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Structure of Submodels
Diagonal indiscernibility in Models of Arithmetic

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Abstract

In this thesis, we study a range of questions concerning submodels of models of Peano arithmetic (PA) or its fragments. Our study focuses on three different areas that share a common central topic, namely diagonally indiscernible elements.

In the first part, we explore a diagonal version of the Infinite Ramsey Theorem provable in PA and partially provable in fragments of PA. We provide a detailed level-by-level analysis of the principle in terms of the arithmetical hierarchy and the corresponding fragments of the schemes of induction and collection. Then we derive a theorem characterizing Σ_n-elementary initial segments of a given model that satisfy (fragments of) PA as cuts on certain systems of diagonal indiscernibles.

In the second part, we study initial segments with some specific properties, and especially their distribution in a given countable model $M \models \mathrm{PA}$. We provide a theorem that gathers general topological consequences of the method of indicators. We then extend the theorem with further results about some prominent families of Σ_n-elementary initial segments (among others, those satisfying PA or $\mathrm{I}\Sigma_{n+k}$, and those isomorphic to M). For example, by applying the results from the first part, we prove that every interval that contains an Σ_n-elementary initial segment satisfying $\mathrm{I}\Sigma_{n+k}$ (or PA) contains a closed subset of such initial segments that is order-isomorphic to the Cantor set. We conclude by proving some strict inclusions between the topological closures of the studied families.

In the last part, we study the properties of the Stone space of the algebra of definable subsets of a given countable model $M \models \mathrm{PA}$. We present the topic from a non-standard viewpoint, situating the countable base model M into some $\aleph_1$-saturated elementary extension C, under which ultrafilters from the Stone space appear as sets of 1-indiscernible elements, called monads. Our main tool here is the Rudin-Keisler (RK) pre-order on monads. We investigate monads of diagonally indiscernible elements and diagonal partition properties on monads. Among other results, we prove that RK-minimal monads (which correspond to selective ultrafilters), p-monads (which correspond to p-points), and regular monads, in this order, are properties of strictly decreasing strengths. Furthermore, we show that the counter-examples (e.g. p-points that are not RK-minimal) form dense subsets in the the corresponding subspaces of the Stone space.

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Contents

Introduction

The first-order theory of Peano arithmetic (PA) is attractive for many reasons: it is probably the most direct formalization of our understanding of the concept of natural numbers; it is a fundamental axiomatic system for arithmetic and number theory; it interprets (and is interpreted in) the Finite Set Theory. With all these features, it necessarily plays a central role in the study of the foundations of mathematics.

Although the intended model for PA is $\mathbb{N}$ (the standard model), Skolem [Sko34] in the 1930's discovered that it is not (up to isomorphism) the only model; at about the same time, Gödel proved his First Theorem [Göd31], by which PA is incomplete and has no recursively axiomatized completion. Gödel's remarkable result has many notable consequences, namely that there is no consistent decidable extension of Peano arithmetic (Church [Chu36]), and that consistency of any reasonably rich axiomatic system cannot be established within the very same system (Gödel's Second Theorem). Ryll-Nardzewski ([RN52]) showed that PA is not finitely axiomatizable, and Rabin ([Rab61]) further strengthened this result by proving that PA cannot be reformulated as an axiomatic system with a bounded quantifier complexity.

The existence of non-standard models of PA, implied by both Skolem's and Gödel's results, attracted further attention: Tennenbaum [Ten59] proved that non-standard models are not recursive, McDowell and Specker proved that every countable non-standard model of Peano arithmetic has an elementary end-extension; it turned out that this property characterizes PA among theories that include a certain weak fragment of PA, as follows from the results of Paris and Kirby [PK78]. Further characterizations of PA of this type were later given by Kaye [Kay91a]. Other remarkable results were the theorems of Gaifman and Friedman, formulated elsewhere in this thesis. Thanks to work by Paris, Kirby, Clote, Kay, Dimitrocopulos, Slaman, and many others, much is known about various fragments of PA and alternatives of the induction scheme.

A substantial result was accomplished by Paris and Harrington [PH77], who proved that besides the independent formulae obtained in the Gödel's

First Theorem, which may not have a direct mathematical interpretation, there are mathematically 'interesting statements', true in $\mathbb{N}$, but unprovable in PA. The statement, called Paris-Harrington principle (PH), is basically a strengthening of the Finite Ramsey Theorem (FRT) (which is provable in PA). FRT asserts that for given k, n, l there is a large-enough finite set u such that for every k-coloring of all increasing n-tuples from u there is a subset $v \subseteq u$ of cardinality at least l such that all increasing n-tuples from v have the same color. Then v is referred to as being homogeneous for the given coloring. Now, PH is a similar statement with the additional requirement that v is relatively large, i.e. $|v| \geq \min(v)$. Paris and Harrington demonstrated that while the principle is provable in PA for every k, l and every concrete numeral n, it is unprovable when n is quantified in the theory.

