x} \le (b-a)^2 \int_{\Omega} \left| \frac{\partial \boldsymbol{v}}{\partial x_1}(\boldsymbol{x}) \right|^2 \mathrm{d}\boldsymbol{x} \\
&\le (b-a)^2 \, |||\boldsymbol{v}|||^2.
\end{aligned}
$$

These estimates and (29) imply that the first inequality in (28) is satisfied for $\boldsymbol{v} \in \mathcal{X}$. Due to the density of $\mathcal{X}$ in X, the inequality holds for all functions $\boldsymbol{v} \in X$. □

I.2.3 From a classical to a weak solution in the domain D. We assume that $\boldsymbol{u} = (u_1, u_2)$, p is a solution of the classical problem I.1.3 in this subsection. We shall formally derive the weak formulation of problem I.1.3.

Let us multiply equation (3) by an arbitrary function $\boldsymbol{w} = (w_1, w_2) \in H^1(D)^2$ that has zero traces on G_i and G_w, its divergence equals zero a.e. in D and it has a compact support in $D \cup G_o$. (We denote the space of such functions as $\mathcal{W}$). Let us integrate the equation (3) over D and use Green's theorem. We get

$$
\begin{aligned}
\int_D \boldsymbol{f} \cdot \boldsymbol{w} \, \mathrm{d}\boldsymbol{x} &= \int_D \left(-\nu \, \Delta \boldsymbol{u} + \sum_{j=1}^2 u_j \frac{\partial \boldsymbol{u}}{\partial x_j} + \nabla p \right) \cdot \boldsymbol{w} \, \mathrm{d}\boldsymbol{x} \\
&= -\nu \int_{\partial D} \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}} \cdot \boldsymbol{w} \, \mathrm{d}S + \nu \int_D \sum_{i,j=1}^2 \frac{\partial u_i}{\partial x_j} \frac{\partial w_i}{\partial x_j} \, \mathrm{d}\boldsymbol{x} + \int_D \sum_{i,j=1}^2 u_j \frac{\partial u_i}{\partial x_j} w_i \, \mathrm{d}\boldsymbol{x} \\
&\quad - \int_D p \, \mathrm{div} \, \boldsymbol{w} \, \mathrm{d}\boldsymbol{x} + \int_{\partial D} p \, \boldsymbol{w} \cdot \boldsymbol{n} \, \mathrm{d}S. \qquad (30)
\end{aligned}
$$

Obviously, the integrals on ∂D equal to the sum of the integrals along G_i, G_w and G_o. Since the function $\boldsymbol{w}$ has the zero trace on $G_i \cup G_w$, the integrals on ∂D in (30) are equal to the integrals on G_o. These integrals can further be transformed by means of the boundary condition (6):

$$
-\nu \int_{G_o} \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}} \cdot \boldsymbol{w} \, \mathrm{d}S + \int_{G_o} p \, \boldsymbol{w} \cdot \boldsymbol{n} \, \mathrm{d}S = \int_{G_o} \frac{1}{2} \, (\boldsymbol{u} \cdot \boldsymbol{n})^- \, \boldsymbol{u} \cdot \boldsymbol{w} \, \mathrm{d}S + \int_{G_o} \boldsymbol{h} \cdot \boldsymbol{w} \, \mathrm{d}S.
$$

Substituting this into (30) and using the equation $\mathrm{div} \, \boldsymbol{w} = 0$, we obtain

$$
\begin{aligned}
\int_D \boldsymbol{f} \cdot \boldsymbol{w} \, \mathrm{d}\boldsymbol{x} &= \nu \int_D \sum_{i,j=1}^2 \frac{\partial u_i}{\partial x_j} \frac{\partial w_i}{\partial x_j} \, \mathrm{d}\boldsymbol{x} + \int_D \sum_{i,j=1}^2 u_j \frac{\partial u_i}{\partial x_j} w_i \, \mathrm{d}\boldsymbol{x} \\
&\quad + \int_{G_o} \frac{1}{2} \, (\boldsymbol{u} \cdot \boldsymbol{n})^- \, \boldsymbol{u} \cdot \boldsymbol{w} \, \mathrm{d}S + \int_{G_o} \boldsymbol{h} \cdot \boldsymbol{w} \, \mathrm{d}S. \qquad (31)
\end{aligned}
$$

I.2.4 Weak formulation of the problem in the domain D. Let the function $\boldsymbol{g}$ in the boundary condition (4) be periodic in the variable x_2 with period τ and let there exist $s \in (\frac{1}{2}, 1]$ such that the restriction of $\boldsymbol{g}$ onto each bounded line segment G_i' lying on G_i belongs to $H^s(G_i')^2$. (The definition of the space $H^s(G_i')^2$ is recalled in subsection I.2.1.)

Suppose that $\boldsymbol{f}$ (respectively $\boldsymbol{h}$) is a given function in D (respectively in G_o), periodic in the variable x_2 with period τ and such that its restriction to each bounded domain D' in D (respectively to each bounded line segment G_o' in G_o) belongs to $L^2(D')^2$ (respectively to $L^2(G_o')^2$).

Let us denote by $\mathcal{W}$ the space of all admissible test functions $\boldsymbol{w}$ with the properties mentioned in the preceding subsection.

We look for a vector function $\boldsymbol{u}$ defined a.e. in D, periodic in the variable x_2 with the period τ such that

a) the restriction of $\boldsymbol{u}$ to each bounded subdomain D' in D belongs to $H^1(D')^2$,

b) $\operatorname{div} \boldsymbol{w} = 0$ a.e. in D,

c) the integral identity (31) is fulfilled for all test functions $\boldsymbol{w} \in \mathcal{W}$,

d) the boundary conditions (4) and (5) are fulfilled in the sense of traces.

The solution of this problem will be called a **weak solution in the domain** D.

I.2.5 From a classical to a weak solution in the domain Ω. In this subsection we suppose that $\boldsymbol{u} = (u_1, u_2)$ and p form a classical solution of a classical problem I.1.4. We shall formally derive the weak formulation of problem I.1.4 in a similar way as in I.2.3.

Let us multiply equation (3) by an arbitrary test function $\boldsymbol{v} = (v_1, v_2) \in V$, integrate over Ω and use Green's theorem. We get

$$\begin{aligned} \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \, \mathrm{d}\boldsymbol{x} \;&=\; \int_\Omega \Big(-\nu\, \Delta \boldsymbol{u} + \sum_{j=1}^{2} u_j \, \frac{\partial \boldsymbol{u}}{\partial x_j} + \nabla p \Big) \cdot \boldsymbol{v} \, \mathrm{d}\boldsymbol{x} \\ &=\; \nu \int_\Omega \sum_{i,j=1}^{2} \frac{\partial u_i}{\partial x_j} \, \frac{\partial v_i}{\partial x_j} \, \mathrm{d}\boldsymbol{x} \;-\; \nu \int_{\partial\Omega} \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}} \cdot \boldsymbol{v} \, \mathrm{d}S \;+\; \int_\Omega \sum_{i,j=1}^{2} u_j \, \frac{\partial u_i}{\partial x_j} \, v_i \, \mathrm{d}\boldsymbol{x} \\ &\quad - \int_\Omega p \operatorname{div} \boldsymbol{v} \, \mathrm{d}\boldsymbol{x} \;+\; \int_{\partial\Omega} p \, \boldsymbol{v} \cdot \boldsymbol{n} \, \mathrm{d}S. \end{aligned} \tag{32}$$

Each of the integrals along $\partial\Omega$ is equal to the sum of integrals along the parts Γ_i, Γ_o, Γ_-, Γ_+ and Γ_w:

$$\int_{\partial\Omega} \ldots = \int_{\Gamma_i} \ldots + \int_{\Gamma_o} \ldots + \int_{\Gamma_-} \ldots + \int_{\Gamma_+} \ldots + \int_{\Gamma_w} \ldots \; .$$

Since the function $\boldsymbol{v} \in V$ has a zero trace on $\Gamma_i \cup \Gamma_w$, the integrals on Γ_i and Γ_w vanish. Using the identities

$$\boldsymbol{n}(x_1, x_2) \;=\; -\boldsymbol{n}(x_1, x_2 + \tau) \qquad \text{for } (x_1, x_2) \in \Gamma_-, \tag{33}$$

the conditions of periodicity (14), (15) and the properties of the test function $\boldsymbol{v}$, we obtain

$$\int_{\Gamma_- \cup \Gamma_+} \left(-\nu \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}} \cdot \boldsymbol{v} + p\, \boldsymbol{v} \cdot \boldsymbol{n} \right) \mathrm{d}S = 0. \tag{34}$$

The integral on Γ_o can be transformed by means of (12):

$$\int_{\Gamma_o} \left(-\nu \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}} \cdot \boldsymbol{v} + p\, \boldsymbol{v} \cdot \boldsymbol{n} \right) \mathrm{d}S = \int_{\Gamma_o} \frac{1}{2} (\boldsymbol{u} \cdot \boldsymbol{n})^- \, \boldsymbol{u} \cdot \boldsymbol{v} \, \mathrm{d}S + \int_{\Gamma_o} \boldsymbol{h} \cdot \boldsymbol{v} \, \mathrm{d}S. \tag{35}$$

The integral of $p \operatorname{div} \boldsymbol{v}$ on Ω vanishes because the test function $\boldsymbol{v}$ is solenoidal. Thus, substituting (33)–(35) into (32), we obtain

$$\begin{aligned} \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \, \mathrm{d}\boldsymbol{x} &= \nu \int_\Omega \sum_{i,j=1}^{2} \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j} \, \mathrm{d}\boldsymbol{x} + \int_\Omega \sum_{i,j=1}^{2} u_j \frac{\partial u_i}{\partial x_j} v_i \, \mathrm{d}\boldsymbol{x} \\ &\quad + \int_{\Gamma_o} \frac{1}{2} (\boldsymbol{u} \cdot \boldsymbol{n})^- \, \boldsymbol{u} \cdot \boldsymbol{v} \, \mathrm{d}S + \int_{\Gamma_o} \boldsymbol{h} \cdot \boldsymbol{v} \, \mathrm{d}S, \qquad \boldsymbol{v} \in V. \end{aligned} \tag{36}$$

We shall often use this integral identity. In order to simplify its form, we introduce the following notation: for $\boldsymbol{u} = (u_1, u_2)$, $\boldsymbol{v} = (v_1, v_2)$, $\boldsymbol{w} = (w_1, w_2) \in H^1(\Omega)^2$, we put

$$\begin{aligned} a_1(\boldsymbol{u}, \boldsymbol{v}) &:= \nu \int_\Omega \sum_{i,j=1}^{2} \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j} \, \mathrm{d}\boldsymbol{x}, \\ a_2(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) &:= \int_\Omega \sum_{i,j=1}^{2} u_j \frac{\partial v_i}{\partial x_j} w_i \, \mathrm{d}\boldsymbol{x}, \\ a_3(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) &:= \int_{\Gamma_o} \frac{1}{2} (\boldsymbol{u} \cdot \boldsymbol{n})^- \, \boldsymbol{v} \cdot \boldsymbol{w} \, \mathrm{d}S, \\ a(\boldsymbol{u}, \boldsymbol{v}) &:= a_1(\boldsymbol{u}, \boldsymbol{v}) + a_2(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) + a_3(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}), \\ (\boldsymbol{f}, \boldsymbol{v}) &:= \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \, \mathrm{d}\boldsymbol{x}, \\ b(\boldsymbol{h}, \boldsymbol{v}) &:= -\int_{\Gamma_o} \boldsymbol{h} \cdot \boldsymbol{v} \, \mathrm{d}S. \end{aligned}$$

Obviously, all these forms are well defined for $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w} \in H^1(\Omega)^2$, $\boldsymbol{f} \in L^2(\Omega)^2$ and $\boldsymbol{h} \in L^2(\Gamma_o)^2$. Now the identity (36) can shortly be written as

$$a(\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}) + b(\boldsymbol{h}, \boldsymbol{v}), \qquad \boldsymbol{v} \in V. \tag{37}$$

I.2.6 Weak formulation of the problem in the domain Ω. Suppose that the function $\boldsymbol{g}$, appearing in the boundary condition (10), belongs to $H^s(\Gamma_i)^2$ for $s \in (\frac{1}{2}, 1]$ and $\boldsymbol{g}(A_1) = \boldsymbol{g}(A_0)$. (Let us recall that A_0 and A_1 are the end–points of Γ_i.) Let $\boldsymbol{f} \in L^2(\Omega)^2$ and $\boldsymbol{h} \in L^2(\Gamma_o)^2$ be given functions. We seek a vector function $\boldsymbol{u} \in H^1(\Omega)^2$ which satisfies the equation of continuity (1) a.e. in Ω, the boundary conditions (10) (respectively (11)) in the sense of traces on Γ_i (respectively on Γ_w), the condition of periodicity (13) a.e. on Γ_- and such that identity (37) holds for all test functions $\boldsymbol{v} \in V$.

The solution $\boldsymbol{u}$ of this problem is called a **weak solution in the domain Ω.**

I.2.7 Space $H^s(\Gamma_i)^2$ and some of its properties. We shall further work with functions from the space $H^s(\Gamma_i)^2$. Let $\boldsymbol{g} \in H^s(\Gamma_i)^2$, where $s \in (\frac{1}{2}, 1]$ and let $H^s(\Gamma_i)$ be the Sobolev–Slobodetskii space with the norm

$$\|\boldsymbol{g}\|_{H^s(\Gamma_i)^2} \;=\; \|\boldsymbol{g}\|_{L^2(\Gamma_i)^2} \;+\; \left(\int_{\Gamma_i} \int_{\Gamma_i} \frac{|\boldsymbol{g}(d_i, y_2) - \boldsymbol{g}(d_i, x_2)|^2}{|y_2 - x_2|^{1+2s}} \, \mathrm{d}y_2 \, \mathrm{d}x_2 \right)^{1/2}. \tag{38}$$

See, for example, [43], Section 9.1.

Due to Theorem 9.8 from [43], the function $\boldsymbol{g} \in H^s(\Gamma_i)^2$ is continuous and, hence the condition $\boldsymbol{g}(A_0) = \boldsymbol{g}(A_1)$ makes sense. Moreover, the imbedding of $H^s(\Gamma_i)^2$ into $C(\Gamma_i)^2$ is continuous. Thus, there exists a constant $c_4 > 0$ independent of $\boldsymbol{g}$ such that

$$\|\boldsymbol{g}\|_{C(\Gamma_i)^2} \;\leq\; c_4 \, \|\boldsymbol{g}\|_{H^s(\Gamma_i)^2} \, . \tag{39}$$

Let us further set $\gamma_k = \gamma(d_i) + k\tau$ (for $k \in \mathbb{Z}$). Hence, we have $A_0 = (d_i, \gamma(d_i)) = (d_i, \gamma_0)$ and $A_1 = (d_i, \gamma(d_i) + \tau) = (d_i, \gamma_1)$.

According to Theorem 11.3 in [43], there exists a constant $c_5 > 0$ such that for each function $\boldsymbol{g} \in H^s(\Gamma_i)^2$ satisfying the condition $\boldsymbol{g}(A_0) = \boldsymbol{g}(A_1)$ (i.e. $\boldsymbol{g}(d_i, \gamma_0) = \boldsymbol{g}(d_i, \gamma_1)$), we have

$$\int_{\gamma_0}^{\gamma_1} \frac{|\boldsymbol{g}(d_i, x_2) - \boldsymbol{g}(d_i, \gamma_0)|^2}{(x_2 - \gamma_0)^{2s}} \, \mathrm{d}x_2 \;\;\leq\;\; c_5 \, \|\boldsymbol{g}\|^2_{H^s(\Gamma_i)^2} \, , \tag{40}$$

$$\int_{\gamma_0}^{\gamma_1} \frac{|\boldsymbol{g}(d_i, \gamma_1) - \boldsymbol{g}(d_i, x_2)|^2}{(\gamma_1 - x_2)^{2s}} \, \mathrm{d}x_2 \;\;\leq\;\; c_5 \, \|\boldsymbol{g}\|^2_{H^s(\Gamma_i)^2} \, . \tag{41}$$

I.2.8 Periodic extension of the the function $\boldsymbol{g}$. Suppose that the function $\boldsymbol{g}$ has the properties mentioned in subsection I.2.6. We shall denote the τ–periodic extension of $\boldsymbol{g}$ from the line segment Γ_i onto the whole straight line G_i by the same symbol $\boldsymbol{g}$.

Let us verify that the extended function $\boldsymbol{g}$ fulfills the assumptions from subsection I.2.4 (the weak solution of the problem in the domain D). Thus, we need to prove that $\boldsymbol{g}$ belongs to the space $H^s(G'_i)^2$ on each bounded line segment $G'_i \subset G_i$. Obviously, it is sufficient to show that $\boldsymbol{g}$ belongs to $H^s(A_{-1}A_1)$ where $A_{-1}A_1$ denotes the line segment with the end–points A_{-1} and A_1. Then, by the mathematical induction, $\boldsymbol{g}$ belongs to $H^s(A_{-k}A_l)$ where $A_{-k}A_l$ is the line segment with the end–points A_{-k} and A_l and k, l are arbitrary integers.

The norm of $\boldsymbol{g}$ in $H^s(A_{-1}A_1)^2$ can be expressed similarly as the norm of $\boldsymbol{g}$ in $H^s(\Gamma_i)^2$ in (38). The following identities which concern only the L^2–norms are simple:

$$\|\boldsymbol{g}\|^2_{L^2(A_{-1}A_1)^2} \;=\; 2\, \|\boldsymbol{g}\|^2_{L^2(A_0A_1)^2} \;\equiv\; 2\, \|\boldsymbol{g}\|^2_{L^2(\Gamma_i)^2} \, . \tag{42}$$

The part of the norm of $\boldsymbol{g}$ in $H^s(A_{-1}A_1)^2$, corresponding to the second term on the right hand side of (38), can be treated as follows:

$$\int_{\gamma_{-1}}^{\gamma_1} \int_{\gamma_{-1}}^{\gamma_1} \frac{[\boldsymbol{g}(d_i, y_2) - \boldsymbol{g}(d_i, x_2)]^2}{|y_2 - x_2|^{1+2s}} \, \mathrm{d}y_2 \, \mathrm{d}x_2$$
$$= \int_{\gamma_{-1}}^{\gamma_0} \int_{\gamma_{-1}}^{\gamma_0} \ldots \,+\, 2 \int_{\gamma_{-1}}^{\gamma_0} \int_{\gamma_0}^{\gamma_1} \ldots \,+\, \int_{\gamma_0}^{\gamma_1} \int_{\gamma_0}^{\gamma_1} \ldots \, . \tag{43}$$

The integral over $(\gamma_0,\gamma_1)\times(\gamma_0,\gamma_1)$ is equal to the integral $\int_{\Gamma_i}\int_{\Gamma_i}$. Comparing it with the right-hand side of (38), we observe that it is less than or equal to $\|\boldsymbol{g}\|^2_{H^s(\Gamma_i)^2}$. The integral on $(\gamma_{-1},\gamma_0)\times(\gamma_{-1},\gamma_0)$ is due to the periodicity of function $\boldsymbol{g}$ equal to the integral on $(\gamma_0,\gamma_1)\times(\gamma_0,\gamma_1)$, hence it is less than or equal to $\|\boldsymbol{g}\|^2_{H^s(\Gamma_i)^2}$, as well. Thus, we only need to estimate the integral over $(\gamma_{-1},\gamma_0)\times(\gamma_0,\gamma_1)$. We successively obtain

$$\begin{aligned}
&\int_{\gamma_{-1}}^{\gamma_0}\int_{\gamma_0}^{\gamma_1}\frac{|\boldsymbol{g}(d_i,y_2)-\boldsymbol{g}(d_i,x_2)|^2}{|y_2-x_2|^{1+2s}}\,\mathrm{d}y_2\,\mathrm{d}x_2\\
\leq\;& 2\int_{\gamma_{-1}}^{\gamma_0}\int_{\gamma_0}^{\gamma_1}\frac{|\boldsymbol{g}(d_i,y_2)-\boldsymbol{g}(d_i,\gamma_0)|^2}{(y_2-x_2)^{1+2s}}\,\mathrm{d}y_2\mathrm{d}x_2\\
+\;& 2\int_{\gamma_{-1}}^{\gamma_0}\int_{\gamma_0}^{\gamma_1}\frac{|\boldsymbol{g}(d_i,\gamma_0)-\boldsymbol{g}(d_i,x_2)|^2}{(y_2-x_2)^{1+2s}}\,\mathrm{d}y_2\mathrm{d}x_2\\
=\;& \frac{2}{2s}\int_{\gamma_0}^{\gamma_1}\left(\frac{|\boldsymbol{g}(d_i,y_2)-\boldsymbol{g}(d_i,\gamma_0)|^2}{(y_2-\gamma_0)^{2s}}-\frac{|\boldsymbol{g}(d_i,y_2)-\boldsymbol{g}(d_i,\gamma_0)|^2}{(y_2-\gamma_{-1})^{2s}}\right)\mathrm{d}y_2\\
+\;& \frac{2}{2s}\int_{\gamma_{-1}}^{\gamma_0}\left(\frac{|\boldsymbol{g}(d_i,\gamma_0)-\boldsymbol{g}(d_i,x_2)|^2}{(\gamma_0-x_2)^{2s}}-\frac{|\boldsymbol{g}(d_i,\gamma_0)-\boldsymbol{g}(d_i,x_2)|^2}{(\gamma_1-x_2)^{2s}}\right)\mathrm{d}x_2\\
\leq\;& \frac{2}{2s}\int_{\gamma_0}^{\gamma_1}\frac{|\boldsymbol{g}(d_i,y_2)-\boldsymbol{g}(d_i,\gamma_0)|^2}{(y_2-\gamma_0)^{2s}}\,\mathrm{d}y_2+\frac{2}{2s}\int_{\gamma_{-1}}^{\gamma_0}\frac{|\boldsymbol{g}(d_i,\gamma_0)-\boldsymbol{g}(d_i,x_2)|^2}{(\gamma_0-x_2)^{2s}}\,\mathrm{d}x_2\\
=\;& \frac{2}{2s}\int_{\gamma_0}^{\gamma_1}\frac{|\boldsymbol{g}(d_i,y_2)-\boldsymbol{g}(d_i,\gamma_0)|^2}{(y_2-\gamma_0)^{2s}}\,\mathrm{d}y_2+\frac{2}{2s}\int_{\gamma_0}^{\gamma_1}\frac{|\boldsymbol{g}(d_i,\gamma_1)-\boldsymbol{g}(d_i,x_2')|^2}{(\gamma_1-x_2')^{2s}}\,\mathrm{d}x_2'\\
\leq\;& \frac{2c_5}{s}\,\|\boldsymbol{g}\|^2_{H^s(\Gamma_i)^2}\,.
\end{aligned}$$

(We have used the substitution $x_2=x_2'-\tau$ in order to transform the integral on (γ_{-1},γ_0) to the integral on (γ_0,γ_1). We have also used the periodicity of function $\boldsymbol{g}$ and inequalities (40), (41).) The last inequality and (42), (43) imply that

$$\|\boldsymbol{g}\|_{H^s(A_{-1}A_1)^2}\;\leq\;\sqrt{2+\frac{2c_5}{s}}\;\|\boldsymbol{g}\|_{H^s(\Gamma_i)^2}\,.$$

This confirms that the restriction of the extended function $\boldsymbol{g}$ onto the line segment $A_{-1}A_1$ belongs to $H^s(A_{-1}A_1)^2$.

The next theorem is very important: it shows that the weak solution in the whole unbounded domain D can be obtained by means of an appropriate extension of the weak solution in the domain Ω.

I.2.9 Theorem 3. Let $\boldsymbol{u}$ be a solution of problem I.2.6 (i.e. a weak solution in the domain Ω). Then the periodic extension (with period τ) in the variable x_2 of the function $\boldsymbol{u}$ onto the domain D is a weak solution of problem I.1.2 .

Proof. We can naturally extend all functions appearing in the weak formulation of the problem in the domain Ω as τ–periodic functions in the variable x_2. Because of simplicity the extended functions will be denoted by the same symbols. Thus, the function $\boldsymbol{f}$ is now defined a.e. in D, the function $\boldsymbol{g}$ is defined a.e. in G_i and the function $\boldsymbol{h}$ is defined a.e. in Γ_o. Clearly, the extended functions $\boldsymbol{f}$ and $\boldsymbol{h}$ fulfill the assumptions from subsection I.1.2. We have shown in subsection I.2.8 that the extended function $\boldsymbol{g}$ also fulfills the assumptions from subsection I.1.2.

A function from $H^1(\Omega)$, extended τ–periodically in variable x_2, need not generally belong to $H^1(D')$ for an arbitrary bounded sub–domain D' of D. In the sequel we shall show that the extended function $\boldsymbol{u}$, however, has this property. Obviously, due to the periodicity of $\boldsymbol{u}$ in the x_2–direction, it is sufficient to verify that $\boldsymbol{u}$ (precisely, its restriction on $\Omega_0 \cup \Gamma_- \cup \Omega_{-1}$) belongs to $H^1(\Omega_0 \cup \Gamma_- \cup \Omega_{-1})$.

The function $\boldsymbol{u}$ has generalized first order derivatives $\partial \boldsymbol{u}/\partial x_i$ ($i = 1, 2$) in domains Ω_0 and Ω_{-1} and these derivatives are square integrable both in Ω_0 and Ω_{-1}. To verify that $\boldsymbol{u} \in H^1(\Omega_0 \cup \Gamma_- \cup \Omega_{-1})$, we need to show that the function $D_i\boldsymbol{u}$, defined by

$$D_i\boldsymbol{u}(x) = \begin{cases} \dfrac{\partial \boldsymbol{u}}{\partial x_i}(\boldsymbol{x}) & \text{for } \boldsymbol{x} \in \Omega_0, \\[2mm] \dfrac{\partial \boldsymbol{u}}{\partial x_i}(\boldsymbol{x}) & \text{for } \boldsymbol{x} \in \Omega_{-1}, \end{cases}$$

is a generalized derivative of $\boldsymbol{u}$ in the union $\Omega_0 \cup \Gamma_- \cup \Omega_{-1}$. Thus, suppose that $\boldsymbol{\varphi} \in C_0^\infty(\Omega_0 \cup \Gamma_- \cup \Gamma_{-1})^2$. Then, denoting by $n_{0,i}$ the i–th component of the outer normal vector to the boundary of Ω_0 on the curve Γ_- and by $n_{-1,i}$ the i–th component of the outer normal vector to the boundary of Ω_{-1} on the curve Γ_-, we have

$$\begin{aligned} \int_{\Omega_0\cup\Gamma_-\cup\Omega_{-1}} D_i\boldsymbol{u}\cdot\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x} &= \int_{\Omega_0} \frac{\partial \boldsymbol{u}}{\partial x_i}\cdot\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x} + \int_{\Omega_{-1}} \frac{\partial \boldsymbol{u}}{\partial x_i}\cdot\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x} \\ = \int_{\Gamma_-} T_0\boldsymbol{u}\cdot\boldsymbol{\varphi}\,n_{0,i}\,\mathrm{d}S &- \int_{\Omega_1} \boldsymbol{u}\cdot\frac{\partial \boldsymbol{\varphi}}{\partial x_i}\,\mathrm{d}\boldsymbol{x} + \int_{\Gamma_-} T_{-1}\boldsymbol{u}\cdot\boldsymbol{\varphi}\,n_{-1,i}\,\mathrm{d}S \\ &- \int_{\Omega_{-1}} \boldsymbol{u}\cdot\frac{\partial \boldsymbol{\varphi}}{\partial x_i}\,\mathrm{d}\boldsymbol{x} = -\int_{\Omega_0\cup\Gamma_-\cup\Omega_{-1}} \boldsymbol{u}\cdot\frac{\partial \boldsymbol{\varphi}}{\partial x_i}\,\mathrm{d}\boldsymbol{x}. \end{aligned} \tag{44}$$

(We have denoted by $T_0\boldsymbol{u}$ the trace of $\boldsymbol{u}$ on Γ_- as the trace of a function from $H^1(\Omega_0)^2$ and by $T_{-1}\boldsymbol{u}$ the trace of $\boldsymbol{u}$ on Γ_- as the trace of a function from $H^1(\Omega_{-1})^2$.) Due to the periodicity condition (13), both the traces on Γ_- coincide, i.e. $T_0\boldsymbol{u} = T_{-1}\boldsymbol{u}$ on Γ_-. Moreover, $n_{0,i} = -n_{-1,i}$ on Γ_-. Hence,

$$\int_{\Gamma_-} T_0\boldsymbol{u}\cdot\varphi\,n_{0,i}\,\mathrm{d}S + \int_{\Gamma_-} T_{-1}\boldsymbol{u}\cdot\varphi\,n_{-1,i}\,\mathrm{d}S = 0.$$

If we use this equality in (44), we can observe that $D_i\boldsymbol{u}$ is a generalized derivative of function $\boldsymbol{u}$ with respect to x_i in $\Omega_0 \cup \Gamma_- \cup \Omega_{-1}$. The square integrability of function $D_i\boldsymbol{u}$ in $\Omega_0 \cup \Gamma_- \cup \Omega_{-1}$ now follows from the definition of $D_i\boldsymbol{u}$ and from the square integrability of $\partial\boldsymbol{u}/\partial x_i$ in Ω_0 and in Ω_{-1}.

To prove that $\boldsymbol{u}$ is the solution of problem I.1.2, we must show that function $\boldsymbol{u}$, moreover, fulfills the integral identity (31) for an arbitrary acceptable test function $\boldsymbol{w}$ in D (i.e. a test function $\boldsymbol{w}$ that is an element of $H^1(D')^2$, where $D' \subset D$ is compact, $\boldsymbol{w}$ has zero traces on G_i and G_w, its divergence is equal to zero a.e. in D and it has a compact support in $D \cup G_o$). Due to the periodicity of $\boldsymbol{u}$, $\boldsymbol{f}$, $\boldsymbol{g}$ and $\boldsymbol{h}$ in the x_2–direction, it is sufficient to consider only such test functions $\boldsymbol{w} \in C^\infty(\overline{D})^2$ that have a compact support in $D \cup G_o$. It means that we can work only with test functions $\boldsymbol{w}$ such that $\boldsymbol{w}(x_1, x_2) = \boldsymbol{0}$, if $|x_2| > K(\boldsymbol{w})$, where $K(\boldsymbol{w})$ is a positive constant depending on $\boldsymbol{w}$. The validity of (31) for $\boldsymbol{w} \in H^1(D)^2$ having a compact support in $D \cup G_o$ can afterwards be proven by means of an appropriate limit procedure. (We can use the density of the space of all infinitely differentiable test functions $\boldsymbol{w}$ with the described properties in the space of test functions $\boldsymbol{w} \in H^1(D)^2$.)

First let us show the validity of (31) for a smooth admissible test function $\boldsymbol{w}$ that equals zero outside $\overline{\Omega_0} \cup \overline{\Omega_{-1}}$.

Lemma 1 *Suppose that $\boldsymbol{u}$ is a function which satisfies the assumptions of Theorem 3 (in the subsection I.2.9). Then $\boldsymbol{u}$ fulfills the integral identity (31) for each test function $\boldsymbol{w} \in C^\infty(\overline{D})^2$ such that $\boldsymbol{w} = \boldsymbol{0}$ on $G_i \cup G_w$, $\operatorname{div} \boldsymbol{w} = 0$ in D and $\boldsymbol{w} = \boldsymbol{0}$ in $D - (\overline{\Omega}_0 \cup \overline{\Omega}_{-1})$.*

Proof. Function $\boldsymbol{u}$ is a weak solution in the domain $\Omega \equiv \Omega_0$. Hence,

$$\int_{\Omega_0} \boldsymbol{f} \cdot \boldsymbol{v} \, \mathrm{d}\boldsymbol{x} \, = \, \nu \int_{\Omega_0} \sum_{i,j=1}^{2} \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j} \, \mathrm{d}\boldsymbol{x} \, + \int_{\Omega_0} \sum_{i,j=1}^{2} u_j \frac{\partial u_i}{\partial x_j} v_i \, \mathrm{d}\boldsymbol{x}$$
$$+ \int_{\Gamma_o} \frac{1}{2} (\boldsymbol{u} \cdot \boldsymbol{n})^- \, \boldsymbol{u} \cdot \boldsymbol{v} \, \mathrm{d}S \, + \int_{\Gamma_o} \boldsymbol{h} \cdot \boldsymbol{v} \, \mathrm{d}S. \tag{45}$$

for each test function $\boldsymbol{v} \in V$. We shall prove the validity of (31) with a test function $\boldsymbol{w}$, satisfying the assumptions of Lemma 1. Since $\boldsymbol{w}$ can differ from zero only in $\overline{\Omega}_0 \cup \overline{\Omega}_{-1}$, we can integrate only over $\Omega_0 \cup \Omega_{-1}$ instead of D. Thus, (31) takes the form

$$\int_{\Omega_0 \cup \Omega_{-1}} \boldsymbol{f} \cdot \boldsymbol{w} \, \mathrm{d}\boldsymbol{x} \, = \, \nu \int_{\Omega_0 \cup \Omega_{-1}} \sum_{i,j=1}^{2} \frac{\partial u_i}{\partial x_j} \frac{\partial w_i}{\partial x_j} \, \mathrm{d}\boldsymbol{x} \, + \int_{\Omega_0 \cup \Omega_{-1}} \sum_{i,j=1}^{2} u_j \frac{\partial u_i}{\partial x_j} w_i \, \mathrm{d}\boldsymbol{x}$$
$$+ \int_{B_{-1}B_1} \frac{1}{2} (\boldsymbol{u} \cdot \boldsymbol{n})^- \, \boldsymbol{u} \cdot \boldsymbol{w} \, \mathrm{d}S \, + \int_{B_{-1}B_1} \boldsymbol{h} \cdot \boldsymbol{w} \, \mathrm{d}S. \tag{46}$$

The integrals on $\Omega_0 \cup \Omega_{-1}$ are equal to the sum of two integrals on Ω_0 and on Ω_{-1}. Similarly, the integrals on the line segment $B_{-1}B_1$ are equal to the sum of two integrals on line segments $B_{-1}B_0$ and B_0B_1. (The line segment B_0B_1 coincides with Γ_o.) The integrals over Ω_{-1} (respectively along $B_{-1}B_0$) can easily be transformed (just shifting the system of coordinates) to the integrals over Ω_0 (respectively along B_0B_1). We can show it, for example, in the case of the first integral on the right-hand side of (46):

$$\int_{\Omega_{-1}} \sum_{i,j=1}^{2} \frac{\partial u_i}{\partial x_j}(x_1, x_2) \frac{\partial w_i}{\partial x_j}(x_1, x_2) \, \mathrm{d}\boldsymbol{x} \, = \int_{\Omega_0} \sum_{i,j=1}^{2} \frac{\partial u_i}{\partial x_j}(x_1, x_2 - \tau) \frac{\partial w_i}{\partial x_j}(x_1, x_2 - \tau) \, \mathrm{d}\boldsymbol{x}$$
$$= \int_{\Omega_0} \sum_{i,j=1}^{2} \frac{\partial u_i}{\partial x_j}(x_1, x_2) \frac{\partial w_i}{\partial x_j}(x_1, x_2 - \tau) \, \mathrm{d}\boldsymbol{x}.$$

(We have used the τ–periodicity of function $\boldsymbol{u}$ in variable x_2.) If we apply the same procedure to all integrals over Ω_{-1} or along $B_{-1}B_0$ in (46) and then sum the integrals over Ω_0 and along B_0B_1, we obtain

$$\int_{\Omega_0} \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} \,=\, \nu\int_{\Omega_0}\sum_{i,j=1}^{2}\frac{\partial u_i}{\partial x_j}\,\frac{\partial v_i}{\partial x_j}\,\mathrm{d}\boldsymbol{x} \,+\, \int_{\Omega_0}\sum_{i,j=1}^{2} u_j\,\frac{\partial u_i}{\partial x_j}\,v_i\,\mathrm{d}\boldsymbol{x}$$
$$+\int_{\Gamma_o}\frac{1}{2}\,(\boldsymbol{u}\cdot\boldsymbol{n})^-\,\boldsymbol{u}\cdot\boldsymbol{v}\,\mathrm{d}S \,+\, \int_{\Gamma_o}\boldsymbol{h}\cdot\boldsymbol{v}\,\mathrm{d}S, \tag{47}$$

where $\boldsymbol{v}(x_1,x_2) = \boldsymbol{w}(x_1,x_2)+\boldsymbol{w}(x_1,x_2-\tau)$ for $(x_1,x_2)\in\Omega_0$. This identity is of the same form as (36). In order to verify that (47) holds, we need to show that function $\boldsymbol{v}$ used in (47) has all the properties required in (36), i.e. that $\boldsymbol{v}\in V$ and then the validity of (47) will immediately follow from (36). Due to the assumption that $\boldsymbol{w}=\boldsymbol{0}$ outside $(\overline{\Omega}_0\cup\overline{\Omega}_{-1})$ and the continuity of $\boldsymbol{w}$, we have $\boldsymbol{w}(x_1,x_2\pm\tau)=\boldsymbol{0}$ for $(x_1,x_2)\in\Gamma_-$. This implies that $\boldsymbol{v}$ satisfies, for $(x_1,x_2)\in\Gamma_-$, the condition of periodicity in the x_2–direction:

$$\begin{aligned}\boldsymbol{v}(x_1,x_2+\tau) &= \boldsymbol{w}(x_1,x_2+\tau)+\boldsymbol{w}(x_1,x_2)\\ &= \boldsymbol{w}(x_1,x_2)+\boldsymbol{w}(x_1,x_2-\tau) \,=\, \boldsymbol{v}(x_1,x_2).\end{aligned}$$

Thus, $\boldsymbol{v}\in V$, (47) is satisfied and consequently, the identities (46) and (31) are also satisfied. □

Suppose further that $\boldsymbol{w}$ is an infinitely differentiable divergence–free vector function in $\overline{D}$, equal to zero on G_i and on G_w, and such that $\boldsymbol{w}(x_1,x_2)=\boldsymbol{0}$ if $|x_2|>K(\boldsymbol{w})$. In order to complete the proof of Theorem 3 (I.2.9), we need one more lemma.

Lemma 2 *Let a function $\boldsymbol{w}$ satisfy the above assumptions. Then there exists a function $\psi\in C^\infty(\overline{D})$ (the so-called "stream function") such that*

a) $\psi(x_1,x_2)=0$ *for* $|x_2|>K(\boldsymbol{w})$,

b) $w_1 \,=\, \dfrac{\partial\psi}{\partial x_2},\quad w_2 \,=\, -\dfrac{\partial\psi}{\partial x_1}$ *in D.*

Proof. We can choose an integer N so large that $\boldsymbol{w}=\boldsymbol{0}$ in $\overline{\Omega}_i$ for all $i\in\mathbb{Z}$ such that $|i|\geq N$. Let us denote

$$D_N \,=\, \Omega_{-N}\,\cup\,\bigcup_{k=N-1}^{N}\Big(\Omega_k\cup\Gamma^k\Big).$$

D_N is a bounded domain. Its boundary ∂D_N has the following components:

a) the line segment $A_{-N}A_N$ (which lies on the straight line G_i),

b) the curves Γ^{-N} and Γ^{N+1},

c) the line segment $B_{-N}B_N$ (which lies on the straight line G_o)

d) and the curves C_i for $i = -N, \dots, N$.

Theorem 3.1 in [23], page 37, provides the existence of a function $\psi \in C^\infty(\overline{D_N})$ such that

$$w_1 = \frac{\partial \psi}{\partial x_2}, \qquad w_2 = -\frac{\partial \psi}{\partial x_1} \qquad \text{in } D_N. \tag{48}$$

(An analogous theorem can be found in [7].) Due to the smoothness of the function ψ in $\overline{D_N}$, that first formula in (48) also holds on the open line segment $A_{-N}A_N$. Since the function w_1 equals zero on this line segment, the derivative of ψ with respect to x_2 also equals zero and consequently, the function ψ is constant on the line segment $A_{-N}A_N$. The constant can be chosen to be zero because the function ψ is given uniquely up to an additive constant. Using the identity $\boldsymbol{w} = \boldsymbol{0}$ and the second formula in (48) in Ω_N and in Ω_{-N}, we can derive that $\psi = 0$ in both domains Ω_N and Ω_{-N}. (Since $w_2 = 0 = -\frac{\partial \psi}{\partial x_1}$ and $\psi = 0$ on $A_{-N}A_N$).

If we extend function ψ from domain D_N onto the whole domain D by zero, we obtain a function with all the properties stated in Lemma 2. □

Continuation of the proof of Theorem 3 (I.2.9). Let us denote by η an infinitely differentiable function of one variable defined in the interval $(-\infty, +\infty)$ such that its support is contained in $(-\tau, \tau)$, its range is $[0, 1]$ and

$$\eta(x_2) + \eta(x_2 + \tau) = 1 \qquad \text{for } x_2 \in [-\tau, 0]. \tag{49}$$

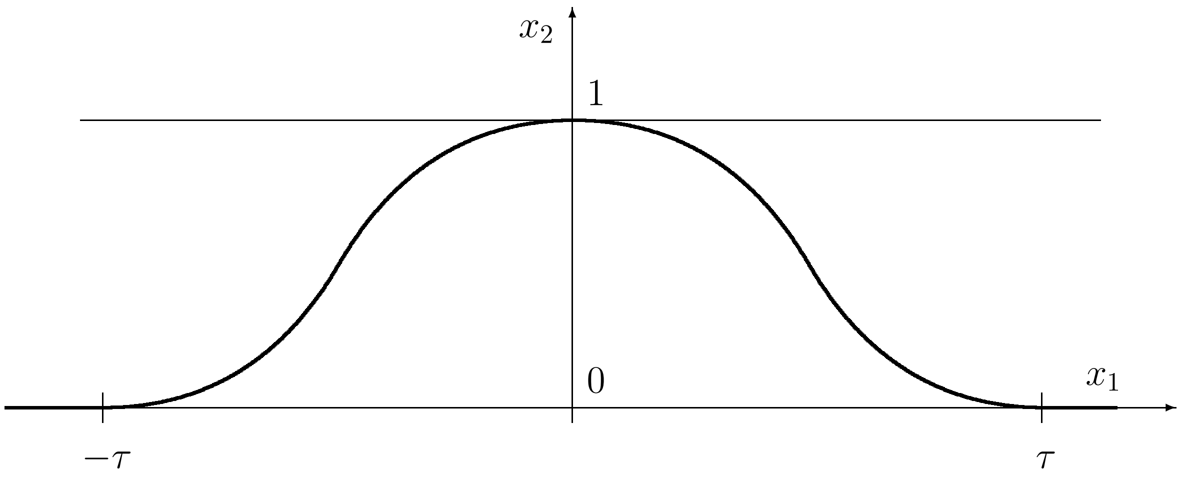

Fig. 3 (the example of function η)

If N is an integer then

$$\sum_{k=-N}^{+N} \eta(x_2 + k\tau) \quad \begin{cases} = 0 & \text{for } x_2 \in \big(-\infty, -(N+1)\tau\big] \cup \big[(N+1)\tau, +\infty\big), \\ \in [0,1] & \text{for } x_2 \in \big[-(N+1)\tau, -N\tau\big] \cup \big[N\tau, (N+1)\tau\big], \\ = 1 & \text{for } x_2 \in \big[-N\tau, N\tau\big]. \end{cases}$$

(In the first case, $x_2 + k\tau$ is outside the interval $(-\tau, \tau)$ for all $k = -N, \dots, N$. In the second case, just one of the points $x_2 + k\tau$ belongs to $(-\tau, \tau)$. In the third case, just two of the points $x_2 + k\tau$ find themselves in the region where $\eta \neq 0$ and the sum of the function values of η at these points equals one due to (49).)

Further, we put $\zeta(x_1, x_2) := \eta(x_2 - \gamma(x_1))$. We can observe that

$$\sum_{k=-N}^{+N} \zeta(x_1, x_2 + k\tau) \quad \begin{cases} = 0 & \text{for } (x_1, x_2) \in \overline{\Omega}_i; \quad |i| \geq N + 1, \\ \in [0, 1] & \text{for } (x_1, x_2) \in \overline{\Omega}_N \cup \overline{\Omega}_{-N}, \\ = 1 & \text{for } (x_1, x_2) \in \overline{\Omega}_i; \quad |i| < N. \end{cases} \tag{50}$$

(In the first case, $x_2 + k\tau - \gamma(x_1)$ is outside the interval $(-\tau, \tau)$ for all $k = -N, \dots, N$. In the second case, just one of the points $x_2 + k\tau - \gamma(x_1)$ belongs to $(-\tau, \tau)$. In the third case, just two of the points $x_2 + k\tau - \gamma(x_1)$ are in the region where $\eta \neq 0$.) Obviously, the functions $\zeta(x_1, x_2 + k\tau)$ represent an appropriate partition of unity.

If N is the same number as in the proof of Lemma 2, then $\psi = 0$ in $\overline{\Omega}_i$ for $|i| \geq N$ and

$$\psi(x_1, x_2) \,=\, \psi(x_1, x_2) \sum_{k=-N}^{+N} \zeta(x_1, x_2 + k\tau) \qquad \text{for } (x_1, x_2) \in D. \tag{51}$$

For $k \in \mathbb{Z}$ we define, a vector function $\boldsymbol{w}^k \equiv (w_1^k, w_2^k)$ by the formulas

$$\begin{aligned} w_1^k(x_1, x_2) \;&:=\; \frac{\partial}{\partial x_2} \left[\, \psi(x_1, x_2)\, \zeta(x_1, x_2 + k\tau) \,\right], \\ w_2^k(x_1, x_2) \;&:=\; -\frac{\partial}{\partial x_1} \left[\, \psi(x_1, x_2)\, \zeta(x_1, x_2 + k\tau) \,\right]. \end{aligned}$$

The function $\boldsymbol{w}^0$ differs from zero only in $\overline{\Omega}_0 \cup \overline{\Omega}_1$. By analogy, the function $\boldsymbol{w}^k$ (for a general $k \in \mathbb{Z}$) differs from zero only in $\overline{\Omega}_k \cup \overline{\Omega}_{k+1}$ and $\boldsymbol{w}^{-k}$ differs from zero only in $\overline{\Omega}_{-k} \cup \overline{\Omega}_{-k-1}$. From (51) it follows that

$$\boldsymbol{w}(x_1, x_2) \,=\, \sum_{k=-N}^{+N} \boldsymbol{w}^k(x_1, x_2)$$

for $(x_1, x_2) \in D$.

Now we use this function $\boldsymbol{w}$ in the integral identity (31). Obviously, if this identity is separately satisfied for each function $\boldsymbol{w}^k$ ($k = -N, -N+1, \dots, N$), then it is also satisfied for the test function $\boldsymbol{w}$.

The identity (31), with the test function $\boldsymbol{w}^k$, has the form

$$\begin{aligned} \int_D \boldsymbol{f} \cdot \boldsymbol{w}^k \, \mathrm{d}\boldsymbol{x} \,&=\, \nu \int_D \sum_{i,j=1}^{2} \frac{\partial u_i}{\partial x_j} \frac{\partial w_i^k}{\partial x_j} \, \mathrm{d}\boldsymbol{x} \,+\, \int_D \sum_{i,j=1}^{2} u_j \frac{\partial u_i}{\partial x_j} w_i^k \, \mathrm{d}\boldsymbol{x} \\ &\quad + \int_{G_o} \frac{1}{2} (\boldsymbol{u} \cdot \boldsymbol{n})^- \boldsymbol{u} \cdot \boldsymbol{w}^k \, \mathrm{d}S \,+\, \int_{G_o} \boldsymbol{h} \cdot \boldsymbol{w}^k \, \mathrm{d}S. \end{aligned}$$

In order to simplify the integrals, we use the substitution $x_1 = \bar{x}_1,\ x_2 = \bar{x}_2 + k\tau$. If we denote $\bar{x} = (\bar{x}_1, \bar{x}_2)$, $\bar{\boldsymbol{w}}^k(\bar{x}_1, \bar{x}_2) = \boldsymbol{w}^k(x_1, x_2 - k\tau)$ and use the equality $\boldsymbol{u}(\bar{x}_1, \bar{x}_2) =$

$\boldsymbol{u}(x_1, x_2 - k\tau) = \boldsymbol{u}(x_1, x_2)$ (following from the periodicity of the function $\boldsymbol{u}$), we obtain,

$$\int_D \boldsymbol{f} \cdot \bar{\boldsymbol{w}}^k \, \mathrm{d}\bar{\boldsymbol{x}} = \nu \int_D \sum_{i,j=1}^{2} \frac{\partial u_i}{\partial \bar{x}_j} \frac{\partial \bar{w}_i^k}{\partial \bar{x}_j} \, \mathrm{d}\bar{\boldsymbol{x}} + \int_D \sum_{i,j=1}^{2} u_j \frac{\partial u_i}{\partial \bar{x}_j} \bar{w}_i^k \, \mathrm{d}\bar{\boldsymbol{x}}$$
$$+ \int_{G_o} \frac{1}{2} (\boldsymbol{u} \cdot \boldsymbol{n})^- \boldsymbol{u} \cdot \bar{\boldsymbol{w}}^k \, \mathrm{d}S + \int_{G_o} \boldsymbol{h} \cdot \bar{\boldsymbol{w}}^k \, \mathrm{d}S. \tag{52}$$

Function $\bar{\boldsymbol{w}}^k$ differs from zero only in $\overline{\Omega}_0 \cup \overline{\Omega}_{-1}$ and fulfills all the assumptions put on a test function in Lemma 1. Thus, we can apply Lemma 1 and we see that identity (52) holds. This completes the proof. □

Further, in order to verify that no important information on the nature of our boundary value problem was lost during the derivation of the weak formulation in the domain Ω from the classical formulation in subsection (I.2.5), we present the next theorem.

I.2.10 Theorem 4 (from a weak solution to a classical solution in the domain Ω). Let $\boldsymbol{f}$ and $\boldsymbol{g}$, be continuous functions in Ω and on Γ_o, respectively and satisfy the conditions from subsection I.2.5,. Let $\boldsymbol{u}$ be a solution of the weak problem I.2.5 and let $\boldsymbol{u}$ have the regularity required in the classical formulation in subsection I.1.4. Then there exists a function p (the pressure) such that the pair $\boldsymbol{u}$, p is a classical solution of the problem I.1.4.

Proof. Function $\boldsymbol{u}$ satisfies the integral equation (36) for all test functions $\boldsymbol{v}$ which have the properties described in subsection I.2.5.

Let us at first deal only with such test functions $\boldsymbol{v}$ that have zero traces on the boundary of Ω. Then we can apply Green's theorem in (36) and derive the equality

$$\int_\Omega \Big((\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - \nu \, \Delta \boldsymbol{u} - \boldsymbol{f} \Big) \cdot \boldsymbol{v} \, \mathrm{d}\boldsymbol{x} = 0.$$

Using Assertion 1.1, 1.2 and Remark 1.4 in [55], page 15, we can deduce that there exists a function $q \in H^1(\Omega)$ such that

$$(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - \nu \, \Delta \boldsymbol{u} - \boldsymbol{f} = -\nabla q \tag{53}$$

a.e. in Ω. This equation and the regularity of the function $\boldsymbol{u}$ imply that ∇q is (after an eventual change on a set of zero measure) continuous in Ω.

Now, let us return to the integral identity (36) and assume that $\boldsymbol{v}$ is a test function which need not have the zero trace on the whole boundary of Ω. However, as required in subsection I.2.6, $\boldsymbol{v}$ has the zero traces on Γ_i and Γ_w. Let us assume, in addition to these conditions, that $\boldsymbol{v} = \boldsymbol{0}$ on Γ_- and on Γ_+. If we again apply Green's theorem in (36) and use (53), we get

$$0 = \int_\Omega \Big((\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - \nu \, \Delta \boldsymbol{u} - \boldsymbol{f} \Big) \cdot \boldsymbol{v} \, \mathrm{d}\boldsymbol{x} + \int_{\Gamma_o} \Big(\nu \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}} + \frac{1}{2} (\boldsymbol{u} \cdot \boldsymbol{n})^- \boldsymbol{u} + \boldsymbol{h} \Big) \cdot \boldsymbol{v} \, \mathrm{d}S$$
$$= -\int_\Omega \nabla q \cdot \boldsymbol{v} \, \mathrm{d}\boldsymbol{x} + \int_{\Gamma_o} \Big(\nu \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}} + \frac{1}{2} (\boldsymbol{u} \cdot \boldsymbol{n})^- \boldsymbol{u} + \boldsymbol{h} \Big) \cdot \boldsymbol{v} \, \mathrm{d}S. \tag{54}$$

The integral containing ∇q can be transformed:

$$-\int_\Omega \nabla q \cdot \boldsymbol{v} \, \mathrm{d}\boldsymbol{x} \,=\, -\int_{\Gamma_o} q\, \boldsymbol{n} \cdot \boldsymbol{v} \, \mathrm{d}S \,+\, \int_\Omega q \,\mathrm{div}\, \boldsymbol{v} \, \mathrm{d}\boldsymbol{x} \,=\, -\int_{\Gamma_o} q\, \boldsymbol{n} \cdot \boldsymbol{v} \, \mathrm{d}S.$$

If we use this in (54) and take into account (53), we obtain

$$\int_{\Gamma_o} \left(\nu \, \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}} - q\, \boldsymbol{n} + \frac{1}{2}\, (\boldsymbol{u} \cdot \boldsymbol{n})^- \, \boldsymbol{u} + \boldsymbol{h} \right) \cdot \boldsymbol{v} \, \mathrm{d}S \,=\, 0. \tag{55}$$

This identity holds for every function $\boldsymbol{v}$ on Γ_o such that $\boldsymbol{v}(B_0) = \boldsymbol{v}(B_1) = \boldsymbol{0}$ and

$$\int_{\Gamma_o} \boldsymbol{v} \cdot \boldsymbol{n} \, \mathrm{d}S \,=\, 0. \tag{56}$$

($\boldsymbol{v}$ is in fact the trace of a function from V. In order not to complicate the notation, we denote the function in V and its trace on Γ_o by the same letter. The equality (56) follows from the condition of incompressibility $\mathrm{div}\, \boldsymbol{v} = 0$ and the assumption that $\boldsymbol{v} = \boldsymbol{0}$ on the whole boundary of Ω except for Γ_o.) Since $\boldsymbol{n} = (1, 0)$ on Γ_o, (56) is equivalent to

$$\int_{\Gamma_o} v_1 \, \mathrm{d}S \,=\, 0. \tag{57}$$

Lemma 3 *Let (55) and (57) hold. Then there exists a constant c_6 such that*

$$\nu \, \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}} - q\, \boldsymbol{n} + \frac{1}{2}\, (\boldsymbol{u} \cdot \boldsymbol{n})^- \, \boldsymbol{u} + \boldsymbol{h} \,=\, (c_6,\, 0) \,=\, c_6\, \boldsymbol{n} \ \text{ on } \Gamma_o. \tag{58}$$

Proof. The set of traces on Γ_o of all functions from V is dense in $\{(v_1, v_2) \in L^2(\Gamma_o)^2;$ $\int_{\Gamma_o} v_1 \, \mathrm{d}S = 0\}$. Thus, (55) is equivalent to

$$\int_{\Gamma_o} \left(\nu \, \frac{\partial u_1}{\partial \boldsymbol{n}} - q + \frac{1}{2}\, (\boldsymbol{u} \cdot \boldsymbol{n})^- \, u_1 + h_1 \right) v_1 \, \mathrm{d}S \,=\, 0, \tag{59}$$

$$\int_{\Gamma_o} \left(\nu \, \frac{\partial u_2}{\partial \boldsymbol{n}} + \frac{1}{2}\, (\boldsymbol{u} \cdot \boldsymbol{n})^- \, u_2 + h_2 \right) v_2 \, \mathrm{d}S \,=\, 0 \tag{60}$$

for all v_1, v_2 in $L^2(\Gamma_o)$ such that $\int_{\Gamma_o} v_1 \, \mathrm{d}S = 0$. Now the identity

$$\nu \, \frac{\partial u_2}{\partial \boldsymbol{n}} + \frac{1}{2}\, (\boldsymbol{u} \cdot \boldsymbol{n})^- \, u_2 + h_2 \,=\, 0$$

is obvious and it remains to verify the equality in the first component of (58). Denote

$$\zeta \,=\, \nu \, \frac{\partial u_1}{\partial \boldsymbol{n}} - q + \frac{1}{2}\, (\boldsymbol{u} \cdot \boldsymbol{n})^- \, u_1 + h_1.$$

We need to show that function ζ is constant on Γ_o. Put

$$\overline{\zeta} \,:=\, \zeta - \frac{1}{|\Gamma_o|} \int_{\Gamma_o} \zeta \, \mathrm{d}S$$

where $|\Gamma_o|$ denotes the length of Γ_o. Thus, the mean value of $\overline{\zeta}$ on Γ_o equals zero. (57) and (59) yield

$$\int_{\Gamma_o} \overline{\zeta}\, v_1 \, \mathrm{d}S \;=\; \int_{\Gamma_o} \zeta\, v_1 \, \mathrm{d}S \;-\; \left(\frac{1}{|\Gamma_o|} \int_{\Gamma_o} \zeta \, \mathrm{d}S \right) \left(\int_{\Gamma_o} v_1 \, \mathrm{d}S \right) \;=\; 0. \tag{61}$$

Suppose now that w is an arbitrary function from $L^2(\Gamma_o)$ (not necessarily having the mean value equal to zero). We define $\overline{w}$ by analogy with $\overline{\zeta}$. We have

$$\begin{aligned} \int_{\Gamma_o} \overline{\zeta}\, w \, \mathrm{d}S \;&=\; \int_{\Gamma_o} \overline{\zeta} \left(\overline{w} + \frac{1}{|\Gamma_o|} \int_{\Gamma_o} w \, \mathrm{d}S \right) \mathrm{d}S \\ &=\; \int_{\Gamma_o} \overline{\zeta}\, \overline{w} \, \mathrm{d}S \;+\; \left(\int_{\Gamma_o} \overline{\zeta} \, \mathrm{d}S \right) \left(\frac{1}{|\Gamma_o|} \int_{\Gamma_o} w \, \mathrm{d}S \right) \;=\; 0. \end{aligned} \tag{62}$$

The first integral on the right hand side equals zero due to (55) and the second term equals zero because the mean value of $\overline{\zeta}$ is zero. We have shown that the scalar product of $\overline{\zeta}$ with an arbitrary function w in $L^2(\Gamma_o)$ equals zero. Thus, $\overline{\zeta} = 0$ in Γ_o and consequently,

$$\zeta(d_o, x_2) \;=\; \frac{1}{|\Gamma_o|} \int_{\Gamma_o} \zeta(d_o, s_2) \, \mathrm{d}s_2 \;=\; \mathrm{const.}$$

This formula shows that ζ is constant on Γ_o. The lemma is proved. $\square$

Continuation of the proof of Theorem 4 (I.2.10). If we use Lemma 3 and define $p := q + c_6$, we obtain:

$$\nu\, \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}} - p\, \boldsymbol{n} + \frac{1}{2}\, (\boldsymbol{u} \cdot \boldsymbol{n})^- \, \boldsymbol{u} + \boldsymbol{h} \;=\; 0 \qquad \text{on } \Gamma_o. \tag{63}$$

Because p differs from q only by the additive constant c_6, the equation (53) holds with p instead of q, too:

$$(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} \;-\; \nu\, \Delta \boldsymbol{u} \;-\; \boldsymbol{f} \;=\; -\nabla p \qquad \text{in } \Omega. \tag{64}$$

Let us again use the integral identity (36). Suppose that the test function $\boldsymbol{v}$ has zero traces only on Γ_i and Γ_w, in accordance with subsection I.2.6. Applying Green's theorem to (36) and using equation (64), we get

$$0 = \nu \int_{\Gamma_- \cup \Gamma_+} \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}} \cdot \boldsymbol{v} \, \mathrm{d}S - \int_{\Omega} \nabla p \cdot \boldsymbol{v} \, \mathrm{d}\boldsymbol{x} + \int_{\Gamma_o} \left(\nu\, \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}} + \frac{1}{2}\, (\boldsymbol{u} \cdot \boldsymbol{n})^- \, \boldsymbol{u} + \boldsymbol{h} \right) \cdot \boldsymbol{v} \, \mathrm{d}S. \tag{65}$$

Using the equation $\operatorname{div} \boldsymbol{v} = 0$, we can treat the second integral on the right hand side as follows:

$$\int_{\Omega} \nabla p \cdot \boldsymbol{v} \, \mathrm{d}\boldsymbol{x} \;=\; -\int_{\Gamma_- \cup \Gamma_+} p\, \boldsymbol{v} \cdot \boldsymbol{n} \, \mathrm{d}S \;-\; \int_{\Gamma_o} p\, \boldsymbol{v} \cdot \boldsymbol{n} \, \mathrm{d}S.$$

If we substitute this into (65) and also use (63), we obtain

$$\int_{\Gamma_- \cup \Gamma_+} \left(-p\, \boldsymbol{n} + \nu\, \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}} \right) \cdot \boldsymbol{v} \, \mathrm{d}S \;=\; 0. \tag{66}$$

This equality is satisfied for all admissible test functions $\boldsymbol{v}$, whose traces satisfy the condition of periodicity

$$\boldsymbol{v}(x_1, x_2+\tau) \;=\; \boldsymbol{v}(x_1, x_2) \qquad \text{for } (x_1, x_2) \in \Gamma_-.$$

From this, we can deduce that

$$\left(-p\,\boldsymbol{n} + \nu\,\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}}\right)\bigg|_{\Gamma_-} \;=\; -\left(-p\,\boldsymbol{n} + \nu\,\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}}\right)\bigg|_{\Gamma_+}. \tag{67}$$

Let us further denote by $\mathbf{t}$ is unit tangent vector to the curves Γ_- and Γ_+ and by $\mathbf{m}$ the unit perpendicular vector to these curves. Using the analytical expression of these curves, we can derive that

$$\boldsymbol{t} = (t_1, t_2) = \frac{(1,\, \gamma'(x_1))}{\sqrt{1+\gamma'^2(x_1)}}, \qquad \boldsymbol{m} = (m_1, m_2) = \frac{(-\gamma'(x_1),\, 1)}{\sqrt{1+\gamma'^2(x_1)}} = (-t_2, t_1)$$

at points (x_1, x_2) on $(\Gamma_-)^\circ$ and on $(\Gamma_+)^\circ$. Naturally, $\boldsymbol{m} = \boldsymbol{n}$ on Γ_+ and $\boldsymbol{m} = -\boldsymbol{n}$ na Γ_-. Thus, the derivative in the direction $\boldsymbol{n}$ on Γ_- can be expressed as

$$\frac{\partial}{\partial \boldsymbol{n}}\bigg|_{\Gamma_-} \;=\; -\frac{\partial}{\partial \mathbf{m}}\bigg|_{\Gamma_-} \;=\; -\left(m_1\,\frac{\partial}{\partial x_1} + m_2\,\frac{\partial}{\partial x_2}\right) \;=\; t_2\,\frac{\partial}{\partial x_1} - t_1\,\frac{\partial}{\partial x_2}.$$

Similarly, the derivative in the direction $\boldsymbol{n}$ on Γ_+ equals

$$\frac{\partial}{\partial \boldsymbol{n}}\bigg|_{\Gamma_+} \;=\; \frac{\partial}{\partial \boldsymbol{m}}\bigg|_{\Gamma_+} \;=\; m_1\,\frac{\partial}{\partial x_1} + m_2\,\frac{\partial}{\partial x_2} \;=\; -t_2\,\frac{\partial}{\partial x_1} + t_1\,\frac{\partial}{\partial x_2}.$$

If we write the equality (67) in coordinates and express the derivatives with respect to $\boldsymbol{n}$ by mens of the above formulas, we obtain

$$\begin{aligned}
-\nu\,t_2\,\frac{\partial u_1}{\partial x_1}\bigg|_{\Gamma_-} + \nu\,t_1\,\frac{\partial u_1}{\partial x_2}\bigg|_{\Gamma_-} + t_2\,p\Big|_{\Gamma_-} \;&=\; -\nu\,t_2\,\frac{\partial u_1}{\partial x_1}\bigg|_{\Gamma_+} + \nu\,t_1\,\frac{\partial u_1}{\partial x_2}\bigg|_{\Gamma_+} + t_2\,p\Big|_{\Gamma_+},\\
-\nu\,t_2\,\frac{\partial u_2}{\partial x_1}\bigg|_{\Gamma_-} + \nu\,t_1\,\frac{\partial u_2}{\partial x_2}\bigg|_{\Gamma_-} - t_1\,p\Big|_{\Gamma_-} \;&=\; -\nu\,t_2\,\frac{\partial u_2}{\partial x_1}\bigg|_{\Gamma_+} + \nu\,t_1\,\frac{\partial u_2}{\partial x_2}\bigg|_{\Gamma_+} - t_1\,p\Big|_{\Gamma_+}.
\end{aligned}$$

If we denote

$$\begin{aligned}
\Delta_{ij} \;&=\; \frac{\partial u_i}{\partial x_j}\bigg|_{\Gamma_-} - \frac{\partial u_i}{\partial x_j}\bigg|_{\Gamma_+} \quad \text{for } i, j = 1, 2 \text{ and}\\
\Delta_p \;&=\; p\Big|_{\Gamma_-} - p\Big|_{\Gamma_+},
\end{aligned}$$

we obtain the equations

$$-\nu\,t_2\,\Delta_{11} + \nu\,t_1\,\Delta_{12} + t_2\,\Delta_p \;=\; 0, \tag{68}$$

$$-\nu\,t_2\,\Delta_{21} + \nu\,t_1\,\Delta_{22} - t_1\,\Delta_p \;=\; 0. \tag{69}$$

Due to the τ–periodicity of function $\boldsymbol{u}$ in the x_2–direction, we have

$$\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{t}}\bigg|_{\Gamma_-} = \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{t}}\bigg|_{\Gamma_+}.$$

Writing this equation in the coordinates, we get

$$\begin{aligned} t_1 \frac{\partial u_1}{\partial x_1}\bigg|_{\Gamma_-} + t_2 \frac{\partial u_1}{\partial x_2}\bigg|_{\Gamma_-} &= t_1 \frac{\partial u_1}{\partial x_1}\bigg|_{\Gamma_+} + t_2 \frac{\partial u_1}{\partial x_2}\bigg|_{\Gamma_+}, \\ t_1 \frac{\partial u_2}{\partial x_1}\bigg|_{\Gamma_-} + t_2 \frac{\partial u_2}{\partial x_2}\bigg|_{\Gamma_-} &= t_1 \frac{\partial u_2}{\partial x_1}\bigg|_{\Gamma_+} + t_2 \frac{\partial u_2}{\partial x_2}\bigg|_{\Gamma_+}. \end{aligned}$$

Subtracting the right-hand sides from the left-hand sides, we obtain

$$t_1 \Delta_{11} + t_2 \Delta_{12} = 0, \tag{70}$$

$$t_1 \Delta_{21} + t_2 \Delta_{22} = 0. \tag{71}$$

Finally, from the equation of continuity, we have

$$\frac{\partial u_1}{\partial x_1}\bigg|_{\Gamma_-} + \frac{\partial u_2}{\partial x_2}\bigg|_{\Gamma_-} = \frac{\partial u_1}{\partial x_1}\bigg|_{\Gamma_+} + \frac{\partial u_2}{\partial x_2}\bigg|_{\Gamma_+} = 0,$$

which implies that

$$\Delta_{11} + \Delta_{22} = 0. \tag{72}$$

The equations (68)–(72) represent the homogenous system of five linear algebraic equations for the unknowns Δ_{11}, Δ_{12}, Δ_{21}, Δ_{22} and Δ_p. Calculating the determinant of the matrix of this system, we find out that the determinant is equal to $\nu\,(t_1^2 + t_2^2)^2 \equiv \nu$. Thus, the determinant is different from zero. Consequently, the unique solution of the system (68)–(72) is $\Delta_{11} = \Delta_{12} = \Delta_{21} = \Delta_{22} = \Delta_p = 0$. From these identities and from the definitions of Δ_{11}–Δ_{22} and Δ_p, we observe that the first order derivatives of functions $\boldsymbol{u}$ and p in an arbitrary direction have the same values in corresponding points on Γ_- and Γ_+. Hence functions $\boldsymbol{u}$ and p satisfy conditions (14) and (15) from the subsection I.1.4 (the classical problem in domain Ω). The condition (13) from subsection I.1.4 is satisfied, due to the assumptions of Theorem 4 (I.2.10), automatically.

Thus, we have proved that a solution of the problem I.2.6 that is "smooth enough" is a solution of the problem I.1.4. In other words, a sufficiently smooth weak solution in the domain Ω is a classical solution in Ω. This completes the proof of Theorem 4 (I.2.10).

□

I.2.11 Remark 1. Theorems 1 (I.1.5) and 3 (I.2.9) show that instead of studying the problem in the whole unbounded domain D, it is reasonable to restrict ourselves to the study of the flow on one spatial period Ω only. Let us recall the arguments which support this approach:

a) The τ–periodic extension of a classical solution in the domain Ω in the x_2–direction is a classical solution in the domain D. (Theorem I.1.5.)

b) The τ–periodic extension of a weak solution in the domain Ω in the x_2–direction is a weak solution in the domain D. (Theorem 3 (I.2.9).)

Furthermore, Theorem 4 (I.2.10) confirms that problem I.2.6 represents a reasonable weak formulation of problem I.1.4. This means that every classical solution is a weak solution and on the other hand, every sufficiently smooth weak solution is a classical solution.

Since the physical nature of the studied boundary value problem and the insufficient smoothness of the input data (functions $\boldsymbol{f}$, $\boldsymbol{g}$, $\boldsymbol{h}$ and especially the shape of domain Ω) do not generally allow us to expect the existence of a classical solution, we shall further focus on the study of the weak formulation of the problem in the domain Ω, introduced in subsection I.2.6. □

I.3 Existence of a weak solution

I.3.1 Extension of the function $\boldsymbol{g}$ from Γ_i onto $\partial\Omega$. Suppose that $s \in (\frac{1}{2}, 1]$. We recall that function $\boldsymbol{g}$ in the boundary condition (10) belongs to the space $H^s(\Gamma_i)^2$, which implies the continuity of $\boldsymbol{g}$ in Γ_i, and $\boldsymbol{g}(A_0) = \boldsymbol{g}(A_1)$. (See subsections I.2.6 and I.2.7.) At first, we extend function $\boldsymbol{g}$ onto the whole boundary $\partial\Omega$ in this way: we put

$$\boldsymbol{g}(x_1, x_2) := \begin{cases} \boldsymbol{g}(A_1) & \text{for } (x_1, x_2) \in \Gamma_+ \, , \\ \boldsymbol{g}(A_0) & \text{for } (x_1, x_2) \in \Gamma_- \, , \\ \boldsymbol{g}(d_i, \gamma(d_i) + x_2 - \gamma(d_o)) & \text{for } (x_1, x_2) = (d_o, x_2) \in \Gamma_o \, , \\ \boldsymbol{0} & \text{for } (x_1, x_2) \in \Gamma_w \, . \end{cases} \tag{73}$$

Thus, the extended function $\boldsymbol{g}$ is equal to the constant $\boldsymbol{g}(A_0)$ (which is the same as $\boldsymbol{g}(A_1)$) on $\Gamma_+ \cup \Gamma_-$, to zero on Γ_w and it has the same shape on Γ_o as on Γ_i.

Lemma 4 *The extension (73) of function $\boldsymbol{g}$ from Γ_i onto $\partial\Omega$ is an element of the space $H^{1/2}(\partial\Omega)^2$. Moreover, there exists a constant $c_7 > 0$ independent of $\boldsymbol{g}$ such that*

$$\|\boldsymbol{g}\|_{H^{1/2}(\partial\Omega)^2} \, \leq \, c_7 \, \|\boldsymbol{g}\|_{H^s(\Gamma_i)^2} \, . \tag{74}$$

Proof. The norm of the extended function $\boldsymbol{g}$ in $H^{1/2}(\partial\Omega)^2$ is defined by

$$\|\boldsymbol{g}\|_{H^{1/2}(\partial\Omega)^2} \, = \, \|\boldsymbol{g}\|_{L^2(\partial\Omega)^2} \, + \, \left(\int_{\partial\Omega} \int_{\partial\Omega} \frac{|\boldsymbol{g}(\boldsymbol{y}) - \boldsymbol{g}(\boldsymbol{x})|^2}{|\boldsymbol{x} - \boldsymbol{y}|^2} \, \mathrm{d}S_y \, \mathrm{d}S_x \right)^{1/2} . \tag{75}$$

(See G. P. Galdi [20], page 43.) We want to show that the expression on the right-hand side of (75) is less than or equal to the right-hand side of (38). From the definition of $\boldsymbol{g}$ on $\partial\Omega$ and inequality (39) we deduce that there exists a constant $c_8 > 0$ independent of $\boldsymbol{g}$ such that

$$\|\boldsymbol{g}\|_{L^2(\partial\Omega)^2} \, \leq \, c_8 \, \|\boldsymbol{g}\|_{H^s(\Gamma_i)^2} \, . \tag{76}$$

The integral over $\partial\Omega \times \partial\Omega$ on the right-hand side of (75) can be written as the sum of integrals over $\Gamma_i \times \Gamma_i$, $\Gamma_i \times \Gamma_+$, $\Gamma_i \times \Gamma_o$, $\Gamma_i \times \Gamma_-$, $\Gamma_i \times \Gamma_w$, $\Gamma_+ \times \Gamma_+$, $\Gamma_+ \times \Gamma_o$, $\Gamma_+ \times \Gamma_-$, $\Gamma_+ \times \Gamma_w$, $\Gamma_o \times \Gamma_o$, $\Gamma_o \times \Gamma_-$, $\Gamma_o \times \Gamma_w$, $\Gamma_- \times \Gamma_-$, $\Gamma_- \times \Gamma_w$ and $\Gamma_w \times \Gamma_w$, and the integrals over the Cartesian products of nonequal sets have to be considered twice. The integrals on $\Gamma_i \times \Gamma_i$ and $\Gamma_o \times \Gamma_o$ can be estimated by $\|\boldsymbol{g}\|^2_{H^s(\Gamma_i)^2}$. The integrals on $\Gamma_+ \times \Gamma_+$, $\Gamma_+ \times \Gamma_-$, $\Gamma_- \times \Gamma_-$ and $\Gamma_w \times \Gamma_w$ vanish because $\boldsymbol{g}$ is constant on these sets. The integrals on $\Gamma_i \times \Gamma_w$, $\Gamma_+ \times \Gamma_w$, $\Gamma_o \times \Gamma_w$ and $\Gamma_- \times \Gamma_w$ can be estimated by $c_9\, \|\boldsymbol{g}\|_{L^2(\Gamma_i)^2}$ (where c_9 is a suitable constant) because Γ_w has a positive distance from Γ_i, Γ_+, Γ_o and Γ_-. Thus, it only remains to estimate the integrals on $\Gamma_i \times \Gamma_+$, $\Gamma_i \times \Gamma_-$, $\Gamma_+ \times \Gamma_o$ and $\Gamma_o \times \Gamma_-$. We shall deal only with the integral on $\Gamma_i \times \Gamma_+$, the other integrals can be treated similarly.

Using the notation $\gamma_0 = \gamma(d_i)$ and $\gamma_1 = \gamma(d_i) + \tau$ from subsection I.2.7 , we can write

$$\int_{\Gamma_i}\int_{\Gamma_+} \frac{|\boldsymbol{g}(\boldsymbol{y}) - \boldsymbol{g}(\boldsymbol{x})|^2}{|\boldsymbol{y} - \boldsymbol{x}|^2}\, \mathrm{d}S_y\, \mathrm{d}S_x \;=\; \int_{\Gamma_i}\int_{\Gamma_+} \frac{|\boldsymbol{g}(d_i,\gamma_1) - \boldsymbol{g}(d_i,x_2)|^2}{|\boldsymbol{y} - \boldsymbol{x}|^2}\, \mathrm{d}S_y\, \mathrm{d}S_x.$$

Using the relations $\boldsymbol{x} = [d_i, x_2]$, $\mathrm{d}S_x = dx_2$, $\boldsymbol{y} = [y_1, \gamma(y_1)]$, $\mathrm{d}S_y = \sqrt{1+\gamma'(y_1)^2}\, dy_1$, we find that

$$\begin{aligned}
&\int_{\Gamma_i}\int_{\Gamma_+} \frac{|\boldsymbol{g}(d_i,\gamma_1) - \boldsymbol{g}(d_i,x_2)|^2}{|\boldsymbol{y} - \boldsymbol{x}|^2}\, \mathrm{d}S_y\, \mathrm{d}S_x \\
&= \int_{\gamma_0}^{\gamma_1}\int_{d_i}^{d_o} \frac{|\boldsymbol{g}(d_i,\gamma_1) - \boldsymbol{g}(d_i,x_2)|^2}{(y_1-d_i)^2 + (\gamma(y_1)-x_2)^2}\, \sqrt{1+\gamma'(y_1)^2}\, \mathrm{d}y_1 \mathrm{d}x_2.
\end{aligned}$$

Since the function γ has a continuous derivative on the closed interval $[d_i, d_o]$, there exist positive constants c_{10}, c_{11} such that

$$\begin{aligned}
&\sqrt{1+\gamma'(y_1)^2} \;\leq\; c_{10} \quad \forall\, y_1 \in (d_i, d_o), \\
&|\boldsymbol{y} - \boldsymbol{x}| \;\geq\; c_{11}\, \big[\mathrm{dist}(\boldsymbol{y}; A_1) + \mathrm{dist}(A_1; \boldsymbol{x})\big] \;\geq\; c_{11}\, \big[(y_1 - d_i) + (\gamma_1 - x_2)\big]
\end{aligned}$$

for all $\boldsymbol{x} = (d_i, x_2) \in \Gamma_i$ and $\boldsymbol{y} = (y_1, \gamma(y_1)) \in \Gamma_+$. These inequalities and (41) imply that the last integral can be estimated from above by

$$\begin{aligned}
&\frac{c_{10}}{c_{11}} \int_{\gamma_0}^{\gamma_1}\int_{d_i}^{d_o} \frac{|\boldsymbol{g}(d_i,\gamma_1) - \boldsymbol{g}(d_i,x_2)|^2}{[(y_1-d_i) + (\gamma_1 - x_2)]^2}\, \mathrm{d}y_1\, \mathrm{d}x_2 \\
&= \frac{c_{10}}{c_{11}} \int_{\gamma_0}^{\gamma_1} |\boldsymbol{g}(d_i,\gamma_1) - \boldsymbol{g}(d_i,x_2)|^2 \left(\frac{1}{\gamma_1 - x_2} - \frac{1}{(d_o - d_i) + (\gamma_1 - x_2)} \right) \mathrm{d}x_2 \\
&\leq \frac{c_{10}}{c_{11}} \int_{\gamma_0}^{\gamma_1} \frac{|\boldsymbol{g}(d_i,\gamma_1) - \boldsymbol{g}(d_i,x_2)|^2}{\gamma_1 - x_2}\, \mathrm{d}x_2 \;\leq\; c_{12} \int_{\gamma_0}^{\gamma_1} \frac{|\boldsymbol{g}(d_i,\gamma_1) - \boldsymbol{g}(d_i,x_2)|^2}{(\gamma_1 - x_2)^{2s}}\, \mathrm{d}x_2 \\
&\leq c_{12}\, c_5\, \|\boldsymbol{g}\|^2_{H^s(\Gamma_1)^2}\,,
\end{aligned}$$

which we wanted to prove. □

Lemma 5 *The extension (73) of function* $\boldsymbol{g}$ *satisfies the condition*

$$\int_{\partial\Omega} \boldsymbol{g}\cdot\boldsymbol{n}\, \mathrm{d}S \;=\; 0. \tag{77}$$

Proof. The boundary of the domain Ω is the union of the curves Γ_i, Γ_+, Γ_o, Γ_- and Γ_w. The intersection of any two of them contains at most one point. The prolonged function $\boldsymbol{g}$ has the same shape on Γ_i as on Γ_o. On the other hand, the normal vector $\boldsymbol{n}$ has the opposite direction on Γ_i and Γ_o, and hence,

$$\int_{\Gamma_i} \boldsymbol{g} \cdot \boldsymbol{n} \, \mathrm{d}S + \int_{\Gamma_o} \boldsymbol{g} \cdot \boldsymbol{n} \, \mathrm{d}S = 0. \tag{78}$$

The prolonged function $\boldsymbol{g}$ is on Γ_+ and Γ_- equal to the same constant vector, but the outer normal vector $\boldsymbol{n}$ satisfies $\boldsymbol{n}(x_1, \gamma(x_1)) = -\boldsymbol{n}(x_1, \gamma(x_1) + \tau)$ at all points $(x_1, \gamma(x_1)) \in \Gamma_-$ and $(x_1, \gamma(x_1) + \tau) \in \Gamma_+$. Hence,

$$\int_{\Gamma_+} \boldsymbol{g} \cdot \boldsymbol{n} \, \mathrm{d}S + \int_{\Gamma_-} \boldsymbol{g} \cdot \boldsymbol{n} \, \mathrm{d}S = 0. \tag{79}$$

The integral of $\boldsymbol{g} \cdot \boldsymbol{n}$ along Γ_w vanishes because $\boldsymbol{g}$ is zero on Γ_w. Identity (77) now follows from (78), (79). □

I.3.2 Extension of the function $\boldsymbol{g}$ from $\partial\Omega$ onto Ω. The next lemma, which immediately follows from Lemma 4.1 in [21] or from Lemma 8.3.19 in [7], shows that the function $\boldsymbol{g}$, constructed in the previous subsection, can be suitably extended onto the whole domain Ω.

Lemma 6 *Let $\boldsymbol{g} \in H^{1/2}(\partial\Omega)^2$ and let it satisfy (77). Then there exists a function $\boldsymbol{g}^* \in H^1(\Omega)^2$ such that*

$$\boldsymbol{g}^*\Big|_{\partial\Omega} = \boldsymbol{g} \quad \textit{(in the sense of traces)}, \tag{80}$$

$$\mathrm{div}\, \boldsymbol{g}^* = 0 \quad \text{a.e. in } \Omega, \tag{81}$$

$$\|\boldsymbol{g}^*\|_{H^1(\Omega)^2} \leq c_{13} \, \|\boldsymbol{g}\|_{H^{1/2}(\partial\Omega)^2}, \tag{82}$$

where the positive constant c_{13} is independent of $\boldsymbol{g}$.

I.3.3 Realization of the boundary condition on Γ_i. The solution $\boldsymbol{u} \in H^1(\Omega)^2$ of the weak problem in the domain Ω (subsection I.2.6) will be constructed in the form $\boldsymbol{u} := \boldsymbol{g}^* + \boldsymbol{z}$ where $\boldsymbol{z}$ is an appropriate function from space V. Then, obviously, the sum $\boldsymbol{u} \equiv \boldsymbol{g}^* + \boldsymbol{z}$ belongs to $H^1(\Omega)^2$ and

$$\boldsymbol{u}|_{\Gamma_i} = \boldsymbol{g}, \tag{83}$$

$$\boldsymbol{u}|_{\Gamma_w} = \boldsymbol{0}, \tag{84}$$

$$\boldsymbol{u}(x_1, x_2 + \tau) = \boldsymbol{u}(x_1, x_2) \qquad \text{for } (x_1, x_2) \in \Gamma_-, \tag{85}$$

$$\mathrm{div}\, \boldsymbol{u} = 0 \qquad \text{a.e. in } \Omega. \tag{86}$$

(Equalities (83)–(85) are satisfied in the sense of traces.) Substituting $\boldsymbol{u} = \boldsymbol{g}^* + \boldsymbol{z}$ into equation (42), we obtain a problem for the new unknown function $\boldsymbol{z} \in V$:

$$a(\boldsymbol{g}^* + \boldsymbol{z}, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}) + b(\boldsymbol{h}, \boldsymbol{v}), \quad \forall\, \boldsymbol{v} \in V. \tag{87}$$

The next theorem will play a fundamental role in the proof of the existence of a weak solution in subsections I.3.6–I.3.8.

I.3.4 Theorem 5. There exist positive constants c_{14} and c_{15} such that

$$\begin{aligned} a(\boldsymbol{g}^* + \boldsymbol{z}, \boldsymbol{z}) \;\geq\; & |||\boldsymbol{z}||| \Big(\nu\, |||\boldsymbol{z}||| - \nu\, c_{13}\, c_7\, \|\boldsymbol{g}\|_{H^s(\Gamma_i)^2} - c_{14}\, \|\boldsymbol{g}\|^2_{H^s(\Gamma_i)^2} \\ & - c_{15}\, \|\boldsymbol{g}\|_{H^s(\Gamma_i)^2}\, |||\boldsymbol{z}||| \Big) \end{aligned} \tag{88}$$

for all $\boldsymbol{z} \in V$. (c_7 is the constant from Lemma 4 and c_{13} is the constant from Lemma 6.)

Proof. Using the definitions of the forms a, a_1, a_2 and a_3 in subsection I.2.5, we find that

$$\begin{aligned} a(\boldsymbol{g}^* + \boldsymbol{z}, \boldsymbol{z}) \;=\; & a_1(\boldsymbol{z}, \boldsymbol{z}) + a_1(\boldsymbol{g}^*, \boldsymbol{z}) \\ & + a_2(\boldsymbol{g}^*, \boldsymbol{g}^*, \boldsymbol{z}) + a_2(\boldsymbol{g}^*, \boldsymbol{z}, \boldsymbol{z}) + a_2(\boldsymbol{z}, \boldsymbol{g}^*, \boldsymbol{z}) + a_2(\boldsymbol{z}, \boldsymbol{z}, \boldsymbol{z}) \\ & + a_3(\boldsymbol{g}^*, \boldsymbol{g}^*, \boldsymbol{z}) + a_3(\boldsymbol{g}^*, \boldsymbol{z}, \boldsymbol{z}) + a_3(\boldsymbol{z}, \boldsymbol{g}^*, \boldsymbol{z}) + a_3(\boldsymbol{z}, \boldsymbol{z}, \boldsymbol{z}). \end{aligned}$$

This implies that

$$\begin{aligned} a(\boldsymbol{g}^* + \boldsymbol{z}, \boldsymbol{z}) \;\geq\; & a_1(\boldsymbol{z}, \boldsymbol{z}) - |a_1(\boldsymbol{g}^*, \boldsymbol{z})| \\ & - |a_2(\boldsymbol{g}^*, \boldsymbol{g}^*, \boldsymbol{z})| - |a_2(\boldsymbol{g}^*, \boldsymbol{z}, \boldsymbol{z})| - |a_2(\boldsymbol{z}, \boldsymbol{g}^*, \boldsymbol{z})| + a_2(\boldsymbol{z}, \boldsymbol{z}, \boldsymbol{z}) \\ & - |a_3(\boldsymbol{g}^*, \boldsymbol{g}^*, \boldsymbol{z})| - |a_3(\boldsymbol{g}^*, \boldsymbol{z}, \boldsymbol{z})| - |a_3(\boldsymbol{z}, \boldsymbol{g}^*, \boldsymbol{z})| + a_3(\boldsymbol{z}, \boldsymbol{z}, \boldsymbol{z}). \end{aligned} \tag{89}$$

The first term on the right hand side of (89) is, by definition, equal to $\nu\, |||\boldsymbol{z}|||^2$ (see subsection I.2.5). The other terms can be estimated by means of the Cauchy inequality, the continuous imbedding of $H^1(\Omega)$ into $L^4(\Omega)$, inequalities (28), Green's theorem and the theorem on traces:

$$\begin{aligned} |a_1(\boldsymbol{g}^*, \boldsymbol{z})| \;=\; & \left| \nu \int_\Omega \sum_{i,j=1}^2 \frac{\partial g_i^*}{\partial x_j} \frac{\partial z_i}{\partial x_j}\, \mathrm{d}\boldsymbol{x} \right| \;\leq \\ \leq\; & \nu \left(\int_\Omega |\nabla \boldsymbol{g}^*|^2\, \mathrm{d}\boldsymbol{x} \right)^{1/2} \left(\int_\Omega |\nabla \boldsymbol{z}|^2\, \mathrm{d}\boldsymbol{x} \right)^{1/2} \;\leq\; \nu\, \|\boldsymbol{g}^*\|_{H^1(\Omega)^2}\, |||\boldsymbol{z}|||, \end{aligned} \tag{90}$$

$$\begin{aligned} |a_2(\boldsymbol{g}^*, \boldsymbol{g}^*, \boldsymbol{z})| \;=\; & \left| \int_\Omega \sum_{i,j=1}^2 g_j^* \frac{\partial g_i^*}{\partial x_j}\, z_i\, \mathrm{d}\boldsymbol{x} \right| \\ \leq\; & \sum_{i,j=1}^2 \left(\int_\Omega \left| \frac{\partial g_i^*}{\partial x_j} \right|^2 \mathrm{d}\boldsymbol{x} \right)^{1/2} \left(\int_\Omega g_j^{*2}\, z_i^2\, \mathrm{d}\boldsymbol{x} \right)^{1/2} \\ \leq\; & c_{16}\, \|\boldsymbol{g}^*\|_{H^1(\Omega)^2} \sum_{i,j=1}^2 \left(\int_\Omega g_j^{*4}\, \mathrm{d}\boldsymbol{x} \right)^{1/4} \left(\int_\Omega z_i^4\, \mathrm{d}\boldsymbol{x} \right)^{1/4} \\ \leq\; & c_{17}\, \|\boldsymbol{g}^*\|^2_{H^1(\Omega)^2}\, |||\boldsymbol{z}|||, \end{aligned} \tag{91}$$

$$\begin{aligned} |a_2(\boldsymbol{g}^*, \boldsymbol{z}, \boldsymbol{z})| \;=\; & \left| \int_\Omega \sum_{i,j=1}^2 g_j^* \frac{\partial z_i}{\partial x_j}\, z_i\, \mathrm{d}\boldsymbol{x} \right| \\ \leq\; & \sum_{i,j=1}^2 \left(\int_\Omega \left| \frac{\partial z_i}{\partial x_j} \right|^2 \mathrm{d}\boldsymbol{x} \right)^{1/2} \left(\int_\Omega (g_j^*)^2 z_i^2\, \mathrm{d}\boldsymbol{x} \right)^{1/2} \end{aligned}$$

$$
\begin{aligned}
&\le \sum_{i,j=1}^{2} \left(\int_\Omega \Big|\frac{\partial z_i}{\partial x_j}\Big|^2 \,\mathrm{d}\boldsymbol{x} \right)^{1/2} \left(\int_\Omega (g_j^*)^4 \,\mathrm{d}\boldsymbol{x} \right)^{1/4} \left(\int_\Omega z_i^4 \,\mathrm{d}\boldsymbol{x} \right)^{1/4} \\
&\le c_{18} |\!|\!| \boldsymbol{z} |\!|\!| \, \|\boldsymbol{g}^*\|_{L^4(\Omega)} \|\boldsymbol{z}\|_{L^4(\Omega)} \le c_{19} |\!|\!| \boldsymbol{z} |\!|\!|^2 \|\boldsymbol{g}^*\|_{H^1(\Omega)}. \qquad (92)
\end{aligned}
$$

$$
\begin{aligned}
|a_2(\boldsymbol{z}, \boldsymbol{g}^*, \boldsymbol{z})| &= \left| \int_\Omega \sum_{i,j=1}^{2} z_j \frac{\partial g_i^*}{\partial x_j} z_i \,\mathrm{d}\boldsymbol{x} \right| \le \left(\int_\Omega |\boldsymbol{z}|^4 \,\mathrm{d}\boldsymbol{x} \right)^{1/2} \left(\int_\Omega |\nabla \boldsymbol{g}^*|^2 \,\mathrm{d}\boldsymbol{x} \right)^{1/2} \\
&\le \|\boldsymbol{z}\|^2_{L^4(\Omega)^2} \|\boldsymbol{g}^*\|_{H^1(\Omega)^2} \le c_{20} |\!|\!| \boldsymbol{z} |\!|\!|^2 \|\boldsymbol{g}^*\|_{H^1(\Omega)^2}, \qquad (93)
\end{aligned}
$$

$$
\begin{aligned}
a_2(\boldsymbol{z}, \boldsymbol{z}, \boldsymbol{z}) + a_3(\boldsymbol{z}, \boldsymbol{z}, \boldsymbol{z}) &= \int_\Omega \sum_{i,j=1}^{2} z_j \frac{\partial z_i}{\partial x_j} z_i \,\mathrm{d}\boldsymbol{x} + \frac{1}{2} \int_{\Gamma_o} (\boldsymbol{z} \cdot \boldsymbol{n})^- |\boldsymbol{z}|^2 \,\mathrm{d}S \\
&= \frac{1}{2} \int_\Omega \sum_{i,j=1}^{2} z_j \frac{\partial z_i^2}{\partial x_j} \,\mathrm{d}\boldsymbol{x} + \frac{1}{2} \int_{\Gamma_o} (\boldsymbol{z} \cdot \boldsymbol{n})^- |\boldsymbol{z}|^2 \,\mathrm{d}S \\
&= \frac{1}{2} \int_{\partial\Omega} (\boldsymbol{z} \cdot \boldsymbol{n}) |\boldsymbol{z}|^2 \,\mathrm{d}S - \frac{1}{2} \int_\Omega \operatorname{div} \boldsymbol{z} \, |\boldsymbol{z}|^2 \,\mathrm{d}\boldsymbol{x} + \frac{1}{2} \int_{\Gamma_o} (\boldsymbol{z} \cdot \boldsymbol{n})^- |\boldsymbol{z}|^2 \,\mathrm{d}S \\
&= \frac{1}{2} \int_{\Gamma_o} \left[(\boldsymbol{z} \cdot \boldsymbol{n}) + (\boldsymbol{z} \cdot \boldsymbol{n})^- \right] |\boldsymbol{z}|^2 \,\mathrm{d}S = \frac{1}{2} \int_{\Gamma_o} (\boldsymbol{z} \cdot \boldsymbol{n})^+ |\boldsymbol{z}|^2 \,\mathrm{d}S \ge 0, \qquad (94)
\end{aligned}
$$

$$
\begin{aligned}
|a_3(\boldsymbol{g}^*, \boldsymbol{g}^*, \boldsymbol{z})| &= \left| \frac{1}{2} \int_{\Gamma_o} (\boldsymbol{g}^* \cdot \boldsymbol{n})^- (\boldsymbol{g}^* \cdot \boldsymbol{z}) \,\mathrm{d}S \right| \le \frac{1}{2} \int_{\Gamma_o} |\boldsymbol{g}^*|^2 |\boldsymbol{z}| \,\mathrm{d}S \\
&\le \frac{1}{2} \left(\int_{\Gamma_o} |\boldsymbol{g}^*|^4 \,\mathrm{d}S \right)^{1/2} \left(\int_{\Gamma_o} |\boldsymbol{z}|^2 \,\mathrm{d}S \right)^{1/2} \le c_{21} \|\boldsymbol{g}^*\|^2_{H^1(\Omega)^2} \|\boldsymbol{z}\|_{H^1(\Omega)^2} \\
&\le c_{22} \|\boldsymbol{g}^*\|^2_{H^1(\Omega)^2} |\!|\!| \boldsymbol{z} |\!|\!|, \qquad (95)
\end{aligned}
$$

$$
\begin{aligned}
|a_3(\boldsymbol{g}^*, \boldsymbol{z}, \boldsymbol{z})| &= \left| \frac{1}{2} \int_{\Gamma_o} (\boldsymbol{g}^* \cdot \boldsymbol{n})^- |\boldsymbol{z}|^2 \,\mathrm{d}S \right| \le \frac{1}{2} \int_{\Gamma_o} |\boldsymbol{g}^*| \, |\boldsymbol{z}|^2 \,\mathrm{d}S \\
&\le \frac{1}{2} \left(\int_{\Gamma_o} |\boldsymbol{g}^*|^2 \,\mathrm{d}S \right)^{1/2} \left(\int_{\Gamma_o} |\boldsymbol{z}|^4 \,\mathrm{d}S \right)^{1/2} \le c_{23} \|\boldsymbol{g}^*\|_{H^1(\Omega)^2} \|\boldsymbol{z}\|^2_{H^1(\Omega)^2} \\
&\le c_{24} \|\boldsymbol{g}^*\|_{H^1(\Omega)^2} |\!|\!| \boldsymbol{z} |\!|\!|^2, \qquad (96)
\end{aligned}
$$

$$
\begin{aligned}
|a_3(\boldsymbol{z}, \boldsymbol{g}^*, \boldsymbol{z})| &= \left| \frac{1}{2} \int_{\Gamma_o} (\boldsymbol{z} \cdot \boldsymbol{n})^- \boldsymbol{g}^* \cdot \boldsymbol{z} \,\mathrm{d}S \right| \le \frac{1}{2} \int_{\Gamma_o} |\boldsymbol{g}^*| \, |\boldsymbol{z}|^2 \,\mathrm{d}S \\
&\le \frac{1}{2} \left(\int_{\Gamma_o} |\boldsymbol{g}^*|^2 \,\mathrm{d}S \right)^{1/2} \left(\int_{\Gamma_o} |\boldsymbol{z}|^4 \,\mathrm{d}S \right)^{1/2} \le c_{25} \|\boldsymbol{g}^*\|_{H^1(\Omega)^2} \|\boldsymbol{z}\|^2_{H^1(\Omega)^2} \\
&\le c_{26} \|\boldsymbol{g}^*\|_{H^1(\Omega)^2} |\!|\!| \boldsymbol{z} |\!|\!|^2. \qquad (97)
\end{aligned}
$$

Substituting (90)–(97) into (89) and using (74), (82), we get

$$
\begin{aligned}
a(\boldsymbol{g}^* + \boldsymbol{z}, \boldsymbol{z}) \ge \nu |\!|\!| \boldsymbol{z} |\!|\!|^2 - \nu \|\boldsymbol{g}^*\|_{H^1(\Omega)^2} |\!|\!| \boldsymbol{z} |\!|\!| - (c_{17} + c_{22}) \|\boldsymbol{g}^*\|^2_{H^1(\Omega)^2} |\!|\!| \boldsymbol{z} |\!|\!| \\
- (c_{19} + c_{20} + c_{24} + c_{26}) \|\boldsymbol{g}^*\|_{H^1(\Omega)^2} |\!|\!| \boldsymbol{z} |\!|\!|^2
\end{aligned}
$$

$$\geq \; |||\boldsymbol{z}||| \Big(\nu \, |||\boldsymbol{z}||| - \nu \, \|\boldsymbol{g}^*\|_{H^1(\Omega)^2} - (c_{17} + c_{22}) \, \|\boldsymbol{g}^*\|^2_{H^1(\Omega)^2}$$
$$- (c_{19} + c_{20} + c_{24} + c_{26}) \, \|\boldsymbol{g}^*\|_{H^1(\Omega)^2} \, |||\boldsymbol{z}||| \Big).$$
$$\geq \; |||\boldsymbol{z}||| \Big(\nu \, |||\boldsymbol{z}||| - \nu \, c_{13} \, c_7 \, \|\boldsymbol{g}\|_{H^s(\Gamma_i)^2} - (c_{17} + c_{22}) \, c_{13}^2 \, c_7^2 \, \|\boldsymbol{g}\|^2_{H^s(\Gamma_i)^2}$$
$$- (c_{19} + c_{20} + c_{24} + c_{26}) \, c_{13} \, c_7 \, \|\boldsymbol{g}\|_{H^s(\Gamma_i)^2} \, |||\boldsymbol{z}||| \Big). \tag{98}$$

Finally, putting $c_{14} = (c_{17} + c_{22}) \, c_{13}^2 \, c_7^2$ and $c_{15} = (c_{19} + c_{20} + c_{24} + c_{26}) \, c_{13} \, c_7$, we arrive at inequality (88). □

I.3.5 Theorem 6. If the function $\boldsymbol{g}$, given originally on Γ_i, is so small that

$$c_{15} \, \|\boldsymbol{g}\|_{H^s(\Gamma_i)^2} \; < \; \nu \tag{99}$$

(where the constant c_{15} is defined at the end of the proof of Theorem 5 (I.3.4)) then the form $a(\boldsymbol{g}^* + \boldsymbol{z}, \boldsymbol{z})$ is coercive in the space V. It means that

$$\lim_{|||z||| \to +\infty} a(\boldsymbol{g}^* + \boldsymbol{z}, \boldsymbol{z}) \; = \; +\infty. \tag{100}$$

Proof. It follows from inequality (88) that if the norm $\|\boldsymbol{g}\|_{H^s(\Gamma_i)^2}$ satisfies (99), then for $|||\boldsymbol{z}|||$ large enough, we have

$$\Big(\nu - c_{15} \, \|\boldsymbol{g}\|_{H^s(\Gamma_i)^2} \Big) \, |||\boldsymbol{z}||| \; - \; \nu \, c_{13} \, c_7 \, \|\boldsymbol{g}\|_{H^s(\Gamma_i)^2} \; - \; c_{14} \, \|\boldsymbol{g}\|^2_{H^s(\Gamma_i)^2} \; \geq \; c_{27},$$

where c_{27} is a positive constant. This inequality and (88) imply that

$$\lim_{|||z||| \to +\infty} a(\boldsymbol{g}^* + \boldsymbol{z}, \boldsymbol{z}) \; \geq \; \lim_{|||z||| \to +\infty} c_{27} \, |||\boldsymbol{z}||| \; = \; +\infty. \qquad \square$$

I.3.6 Construction of approximations and their estimates. The space V is, as a closed subspace of $H^1(\Omega)^2$, a separable Hilbert space with the same scalar product as in $H^1(\Omega)^2$. Using the theorem on equivalent norms in V (subsection I.2.2), it can easily be verified that

$$(\boldsymbol{u}, \boldsymbol{v})_V \; := \; \int_\Omega \sum_{i,j=1}^{2} \frac{\partial u_i}{\partial x_j} \, \frac{\partial v_i}{\partial x_j} \, \mathrm{d}\boldsymbol{x}. \qquad ((\boldsymbol{u}, \boldsymbol{v}))$$

is also a scalar product in V, equivalent with the scalar product $(.\,,\,.)_{H^1(\Omega)^2}$. The norm $|||\,.\,|||$ is induced by the scalar product $(.\,,\,.)_V$.