Ramsey-like theorems are widely used in model theory for constructing sets of indiscernible elements, i.e. such that no two increasing tuples of the elements can be distinguished by a formula without parameters. The crucial point in the Paris and Harrington's unprovability result lies in the notion of diagonally indiscernible elements which are one of the main themes of this thesis. The notion comes probably from Harrington. X is a set of diagonally indiscernible elements if no two increasing tuples from X can be distinguished by a formula with parameters that are smaller than the minimum of the tuples and separated from the tuples by an element from X.

Obtaining diagonally indiscernible elements from PH requires some effort and cleverness. Kanamori and McAloon [KM87] found a principle, which produces sets of (almost) diagonally indiscernible elements in a more straightforward way.

In their usual form, the independence results take advantage of the full induction scheme in PA, which makes PA quite a strong theory (once means of coding are developed, theorems about finite sets are obtained as easily in PA as in the Finite Set Theory). However, for fragments of PA, where the induction is restricted to formulae of some bounded complexity, say Σ_n, the situation becomes more intricate. In [Par80], Paris meticulously analyzed the strength of PH restricted to $(n+1)$-tuples and showed how it relates to the induction scheme for Σ_n formulae. Most of his arguments were model-theoretic; proof-theoretic versions of his results and a similar analysis of the Infinite Ramsey Theorem in arithmetic can be found in Hájek and Pudlák's [HP93].

As Kanamori and McAloon noted in their paper, Paris' detailed analysis relating PH and fragments of PA can be translated for the Kanamori-McAloon principle, too, but doing so requires clever and rather complicated methods based on Mills's analysis of arboreal combinatorial properties [Mil80].

In Section 2.1, we consider a combinatorial principle that, as we show, can be viewed as the infinite version of the Kanamori-McAloon principle with some further generalization. The principle is provable in PA in the same sense in which the Infinite Ramsey Theorem is provable in PA. By following the example of [HP93], we are able to give a detailed level-by-level analysis of the strength of the principle, avoiding complicated combinatorics. As an immediate consequence of the results we get the existence of bounded and unbounded systems of diagonally indiscernible elements in fragments of PA, which we further exploit in Section 2.3.

The achievements of Paris and Harrington and the above-mentioned work of Paris and Kirby introduced several other fruitful ideas, such as the method of indicators, which we make use of in Chapter 3 of this thesis. Using their and similar methods, Smoryński, Lessan, Wilkie, McAloon, Kotlarski, Murawski, and many others, presented various interesting results about initial segments, cofinal extensions, and various other independent combinatorial statements in PA.

One of the open problems about models of arithmetic concerns the order of the set of initial segments satisfying PA. Given a countable model $M \models \mathrm{PA}$, let $\mathcal{P}_0$ be the family of all its initial segments that satisfy PA and let $\overline{\mathcal{P}}_0$ be its closure under unions and intersections of arbitrary subsystems. It is known, and in fact easily proved, that $\overline{\mathcal{P}}_0$ ordered by inclusion has the order type of the Cantor set. But what is the order type of $\mathcal{P}_0$ itself? How does it relate to $\overline{\mathcal{P}}_0$ in topological terms (beside being a dense subset)? What types of initial segments can we expect to find in $\overline{\mathcal{P}}_0 \setminus \mathcal{P}_0$? Are there any 'interesting' families $\mathcal{R} \neq \mathcal{P}_0$ of initial segments of M that are *symbiotic* with $\mathcal{P}_0$, i.e. $\overline{\mathcal{R}} = \overline{\mathcal{P}}_0$? Similar and various related questions were asked by Kotlarski in [Kot84b] with some interesting partial results.

In Chapter 3, we provide some general answers to these questions for an arbitrary family of initial segments of M that has an indicator in M. We then look in more detail at some prominent families of Σ_n-elementary initial segments; we prove the considered families to be non-symbiotic and give some further results about each of them. For example, by applying our results from Chapter 2 about systems of diagonal indiscernibles, we learn that every interval in the ordering of all initial segments of M that contains a Σ_n-elementary initial segment satisfying $\mathrm{I}\Sigma_{n+k}$ includes a closed subset order-isomorphic to the Cantor set of initial segments of the same kind.

This is an example of a theorem of a 'one-means-many' type, when the existence of one witness for some property implies the existence of some 'large' set of such witnesses. Theorems of this sort are quite common in the model theory of PA, although they may sometimes differ in the definition of 'many' or 'large'. Sometimes many in one sense may turn out to be not that many in some other sense (for example, $\mathcal{P}_0$ is dense but meager in $\overline{\mathcal{P}}_0$).