Space V has a separable basis. Since $\mathcal{V}$ is dense in V, the basis can be chosen so that all its elements are from $\mathcal{V}$. Thus, each function of the basis is infinitely differentiable. Applying the Schmidt orthogonalization procedure, we can modify the basis so that it becomes orthonormal with respect to the scalar product $(.\,,\,.)_V$. We shall further denote the functions of this basis by $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, ... For any $n \in \mathbb{N}$ we put

$$V_n \; := \; \mathcal{L}\{\boldsymbol{e}_1, \boldsymbol{e}_2, \; \dots \; , \boldsymbol{e}_n\}$$

(the linear hull of the first n functions $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\ldots$, $\boldsymbol{e}_n$). The approximate solution $\boldsymbol{z}_n$ of problem (87) in space V_n has the form

$$\boldsymbol{z}_n \,=\, \sum_{k=1}^{n} \vartheta_k\, \boldsymbol{e}_k\,. \tag{101}$$

If we denote $\boldsymbol{\vartheta} = [\vartheta_1, \ldots, \vartheta_n]$ and $|\boldsymbol{\vartheta}| := \left(\sum_{k=1}^{n} \vartheta_k^2\right)^{1/2}$, then

$$|||\boldsymbol{z}_n||| \,=\, \sqrt{(\boldsymbol{z}_n, \boldsymbol{z}_n)_V} \,=\, \left(\sum_{k,l=1}^{n} \vartheta_k\, \vartheta_l\, (\boldsymbol{e}_k, \boldsymbol{e}_l)_V\right)^{1/2} \,=\, |\boldsymbol{\vartheta}|. \tag{102}$$

The approximate solution $\boldsymbol{z}_n \in V_n$ is defined by the requirement that

$$a(\boldsymbol{g}^* + \boldsymbol{z}_n, \boldsymbol{v}) \,=\, (\boldsymbol{f}, \boldsymbol{v}) + b(\boldsymbol{h}, \boldsymbol{v}) \qquad \forall\, \boldsymbol{v} \in V_n\,. \tag{103}$$

This is equivalent with n equations

$$a(\boldsymbol{g}^* + \boldsymbol{z}_n, \boldsymbol{e}_k) \,=\, (\boldsymbol{f}, \boldsymbol{e}_k) + b(\boldsymbol{h}, \boldsymbol{e}_k) \qquad (k = 1, 2, \ldots, n). \tag{104}$$

Since $a = a_1 + a_2 + a_3$, we get

$$\begin{aligned} &a_1(\boldsymbol{g}^* + \boldsymbol{z}_n, \boldsymbol{e}_k) \,+\, a_2(\boldsymbol{g}^* + \boldsymbol{z}_n, \boldsymbol{g}^* + \boldsymbol{z}_n, \boldsymbol{e}_k) \,+\, a_3(\boldsymbol{g}^* + \boldsymbol{z}_n, \boldsymbol{g}^* + \boldsymbol{z}_n, \boldsymbol{e}_k) \\ &\quad =\, (\boldsymbol{f}, \boldsymbol{e}_k) + b(\boldsymbol{h}, \boldsymbol{e}_k) \qquad (k = 1, 2, \ldots, n). \end{aligned} \tag{105}$$

Substituting $\boldsymbol{z}_n$ in the form (101) into (105), we obtain

$$\begin{aligned} &a_1(\boldsymbol{g}^*, \boldsymbol{e}_k) + \sum_{l=1}^{n} \vartheta_l\, a_1(\boldsymbol{e}_l, \boldsymbol{e}_k) + a_2(\boldsymbol{g}^*, \boldsymbol{g}^*, \boldsymbol{e}_k) + \sum_{l=1}^{n} \vartheta_l\, \big[a_2(\boldsymbol{g}^*, \boldsymbol{e}_l, \boldsymbol{e}_k) + a_2(\boldsymbol{e}_l, \boldsymbol{g}^*, \boldsymbol{e}_k)\big] \\ &\quad + \sum_{l,m=1}^{n} \vartheta_l \vartheta_m\, a_2(\boldsymbol{e}_l, \boldsymbol{e}_m, \boldsymbol{e}_k) + a_3(\boldsymbol{g}^*, \boldsymbol{g}^*, \boldsymbol{e}_k) + \sum_{l=1}^{n} \vartheta_l\, \big[a_3(\boldsymbol{g}^*, \boldsymbol{e}_l, \boldsymbol{e}_k) + a_3(\boldsymbol{e}_l, \boldsymbol{g}^*, \boldsymbol{e}_k)\big] \\ &\quad + \sum_{l,m=1}^{n} \vartheta_l \vartheta_m\, a_3(\boldsymbol{e}_l, \boldsymbol{e}_m, \boldsymbol{e}_k) - (\boldsymbol{f}, \boldsymbol{e}_k) - b(\boldsymbol{h}, \boldsymbol{e}_k) \,=\, 0, \qquad (k = 1, 2, \ldots, n). \end{aligned} \tag{106}$$

This is a system of n quadratic equations for the unknown coefficients ϑ_1, ϑ_2, $\ldots$, ϑ_n. If we denote by $A_k(\boldsymbol{\vartheta})$ the left–hand side of the k–th equation and put

$$\mathcal{A}(\boldsymbol{\vartheta}) \,:=\, \big[\mathcal{A}_1(\boldsymbol{\vartheta}),\, \ldots,\, \mathcal{A}_n(\boldsymbol{\vartheta})\big], \tag{107}$$

system (106) can simply be written in the form of one equation

$$\mathcal{A}(\boldsymbol{\vartheta}) \,=\, \boldsymbol{0} \tag{108}$$

in $\mathbb{R}^n$. In order to prove the existence of a solution to this equation, we shall apply the following lemma which can be found in [7] (Lemma 4.1.53, p. 230) or in [55] (Lemma II.1.4, p. 164).

Lemma 7 *Let $\mathcal{A}$ be a continuous mapping of $\mathbb{R}^n$ into $\mathbb{R}^n$. If there exists $R > 0$ such that*

$$\mathcal{A}(\boldsymbol{\vartheta}) \cdot \boldsymbol{\vartheta} \, > \, 0 \tag{109}$$

for all $\boldsymbol{\vartheta} \in \mathbb{R}^n$ such that $|\boldsymbol{\vartheta}| = R$, then the equation

$$\mathcal{A}(\boldsymbol{\vartheta}) \, = \, \mathbf{0} \tag{110}$$

has a solution $\boldsymbol{\vartheta}$ in the closed ball $\overline{B_R(\mathbf{0})}$.

Clearly, our mapping $\mathcal{A}$ given by (107) is a continuous mapping of $\mathbb{R}^n$ to $\mathbb{R}^n$. Using inequalities (88), (28) and the theorem on traces, we successively deduce that

$$\begin{aligned}
\mathcal{A}(\boldsymbol{\vartheta}) \cdot \boldsymbol{\vartheta} \, &= \, \sum_{k=1}^{n} \mathcal{A}_k(\boldsymbol{\vartheta}) \, \vartheta_k \, = \, \sum_{k=1}^{n} \vartheta_k \, a_1(\boldsymbol{g}^*, \boldsymbol{e}_k) + \sum_{k,l=1}^{n} \vartheta_k \, \vartheta_l \, a_1(\boldsymbol{e}_l, \boldsymbol{e}_k) \\
&\quad + \sum_{k=1}^{n} \vartheta_k \, a_2(\boldsymbol{g}^*, \boldsymbol{g}^*, \boldsymbol{e}_k) + \sum_{k,l=1}^{n} \vartheta_k \, \vartheta_l \, [a_2(\boldsymbol{g}^*, \boldsymbol{e}_l, \boldsymbol{e}_k) + a_2(\boldsymbol{e}_l, \boldsymbol{g}^*, \boldsymbol{e}_k)] \\
&\quad + \sum_{k,l,m=1}^{n} \vartheta_k \, \vartheta_l \, \vartheta_m \, a_2(\boldsymbol{e}_l, \boldsymbol{e}_m, \boldsymbol{e}_k) + \sum_{k=1}^{n} \vartheta_k \, a_3(\boldsymbol{g}^*, \boldsymbol{g}^*, \boldsymbol{e}_k) \\
&\quad + \sum_{k,l=1}^{n} \vartheta_k \, \vartheta_l \, [a_3(\boldsymbol{g}^*, \boldsymbol{e}_l, \boldsymbol{e}_k) + a_3(\boldsymbol{e}_l, \boldsymbol{g}^*, \boldsymbol{e}_k)] + \sum_{k,l,m=1}^{n} \vartheta_k \, \vartheta_l \, \vartheta_m \, a_3(\boldsymbol{e}_l, \boldsymbol{e}_m, \boldsymbol{e}_k) \\
&\quad - \sum_{k=1}^{n} \vartheta_k \, (\boldsymbol{f}, \boldsymbol{e}_k) - \sum_{k=1}^{n} \vartheta_k \, b(\boldsymbol{h}, \boldsymbol{e}_k) \\
&= \, a_1(\boldsymbol{g}^*, \boldsymbol{z}_n) + a_1(\boldsymbol{z}_n, \boldsymbol{z}_n) + a_2(\boldsymbol{g}^*, \boldsymbol{g}^*, \boldsymbol{z}_n) + a_2(\boldsymbol{g}^*, \boldsymbol{z}_n, \boldsymbol{z}_n) + a_2(\boldsymbol{z}_n, \boldsymbol{g}^*, \boldsymbol{z}_n) \\
&\quad + a_3(\boldsymbol{g}^*, \boldsymbol{g}^*, \boldsymbol{z}_n) + a_3(\boldsymbol{g}^*, \boldsymbol{z}_n, \boldsymbol{z}_n) + a_3(\boldsymbol{z}_n, \boldsymbol{g}^*, \boldsymbol{z}_n) - (\boldsymbol{f}, \boldsymbol{z}_n) - b(\boldsymbol{h}, \boldsymbol{z}_n) \\
&= \, a(\boldsymbol{g}^* + \boldsymbol{z}_n, \boldsymbol{z}_n) - (\boldsymbol{f}, \boldsymbol{z}_n) - b(\boldsymbol{h}, \boldsymbol{z}_n) \\
&\geq \, |||\boldsymbol{z}_n||| \left(\nu \, |||\boldsymbol{z}_n||| - \nu \, c_{13} \, c_7 \, \|\boldsymbol{g}\|_{H^s(\Gamma_i)^2} - c_{14} \, \|\boldsymbol{g}\|^2_{H^s(\Gamma_i)^2} \, - \, c_{15} \, \|\boldsymbol{g}\|_{H^s(\Gamma_i)^2} \, |||\boldsymbol{z}_n||| \right) \\
&\quad - c_1 \, \|\boldsymbol{f}\|_{L^2(\Omega)^2} \, |||\boldsymbol{z}_n||| - c_{28} \, \|\boldsymbol{h}\|_{L^2(\Gamma_o)^2} \, |||\boldsymbol{z}_n||| \\
&= \, |\boldsymbol{\vartheta}| \left(\nu \, |\boldsymbol{\vartheta}| - \nu \, c_{13} \, c_7 \, \|\boldsymbol{g}\|_{H^s(\Gamma_i)^2} - c_{14} \, \|\boldsymbol{g}\|^2_{H^s(\Gamma_i)^2} \, - \, c_{15} \, \|\boldsymbol{g}\|_{H^s(\Gamma_i)^2} \, |\boldsymbol{\vartheta}| \right) \\
&\quad - c_1 \, \|\boldsymbol{f}\|_{L^2(\Omega)^2} \, |\boldsymbol{\vartheta}| - c_{28} \, \|\boldsymbol{h}\|_{L^2(\Gamma_o)^2} \, |\boldsymbol{\vartheta}| \\
&= \, |\boldsymbol{\vartheta}| \, \Big[\, |\boldsymbol{\vartheta}| \left(\nu - c_{15} \, \|\boldsymbol{g}\|_{H^s(\Gamma_i)^2} \right) \, - \, \nu \, c_{13} \, c_7 \, \|\boldsymbol{g}\|_{H^s(\Gamma_i)^2} \, - \, c_{14} \, \|\boldsymbol{g}\|^2_{H^s(\Gamma_i)^2} \\
&\quad - c_1 \, \|\boldsymbol{f}\|_{L^2(\Omega)^2} \, - \, c_{28} \, \|\boldsymbol{h}\|_{L^2(\Gamma_o)^2} \, \Big].
\end{aligned} \tag{111}$$

Assume that the norm $\|\boldsymbol{g}\|_{H^s(\Gamma_i)^2}$ is so small that (99) holds. Estimates (111) now show that if $|\boldsymbol{\vartheta}| = R_0$, where

$$R_0 \, = \, \frac{\nu \, c_{13} \, c_7 \, \|\boldsymbol{g}\|_{H^s(\Gamma_i)^2} + c_{14} \, \|\boldsymbol{g}\|^2_{H^s(\Gamma_i)^2} + c_1 \, \|\boldsymbol{f}\|_{L^2(\Omega)^2} + c_{28} \, \|\boldsymbol{h}\|_{L^2(\Gamma_o)^2}}{\nu - c_{15} \, \|\boldsymbol{g}\|_{H^s(\Gamma_i)^2}}, \tag{112}$$

then (109) holds. Due to Lemma 7, equation (108) has at least one solution $\boldsymbol{\vartheta}$ such that $|\boldsymbol{\vartheta}| \le R_0$. Thus, the function $\boldsymbol{z}_n$ given by (101) is a solution of problem (103). Since $|||\boldsymbol{z}_n||| = |\boldsymbol{\vartheta}|$ (see (102)), we have the estimate

$$|||\boldsymbol{z}_n||| \le R_0. \tag{113}$$

Thus, we have proven the theorem:

Theorem 7. If function $\boldsymbol{g}$ satisfies the condition of sufficient smallness (99) and $n \in \mathbb{N}$, then problem (103) has a solution $\boldsymbol{z}_n \in V_n$, which satisfies the estimate (113).

I.3.7 Convergence of approximate solutions. In what follows we shall assume that the norm
$\|\boldsymbol{g}\|_{H^s(\Gamma_i)^2}$ satisfies condition (99). Since space V is reflexive, the boundedness of the sequence $\{\boldsymbol{z}_n\}$ implies that there exist a subsequence which is weakly convergent in V. Because of simplicity of notation, we shall denote this subsequence again by $\{\boldsymbol{z}_n\}$. Thus, if we denote the limit function by $\boldsymbol{z}$, we have

$$\boldsymbol{z}_n \longrightarrow \boldsymbol{z} \quad \text{for } n \to +\infty \quad \text{weakly in } V. \tag{114}$$

The norm $|||\cdot|||$, used in V, is equivalent with the norm in $H^1(\Omega)^2$. Hence,

$$\boldsymbol{z}_n \longrightarrow \boldsymbol{z} \quad \text{for } n \to +\infty \quad \text{weakly in } H^1(\Omega)^2. \tag{115}$$

The space $H^1(\Omega)^2$ is compactly imbedded into $L^q(\Omega)^2$ for $1 \le q < +\infty$. (See, for example, [23], Theorem 1.3.) This implies that

$$\boldsymbol{z}_n \longrightarrow \boldsymbol{z} \quad \text{for } n \to +\infty \quad \text{strongly in } L^q(\Omega)^2 \tag{116}$$

for each $q \in [1, +\infty)$. The operator of traces from $H^1(\Omega)^2$ into $L^q(\partial\Omega)^2$ is compact for each $q \in [1, +\infty)$. (See, e.g. [40], Theorem 6.4.2.) Hence, it follows from (114) that

$$\boldsymbol{z}_n \longrightarrow \boldsymbol{z} \quad \text{for } n \to +\infty \quad \text{strongly in } L^q(\partial\Omega)^2$$

and, in particular,

$$\boldsymbol{z}_n \longrightarrow \boldsymbol{z} \quad \text{for } n \to +\infty \quad \text{strongly in } L^q(\Gamma_o)^2 \tag{117}$$

for each $q \in [1, +\infty)$. This implies that

$$\begin{aligned} \boldsymbol{z}_n \cdot \boldsymbol{n} \quad &\longrightarrow \quad \boldsymbol{z} \cdot \boldsymbol{n} \qquad \text{for } n \to +\infty \quad \text{strongly in } L^q(\Gamma_o), \\ (\boldsymbol{z}_n \cdot \boldsymbol{n})^- \quad &\longrightarrow \quad (\boldsymbol{z} \cdot \boldsymbol{n})^- \qquad \text{for } n \to +\infty \quad \text{strongly in } L^q(\Gamma_o)^2. \end{aligned} \tag{118}$$

Due to (117) and (118), the sequences $\{\boldsymbol{z}_n\}$ and $\{(\boldsymbol{z}_n \cdot \boldsymbol{n})^-\}$ are bounded in $L^q(\Gamma_o)^2$ and in $L^q(\Gamma_o))$ respectively, for each $q \in [1, +\infty)$.

I.3.8 The limit process in equation (103) . Let $\boldsymbol{g}$ satisfy the condition of sufficient smallness (99). Using the definition of the form a (subsection I.2.5), equation (103) can be written in the form

$$\begin{aligned} &a_1(\boldsymbol{g}^*, \boldsymbol{v}) + a_1(\boldsymbol{z}_n, \boldsymbol{v}) + a_2(\boldsymbol{g}^*, \boldsymbol{g}^*, \boldsymbol{v}) + a_2(\boldsymbol{z}_n, \boldsymbol{g}^*, \boldsymbol{v}) + a_2(\boldsymbol{g}^*, \boldsymbol{z}_n, \boldsymbol{v}) \\ &\quad + a_2(\boldsymbol{z}_n, \boldsymbol{z}_n, \boldsymbol{v}) + a_3(\boldsymbol{g}^*, \boldsymbol{g}^*, \boldsymbol{v}) + a_3(\boldsymbol{z}_n, \boldsymbol{g}^*, \boldsymbol{v}) + a_3(\boldsymbol{g}^*, \boldsymbol{z}_n, \boldsymbol{v}) \\ &\quad + a_3(\boldsymbol{z}_n, \boldsymbol{z}_n, \boldsymbol{v}) \;=\; (\boldsymbol{f}, \boldsymbol{v}) + b(\boldsymbol{h}, \boldsymbol{v}). \end{aligned} \tag{119}$$

Let $\boldsymbol{v} \in V_m$ for a fixed number $m \in \mathbb{N}$. (Then, $\boldsymbol{v}$ is infinitely differentiable in $\overline{\Omega}$). Further, let us assume that $n \in \mathbb{N}$, $n \geq m$. We shall use the notation $\boldsymbol{z}_n = (z_{n1}, z_{n2})$, $\boldsymbol{z} = (z_1, z_2)$, $\boldsymbol{g}^* = (g_1^*, g_2^*)$ and $\boldsymbol{v} = (v_1, v_2)$. It follows from (115) that

$$a_1(\boldsymbol{z}_n, \boldsymbol{v}) \;\longrightarrow\; a_1(\boldsymbol{z}, \boldsymbol{v}) \quad \text{for } n \to +\infty. \tag{120}$$

The products $(\partial g_i^*/\partial x_j)\, v_i$ $(i, j = 1, 2)$, appearing in the terms $a_2(\boldsymbol{z}_n, \boldsymbol{g}^*, \boldsymbol{v})$ and $a_2(\boldsymbol{z}, \boldsymbol{g}^*, \boldsymbol{v})$, belong to $L^2(\Omega)$. Thus, in view of (116), we have

$$\begin{aligned} \big|a_2(\boldsymbol{z}_n, \boldsymbol{g}^*, \boldsymbol{v}) - a_2(\boldsymbol{z}, \boldsymbol{g}^*, \boldsymbol{v})\big| \;&=\; \left| \int_\Omega \sum_{i,j=1}^2 (z_{nj} - z_j)\, \frac{\partial g_i^*}{\partial x_j}\, v_i \, \mathrm{d}\boldsymbol{x} \right| \\ &\leq\; c_{29} \left(\int_\Omega |\boldsymbol{z}_n - \boldsymbol{z}|^2 \, \mathrm{d}\boldsymbol{x} \right)^{1/2} \left(\int_\Omega \sum_{i,j=1}^2 \left| \frac{\partial g_i^*}{\partial x_j} \right|^2 |v_i|^2 \, \mathrm{d}\boldsymbol{x} \right)^{1/2} \end{aligned}$$

where the constant c_{29} is independent of n. This implies that

$$a_2(\boldsymbol{z}_n, \boldsymbol{g}^*, \boldsymbol{v}) \;\longrightarrow\; a_2(\boldsymbol{z}, \boldsymbol{g}^*, \boldsymbol{v}) \quad \text{for } n \to +\infty. \tag{121}$$

The term $a_2(\boldsymbol{g}^*, \boldsymbol{z}_n, \boldsymbol{v})$ can be treated similarly: the products $g_j^*\, v_i$ $(i, j = 1, 2)$ belong to $H^1(\Omega)^2$ and so

$$\begin{aligned} \big|a_2(\boldsymbol{g}^*, \boldsymbol{z}_n, \boldsymbol{v}) - a_2(\boldsymbol{g}^*, \boldsymbol{z}, \boldsymbol{v})\big| \;&=\; \left| \int_\Omega \sum_{i,j=1}^2 g_j^* \left(\frac{\partial z_{ni}}{\partial x_j} - \frac{\partial z_i}{\partial x_j} \right) v_i \, \mathrm{d}\boldsymbol{x} \right| \\ &\leq\; c_{30}\, |||\boldsymbol{z}_n - \boldsymbol{z}||| \left(\int_\Omega \sum_{i,j=1}^2 |g_j^*|^2\, |v_i|^2 \, \mathrm{d}\boldsymbol{x} \right)^{1/2}. \end{aligned}$$

Thus,

$$a_2(\boldsymbol{g}^*, \boldsymbol{z}_n, \boldsymbol{v}) \;\longrightarrow\; a_2(\boldsymbol{g}^*, \boldsymbol{z}, \boldsymbol{v}) \quad \text{for } n \to +\infty. \tag{122}$$

By analogy,

$$\begin{aligned} &\big|a_2(\boldsymbol{z}_n, \boldsymbol{z}_n, \boldsymbol{v}) - a_2(\boldsymbol{z}, \boldsymbol{z}, \boldsymbol{v})\big| \;\leq\; \big|a_2(\boldsymbol{z}_n, \boldsymbol{z}_n, \boldsymbol{v}) - a_2(\boldsymbol{z}, \boldsymbol{z}_n, \boldsymbol{v})\big| \\ &\qquad + \big|a_2(\boldsymbol{z}, \boldsymbol{z}_n, \boldsymbol{v}) - a_2(\boldsymbol{z}, \boldsymbol{z}, \boldsymbol{v})\big| \\ &\;=\; \left| \int_\Omega \sum_{i,j=1}^2 (z_{nj} - z_j)\, \frac{\partial z_{ni}}{\partial x_j}\, v_i \, \mathrm{d}\boldsymbol{x} \right| + \left| \int_\Omega \sum_{i,j=1}^2 z_j \left(\frac{\partial z_{ni}}{\partial x_j} - \frac{\partial z_i}{\partial x_j} \right) v_i \, \mathrm{d}\boldsymbol{x} \right| \end{aligned}$$

$$\leq \quad c_{31} \, \|\boldsymbol{z}_n - \boldsymbol{z}\|_{L^2(\Omega)^2} \, |||\boldsymbol{z}_n||| \; + \; \left| \int_\Omega \sum_{i,j=1}^{2} z_j \left(\frac{\partial z_{ni}}{\partial x_j} - \frac{\partial z_i}{\partial x_j} \right) v_i \, \mathrm{d}\boldsymbol{x} \right|$$

where c_{31} is a constant independent of n. The first term on the right–hand side tends to zero as $n \to +\infty$ due to (116) and to the boundedness of the sequence $\{\boldsymbol{z}_n\}$ in V. The second term tends to zero for $n \to +\infty$ as a result of (115). Thus,

$$a_2(\boldsymbol{z}_n, \boldsymbol{z}_n, \boldsymbol{v}) \; \longrightarrow \; a_2(\boldsymbol{z}, \boldsymbol{z}, \boldsymbol{v}) \quad \text{for } n \to +\infty. \tag{123}$$

In accordance with the definition of the form a_3, we have

$$a_3(\boldsymbol{g}^*, \boldsymbol{z}_n, \boldsymbol{v}) \; = \; \frac{1}{2} \int_{\Gamma_o} (\boldsymbol{g}^* \cdot \boldsymbol{n})^- \, \boldsymbol{z}_n \cdot \boldsymbol{v} \, \mathrm{d}S.$$

Term $(\boldsymbol{g}^* \cdot \boldsymbol{n})^- \, \boldsymbol{v} \in L^q(\Gamma_o)^2$ for each $q \geq 1$. Thus, using (117), we obtain

$$a_3(\boldsymbol{g}^*, \boldsymbol{z}_n, \boldsymbol{v}) \; \longrightarrow \; a_3(\boldsymbol{g}^*, \boldsymbol{z}, \boldsymbol{v}) \quad \text{for } n \to +\infty. \tag{124}$$

Similarly, using (118), we get

$$a_3(\boldsymbol{z}_n, \boldsymbol{g}^*, \boldsymbol{v}) \; = \; \frac{1}{2} \int_{\Gamma_o} (\boldsymbol{z}_n \cdot \boldsymbol{n})^- \, \boldsymbol{g}^* \cdot \boldsymbol{v} \, \mathrm{d}S \; \longrightarrow \; a_3(\boldsymbol{z}, \boldsymbol{g}^*, \boldsymbol{v}) \quad \text{for } n \to +\infty. \tag{125}$$

because $\boldsymbol{g}^* \cdot \boldsymbol{v} \in L^q(\Gamma_o)$ for each $q \geq 1$. Finally,

$$a_3(\boldsymbol{z}_n, \boldsymbol{z}_n, \boldsymbol{v}) \; = \; \frac{1}{2} \int_{\Gamma_o} (\boldsymbol{z}_n \cdot \boldsymbol{n})^- \, \boldsymbol{z}_n \cdot \boldsymbol{v} \, \mathrm{d}S.$$

This implies that

$$\begin{aligned}
|a_3(\boldsymbol{z}_n, \boldsymbol{z}_n, \boldsymbol{v}) - a_3(\boldsymbol{z}, \boldsymbol{z}, \boldsymbol{v})| \; &\leq \; |a_3(\boldsymbol{z}_n, \boldsymbol{z}_n, \boldsymbol{v}) - a_3(\boldsymbol{z}_n, \boldsymbol{z}, \boldsymbol{v})| \\
+ \quad & |a_3(\boldsymbol{z}_n, \boldsymbol{z}, \boldsymbol{v}) - a_3(\boldsymbol{z}, \boldsymbol{z}, \boldsymbol{v})| \\
= \quad & \left| \frac{1}{2} \int_{\Gamma_o} (\boldsymbol{z}_n \cdot \boldsymbol{n})^- \, (\boldsymbol{z}_n - \boldsymbol{z}) \cdot \boldsymbol{v} \, \mathrm{d}S \right| + \left| \frac{1}{2} \int_{\Gamma_o} \big((\boldsymbol{z}_n - \boldsymbol{z}) \cdot \boldsymbol{n} \big)^- \, \boldsymbol{z} \cdot \boldsymbol{v} \, \mathrm{d}S \right| \\
\leq \quad & c_{32} \left(\int_{\Gamma_o} \left| (\boldsymbol{z}_n \cdot \boldsymbol{n})^- \right|^2 \mathrm{d}S \right)^{1/2} \left(\int_{\Gamma_o} |\boldsymbol{z}_n - \boldsymbol{z}|^2 \, \mathrm{d}S \right)^{1/2} \\
& + c_{33} \left(\int_{\Gamma_o} \left| \big((\boldsymbol{z}_n - \boldsymbol{z}) \cdot \boldsymbol{n} \big)^- \right|^2 \mathrm{d}S \right)^{1/2} \left(\int_{\Gamma_o} |\boldsymbol{z}|^2 \, \mathrm{d}S \right)^{1/2}.
\end{aligned}$$

Using the boundedness of the sequence $\{(\boldsymbol{z}_n \cdot \boldsymbol{n})^-\}$ in $L^2(\Gamma_o)$, (117) (which implies that the first term on the right hand side tends to zero) and (118) (which implies that the second term on the right hand side tends to zero), we get

$$a_3(\boldsymbol{z}_n, \boldsymbol{z}_n, \boldsymbol{v}) \; \longrightarrow \; a_3(\boldsymbol{z}, \boldsymbol{z}, \boldsymbol{v}) \quad \text{for } n \to +\infty. \tag{126}$$

Thus, letting n tend to $+\infty$ in (119), we deduce that the equation

$$a(\boldsymbol{g}^* + \boldsymbol{z}, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}) + b(\boldsymbol{h}, \boldsymbol{v}) \tag{127}$$

holds for all $\boldsymbol{v} \in V_m$, where m is an arbitrary number from $\mathbb{N}$. It means that (127) is satisfied for all test functions $\boldsymbol{v} \in \cup_{m=1}^{+\infty} V_m$. The union of all spaces V_m is dense in V and terms in (127) are bounded linear functionals in dependence on $\boldsymbol{v} \in V$. From this we deduce that (127) holds for all $\boldsymbol{v} \in V$.

The function $\boldsymbol{z}$ is a solution of problem (87). Consequently, due to the explanation given in subsection I.3.3, the function $\boldsymbol{u} = \boldsymbol{g}^* + \boldsymbol{z}$ is a weak solution of the problem defined in subsection I.2.6. Solution $\boldsymbol{u}$ corresponds to a stationary flow through the cascade of profiles. It satisfies in a weak sense the Navier–Stokes equation (3), the equation of continuity (1), the boundary conditions (10)–(12) and the conditions of periodicity in the x_2–direction (13), (14). Due to (113), (82) and (74), $\boldsymbol{u}$ satisfies the inequality

$$\|\nabla \boldsymbol{u}\|_{L^2(\Omega)^2} \leq R_0 + \|\nabla \boldsymbol{g}^*\|_{L^2(\Omega)^2} \leq R_0 + c_{13}\, c_7\, \|\boldsymbol{g}\|_{H^s(\Gamma_i)^2} := R_1 \tag{128}$$

where R_0 is given by (112). Thus, we have proven the theorem:

Theorem 8 (on the existence of a weak solution). Suppose that the norm $\|\boldsymbol{g}\|_{H^s(\Gamma_i)^2}$ is so small that it satisfies inequality (99). Then the weak problem of a flow through the cascade, has a weak solution, defined in I.2.6, which satisfies the estimate (128).

I.4 Uniqueness of the weak solution

In this section, we study the question of uniqueness of the weak solution defined in subsection I.2.6. Let us recall that the uniqueness of a weak solution of the stationary Navier–Stokes equation with nonhomogeneous Dirichlet–type boundary data is known to hold only if certain norm of the boundary data and the external body force is "sufficiently small" in comparison with the viscosity. Then the weak solution can be constructed so that it lies in a "sufficiently small" ball and the theorem on uniqueness says that the solution is unique, not only among solutions in the same ball, but in the class of all weak solutions. (See e.g. the books by R. Temam [55] and G. P. Galdi [21]) The following theorem contains a result of a similar type. However, due to the additional nonlinearity in the boundary condition on Γ_o, the theorem brings the information only on uniqueness in the class of "sufficiently small" solutions.

I.4.1 Theorem 9 (on the uniqueness of a weak solution). There exists $R > 0$ such that if $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are two solutions of the problem I.2.6 such that $\|\nabla \boldsymbol{u}_1\|_{L^2(\Omega)^4} \leq R$ and $\|\nabla \boldsymbol{u}_2\|_{L^2(\Omega)^4} \leq R$ then $\boldsymbol{u}_1 = \boldsymbol{u}_2$.

Proof. Since $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are the solutions of the problem I.2.6, they fulfil the equations

$$\begin{aligned} a(\boldsymbol{u}_1, \boldsymbol{v}) &= (\boldsymbol{f}, \boldsymbol{v}) + b(\boldsymbol{h}, \boldsymbol{v}). \\ a(\boldsymbol{u}_2, \boldsymbol{v}) &= (\boldsymbol{f}, \boldsymbol{v}) + b(\boldsymbol{h}, \boldsymbol{v}) \end{aligned}$$

for all $\boldsymbol{v} \in V$. Subtracting these equations, we get

$$a(\boldsymbol{u}_1, \boldsymbol{v}) - a(\boldsymbol{u}_2, \boldsymbol{v}) = 0.$$

Expressing the bilinear form a by means of the forms a_1, a_2 and a_3 defined in subsection I.2.5, we obtain

$$a_1(\boldsymbol{u}_1, \boldsymbol{v}) - a_1(\boldsymbol{u}_2, \boldsymbol{v}) + a_2(\boldsymbol{u}_1, \boldsymbol{u}_1, \boldsymbol{v}) - a_2(\boldsymbol{u}_2, \boldsymbol{u}_2, \boldsymbol{v}) \\ + a_3(\boldsymbol{u}_1, \boldsymbol{u}_1, \boldsymbol{v}) - a_3(\boldsymbol{u}_2, \boldsymbol{u}_2, \boldsymbol{v}) = 0.$$

This holds for all $\boldsymbol{v} \in V$. If we choose $\boldsymbol{v} = \boldsymbol{u}_1 - \boldsymbol{u}_2$ then this identity yields

$$a_1(\boldsymbol{u}_1 - \boldsymbol{u}_2, \boldsymbol{u}_1 - \boldsymbol{u}_2) + a_2(\boldsymbol{u}_1, \boldsymbol{u}_1, \boldsymbol{u}_1 - \boldsymbol{u}_2) - a_2(\boldsymbol{u}_2, \boldsymbol{u}_2, \boldsymbol{u}_1 - \boldsymbol{u}_2) \\ + a_3(\boldsymbol{u}_1, \boldsymbol{u}_1, \boldsymbol{u}_1 - \boldsymbol{u}_2) - a_3(\boldsymbol{u}_2, \boldsymbol{u}_2, \boldsymbol{u}_1 - \boldsymbol{u}_2) = 0. \tag{129}$$

If we denote

$$\begin{aligned} I_1 &:= a_1(\boldsymbol{u}_1 - \boldsymbol{u}_2, \boldsymbol{u}_1 - \boldsymbol{u}_2) = \nu \int_\Omega |\nabla(\boldsymbol{u}_1 - \boldsymbol{u}_2)|^2 \, \mathrm{d}\boldsymbol{x} = \nu \, |||\boldsymbol{u}_1 - \boldsymbol{u}_2|||^2, \\ I_2 &:= a_2(\boldsymbol{u}_1, \boldsymbol{u}_1, \boldsymbol{u}_1 - \boldsymbol{u}_2) - a_2(\boldsymbol{u}_2, \boldsymbol{u}_2, \boldsymbol{u}_1 - \boldsymbol{u}_2), \\ I_3 &:= a_3(\boldsymbol{u}_1, \boldsymbol{u}_1, \boldsymbol{u}_1 - \boldsymbol{u}_2) - a_3(\boldsymbol{u}_2, \boldsymbol{u}_2, \boldsymbol{u}_1 - \boldsymbol{u}_2) \end{aligned}$$

then (129) takes the form

$$I_1 = -I_2 - I_3. \tag{130}$$

We shall further estimate the terms on the right hand side of (130).

$$\begin{aligned} |I_2| &= \left| \int_\Omega \boldsymbol{u}_1 \cdot \nabla \boldsymbol{u}_1 \cdot (\boldsymbol{u}_1 - \boldsymbol{u}_2) \, \mathrm{d}\boldsymbol{x} - \int_\Omega \boldsymbol{u}_2 \cdot \nabla \boldsymbol{u}_2 \cdot (\boldsymbol{u}_1 - \boldsymbol{u}_2) \, \mathrm{d}\boldsymbol{x} \right| \\ &\le \left| \int_\Omega (\boldsymbol{u}_1 - \boldsymbol{u}_2) \cdot \nabla \boldsymbol{u}_1 \cdot (\boldsymbol{u}_1 - \boldsymbol{u}_2) \, \mathrm{d}\boldsymbol{x} \right| + \left| \int_\Omega \boldsymbol{u}_2 \cdot \nabla(\boldsymbol{u}_1 - \boldsymbol{u}_2) \cdot (\boldsymbol{u}_1 - \boldsymbol{u}_2) \, \mathrm{d}\boldsymbol{x} \right| \\ &\le \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_{L^4(\Omega)^2}^2 \, \|\nabla \boldsymbol{u}_1\|_{L^2(\Omega)^4} + \|\boldsymbol{u}_2\|_{L^4(\Omega)^2} \, \|\nabla(\boldsymbol{u}_1 - \boldsymbol{u}_2)\|_{L^2(\Omega)^4} \, \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_{L^4(\Omega)^2} \\ &\le 2c_{34}^2 \, R \, \|\nabla(\boldsymbol{u}_1 - \boldsymbol{u}_2)\|_{L^2(\Omega)^4}^2 = 2c_{34}^2 \, R \, |||\boldsymbol{u}_1 - \boldsymbol{u}_2|||^2 \end{aligned} \tag{131}$$

where the constant c_{34} comes from the inequality

$$\|\boldsymbol{u}\|_{L^4(\Omega)^2} \le c_{34} \, \|\nabla \boldsymbol{u}\|_{L^2(\Omega)^4} \tag{132}$$

for functions $\boldsymbol{u}$ from $H^1(\Omega)^2$, satisfying the boundary condition (11); see Appendix Lemma A1. The term I_3 is

$$I_3 = \int_{\Gamma_o} \frac{1}{2} (\boldsymbol{u}_1 \cdot \boldsymbol{n})^- \, \boldsymbol{u}_1 \cdot (\boldsymbol{u}_1 - \boldsymbol{u}_2) \, \mathrm{d}S - \int_{\Gamma_o} \frac{1}{2} (\boldsymbol{u}_2 \cdot \boldsymbol{n})^- \, \boldsymbol{u}_2 \cdot (\boldsymbol{u}_1 - \boldsymbol{u}_2) \, \mathrm{d}S.$$

According to the signs of $\boldsymbol{u}_1 \cdot \boldsymbol{n}$ and $\boldsymbol{u}_2 \cdot \boldsymbol{n}$ on Γ_o, we must split Γ_o into four parts

$$\Gamma_o = \Gamma_{o1} \cup \Gamma_{o2} \cup \Gamma_{o3} \cup \Gamma_{o4},$$

where

a) on Γ_{o1} ... $\boldsymbol{u}_1 \cdot \boldsymbol{n} \geq 0, \ \ \boldsymbol{u}_2 \cdot \boldsymbol{n} \geq 0$ ... $(\boldsymbol{u}_1 \cdot \boldsymbol{n})^- = 0, \ \ (\boldsymbol{u}_2 \cdot \boldsymbol{n})^- = 0,$

b) on Γ_{o2} ... $\boldsymbol{u}_1 \cdot \boldsymbol{n} < 0, \ \ \boldsymbol{u}_2 \cdot \boldsymbol{n} \geq 0$ ... $(\boldsymbol{u}_1 \cdot \boldsymbol{n})^- = \boldsymbol{u}_1 \cdot \boldsymbol{n}, \ \ (\boldsymbol{u}_2 \cdot \boldsymbol{n})^- = 0,$

c) on Γ_{o3} ... $\boldsymbol{u}_1 \cdot \boldsymbol{n} \geq 0, \ \ \boldsymbol{u}_2 \cdot \boldsymbol{n} < 0$ ... $(\boldsymbol{u}_1 \cdot \boldsymbol{n})^- = 0, \ \ (\boldsymbol{u}_2 \cdot \boldsymbol{n})^- = \boldsymbol{u}_2 \cdot \boldsymbol{n},$

d) on Γ_{o4} ... $\boldsymbol{u}_1 \cdot \boldsymbol{n} < 0, \ \ \boldsymbol{u}_2 \cdot \boldsymbol{n} < 0$ ... $(\boldsymbol{u}_1 \cdot \boldsymbol{n})^- = \boldsymbol{u}_1 \cdot \boldsymbol{n}, \ \ (\boldsymbol{u}_2 \cdot \boldsymbol{n})^- = \boldsymbol{u}_2 \cdot \boldsymbol{n}.$

Let us denote by I_3^{o1}, I_3^{o2}, I_3^{o3} and I_3^{o4} the same integrals as in I_3, however considered successively on Γ_{o1}, Γ_{o2}, Γ_{o3} and Γ_{o4}. Obviously, $I_3^{o1} = 0$ because the integrands are equal to zero on Γ_{o1}. On Γ_{o2}, we use the inequality $|\boldsymbol{u}_1 \cdot \boldsymbol{n}| \leq |\boldsymbol{u}_1 \cdot \boldsymbol{n} - \boldsymbol{u}_2 \cdot \boldsymbol{n}|$, which holds because $\boldsymbol{u}_1 \cdot \boldsymbol{n} < 0$ and $\boldsymbol{u}_2 \cdot \boldsymbol{n} \geq 0$. Hence

$$
\begin{aligned}
|I_3^{o2}| &= \left| \int_{\Gamma_{o2}} (\boldsymbol{u}_1 \cdot \boldsymbol{n})^- \boldsymbol{u}_1 \cdot (\boldsymbol{u}_1 - \boldsymbol{u}_2) \, \mathrm{d}S \right| \\
&\leq \int_{\Gamma_{o2}} |\boldsymbol{u}_1 \cdot \boldsymbol{n} - \boldsymbol{u}_2 \cdot \boldsymbol{n}| \, |\boldsymbol{u}_1| \, |\boldsymbol{u}_1 - \boldsymbol{u}_2| \, \mathrm{d}S \leq \int_{\Gamma_{o2}} |\boldsymbol{u}_1 - \boldsymbol{u}_2|^2 \, |\boldsymbol{u}_1| \, \mathrm{d}S \\
&\leq \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|^2_{L^4(\Gamma_{o2})^2} \, \|\boldsymbol{u}_1\|_{L^2(\Gamma_{o2})^2} \leq \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|^2_{L^4(\Gamma_o)^2} \, \|\boldsymbol{u}_1\|_{L^2(\Gamma_o)^2} \\
&\leq c_{35} \, \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|^2_{H^1(\Omega)^2} \, \|\boldsymbol{u}_1\|_{H^1(\Omega)^2} \leq c_{36} \, |\!|\!|\boldsymbol{u}_1 - \boldsymbol{u}_2|\!|\!|^2 \, \|\nabla \boldsymbol{u}_1\|_{L^2(\Omega)^4} \\
&\leq c_{36} \, R \, |\!|\!|\boldsymbol{u}_1 - \boldsymbol{u}_2|\!|\!|^2
\end{aligned}
\tag{133}
$$

where the constants c_{35} and c_{36} come from the inequalities

$$
\|\boldsymbol{u}\|_{L^2(\Gamma_o)^2} \leq c_{35} \, \|\boldsymbol{u}\|_{H^1(\Omega)^2} \leq c_{36} \, \|\nabla \boldsymbol{u}\|_{L^2(\Omega)^4}
$$

for functions $\boldsymbol{u}$ from $H^1(\Omega)^2$, satisfying the boundary condition (11); see Appendix Lemma A1. The term I_3^{o3} can estimated in the same way as I_3^{o2}. The term I_3^{o4} can be treated as follows:

$$
\begin{aligned}
|I_3^{o4}| &= \left| \int_{\Gamma_{o4}} (\boldsymbol{u}_1 \cdot \boldsymbol{n}) \, \boldsymbol{u}_1 \cdot (\boldsymbol{u}_1 - \boldsymbol{u}_2) \, \mathrm{d}S - \int_{\Gamma_{o4}} (\boldsymbol{u}_2 \cdot \boldsymbol{n}) \, \boldsymbol{u}_2 \cdot (\boldsymbol{u}_1 - \boldsymbol{u}_2) \, \mathrm{d}S \right| \\
&= \left| \int_{\Gamma_{o4}} [(\boldsymbol{u}_1 - \boldsymbol{u}_2) \cdot \boldsymbol{n}] \, \boldsymbol{u}_1 \cdot (\boldsymbol{u}_1 - \boldsymbol{u}_2) \, \mathrm{d}S + \int_{\Gamma_{o4}} (\boldsymbol{u}_2 \cdot \boldsymbol{n}) \, (\boldsymbol{u}_1 - \boldsymbol{u}_2) \cdot (\boldsymbol{u}_1 - \boldsymbol{u}_2) \, \mathrm{d}S \right| \\
&\leq \int_{\Gamma_{o4}} |\boldsymbol{u}_1 - \boldsymbol{u}_2| \, |\boldsymbol{u}_1| \, |\boldsymbol{u}_1 - \boldsymbol{u}_2| \, \mathrm{d}S + \int_{\Gamma_{o4}} |\boldsymbol{u}_2| \, |\boldsymbol{u}_1 - \boldsymbol{u}_2| \, |\boldsymbol{u}_1 - \boldsymbol{u}_2| \, \mathrm{d}S \\
&\leq \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|^2_{L^4(\Gamma_{o4})^2} \left(\|\boldsymbol{u}_1\|_{L^2(\Gamma_{o4})^2} + \|\boldsymbol{u}_2\|_{L^2(\Gamma_{o4})^2} \right) \\
&\leq \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|^2_{L^4(\Gamma_o)^2} \left(\|\boldsymbol{u}_1\|_{L^2(\Gamma_o)^2} + \|\boldsymbol{u}_2\|_{L^2(\Gamma_o)^2} \right) \\
&\leq c_{37} \, \|\nabla \boldsymbol{u}_1 - \nabla \boldsymbol{u}_2\|_{L^2(\Omega)^4} \, c_{36} \left(\|\nabla \boldsymbol{u}_1\|_{L^2(\Omega)^4} + \|\nabla \boldsymbol{u}_2\|_{L^2(\Omega)^4} \right) \\
&\leq 2 c_{37} \, |\!|\!|\boldsymbol{u}_1 - \boldsymbol{u}_2|\!|\!|^2 \, c_{36} \, R
\end{aligned}
\tag{134}
$$

where the constant c_{37} comes from the inequality

$$
\|\boldsymbol{u}\|_{L^4(\Gamma_o)^2} \leq c_{37} \, \|\nabla \boldsymbol{u}\|_{L^2(\Omega)^4}
$$

for functions $\boldsymbol{u}$ from $H^1(\Omega)^2$, satisfying the boundary condition (11); see Appendix Lemma A1.

Substituting from (131), (133) and (134) into (130), we obtain

$$\nu\, |||\boldsymbol{u}_1 - \boldsymbol{u}_2|||^2 \;\leq\; (2c_{34}^2 + 2c_{36} + 2c_{36}c_{37})\, R\, |||\boldsymbol{u}_1 - \boldsymbol{u}_2|||^2.$$

Now it is seen that if R is so small that

$$\nu \;>\; (2c_{34}^2 + 2c_{36} + 2c_{36}c_{37})\, R$$

then $\boldsymbol{u}_1 = \boldsymbol{u}_2$. The theorem is proved. □

Corollary. If the given functions $\boldsymbol{g}$, $\boldsymbol{f}$ and $\boldsymbol{h}$ are so small in comparison with ν that the number R_1 (given by (128)) is less than or equal to R then the weak solution $\boldsymbol{u}$, constructed in subsections I.3.6–I.3.8, is unique in the class of weak solutions satisfying (128).

Chapter II

Stationary problem with modified boundary conditions on the outflow

II.1 The problem with a linear mixed boundary condition involving Bernoulli's pressure on the outflow

II.1.1 Equations of motion. In this section, we write the 2D Navier–Stokes system (I.2) in the form

$$-\,\omega(\boldsymbol{u})\,u_2 \;=\; -\frac{\partial q}{\partial x_1}+\nu\Delta u_1+f_1\,, \tag{1}$$

$$\omega(\boldsymbol{u})\,u_1 \;=\; -\frac{\partial q}{\partial x_2}+\nu\Delta u_2+f_2\,. \tag{2}$$

where

$$\omega(\boldsymbol{u}) \;:=\; \frac{\partial u_2}{\partial x_1}-\frac{\partial u_1}{\partial x_2}\,, \qquad q \;:=\; p+\frac{|\boldsymbol{u}|^2}{2}\,. \tag{3}$$

$\omega(\boldsymbol{u})$ is the vorticity of the flow and q is the so called Bernoulli pressure. If we denote $\boldsymbol{f}=(f_1,f_2)$ and

$$\boldsymbol{u}^{\perp} \;=\; (-u_2,u_1), \tag{4}$$

we can write the system (1), (2) as one vector equation:

$$\omega(\boldsymbol{u})\,\boldsymbol{u}^{\perp} \;=\; -\nabla q+\nu\Delta\boldsymbol{u}+\boldsymbol{f}. \tag{5}$$

As in Chapter I, this equation is accompanied by the condition of incompressibility

$$\operatorname{div}\boldsymbol{u} \;=\; 0. \tag{6}$$

II.1.2 Boundary conditions. We assume that the velocity $\boldsymbol{u}$ satisfies the same boundary conditions on Γ_i, Γ_+, Γ_- and Γ_w as in Chapter I, i.e. conditions (I.10), (I.11), (I.13) and (I.14):

$$\boldsymbol{u}\,|_{\Gamma_i} \;=\; \boldsymbol{g}, \tag{7}$$

$$\boldsymbol{u}\,|_{\Gamma_w} \;=\; \boldsymbol{0}, \tag{8}$$

$$\boldsymbol{u}(x_1,x_2+\tau) \;=\; \boldsymbol{u}(x_1,x_2) \qquad \text{for } (x_1,x_2)\in\Gamma_-, \tag{9}$$

$$\frac{\partial\boldsymbol{u}}{\partial\boldsymbol{n}}(x_1,x_2+\tau) \;=\; -\frac{\partial\boldsymbol{u}}{\partial\boldsymbol{n}}(x_1,x_2). \qquad \text{for } (x_1,x_2)\in\Gamma_-. \tag{10}$$

The Bernoulli pressure q is naturally supposed to be τ–periodic in the x_2–direction, i.e.

$$q(x_1, x_2 + \tau) \;=\; q(x_1, x_2) \qquad \text{for } (x_1, x_2) \in \Gamma_-. \tag{11}$$

The boundary condition used on the outflow Γ_o is

$$-\nu \,\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}} + q\boldsymbol{n} \;=\; \boldsymbol{h} \qquad \text{on } \Gamma_o. \tag{12}$$

We shall see in the next subsection that this condition naturally arises (as a boundary condition of the "do nothing" type) from the weak formulation of the problem.

II.1.3 Weak formulation of the problem in the domain Ω. In order to arrive formally at the weak formulation of the problem (5)–(11), we multiply equation (5) by an arbitrary test function $\boldsymbol{v} = (v_1, v_2) \in V$, integrate over Ω, apply Green's theorem and use the boundary conditions and the conditions of periodicity (7)–(11). We finally obtain the equation

$$\int_\Omega \omega(\boldsymbol{u})\, \boldsymbol{u}^\perp \cdot \boldsymbol{v}\, \mathrm{d}\boldsymbol{x} \;=\; \int_{\Gamma_o} \Big[-q\boldsymbol{n} + \nu\, \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}}\Big] \cdot \boldsymbol{v}\, \mathrm{d}S - \int_\Omega \nu\, \nabla \boldsymbol{u} \cdot \nabla \boldsymbol{v}\, \mathrm{d}\boldsymbol{x} + \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, \mathrm{d}\boldsymbol{x}.$$

This can be written in the form

$$a(\boldsymbol{u}, \boldsymbol{v}) \;=\; (\boldsymbol{f}, \boldsymbol{v}) + b(\boldsymbol{h}, \boldsymbol{v}) \tag{13}$$

where $\boldsymbol{h}$ is given by (12) and the form a has a similar structure as in subsection I.2.5:

$$a_1(\boldsymbol{u}, \boldsymbol{v}) \;=\; \nu\, (\nabla \boldsymbol{u}, \nabla \boldsymbol{v}), \tag{14}$$

$$a_2(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) \;=\; \int_\Omega \omega(\boldsymbol{u})\, \boldsymbol{v}^\perp \cdot \boldsymbol{w}\; \mathrm{d}\boldsymbol{x}, \tag{15}$$

$$a(\boldsymbol{u}, \boldsymbol{v}) \;=\; a_1(\boldsymbol{u}, \boldsymbol{v}) + a_2(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}), \tag{16}$$

$$b(\boldsymbol{h}, \boldsymbol{v}) \;=\; -\int_{\Gamma_o} \boldsymbol{h} \cdot \boldsymbol{v}\; \mathrm{d}S. \tag{17}$$

The weak problem now reads as follows:

Definition 1. Let function $\boldsymbol{g} \in H^s(\Gamma_i)^2$ (for some $s \in (\frac{1}{2}, 1]$) satisfy the condition $\boldsymbol{g}(A_1) = \boldsymbol{g}(A_0)$. (Recall that A_0 and A_1 are the end points of Γ_i.) Let $\boldsymbol{f} \in L^2(\Omega)^2$ and $\boldsymbol{h} \in L^2(\Gamma_o)^2$. The **weak solution** of the problem (5)–(11) is a vector function $\boldsymbol{u} \in H^1(\Omega)^2$ which satisfies the condition of incompressibility (6) a.e. in Ω, the identity (13) for all test functions $\boldsymbol{v} \in V$, the boundary conditions (7), (8) in the sense of traces on Γ_i and Γ_w and the condition of periodicity (9) in the sense of traces on Γ_- and Γ_+.

The weak solution $\boldsymbol{u}$ need not have the normal derivative on

Γ_o, hence the condition (12) does not generally make a sense. However, if $\boldsymbol{u}$ is smooth, we can reverse the procedure leading to equation (13) and show, by analogy with subsection I.2.10, that $\boldsymbol{u}$ satisfies (12) on Γ_o.

Similarly as in Theorem 3 (I.2.9), it can be shown that the weak solution in the domain Ω, extended periodically in the direction x_2 with the period τ, is a weak solution in the infinite and unbounded (in the direction of x_2) domain D.

The pressure does not explicitly appear in the definition of the weak solution, however, as it is usual in the theory of the Navier–Stokes equations, it can be defined on the level of distributions or it can be recovered as a function from $L^2(\Omega)$, if the weak solution $\boldsymbol{u}$ is sufficiently smooth.

Let us further denote by $\boldsymbol{g}^*$ the extension of the function $\boldsymbol{g}$ from the line segment Γ_i into Ω, constructed in subsections I.3.1 and I.3.2. Thus, $\boldsymbol{g}^*$ satisfies the estimates (I.74) and (I.82) and the conditions (I.80) and (I.81).

We can further seek for the weak solution $\boldsymbol{u}$ in the form $\boldsymbol{u} = \boldsymbol{g}^* + \boldsymbol{z}$ where $\boldsymbol{z} \in V$ is a new unknown function. This form of $\boldsymbol{u}$ guarantees that $\boldsymbol{u}$ satisfies the equation (7) and the boundary and periodicity conditions (8)–(10). Substituting the sum $\boldsymbol{g}^* + \boldsymbol{z}$ for $\boldsymbol{u}$ into the equation (5), we arrive at the following problem: Find a function $\boldsymbol{z} \in V$ such that it satisfies the equation

$$a(\boldsymbol{g}^* + \boldsymbol{z}, \boldsymbol{v}) \,=\, (\boldsymbol{f}, \boldsymbol{v}) \,+\, b(\boldsymbol{h}, \boldsymbol{v}) \tag{18}$$

for all $\boldsymbol{v} \in V$.

II.1.4 Coercivity of the form $a(\boldsymbol{g}^* + \boldsymbol{z}, \boldsymbol{z})$. The next two lemmas bring a sufficient condition for coercivity of the form $a(\boldsymbol{g}^* + \boldsymbol{z}, \boldsymbol{z})$.

Lemma 8 *There exist positive constants c_{38} and c_{39} such that*

$$\begin{aligned} a(\boldsymbol{g}^* + \boldsymbol{z}, \boldsymbol{z}) \,\geq\, |||\boldsymbol{z}||| \Big(& \nu\, |||\boldsymbol{z}||| - \nu\, c_{13}\, c_7\, \|\boldsymbol{g}\|_{H^s(\Gamma_i)^2} - c_{38}\, \|\boldsymbol{g}\|^2_{H^s(\Gamma_i)^2} \\ & - c_{39}\, \|\boldsymbol{g}\|_{H^s(\Gamma_i)^2}\, |||\boldsymbol{z}||| \Big) \end{aligned} \tag{19}$$

for all $\boldsymbol{z} \in V$.

Proof. Using the definition of the forms a, a_1 and a_2, we find that

$$\begin{aligned} a(\boldsymbol{g}^* + \boldsymbol{z}, \boldsymbol{z}) \,=\, & a_1(\boldsymbol{z}, \boldsymbol{z}) + a_1(\boldsymbol{g}^*, \boldsymbol{z}) + a_2(\boldsymbol{g}^*, \boldsymbol{g}^*, \boldsymbol{z}) \\ & + a_2(\boldsymbol{g}^*, \boldsymbol{z}, \boldsymbol{z}) + a_2(\boldsymbol{z}, \boldsymbol{g}^*, \boldsymbol{z}) + a_2(\boldsymbol{z}, \boldsymbol{z}, \boldsymbol{z}). \end{aligned}$$

Since $\boldsymbol{z}^\perp \cdot \boldsymbol{z} = 0$ in Ω, the terms $a_2(\boldsymbol{g}^*, \boldsymbol{z}, \boldsymbol{z})$ and $a_2(\boldsymbol{z}, \boldsymbol{z}, \boldsymbol{z})$ vanish. Hence

$$a(\boldsymbol{g}^* + \boldsymbol{z}, \boldsymbol{z}) \,\geq\, a_1(\boldsymbol{z}, \boldsymbol{z}) - |a_1(\boldsymbol{g}^*, \boldsymbol{z})| \,-\, |a_2(\boldsymbol{g}^*, \boldsymbol{g}^*, \boldsymbol{z})| - |a_2(\boldsymbol{z}, \boldsymbol{g}^*, \boldsymbol{z})|. \tag{20}$$

We obviously have

$$a_1(\boldsymbol{z}, \boldsymbol{z}) \,=\, \nu(\nabla \boldsymbol{z}, \nabla \boldsymbol{z}) \,=\, \nu\, |||\boldsymbol{z}|||^2. \tag{21}$$

Let us further estimate the terms on the right-hand side of (20). If we use the Cauchy inequality, the continuous imbedding of $H^1(\Omega)$ into $L^4(\Omega)$, Green's theorem and the theorem on traces, we successively obtain

$$|a_1(\boldsymbol{g}^*, \boldsymbol{z})| \,=\, \nu\, (\nabla \boldsymbol{g}^*, \nabla \boldsymbol{z}) \,\leq\, \nu\, \|\boldsymbol{g}^*\|_{H^1(\Omega)^2}\, |||\boldsymbol{z}|||, \tag{22}$$

$$|a_2(\boldsymbol{g}^*, \boldsymbol{g}^*, \boldsymbol{z})| \,=\, \left| \int_\Omega \omega(\boldsymbol{g}^*)\, (\boldsymbol{g}^*)^\perp \cdot \boldsymbol{z}\; \mathrm{d}\boldsymbol{x} \right|$$

$$\leq \ \|\omega(\boldsymbol{g}^*)\|_{L^2(\Omega)}\, \|\boldsymbol{g}^*\|_{L^4(\Omega)^2}\, \|\boldsymbol{z}\|_{L^4(\Omega)^2} \leq c_{40}\, \|\boldsymbol{g}^*\|^2_{H^1(\Omega)^2}\, |||\boldsymbol{z}|||, \tag{23}$$

$$|a_2(\boldsymbol{z},\boldsymbol{g}^*,\boldsymbol{z})| \ = \ \left| \int_\Omega \omega(\boldsymbol{z})\,(\boldsymbol{g}^*)^\perp \cdot \boldsymbol{z}\, \mathrm{d}\boldsymbol{x} \right| \leq \ \|\omega(\boldsymbol{z})\|_{L^2(\Omega)}\, \|\boldsymbol{g}^*\|_{L^4(\Omega)^2}\, \|\boldsymbol{z}\|_{L^4(\Omega)^2}$$

$$\leq \ c_{41}\, |||\boldsymbol{z}|||^2\, \|\boldsymbol{g}^*\|_{H^1(\Omega)^2}. \tag{24}$$

(The norm $\|\boldsymbol{z}\|_{L^4(\Omega)^2}$ can be estimated from above by the norm $|||\boldsymbol{z}|||$ due to inequality (Lemma A2 in Appendix); see p. 89.) Substituting (21)–(24) into (20) and using (I.74), (I.82), we get

$$\begin{aligned} a(\boldsymbol{g}^*+\boldsymbol{z},\boldsymbol{z}) \ &\geq \ \nu\, |||\boldsymbol{z}|||^2 - \nu\, \|\boldsymbol{g}^*\|_{H^1(\Omega)^2}\, |||\boldsymbol{z}||| - c_{40}\, \|\boldsymbol{g}^*\|^2_{H^1(\Omega)^2}\, |||\boldsymbol{z}||| \\ &\quad - c_{41}\, \|\boldsymbol{g}^*\|_{H^1(\Omega)^2}\, |||\boldsymbol{z}|||^2 \\ &\geq \ |||\boldsymbol{z}||| \Big(\nu\, |||\boldsymbol{z}||| - \nu\, c_7\, c_{13}\, \|\boldsymbol{g}\|_{H^s(\Gamma_i)^2} - c_{40}\, c_7^2\, c_{13}^2\, \|\boldsymbol{g}\|^2_{H^s(\Gamma_i)^2} \\ &\quad - c_{41}\, c_7\, c_{13}\, \|\boldsymbol{g}\|_{H^s(\Gamma_i)^2}\, |||\boldsymbol{z}||| \Big). \end{aligned} \tag{25}$$

If we denote $c_{38} := c_{40}\, c_7^2\, c_{13}^2$ and $c_{39} := c_{41}\, c_7\, c_{13}$, we obtain (19). □

Lemma 9 *If $\boldsymbol{g}$ is so small that*

$$\nu \ \geq \ c_{39}\, \|\boldsymbol{g}\|_{H^s(\Gamma_i)^2} \tag{26}$$

then the form $a(\boldsymbol{g}^*+\boldsymbol{z},\boldsymbol{z})$ is coercive on the space V. It means that

$$\lim_{|||\boldsymbol{z}|||\to+\infty} a(\boldsymbol{g}^*+\boldsymbol{z},\boldsymbol{z}) \ = \ +\infty. \tag{27}$$

The statement of Lemma 9 immediately follows from the inequality (25).

Lemmas 8 and 9 are analogous to Theorems 5 (I.3.4) and 6 (I.3.5) from Chapter I. We needed the form a_3 in Chapter I because $a_3(\boldsymbol{z},\boldsymbol{z},\boldsymbol{z})$ enabled us to control $a_2(\boldsymbol{z},\boldsymbol{z},\boldsymbol{z})$ (see (I.94)) in order to prove the coercivity of the form $a(\boldsymbol{g}^*+\boldsymbol{z},\boldsymbol{z})$ (in dependence on $\boldsymbol{z}$). Now, due to the special form of the nonlinear term in equation (5), the form a_2 has the property that $a_2(\boldsymbol{w},\boldsymbol{z},\boldsymbol{z}) = 0$ for an arbitrary $\boldsymbol{w} \in H^1(\Omega)^2$. Hence the form a in (13) may consist only of a_1 and a_2 and it need not include any other norm analogous to a_3.

II.1.5 Construction of a weak solution. We can use the Galerkin method and follow the procedure described in subsections I.3.6–I.3.8. The approximations $\boldsymbol{z}_n$ have again the form (I.101) and they satisfy (I.103) (of course, now with the form a defined by (16)):

$$a(\boldsymbol{g}^*+\boldsymbol{z}_n,\boldsymbol{v}) \ = \ (\boldsymbol{f},\boldsymbol{v}) + b(\boldsymbol{h},\boldsymbol{v}) \qquad \forall\, \boldsymbol{v} \in V_n\,. \tag{28}$$

As in subsection I.3.6, we have $|\boldsymbol{\vartheta}| = |||\boldsymbol{z}_n|||$. Using Lemma 8, we derive the inequality, corresponding to (I.111):

$$\begin{aligned} A(\boldsymbol{\vartheta})\cdot\boldsymbol{\vartheta} \ &= \ \ldots \ = \ a(\boldsymbol{g}^*+\boldsymbol{z}_n,\boldsymbol{z}_n) - (\boldsymbol{f},\boldsymbol{z}_n) - b(\boldsymbol{h},\boldsymbol{z}_n) \\ &\geq \ |||\boldsymbol{z}||| \Big(\nu\, |||\boldsymbol{z}||| - \nu\, c_{13}\, c_7\, \|\boldsymbol{g}\|_{H^s(\Gamma_i)^2} - c_{38}\, \|\boldsymbol{g}\|^2_{H^s(\Gamma_i)^2} - c_{39}\, \|\boldsymbol{g}\|_{H^s(\Gamma_i)^2}\, |||\boldsymbol{z}||| \Big) \end{aligned}$$

$$
\begin{aligned}
&\quad - c_1 \, \|\boldsymbol{f}\|_{L^2(\Omega)^2} \, |||\boldsymbol{z}_n||| - c_{28} \, \|\boldsymbol{h}\|_{L^2(\Gamma_o)^2} \, |||\boldsymbol{z}_n||| \\
&= \; |||\boldsymbol{z}_n||| \, \Big(|||\boldsymbol{z}_n||| \, \big[\nu - c_{39} \, \|\boldsymbol{g}\|_{H^s(\Gamma_i)^2}\big] - \nu \, c_{13} \, c_7 \, \|\boldsymbol{g}\|_{H^s(\Gamma_i)^2} - c_{38} \, \|\boldsymbol{g}\|^2_{H^s(\Gamma_i)^2} \\
&\quad - c_1 \, \|\boldsymbol{f}\|_{L^2(\Omega)^2} - c_{28} \, \|\boldsymbol{h}\|_{L^2(\Gamma_o)^2} \Big) \\
&= \; |\boldsymbol{\vartheta}| \, \Big(|\boldsymbol{\vartheta}| \, \big[\nu - c_{39} \, \|\boldsymbol{g}\|_{H^s(\Gamma_i)^2}\big] - \nu \, c_{13} \, c_7 \, \|\boldsymbol{g}\|_{H^s(\Gamma_i)^2} - c_{38} \, \|\boldsymbol{g}\|^2_{H^s(\Gamma_i)^2} \\
&\quad - c_1 \, \|\boldsymbol{f}\|_{L^2(\Omega)^2} - c_{28} \, \|\boldsymbol{h}\|_{L^2(\Gamma_o)^2} \Big). \qquad (29)
\end{aligned}
$$

Thus, if $\boldsymbol{g}$ satisfies (26) and if $|\boldsymbol{\vartheta}| = R_2$ where

$$
R_2 \, = \, \frac{\nu \, c_{13} \, c_7 \, \|\boldsymbol{g}\|_{H^s(\Gamma_i)^2} + c_{38} \, \|\boldsymbol{g}\|^2_{H^s(\Gamma_i)^2} + c_1 \, \|\boldsymbol{f}\|_{L^2(\Omega)^2} + c_{28} \, \|\boldsymbol{h}\|_{L^2(\Gamma_o)^2}}{\nu - c_{39} \, \|\boldsymbol{g}\|_{H^s(\Gamma_i)^2}} \qquad (30)
$$

then $A(\boldsymbol{\vartheta}) \cdot \boldsymbol{\vartheta} > 0$. Consequently, by means of Lemma 7, we deduce that the problem for the approximations (which is formally identical with (I.103)) has at least one solution $\boldsymbol{z}_n$ such that

$$
|||\boldsymbol{z}_n||| \, \le \, R_2. \qquad (31)
$$

As in subsection I.3.7, we can now obtain $\boldsymbol{z} \in V$ and a subsequence of $\{\boldsymbol{z}_n\}$ (we shall further also denote the subsequence by $\{\boldsymbol{z}_n\}$ in order to preserve a simple notation) such that

$$
\begin{aligned}
\boldsymbol{z}_n \; &\longrightarrow \; \boldsymbol{z} \quad \text{for } n \to +\infty \quad \text{weakly in } V, \qquad (32) \\
\boldsymbol{z}_n \; &\longrightarrow \; \boldsymbol{z} \quad \text{for } n \to +\infty \quad \text{strongly in } L^q(\Gamma_o)^2 \qquad (33)
\end{aligned}
$$

for each $q \in [1, +\infty)$. These types of convergence enable us to make the limit procedure in (28). Concretely, they enable us to prove that

$$
\begin{aligned}
a_1(\boldsymbol{z}_n, \boldsymbol{v}) \; &\longrightarrow \; a_1(\boldsymbol{z}, \boldsymbol{v}) && \text{for } n \to +\infty, \\
a_2(\boldsymbol{z}_n, \boldsymbol{g}^*, \boldsymbol{v}) \; &\longrightarrow \; a_2(\boldsymbol{z}, \boldsymbol{g}^*, \boldsymbol{v}) && \text{for } n \to +\infty, \\
a_2(\boldsymbol{g}^*, \boldsymbol{z}_n, \boldsymbol{v}) \; &\longrightarrow \; a_2(\boldsymbol{g}^*, \boldsymbol{z}, \boldsymbol{v}) && \text{for } n \to +\infty, \\
a_2(\boldsymbol{z}_n, \boldsymbol{z}_n, \boldsymbol{v}) \; &\longrightarrow \; a_2(\boldsymbol{z}, \boldsymbol{z}, \boldsymbol{v}) && \text{for } n \to +\infty
\end{aligned}
$$

for each $\boldsymbol{v} \in V$. Thus, the limit function $\boldsymbol{z}$ satisfies (18). We have proved the theorem

Theorem 10 (on the existence of a weak solution). Suppose that the norm $\|\boldsymbol{g}\|_{H^s(\Gamma_i)^2}$ is so small that it fulfills (26). Then the weak problem (18) has a solution $\boldsymbol{z}$ that satisfies the estimate

$$
|||\boldsymbol{z}||| \, \le \, R_2. \qquad (34)
$$

Consequently, the weak problem (13) has a solution $\boldsymbol{u}$ $(= \boldsymbol{z} + \boldsymbol{g}^*)$ that satisfies

$$
\|\nabla \boldsymbol{u}\|_{L^2(\Omega)^2} \, \le \, R_2 + \|\nabla \boldsymbol{g}^*\|_{L^2(\Omega)^2} \, \le \, R_2 + c_{13} \, c_7 \, \|\boldsymbol{g}\|_{H^s(\Gamma_i)^2} \, := \, R_3. \qquad (35)
$$

II.1.6 On uniqueness of a weak solution. Suppose that $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are two solutions of the weak problem (13). We shall prove that the solutions coincide if at least one of them is "sufficiently small" in the norm $|||\,.\,|||$. This is a stronger result than that one formulated in subsection I.4.1 (where we needed both solutions $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ to be "small enough".) The stronger result is now enabled by the absence of the nonlinear form a_3 in the definition of the form a (see (16)) and by the special structure of the form a_2 which satisfies

$$a_2(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{v}) \;=\; \int_\Omega \omega(\boldsymbol{u})\, \boldsymbol{v}^\perp \cdot \boldsymbol{v} \; \mathrm{d}\boldsymbol{x} \;=\; 0$$

for all $\boldsymbol{u}$ and $\boldsymbol{v}$ from $H^1(\Omega)^2$.

Theorem 11 (on the uniqueness of a weak solution). There exists $R > 0$ such that if $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are two solutions of the problem (13) such that $\|\nabla \boldsymbol{u}_1\|_{L^2(\Omega)^4} \leq R$ then $\boldsymbol{u}_1 \;=\; \boldsymbol{u}_2$.

Proof. The solutions $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ fulfil

$$\begin{aligned} a(\boldsymbol{u}_1, \boldsymbol{v}) \;&=\; (\boldsymbol{f}, \boldsymbol{v}) \,+\, b(\boldsymbol{h}, \boldsymbol{v}). \\ a(\boldsymbol{u}_2, \boldsymbol{v}) \;&=\; (\boldsymbol{f}, \boldsymbol{v}) \,+\, b(\boldsymbol{h}, \boldsymbol{v}) \end{aligned}$$

for all $\boldsymbol{v} \in V$. Subtracting these equations, we get

$$a(\boldsymbol{u}_1, \boldsymbol{v}) \,-\, a(\boldsymbol{u}_2, \boldsymbol{v}) \;=\; 0.$$

Substituting here the form of a from (16) and choosing $\boldsymbol{v} = \boldsymbol{u}_1 - \boldsymbol{u}_2$, we get

$$a_1(\boldsymbol{u}_1 - \boldsymbol{u}_2, \boldsymbol{u}_1 - \boldsymbol{u}_2) + a_2(\boldsymbol{u}_1, \boldsymbol{u}_1, \boldsymbol{u}_1 - \boldsymbol{u}_2) - a_2(\boldsymbol{u}_2, \boldsymbol{u}_2, \boldsymbol{u}_1 - \boldsymbol{u}_2) \;=\; 0. \tag{36}$$

The difference $a_2(\boldsymbol{u}_1, \boldsymbol{u}_1, \boldsymbol{u}_1 - \boldsymbol{u}_2) - a_2(\boldsymbol{u}_2, \boldsymbol{u}_2, \boldsymbol{u}_1 - \boldsymbol{u}_2)$ can be estimated as follows:

$$\begin{aligned} &\big| a_2(\boldsymbol{u}_1, \boldsymbol{u}_1, \boldsymbol{u}_1 - \boldsymbol{u}_2) - a_2(\boldsymbol{u}_2, \boldsymbol{u}_2, \boldsymbol{u}_1 - \boldsymbol{u}_2) \big| \\ &\quad = \; \big| a_2(\boldsymbol{u}_1 - \boldsymbol{u}_2, \boldsymbol{u}_1, \boldsymbol{u}_1 - \boldsymbol{u}_2) - a_2(\boldsymbol{u}_2, \boldsymbol{u}_1 - \boldsymbol{u}_2, \boldsymbol{u}_1 - \boldsymbol{u}_2) \big| \\ &\quad = \; \big| a_2(\boldsymbol{u}_1 - \boldsymbol{u}_2, \boldsymbol{u}_1, \boldsymbol{u}_1 - \boldsymbol{u}_2) \big| \;=\; \left| \int_\Omega \omega(\boldsymbol{u}_1 - \boldsymbol{u}_2)\, \boldsymbol{u}_1^\perp \cdot (\boldsymbol{u}_1 - \boldsymbol{u}_2) \; \mathrm{d}\boldsymbol{x} \right| \\ &\quad \leq \; \|\omega(\boldsymbol{u}_1 - \boldsymbol{u}_2)\|_{L^2(\Omega)} \, \|\boldsymbol{u}_1\|_{L^4(\Omega)^2} \, \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_{L^4(\Omega)^2} \\ &\quad \leq \; c_{42} \, \|\nabla(\boldsymbol{u}_1 - \boldsymbol{u}_2)\|_{L^2(\Omega)} \, \|\boldsymbol{u}_1\|_{L^4(\Omega)^2} \, \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_{L^4(\Omega)^2} \\ &\quad \leq \; c_{42} \, \|\nabla(\boldsymbol{u}_1 - \boldsymbol{u}_2)\|_{L^2(\Omega)} \, c_{34}^2 \, \|\nabla \boldsymbol{u}_1\|_{L^2(\Omega)^2} \, \|\nabla(\boldsymbol{u}_1 - \boldsymbol{u}_2)\|_{L^2(\Omega)^2} \\ &\quad \leq \; c_{42} \, c_{34}^2 \, R \, |||\boldsymbol{u}_1 - \boldsymbol{u}_2|||^2 \end{aligned} \tag{37}$$

where the constant c_{34} comes from (I.132). Substituting now to (36), we get

$$\nu \, |||\boldsymbol{u}_1 - \boldsymbol{u}_2|||^2 \;\leq\; c_{42} \, c_{34}^2 \, R \, |||\boldsymbol{u}_1 - \boldsymbol{u}_2|||^2.$$

This inequality easily implies the statement of the theorem. □

Corollary. If the given functions $\boldsymbol{g}$, $\boldsymbol{f}$ and $\boldsymbol{h}$ are so small in comparison with ν that the number R_1 (given by (35)) is less than or equal to R then the weak solution $\boldsymbol{u}$, constructed in subsection II.1.5, is unique in the class of all solutions of the problem (13).

II.2 The problem with linear separated boundary conditions for vorticity and Bernoulli's pressure on the outflow

II.2.1 Equations of motion. In this section, we write the 2D Navier–Stokes system (I.2) in the form

$$\omega(\boldsymbol{u})\,\boldsymbol{u}^{\perp} \;=\; -\nabla q + \nu\,(-\partial_2, \partial_1)\,\omega(\boldsymbol{u}) + \boldsymbol{f} \tag{38}$$

where $\boldsymbol{u}^{\perp}$, $\omega(\boldsymbol{u})$ and q are defined in the same way as in Section II.1:

$$\boldsymbol{u}^{\perp} \;:=\; (-u_2, u_1)\,, \qquad \omega(\boldsymbol{u}) \;:=\; \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\,, \qquad q \;:=\; p + \frac{|\boldsymbol{u}|^2}{2}\,.$$

$\omega(\boldsymbol{u})$ is the vorticity of the flow and q is the Bernoulli pressure. We also consider the condition of incompressibility

$$\operatorname{div}\boldsymbol{u} \;=\; 0. \tag{39}$$

II.2.2 Boundary conditions. We assume that the velocity $\boldsymbol{u}$ satisfies the same boundary conditions on Γ_i, Γ_+, Γ_- and Γ_w as in Chapter I and in the previous section, i.e. conditions (I.10), (I.11), (I.13) and (I.14) (which are identical with the conditions (7)–(10) in Section II.1).

The Bernoulli pressure q is (as before) naturally supposed to be τ–periodic in the x_2–direction, i.e.

$$q(x_1, x_2 + \tau) \;=\; q(x_1, x_2) \qquad\qquad \text{for } (x_1, x_2) \in \Gamma_-. \tag{40}$$

The boundary condition used on the outflow Γ_o is

$$q = h_1, \qquad -\nu\,\omega(\boldsymbol{u}) = h_2 \tag{41}$$

where $\boldsymbol{h} = (h_1, h_2)$ is a given function on Γ_o. This condition will again naturally arise (as a boundary condition of the "do nothing" type) from an appropriate weak formulation.

II.2.3 Weak formulation of the problem in the domain Ω. In order to arrive formally at the weak formulation of the problem (38)–(41), we multiply equation (38) by an arbitrary test function $\boldsymbol{v} = (v_1, v_2) \in V$, integrate over Ω, apply Green's theorem and use the condition of incompressibility (39), and the boundary conditions and conditions of periodicity (9), (10), (40). We obtain the equation

$$\nu \int_{\Omega} \omega(\boldsymbol{u}) \cdot \omega(\boldsymbol{v})\,\mathrm{d}\boldsymbol{x} + \int_{\Omega} \omega(\boldsymbol{u})\,\boldsymbol{u}^{\perp} \cdot \boldsymbol{v}\,\mathrm{d}\boldsymbol{x} \;-\; \nu \int_{\Gamma_o} \omega(\boldsymbol{u})\,(v_2 n_1 - v_1 n_2)\,\mathrm{d}S$$

$$+ \int_{\Gamma_o} q\, \boldsymbol{v} \cdot \boldsymbol{n}\, \mathrm{d}\boldsymbol{x} \;=\; \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, \mathrm{d}\boldsymbol{x}.$$

Using the identities $n_1 = 1$ and $n_2 = 0$ on Γ_o we have

$$\nu \int_\Omega \omega(\boldsymbol{u}) \cdot \omega(\boldsymbol{v})\, \mathrm{d}\boldsymbol{x} + \int_\Omega \omega(\boldsymbol{u})\, \boldsymbol{u}^\perp \cdot \boldsymbol{v}\, \mathrm{d}\boldsymbol{x} \;-\; \int_{\Gamma_o} [\nu\, \omega(\boldsymbol{u})\, v_2 - q\, v_1]\; \mathrm{d}S \;=\; \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, \mathrm{d}\boldsymbol{x}.$$

Substituting here for the terms in the integrand on Γ_o from (41), we obtain

$$\nu \int_\Omega \omega(\boldsymbol{u}) \cdot \omega(\boldsymbol{v})\, \mathrm{d}\boldsymbol{x} + \int_\Omega \omega(\boldsymbol{u})\, \boldsymbol{u}^\perp \cdot \boldsymbol{v}\, \mathrm{d}\boldsymbol{x} \;+\; \int_{\Gamma_o} [h_2\, v_2 + h_1\, v_1]\; \mathrm{d}S \;=\; \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, \mathrm{d}\boldsymbol{x}.$$

This integral equation can be written in the form

$$a(\boldsymbol{u}, \boldsymbol{v}) \;=\; (\boldsymbol{f}, \boldsymbol{v}) + b(\boldsymbol{h}, \boldsymbol{v}) \tag{42}$$

where $\boldsymbol{h}$ is given by (41) and the forms a and b are defined similarly as in subsection I.2.5:

$$a_1(\boldsymbol{u}, \boldsymbol{v}) \;=\; \nu \big(\omega(\boldsymbol{u}), \omega(\boldsymbol{v})\big)_{L^2(\Omega)}, \tag{43}$$

$$a_2(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) \;=\; \int_\Omega \omega(\boldsymbol{u})\, \boldsymbol{v}^\perp \cdot \boldsymbol{w}\; \mathrm{d}\boldsymbol{x}, \tag{44}$$

$$a(\boldsymbol{u}, \boldsymbol{v}) \;=\; a_1(\boldsymbol{u}, \boldsymbol{v}) + a_2(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}), \tag{45}$$

$$b(\boldsymbol{h}, \boldsymbol{v}) \;=\; -\int_{\Gamma_o} \boldsymbol{h} \cdot \boldsymbol{v}\; \mathrm{d}S. \tag{46}$$

The weak problem now reads as follows:

Definition 2. Let function $\boldsymbol{g} \in H^s(\Gamma_i)^2$ (for some $s \in (\frac{1}{2}, 1]$) satisfy the condition $\boldsymbol{g}(A_1) = \boldsymbol{g}(A_0)$ (where A_0 and A_1 are the end points of Γ_i). Let $\boldsymbol{f} \in L^2(\Omega)^2$ and $\boldsymbol{h} \in L^2(\Gamma_o)^2$. The **weak solution** of the problem (38), (39), (7)–(10), (40), (41) is a vector function $\boldsymbol{u} \in H^1(\Omega)^2$ which satisfies the condition of incompressibility (39) a.e. in Ω, the identity (42) for all test functions $\boldsymbol{v} \in V$, the boundary conditions (7), (8) in the sense of traces on Γ_i and Γ_w and the condition of periodicity (9) in the sense of traces on Γ_- and Γ_+.

Similarly as in Theorem 3 (I.2.9), one can show that the weak solution in the domain Ω, extended periodically in the direction x_2 with the period τ, is a weak solution in the infinite and unbounded domain D (i.e. the domain around the infinite cascade of profiles defined on p. 5).

The pressure again does not explicitly appear in the definition of the weak solution, however it can be introduced as a function from $L^2(\Omega)$ if the weak solution $\boldsymbol{u}$ is sufficiently smooth.

Let us further denote by $\boldsymbol{g}^*$ the extension of the function $\boldsymbol{g}$ from the line segment Γ_i into Ω, constructed in subsections I.3.1 and I.3.2. Thus, $\boldsymbol{g}^*$ satisfies the estimates (I.74) and (I.82) and the conditions (I.80) and (I.81).

The weak solution $\boldsymbol{u}$ can be sought for in the form $\boldsymbol{u} = \boldsymbol{g}^* + \boldsymbol{z}$ where $\boldsymbol{z} \in V$ becomes a new unknown function. This form of $\boldsymbol{u}$ guarantees that $\boldsymbol{u}$ satisfies the boundary and periodicity conditions (7)–(9). Substituting the new form of $\boldsymbol{u}$ into the equation (38), we derive the following problem: Find a function $\boldsymbol{z} \in V$ such that it satisfies the equation

$$a(\boldsymbol{g}^* + \boldsymbol{z}, \boldsymbol{v}) \;=\; (\boldsymbol{f}, \boldsymbol{v}) \;+\; b(\boldsymbol{h}, \boldsymbol{v}) \tag{47}$$

for all $\boldsymbol{v} \in V$.

II.2.4 Coercivity of the form $a(\boldsymbol{g}^* + \boldsymbol{z}, \boldsymbol{z})$. The next estimate can be derived in the same way as in the previous section (and in the proofs of Theorems 5 (I.3.4) and 6 (I.3.5) in Chapter I):

$$\begin{aligned} a(\boldsymbol{g}^* + \boldsymbol{z}, \boldsymbol{z}) \;\geq\; |||\boldsymbol{z}||| \Big(\nu\, |||\boldsymbol{z}||| - \nu\, c_{13}\, c_7\, \|\boldsymbol{g}\|_{H^s(\Gamma_i)^2} - c_{38}\, \|\boldsymbol{g}\|^2_{H^s(\Gamma_i)^2} \\ -c_{39}\, \|\boldsymbol{g}\|_{H^s(\Gamma_i)^2}\; |||\boldsymbol{z}|||\Big) \end{aligned} \tag{48}$$

holds for all $\boldsymbol{z} \in V$. The inequality (48) enables us to prove the coercivity of the form $a(\boldsymbol{g}^* + \boldsymbol{z}, \boldsymbol{z})$ in the case when $\boldsymbol{g}$ fulfills the condition

$$\nu \;\geq\; c_{39}\, \|\boldsymbol{g}\|_{H^s(\Gamma_i)^2}. \tag{49}$$

In other words, the condition (49) implies that

$$\lim_{|||z|||\to+\infty} a(\boldsymbol{g}^* + \boldsymbol{z}, \boldsymbol{z}) \;=\; +\infty. \tag{50}$$

As in Section II.1 we again do not need the form a_3 used in Chapter I. (This is due to the special form of the form a_2 which satisfies the identity $a_2(\boldsymbol{w}, \boldsymbol{z}, \boldsymbol{z}) = 0$ for an arbitrary $\boldsymbol{w} \in H^1(\Omega)^2$). Hence the form a consists only of a_1 and a_2.

II.2.5 Construction of a weak solution. The existence of a weak solution can again be proved by means of the Galerkin method. The approximations $\boldsymbol{z}_n$ can be constructed exactly in the same way as in Section II.1 and the estimate

$$|||\boldsymbol{z}_n||| \;\leq\; R_2. \tag{51}$$

(with R_2 given by 30)) can also be derived in the same way. Hence there exists $\boldsymbol{z} \in V$ such that a subsequence of $\{\boldsymbol{z}_n\}$ (denoted again by $\{\boldsymbol{z}_n\}$ converges to $\boldsymbol{z}$ (for $n \to +\infty$) as in (32) and (33). The validity of the types of convergence

$$\begin{aligned} a_1(\boldsymbol{z}_n, \boldsymbol{v}) \;&\longrightarrow\; a_1(\boldsymbol{z}, \boldsymbol{v}) && \text{for } n \to +\infty, \\ a_2(\boldsymbol{z}_n, \boldsymbol{g}^*, \boldsymbol{v}) \;&\longrightarrow\; a_2(\boldsymbol{z}, \boldsymbol{g}^*, \boldsymbol{v}) && \text{for } n \to +\infty, \\ a_2(\boldsymbol{g}^*, \boldsymbol{z}_n, \boldsymbol{v}) \;&\longrightarrow\; a_2(\boldsymbol{g}^*, \boldsymbol{z}, \boldsymbol{v}) && \text{for } n \to +\infty, \\ a_2(\boldsymbol{z}_n, \boldsymbol{z}_n, \boldsymbol{v}) \;&\longrightarrow\; a_2(\boldsymbol{z}, \boldsymbol{z}, \boldsymbol{v}) && \text{for } n \to +\infty \end{aligned}$$

(for each $\boldsymbol{v} \in V$) can be verified as in subsection I.3.7. Consequently, the limit function $\boldsymbol{z}$ satisfies (47). Thus, we are in the position that we can formulate the theorem:

Theorem 12 (on the existence of a weak solution). Suppose that the norm $\|\boldsymbol{g}\|_{H^s(\Gamma_i)^2}$ is so small that it fulfills (49). Then the weak problem (47) has a solution $\boldsymbol{z}$ that satisfies the estimate

$$|||\boldsymbol{z}||| \le R_2. \tag{52}$$

Consequently, the weak problem (42) has a solution $\boldsymbol{u}$ $(= \boldsymbol{z} + \boldsymbol{g}^*)$ that satisfies

$$\|\nabla \boldsymbol{u}\|_{L^2(\Omega)^2} \le R_2 + \|\nabla \boldsymbol{g}^*\|_{L^2(\Omega)^2} \le R_2 + c_{13}\, c_7\, \|\boldsymbol{g}\|_{H^s(\Gamma_i)^2} := R_3. \tag{53}$$

II.2.6 On uniqueness of a weak solution. The next theorem on uniqueness can be proved in the same way as Theorem 11 in subsection II.1.6. Thus, we present it here without proof.

Theorem 13 (on the uniqueness of a weak solution). There exists $R > 0$ such that if $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are two solutions of the problem (42) such that $\|\nabla \boldsymbol{u}_1\|_{L^2(\Omega)^4} \le R$ then $\boldsymbol{u}_1 = \boldsymbol{u}_2$.

Corollary. If the given functions $\boldsymbol{g}$, $\boldsymbol{f}$ and $\boldsymbol{h}$ are so small in comparison with ν that the number R_1 (given by (53)) is less than or equal to R then the weak solution $\boldsymbol{u}$, constructed in subsection II.2.5, is unique in the class of all solutions of the problem (42).

II.3 The problem with a modified nonlinear boundary condition on the outflow and a large inflow

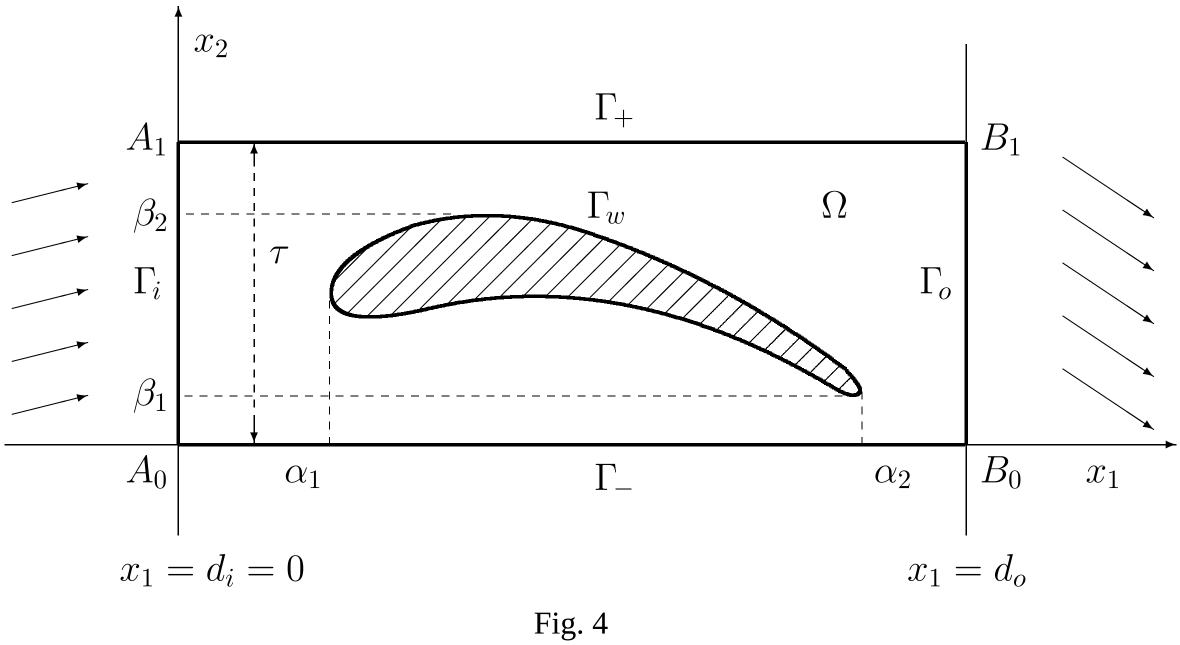

Fig. 4

II.3.1 A special shape of domain Ω. We suppose that the profiles in the cascade have a shape which enables us to choose the function γ to be constant. Furthermore, we choose for simplicity the origin of the system of coordinates so that $A_0 = [0,0]$. Then $d_i = 0$ and $A_1 = [0,\tau]$, $B_0 = [d_o, 0]$ and $B_1 = [d_o, \tau]$. The corresponding shape of domain Ω is now obvious from Fig. 4. For technical reasons we assume that the curve Γ_w is of the class C^2.

II.3.2 The auxiliary cut–off function θ_ϵ. Suppose that $\epsilon > 0$ is a small positive number. We shall specify later how small it must be, nevertheless we assume already from the beginning that

$$\mathrm{e}^{-1/\epsilon} < \max\left\{\tfrac{1}{6}\alpha_1;\ \tfrac{1}{3}\,[\tau - \beta_2];\ \tfrac{1}{3}\,\beta_1\right\}. \tag{54}$$

where $\alpha_1,\ \alpha_2$ (respectively $\beta_1,\ \beta_2$) are minimum and maximum values of the x–coordinate (y–coordinate) for points on the profile Γ_w (see Fig. 4).
We set $\kappa = \mathrm{e}^{-1/\epsilon}$ (hence $\mathrm{e}^{-2/\epsilon} = \kappa^2$) and

$$\vartheta_\epsilon(x_1) := \begin{cases} 0 & \text{for } x_1 \le \tfrac{1}{2}\kappa^2, \\ \text{linear} & \text{for } \tfrac{1}{2}\kappa^2 \le x_1 \le \kappa^2, \\ \epsilon/x_1 & \text{for } \kappa^2 \le x_1 \le \kappa, \\ \text{linear} & \text{for } \kappa \le x_1 \le \tfrac{3}{2}\kappa, \\ 0 & \text{for } \tfrac{3}{2}\kappa \le x_1. \end{cases}$$

The graph of the function ϑ_ϵ is in Fig. 5a. Further, we define

$$\theta_\epsilon(x_1) = \frac{1}{K}\int_{x_1}^{\frac{3}{2}\kappa} \vartheta_\epsilon(t)\,\mathrm{d}t \qquad \text{where} \qquad K := \int_0^{\frac{3}{2}\kappa} \vartheta_\epsilon(t)\,\mathrm{d}t.$$

An elementary calculation shows that $K = \frac{1}{2}\epsilon + 1 \ge 1$. The graph of the function θ_ϵ is in Fig. 5b.