We see several examples of this type of theorems also in Chapter 4, where we study the algebra of definable subsets of a given countable model $M \models \mathrm{PA}$ and the associated Stone space. The topic relates to previous work, especially that of Kirby [Kir84] for models of fragments of the second-order arithmetic and various mathematicians studying indiscernibility in the context of Vopěnka's Alternative Set Theory. The latter connection is more than apparent when we study the situation in an enlarged context of an $\aleph_1$-saturated elementary extension C of M; C has features of a 'big model', a tool widely exploited in modern model theory. In this enlarged context, the studied ultrafilters can be identified with their non-empty intersections, called monads. After developing the necessary tools such as Rudin-Keisler (RK) ordering, we confront the properties of monads with properties of the corresponding initial segments of M. Although this introductory part does not contain any particularly novel or surprising results, we develop a tool for obtaining results of the 'one-means-many' type mentioned before.

The central and largely novel part of Chapter 4 is in Section 4.4, where we return to the main theme of diagonally indiscernible elements, proving several results about diagonal partition properties of monads. We also introduce the notion of a p-monad and prove by counter-examples of the 'one-means-many' type that this notion lies strictly between RK-minimality and regularity.

The text of this thesis is accompanied by two appendices. In Appendix A we reprove McDowell's and Specker's theorem by means of the combinatorial principle from Section 2.1; the proof includes a 'one-means-many' addendum, attributed to Gaifman. Appendix B, reproving certain results about strong cuts, is intended to complement Chapter 4 and make it largely self-contained.

CHAPTER 1

PRELIMINARIES AND FACTS

This chapter introduces basic notions and notational conventions used in this theses. In the last section of this chapter, we recall some useful facts about models of Peano arithmetic so that we may refer to them later. A reader familiar with the model theory of Peano arithmetic and its fragments may skip this chapter and use it only as a reference. For the most part our language and notation follows the mainstream publications in this area, namely [Kay91b], [HP93], and [KS06].

1.1 Language and notation

We use $\stackrel{\text{df}}{=}$ to define the symbol on the left to be what is on the right; similarly we use $\stackrel{\text{df}}{\iff}$ in definitions of formulae. We reserve the letters $\varphi, \psi, \theta, \xi, \dots$ for formulae, the letters $x, y, z, \dots$ for first-order variables, and n, m, k, l for natural numbers; in each case we allow subscripts or primes. For a language $\mathcal{L}$, $\mathrm{Fm}(\mathcal{L})$ denotes the set of all $\mathcal{L}$-formulae.

The language of first-order arithmetic, denoted by $\mathcal{L}^{Ar}$, is the first-order language with the signature $\langle 0, S, +, \cdot, <, = \rangle$.

For every natural n, we define

$$\underline{n} \stackrel{\text{df}}{=} \underbrace{S(\dots(S}_{n\text{-times}}(0))\dots),$$

the n-th *numeral*; concrete numerals are written plainly, as $1, 2, 3, \dots$, etc.

We use the following symbols for propositional connectives: $\rightarrow$ implication, $\leftrightarrow$ equivalence, $\wedge$ conjunction, $\vee$ disjunction. The symbol $(\exists! x)$ reads 'there exists a unique x'. Formally, $(\exists! x_1, \dots, x_n)\varphi(x_1, \dots, x_n)$ abbreviates

$$(\exists x_1, \dots, x_n)\varphi(x_1, \dots, x_n) \wedge \\ (\forall y_1, \dots, y_n)(\varphi(y_1, \dots, y_n) \rightarrow (y_1 = x_1 \wedge \dots \wedge y_n = x_n)).$$

The symbol $\bar{x}$ abbreviates the finite list $x_1,\dots,x_n$ of variables or elements. We use it in contexts where n—the length of the list—is known or not specifically important. Sometimes we write $\vec{x}$ instead of $\bar{x}$ to express the assumption that the tuple $\bar{x}$ is ordered increasingly with respect to $<$. The notation $\bar{x} \in X$ reads $x_1 \in X \wedge \dots \wedge x_n \in X$; similarly, $\vec{x} \in X$ reads $x_1 < \dots < x_n \wedge \bar{x} \in X$.

1.2 Arithmetic hierarchy, induction, and further schemes

We write $(\forall x < t)\varphi$ and $(\exists x < t)\varphi$ to abbreviate the formulae $(\forall x)(x < y \to \varphi)$ and $(\exists x)(x < y \wedge \varphi)$, respectively, and it is assumed that x has no occurrence in the term t.