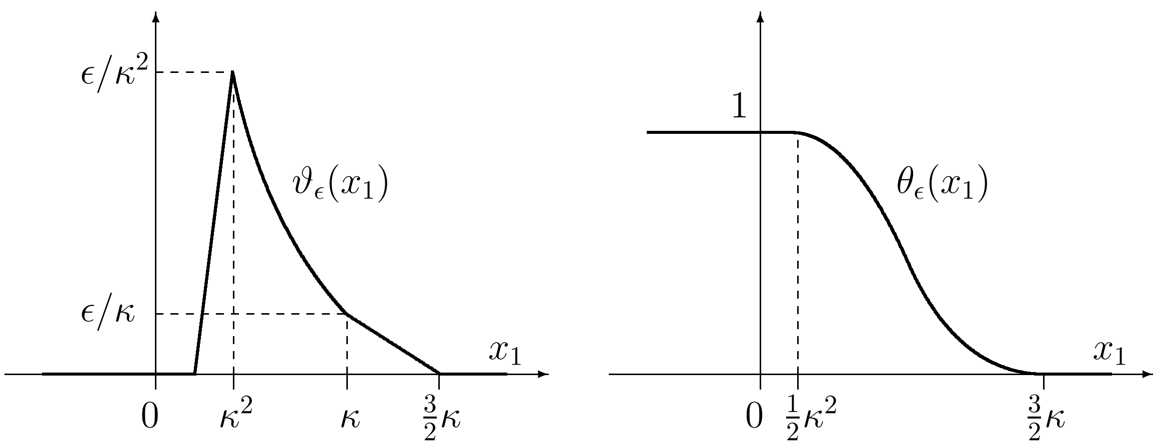

Fig. 5a Fig. 5b

The definition of the functions ϑ_ϵ and θ_ϵ yields: $\theta_\epsilon'(x_1) = -\vartheta_\epsilon(x_1)/K$ for all $x_1 \in \mathbb{R}$. Hence

$$|\theta_\epsilon'(x_1)| \;=\; \begin{cases} 0 & \text{for} \quad x_1 < \frac{1}{2}\kappa^2, \\ \dfrac{2\epsilon}{K\kappa^4}\left(x_1 - \frac{1}{2}\kappa^2\right) \;\leq\; \dfrac{\epsilon}{K\kappa^2} \;\leq\; \dfrac{\epsilon}{Kx_1} & \text{for} \quad \frac{1}{2}\kappa^2 < x_1 < \kappa^2, \\ \dfrac{\epsilon}{Kx_1} & \text{for} \quad \kappa^2 < x_1 < \kappa, \\ \dfrac{2\epsilon}{K\kappa^2}\left(\frac{3}{2}\kappa - x_1\right) \;\leq\; \dfrac{\epsilon}{K\kappa} \;\leq\; \dfrac{3\epsilon}{2Kx_1} & \text{for} \quad \kappa < x_1 < \frac{3}{2}\kappa, \\ 0 & \text{for} \quad \frac{3}{2}\kappa < x_1. \end{cases}$$

We observe that

$$\sup_{x_1>0}\, |\theta_\epsilon'(x_1)| \;=\; \frac{\epsilon}{K\kappa^2} \;\leq\; \frac{\epsilon}{\kappa^2}, \tag{55}$$

$$|\theta_\epsilon'(x_1)| \;\leq\; \frac{3\epsilon}{2Kx_1} \;\leq\; \frac{3\epsilon}{2x_1} \qquad \text{for } x_1 > 0, \quad x_1 \neq \tfrac{1}{2}\kappa^2,\ \kappa^2,\ \kappa,\ \tfrac{3}{2}\kappa. \tag{56}$$

Furthermore, the second derivative of the function θ_ϵ satisfies

$$|\theta_\epsilon''(x_1)| \;=\; \begin{cases} 0 & \text{for} \quad x_1 < \frac{1}{2}\kappa^2, \\ \dfrac{2\epsilon}{K\kappa^4} \;\leq\; \dfrac{2\epsilon}{Kx_1^2} & \text{for} \quad \frac{1}{2}\kappa^2 < x_1 < \kappa^2, \\ \dfrac{\epsilon}{Kx_1^2} & \text{for} \quad \kappa^2 < x_1 < \kappa, \\ \dfrac{2\epsilon}{K\kappa^2} \;\leq\; \dfrac{9\epsilon}{2Kx_1^2} & \text{for} \quad \kappa < x_1 < \frac{3}{2}\kappa, \\ 0 & \text{for} \quad \frac{3}{2}\kappa < x_1. \end{cases}$$

Hence

$$\sup_{x_1>0}\, |\theta_\epsilon''(x_1)| \;=\; \frac{2\epsilon}{K\kappa^4} \;\leq\; \frac{2\epsilon}{\kappa^4}, \tag{57}$$

$$|\theta_\epsilon''(x_1)| \;\leq\; \frac{9\epsilon}{2Kx_1^2} \;\leq\; \frac{9\epsilon}{2x_1^2} \qquad \text{for } x_1 > 0, \quad x_1 \neq \tfrac{1}{2}\kappa^2,\ \kappa^2,\ \kappa,\ \tfrac{3}{2}\kappa. \tag{58}$$

II.3.3 The auxiliary cut–off function χ_ϵ. By analogy with R. Temam [55], p. 175, we define $\rho(\boldsymbol{x}) := \mathrm{dist}(\boldsymbol{x}, \Gamma_w)$ and put

$$\chi_\epsilon(\boldsymbol{x}) \;:=\; 1 - \theta_\epsilon\big(\rho(\boldsymbol{x})\big) \qquad \text{for } \boldsymbol{x} \in \Omega. \tag{59}$$

The function χ_ϵ equals zero in a neighborhood of the profile Γ_w (for $\rho(\boldsymbol{x}) < \frac{1}{2}\kappa^2$ and equals one far from Γ_w (for $\rho(\boldsymbol{x}) > \frac{3}{2}\kappa$).

As the function $\rho(\boldsymbol{x})$ is twice continuously differentiable in $\overline{\Omega}$ (see [55], p. 175), we can derive from (55), (56), (57) and (58) that there exist positive constants c_{43}, c_{44}, c_{45} and c_{46} (independent of ϵ) such that

$$\sup_{x\in\Omega}\, |\nabla\chi_\epsilon| \;=\; c_{43}\,\frac{\epsilon}{\kappa^2}, \tag{60}$$

$$|\nabla\chi_\epsilon(\boldsymbol{x})| \;\leq\; c_{44}\,\frac{\epsilon}{\rho(\boldsymbol{x})} \qquad \text{for a.a. } \boldsymbol{x}\in\Omega, \tag{61}$$

$$\sup_{x\in\Omega}\,|\nabla^2\chi_\epsilon(\boldsymbol{x})| \;\leq\; c_{45}\,\frac{\epsilon}{\kappa^4}, \tag{62}$$

$$|\nabla^2\chi_\epsilon(\boldsymbol{x})| \;\leq\; c_{46}\,\frac{\epsilon}{\rho^2(\boldsymbol{x})} \qquad \text{for a.a. } \boldsymbol{x}\in\Omega. \tag{63}$$

II.3.4 A special extension of the inflow profile $\boldsymbol{g}$ to domain Ω. We suppose as in the previous sections that function $\boldsymbol{g}$ represents the given velocity profile on the inflow (i.e. on the line segment Γ_i whose end points are A_0 and A_1). Function $\boldsymbol{g}$ satisfies the condition $\boldsymbol{g}(A_0)=\boldsymbol{g}(A_1)$. We assume that $\boldsymbol{g}^*$ is the extension of the function $\boldsymbol{g}$ from Γ_i onto Ω, constructed in subsections I.3.1 and I.3.2. Thus, it satisfies (I.74) and (I.80)–(I.82). By analogy with Lemma 2, there exists a stream function $\psi^*\in H^2(\Omega)$ such that

$$\boldsymbol{g}^* \;=\; \Big(\frac{\partial\psi^*}{\partial x_2},\,-\frac{\partial\psi^*}{\partial x_1}\Big)$$

in Ω. (This can be deduced from Theorem 3.1 in [23], p. 37. Since domain Ω is not simply connected, it is here important that the trace of $\boldsymbol{g}^*$ on Γ_w is zero.) Moreover, there exists a constant $c_{47}>0$ (independent of $\boldsymbol{g}^*$) such that

$$\|\psi^*\|_{H^2(\Omega)} \;\leq\; c_{47}\,\|\boldsymbol{g}^*\|_{H^1(\Omega)^2}. \tag{64}$$

Now we modify the stream function ψ^* by means of the cut–off functions θ_ϵ and χ_ϵ and we define

$$\psi^{**}(x_1,x_2) \;:=\; \psi^*(x_1,x_2)\,\theta_\epsilon(x_1)+\frac{\Phi}{\tau}\,x_2\,\big[1-\theta_\epsilon(x_1)\big]\,\chi_\epsilon(x_1,x_2), \tag{65}$$

$$\boldsymbol{g}^{**}(x_1,x_2) \;:=\; \Big(\frac{\partial\psi^{**}}{\partial x_2}(x_1,x_2),\,-\frac{\partial\psi^{**}}{\partial x_1}(x_1,x_2)\Big) \tag{66}$$

where $\Phi := \int_0^\tau g_1(s)\,\mathrm{d}s$ (the flux into Ω through Γ_i). Thus, $\boldsymbol{g}^{**}$ is the divergence–free vector function whose stream function is ψ^{**}.

The idea of the definition of the stream function ψ^{**} is as follows: We first use the cut–off function θ_ϵ in order to interpolate between the stream function ψ^* (generating the flow $\boldsymbol{g}^*$) and the stream function $(\Phi/\tau)\,x_2$ (generating the constant one–dimensional flow $(\Phi/\tau,\,0)$). The interpolation in fact takes place in the area $\frac{1}{2}\kappa^2 < x_1 < \frac{3}{2}\kappa$. Then we multiply the stream function of the constant flow $(\Phi/\tau,\,0)$ in the area $x_1 > \frac{3}{2}\kappa$ by the cut–off function $\chi_\epsilon(\boldsymbol{x})$ in order to modify the flow in the neighborhood of the profile Γ_w.

Substituting to (66) for ψ^{**} from (65), we can express the components of $\boldsymbol{g}^{**}$ as follows:

$$\begin{aligned} g_1^{**}(x_1,x_2) \;&=\; g_1^*(x_1,x_2)\,\theta_\epsilon(x_1)+\frac{\Phi}{\tau}\,\big[1-\theta_\epsilon(x_1)\big]\,\chi_\epsilon(x_1,x_2) \\ &\quad +\frac{\Phi}{\tau}\,x_2\,\big[1-\theta_\epsilon(x_1)\big]\,\frac{\partial\chi_\epsilon}{\partial x_2}(x_1,x_2), \end{aligned} \tag{67}$$

$$g_2^{**}(x_1,x_2) \;=\; g_2^*(x_1,x_2)\,\theta_\epsilon(x_1)-\psi^*(x_1,x_2)\,\theta'_\epsilon(x_1)+\frac{\Phi}{\tau}\,x_2\,\theta'_\epsilon(x_1)\,\chi_\epsilon(x_1,x_2)$$

$$-\frac{\Phi}{\tau}\, x_2 \left[1-\theta_\epsilon(x_1)\right] \frac{\partial \chi_\epsilon}{\partial x_1}(x_1,x_2) \tag{68}$$

This means that

$$\boldsymbol{g}^{**}(x_1,x_2) \;=\; \big(g_1^*(x_1,x_2),\; g_2^*(x_1,x_2)\big) \tag{69}$$

for $0 \le x_1 \le \frac{1}{2}\kappa^2 = \frac{1}{2}\,\mathrm{e}^{-2/\epsilon}$ and

$$\boldsymbol{g}^{**}(x_1,x_2) \;=\; \big(\Phi/\tau,\; 0\big) \tag{70}$$

in the region where $\frac{3}{2}\kappa = \frac{3}{2}\,\mathrm{e}^{-1/\epsilon} \le x_1 \le d_o$ and $\rho(\boldsymbol{x}) > \frac{3}{2}\kappa$. It can also be deduced from (67), (68), (64), (I.82), (60)–(63) and (I.74) that

$$\begin{aligned}
\|\boldsymbol{g}^{**}\|_{L^2(\Omega)^2} \;&\le\; \|\boldsymbol{g}^*\,\theta_\epsilon\|_{L^2(\Omega)} + \|\psi^*\,\theta_\epsilon'\|_{L^2(\Omega)} + \frac{|\Phi|}{\tau}\,\|(1-\theta_\epsilon)\,\chi_\epsilon\|_{L^2(\Omega)} \\
&\quad + \frac{|\Phi|}{\tau}\,\|x_2\,\theta_\epsilon'\,\chi_\epsilon\|_{L^2(\Omega)} + \frac{|\Phi|}{\tau}\,\|x_2(1-\theta_\epsilon)\,\nabla\chi_\epsilon\|_{L^2(\Omega)^2} \\
&\le\; \|\boldsymbol{g}^*\|_{L^2(\Omega)} + \frac{\epsilon}{\kappa^2}\,\|\psi^*\|_{L^2(\Omega)} + C\,\frac{|\Phi|}{\tau} + C\,\frac{|\Phi|}{\tau}\,\frac{\epsilon}{\kappa^2} \\
&\le\; \left(1+\frac{c_{47}\,\epsilon}{\kappa^2}\right)\|\boldsymbol{g}^*\|_{H^1(\Omega)^2} + C\left(1+\frac{\epsilon}{\kappa}\right)\|\boldsymbol{g}\|_{L^1(\Gamma_i)^2} \\
&=\; \left(1+\frac{c_{47}\,\epsilon}{\mathrm{e}^{-2/\epsilon}}\right)\|\boldsymbol{g}^*\|_{H^1(\Omega)^2} + C\left(1+\frac{\epsilon}{\mathrm{e}^{-2/\epsilon}}\right)\|\boldsymbol{g}\|_{L^1(\Gamma_i)^2} \\
&\le\; \left(1+\frac{c_{47}\,\epsilon}{\mathrm{e}^{-2/\epsilon}}\right) c_{13}\,c_7\,\|\boldsymbol{g}\|_{H^s(\Gamma_i)^2} + C\left(1+\frac{\epsilon}{\mathrm{e}^{-2/\epsilon}}\right)\|\boldsymbol{g}\|_{H^s(\Gamma_i)^2} \\
&=\; c_{48}(\epsilon)\,\|\boldsymbol{g}\|_{H^s(\Gamma_i)^2}\,.
\end{aligned} \tag{71}$$

Furthermore, the partial derivatives of the components of $\boldsymbol{g}^{**}$ equal

$$\begin{aligned}
\frac{\partial g_1^{**}}{\partial x_1}(x_1,x_2) \;&=\; \frac{\partial g_1^*}{\partial x_1}(x_1,x_2)\,\theta_\epsilon(x_1) + g_1^*(x_1,x_2)\,\theta_\epsilon'(x_1) - \frac{\Phi}{\tau}\,\theta_\epsilon'(x_1)\,\chi_\epsilon(x_1,x_2) \\
&\quad + \frac{\Phi}{\tau}\left[1-\theta_\epsilon(x_1)\right]\frac{\partial\chi_\epsilon}{\partial x_1}(x_1,x_2) - \frac{\Phi}{\tau}\,x_2\,\theta_\epsilon'(x_1)\,\frac{\partial\chi_\epsilon}{\partial x_2}(x_1,x_2), \\
\frac{\partial g_2^{**}}{\partial x_1}(x_1,x_2) \;&=\; \frac{\partial g_2^*}{\partial x_1}(x_1,x_2)\,\theta_\epsilon(x_1) + g_2^*(x_1,x_2)\,\theta_\epsilon'(x_1) - \frac{\partial\psi^*}{\partial x_1}(x_1,x_2)\,\theta_\epsilon'(x_1) \\
&\quad - \psi^*(x_1,x_2)\,\theta_\epsilon''(x_1) + \frac{\Phi}{\tau}\,x_2\,\theta_\epsilon''(x_1)\,\chi_\epsilon(x_1,x_2) + 2\,\frac{\Phi}{\tau}\,x_2\,\theta_\epsilon'(x_1)\,\frac{\partial\chi_\epsilon}{\partial x_1}(x_1,x_2) \\
&\quad - \frac{\Phi}{\tau}\,x_2\left[1-\theta_\epsilon(x_1)\right]\frac{\partial^2\chi_\epsilon}{\partial x_1^2}(x_1,x_2), \\
\frac{\partial g_1^{**}}{\partial x_2}(x_1,x_2) \;&=\; \frac{\partial g_1^*}{\partial x_2}(x_1,x_2)\,\theta_\epsilon(x_1) + \frac{\Phi}{\tau}\left[1-\theta_\epsilon(x_1)\right]\frac{\partial\chi_\epsilon}{\partial x_2}(x_1,x_2), \\
\frac{\partial g_2^{**}}{\partial x_2}(x_1,x_2) \;&=\; \frac{\partial g_2^*}{\partial x_2}(x_1,x_2)\,\theta_\epsilon(x_1) - \frac{\partial\psi^*}{\partial x_2}(x_1,x_2)\,\theta_\epsilon'(x_1) + \frac{\Phi}{\tau}\,\theta_\epsilon'(x_1)\,\chi_\epsilon(x_1,x_2) \\
&\quad - \frac{\Phi}{\tau}\left[1-\theta_\epsilon(x_1)\right]\frac{\partial\chi_\epsilon}{\partial x_1}(x_1,x_2) - \frac{\Phi}{\tau}\,x_2\left[1-\theta_\epsilon(x_1)\right]\frac{\partial^2\chi_\epsilon}{\partial x_1\,\partial x_2}(x_1,x_2)
\end{aligned}$$

a.e. in Ω. The terms containing the product of the derivatives of θ_ϵ and χ_ϵ are equal to zero because the supports of these derivatives are disjoint. For the same reason, the terms containing the product of θ_ϵ with a derivative of χ_ϵ are also equal to zero. Thus, we can derive from the above formulas that

$$\nabla \boldsymbol{g}^{**} \;=\; \nabla \boldsymbol{g}^{*}\,\theta_\epsilon + \mathcal{A}\,\theta'_\epsilon + \mathcal{B}\,\theta''_\epsilon + \mathcal{C} + \mathcal{D} \tag{72}$$

where

$$\mathcal{A} \;=\; \begin{pmatrix} g_1^* - \dfrac{\Phi}{\tau}\,\chi_\epsilon, & 0 \\ g_2^* - \dfrac{\partial \psi^*}{\partial x_1}, & -\dfrac{\partial \psi^*}{\partial x_2} + \dfrac{\Phi}{\tau}\,\chi_\epsilon \end{pmatrix}, \qquad \mathcal{B} \;=\; \begin{pmatrix} 0, & 0 \\ -\psi^* + \dfrac{\Phi}{\tau}\,x_2\,\chi_\epsilon, & 0 \end{pmatrix},$$

$$\mathcal{C} \;=\; \begin{pmatrix} \dfrac{\Phi}{\tau}\,\dfrac{\partial \chi_\epsilon}{\partial x_1}, & \dfrac{\Phi}{\tau}\,\dfrac{\partial \chi_\epsilon}{\partial x_2} \\ 0, & -\dfrac{\Phi}{\tau}\,\dfrac{\partial \chi_\epsilon}{\partial x_1} \end{pmatrix}, \qquad \mathcal{D} \;=\; \begin{pmatrix} 0, & 0 \\ -\dfrac{\Phi}{\tau}\,x_2\,\dfrac{\partial^2 \chi_\epsilon}{\partial x_1^2}, & \dfrac{\Phi}{\tau}\,x_2\,\dfrac{\partial^2 \chi_\epsilon}{\partial x_1\,\partial x_2} \end{pmatrix}.$$

Hence, using (64), (I.82), (I.74) and the inequalities $0 \le \chi_\epsilon \le 1$, we obtain for $r \in \mathbb{R}$, $r > 1$ that

$$\begin{aligned} \|\mathcal{A}\|_{L^r(\Omega)^4} \;&\le\; \|\boldsymbol{g}^*\|_{L^r(\Omega)^2} + \|\nabla \psi^*\|_{L^r(\Omega)} + 2\,\frac{|\Phi|}{\tau}\,\mathrm{meas}(\Omega) \\ &\le\; C(r)\,\|\boldsymbol{g}^*\|_{H^1(\Omega)^2} + C(r)\,\|\nabla \psi^*\|_{H^1(\Omega)} + 2\,\frac{|\Phi|}{\tau}\,\mathrm{meas}(\Omega) \\ &\le\; C(r)\,c_{13}\,c_7\,(1 + c_{47})\,\|\boldsymbol{g}\|_{H^s(\Gamma_i)^2} + C\,\|\boldsymbol{g}\|_{H^s(\Gamma_i)^2} \\ &=\; c_{49}(r)\,\|\boldsymbol{g}\|_{H^s(\Gamma_i)^2}\,. \end{aligned} \tag{73}$$

We can similarly estimate the norm of $\mathcal{B}$ in $L^\infty(\Omega)^4$:

$$\begin{aligned} \|\mathcal{B}\|_{L^\infty(\Omega)^4} \;&\le\; \|\psi^*\|_{L^\infty(\Omega)} + \frac{|\Phi|}{\tau}\,\|x_2\|_{L^\infty(\Omega)} \\ &\le\; C\,\|\psi^*\|_{H^2(\Omega)} + C\,\|\boldsymbol{g}\|_{L^1(\Gamma_i)^2} \;\le\; C\,c_{47}\,\|\boldsymbol{g}^*\|_{H^1(\Omega)^2} + C\,\|\boldsymbol{g}\|_{H^s(\Gamma_i)^2} \\ &\le\; C\,c_{47}\,c_{13}\,c_7\,\|\boldsymbol{g}\|_{H^s(\Gamma_i)^2} + C\,\|\boldsymbol{g}\|_{H^s(\Gamma_i)^2} \;=\; c_{50}\,\|\boldsymbol{g}\|_{H^s(\Gamma_i)^2}\,. \end{aligned} \tag{74}$$

$\mathcal{C}$ and $\mathcal{D}$ can be estimated by means of (61) and (63):

$$|\mathcal{C}(\boldsymbol{x})| \;\le\; \frac{|\Phi|}{\tau}\,|\nabla \chi_\epsilon(\boldsymbol{x})| \;\le\; c_{51}\,\|\boldsymbol{g}\|_{H^s(\Gamma_i)^2}\,\frac{\epsilon}{\rho(\boldsymbol{x})}, \tag{75}$$

$$|\mathcal{D}(\boldsymbol{x})| \;\le\; \frac{|\Phi|}{\tau}\,\tau\,|\nabla^2 \chi_\epsilon(\boldsymbol{x})| \;\le\; c_{52}\,\|\boldsymbol{g}\|_{H^s(\Gamma_i)^2}\,\frac{\epsilon}{\rho^2(\boldsymbol{x})}. \tag{76}$$

Now we obtain that

$$\|\nabla \boldsymbol{g}^{**}\|_{L^2(\Omega)^4} \;\le\; \|\nabla \boldsymbol{g}^*\,\theta_\epsilon\|_{L^2(\Omega)^4} + \|\mathcal{A}\,\theta'_\epsilon\|_{L^2(\Omega)^4} + \|\mathcal{B}\,\theta''_\epsilon\|_{L^2(\Omega)^4} + \|\mathcal{C} + \mathcal{D}\|_{L^2(\Omega)^2}$$

$$
\begin{aligned}
&\leq \; \|\nabla \boldsymbol{g}^*\|_{L^2(\Omega)^4} + \frac{\epsilon}{\kappa^2}\,\|\mathcal{A}\|_{L^2(\Omega)^4} + \frac{2\epsilon}{\kappa^4}\,\|\mathcal{B}\|_{L^2(\Omega)^4} + \|\mathcal{C}\|_{L^2(\Omega)^4} + \|\mathcal{D}\|_{L^2(\Omega)^4} \\
&\leq \; \Big(c_{13}\, c_7 + c_{49}\,\frac{\epsilon}{\kappa^2} + c_{50}\,\frac{2\epsilon}{\kappa^4}\Big)\,\|\boldsymbol{g}\|_{H^s(\Gamma_i)^2} + \big[c_{51}(\epsilon) + c_{52}(\epsilon)\big]\,\|\boldsymbol{g}\|_{H^s(\Gamma_i)^2} \\
&= \; \Big(c_{13}\, c_7 + c_{49}\,\frac{\epsilon}{\mathrm{e}^{-2/\epsilon}} + c_{50}\,\frac{2\epsilon}{\mathrm{e}^{-4/\epsilon}}\Big)\,\|\boldsymbol{g}\|_{H^s(\Gamma_i)^2} + \big[c_{51}(\epsilon) + c_{52}(\epsilon)\big]\,\|\boldsymbol{g}\|_{H^s(\Gamma_i)^2} \\
&= \; c_{53}(\epsilon)\,\|\boldsymbol{g}\|_{H^s(\Gamma_i)^2}\,. \qquad (77)
\end{aligned}
$$

These inequalities and (71) yield the important estimate

$$
\|\boldsymbol{g}^{**}\|_{H^1(\Omega)^2} \;\leq\; c_{54}(\epsilon)\,\|\boldsymbol{g}\|_{H^s(\Gamma_i)^2} \qquad (78)
$$

where $c_{54}(\epsilon) = c_{48}(\epsilon) + c_{53}(\epsilon)$.

II.3.5 The boundary condition on Γ_o. In this section, we solve the same problem as in Chapter I. We use the special extension of the given inflow velocity profile $\boldsymbol{g}$ from the line segment Γ_i to domain Ω, described in the preceding subsection II.3.2. Recall that the extended function is denoted by $\boldsymbol{g}^{**}$. The form of $\boldsymbol{g}^{**}$ will enable us to derive similar estimates as in subsection I.3.4 and to make the same conclusion on coerciveness of the form a as in subsection I.3.5, **without** the requirement that $\boldsymbol{g}$ satisfies the condition of sufficient smallness (I.99). However, we pay for this advantage by a necessary modification of the nonlinear boundary condition on the outflow Γ_o: instead of the boundary condition (I.12) from Chapter I, now we use the condition

$$
-\nu\,\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}} + p\,\boldsymbol{n} - \frac{1}{2}\,(\boldsymbol{u}\cdot\boldsymbol{n})^-\,(\boldsymbol{u}-\boldsymbol{g}^{**}) \;=\; \boldsymbol{h} \qquad \text{on } (\Gamma_o)^\circ \qquad (79)
$$

which can, due to the special form of $\boldsymbol{g}^{**}$ on the outflow (following from (70)), be written in the form

$$
-\nu\,\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}} + p\,\boldsymbol{n} - \frac{1}{2}\,(\boldsymbol{u}\cdot\boldsymbol{n})^-\,\boldsymbol{u} + \frac{1}{2}\,(\boldsymbol{u}\cdot\boldsymbol{n})^-\,\big(\Phi/\tau,\,0\big) \;=\; \boldsymbol{h} \qquad \text{on } (\Gamma_o)^\circ.
$$

We observe that the difference between (I.12) and (79) is in the term $\frac{1}{2}\,(\boldsymbol{u}\cdot\boldsymbol{n})^-\,(\Phi/\tau,0)$ which does not appear in (I.12), but it appears on the left hand side of (79). This modification of the boundary condition is needed in the proof of coerciveness of the form a: without the condition (I.99) (the smallness of $\boldsymbol{g}$) we are able to verify that the form a is coercive if its part a_3 has the special form

$$
a_3(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) \;:=\; \int_{\Gamma_o} \frac{1}{2}\,(\boldsymbol{u}\cdot\boldsymbol{n})^-\,(\boldsymbol{v}-\boldsymbol{g}^{**})\cdot\boldsymbol{w}\;\mathrm{d}S \qquad (80)
$$

(which will be shown in subsection II.3.7) and this special form of a_3 induces the boundary condition (79) (which is explained in the next subsection II.3.6).

II.3.6 Weak formulation of the problem in domain Ω. We proceed in the same way as in subsection I.2.5, however with the boundary condition (79) instead of (I.12) on Γ_o, and we arrive at the integral equation which is formally identical with (I.37):

$$
a(\boldsymbol{u},\boldsymbol{v}) \;=\; (\boldsymbol{f},\boldsymbol{v}) + b(\boldsymbol{h},\boldsymbol{v}). \qquad (81)
$$

The form $a(\boldsymbol{u},\boldsymbol{v})$ equals the sum $a_1(\boldsymbol{u},\boldsymbol{v}) + a_2(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}) + a_3(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v})$ where a_1, a_2 and $b(\boldsymbol{h},\boldsymbol{v})$ are the same as in subsection I.2.5:

$$\begin{aligned} a_1(\boldsymbol{u},\boldsymbol{v}) &:= \nu \int_\Omega \sum_{i,j=1}^{2} \frac{\partial u_i}{\partial x_j}\frac{\partial v_i}{\partial x_j}\,\mathrm{d}\boldsymbol{x} \;=\; \nu\int_\Omega \nabla\boldsymbol{u}\cdot\nabla\boldsymbol{v}\,\mathrm{d}\boldsymbol{x},\\ a_2(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}) &:= \int_\Omega \sum_{i,j=1}^{2} u_j\,\frac{\partial u_i}{\partial x_j}\,v_i\,\mathrm{d}\boldsymbol{x} \;=\; \int_\Omega (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x},\\ b(\boldsymbol{h},\boldsymbol{v}) &:= -\int_{\Gamma_o} \boldsymbol{h}\cdot\boldsymbol{v}\,\mathrm{d}S. \end{aligned}$$

The only difference is in the form a_3 which is now given by (80). The weak problem now reads as follows:

Definition 3. Let function $\boldsymbol{g}\in H^s(\Gamma_i)^2$ (for some $s\in(\frac{1}{2},1]$) satisfy the condition $\boldsymbol{g}(A_1)=\boldsymbol{g}(A_0)$ (where A_0 and A_1 are the end points of Γ_i). Let $\boldsymbol{f}\in L^2(\Omega)^2$ and $\boldsymbol{h}\in L^2(\Gamma_o)^2$. We seek a vector function $\boldsymbol{u}\in H^1(\Omega)^2$ which satisfies the equation of continuity (I.1) a.e. in Ω, the boundary conditions (I.10) (respectively (I.11)) in the sense of traces on Γ_i (respectively on Γ_w), the condition of periodicity (I.13) a.e. on Γ_- and such that identity (81) holds for all test functions $\boldsymbol{v}\in V$. The solution of this problem is called a **weak solution in the domain Ω.**

Using the same reverse procedure as in subsection I.2.10, we can show that if the weak solution $\boldsymbol{u}$ is sufficiently smooth then there exists an appropriate pressure p such that the pair $\boldsymbol{u}$, p satisfies the condition (79). Thus, the boundary condition (79), although it does not explicitly appear in the definition of the weak solution, follows from the definition as a natural boundary condition in the case that the solution is smooth. The same was already several times observed in previous sections, each times in the case of the boundary condition on the outlet Γ_o.

As in the subsections I.3.3, II.1.3 and II.2.3, it is now again logical to seek for the weak solution $\boldsymbol{u}$ in the form $\boldsymbol{u}=\boldsymbol{g}^{**}+\boldsymbol{z}$ where $\boldsymbol{z}\in V$ is a new unknown function. This guarantees that $\boldsymbol{u}$ satisfies all the boundary and periodicity conditions (7)–(9). Substituting this form of $\boldsymbol{u}$ into the equation (81), we derive the following problem: Find a function $\boldsymbol{z}\in V$ such that it satisfies the equation

$$a(\boldsymbol{g}^{**}+\boldsymbol{z},\boldsymbol{v}) \;=\; (\boldsymbol{f},\boldsymbol{v}) \;+\; b(\boldsymbol{h},\boldsymbol{v}) \tag{82}$$

for all $\boldsymbol{v}\in V$.

II.3.7 Coercivity of the form $a(\boldsymbol{g}^{**}+\boldsymbol{z},\boldsymbol{z})$. Using the definition of $a(\boldsymbol{g}^{**}+\boldsymbol{z},\boldsymbol{z})$ from the previous subsection II.3.6, we obtain:

$$\begin{aligned} a(\boldsymbol{g}^{**}+\boldsymbol{z},\boldsymbol{z}) \;=\; & a_1(\boldsymbol{g}^{**},\boldsymbol{z}) + a_1(\boldsymbol{z},\boldsymbol{z}) + a_2(\boldsymbol{g}^{**},\boldsymbol{g}^{**},\boldsymbol{z})\\ & + a_2(\boldsymbol{z},\boldsymbol{g}^{**},\boldsymbol{z}) + a_2(\boldsymbol{g}^{**}+\boldsymbol{z},\boldsymbol{z},\boldsymbol{z}) + a_3(\boldsymbol{g}^{**}+\boldsymbol{z},\boldsymbol{g}^{**}+\boldsymbol{z},\boldsymbol{z}). \end{aligned} \tag{83}$$

The first term on the right hand side can be estimated by means of (78):

$$|a_1(\boldsymbol{g}^{**},\boldsymbol{z})| \;\le\; \nu\,\|\boldsymbol{g}^{**}\|_{H^1(\Omega)^2}\,|||\boldsymbol{z}||| \;\le\; \nu\,c_{54}(\epsilon)\,|||\boldsymbol{z}|||\,\|\boldsymbol{g}\|_{H^s(\Gamma_i)^2}\,. \tag{84}$$

The second term equals $\nu\,|||\boldsymbol{z}|||^2$. The third term can be estimated by means of (I.91), and (78):

$$|a_2(\boldsymbol{g}^{**},\boldsymbol{g}^{**},\boldsymbol{z})| \;\leq\; c_{17}\,\|\boldsymbol{g}^{**}\|^2_{H^1(\Omega)^2}\;|||\boldsymbol{z}|||\;\leq\; c_{17}\,c_{54}(\epsilon)^2\,\|\boldsymbol{g}\|^2_{H^s(\Gamma_i)^2}\;|||\boldsymbol{z}|||. \tag{85}$$

The estimate of the fourth term on the right hand side of (83) is fundamental. The special extension of the input velocity profile $\boldsymbol{g}$ from Γ_i to Ω (denoted by $\boldsymbol{g}^{**}$) was constructed mainly in order to obtain this estimate. Thus, using (73), (74), (72), (75) and (76), we have

$$\begin{aligned}
|a_2(\boldsymbol{z},\boldsymbol{g}^{**},\boldsymbol{z})| \;&=\; \left|\int_\Omega \boldsymbol{z}\cdot\big(\nabla\boldsymbol{g}^*\,\theta_\epsilon+\mathcal{A}\,\theta'_\epsilon+\mathcal{B}\,\theta''_\epsilon+\mathcal{C}+\mathcal{D}\big)\cdot\boldsymbol{z}\;\mathrm{d}\boldsymbol{x}\right|\\
&\leq\; \|\boldsymbol{z}\|^2_{L^4(\Omega)^2}\int_\Omega|\nabla\boldsymbol{g}^*|^2\,\theta_\epsilon^2\;\mathrm{d}\boldsymbol{x}+\|\mathcal{A}\|_{L^4(\Omega)^4}\,\|\boldsymbol{z}\|_{L^4(\Omega)^2}\left(\int_\Omega|\theta'_\epsilon|^2\,|\boldsymbol{z}|^2\;\mathrm{d}\boldsymbol{x}\right)^{1/2}\\
&\quad+\|\mathcal{B}\|_{L^\infty(\Omega)^4}\int_\Omega|\theta''_\epsilon|\,|\boldsymbol{z}|^2\;\mathrm{d}\boldsymbol{x}+\int_\Omega(|\mathcal{C}|+|\mathcal{D}|)\,|\boldsymbol{z}|^2\;\mathrm{d}\boldsymbol{x}\\
&\leq\; C\,|||\boldsymbol{z}|||^2\int_\Omega|\nabla\boldsymbol{g}^*|^2\,\theta_\epsilon^2\;\mathrm{d}\boldsymbol{x}+C\,c_{49}\,\|\boldsymbol{g}\|_{H^s(\Gamma_i)^2}\,|||\boldsymbol{z}|||\left(\int_\Omega\frac{\epsilon^2}{x_1^2}\,|\boldsymbol{z}|^2\;\mathrm{d}\boldsymbol{x}\right)^{1/2}\\
&\quad+c_{50}\,\|\boldsymbol{g}\|_{H^s(\Gamma_i)^2}\int_\Omega\frac{2\epsilon}{x_1^2}\,|\boldsymbol{z}|^2\;\mathrm{d}\boldsymbol{x}+c_{51}\,\|\boldsymbol{g}\|_{H^s(\Gamma_i)^2}\,\epsilon\int_\Omega\frac{|\boldsymbol{z}|^2}{\rho(\boldsymbol{x})}\;\mathrm{d}\boldsymbol{x}\\
&\quad+c_{52}\,\|\boldsymbol{g}\|_{H^s(\Gamma_i)^2}\,\epsilon\int_\Omega\frac{|\boldsymbol{z}|^2}{\rho^2(\boldsymbol{x})}\;\mathrm{d}\boldsymbol{x}.
\end{aligned} \tag{86}$$

Lemma 10 *There exists a constant c_{55} such that all functions $\boldsymbol{z}\in V$ satisfy the inequality*

$$\int_\Omega\frac{|\boldsymbol{z}|^2}{x_1^2}\;\mathrm{d}\boldsymbol{x}\;\leq\;c_{55}\,|||\boldsymbol{z}|||^2. \tag{87}$$

Proof. Let us denote by $\varphi(x_1)$ an infinitely differentiable cut–off function, defined for $0\leq x_1\leq d_o$ and such that

$$\varphi(x_1)\;=\;\begin{cases}1 & \text{for}\quad 0\leq x_1<\frac13\,\alpha_1,\\ 0 & \text{for}\quad \frac23\,\alpha_1<x_1\leq d_o\end{cases}$$

and $0\;\leq\;\varphi(x_1)\;\leq\;1$ for $\frac13\,\alpha_1\;\leq\;x_1\;\leq\;\frac23\,\alpha_1$. Function $\boldsymbol{z}$ can be written in the form $\boldsymbol{z}=\boldsymbol{z}_1+\boldsymbol{z}_2$ where $\boldsymbol{z}_1=\varphi\,\boldsymbol{z}$ and $\boldsymbol{z}_2=(1-\varphi)\,\boldsymbol{z}$. The integral on the left hand side of (87) can be obviously estimated as follows:

$$\begin{aligned}
&\int_\Omega\frac{|\boldsymbol{z}|^2}{x_1^2}\;\mathrm{d}\boldsymbol{x}\leq 2\int_\Omega\frac{|\boldsymbol{z}_1|^2}{x_1^2}\,\mathrm{d}\boldsymbol{x}+2\int_\Omega\frac{|\boldsymbol{z}_2|^2}{x_1^2}\,\mathrm{d}\boldsymbol{x}\leq 2\int_\Omega\frac{|\boldsymbol{z}_1|^2}{x_1^2}\,\mathrm{d}\boldsymbol{x}+2\int_{\Omega;\,x_1\geq 1/3\alpha_1}\frac{|\boldsymbol{z}_2|^2}{x_1^2}\;\mathrm{d}\boldsymbol{x}\\
&\leq\;2\int_\Omega\frac{|\boldsymbol{z}_1|^2}{x_1^2}\;\mathrm{d}\boldsymbol{x}+\frac{2}{(\frac13\alpha_1)^2}\int_\Omega|\boldsymbol{z}_2|^2\;\mathrm{d}\boldsymbol{x}\leq\;2\int_\Omega\frac{|\boldsymbol{z}_1|^2}{x_1^2}\;\mathrm{d}\boldsymbol{x}+C\,|||\boldsymbol{z}_2|||^2.
\end{aligned} \tag{88}$$

In order to estimate the first integral on the right hand side of (88), we use the one–dimensional Hardy inequality

$$\int_0^{+\infty}\left|\frac{\boldsymbol{w}(s)}{s}\right|^2\,\mathrm{d}s\;\leq\;2\int_0^{+\infty}|\boldsymbol{w}'(s)|^2\;\mathrm{d}s \tag{89}$$

for $\boldsymbol{w}$ infinitely differentiable and with a compact support in $(0,+\infty)$, proved in [55], p. 176. Due to the density of these functions $\boldsymbol{w}$ in $W_0^{1,2}((0,+\infty))^2$, the inequality (89) also holds for $\boldsymbol{w} \in W_0^{1,2}((0,+\infty))^2$ and it can therefore be also used with $\boldsymbol{w} = \boldsymbol{z}_1(\,.\,,x_2)$ (for a.a. $x_2 \in (0,\tau)$). Thus, we obtain

$$\int_0^{d_o} \Big| \frac{\boldsymbol{z}_1(x_1,x_2)}{x_1} \Big|^2 \, \mathrm{d}x_1 \;\leq\; 2 \int_0^{d_o} \Big| \frac{\partial \boldsymbol{z}_1}{\partial x_1}(x_1,x_2) \Big|^2 \, \mathrm{d}x_1$$

for a.a. $x_2 \in (0,\tau)$. Integrating this inequality with respect to x_2 from 0 to τ, we obtain

$$\int_\Omega \frac{|\boldsymbol{z}_1|^2}{x_1^2} \, \mathrm{d}\boldsymbol{x} \;\leq\; 2\, |||\boldsymbol{z}_1|||^2$$

Using this inequality in (88) and writing $\boldsymbol{z}_1$ and $\boldsymbol{z}_2$ in the forms $\varphi\boldsymbol{z}$ and $(1-\varphi)\boldsymbol{z}$, we obtain (87). □

Lemma 11 *There exists a constant c_{56} such that all functions $\boldsymbol{z} \in V$ satisfy the inequality*

$$\int_\Omega \frac{|\boldsymbol{z}|^2}{\rho^2(\boldsymbol{x})} \, \mathrm{d}\boldsymbol{x} \;\leq\; c_{56}\, |||\boldsymbol{z}|||^2. \tag{90}$$

Proof. Let us choose a positive number δ such that $\delta \leq \frac{1}{3}\, \mathrm{dist}(\Gamma_w,\, \Gamma_i \cup \Gamma_+ \cup \Gamma_o \cup \Gamma_-)$. Let $\varphi(\boldsymbol{x})$ now denote an infinitely differentiable cut–off function, defined in Ω and such that

$$\varphi(\boldsymbol{x}) \;=\; \begin{cases} 1 & \text{for} \quad 0 \leq \rho(\boldsymbol{x}) < \delta, \\ 0 & \text{for} \quad 2\,\delta < \rho(\boldsymbol{x}) \end{cases}$$

and $0 \leq \varphi(\boldsymbol{x}) \leq 1$ if $\delta \leq \rho(\boldsymbol{x}) \leq 2\,\delta$. Writing again function $\boldsymbol{z}$ in the form $\boldsymbol{z} = \boldsymbol{z}_1 + \boldsymbol{z}_2$, where $\boldsymbol{z}_1 = \varphi\,\boldsymbol{z}$ and $\boldsymbol{z}_2 = (1-\varphi)\,\boldsymbol{z}$, we can estimate the integral on the left hand side of (90) in this way:

$$\int_\Omega \frac{|\boldsymbol{z}|^2}{\rho^2(\boldsymbol{x})} \, \mathrm{d}\boldsymbol{x} \;\leq\; 2 \int_\Omega \frac{|\boldsymbol{z}_1|^2}{\rho^2(\boldsymbol{x})} \, \mathrm{d}\boldsymbol{x} + 2 \int_\Omega \frac{|\boldsymbol{z}_2|^2}{\rho^2(\boldsymbol{x})} \, \mathrm{d}\boldsymbol{x} \;\leq\; 2 \int_\Omega \frac{|\boldsymbol{z}_1|^2}{\rho^2(\boldsymbol{x})} \, \mathrm{d}\boldsymbol{x} \tag{91}$$
$$+2 \int_{\Omega;\, \rho(x) \geq \delta} \frac{|\boldsymbol{z}_2|^2}{\rho^2(\boldsymbol{x})} \, \mathrm{d}\boldsymbol{x} \leq 2 \int_\Omega \frac{|\boldsymbol{z}_1|^2}{\rho^2(\boldsymbol{x})} \, \mathrm{d}\boldsymbol{x} + \frac{2}{\delta^2} \int_\Omega |\boldsymbol{z}_2|^2 \, \mathrm{d}\boldsymbol{x} \leq 2 \int_\Omega \frac{|\boldsymbol{z}_1|^2}{\rho^2(\boldsymbol{x})} \, \mathrm{d}\boldsymbol{x} + C\, |||\boldsymbol{z}_2|||^2.$$

The first integral on the right hand side of (91) can be estimated by means of Lemma 1.10 in [55], p. 175:

$$\int_\Omega \frac{|\boldsymbol{z}_1|^2}{\rho^2(\boldsymbol{x})} \, \mathrm{d}\boldsymbol{x} \;\leq\; C\, |||\boldsymbol{z}_1|||^2.$$

Using this inequality in (91), we obtain (90). □

Substituting from the inequalities (87) and (90) to (86), we get

$$|a_2(\boldsymbol{z}, \boldsymbol{g}^{**}, \boldsymbol{z})| \;\leq\; c_{57}\, |||\boldsymbol{z}|||^2 \int_\Omega |\nabla \boldsymbol{g}^*|^2\, \theta_\epsilon^2 \, \mathrm{d}\boldsymbol{x} + c_{58}\, \epsilon\, \|\boldsymbol{g}\|_{H^s(\Gamma_i)^2}\, |||\boldsymbol{z}|||^2. \tag{92}$$

The function θ_ϵ is supported for $0 \leq x_1 \leq \frac{3}{2}\kappa = \frac{3}{2}\,\mathrm{e}^{-1/\epsilon}$. Thus, we can choose ϵ so small that

$$\left(c_{57} \int_\Omega |\nabla \boldsymbol{g}^*|^2\, \theta_\epsilon^2\, \mathrm{d}\boldsymbol{x} + c_{58}\, \epsilon\, \|\boldsymbol{g}\|_{H^s(\Gamma_i)^2} \right) \;<\; \frac{\nu}{2}\,. \tag{93}$$

Then we obtain

$$|a_2(\boldsymbol{z}, \boldsymbol{g}^{**}, \boldsymbol{z})| \;\leq\; \frac{\nu}{2}\, |||\boldsymbol{z}|||^2. \tag{94}$$

Finally, the fifth and the sixth term on the right hand side of (83) can be treated in this way:

$$\begin{aligned} & a_2(\boldsymbol{g}^{**} + \boldsymbol{z}, \boldsymbol{z}, \boldsymbol{z}) + a_3(\boldsymbol{g}^{**} + \boldsymbol{z}, \boldsymbol{g}^{**} + \boldsymbol{z}, \boldsymbol{z}) \\ &= \int_\Omega (\boldsymbol{g}^{**} + \boldsymbol{z}) \cdot \nabla \boldsymbol{z} \cdot \boldsymbol{z}\, \mathrm{d}\boldsymbol{x} + \int_{\Gamma_o} \frac{1}{2} \big((\boldsymbol{g}^{**} + \boldsymbol{z}) \cdot \boldsymbol{n} \big)^- \boldsymbol{z} \cdot \boldsymbol{z}\, \mathrm{d}S \\ &= \int_{\Gamma_o} \big((\boldsymbol{g}^{**} + \boldsymbol{z}) \cdot \boldsymbol{n} \big) \frac{1}{2} |\boldsymbol{z}|^2\, \mathrm{d}S + \int_{\Gamma_o} \frac{1}{2} \big((\boldsymbol{g}^{**} + \boldsymbol{z}) \cdot \boldsymbol{n} \big)^- |\boldsymbol{z}|^2\, \mathrm{d}S \;\geq\; 0. \end{aligned} \tag{95}$$

Substituting now the estimates (84), (85), (94) and (95) into (83), we get

$$\begin{aligned} a(\boldsymbol{g}^{**} + \boldsymbol{z}, \boldsymbol{z}) \;&\geq\; \frac{\nu}{2}\, |||\boldsymbol{z}|||^2 - \nu\, c_{54}(\epsilon)\, |||\boldsymbol{z}|||\, \|\boldsymbol{g}\|_{H^s(\Gamma_i)^2} - c_{17}\, c_{54}(\epsilon)^2\, \|\boldsymbol{g}\|^2_{H^s(\Gamma_i)^2}\, |||\boldsymbol{z}||| \\ &=\; |||\boldsymbol{z}||| \left(\frac{\nu}{2}\, |||\boldsymbol{z}||| - \nu\, c_{54}(\epsilon)\, \|\boldsymbol{g}\|_{H^s(\Gamma_i)^2} - c_{17}\, c_{54}(\epsilon)^2\, \|\boldsymbol{g}\|^2_{H^s(\Gamma_i)^2} \right). \end{aligned} \tag{96}$$

We have proved the lemma:

Lemma 12 *If the positive number ϵ is so small that the condition (93) is satisfied then the form a (defined in the subsection II.3.6) is coercive.*

II.3.8 Existence of a weak solution of the boundary–value problem. As we have already observed in Chapter I, the coercivity of the form a plays the fundamental role in the proof of the existence of a weak solution of the problem defined by the equation (82) and of its estimate. Since we have already proved the coerciveness and we also have the estimate (96), we can complete the proof of the existence in the same way as in the subsections I.3.6 –I.3.8 in Chapter I and we obtain the theorem:

Theorem 14 (on the existence of a weak solution). Suppose that $\epsilon > 0$ is chosen so small that (93) holds. Then the weak problem (82) has a solution $\boldsymbol{z}$ that satisfies the estimate

$$|||\boldsymbol{z}||| \;\leq\; R_4 \tag{97}$$

where

$$R_4 \;:=\; \frac{2}{\nu} \left(\nu\, c_{54}(\epsilon)\, \|\boldsymbol{g}\|_{H^s(\Gamma_i)^2} + c_{17}\, c_{54}(\epsilon)^2\, \|\boldsymbol{g}\|^2_{H^s(\Gamma_i)^2} \right). \tag{98}$$

Consequently, the weak problem (81) has a solution $\boldsymbol{u}$ ($= \boldsymbol{z} + \boldsymbol{g}^{**}$) that satisfies

$$\|\nabla \boldsymbol{u}\|_{L^2(\Omega)^2} \;\leq\; R_4 + \|\nabla \boldsymbol{g}^{**}\|_{L^2(\Omega)^2} \;\leq\; R_4 + c_{53}(\epsilon)\, \|\boldsymbol{g}\|_{H^s(\Gamma_i)^2} \;:=\; R_5. \tag{99}$$

Let us note that the assumption on the smallness of the number ϵ represents no restriction on the data of the problem because ϵ is an auxiliary parameter which can be chosen as small as we wish. (See the subsection II.3.7.) The size of ϵ only influences the value of the number R_5 on the right hand side (99). The smaller is ϵ, the larger is R_5.

II.3.9 Uniqueness of the weak solution of the boundary–value problem. Suppose that $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are two solutions of the weak problem (13). We shall prove that the solutions coincide if both solutions are "sufficiently small" in the norm $|||\,.\,|||$. This is due to the nonlinearity of the boundary condition.

Theorem 15 (on the uniqueness of a weak solution). There exists $R > 0$ and $\xi > 0$ such that if $\|\boldsymbol{g}\|_{H^s(\Gamma_i)} < \xi$ and $\boldsymbol{u}_1$, $\boldsymbol{u}_2$ are two solutions of the problem II.3.6 such that $\|\nabla\boldsymbol{u}_1\|_{L^2(\Omega)^4} \leq R$, $\|\nabla\boldsymbol{u}_2\|_{L^2(\Omega)^4} \leq R$ then $\boldsymbol{u}_1 = \boldsymbol{u}_2$.

Proof. Since $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are the solutions of the problem II.3.6, they fulfil the equations

$$a(\boldsymbol{u}_1,\boldsymbol{v}) = (\boldsymbol{f},\boldsymbol{v}) + b(\boldsymbol{h},\boldsymbol{v}), \qquad a(\boldsymbol{u}_2,\boldsymbol{v}) = (\boldsymbol{f},\boldsymbol{v}) + b(\boldsymbol{h},\boldsymbol{v})$$

for all $\boldsymbol{v} \in V$. Subtracting these equations, we get

$$a(\boldsymbol{u}_1,\boldsymbol{v}) - a(\boldsymbol{u}_2,\boldsymbol{v}) = 0.$$

Expressing the bilinear form a by means of the forms a_1, a_2 defined in subsection II.3.6 and a_3 defined by (80), we obtain

$$\begin{aligned} & a_1(\boldsymbol{u}_1,\boldsymbol{v}) - a_1(\boldsymbol{u}_2,\boldsymbol{v}) + a_2(\boldsymbol{u}_1,\boldsymbol{u}_1,\boldsymbol{v}) - a_2(\boldsymbol{u}_2,\boldsymbol{u}_2,\boldsymbol{v}) \\ & \quad + a_3(\boldsymbol{u}_1,\boldsymbol{u}_1,\boldsymbol{v}) - a_3(\boldsymbol{u}_2,\boldsymbol{u}_2,\boldsymbol{v}) = 0. \end{aligned}$$

This holds for all $\boldsymbol{v} \in V$. If we choose $\boldsymbol{v} = \boldsymbol{u}_1 - \boldsymbol{u}_2$ then this identity yields

$$\begin{aligned} & a_1(\boldsymbol{u}_1-\boldsymbol{u}_2,\boldsymbol{u}_1-\boldsymbol{u}_2) + a_2(\boldsymbol{u}_1,\boldsymbol{u}_1,\boldsymbol{u}_1-\boldsymbol{u}_2) - a_2(\boldsymbol{u}_2,\boldsymbol{u}_2,\boldsymbol{u}_1-\boldsymbol{u}_2) \\ & \quad + a_3(\boldsymbol{u}_1,\boldsymbol{u}_1,\boldsymbol{u}_1-\boldsymbol{u}_2) - a_3(\boldsymbol{u}_2,\boldsymbol{u}_2,\boldsymbol{u}_1-\boldsymbol{u}_2) = 0. \end{aligned} \tag{100}$$

If we denote

$$\begin{aligned} I_1 &:= a_1(\boldsymbol{u}_1-\boldsymbol{u}_2,\boldsymbol{u}_1-\boldsymbol{u}_2) = \nu\int_\Omega |\nabla(\boldsymbol{u}_1-\boldsymbol{u}_2)|^2\,\mathrm{d}\boldsymbol{x} = \nu\,|||\boldsymbol{u}_1-\boldsymbol{u}_2|||^2, \\ I_2 &:= a_2(\boldsymbol{u}_1,\boldsymbol{u}_1,\boldsymbol{u}_1-\boldsymbol{u}_2) - a_2(\boldsymbol{u}_2,\boldsymbol{u}_2,\boldsymbol{u}_1-\boldsymbol{u}_2), \\ I_3 &:= a_3(\boldsymbol{u}_1,\boldsymbol{u}_1,\boldsymbol{u}_1-\boldsymbol{u}_2) - a_3(\boldsymbol{u}_2,\boldsymbol{u}_2,\boldsymbol{u}_1-\boldsymbol{u}_2) \end{aligned}$$

then (100) takes the form

$$I_1 = -I_2 - I_3. \tag{101}$$

We shall further estimate the terms on the right hand side of (101).

$$\begin{aligned} |I_2| &= \left| \int_\Omega \boldsymbol{u}_1\cdot\nabla\boldsymbol{u}_1\cdot(\boldsymbol{u}_1-\boldsymbol{u}_2)\,\mathrm{d}\boldsymbol{x} - \int_\Omega \boldsymbol{u}_2\cdot\nabla\boldsymbol{u}_2\cdot(\boldsymbol{u}_1-\boldsymbol{u}_2)\,\mathrm{d}\boldsymbol{x} \right| \\ &\leq \left| \int_\Omega (\boldsymbol{u}_1-\boldsymbol{u}_2)\cdot\nabla\boldsymbol{u}_1\cdot(\boldsymbol{u}_1-\boldsymbol{u}_2)\,\mathrm{d}\boldsymbol{x} \right| + \left| \int_\Omega \boldsymbol{u}_2\cdot\nabla(\boldsymbol{u}_1-\boldsymbol{u}_2)\cdot(\boldsymbol{u}_1-\boldsymbol{u}_2)\,\mathrm{d}\boldsymbol{x} \right| \\ &\leq \|\boldsymbol{u}_1-\boldsymbol{u}_2\|^2_{L^4(\Omega)^2}\,\|\nabla\boldsymbol{u}_1\|_{L^2(\Omega)^4} + \|\boldsymbol{u}_2\|_{L^4(\Omega)^2}\,\|\nabla(\boldsymbol{u}_1-\boldsymbol{u}_2)\|_{L^2(\Omega)^4}\,\|\boldsymbol{u}_1-\boldsymbol{u}_2\|_{L^4(\Omega)^2} \end{aligned}$$

$$\leq \; 2c_{34}^2\, R\, \|\nabla(\boldsymbol{u}_1 - \boldsymbol{u}_2)\|^2_{L^2(\Omega)^4} \;=\; 2c_{34}^2\, R\, |||\boldsymbol{u}_1 - \boldsymbol{u}_2|||^2 \tag{102}$$

where the constant c_{34} comes from the inequality

$$\|\boldsymbol{u}\|_{L^4(\Omega)^2} \;\leq\; c_{34}\, \|\nabla \boldsymbol{u}\|_{L^2(\Omega)^4} \tag{103}$$

for functions $\boldsymbol{u}$ from $H^1(\Omega)^2$, satisfying the boundary condition (11); see Appendix Lemma A1. (The above estimates are analogous to the estimates in the proof of Theorem 9.) The term I_3 is

$$\begin{aligned} I_3 \;=\; & \int_{\Gamma_o} \frac{1}{2}\, (\boldsymbol{u}_1 \cdot \boldsymbol{n})^-\, (\boldsymbol{u}_1 - \boldsymbol{g}^{**}) \cdot (\boldsymbol{u}_1 - \boldsymbol{u}_2)\, \mathrm{d}S \\ & - \int_{\Gamma_o} \frac{1}{2}\, (\boldsymbol{u}_2 \cdot \boldsymbol{n})^-\, (\boldsymbol{u}_2 - \boldsymbol{g}^{**}) \cdot (\boldsymbol{u}_1 - \boldsymbol{u}_2)\, \mathrm{d}S. \end{aligned}$$

According to the signs of $\boldsymbol{u}_1 \cdot \boldsymbol{n}$ and $\boldsymbol{u}_2 \cdot \boldsymbol{n}$ on Γ_o, we must split Γ_o into four parts

$$\Gamma_o = \Gamma_{o1} \cup \Gamma_{o2} \cup \Gamma_{o3} \cup \Gamma_{o4},$$

where

a) on Γ_{o1} $\ldots$ $\boldsymbol{u}_1 \cdot \boldsymbol{n} \geq 0$, $\boldsymbol{u}_2 \cdot \boldsymbol{n} \geq 0$ $\ldots$ $(\boldsymbol{u}_1 \cdot \boldsymbol{n})^- = 0$, $(\boldsymbol{u}_2 \cdot \boldsymbol{n})^- = 0$,

b) on Γ_{o2} $\ldots$ $\boldsymbol{u}_1 \cdot \boldsymbol{n} < 0$, $\boldsymbol{u}_2 \cdot \boldsymbol{n} \geq 0$ $\ldots$ $(\boldsymbol{u}_1 \cdot \boldsymbol{n})^- = \boldsymbol{u}_1 \cdot \boldsymbol{n}$, $(\boldsymbol{u}_2 \cdot \boldsymbol{n})^- = 0$,

c) on Γ_{o3} $\ldots$ $\boldsymbol{u}_1 \cdot \boldsymbol{n} \geq 0$, $\boldsymbol{u}_2 \cdot \boldsymbol{n} < 0$ $\ldots$ $(\boldsymbol{u}_1 \cdot \boldsymbol{n})^- = 0$, $(\boldsymbol{u}_2 \cdot \boldsymbol{n})^- = \boldsymbol{u}_2 \cdot \boldsymbol{n}$,

d) on Γ_{o4} $\ldots$ $\boldsymbol{u}_1 \cdot \boldsymbol{n} < 0$, $\boldsymbol{u}_2 \cdot \boldsymbol{n} < 0$ $\ldots$ $(\boldsymbol{u}_1 \cdot \boldsymbol{n})^- = \boldsymbol{u}_1 \cdot \boldsymbol{n}$, $(\boldsymbol{u}_2 \cdot \boldsymbol{n})^- = \boldsymbol{u}_2 \cdot \boldsymbol{n}$.

Let us denote by I_3^{o1}, I_3^{o2}, I_3^{o3} and I_3^{o4} the same integrals as in I_3, however considered successively on Γ_{o1}, Γ_{o2}, Γ_{o3} and Γ_{o4}. Obviously, $I_3^{o1} = 0$ because the integrands are equal to zero on Γ_{o1}. On Γ_{o2}, we use the inequality $|\boldsymbol{u}_1 \cdot \boldsymbol{n}| \leq |\boldsymbol{u}_1 \cdot \boldsymbol{n} - \boldsymbol{u}_2 \cdot \boldsymbol{n}|$, which holds because $\boldsymbol{u}_1 \cdot \boldsymbol{n} < 0$ and $\boldsymbol{u}_2 \cdot \boldsymbol{n} \geq 0$. Moreover we use (75). Hence

$$\begin{aligned} |I_3^{o2}| \;&=\; \left| \int_{\Gamma_{o2}} (\boldsymbol{u}_1 \cdot \boldsymbol{n})^- (\boldsymbol{u}_1 - \boldsymbol{g}^{**}) \cdot (\boldsymbol{u}_1 - \boldsymbol{u}_2)\, \mathrm{d}S \right| \\ &\leq\; \int_{\Gamma_{o2}} |\boldsymbol{u}_1 \cdot \boldsymbol{n} - \boldsymbol{u}_2 \cdot \boldsymbol{n}|\, |\boldsymbol{u}_1 - \boldsymbol{g}^{**}|\, |\boldsymbol{u}_1 - \boldsymbol{u}_2|\, \mathrm{d}S \\ &\leq\; \int_{\Gamma_{o2}} |\boldsymbol{u}_1 - \boldsymbol{u}_2|^2\, |\boldsymbol{u}_1 - \boldsymbol{g}^{**}|\, \mathrm{d}S \\ &\leq\; \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|^2_{L^4(\Gamma_{o2})^2}\, (\|\boldsymbol{u}_1\|_{L^2(\Gamma_{o2})^2} \;+\; \|\boldsymbol{g}^{**}\|_{L^2(\Gamma_{o2})^2}) \\ &\leq\; \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|^2_{L^4(\Gamma_o)^2}\, (\|\boldsymbol{u}_1\|_{L^2(\Gamma_o)^2} \;+\; \|\boldsymbol{g}^{**}\|_{L^2(\Gamma_o)^2}) \\ &\leq\; c_{35}\, \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|^2_{H^1(\Omega)^2}\, (\|\boldsymbol{u}_1\|_{H^1(\Omega)^2} \;+\; \|\boldsymbol{g}^{**}\|_{H^1(\Omega)^2}) \\ &\leq\; |||\boldsymbol{u}_1 - \boldsymbol{u}_2|||^2\, (c_{36}\, \|\nabla \boldsymbol{u}_1\|_{L^2(\Omega)^4} \;+\; c_{54}(\epsilon)\, \|\boldsymbol{g}\|_{H^s(\Gamma_i)^2}) \\ &\leq\; (c_{36}\, R \;+\; c_{54}(\epsilon)\, \|\boldsymbol{g}\|_{H^s(\Gamma_i)^2})\, |||\boldsymbol{u}_1 - \boldsymbol{u}_2|||^2 \end{aligned} \tag{104}$$

where the constants c_{35} and c_{36} come from the inequalities

$$\|\boldsymbol{u}\|_{L^2(\Gamma_o)^2} \;\leq\; c_{35}\,\|\boldsymbol{u}\|_{H^1(\Omega)^2} \;\leq\; c_{36}\,\|\nabla\boldsymbol{u}\|_{L^2(\Omega)^4}$$

for functions $\boldsymbol{u}$ from $H^1(\Omega)^2$, satisfying the boundary condition (11); see Appendix Lemma A1. The term I_3^{o3} can be estimated in the same way as I_3^{o2}. The term I_3^{o4} can be treated as follows:

$$\begin{aligned}
|I_3^{o4}| \;&=\; \left|\int_{\Gamma_{o4}} (\boldsymbol{u}_1\cdot\boldsymbol{n})\,(\boldsymbol{u}_1-\boldsymbol{g}^{**})\cdot(\boldsymbol{u}_1-\boldsymbol{u}_2)\;\mathrm{d}S - \int_{\Gamma_{o4}} (\boldsymbol{u}_2\cdot\boldsymbol{n})\,(\boldsymbol{u}_2-\boldsymbol{g}^{**})\cdot(\boldsymbol{u}_1-\boldsymbol{u}_2)\;\mathrm{d}S\right| \\
&=\; \left|\int_{\Gamma_{o4}} [(\boldsymbol{u}_1-\boldsymbol{u}_2)\cdot\boldsymbol{n}]\,(\boldsymbol{u}_1-\boldsymbol{g}^{**})\cdot(\boldsymbol{u}_1-\boldsymbol{u}_2)\;\mathrm{d}S\right. \\
&\qquad\left. + \int_{\Gamma_{o4}} (\boldsymbol{u}_2\cdot\boldsymbol{n})\,(\boldsymbol{u}_1-\boldsymbol{u}_2)\cdot(\boldsymbol{u}_1-\boldsymbol{u}_2)\;\mathrm{d}S\right| \\
&\leq\; \int_{\Gamma_{o4}} |\boldsymbol{u}_1-\boldsymbol{u}_2|\,(|\boldsymbol{u}_1| \,+\, |\boldsymbol{g}^{**}|)\,|\boldsymbol{u}_1-\boldsymbol{u}_2|\;\mathrm{d}S + \int_{\Gamma_{o4}} |\boldsymbol{u}_2|\,|\boldsymbol{u}_1-\boldsymbol{u}_2|\,|\boldsymbol{u}_1-\boldsymbol{u}_2|\;\mathrm{d}S \\
&\leq\; \|\boldsymbol{u}_1-\boldsymbol{u}_2\|^2_{L^4(\Gamma_{o4})^2}\,\big(\|\boldsymbol{u}_1\|_{L^2(\Gamma_{o4})^2} \,+\, \|\boldsymbol{g}^{**}\|_{L^2(\Gamma_{o2})^2} + \|\boldsymbol{u}_2\|_{L^2(\Gamma_{o4})^2}\big) \\
&\leq\; \|\boldsymbol{u}_1-\boldsymbol{u}_2\|^2_{L^4(\Gamma_o)^2}\,\big(\|\boldsymbol{u}_1\|_{L^2(\Gamma_o)^2} \,+\, \|\boldsymbol{g}^{**}\|_{L^2(\Gamma_o)^2} + \|\boldsymbol{u}_2\|_{L^2(\Gamma_o)^2}\big) \\
&\leq\; c_{37}\,\|\nabla\boldsymbol{u}_1-\nabla\boldsymbol{u}_2\|_{L^2(\Omega)^4}\,c_{36}\,\big(\|\nabla\boldsymbol{u}_1\|_{L^2(\Omega)^4} \,+\, c_{54}(\epsilon)\,\|\boldsymbol{g}\|_{H^s(\Gamma_i)^2} \,+\, \|\nabla\boldsymbol{u}_2\|_{L^2(\Omega)^4}\big) \\
&\leq\; c_{37}\,|||\boldsymbol{u}_1-\boldsymbol{u}_2|||^2\,c_{36}\,(2R \,+\, c_{54}(\epsilon)\,\|\boldsymbol{g}\|_{H^s(\Gamma_i)^2})
\end{aligned} \tag{105}$$

where the constant c_{37} comes from the inequality

$$\|\boldsymbol{u}\|_{L^4(\Gamma_o)^2} \;\leq\; c_{37}\,\|\nabla\boldsymbol{u}\|_{L^2(\Omega)^4}$$

for functions $\boldsymbol{u}$ from $H^1(\Omega)^2$, satisfying the boundary condition (11); see Appendix Lemma A1.

Substituting from (102), (104) and (105) into (101), we obtain

$$\begin{aligned}
\nu\,|||\boldsymbol{u}_1-\boldsymbol{u}_2|||^2 \;\leq\; &(2c_{34}^2R + 2c_{36}R \,+\, 2c_{54}(\epsilon)\,\|\boldsymbol{g}\|_{H^s(\Gamma_i)^2} \\
&+2c_{36}c_{37}R \,+\, c_{36}\,c_{37}\,c_{54}(\epsilon)\,\|\boldsymbol{g}\|_{H^s(\Gamma_i)^2})\,|||\boldsymbol{u}_1-\boldsymbol{u}_2|||^2.
\end{aligned}$$

Now it is seen that if

$$\nu \;>\; (2c_{34}^2R + 2c_{36}R \,+\, 2c_{54}(\epsilon)\,\|\boldsymbol{g}\|_{H^s(\Gamma_i)^2} \,+\, 2c_{36}c_{37}R \,+\, c_{36}\,c_{37}\,c_{54}(\epsilon)\,\|\boldsymbol{g}\|_{H^s(\Gamma_i)^2})$$

then $\boldsymbol{u}_1 = \boldsymbol{u}_2$. The statement of the theorem now easily follows from this inequality. □

II.3.10 Remark 2. We have studied the boundary value problems which contain the Bernoulli pressure q instead the “static” pressure p in Sections II.1 and II.2. In these sections, the weak formulation of the boundary–value problem was formally derived from the Navier–Stokes equation with the nonlinear term written in the form $\omega(\boldsymbol{u})\cdot\boldsymbol{u}^{\perp}$ (instead of the usual form $\boldsymbol{u}\cdot\nabla\boldsymbol{u}$). By analogy with the present Section II.3, it would also be possible to study the existence and uniqueness of a weak solution of the same boundary–value problems as in Sections II.1 and II.2 without the restriction to small inflows, however it would also require certain modifications of the boundary conditions on the outflow, as in the Section II.3. □

Chapter III

Nonstationary problem with a nonlinear mixed boundary condition on the outflow

III.1 Classical Formulation

III.1.1 Governing equations. We study the same problem as in Chapters I and II, however we admit that the solution may also depend on time. We come from the same setting of the problem as in Chapter I, the only formal difference is that the equations (I.1) and (I.2) also contain the derivative with respect to time. Thus, the governing system of equations has the form

$$\operatorname{div} \boldsymbol{u} = \boldsymbol{0}, \tag{1}$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = \boldsymbol{f} - \nabla p + \nu \Delta \boldsymbol{u}. \tag{2}$$

As in Chapter I, the vector equation (2) can be written down as a system of two scalar equations

$$\frac{\partial u_i}{\partial t} + \sum_{j=1}^{2} u_j \frac{\partial u_i}{\partial x_j} = f_i - \frac{\partial p}{\partial x_i} + \nu \Delta u_i, \qquad (i = 1, 2).$$

We assume that T is a given positive number and that $t \in (0, T)$.

III.1.2 Boundary conditions and initial condition. In order to obtain a well–posed problem in the domain D, we apply the same boundary conditions as in Chapter I (see the subsection I.1.2) on Γ_i, Γ_w and Γ_o:

$$\boldsymbol{u}|_{G_i} = \boldsymbol{g}, \tag{3}$$

$$\boldsymbol{u}|_{G_w} = \boldsymbol{0}, \tag{4}$$

$$-\nu \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}} + p\boldsymbol{n} - \frac{1}{2} (\boldsymbol{u} \cdot \boldsymbol{n})^- \boldsymbol{u} = \boldsymbol{h} \qquad \text{for } (x_1, x_2) \in G_o\,. \tag{5}$$

Moreover, we assume that the velocity distribution in D is known at the initial time $t = 0$:

$$\boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{u}_0(\boldsymbol{x}), \qquad \text{for } \boldsymbol{x} \in D. \tag{6}$$

III.1.3 Classical formulation of the problem in the domain D. Let functions $\boldsymbol{f} : D \times (0, T) \to \mathbb{R}^2$, $\boldsymbol{u}_0 : D \to \mathbb{R}^2$, $\boldsymbol{g} : G_i \to \mathbb{R}^2$ and $\boldsymbol{h} : G_o \to \mathbb{R}^2$ be continuous and periodic in the direction x_2 with period τ. The *classical solution* of the initial–boundary value problem in domain D can be defined as a pair of functions $\boldsymbol{u} = (u_1, u_2)$ and p such that

a) $\boldsymbol{u}$ is continuous in $\overline{D} \times [0, T)$,

b) the 1st order partial derivative of $\boldsymbol{u}$ with respect to time and the 2nd order partial derivatives of $\boldsymbol{u}$ with respect to the space variables are continuous in $D \times (0, T)$,

c) p is continuous in $[D \cup G_o] \times (0, T)$,

d) p has continuous the 1st order partial derivatives with respect to the space variables in $D \times (0, T)$,

and furthermore, $\boldsymbol{u}$ and p satisfy the equations (2) and (1) in $D \times (0, T)$, the boundary conditions (3), (4), (5) and the initial condition (6).

III.1.4 Classical formulation of the problem in the domain Ω. The domain D is unbounded, but due to the τ–periodicity of all the given data in the x_2–direction, it is logical to search for the solution in the class of functions which are also τ–periodic in the variable x_2. This approach enables us, like in Chapter I, to reduce the problem only to one spatial period Ω of domain D. The assumptions on the given data can therefore be also reduced only onto the one spatial period Ω:

Given continuous vector functions $\boldsymbol{f}$ (on $\overline{\Omega} \times (0, T)$), $\boldsymbol{g}$ (on Γ_i) and $\boldsymbol{h}$ (on Γ_o) such that

$$\begin{aligned} \boldsymbol{f}(x_1, x_2) &= \boldsymbol{f}(x_1, x_2 + \tau) \qquad \text{for } (x_1, x_2) \in \Gamma_-, \\ \boldsymbol{g}(A_0) &= \boldsymbol{g}(A_1), \\ \boldsymbol{h}(B_0) &= \boldsymbol{h}(B_1). \end{aligned} \tag{7}$$

Let us set $Q_T = \Omega \times (0, T)$.

We want to find $\boldsymbol{u} = (u_1, u_2)$ and p such that

a) $\boldsymbol{u} \in C^2(\overline{Q_T})$,

b) $p \in C^1(\overline{Q_T})$,

c) $\boldsymbol{u}$ and p satisfy the equations (1) and (2) in Q_T,

d) $\boldsymbol{u}$ and p satisfy the boundary conditions (3), (4), (5) restricted to the boundary of the domain Ω:

$$\boldsymbol{u}|_{\Gamma_i} = \boldsymbol{g} \qquad \text{for } t \in (0, T), \tag{8}$$

$$\boldsymbol{u}|_{\Gamma_w} = \boldsymbol{0} \qquad \text{for } t \in (0, T), \tag{9}$$

$$-\nu \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}} + p\boldsymbol{n} - \frac{1}{2} (\boldsymbol{u} \cdot \boldsymbol{n})^- \boldsymbol{u} = \boldsymbol{h} \qquad \text{on } \Gamma_o \times (0, T), \tag{10}$$

e) $\boldsymbol{u}$ and p satisfy the periodicity conditions

$$\boldsymbol{u}(x_1, x_2 + \tau, t) = \boldsymbol{u}(x_1, x_2, t) \qquad \text{for } (x_1, x_2) \in \Gamma_-,\ t \in (0, T), \tag{11}$$

$$\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}}(x_1, x_2 + \tau, t) = -\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}}(x_1, x_2, t) \qquad \text{for } (x_1, x_2) \in \Gamma_-,\ t \in (0, T), \tag{12}$$

$$p(x_1, x_2 + \tau, t) = p(x_1, x_2, t) \qquad \text{for } (x_1, x_2) \in \Gamma_-,\ t \in (0, T), \tag{13}$$

f) $\boldsymbol{u}$ satisfies the initial condition initial condition (6) restricted to the domain Ω:

$$\boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{u}_0(\boldsymbol{x}) \qquad \text{for } \boldsymbol{x} \in \Omega. \tag{14}$$

The problem above will be further for simplicity denoted by $\mathcal{P}_{\text{class}}(\Omega)$.

Remark 3. In the same way as in Section (I. 1)), it can be proved that the restriction of the classical solution in domain D onto Ω is a classical solution in domain Ω. On the other hand, the τ–periodic extension of a classical solution in Ω onto D is a classical solution in domain D. □

III.2 Weak formulation

III.2.1 Definition of function spaces. In addition to all the function spaces defined in the subsection I.2.1, we shall also need spaces which are usual in the theory of evolutionary equations.

Let $\mathcal{B}$ be a Banach space. For $r \in [1, \infty]$ we denote by $L^r(0,T;\,\mathcal{B})$ the Bochner space of strongly measurable functions $\boldsymbol{u} : (0,T) \to \mathcal{B}$ such that

$$\begin{aligned} \|\boldsymbol{u}\|_{L^r(0,T;\,\mathcal{B})} &:= \left[\int_a^b \|\boldsymbol{u}(t)\|_{\mathcal{B}}^r \,\mathrm{d}t\right]^{1/r} < +\infty \qquad \text{if } 1 \le r < +\infty, \\ \|\boldsymbol{u}\|_{L^r(0,T;\,\mathcal{B})} &:= \operatorname*{ess\,sup}_{t\in(0,T)} \|\boldsymbol{u}(t)\|_{\mathcal{B}} < +\infty \qquad \text{if } r = +\infty. \end{aligned}$$

III.2.2 Weak formulation of the initial–boundary value problem in the domain Ω. We shall formally derive the weak formulation from the classical formulation in this subsection. Suppose that $\boldsymbol{u} = (u_1, u_2)$ and p is a solution of the problem $\mathcal{P}_{\text{class}}(\Omega)$. Let us multiply equation (2) by an arbitrary test function $\boldsymbol{v} = (v_1, v_2) \in V$, integrate over Ω and use Green's theorem. We obtain

$$\begin{aligned} \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} &= \int_\Omega \left(\frac{\partial \boldsymbol{u}}{\partial t} - \nu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\,\boldsymbol{u} + \nabla p\right)\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} \\ &= \int_\Omega \frac{\partial \boldsymbol{u}}{\partial t}\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} + \nu\int_\Omega \sum_{i,j=1}^{2} \frac{\partial u_i}{\partial x_j}\frac{\partial v_i}{\partial x_j}\,\mathrm{d}\boldsymbol{x} - \nu\int_{\partial\Omega} \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}}\cdot\boldsymbol{v}\,\mathrm{d}S \\ &\quad + \int_\Omega \sum_{i,j=1}^{2} u_j\frac{\partial u_i}{\partial x_j}\,v_i\,\mathrm{d}\boldsymbol{x} - \int_\Omega p\,\operatorname{div}\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} + \int_{\partial\Omega} p\,\boldsymbol{v}\cdot\boldsymbol{n}\,\mathrm{d}S. \end{aligned} \tag{15}$$

Using the properties of the function $\boldsymbol{v}$, the boundary conditions (10)–(13) and the relation

$$\boldsymbol{n}(x_1,x_2) = -\boldsymbol{n}(x_1,x_2+\tau) \qquad (\text{for } (x_1,x_2)\in\Gamma_-), \tag{16}$$

we obtain the identity

$$\begin{aligned} \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} &= \int_\Omega \frac{\partial \boldsymbol{u}}{\partial t}\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} + \nu\int_\Omega \sum_{i,j=1}^{2} \frac{\partial u_i}{\partial x_j}\frac{\partial v_i}{\partial x_j}\,\mathrm{d}\boldsymbol{x} + \\ &+ \int_\Omega \sum_{i,j=1}^{2} u_j\frac{\partial u_i}{\partial x_j}\,v_i\,\mathrm{d}\boldsymbol{x} + \int_{\Gamma_o} \frac{1}{2}(\boldsymbol{u}\cdot\boldsymbol{n})^-\boldsymbol{u}\cdot\boldsymbol{v}\,\mathrm{d}S + \int_{\Gamma_o} \boldsymbol{h}\cdot\boldsymbol{v}\,\mathrm{d}S. \end{aligned} \tag{17}$$

For $\boldsymbol{u} = (u_1, u_2)$, $\boldsymbol{v} = (v_1, v_2)$, $\boldsymbol{w} = (w_1, w_2) \in H^1(\Omega)^2$ we use the same scalar product $(\,.\,,\,.\,)$ and forms as in the subsection I.2.5:

$$
\begin{aligned}
(\boldsymbol{v}, \boldsymbol{w}) &= \int_\Omega \boldsymbol{v} \cdot \boldsymbol{w} \, \mathrm{d}\boldsymbol{x}, \\
a_1(\boldsymbol{u}, \boldsymbol{v}) &= \nu \int_\Omega \sum_{i,j=1}^{2} \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j} \, \mathrm{d}\boldsymbol{x}, \\
a_2(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) &= \int_\Omega \sum_{i,j=1}^{2} u_j \frac{\partial v_i}{\partial x_j} \, w_i \, \mathrm{d}\boldsymbol{x}, \\
a_3(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) &= \int_{\Gamma_o} \frac{1}{2} (\boldsymbol{u} \cdot \boldsymbol{n})^- \boldsymbol{v} \cdot \boldsymbol{w} \, \mathrm{d}S, \\
b(\boldsymbol{h}, \boldsymbol{v}) &= -\int_{\Gamma_o} \boldsymbol{h} \cdot \boldsymbol{v} \, \mathrm{d}S, \\
a(\boldsymbol{u}, \boldsymbol{v}) &= a_1(\boldsymbol{u}, \boldsymbol{v}) + a_2(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) + a_3(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}).
\end{aligned} \tag{18}
$$

For the classical solution $\boldsymbol{u}$ and a function $\boldsymbol{v} \in V$ we can write

$$
\int_\Omega \frac{\partial \boldsymbol{u}}{\partial t} \boldsymbol{v} \, \mathrm{d}\boldsymbol{x} \;=\; \Big(\frac{\partial \boldsymbol{u}(t)}{\partial t}, \boldsymbol{v} \Big) \;=\; \frac{\mathrm{d}}{\mathrm{d}t} \big(\boldsymbol{u}(t), \boldsymbol{v} \big).
$$

Using the above notation, we can rewrite the integral identity (17) in the form

$$
\frac{\mathrm{d}}{\mathrm{d}t} (\boldsymbol{u}(t), \boldsymbol{v}) + a(\boldsymbol{u}(t), \boldsymbol{v}) \;=\; (\boldsymbol{f}(t), \boldsymbol{v}) + b(\boldsymbol{h}(t), \boldsymbol{v}), \quad \boldsymbol{v} \in V, \; t \in (0, T). \tag{19}
$$

Definition 4. Let $\boldsymbol{f} \in L^2(0, T;\, L^2(\Omega)^2)$, $\boldsymbol{u}_0 \in L^2(\Omega)^2$ and $\boldsymbol{h} \in L^2(0, T;\, L^2(\Gamma_o))$. Further, let $\boldsymbol{g} \in H^s(\Gamma_i)^2$ for some $s \in \big(\frac{1}{2}, 1\big)$ and $\boldsymbol{g}(A_1) = \boldsymbol{g}(A_0)$. We look for a function $\boldsymbol{u}$ with following properties:

a) $\boldsymbol{u} \in L^2(0, T;\, H^1(\Omega)^2) \cap L^\infty(0, T;\, L^2(\Omega)^2)$,

b) $\boldsymbol{u}$ fulfills identity (19) for all test functions $\boldsymbol{v} \in V$ in the sense of scalar distributions in the interval $(0, T)$ and the continuity equation (1) in Ω,

c) the conditions

$$
\begin{aligned}
\boldsymbol{u}|_{\Gamma_i} &= \boldsymbol{g}, \\
\boldsymbol{u}|_{\Gamma_w} &= \boldsymbol{0}, \\
\boldsymbol{u}(x_1, x_2 + \tau, t) &= \boldsymbol{u}(x_1, x_2, t) \qquad \text{for } (x_1, x_2) \in \Gamma_-, \; t \in (0, T)
\end{aligned} \tag{20}
$$

are satisfied in the sense of traces,

d) $\boldsymbol{u}$ satisfies the initial condition

$$
\boldsymbol{u}(0) = \boldsymbol{u}_0. \tag{21}
$$

This problem will be further denoted by $\mathcal{P}_{\mathrm{weak}}(\Omega)$. Its solution is called the **weak solution**.

III.2.3 Some deeper properties of the forms a_1, a_2, a_3. It can be verified that the forms a_1, a_2, a_3 and b, defined by (17), make sense for $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w} \in H^1(\Omega)^2$ and $\boldsymbol{h} \in L^2(\Gamma_o)^2$. By $\langle\, .\, ,\, .\,\rangle$ we shall denote the duality between the spaces V^* and V or between the spaces $(H^1(\Omega)^2)^*$ and $H^1(\Omega)^2$.

For a fixed $\boldsymbol{z} \in H^1(\Omega)^2$, let us consider the linear mappings

$$\begin{array}{llcl} \text{a)} & \boldsymbol{v} \in V & \longrightarrow & a_1(\boldsymbol{z}, \boldsymbol{v}) \in \mathbb{R}, \\ \text{b)} & \boldsymbol{v} \in V & \longrightarrow & a_2(\boldsymbol{z}, \boldsymbol{z}, \boldsymbol{v}) \in \mathbb{R}, \\ \text{c)} & \boldsymbol{v} \in V & \longrightarrow & a_3(\boldsymbol{z}, \boldsymbol{z}, \boldsymbol{v}) \in \mathbb{R}. \end{array} \tag{22}$$

Lemma 13 *The mappings defined by (22) are continuous linear functionals on V. Thus, there exist operators*

$$A_1 : H^1(\Omega)^2 \longrightarrow V^*, \quad A_2 : H^1(\Omega)^2 \longrightarrow V^*, \quad A_3 : H^1(\Omega)^2 \longrightarrow V^*$$

such that

$$\begin{aligned} \langle A_1 \boldsymbol{z}, \boldsymbol{v} \rangle &= a_1(\boldsymbol{z}, \boldsymbol{v}), \\ \langle A_2 \boldsymbol{z}, \boldsymbol{v} \rangle &= a_2(\boldsymbol{z}, \boldsymbol{z}, \boldsymbol{v}), \\ \langle A_3 \boldsymbol{z}, \boldsymbol{v} \rangle &= a_3(\boldsymbol{z}, \boldsymbol{z}, \boldsymbol{v}) \end{aligned} \tag{23}$$

for $\boldsymbol{z} \in H^1(\Omega)^2$ and $\boldsymbol{v} \in V$.

Further, for $\boldsymbol{z} \in L^2(0,T; H^1(\Omega)^2) \cap L^\infty(0,T; L^2(\Omega)^2)$, the functions $A_1\boldsymbol{z}(t)$, $A_2\boldsymbol{z}(t)$, $A_3\boldsymbol{z}(t)$ are defined for almost all $t \in (0,T)$ and

$$A_1 \boldsymbol{z} \in L^2(0,T; V^*), \quad A_2 \boldsymbol{z},\ A_3 \boldsymbol{z} \in L^{4/3}(0,T; V^*).$$

Namely, there exist $c_{59} > 0$ and $c_{60} > 0$ such that

$$\|A_1 \boldsymbol{z}\|_{L^2(0,T;V^*)} \leq \nu\, \|\boldsymbol{z}\|_{L^2(0,T;H^1(\Omega)^2)}, \tag{24}$$

$$\|A_2 \boldsymbol{z}\|_{L^{4/3}(0,T;V^*)} \leq c_{59}\, \|\boldsymbol{z}\|^{1/2}_{L^\infty(0,T;L^2(\Omega)^2)}\, \|\boldsymbol{z}\|^{3/2}_{L^2(0,T;H^1(\Omega)^2)}, \tag{25}$$

$$\|A_3 \boldsymbol{z}\|_{L^{4/3}(0,T;V^*)} \leq c_{60}\, \|\boldsymbol{z}\|^{1/2}_{L^\infty(0,T;L^2(\Omega)^2)}\, \|\boldsymbol{z}\|^{3/2}_{L^2(0,T;H^1(\Omega)^2)}. \tag{26}$$

Proof. If $\boldsymbol{z} \in H^1(\Omega)^2$ and $\boldsymbol{v} \in V$ then

$$|a_1(\boldsymbol{z}, \boldsymbol{v})| = \nu\, |((\boldsymbol{z}, \boldsymbol{v}))| \leq \nu\, |||\boldsymbol{z}|||\, |||\boldsymbol{v}|||.$$

Consequently, the functional (22a) is continuous and we obtain the existence of an element $A_1 \boldsymbol{z} \in V^*$ such that the first identity in (23) holds for all $\boldsymbol{v} \in V$. Moreover, if $\boldsymbol{z} \in L^2(0,T; H^1(\Omega)^2)$, then

$$\|A_1 \boldsymbol{z}(t)\|_{V^*} \leq \nu |||\boldsymbol{z}(t)|||$$

for almost all $t \in (0,T)$ and

$$\|A_1 \boldsymbol{z}\|^2_{L^2(0,T;V^*)} = \int_0^T \|A_1 \boldsymbol{z}(t)\|^2_{V^*}\, \mathrm{d}t \leq \nu \int_0^T |||\boldsymbol{z}(t)|||^2\, \mathrm{d}t \leq \nu \|\boldsymbol{z}\|^2_{L^2(0,T;H^1(\Omega)^2)},$$

which yields (24).

Let us further consider the form a_2. Using the Cauchy inequality, the continuous imbedding $H^1(\Omega) \hookrightarrow L^4(\Omega)$, (I.28) and (8) (Lemma A2 in Appendix), for $\boldsymbol{z} \in H^1(\Omega)^2$ and $\boldsymbol{v} \in V$ we can deduce that

$$\begin{aligned} a_2(\boldsymbol{z},\boldsymbol{z},\boldsymbol{v}) \;&=\; \int_\Omega \sum_{i,j=1,2} z_j \frac{\partial z_i}{\partial x_j} v_i \,\mathrm{d}\boldsymbol{x} \;\le\; C\,\|\boldsymbol{z}\|_{L^4(\Omega)^2} |||\boldsymbol{z}|||\, \|\boldsymbol{v}\|_{L^4(\Omega)^2} \\ &\le\; c_{61}\,\|\boldsymbol{z}\|^{1/2}_{L^2(\Omega)^2}\,\|\boldsymbol{z}\|^{3/2}_{H^1(\Omega)^2}\,|||\boldsymbol{v}||| \end{aligned}$$

for all $\boldsymbol{z} \in H^1(\Omega)^2$ and $\boldsymbol{v} \in V$. (Here C is a generic constant and c_{61} is a fixed constant independent of $\boldsymbol{z}$ and $\boldsymbol{v}$.) Hence, for each $\boldsymbol{z} \in H^1(\Omega)^2$ there exists $A_2\boldsymbol{z} \in V^*$ such that

$$a_2(\boldsymbol{z},\boldsymbol{z},\boldsymbol{v}) \;=\; \langle A_2\boldsymbol{z}, \boldsymbol{v}\rangle$$

for all $\boldsymbol{v} \in V$ and

$$\|A_2\boldsymbol{z}\|_{V^*} \;\le\; c_{61}\,\|\boldsymbol{z}\|^{1/2}_{L^2(\Omega)^2}\,\|\boldsymbol{z}\|^{3/2}_{H^1(\Omega)^2}.$$

Furthermore, if $\boldsymbol{z} \in L^\infty(0,T;\, L^2(\Omega)^2) \cap L^2(0,T;\, H^1(\Omega)^2)$, we have

$$\begin{aligned} \int_0^T \|A_2\boldsymbol{z}\|^{4/3}_{V^*}\,\mathrm{d}t \;&\le\; c_{61} \int_0^T \|\boldsymbol{z}(t)\|^{\frac{2}{3}}_{L^2(\Omega)^2} \|\boldsymbol{z}(t)\|^2_{H^1(\Omega)^2}\,\mathrm{d}t \\ &\le c_{62}\,\|\boldsymbol{z}\|^{2/3}_{L^\infty(0,T;L^2(\Omega)^2)}\,\|\boldsymbol{z}\|^2_{L^2(0,T;H^1(\Omega)^2)}, \end{aligned}$$

which implies that

$$\|A_2\boldsymbol{z}\|_{L^{4/3}(0,T;V^*)} \le c_{62}\,\|\boldsymbol{z}\|^{1/2}_{L^\infty(0,T;\,L^2(\Omega)^2)}\,\|\boldsymbol{z}\|^{3/2}_{L^2(0,T;\,H^1(\Omega)^2)}. \tag{27}$$

This proves (25).

Let us finally consider the form a_3. For $\boldsymbol{z} \in H^1(\Omega)^2$ and $\boldsymbol{v} \in V$, using the Cauchy inequality, the multiplicative trace inequality (2) (Lemma A1 in Appendix), the trace inequality $\|\boldsymbol{z}\|_{L^4(\partial\Omega)^2} \le C\,\|\boldsymbol{z}\|_{H^1(\Omega)^2}$ and (I.28), we obtain

$$\begin{aligned} a_3(\boldsymbol{z},\boldsymbol{z},\boldsymbol{v}) \;&=\; \frac{1}{2}\int_{\Gamma_o} (\boldsymbol{z}\cdot\boldsymbol{n})^- \boldsymbol{z}\cdot\boldsymbol{v}\,\mathrm{d}S \;\le\; \frac{1}{2}\int_{\Gamma_o} |\boldsymbol{z}|^2\,|\boldsymbol{v}|\,\mathrm{d}S \\ &\le\; \frac{1}{2}\|\boldsymbol{z}\|_{L^2(\Gamma_o)^2}\,\|\boldsymbol{z}\|_{L^4(\Gamma_o)}\,\|\boldsymbol{v}\|_{L^4(\Gamma_o)^2} \;\le\; c_{63}\,\|\boldsymbol{z}\|^{1/2}_{L^2(\Omega)^2}\,\|\boldsymbol{z}\|^{3/2}_{H^1(\Omega)^2}\,|||\boldsymbol{v}|||. \end{aligned}$$

This implies that there exists $A_3\boldsymbol{z} \in V^*$ such that

$$a_3(\boldsymbol{z},\boldsymbol{z},\boldsymbol{v}) \;=\; \langle A_3\boldsymbol{z}, \boldsymbol{v}\rangle,$$

for all $\boldsymbol{v} \in V$ and

$$\|A_3\boldsymbol{z}\|_{V^*} \;\le\; c_{63}\,\|\boldsymbol{z}\|^{1/2}_{L^2(\Omega)^2}\,\|\boldsymbol{z}\|^{3/2}_{H^1(\Omega)^2}.$$

Now we can proceed similarly as in (27) and show that for $\boldsymbol{z} \in L^\infty(0,T;\, L^2(\Omega)^2) \cap L^2(0,T;\, H^1(\Omega)^2)$ one has

$$\|A_3\boldsymbol{z}\|_{L^{4/3}(0,T;V^*)} \;\le\; c_{64}\,\|\boldsymbol{z}\|^{1/2}_{L^\infty(0,T;L^2(\Omega)^2)}\,\|\boldsymbol{z}\|^{3/2}_{L^2(0,T;H^1(\Omega)^2)}.$$

Hence (26) holds. This completes the proof. $\square$

Remark 4. The operators A_1, A_2, A_3 can also be considered as mappings of $H^1(\Omega)^2$ not only into V^*, but into $(H^1(\Omega)^2)^*$ because the expressions $\langle A_i \boldsymbol{z}, \boldsymbol{v}\rangle$ make sense for $\boldsymbol{z}$, $\boldsymbol{v} \in H^1(\Omega)^2$. In a similar way as above, we can prove the estimates

$$\|A_1\boldsymbol{z}\|^*_{L^2(0,T;(H^1(\Omega)^2)}) \;\leq\; \nu\,\|\boldsymbol{z}\|_{L^2(0,T;H^1(\Omega)^2)}, \tag{28}$$

$$\|A_2\boldsymbol{z}\|_{L^{4/3}(0,T;(H^1(\Omega)^2))^*)} \;\leq\; c_{65}\,\|\boldsymbol{z}\|^{1/2}_{L^\infty(0,T;L^2(\Omega)^2)}\,\|\boldsymbol{z}\|^{3/2}_{L^2(0,T;H^1(\Omega)^2)}, \tag{29}$$

$$\|A_3\boldsymbol{z}\|_{L^{4/3}(0,T;(H^1(\Omega)^2))^*)} \;\leq\; c_{66}\,\|\boldsymbol{z}\|^{1/2}_{L^\infty(0,T;L^2(\Omega)^2)}\,\|\boldsymbol{z}\|^{3/2}_{L^2(0,T;H^1(\Omega)^2)}. \tag{30}$$

$\square$

By analogy with the procedures used in the books [55] or [7], we can write the term containing the derivative with respect to time in (19) in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\boldsymbol{u}(t),\boldsymbol{v}\big) \;=\; \langle \boldsymbol{u}', \boldsymbol{v}\rangle, \qquad \text{for } \boldsymbol{v} \in V, \tag{31}$$

where $\boldsymbol{u}'$ is the generalized time derivative of $\boldsymbol{u}$. Thus, the identity (19), satisfied by the weak solution, can be written in the form of an abstract differential equation

$$\boldsymbol{u}' + A_1\boldsymbol{u} + A_2\boldsymbol{u} + A_3\boldsymbol{u} \;=\; \boldsymbol{f}. \tag{32}$$

The properties of the operators A_1, A_2, A_3 and the function $\boldsymbol{f}$ imply that

$$\boldsymbol{u}' \in L^{4/3}(0,T;\, V^*). \tag{33}$$

This implies that $\boldsymbol{u} \in C([0,T];V^*)$, which gives a reasonable sense to the initial condition (6).

III.3 Existence of a weak solution

We shall be concerned with the proof of the existence of a solution to problem $\mathcal{P}_{\text{weak}}(\Omega)$ in the rest of this chapter. As in Chapter I, we can get rid of the inhomogeneous Dirichlet boundary condition (8) on Γ_i. Lemma 6 in the subsection I.3.2 guarantees the existence of a function $\boldsymbol{g}^*$ with the properties:

$$\begin{aligned}
&\boldsymbol{g}^* \in H^1(\Omega)^2, \qquad \operatorname{div}\boldsymbol{g}^* = \boldsymbol{0} \ \text{ in } \Omega,\\
&\boldsymbol{g}^*|_{\Gamma_i} = \boldsymbol{g}, \qquad \boldsymbol{g}^*|_{\Gamma_w} = 0,\\
&\boldsymbol{g}^*(x_1, x_2+\tau) \;=\; \boldsymbol{g}^*(x_1,x_2),\ (x_1,x_2) \in \Gamma_-.
\end{aligned} \tag{34}$$

This provides the opportunity to search for the weak solution in the form

$$\boldsymbol{u}(t) = \boldsymbol{g}^* + \boldsymbol{z}(t), \qquad t \in (0,T),$$

where $\boldsymbol{z}(t) \in V$ for almost all $t \in (0,T)$.

Remark 5. In order to solve the stationary problem for approximations in the subsection I.3.6 and to obtain their appropriate estimates, it was necessary to assume that the inflow velocity profile $\boldsymbol{g}$ on Γ_i is "sufficiently small". As we shall see in this section, solution of the nonstationary problem does not require this condition and we may therefore assume that the norm $\|\boldsymbol{g}\|_{H^s(\Gamma_i)^2}$, and consequently also $\|\boldsymbol{g}^*\|_{H^1(\Omega)^2}$, are arbitrarily large. $\square$

III.3.1 Time semidiscretization. For the proof of the existence of a weak solution we shall apply the method of discretization in time, also called the Rothe method; see [32], [53]. This method converts the nonstationary Navier-Stokes problem to a sequence of modified stationary problems.

For any $n > 0$ we put $\vartheta = T/n$ and consider the partition of the interval $[0, T]$ formed by the time instants $t_k = k\vartheta$, $k = 0, 1, \dots, n$.

Let us find the sequence of functions $\boldsymbol{u}^0, \boldsymbol{u}^1, \boldsymbol{u}^2, \dots, \boldsymbol{u}^n$, which are solutions of the modified stationary problems:

$$\begin{aligned} a)&\quad \boldsymbol{u}^0 = \boldsymbol{u}_0 \in H, \\ b)&\quad \boldsymbol{u}^k - \boldsymbol{g}^* \in V, \\ c)&\quad \Big(\frac{\boldsymbol{u}^k - \boldsymbol{u}^{k-1}}{\vartheta}, \boldsymbol{v}\Big) + a(\boldsymbol{u}^k, \boldsymbol{v}) = (\boldsymbol{f}^k, \boldsymbol{v}), \quad \forall\, \boldsymbol{v} \in V, \end{aligned} \tag{35}$$

where

$$\boldsymbol{f}^k := \frac{1}{\vartheta} \int_{t_{k-1}}^{t_k} \boldsymbol{f}(t)\, \mathrm{d}t \in L^2(\Omega)^2, \quad k = 1, 2, \dots, n. \tag{36}$$

We shall call the sequence $\{\boldsymbol{u}^k\}_{k=0}^n$ an *approximate solution*. The solution $\boldsymbol{u}^k$ of problem (35c) can be sought in the form

$$\boldsymbol{u}^k = \boldsymbol{g}^* + \boldsymbol{z}^k \tag{37}$$

where the new unknown $\boldsymbol{z}^k$ belongs to V. Let us set

$$A(\boldsymbol{u}, \boldsymbol{v}) = \frac{1}{\vartheta}(\boldsymbol{u}, \boldsymbol{v}) + a(\boldsymbol{u}, \boldsymbol{v}). \tag{38}$$

Then problem (35) is equivalent to finding $\boldsymbol{z}^k$ such that

$$\begin{aligned} a)&\quad \boldsymbol{z}^k \in V, \\ b)&\quad A(\boldsymbol{g}^* + \boldsymbol{z}^k, \boldsymbol{v}) = (\boldsymbol{f}^k, \boldsymbol{v}) + \frac{1}{\vartheta}(\boldsymbol{u}^{k-1}, \boldsymbol{v}) \quad \forall \boldsymbol{v} \in V. \end{aligned} \tag{39}$$

III.3.2 Existence of an approximate solution. The following theorem provides the information on the existence of the approximate solution. Since $\vartheta = T/n$ for $n \in \mathbb{N}$, ϑ can in fact take only a discrete set of values corresponding to $n \in \mathbb{N}$.

Theorem 16. There exists a constant $\tilde{\vartheta} > 0$ such that for any $\vartheta \in (0, \tilde{\vartheta})$ there exists at least one solution $\boldsymbol{z}^k$ of problem (39), and consequently, there exists at least one solution $\boldsymbol{u}^k$ of problem (35).

The proof of this theorem will be carried out in several steps. We can use the ideas from Chapter I, but we can also use the advantage of having the additive term $(1/\vartheta)\,(\boldsymbol{u}, \boldsymbol{v})$ in the form A (see (38)) which was not present in problems treated in Chapter I.

Lemma 14 *For all $z \in V$ we have*

$$a(\boldsymbol{g}^* + \boldsymbol{z}, \boldsymbol{z}^k) \ \geq \ \frac{\nu}{2} |||\boldsymbol{z}|||^2 - c_{67}\, \|\boldsymbol{z}\|^2_{L^2(\Omega)^2} - c_{68} \tag{40}$$

where

$$\begin{aligned} c_{67} &= \frac{C}{\nu}\, \|\boldsymbol{g}^*\|^2_{H^1(\Omega)^2} + \frac{C}{\nu^3}\, \|\boldsymbol{g}^*\|^4_{H^1(\Omega)^2}, \qquad (41) \\ c_{68} &= C\,\nu \|\boldsymbol{g}^*\|^2_{H^1(\Omega)^2} + \frac{C}{\nu}\, \|\boldsymbol{g}^*\|^4_{H^1(\Omega)^2} + \frac{C}{\nu^3}\, \|\boldsymbol{g}^*\|^4_{H^1(\Omega)^2}. \end{aligned}$$

The constant C is independent of $\boldsymbol{z}$, $\boldsymbol{g}^*$ and ν.

Proof. By (18),

$$\begin{aligned} a(\boldsymbol{g}^* + \boldsymbol{z}, \boldsymbol{z}) &= a_1(\boldsymbol{z}, \boldsymbol{z}) + a_1(\boldsymbol{g}^*, \boldsymbol{z}) \qquad (42) \\ &\quad + a_2(\boldsymbol{z}, \boldsymbol{z}, \boldsymbol{z}) + a_2(\boldsymbol{g}^*, \boldsymbol{g}^*, \boldsymbol{z}) + a_2(\boldsymbol{g}^*, \boldsymbol{z}, \boldsymbol{z}) + a_2(\boldsymbol{z}, \boldsymbol{g}^*, \boldsymbol{z}) \\ &\quad + a_3(\boldsymbol{z}, \boldsymbol{z}, \boldsymbol{z}) + a_3(\boldsymbol{g}^*, \boldsymbol{g}^*, \boldsymbol{z}) + a_3(\boldsymbol{g}^*, \boldsymbol{z}, \boldsymbol{z}) + a_3(\boldsymbol{z}, \boldsymbol{g}^*, \boldsymbol{z}). \end{aligned}$$

Hence,

$$\begin{aligned} a(\boldsymbol{g}^* + \boldsymbol{z}, \boldsymbol{z}) &\geq a_1(\boldsymbol{z}, \boldsymbol{z}) - |a_1(\boldsymbol{g}^*, \boldsymbol{z})| \qquad (43) \\ &\quad + a_2(\boldsymbol{z}, \boldsymbol{z}, \boldsymbol{z}) - |a_2(\boldsymbol{g}^*, \boldsymbol{g}^*, \boldsymbol{z})| - |a_2(\boldsymbol{g}^*, \boldsymbol{z}, \boldsymbol{z})| - |a_2(\boldsymbol{z}, \boldsymbol{g}^*, \boldsymbol{z})| \\ &\quad + a_3(\boldsymbol{z}, \boldsymbol{z}, \boldsymbol{z}) - |a_3(\boldsymbol{g}^*, \boldsymbol{g}^*, \boldsymbol{z})| - |a_3(\boldsymbol{g}^*, \boldsymbol{z}, \boldsymbol{z})| - |a_3(\boldsymbol{z}, \boldsymbol{g}^*, \boldsymbol{z})|. \end{aligned}$$

Now we shall estimate each term in (43). We shall use the Cauchy inequality, the continuous imbedding $H^1(\Omega) \hookrightarrow L^4(\Omega)$, the equivalence (I.28) of the norms $\|\cdot\|_{H^1(\Omega)^2}$ and $|||\cdot|||$ on V, the theorems on traces, the multiplicative trace inequality (2) (Lemma A1 in Appendix) and Young's inequality (23) (Lemma A3 in Appendix). We arrive at

$$a_1(\boldsymbol{z}, \boldsymbol{z}) = \nu |||\boldsymbol{z}|||^2, \tag{44}$$

$$|a_1(\boldsymbol{g}^*, \boldsymbol{z})| = \leq \nu\, \|\boldsymbol{g}^*\|_{H^1(\Omega)^2} |||\boldsymbol{z}||| \leq \frac{\nu}{12} |||\boldsymbol{z}|||^2 + c\nu \|\boldsymbol{g}^*\|^2_{H^1(\Omega)^2}, \tag{45}$$

$$\begin{aligned} |a_2(\boldsymbol{g}^*, \boldsymbol{g}^*, \boldsymbol{z})| &= \left| \int_\Omega \sum_{i,j=1}^2 g_j^* \frac{\partial g_i^*}{\partial x_j}\, z_i \, \mathrm{d}\boldsymbol{x} \right| \\ &\leq \left(\sum_{i,j=1}^2 \int_\Omega \left| \frac{\partial g_i^*}{\partial x_j} \right|^2 \mathrm{d}x \right)^{1/2} \left(\sum_{i,j=1}^2 \int_\Omega g_j^{*^2} (z_i)^2 \mathrm{d}\boldsymbol{x} \right)^{1/2} \\ &\leq \|\boldsymbol{g}^*\|_{H^1(\Omega)^2} \|\boldsymbol{g}^*\|_{L^4(\Omega)}\, \|\boldsymbol{z}\|_{L^4(\Omega)} \leq C\, \|\boldsymbol{g}^*\|^2_{H^1(\Omega)^2}\, |||\boldsymbol{z}||| \\ &\leq \frac{\nu}{12}\, |||\boldsymbol{z}|||^2 + \frac{C}{\nu}\, \|\boldsymbol{g}^*\|^4_{H^1(\Omega)^2}. \qquad (46) \end{aligned}$$

Further, using the Cauchy inequality, (8) (Lemma A2 in Appendix) and (23) (Lemma A3 in Appendix), we get

$$|a_2(\boldsymbol{z}, \boldsymbol{g}^*, \boldsymbol{z})| \leq \left| \int_\Omega |\nabla \boldsymbol{g}^*|\, |\boldsymbol{z}|^2 \, \mathrm{d}\boldsymbol{x} \right| \tag{47}$$

$$
\begin{aligned}
&\leq |\boldsymbol{g}^*|_{H^1(\Omega)} \|\boldsymbol{z}\|^2_{L^4(\Omega)} \leq c\, |\boldsymbol{g}^*|_{H^1(\Omega)} \|\boldsymbol{z}\|_{L^2(\Omega)} |||\boldsymbol{z}||| \\
&\leq \frac{\nu}{12} |||\boldsymbol{z}|||^2 + \frac{c}{\nu} \|\boldsymbol{z}\|^2_{L^2(\Omega)^2} \|\boldsymbol{g}^*\|^2_{H^1(\Omega)^2}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
|a_2(\boldsymbol{g}^*, \boldsymbol{z}, \boldsymbol{z}) &\leq \int_\Omega |\boldsymbol{g}^*||\boldsymbol{z}||\nabla \boldsymbol{z}| \,\mathrm{d}\boldsymbol{x} \qquad (48)\\
&\leq |||\boldsymbol{z}|||_{H^1(\Omega)} \|\boldsymbol{g}^*\|_{L^4(\Omega)^2} \|\boldsymbol{z}\|_{L^4(\Omega)^2} \leq c\, |||\boldsymbol{z}|||^{\frac{3}{2}} \|\boldsymbol{z}\|^{\frac{1}{2}}_{L^2(\Omega)^2} \|\boldsymbol{g}^*\|_{H^1(\Omega)^2} \\
&\leq \frac{\nu}{12} |||\boldsymbol{z}|||^2 + \frac{c}{\nu^3} \|\boldsymbol{z}\|^2_{L^2(\Omega)^2} \|\boldsymbol{g}^*\|^4_{H^1(\Omega)^2}.
\end{aligned}
$$

By (18) and Green's theorem, we obtain

$$
\begin{aligned}
a_2(\boldsymbol{z}, \boldsymbol{z}, \boldsymbol{z}) + a_3(\boldsymbol{z}, \boldsymbol{z}, \boldsymbol{z}) &= \int_\Omega \sum_{i,j=1}^2 z_j \frac{\partial z_i}{\partial x_j} z_i \,\mathrm{d}\boldsymbol{x} + \frac{1}{2}\int_{\Gamma_o} (\boldsymbol{z}\cdot\boldsymbol{n})^- |\boldsymbol{z}|^2 \,\mathrm{d}S \\
&= \frac{1}{2}\int_\Omega \sum_{i,j=1}^2 z_j \frac{\partial (z_i)^2}{\partial x_j} \,\mathrm{d}\boldsymbol{x} + \frac{1}{2}\int_{\Gamma_o} (\boldsymbol{z}\cdot\boldsymbol{n})^- |\boldsymbol{z}|^2 \,\mathrm{d}S \\
&= \frac{1}{2}\int_{\partial\Omega} (\boldsymbol{z}\cdot\boldsymbol{n})\, |\boldsymbol{z}|^2 \,\mathrm{d}S - \frac{1}{2}\int_\Omega \operatorname{div} \boldsymbol{z} |\boldsymbol{z}|^2 \,\mathrm{d}\boldsymbol{x} + \frac{1}{2}\int_{\Gamma_o} (\boldsymbol{z}\cdot\boldsymbol{n})^- \,|\boldsymbol{z}|^2 \,\mathrm{d}S.
\end{aligned}
$$

Using the properties of $\boldsymbol{z} \in V$, we get

$$
\begin{aligned}
a_2(\boldsymbol{z}, \boldsymbol{z}, \boldsymbol{z}) &+ a_3(\boldsymbol{z}, \boldsymbol{z}, \boldsymbol{z}) \qquad (49)\\
&= \frac{1}{2}\int_{\partial\Omega} (\boldsymbol{z}\cdot\boldsymbol{n})\, |\boldsymbol{z}|^2 \,\mathrm{d}S + \frac{1}{2}\int_{\Gamma_o} (\boldsymbol{z}\cdot\boldsymbol{n})^- \,|\boldsymbol{z}|^2 \,\mathrm{d}S \\
&= \frac{1}{2}\int_{\Gamma_o} (\boldsymbol{z}\cdot\boldsymbol{n})^+ \,|\boldsymbol{z}|^2 \,\mathrm{d}S \geq 0.
\end{aligned}
$$

Further, we have

$$
\begin{aligned}
|a_3(\boldsymbol{g}^*, \boldsymbol{g}^*, \boldsymbol{z})| &= \left|\frac{1}{2}\int_{\Gamma_o} (\boldsymbol{g}^*\cdot\boldsymbol{n})^- (\boldsymbol{g}^*\cdot\boldsymbol{z}) \,\mathrm{d}S\right| \leq \frac{1}{2}\int_{\Gamma_o} |\boldsymbol{g}^*|^2 \,|\boldsymbol{z}| \,\mathrm{d}S \qquad (50)\\
&\leq \frac{1}{2}\left(\int_{\Gamma_o} |\boldsymbol{g}^*|^4 \,\mathrm{d}S\right)^{1/2} \left(\int_{\Gamma_o} |\boldsymbol{z}|^2 \,\mathrm{d}S\right)^{1/2} \leq c\,\|\boldsymbol{g}^*\|^2_{H^1(\Omega)^2} \|\boldsymbol{z}\|_{H^1(\Omega)^2} \\
&\leq c\,\|\boldsymbol{g}^*\|^2_{H^1(\Omega)^2} |||\boldsymbol{z}||| \leq \frac{\nu}{12} |||\boldsymbol{z}|||^2 + \frac{c}{\nu^3} \|\boldsymbol{g}^*\|^4_{H^1(\Omega)^2} \|\boldsymbol{z}\|^2_{L^2(\Omega)^2},
\end{aligned}
$$

$$
\begin{aligned}
|a_3(\boldsymbol{g}^*, \boldsymbol{z}, \boldsymbol{z})| &\leq \frac{1}{2}\int_{\Gamma_o} |\boldsymbol{g}^*|\,|\boldsymbol{z}|^2 \,\mathrm{d}S \qquad (51)\\
&\leq \frac{1}{2}\|\boldsymbol{z}\|_{L^2(\Gamma_o)^2} \|\boldsymbol{g}^*\|_{L^4(\Gamma_o)^2} \|\boldsymbol{z}\|_{L^4(\Gamma_o)^2} \\
&\leq c\,\|\boldsymbol{g}^*\|_{H^1(\Omega)} |||\boldsymbol{z}|||^{\frac{3}{2}} \|\boldsymbol{z}\|^{\frac{1}{2}}_{L^2(\Omega)} \leq \frac{\nu}{12} |||\boldsymbol{z}|||^2 + \frac{c}{\nu^3} \|\boldsymbol{g}^*\|^4_{H^1(\Omega)} \|\boldsymbol{z}\|^2_{L^2(\Omega)}.
\end{aligned}
$$

In the same way we prove that

$$|a_3(\boldsymbol{z}, \boldsymbol{g}^*, \boldsymbol{z})| \le c\, \|\boldsymbol{g}^*\|_{H^1(\Omega)}\, |||\boldsymbol{z}|||^{\frac{3}{2}}\, \|\boldsymbol{z}\|^{\frac{1}{2}}_{L^2(\Omega)} \le \frac{\nu}{12}\, |||\boldsymbol{z}|||^2 + \frac{c}{\nu^3}\, \|\boldsymbol{g}^*\|^4_{H^1(\Omega)}\, \|\boldsymbol{z}\|^2_{L^2(\Omega)}. \tag{52}$$

If we substitute the above estimates (44) – (52) into (43), we get (40). □

Now we are ready to prove the **coercivity of the form** A.

Lemma 15 *The form A satisfies the estimates*

$$A(\boldsymbol{g}^* + \boldsymbol{z}, \boldsymbol{z}) - (\boldsymbol{f}^k, \boldsymbol{z}) - \frac{1}{\vartheta}(\boldsymbol{u}^{k-1}, \boldsymbol{z}) \ \ge\ \frac{1}{2\vartheta}\, \|\boldsymbol{z}\|^2_{L^2(\Omega)^2} + \frac{\nu}{2}|||\boldsymbol{z}|||^2 - \beta \quad \forall \boldsymbol{z} \in V \tag{53}$$

where

$$\beta = \beta(\vartheta) = \frac{2}{\vartheta}\left(\|\boldsymbol{g}^*\|^2_{L^2(\Omega)^2} + \|\boldsymbol{u}^{k-1}\|^2_{L^2(\Omega)^2}\right) + 2\vartheta\, \|\boldsymbol{f}^k\|^2_{L^2(\Omega)^2} + c_{68}, \tag{54}$$

provided $0 < \vartheta \le \tilde{\vartheta} := 1/8c_{67}$*, where* c_{67} *and* c_{68} *are the constants from Lemma 14.*

Proof. By Young's inequality we obtain the estimates

$$\begin{aligned}
\frac{1}{\vartheta}\,(\boldsymbol{g}^* + \boldsymbol{z}, \boldsymbol{z}) \ &\ge\ \frac{1}{\vartheta}\, \|\boldsymbol{z}\|^2_{L^2(\Omega)^2} - \frac{1}{8\vartheta}\, \|\boldsymbol{z}\|^2_{L^2(\Omega)^2} - \frac{2}{\vartheta}\, \|\boldsymbol{g}^*\|^2_{L^2(\Omega)^2},\\
\frac{1}{\vartheta}\,(\boldsymbol{u}^{k-1}, \boldsymbol{z}) \ &\le\ \frac{1}{\vartheta}\, \|\boldsymbol{u}^{k-1}\|_{L^2(\Omega)^2}\, \|\boldsymbol{z}\|_{L^2(\Omega)^2} \le \frac{1}{8\vartheta}\, \|\boldsymbol{z}\|^2_{L^2(\Omega)^2} + \frac{2}{\vartheta}\, \|\boldsymbol{u}^{k-1}\|^2_{L^2(\Omega)^2},\\
(\boldsymbol{f}^k, \boldsymbol{z}) \ &\le\ \|\boldsymbol{f}^k\|_{L^2(\Omega)^2}\|\boldsymbol{z}\|_{L^2(\Omega)^2} \le \frac{1}{8\vartheta}\, \|\boldsymbol{z}\|^2_{L^2(\Omega)^2} + 2\vartheta\, \|\boldsymbol{f}^k\|^2_{L^2(\Omega)^2}.
\end{aligned}$$

Now, using this and Lemma 14, we immediately obtain the inequality (53). □

<u>Construction of approximations.</u> For simplicity, we shall write $\boldsymbol{z}$ instead of $\boldsymbol{z}^k$ in (39) for a while. Thus, we wish to find $\boldsymbol{z} \in V$ such that

$$A(\boldsymbol{g}^* + \boldsymbol{z}, \boldsymbol{v}) = \frac{1}{\vartheta}(\boldsymbol{u}^{k-1}, \boldsymbol{v}) + (\boldsymbol{f}^k, \boldsymbol{v}) \quad \forall\, \boldsymbol{v} \in V. \tag{55}$$

We shall use the Galerkin method, which will be applied in several steps. The space V, as a subspace of $H^1(\Omega)^2$, is a separable Hilbert space and $\mathcal{V}$ is dense in V. Therefore, there exists an orthonormal basis $\{\boldsymbol{e}_i\}_{i=1}^{\infty}$ in V with $e_i \in \mathcal{V}$. Let us set

$$V_n = \mathcal{L}\{\boldsymbol{e}_1, \boldsymbol{e}_2, \dots, \boldsymbol{e}_n\},$$

a linear space spanned by the elements $\boldsymbol{e}_1, \dots, \boldsymbol{e}_n$. In this space we shall define an approximate solution of problem (35):

$$\boldsymbol{z}_n = \sum_{r=1}^{n} \xi_r\, \boldsymbol{e}_r\,. \tag{56}$$

If we set $\boldsymbol{\zeta} = [\xi_1, \dots, \xi_n] \in \mathbb{R}^n$ and

$$|\xi| \;=\; \left(\sum_{r=1}^{n} \xi_r^2\right)^{1/2},$$

then

$$|||\boldsymbol{z}_n||| \;=\; \sqrt{(\boldsymbol{z}_n, \boldsymbol{z}_n)_V} \;=\; \left(\sum_{r,l=1}^{n} \xi_r\, \xi_l\, (\boldsymbol{e}_k, \boldsymbol{e}_l)_V\right)^{1/2} \;=\; |\xi|. \tag{57}$$

The approximation $\boldsymbol{z}_n \in V_n$ of $\boldsymbol{z}$ is defined by the condition

$$A(\boldsymbol{g}^* + \boldsymbol{z}_n, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}) + \frac{1}{\vartheta}(\boldsymbol{u}^{k-1}, \boldsymbol{v}), \quad \text{for } \forall\, \boldsymbol{v} \in V_n. \tag{58}$$

This is equivalent to the equations

$$A(\boldsymbol{g}^* + \boldsymbol{z}_n, \boldsymbol{e}_r) = (\boldsymbol{f}, \boldsymbol{e}_r) + \frac{1}{\vartheta}(\boldsymbol{u}^{k-1}, \boldsymbol{e}_r) \quad \text{for } r = 1, \dots, n, \tag{59}$$

which means that

$$\begin{aligned}&\frac{1}{\vartheta}(\boldsymbol{g}^* + \boldsymbol{z}_n, \boldsymbol{e}_r) + a_1(\boldsymbol{g}^* + \boldsymbol{z}_n, \boldsymbol{e}_r) + a_2(\boldsymbol{g}^* + \boldsymbol{z}_n, \boldsymbol{g}^* + \boldsymbol{z}_n, \boldsymbol{e}_r)\\ &\quad + a_3(\boldsymbol{g}^* + \boldsymbol{z}_n, \boldsymbol{g}^* + \boldsymbol{z}_n, \boldsymbol{e}_r) = (\boldsymbol{f}, \boldsymbol{e}_r) + \frac{1}{\vartheta}(\boldsymbol{u}^{k-1}, \boldsymbol{e}_r) \quad \text{for } r = 1, \dots, n.\end{aligned}$$

Now using (56), we obtain

$$\begin{aligned}&\frac{1}{\vartheta}(\boldsymbol{g}^*, \boldsymbol{e}_r) + \frac{1}{\vartheta}\sum_{\ell=1}^{n} \xi_\ell(\boldsymbol{e}_\ell, \boldsymbol{e}_r) + a_1(\boldsymbol{g}^*, \boldsymbol{e}_r) + \sum_{\ell=1}^{n} \xi_\ell\, a_1(\boldsymbol{e}_\ell, \boldsymbol{e}_r) \qquad (60)\\ &+ a_2(\boldsymbol{g}^*, \boldsymbol{g}^*, \boldsymbol{e}_r) + \sum_{\ell=1}^{n} \xi_\ell \left[a_2(\boldsymbol{g}^*, \boldsymbol{e}_\ell, \boldsymbol{e}_r) + a_2(\boldsymbol{e}_\ell, \boldsymbol{g}^*, \boldsymbol{e}_r)\right]\\ &+ \sum_{\ell,m=1}^{n} \xi_\ell \xi_m\, a_2(\boldsymbol{e}_\ell, \boldsymbol{e}_m, \boldsymbol{e}_r) + a_3(\boldsymbol{g}^*, \boldsymbol{g}^*, \boldsymbol{e}_r)\\ &+ \sum_{\ell=1}^{n} \xi_\ell \left[a_3(\boldsymbol{g}^*, \boldsymbol{e}_\ell, \boldsymbol{e}_r) + a_3(\boldsymbol{e}_\ell, \boldsymbol{g}^*, \boldsymbol{e}_r)\right]\\ &+ \sum_{\ell,m=1}^{n} \xi_\ell \xi_m\, a_3(\boldsymbol{e}_\ell, \boldsymbol{e}_m, \boldsymbol{e}_r) - (\boldsymbol{f}, \boldsymbol{e}_r) - \frac{1}{\vartheta}(\boldsymbol{u}^{k-1}, \boldsymbol{e}_r) = 0 \quad \text{for } r = 1, \dots, n.\end{aligned}$$

This is a system of quadratic equations for the unknowns $\xi_1, \dots, \xi_n$, which can be written in the form of one equation

$$\mathcal{A}(\boldsymbol{\zeta}) \;=\; \boldsymbol{0}, \tag{61}$$

in $\mathbb{R}^n$, where

$$\mathcal{A}(\boldsymbol{\zeta}) = [\mathcal{A}_1(\boldsymbol{\zeta}), \dots , \mathcal{A}_n(\boldsymbol{\zeta})] \,,$$

$\mathcal{A}_r(\boldsymbol{\zeta})$ are the left-hand sides of (60) and $\mathcal{A} : \mathbb{R}^n \to \mathbb{R}^n$. Obviously, the mapping $\mathcal{A}$ is continuous. The equation (61) can be solved by means of Lemma 7. Let us verify the assumption of this lemma. By a detailed calculation we find that

$$\begin{aligned}
\mathcal{A}(\xi)\cdot\xi &= \sum_{r=1}^{n}\mathcal{A}_r(\xi)\cdot\xi = \sum_{r=1}^{n}\xi_r\cdot\frac{1}{\vartheta}(\boldsymbol{g}^*,\boldsymbol{e}_r) + \sum_{r,\ell=1}^{n}\xi_r\cdot\frac{1}{\vartheta}\xi_\ell(\boldsymbol{e}_\ell,\boldsymbol{e}_r)\\
&\quad + \sum_{r=1}^{n}\xi_r\, a_1(\boldsymbol{g}^*,\boldsymbol{e}_r) + \sum_{r,\ell=1}^{n}\xi_\ell\xi_r\, a_1(\boldsymbol{e}_\ell,\boldsymbol{e}_r) + \sum_{r=1}^{n}\xi_r\, a_2(\boldsymbol{g}^*,\boldsymbol{g}^*,\boldsymbol{e}_r)\\
&\quad + \sum_{r,\ell=1}^{n}\xi_r\xi_\ell\left[a_2(\boldsymbol{g}^*,\boldsymbol{e}_\ell,\boldsymbol{e}_r) + a_2(\boldsymbol{e}_\ell,\boldsymbol{g}^*,\boldsymbol{e}_r)\right]\\
&\quad + \sum_{r,\ell,m}^{n}\xi_r\xi_\ell\xi_m a_2(\boldsymbol{e}_\ell,\boldsymbol{e}_m,\boldsymbol{e}_r) + \sum_{r=1}^{n}\xi_r\, a_3(\boldsymbol{g}^*,\boldsymbol{g}^*,\boldsymbol{e}_r)\\
&\quad + \sum_{r,\ell=1}^{n}\xi_r\xi_\ell\left[a_3(\boldsymbol{g}^*,\boldsymbol{e}_\ell,\boldsymbol{e}_r) + a_3(\boldsymbol{e}_\ell,\boldsymbol{g}^*,\boldsymbol{e}_r)\right]\\
&\quad + \sum_{r,\ell,m=1}^{n}\xi_r\xi_\ell\xi_m\, a_3(\boldsymbol{e}_\ell,\boldsymbol{e}_m,\boldsymbol{e}_r) - \sum_{r=1}^{n}\xi_r(\boldsymbol{f},\boldsymbol{e}_r)\\
&\quad - \sum_{r=1}^{n}\frac{1}{\vartheta}\xi_r(\boldsymbol{u}^{k-1},\boldsymbol{e}_r)\\
&= \frac{1}{\vartheta}(\boldsymbol{g}^*,\boldsymbol{z}_n) + \frac{1}{\vartheta}(\boldsymbol{z}_n,\boldsymbol{z}_n) + a_1(\boldsymbol{g}^*,\boldsymbol{z}_n) + a_1(\boldsymbol{z}_n,\boldsymbol{z}_n) +\\
&+ a_2(\boldsymbol{g}^*,\boldsymbol{g}^*,\boldsymbol{z}_n) + a_2(\boldsymbol{g}^*,\boldsymbol{z}_n,\boldsymbol{z}_n) + a_2(\boldsymbol{z}_n,\boldsymbol{g}^*,\boldsymbol{z}_n) + a_2(\boldsymbol{z}_n,\boldsymbol{z}_n,\boldsymbol{z}_n) +\\
&+ a_3(\boldsymbol{g}^*,\boldsymbol{g}^*,\boldsymbol{z}_n) + a_3(\boldsymbol{g}^*,\boldsymbol{z}_n,\boldsymbol{z}_n) + a_3(\boldsymbol{z}_n,\boldsymbol{g}^*,\boldsymbol{z}_n) + a_3(\boldsymbol{z}_n,\boldsymbol{z}_n,\boldsymbol{z}_n) -\\
&- (\boldsymbol{f},\boldsymbol{z}_n) - \frac{1}{\vartheta}(\boldsymbol{u}^{k-1},\boldsymbol{z}_n) = A(\boldsymbol{g}^* + \boldsymbol{z}_n,\boldsymbol{z}_n) - (\boldsymbol{f},\boldsymbol{z}_n) - \frac{1}{\vartheta}(\boldsymbol{u}^{k-1},\boldsymbol{z}_n).
\end{aligned}$$

Using Lemma 15, (56) and (57) we easily verify that $\mathcal{A}(\xi)\cdot\xi \ge 0$ for all $\xi \in \mathbb{R}^n$ such that $|\xi|^2 = R > 2\,\beta(\vartheta)/\nu$. Then, by Lemma 7, the problem (58) has a solution such that $|||\boldsymbol{z}_n||| \le R$.

<u>The limit process in (58).</u> The space V is a reflexive Hilbert space and $\{\boldsymbol{z}_n\}$ is a bounded sequence. Thus, there exists a subsequence weakly convergent in V. For simplicity, we denote it again by $\{\boldsymbol{z}_n\}$. Hence we can write

$$\boldsymbol{z}_n \to \boldsymbol{z} \quad \text{for} \quad n \to +\infty \quad \text{weakly in } V, \tag{62}$$

which implies that

$$\boldsymbol{z}_n \to \boldsymbol{z} \quad \text{for} \quad n \to +\infty \quad \text{weakly in } H^1(\Omega)^2. \tag{63}$$

Moreover, since V is a subspace of $H^1(\Omega)^2$ and

$$H^1(\Omega)^2 \hookrightarrow\hookrightarrow L^q(\Omega)^2 \quad \text{for } q \in [1,\infty),$$

(62) implies that

$$\boldsymbol{z}_n \to \boldsymbol{z} \quad \text{for} \quad n \to +\infty \quad \text{strongly in } L^q(\Omega)^2 \text{ for } q \in [1,\infty).$$

The operator of traces from $H^1(\Omega)^2$ into $L^q(\partial\Omega)^2$ is compact for $q \in [1,\infty)$. This implies that

$$\boldsymbol{z}_n \to \boldsymbol{z} \quad \text{for} \quad n \to +\infty \quad \text{strongly in } L^q(\partial\Omega)^2$$

and, in particular,

$$\boldsymbol{z}_n \to \boldsymbol{z} \quad \text{for} \quad n \to +\infty \quad \text{strongly in } L^q(\Gamma_o)^2 \tag{64}$$

for each $q \in [1,\infty)$. From this we get

$$\left.\begin{array}{ccc} \boldsymbol{z}_n \cdot \boldsymbol{n} & \longrightarrow & \boldsymbol{z} \cdot \boldsymbol{n} \\ (\boldsymbol{z}_n \cdot \boldsymbol{n})^- & \longrightarrow & (\boldsymbol{z} \cdot \boldsymbol{n})^- \end{array}\right\} \quad \text{for } n \to +\infty, \text{ strongly in } L^q(\Gamma_o)^2. \tag{65}$$

In virtue of (64) and (65), the sequences $\{\boldsymbol{z}_n\}$ and $(\boldsymbol{z}_n \cdot \boldsymbol{n})^-$ are bounded in $L^q(\Gamma_o)^2$ for each $q \in [1,\infty)$.

Due to these facts the limit process in each part of the form $a(\cdot,\cdot)$ is possible and the limit function (which we from now again denote by $\boldsymbol{z}^k$ instead of $\boldsymbol{z}$) is a solution of the modified stationary problem (69). Hence, $\boldsymbol{u}^k := \boldsymbol{g}^* + \boldsymbol{z}^k$ is a solution of the problem (35), parts b)–c). This completes the proof of Theorem 16 in III.3.2.

III.3.3 Estimates of the approximations. We shall derive some useful estimates of the approximate solutions $\boldsymbol{z}^k$ and $\boldsymbol{u}^k$ of problems (39) and (35), respectively, in this subsection. Their existence for $\vartheta \in (0,\tilde{\vartheta})$ is guaranteed by Theorem 16.

We shall need the following discrete version of Gronwall's lemma ([32]). (Here a_i and A_i have nothing to do with our previous notation a_1–a_3 or A_1–A_3.)

Lemma 16 *Let $\{a_i\}$ and $\{A_i\}$ be sequences of nonnegative numbers and $L > 0$. We assume that the sequence $\{A_i\}$ is nondecreasing and $\vartheta < 1/L$. Let us set $L_\vartheta = L/(1-L\vartheta)$. If the inequality*

$$a_i \le A_i + L \sum_{j=1}^{i} a_j \vartheta$$

holds for all $i = 1,2,\ldots$, then

$$a_i \le \frac{L_\vartheta}{L} A_i \exp((i-1)\,\vartheta\, L_\vartheta), \quad i = 1,2,\ldots.$$

Lemma 17 *Suppose that $\boldsymbol{z}^k \in V$ fulfil (35). Then there exist positive constants c_{69}, c_{70} and c_{71}, independent of ϑ (and therefore also independent of k), such that*

$$\max_{k=1,\dots,n} \|\boldsymbol{z}^k\|_{L^2(\Omega)} \;\leq\; c_{69}, \tag{66}$$

$$\vartheta \sum_{k=1}^{n} |||\boldsymbol{z}^k|||^2 \;\leq\; c_{70}, \tag{67}$$

$$\sum_{k=1}^{n} \|\boldsymbol{z}^k - \boldsymbol{z}^{k-1}\|^2_{L^2(\Omega)} \;\leq\; c_{71}. \tag{68}$$

Proof. The substitution $\boldsymbol{u}^k = \boldsymbol{g}^* + \boldsymbol{z}^k$ in (35) yields

$$\left(\frac{\boldsymbol{z}^k - \boldsymbol{z}^{k-1}}{\vartheta},\, \boldsymbol{v}\right) + a(\boldsymbol{g}^* + \boldsymbol{z}^k, \boldsymbol{v}) \;=\; (\boldsymbol{f}^k, \boldsymbol{v}). \tag{69}$$

Choosing $\boldsymbol{v} = \boldsymbol{z}^k$ and using the formula

$$(a-b)a = \frac{1}{2}\,[a^2 - b^2 + (a-b)^2],$$

we get

$$\frac{1}{2\vartheta}\,\left(\|\boldsymbol{z}^k\|^2_{L^2(\Omega)^2} - \|\boldsymbol{z}^{k-1}\|^2_{L^2(\Omega)^2} + \|\boldsymbol{z}^k - \boldsymbol{z}^{k-1}\|^2_{L^2(\Omega)^2}\right) + a(\boldsymbol{g}^* + \boldsymbol{z}^k, \boldsymbol{z}^k) = (\boldsymbol{f}^k, \boldsymbol{z}^k).$$

By Lemma 14, (28) and Young's inequality we have

$$\begin{aligned}
&\frac{1}{2\vartheta}\,\left(\|\boldsymbol{z}^k\|^2_{L^2(\Omega)^2} - \|\boldsymbol{z}^{k-1}\|^2_{L^2(\Omega)^2} + \|\boldsymbol{z}^k - \boldsymbol{z}^{k-1}\|^2_{L^2(\Omega)^2}\right)\\
&+\frac{\nu}{2}\,|||\boldsymbol{z}^k|||^2 - c_{67}\,\|\boldsymbol{z}^k\|^2_{L^2(\Omega)^2} - c_{68}\\
\leq\;\; &\frac{\nu}{4}\,|||\boldsymbol{z}^k|||^2 + C\,\|\boldsymbol{f}^k\|^2_{L^2(\Omega)^2}.
\end{aligned}$$

Now we multiply this inequality by 2ϑ and sum for $k = 1, \dots, r$ with $1 \leq r \leq n$. We get

$$\begin{aligned}
&\|\boldsymbol{z}^r\|^2_{L^2(\Omega)^2} - \|\boldsymbol{z}^0\|^2_{L^2(\Omega)^2} + \sum_{k=1}^{r} \|\boldsymbol{z}^k - \boldsymbol{z}^{k-1}\|^2_{L^2(\Omega)^2} + \frac{\vartheta\nu}{2}\,\sum_{k=1}^{r} |||\boldsymbol{z}^k|||^2\\
\leq\;\; &2\left(C\vartheta\,\sum_{k=1}^{r} \|\boldsymbol{f}^k\|^2_{L^2(\Omega)^2} + c_{67}\vartheta\,\sum_{k=1}^{r} \|\boldsymbol{z}^k\|^2_{L^2(\Omega)^2} + c_{68}\,r\,\vartheta\right).
\end{aligned} \tag{70}$$

Taking into account the definition of $\boldsymbol{f}^k$, we obtain

$$\vartheta\,\sum_{k=1}^{r} \|\boldsymbol{f}^k\|^2_{L^2(\Omega)^2} \;\leq\; \int_0^T \|\boldsymbol{f}(t)\|^2_{L^2(\Omega)^2}\,\mathrm{d}t \;=\; \|\boldsymbol{f}\|^2_{L^2(0,T;L^2(\Omega)^2)}. \tag{71}$$

This and the inequality $r\vartheta \leq T$ for $r = 1, \dots, n$ yield the estimate

$$\|\boldsymbol{z}^r\|^2_{L^2(\Omega)^2} + \sum_{k=1}^{r} \|\boldsymbol{z}^k - \boldsymbol{z}^{k-1}\|^2_{L^2(\Omega)^2} + \frac{\vartheta\nu}{2}\sum_{k=1}^{r} |||\boldsymbol{z}^k|||^2$$

$$\leq \quad K + L\vartheta \sum_{k=1}^{r} \|\boldsymbol{z}^k\|^2_{L^2(\Omega)^2}, \tag{72}$$

where

$$\begin{aligned} K &= \|\boldsymbol{z}^0\|^2_{L^2(\Omega)^2} + 2C \left(\|\boldsymbol{f}\|^2_{L^2(0,T;L^2(\Omega)^2)} + T \right), \\ L &= 2c_{67}. \end{aligned}$$

Using Lemma 16 with $\vartheta < 1/2L$, we get

$$\|\boldsymbol{z}^r\|^2_{L^2(\Omega)^2} \leq \frac{1}{1 - L\vartheta} K \exp\left((r-1)\vartheta \frac{L}{1 - L\vartheta} \right) \leq C \exp(T\tilde{C}) \tag{73}$$

for all $r = 1, \ldots, n$, with constants $C, \tilde{C} > 0$ independent of r and ϑ. This yields estimate (66). Moreover by (73), the right hand side of (72) is also bounded and, thus, we get estimates (67) and (68). □

Corollary. For the functions $\boldsymbol{u}^k = \boldsymbol{z}^k + \boldsymbol{g}^*$ we have the estimates

$$\max_{k=1,\ldots,n} \|\boldsymbol{u}^k\|_{L^2(\Omega)} \leq c_{72}, \tag{74}$$

$$\vartheta \sum_{k=1}^{n} \|\boldsymbol{u}^k\|^2_{H^1(\Omega)^2} \leq c_{73}, \tag{75}$$

$$\sum_{k=1}^{n} \|\boldsymbol{u}^k - \boldsymbol{u}^{k-1}\|^2_{L^2(\Omega)} \leq c_{74} \tag{76}$$

with the constants c_{72}–c_{74} independent of ϑ (and consequently also of k).

The proof immediately follows from Lemma 17, the triangle inequality and (28).

III.3.4 The limit procedure.

Using the functions

$$\boldsymbol{u}^k = \boldsymbol{g}^* + \boldsymbol{z}^k, \; k = 0, \ldots, n, \tag{77}$$

we can now define the functions $\boldsymbol{u}_\vartheta : [0,T] \to H^1(\Omega)^2$ and $\boldsymbol{w}_\vartheta : [0,T] \to L^2(\Omega)$ so that the function $\boldsymbol{u}_\vartheta$ is piecewise constant and

$$\boldsymbol{u}_\vartheta(0) := \boldsymbol{u}^1, \quad \boldsymbol{u}_\vartheta(t) := \boldsymbol{u}^k \quad \text{for } t \in (t_{k-1}, t_k], \; k = 1, \ldots, n.$$

The function $\boldsymbol{w}_\vartheta$ is continuous on $[0,T]$ and linear on each interval $[t_{k-1}, t_k]$ for $k = 1, \ldots, n$ with the values

$$\boldsymbol{w}_\vartheta(t_k) := \boldsymbol{u}^k \quad \text{for}$$

at the points $t_0, t_1, \ldots, t_n$. Furthermore, we define a modification $\tilde{\boldsymbol{w}}_\vartheta : [0,T] \to H^1(\Omega)^2$ of $\boldsymbol{w}_\vartheta$ by the formulas

$$\tilde{\boldsymbol{w}}_\vartheta := \boldsymbol{u}^1 \qquad \text{on } [0, \vartheta],$$

$$\tilde{\boldsymbol{w}}_\vartheta \;:=\; \boldsymbol{w}_\vartheta \qquad \text{on } [\vartheta, T].$$

Finally, we define the function $\boldsymbol{f}_\vartheta$ on $(0,T)$ by the formula

$$\boldsymbol{f}_\vartheta|_{(t_{k-1},t_k]} \;:=\; \boldsymbol{f}^k, \qquad k=1,\dots,n.$$

Recall that ϑ is always connected with some $n\in\mathbb{N}$ via the formula $\vartheta = T/n$. Thus, $\{\boldsymbol{u}_\vartheta\}$ and $\{\boldsymbol{w}_\vartheta\}$, $\{\tilde{\boldsymbol{w}}_\vartheta\}$ and $\{\boldsymbol{f}_\vartheta\}$ can be treated as the sequences of functions and $\vartheta\to 0+$ means the same as $n\to+\infty$.

Lemma 18 *The functions $\boldsymbol{u}_\vartheta$, $\boldsymbol{w}_\vartheta$, $\tilde{\boldsymbol{w}}_\vartheta$, $\boldsymbol{f}_\vartheta$ have the following properties:*

$$\begin{aligned}
&\text{a)}\quad \boldsymbol{u}_\vartheta,\ \boldsymbol{w}_\vartheta \quad \text{are bounded independently of } \vartheta \text{ in } L^\infty(0,T;L^2(\Omega)^2),\\
&\text{b)}\quad \boldsymbol{u}_\vartheta,\ \tilde{\boldsymbol{w}}_\vartheta \quad \text{are bounded independently of } \vartheta \text{ in } L^2(0,T;H^1(\Omega)^2),\\
&\text{c)}\quad \boldsymbol{w}'_\vartheta \quad \text{is bounded independently of } \vartheta \text{ in } L^{4/3}(0,T;V^*),\\
&\text{d)}\quad \|\boldsymbol{w}_\vartheta - \boldsymbol{u}_\vartheta\|_{L^2(0,T;L^2(\Omega)^2)} \;\le\; c_{75}\,\vartheta,\\
&\text{e)}\quad \boldsymbol{f}_\vartheta \ \text{ is bounded independently of } \vartheta \text{ in } L^2(0,T;L^2(\Omega)^2)) \text{ and}\\
&\qquad\ \boldsymbol{f}_\vartheta \to \boldsymbol{f} \ \text{ in } \ L^2(0,T;L^2(\Omega)^2) \ \text{ as } \ \vartheta\to 0+\,.
\end{aligned}$$

Proof Assertion a) is a consequence of (74) and assertion b) follows from (75). Further, it is easy to see that identity (35c) can be written in the form

$$\boldsymbol{w}'_\vartheta + A_1\boldsymbol{u}_\vartheta + A_2\boldsymbol{u}_\vartheta + A_3\boldsymbol{u}_\vartheta \;=\; \boldsymbol{f}_\vartheta \qquad \text{in } (0,T). \tag{78}$$

This, the properties of the operators A_1, A_2, A_3 from Lemma 13 and the continuous imbeddings $L^2(0,T;L^2(\Omega)^2)\hookrightarrow L^{4/3}(0,T;V^*)$ and $L^2(0,T;V^*)\hookrightarrow L^{4/3}(0,T;V^*)$ imply the assertion in item c). Further, we can write

$$\boldsymbol{w}_\vartheta(t) - \boldsymbol{u}_\vartheta(t) \;=\; (t-t_k)\,\frac{(\boldsymbol{u}^k - \boldsymbol{u}^{k-1})}{\vartheta}.$$

Thus,

$$\int_{t_{k-1}}^{t_k} \|\boldsymbol{w}_\vartheta(t) - \boldsymbol{u}_\vartheta(t)\|^2_{L^2(\Omega)^2}\,\mathrm{d}t \;=\; \frac{\vartheta}{3}\,\|\boldsymbol{u}^k - \boldsymbol{u}^{k-1}\|^2_{L^2(\Omega)^2}$$

and, by (76),

$$\begin{aligned}
\|\boldsymbol{w}_\vartheta(t) - \boldsymbol{u}_\vartheta(t)\|^2_{L^2(0,T;L^2(\Omega)^2)} \;&=\; \int_0^T \|\boldsymbol{w}_\vartheta(t) - \boldsymbol{u}_\vartheta(t)\|^2_{L^2(\Omega)^2}\,\mathrm{d}t\\
=\;\; &\sum_{k=1}^n \frac{\vartheta}{3}\|\boldsymbol{u}^k - \boldsymbol{u}^{k-1}\|^2_{L^2(\Omega)^2} \le C\,\vartheta,
\end{aligned}$$

which proves the assertion in item d). Finally, assertion e) can be proven in the same way as in [55], Chap. III, Par. 4, or [7], Lemma 8.7.55.:
The mapping $\boldsymbol{f}\in L^2(0,T;L^2(\Omega)^2)\to \boldsymbol{f}_\vartheta \in L^2(0,T;L^2(\Omega)^2)$ is linear, continuous and

$$\|\boldsymbol{f}_\vartheta\|_{L^2(0,T;\,L^2(\Omega)^2)} \;\le\; \|\boldsymbol{f}\|_{L^2(0,T;\,L^2(\Omega)^2)}. \tag{79}$$

First we will consider $\boldsymbol{f}$ from space $C(0,T;\, L^2(\Omega)^2)$, which is dense in $L^2(0,T;\, L^2(\Omega)^2)$. Let $\varepsilon > 0$. From the uniform continuity of $\boldsymbol{f}$ on $[0,T]$ we find that there exists $\vartheta_\varepsilon > 0$ such that $\|\boldsymbol{f}(t) - \boldsymbol{f}(s)\|_{L^2(\Omega)^2} \le \varepsilon$ for $s,\, t \in [t_{k-1}, t_k]$ and $\vartheta < \vartheta_\varepsilon$. Then for $\vartheta \in (0, \vartheta_\varepsilon)$ and $t \in [t_{k-1}, t_k]$ we have

$$\left\|\boldsymbol{f}(t) - \frac{1}{\vartheta}\int_{t_{k-1}}^{t_k} \boldsymbol{f}(s)\,\mathrm{d}s\right\|_{L^2(\Omega)^2} \le \frac{1}{\vartheta}\int_{t_{k-1}}^{t_k} \|\boldsymbol{f}(t) - \boldsymbol{f}(s)\|_{L^2(\Omega)^2}\,\mathrm{d}s \le \varepsilon$$

and therefore

$$\begin{aligned}\|\boldsymbol{f} - \boldsymbol{f}_\vartheta\|_{L^2(\Omega)^2} &= \left(\int_0^T \|\boldsymbol{f}(t) - \boldsymbol{f}_\vartheta(t)\|^2_{L^2(\Omega)^2}\,\mathrm{d}t\right)^{1/2}\\ &= \left(\sum_{k=1}^n \int_{t_{k-1}}^{t_k} \|\boldsymbol{f}(t) - \frac{1}{\vartheta}\int_{t_{k-1}}^{t_k} \boldsymbol{f}(s)\,\mathrm{d}s\|_{L^2(\Omega)^2}\,\mathrm{d}t\right)^{1/2} \le \sqrt{n\vartheta}\varepsilon = \sqrt{T}\varepsilon\end{aligned}$$

for $\vartheta \in (0, \vartheta_\varepsilon)$. Now let $\boldsymbol{f} \in L^2(0,T;\, L^2(\Omega)^2)$ and $\varepsilon > 0$. Then there exists $\tilde{\boldsymbol{f}} \in C(0,T;\, L^2(\Omega)^2)$ such that

$$\|\boldsymbol{f} - \tilde{\boldsymbol{f}}\|_{L^2(0,T;\, L^2(\Omega)^2)} \le \frac{\varepsilon}{3}. \tag{80}$$

Due to (79), also

$$\|\boldsymbol{f}_\vartheta - \tilde{\boldsymbol{f}}_\vartheta\|_{L^2(0,T;\, L^2(\Omega)^2)} \le \frac{\varepsilon}{3}. \tag{81}$$

To $\tilde{\boldsymbol{f}}$ there exists $\vartheta^* > 0$ such that

$$\|\tilde{\boldsymbol{f}}_\vartheta - \tilde{\boldsymbol{f}}\|_{L^2(0,T;\, L^2(\Omega)^2)} \le \frac{\varepsilon}{3} \qquad \forall\, \vartheta \in (0, \vartheta^*). \tag{82}$$

From (80)–(82) it follows that

$$\|\boldsymbol{f}_\vartheta - \boldsymbol{f}\| \le \|\boldsymbol{f}_\vartheta - \tilde{\boldsymbol{f}}_\vartheta\| + \|\tilde{\boldsymbol{f}}_\vartheta - \tilde{\boldsymbol{f}}\| + \|\tilde{\boldsymbol{f}} - \boldsymbol{f}\| < \varepsilon \qquad \forall\, \varepsilon \in (0, \vartheta^*),$$

with all the norms being the norms in $L^2(0,T;\, L^2(\Omega)^2)$, which we wanted to prove. □

The space $L^\infty(0,T;\, L^2(\Omega)^2)$ is not reflexive, but it is a dual to the separable space $L^1(0,T;\, L^2(\Omega)^2)$. Thus, from a bounded sequence $\{\boldsymbol{u}_\vartheta\}$ in $L^\infty(0,T;\, L^2(\Omega)^2)$, one can choose a subsequence (which will be again denoted $\{\boldsymbol{u}_\vartheta\}$) such that for some $\boldsymbol{u} \in L^2(0,T;\, L^2(\Omega)^2)$

$$\int_0^T \int_\Omega \boldsymbol{u}_\vartheta \cdot \boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \longrightarrow \int_0^T \int_\Omega \boldsymbol{u} \cdot \boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t$$

as $\vartheta \to 0+$. This is the so-called weak–$*$ convergence in $L^\infty(0,T; L^2(\Omega)^2)$.

Thus, from Lemma 18 we can deduce that there exist limit functions $\boldsymbol{u}$ and $\boldsymbol{w}$ and subsequences of $\{\boldsymbol{u}_\vartheta\}$, $\{\boldsymbol{w}_\vartheta\}$ (which will be again denoted by $\{\boldsymbol{u}_\vartheta\}$ and $\{\boldsymbol{w}_\vartheta\}$ in order to keep a simple notation) such that

$$\boldsymbol{u}_\vartheta \longrightarrow \boldsymbol{u} \quad \text{weakly in } L^2(0,T; H^1(\Omega)^2), \tag{83}$$

$$\begin{aligned}
\boldsymbol{u}_\vartheta &\longrightarrow \boldsymbol{u} \text{ weakly} - * \text{ in } L^\infty(0,T;L^2(\Omega)^2),\\
\boldsymbol{w}_\vartheta &\longrightarrow \boldsymbol{w} \text{ weakly} - * \text{ in } L^\infty(0,T;L^2(\Omega)^2),\\
\boldsymbol{w}'_\vartheta &\longrightarrow \boldsymbol{w}' \text{ weakly in } L^{4/3}(0,T;V^*), \qquad (84)\\
\boldsymbol{w}_\vartheta &\longrightarrow \boldsymbol{w} \text{ weakly in } L^2(\varepsilon,T;H^1(\Omega)^2) \text{ for any } \varepsilon > 0 \qquad (85)
\end{aligned}$$

as $\vartheta \to 0+$.

Lemma 19 *The sequences $\{\boldsymbol{w}_\vartheta\}$ and $\{\boldsymbol{u}_\vartheta\}$ converge strongly in $L^2(0,T;\ L^2(\Omega)^2)$ to the same limit $\boldsymbol{w} \equiv \boldsymbol{u}$. Moreover, $\boldsymbol{u} - \boldsymbol{g}^* \in V$.*

Proof. The sequence $\{\boldsymbol{w}_\vartheta\}$ is bounded in $L^\infty(0,T;L^2(\Omega)^2)$. Hence for $\varepsilon \in (0,T)$, we have

$$\begin{aligned}
\int_0^\varepsilon \|\boldsymbol{w}_\vartheta(t)\|^2_{L^2(\Omega)^2}\,\mathrm{d}t &\le \varepsilon\, \|\boldsymbol{w}_\vartheta\|^2_{L^\infty(0,T;L^2(\Omega)^2)} \le c_{76}\,\varepsilon\\
\int_0^\varepsilon \|\boldsymbol{w}(t)\|^2_{L^2(\Omega)^2}\,\mathrm{d}t &\le \varepsilon\, \|\boldsymbol{w}\|^2_{L^\infty(0,T;L^2(\Omega)^2)} \le c_{76}\,\varepsilon,
\end{aligned}$$

where c_{76} is a constant independent of ε. Let $\alpha > 0$ be arbitrary and let us choose $\varepsilon > 0$ so that $c_{76}\,\varepsilon < \frac{\alpha}{8}$. Then

$$\int_0^\varepsilon \|\boldsymbol{w}_\vartheta(t) - \boldsymbol{w}(t)\|^2_{L^2(\Omega)^2}\,\mathrm{d}t \;\le\; 2\left(\int_0^\varepsilon \|\boldsymbol{w}_\vartheta(t)\|^2_{L^2(\Omega)^2}\,\mathrm{d}t + \int_0^\varepsilon \|\boldsymbol{w}(t)\|^2_{L^2(\Omega)^2}\,\mathrm{d}t\right) < \frac{\alpha}{2}. \qquad (86)$$

Further, we can write

$$\begin{aligned}
\|\boldsymbol{w}_\vartheta - \boldsymbol{w}\|^2_{L^2(0,T;L^2(\Omega)^2)} &= \int_0^T \|\boldsymbol{w}_\vartheta(t) - \boldsymbol{w}(t)\|^2_{L^2(0,T;L^2(\Omega)^2)}\,\mathrm{d}t \qquad (87)\\
= \int_0^\varepsilon \|\boldsymbol{w}_\vartheta(t) - \boldsymbol{w}(t)\|^2_{L^2(0,T;L^2(\Omega)^2)}\,\mathrm{d}t &+ \int_\varepsilon^T \|\boldsymbol{w}_\vartheta(t) - \boldsymbol{w}(t)\|^2_{L^2(0,T;L^2(\Omega)^2)}\,\mathrm{d}t.
\end{aligned}$$

Let us define the space

$$W_\varepsilon = W(\varepsilon, T, 2, \tfrac{4}{3}, H^1(\Omega)^2, V^*) = \left\{v \in L^2(\varepsilon,T;H^1(\Omega)^2); v' \in L^{4/3}(\varepsilon,T;V^*)\right\}.$$

Since $H^1(\Omega)^2 \hookrightarrow\hookrightarrow L^2(\Omega)^2 \hookrightarrow V^*$ by the well–known Aubin–Lions theorem ([7], Theorem 8.6.12, or [55], Chap. III, Par. 2) the imbedding of the space W_ε into $L^2(\varepsilon,T;\ L^2(\Omega)^2)$ is compact. Hence, in virtue of (84) and (85), we get the strong convergence

$$\boldsymbol{w}_\vartheta \to \boldsymbol{w} \qquad \text{strongly in } L^2(\varepsilon,T;L^2(\Omega)) \quad \text{for } \vartheta \to 0+.$$

This means that there exists $\vartheta^* > 0$ such that

$$\int_\varepsilon^T \|\boldsymbol{w}_\vartheta(t) - \boldsymbol{w}(t)\|^2_{L^2(\Omega)}\,\mathrm{d}t \le \frac{\alpha}{2}, \qquad \forall\,\vartheta \in (0,\vartheta^*).$$

This inequality, together with (86) and (87), implies that

$$\int_0^T \|\boldsymbol{w}_\vartheta(t) - \boldsymbol{w}(t)\|_{L^2(\Omega)}\,\mathrm{d}t < \alpha, \qquad \forall\,\vartheta \in (0, \vartheta^*).$$

It means that $\boldsymbol{w}_\vartheta \to \boldsymbol{w}$ in $L^2(0,T;\,L^2(\Omega)^2)$ as $\vartheta \to 0+$. Furthermore, assertion d) from Lemma 18 implies that $\boldsymbol{u} = \boldsymbol{w}$ and we thus have the strong convergence $\boldsymbol{w}_\vartheta \to \boldsymbol{u}$ and $\boldsymbol{u}_\vartheta \to \boldsymbol{u}$ in $L^2(0,T;\,L^2(\Omega)^2)$. Moreover, due to (77), we also have $\boldsymbol{u} = \boldsymbol{g}^* + \boldsymbol{z}$, where $\boldsymbol{z} \in V$. □

Finally, it is necessary to verify that the limit process for $\vartheta \to 0+$ is also possible in (78). In order to do it, we show that

$$\left.\begin{array}{lcl} A_1\boldsymbol{u}_\vartheta & \to & A_1\boldsymbol{u} \\ A_2\boldsymbol{u}_\vartheta & \to & A_2\boldsymbol{u} \\ A_3\boldsymbol{u}_\vartheta & \to & A_3\boldsymbol{u} \end{array}\right\} \quad \text{weakly in } L^{4/3}(0,T;V^*) \text{ as } \vartheta \to 0+. \tag{88}$$

Let us at first prove that

$$A_1\boldsymbol{u}_\vartheta \to A_1\boldsymbol{u} \quad \text{weakly in } L^2(0,T;V^*). \tag{89}$$

Due to (83), we have

$$\begin{aligned} &\int_0^T \langle A_1\boldsymbol{u}_\vartheta(t) - A_1\boldsymbol{u}(t), \boldsymbol{v}(t)\rangle\,\mathrm{d}t \\ &\quad = \int_0^T (\boldsymbol{u}_\vartheta(t) - \boldsymbol{u}(t), \boldsymbol{v}(t))_V\,\mathrm{d}t \longrightarrow 0 \quad \text{for } \vartheta \to 0+. \end{aligned} \tag{90}$$

for $\boldsymbol{v} \in L^2(0,T;\,V)$, which confirms (89). Now we shall prove that

$$A_2\boldsymbol{u}_\vartheta \to A_2\boldsymbol{u} \quad \text{weakly in } L^{4/3}(0,T;V^*). \tag{91}$$

Suppose that $\boldsymbol{v} \in L^4(0,T;\,V)$ and $\varepsilon > 0$. Using an appropriate regularization techniques (see e.g. [40], pp. 71–73), we can construct $\boldsymbol{v}_\varepsilon \in C^1(\overline{\Omega} \times [0,T])$ such that

$$\|\boldsymbol{v}_\varepsilon - \boldsymbol{v}\|_{L^4(0,T;H^1(\Omega)^2)} \le \varepsilon.$$

Then we can write

$$\begin{aligned} &\left|\int_0^T \langle A_2\boldsymbol{u}_\vartheta(t) - A_2\boldsymbol{u}(t), \boldsymbol{v}(t)\rangle\,\mathrm{d}t\right| \qquad (92) \\ &\quad = \left|\int_0^T \left(a_2(\boldsymbol{u}_\vartheta, \boldsymbol{u}_\vartheta, \boldsymbol{v}) - a_2(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v})\right)\,\mathrm{d}t\right| \\ &\quad \le \left|\int_0^T \left(a_2(\boldsymbol{u}_\vartheta, \boldsymbol{u}_\vartheta, \boldsymbol{v}) - a_2(\boldsymbol{u}_\vartheta, \boldsymbol{u}_\vartheta, \boldsymbol{v}_\varepsilon)\right)\,\mathrm{d}t\right| \\ &\quad + \left|\int_0^T \left(a_2(\boldsymbol{u}_\vartheta, \boldsymbol{u}_\vartheta, \boldsymbol{v}_\varepsilon) - a_2(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}_\varepsilon)\right)\,\mathrm{d}t\right| \end{aligned}$$

$$+\left|\int_0^T (a_2(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}_\varepsilon)-a_2(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}))\,\mathrm{d}t\right|.$$

Now we shall separately estimate each term in (92) and use Remark 5. Let us consider an arbitrary $\delta>0$. We have

$$\left|\int_0^T a_2(\boldsymbol{u}_\vartheta,\boldsymbol{u}_\vartheta,\boldsymbol{v})-a_2(\boldsymbol{u}_\vartheta,\boldsymbol{u}_\vartheta,\boldsymbol{v}_\varepsilon)\,\mathrm{d}t\right| = \left|\int_0^T \langle A_2\boldsymbol{u}_\vartheta,\boldsymbol{v}-\boldsymbol{v}_\varepsilon\rangle\,\mathrm{d}t\right|$$
$$\leq \|A_2\boldsymbol{u}_\vartheta\|_{L^{\frac{4}{3}}(0,T;(H^1(\Omega)^2)^*)}\,\|\boldsymbol{v}-\boldsymbol{v}_\varepsilon\|_{L^4(0,T;\,H^1(\Omega)^2)}\leq C\varepsilon.$$

Let us choose $\varepsilon_\delta>0$ so small that $c\varepsilon_\delta<\frac{1}{3}\delta$. Further, since $|\boldsymbol{v}_{\varepsilon_\delta}|\leq C(\varepsilon_\delta)$ in $\overline{\Omega}\times[0,T]$,

$$\left|\int_0^T a_2(\boldsymbol{u}_\vartheta,\boldsymbol{u}_\vartheta,\boldsymbol{v}_{\varepsilon_\delta})-a_2(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}_{\varepsilon_\delta})\,\mathrm{d}t\right| \tag{93}$$
$$\leq C(\varepsilon_\delta)\int_0^T\int_\Omega\sum_{i,j=1}^2\left|u_{\vartheta,j}\frac{\partial u_{\vartheta,i}}{\partial x_j}-u_j\frac{\partial u_i}{\partial x_j}\right|\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t$$
$$\leq C(\varepsilon_\delta)\int_\Omega\sum_{i,j=1}^2\left|u_j\left(\frac{\partial u_{\vartheta,i}}{\partial x_j}-\frac{\partial u_i}{\partial x_j}\right)\right|\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t$$
$$+\|\boldsymbol{u}_\vartheta-\boldsymbol{u}\|_{L^2(0,T;L^2(\Omega)^2)}\,\|\boldsymbol{u}_\vartheta\|_{L^2(0,T;H^1(\Omega)^2)}.$$

Due to the weak convergence (83), Lemma 19 and assertion b) from Lemma 18, we can choose $\vartheta_{\varepsilon_\delta}>0$ so small that

$$\left|\int_0^T a_2(\boldsymbol{u}_\vartheta,\boldsymbol{u}_\vartheta,\boldsymbol{v}_{\varepsilon_\delta})-a_2(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}_{\varepsilon_\delta})\,\mathrm{d}t\right|\leq\frac{\delta}{3}\quad\forall\,\vartheta\in(0,\vartheta_{\varepsilon_\delta}). \tag{94}$$

Finally,

$$\left|\int_0^T a_2(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}_{\varepsilon_\delta})-a_2(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v})\,\mathrm{d}t\right| = \left|\int_0^T\langle A_2\boldsymbol{u}_\vartheta,\boldsymbol{v}_{\varepsilon_\delta}-\boldsymbol{v}\rangle\,\mathrm{d}t\right|$$
$$\leq \|A_2\boldsymbol{u}\|_{L^{\frac{4}{3}}(0,T;(H^1(\Omega)^2)^*)}\,\|\boldsymbol{v}_{\varepsilon_\delta}-\boldsymbol{v}\|_{L^4(0,T;H^1(\Omega)^2)}\leq C\,\varepsilon_\delta<\frac{\delta}{3}.$$

Summing the estimates (93)–(95), we obtain

$$\left|\int_0^T a_2(\boldsymbol{u}_\vartheta,\boldsymbol{u}_\vartheta,\boldsymbol{v})-a_2(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}))\,\mathrm{d}t\right|\leq\delta\quad\text{for }\vartheta\in(0,\vartheta_{\varepsilon_\delta}), \tag{95}$$

which implies (91). Now we shall prove that

$$A_3\boldsymbol{u}_\vartheta\to A_3\boldsymbol{u}\quad\text{weakly in }L^{4/3}(0,T;V^*). \tag{96}$$

Let $\boldsymbol{v}\in L^4(0,T;V)$ and $\delta>0$. Similarly as in the case of operator A_2, we can write

$$\int_0^T\langle A_3\boldsymbol{u}_\vartheta(t)-A_3\boldsymbol{u}(t),\boldsymbol{v}(t)\rangle\,\mathrm{d}t=\sigma_1+\sigma_2+\sigma_3 \tag{97}$$

where

$$\sigma_1 := \int_0^T \langle A_3 \boldsymbol{u}_\vartheta, \boldsymbol{v} - \boldsymbol{v}_\varepsilon \rangle \,\mathrm{d}t, \tag{98}$$

$$\sigma_2 := \int_0^T a_3(\boldsymbol{u}_\vartheta, \boldsymbol{u}_\vartheta, \boldsymbol{v}) - a_3(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}))\,\mathrm{d}t,$$

$$\sigma_3 := \int_0^T \langle A_3 \boldsymbol{u}, \boldsymbol{v}_\varepsilon - \boldsymbol{v} \rangle \,\mathrm{d}t.$$

The estimates of σ_1 and σ_3 are easy. From (30) and assertions a), b) of Lemma 18 we have

$$|\sigma_1| \le \|A_3 \boldsymbol{u}_\vartheta\|_{L^{4/3}(0,T;\,(H^1(\Omega)^2)^*)} \|\boldsymbol{v} - \boldsymbol{v}_\varepsilon\|_{L^4(0,T;\,H^1(\Omega)^2)} \le C\varepsilon \le \frac{\delta}{3} \tag{99}$$

if $\varepsilon = \varepsilon_\delta$ is chosen sufficiently small. Similarly we find that $|\sigma_3| \le \frac{\delta}{3}$. The estimate of σ_2 is more complicated. Since

$$|(\boldsymbol{u}_\vartheta \cdot \boldsymbol{n})^- \boldsymbol{u}_\vartheta - (\boldsymbol{u} \cdot \boldsymbol{n})^- \boldsymbol{u}| \le (|\boldsymbol{u}| + |\boldsymbol{u}_\vartheta|)|\boldsymbol{u}_\vartheta - \boldsymbol{u}|,$$

and the function $|\boldsymbol{v}_{\varepsilon_\delta}| \le C(\varepsilon_\delta)$ in $\bar{Q}_T$, the use of the Cauchy inequality and the multiplicative trace inequality implies that

$$\begin{aligned}
|\sigma_2| &\le C(\varepsilon_\delta) \int_0^T \int_{\Gamma_o} (|\boldsymbol{u}| + |\boldsymbol{u}_\vartheta|)\, |\boldsymbol{u} - \boldsymbol{u}_\vartheta| \,\mathrm{d}S\,\mathrm{d}t \\
&\le C(\varepsilon_\delta) \left(\int_0^T (\|\boldsymbol{u}\|^2_{L^2(\Gamma_o)^2} + \|\boldsymbol{u}_\vartheta\|^2_{L^2(\Gamma_o)^2})\,\mathrm{d}t \right)^{1/2} \left(\int_0^T \|\boldsymbol{u} - \boldsymbol{u}_\vartheta\|^2_{L^2(\Gamma_o)^2}\,\mathrm{d}t \right)^{1/2} \\
&\le C_M\, C(\varepsilon_\delta) \left(\int_0^T (\|\boldsymbol{u}\|_{L^2(\Omega)^2} \|\boldsymbol{u}\|_{H^1(\Omega)^2} + \|\boldsymbol{u}_\vartheta\|_{L^2(\Omega)^2} \|\boldsymbol{u}_\vartheta\|_{H^1(\Omega)^2})\,\mathrm{d}t \right)^{1/2} \\
&\qquad \times \left(\int_0^T \|\boldsymbol{u} - \boldsymbol{u}_\vartheta\|_{L^2(\Omega)^2} \|\boldsymbol{u} - \boldsymbol{u}_\vartheta\|_{H^1(\Omega)^2}\,\mathrm{d}t \right)^{1/2} \\
&\le C_M\, C(\varepsilon_\delta) \Big(\|\boldsymbol{u}\|_{L^2(0,T;L^2(\Omega)^2)} \|\boldsymbol{u}\|_{L^2(0,T;H^1(\Omega)^2)} \\
&\qquad + \|\boldsymbol{u}_\vartheta\|_{L^2(0,T;L^2(\Omega)^2)} \|\boldsymbol{u}_\vartheta\|_{L^2(0,T;H^1(\Omega)^2)} \Big)^{1/2} \\
&\qquad \times \|\boldsymbol{u} - \boldsymbol{u}_\vartheta\|^{1/2}_{L^2(0,T;L^2(\Omega)^2)} \|\boldsymbol{u} - \boldsymbol{u}_\vartheta\|^{1/2}_{L^2(0,T;H^1(\Omega)^2)}.
\end{aligned}$$

Due to assertion b) of Lemma 18 and Lemma 19, we can deduce that there exists $\vartheta_\delta > 0$ such that $|\sigma_2| \le \frac{1}{3}\delta$ for all $\vartheta \in (0, \vartheta_\delta)$.

Using the above results, we verify that (96) holds.

Now, (84), (88) and the fact that $\boldsymbol{w} = \boldsymbol{u}$ allow us to carry out the limit process in (78), which implies that the limit function $\boldsymbol{u}$ satisfies equation (60). Since $\boldsymbol{u} - \boldsymbol{g}^* \in V$, the function $\boldsymbol{u}$ satisfies the boundary conditions (20).

We shall finally show that $\boldsymbol{u}$ also satisfies the initial condition (21).

Suppose that $\boldsymbol{v} \in V$ and $\varphi \in C^\infty(0,T)$, $\varphi(T) = 0$, $\varphi(0) \neq 0$. Then obviously $\eta(t) = \boldsymbol{v}\,\varphi(t)$, $\eta'(t) = \boldsymbol{v}\,\varphi'(t) \in L^q(0,T;V)$ $(q \geq 2,\ 1/q + 1/\alpha_1 = 1)$ and, in virtue of (84) and (85), we have

$$a)\int_0^T (\boldsymbol{w}_\vartheta(t) - \boldsymbol{u}(t), \boldsymbol{v})\,\underline{\varphi'}(t)\,\mathrm{d}t \longrightarrow 0$$

and

$$b)\int_0^T \left\langle \frac{d\boldsymbol{w}_\vartheta}{dt}(t) - \frac{d\boldsymbol{u}}{dt}(t), \underline{\eta}(t) \right\rangle \mathrm{d}t \longrightarrow 0,$$

respectively, as $\vartheta \to 0$. Using the integration by parts (see e.g. [22]), we obtain

$$\int_0^T \left\langle \frac{d\boldsymbol{w}_\vartheta}{dt}(t) - \frac{d\boldsymbol{u}}{dt}(t), \underline{\eta}(t) \right\rangle \mathrm{d}t = \int_0^T \left\langle \frac{d\boldsymbol{w}_\vartheta}{dt}(t) - \frac{d\boldsymbol{u}}{dt}(t), \boldsymbol{v} \right\rangle \varphi(t)\,\mathrm{d}t =$$
$$= -\langle \boldsymbol{w}_\vartheta(0) - \boldsymbol{u}(0), \boldsymbol{v}\rangle\,\varphi(0) - \int_0^T \big(\boldsymbol{w}_\vartheta(t) - \boldsymbol{u}(t), \boldsymbol{v}\big)\,\varphi'(t)\,\mathrm{d}t.$$

Now taking into account that $\boldsymbol{w}_\vartheta(0) \to \boldsymbol{u}_0$ for any $\vartheta \to 0+$ (in fact $\boldsymbol{w}_\vartheta(0) = \boldsymbol{u}_0$), we obtain the identity

$$\langle \boldsymbol{u}_0 - \boldsymbol{u}(0), \boldsymbol{v}\rangle = 0 \quad \forall \boldsymbol{v} \in V,$$

which means that $\boldsymbol{u}(0) = \boldsymbol{u}_0$.

Thus, we observe that $\boldsymbol{u}$ is a weak solution of the initial–boundary value problem $\mathcal{P}_{\mathrm{weak}}(\Omega)$. We have proved the main theorem of this chapter:

Theorem 17. Let $\boldsymbol{f} \in L^2(0,T;\, L^2(\Omega)^2)$, $\boldsymbol{u}_0 \in L^2(\Omega)^2$. Let $\boldsymbol{g}^*$ be a function with the properties (34) and let $\nu > 0$ be a constant. Then the weak problem $\mathcal{P}_{\mathrm{weak}}(\Omega)$ has a solution.

Remark 7. Using the results from [23], [49] and [50], it is possible to show that to a weak solution $\boldsymbol{u}$ (i.e. the solution of the problem $\mathcal{P}_{\mathrm{weak}}(\Omega)$), introduced in Definition 4, there exists a pressure function $p : (0,T) \to \mathbb{R}$ such that $p(t) \in L^2(\Omega)$ for almost every $t \in (0,T)$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{u}(t), \boldsymbol{v}) + a(\boldsymbol{u}(t), \boldsymbol{v}) - (p(t), \operatorname{div} \boldsymbol{v}) = (\boldsymbol{f}(t), \boldsymbol{v}) + b(\boldsymbol{h}(t), \boldsymbol{v}) \tag{100}$$

for all $\boldsymbol{v} \in X$ and a.a. $t \in (0,T)$. (See the subsection I.2.1 for the definition of the space X.) This implies the cascade flow initial–boundary value problem can be reformulated as the problem to find a pair $\boldsymbol{u}$, p satisfying the identity (100), the equation

$$-(\operatorname{div} \boldsymbol{u}, q) = 0 \quad \forall q \in L^2(\Omega), \tag{101}$$

the boundary conditions (20) and the initial condition (21). This *velocity–pressure formulation* can be considered as the basis for the finite element discretization of the problem $\mathcal{P}_{\mathrm{weak}}(\Omega)$.

Appendix

Some important inequalities

Definition A1. We say that a bounded domain $\omega \subset \mathbb{R}^N$ has a Lipschitz-continuous boundary if there exist numbers $\gamma > 0$, $\delta > 0$, M Cartesian coordinate systems $(x_{r1}, \cdots, x_{rN-1}, x_{rN}) = (x'_r, x_{rN})$ (for $r = 1, \ldots, M$) and functions $a(x'_r)$, Lipschitz continuous in $\Delta_r := \{x'_r;\ |x_{ri}| \le \gamma$ for $i = 1, \cdots, N-1\}$, such that

$$\begin{aligned}
&\forall x \in \partial\omega\ \exists r\ (1 \le r \le M, r \in \mathbb{Z})\ \exists x'_r \in \Delta_r : x = (x'_r, a_r(x'_r)),\\
&U_r^+ \ := \ \{(x'_r, x_{rN});\ a_r(x'_r) \le x_{rN} \le a_r(x'_r) + \delta,\ x'_r \in \Delta_r\} \ \subset \ \omega,\\
&U_r^- \ := \ \{(x'_r, x_{rN});\ a_r(x'_r) - \delta \le x_{rN} \le a_r(x'_r),\ x'_r \in \Delta_r\} \ \subset \ \mathbb{R}^N - \overline{\omega}.
\end{aligned} \tag{1}$$

This definition is usual, see e.g. [48].

Lemma A1 (multiplicative trace inequality). Let the domain $\omega \subset \mathbb{R}^N$ have a Lipschitz–continuous boundary. Then there exists a constant c_M such that

$$\|u\|^2_{L^2(\partial\omega)} \ \le c_M \, \|u\|_{L^2(\omega)} \, \|u\|_{H^1(\omega)} \quad \forall u \in H^1(\omega). \tag{2}$$

The same problem was already treated on more general level by G. P. Galdi in [20], pp. 42. However, since there are misprints in the corresponding theorem in [20], we present another proof here.

Proof of Lemma A1. Using the notation from Definition A1, we consider an arbitrary $r \in \{1, \ldots, M\}$ and for simplicity denote $\Delta = \Delta_r, a = a_r, x' = x'_r, U^+ = U_r^+$. Let $s \in (a(x'), a(x') + \delta)$. Then

$$\begin{aligned}
|u(x', a(x'))|^2 \ &= \ |u(x', s)|^2 - \int_{a(x')}^{s} \frac{\partial}{\partial x_N} \, |u(x', \xi_N)|^2 \, \mathrm{d}\xi_N \\
&= \ |u(x', s)|^2 - 2 \int_{a(x')}^{s} u(x', \xi_N) \, \frac{\partial u}{\partial x_N} \, (x', \xi_N) \, \mathrm{d}\xi_N.
\end{aligned} \tag{3}$$

Now we integrate (3) with respect to s from $a(x')$ to $a(x') + \delta$ and obtain the inequality

$$\begin{aligned}
&\delta |u(x', a(x'))|^2 \\
&\le \ \int_{a(x')}^{a(x')+\delta} |u(x', s)|^2 \, \mathrm{d}s + 2\delta \int_{a(x')}^{a(x')+\delta} |u(x', \xi_N)| \, \Big| \frac{\partial u}{\partial x_N} (x', \xi_N) \Big| \, \mathrm{d}\xi_N.
\end{aligned} \tag{4}$$

Further, we integrate (4) with respect to x' over Δ, use the Cauchy inequality and obtain

$$\int_{\Delta} |u(x', a(x'))|^2 \, \mathrm{d}\boldsymbol{x}' \le \frac{1}{\delta} \int_{U^+} |u(x', x_N)|^2 \, \mathrm{d}\boldsymbol{x} + 2 \int_{U^+} |u(x', x_N)| \, \Big| \frac{\partial u}{\partial x_N} (x', x_N) \Big| \, \mathrm{d}\boldsymbol{x}$$

$$\leq \frac{1}{\delta}\int_{U^+}|u|^2\,\mathrm{d}\boldsymbol{x}+2\left(\int_{U^+}|u|^2\,\mathrm{d}\boldsymbol{x}\right)^{1/2}\left(\int_{U^+}|\nabla u|^2\,\mathrm{d}\boldsymbol{x}\right)^{1/2}. \tag{5}$$

Thus,

$$\begin{aligned}
\|u\|^2_{L^2(\partial\omega)} &\leq \sum_{r=1}^{M}\int_{\Delta_r}|u|^2\,\mathrm{d}\boldsymbol{x}'_r \\
&\leq \sum_{r=1}^{M}\left[\frac{1}{\delta}\int_{U_r^+}|u|^2\,\mathrm{d}\boldsymbol{x}+2\left(\int_{U_r^+}|u|^2\,\mathrm{d}x\right)^{1/2}\left(\int_{U_r^+}|\nabla u|^2\,\mathrm{d}x\right)^{1/2}\right].
\end{aligned} \tag{6}$$

Using the Cauchy inequality, we find that

$$\begin{aligned}
\|u\|^2_{L^2(\partial\omega)} &\leq C\left(\|u\|^2_{L^2(\omega)}+\|u\|_{L^2(\omega)}\,\|u\|_{H^1(\omega)}\right) \\
&\leq C\,\|u\|_{L^2(\omega)}\,\|u\|_{H^1(\omega)},
\end{aligned} \tag{7}$$

which we wanted to prove. All operations can be made with smooth functions from $H^1(\omega)$; the inequality (7) can be afterwards extended to the whole space $H^1(\omega)$ by means of the density argument. $\square$

Lemma A2. For all $v\in H^1(\Omega)$ such that $v=0$ on Γ_w, we have

$$\|v\|_{L^4(\Omega)} \leq c_{77}\,\|v\|^{1/2}_{L^2(\Omega)}\,\|v\|^{1/2}_{H^1(\Omega)} \tag{8}$$

with a constant c_{77} independent of v.

The inequality (8) is proved in [41] for functions from $H^1_0(\Omega)$, Ω being a domain in $\mathbb{R}^2$. We present the proof which uses the special form of Ω, see Fig. 2, on the other hand we deal with functions that have the trace only on the part Γ_w of the boundary equal to zero.

Proof of Lemma A2. Let us at first prove the inequality (8) for a rectangular domain $\widetilde{\Omega}=(d_i,d_o)\times(\eta_1,\eta_2)$ with $\partial\widetilde{\Omega}=\Gamma_i\cup\Gamma_o\cup\Gamma_-\cup\Gamma_+$, see Fig. 6.

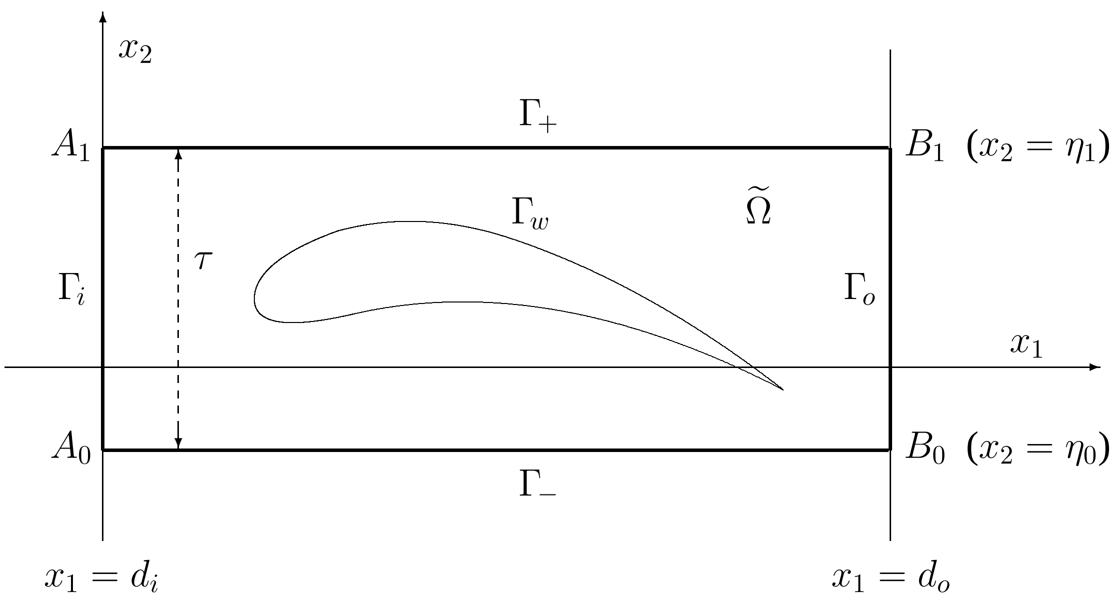

Fig. 6

Suppose that $v \in H^1(\widetilde{\Omega})$ such that $v = 0$ inside Γ_w. Since the space $\{v \in C^\infty(\overline{\widetilde{\Omega}});\ v = 0 \text{ in } Int(\Gamma_w)\}$ is dense in $\{v \in H^1(\widetilde{\Omega});\ v = 0 \text{ a.e. in } Int(\Gamma_w)\}$, it is sufficient to prove inequality (8) for $v \in C^\infty(\overline{\widetilde{\Omega}})$ such that $v = 0$ in $Int(\Gamma_w)$. For such v we have

$$v^2(\boldsymbol{x}) \ = \ v^2(d_i, x_2) + 2\int_{d_i}^{x_1} v(\xi_1, x_2)\frac{\partial v(\xi_1, x_2)}{\partial x_1}\,\mathrm{d}\xi_1 \ \le \ v_1(x_2) \tag{9}$$

where

$$v_1(x_2) \ := \ v^2(d_i, x_2) + 2\int_{d_i}^{d_o} |v(\xi_1, x_2)| \left|\frac{\partial v(\xi_1, x_2)}{\partial x_1}\right| \mathrm{d}\xi_1. \tag{10}$$

Similarly

$$v^2(\boldsymbol{x}) \ = \ v^2(x_1, \eta_0) + 2\int_{\eta_0}^{x_2} \frac{\partial v(x_1, \xi_2)}{\partial x_2} v(x_1, \xi_2)\,\mathrm{d}\xi_2 \ \le \ v_2(x_1) \tag{11}$$

where

$$v_2(x_1) \ := \ v^2(x_1, \eta_0) + 2\int_{\eta_0}^{\eta_1} |v(x_1, \xi_2)| \left|\frac{\partial v(x_1, \xi_2)}{\partial x_2}\right| \mathrm{d}\xi_2. \tag{12}$$

Hence we have

$$\|v\|^4_{L^4(\widetilde{\Omega})} \ = \ \int_{\widetilde{\Omega}} v^4\,\mathrm{d}\boldsymbol{x} \ = \ \int_{\widetilde{\Omega}} v^2(\boldsymbol{x})\, v^2(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \tag{13}$$

$$\begin{aligned}
&\le \ \int_{\widetilde{\Omega}} v_1(x_2)\, v_2(x_1)\,\mathrm{d}x_1\,\mathrm{d}x_2 \ = \ \int_{d_i}^{d_o}\left(\int_{\eta_0}^{\eta_1} v_1(x_2)\,\mathrm{d}x_2\right) v_2(x_1)\,\mathrm{d}x_1 \\
&= \ \int_{\eta_0}^{\eta_1} v_1(x_2)\,\mathrm{d}x_2 \int_{d_i}^{d_o} v_2(x_1)\,\mathrm{d}x_1 \\
&= \ \int_{\eta_0}^{\eta_1}\left(v^2(d_i, x_2) + 2\int_{d_i}^{d_o} |v(\xi_1, x_2)| \left|\frac{\partial v(\xi_1, x_2)}{\partial x_1}\right| \mathrm{d}\xi_1\right) \mathrm{d}x_2 \\
&\quad \cdot \int_{d_i}^{d_o}\left(v^2(x_1, \eta_0) + 2\int_{\eta_0}^{\eta_1} |v(x_1, \xi_2)| \left|\frac{\partial v(x_1, \xi_2)}{\partial x_2}\right| \mathrm{d}\xi_2\right) \mathrm{d}x_1.
\end{aligned} \tag{14}$$

This and the Cauchy inequality imply that

$$\|v\|^4_{L^4(\widetilde{\Omega})} \ \le \ \left(\|v\|^2_{L^2(\Gamma_i)} + 2\,\|v\|_{L^2(\widetilde{\Omega})}\,\|v\|_{H^1(\widetilde{\Omega})}\right)\ \left(\|v\|^2_{L^2(\Gamma_-)} + 2\,\|v\|_{L^2(\widetilde{\Omega})}\,\|v\|_{H^1(\widetilde{\Omega})}\right).$$

Applying now Lemma A1 to the norms $\|v\|^2_{L^2(\Gamma_i)}$ and $\|v\|^2_{L^2(\Gamma_-)}$, we obtain the inequality

$$\|v\|^4_{L^4(\widetilde{\Omega})} \ \le \ C\,\|v\|^2_{L^2(\widetilde{\Omega})}\,\|v\|^2_{H^1(\widetilde{\Omega})}. \tag{15}$$

Further we generalize this result to the domain Ω of a general shape, see Fig. 7, by means of a transformation of coordinates. Similarly as in the case of the rectangular domain $\widetilde{\Omega}$, which also involves Γ_w and its interior, it is convenient to involve Γ_w and its interior to Ω in this proof. However, all functions we deal with are supposed to be zero in $Int(\Gamma_w)$.

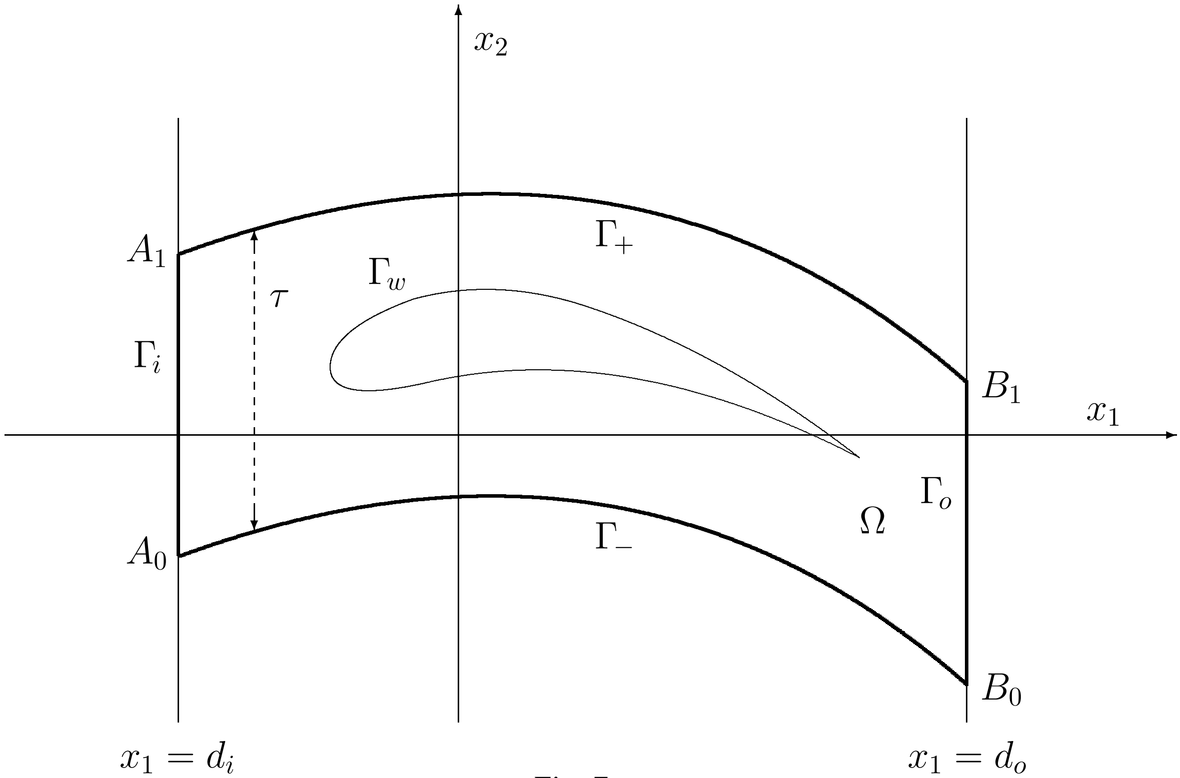

Fig. 7

Let us denote by (x_1, x_2) and $(\widetilde{x}_1, \widetilde{x}_2)$ points in Ω and $\widetilde{\Omega}$, respectively. We use the transformation F of the domain Ω onto $\widetilde{\Omega}$:

$$\begin{aligned}(x_1, x_2) \in \Omega \quad \longrightarrow \quad & (\widetilde{x}_1, \widetilde{x}_2) = F(x_1, x_2) \in \widetilde{\Omega},\\ \widetilde{x}_1 \quad = \quad & x_1 \\ \widetilde{x}_2 \quad = \quad & x_2 - \gamma(x_1).\end{aligned}$$

The Jacobian of this transformation is

$$|J(x_1, x_2)| \;=\; \det \begin{pmatrix} 1 & 0 \\ -\gamma'(x_1) & 0 \end{pmatrix} \;=\; 1 \;\neq\; 0.$$

The inverse transformation F_{-1} and its Jacobian have the form

$$\begin{aligned} x_1 &= \widetilde{x}_1 \\ x_2 &= \widetilde{x}_2 + \gamma(\widetilde{x}_1) \\ |\widetilde{J}(\widetilde{x}_1, \widetilde{x}_2)| &= 1 \;\neq\; 0. \end{aligned} \tag{16}$$

For $v \in H^1(\Omega)$, $v|_{Int(\Gamma_w} = 0$, we set $\widetilde{v}(\widetilde{x}_1, \widetilde{x}_2) = v(\widetilde{x}_1, \widetilde{x}_2 + \gamma(\widetilde{x}_1)$. Then

$$\|v\|_{L^2(\Omega)} \quad = \quad \|\widetilde{v}\|_{L^2(\widetilde{\Omega})}$$

$$\|v\|_{L^4(\Omega)} \;=\; \|\widetilde{v}\|_{L^4(\widetilde{\Omega})}. \tag{17}$$

Further,

$$\begin{aligned}
\frac{\partial v}{\partial x_1} &= \frac{\partial \widetilde{v}}{\partial \widetilde{x}_1}\,\frac{\partial \widetilde{x}_1}{\partial x_1} + \frac{\partial \widetilde{v}}{\partial \widetilde{x}_2}\,\frac{\partial \widetilde{x}_2}{\partial x_1} = \frac{\partial \widetilde{v}}{\partial \widetilde{x}_1} + \frac{\partial \widetilde{v}}{\partial \widetilde{x}_2}(-\gamma'(x_1)),\\
\frac{\partial v}{\partial x_2} &= \frac{\partial \widetilde{v}}{\partial \widetilde{x}_1}\,\frac{\partial \widetilde{x}_1}{\partial x_2} + \frac{\partial \widetilde{v}}{\partial \widetilde{x}_2}\,\frac{\partial \widetilde{x}_2}{\partial x_2} = \frac{\partial \widetilde{v}}{\partial \widetilde{x}_2}.
\end{aligned} \tag{18}$$

Thus, we have

$$\Big|\frac{\partial v}{\partial x_1}\Big|^2 + \Big|\frac{\partial v}{\partial x_2}\Big|^2 = \Big|\frac{\partial \widetilde{v}}{\partial \widetilde{x}_1}\Big|^2 + 2\Big|\frac{\partial \widetilde{v}}{\partial \widetilde{x}_1}\Big|\,\Big|\frac{\partial \widetilde{v}}{\partial \widetilde{x}_2}\Big|\,\Big|\gamma'(x_1)\Big| + \Big|\frac{\partial \widetilde{v}}{\partial \widetilde{x}_2}\Big|^2\left(\Big|\gamma'(x_1)\Big|^2 + 1\right), \tag{19}$$

which yields

$$|\nabla v(\boldsymbol{x})|^2 \;\le\; \widetilde{C}\;|\widetilde{\nabla}\widetilde{v}(F(\boldsymbol{x}))|^2, \quad \boldsymbol{x} \in \Omega. \tag{20}$$

Similarly we find that

$$|\widetilde{\nabla}\widetilde{v}(\widetilde{\boldsymbol{x}})|^2 \;\le\; C\;|\nabla v(F_{-1}(\widetilde{\boldsymbol{x}}))|^2, \quad \widetilde{\boldsymbol{x}} \in \widetilde{\Omega}. \tag{21}$$

Integration in (20) and (21) and the theorem on substitution yield the estimates

$$\|v\|_{H^1(\Omega)} \;\le\; C\;\|\widetilde{v}\|_{H^1(\widetilde{\Omega})} \;\le\; C^*\;\|v\|_{H^1(\Omega)}. \tag{22}$$

Now (15), (17) and (22) imply (8). □

The last lemma contains the well–known Young inequality, see e.g. [20], p. 22.

Lemma A3. Let $a,\, b \ge 0,\; \varepsilon > 0,$ and $\lambda,\, \lambda^* > 1$ satisfy $1/\lambda + 1/\lambda^* = 1$. Then

$$ab \;\le\; \frac{1}{\lambda}\varepsilon^{\lambda}a^{\lambda} + \frac{1}{\lambda^*}\,\frac{1}{\varepsilon^{\lambda^*}}\,b^{\lambda^*}. \tag{23}$$

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