1.2.1 Definition. *Bounded formulae* are formulae whose all quantifiers are bounded, i.e. occur in a context that can be written as $(\forall x < y)\varphi$ or $(\exists x < y)\varphi$, where x and y are distinct variables. The class of all bounded formulae is denoted by Δ_0. The classes of $\mathcal{L}^{Ar}$-formulae Δ_n, Σ_n, and Π_n are defined by induction on $n \in \omega$ as follows:

$\Sigma_0 \stackrel{\mathrm{df}}{=} \Pi_0 \stackrel{\mathrm{df}}{=} \Delta_0$ with Δ_0 as defined above. A formula *is* Σ_{n+1} if it is either Π_n or of the form $(\exists\bar{x})\varphi$ with $\varphi \in \Pi_n$; a formula *is* Π_{n+1} if it is Σ_n or of the form $(\forall\bar{x})\varphi$ with $\varphi \in \Sigma_n$. If Γ is a class of formulae (Σ_n or Π_n) and T is an $\mathcal{L}^{Ar}$-theory, then φ is said to be Γ *in* T, briefly $\Gamma(T)$, if $T \vdash \varphi \leftrightarrow \psi$ for some Γ formula ψ. Analogously is defined $\Gamma(M)$ for a model M.

A formula φ *is* Δ_n, respectively, if there are a Π_n formula ψ and a Σ_n formula θ such that $\vdash \varphi \leftrightarrow \psi \leftrightarrow \theta$. If the equivalence is provable in a theory T, we say that φ *is* $\Delta_n(T)$; similarly for a model. The hierarchy of formula classes $\Sigma_n, \Pi_n, \Delta_n$ for $n \in \omega$ is called the *arithmetic hierarchy*.

Every first-order formula is equivalent in the predicate calculus to a formula in the prenex normal form, thus every $\mathcal{L}^{Ar}$-formula is either Π_n or Σ_n for some $n \in \omega$.

The basic arithmetic theory, denoted by PA^-, is the $\mathcal{L}^{Ar}$-theory of *non-negative parts of discretely ordered rings* whose axioms are the universal closures of the following formulae (the given axiomatic system is not meant to be minimal, but clear and easy to deal with):

Ax1:	$(x+y)+z = x+(y+z)$	Ax9:	$\neg x < x$
Ax2:	$x+y = y+x$	Ax10:	$x < y \vee x = y \vee y < x$
Ax3:	$(x\cdot y)\cdot z = x\cdot(y\cdot z)$	Ax11:	$x < y \to x+z < y+z$
Ax4:	$x\cdot y = y\cdot x$	Ax12:	$(0 < z \wedge x < y) \to x\cdot z < y\cdot z$
Ax5:	$x\cdot(y+z) = x\cdot y + x\cdot z$	Ax13:	$x < y \to (\exists z)(x+z = y)$
Ax6:	$x+0 = x \wedge x\cdot 0 = 0$	Ax14:	$0 < S(0)$
Ax7:	$x\cdot S(0) = x$	Ax15:	$x > 0 \leftrightarrow (\exists z)(x = S(z))$
Ax8:	$x < y \wedge y < z \to x < z$	Ax16:	$0 \le x$

If $\varphi(x,\bar{y})$ is an $\mathcal{L}^{Ar}$-formula, the *axiom of induction on* x *for* $\varphi(x,\bar{y})$, written as $\mathrm{I}_x\varphi$, is the sentence

Ax17: $(\forall\bar{p})[\varphi(0,\bar{p})\wedge(\forall x)(\varphi(x,\bar{p})\rightarrow\varphi(S(x),\bar{p}))\rightarrow(\forall x)\varphi(x,\bar{p})]$

Peano arithmetic, denoted by PA, is the first-order theory containing all axioms of PA^- together with axioms of induction for all $\mathcal{L}^{Ar}$-formulae.

We shall often work with subtheories of PA that include only a limited portion of the induction scheme. Specifically, if Γ ($=\Sigma_n$ or Π_n) is a class of $\mathcal{L}^{Ar}$-formulae, then $\mathrm{I}\Gamma$ will denote the theory containing all axioms of PA^- and the induction axiom $\mathrm{I}_x\varphi$ for every $\varphi\in\Gamma$. We define $\mathrm{I}\Delta_n$ as the theory extending PA^- with the scheme of axioms $(\forall x,\bar{y})(\varphi\leftrightarrow\neg\psi)\rightarrow\mathrm{I}_x\varphi$ for all $\varphi,\psi\in\Sigma_n$ (we assume all free variables of φ,ψ are among $x,\bar{y}$).

Apart from induction, we also refer to the schemes defined below. Let Γ be a class of $\mathcal{L}^{Ar}$-formulae (namely Σ_n or Π_n). We define:

- $\mathrm{L}\Gamma$ is the theory obtained by adding to PA^- the *least element axioms* $\mathrm{L}\varphi$ (we should write $\mathrm{L}_x\varphi$ in fact) for all $\varphi\in\Gamma$, where $\mathrm{L}\varphi$ is the sentence

$$(\forall\bar{p})[(\exists x)\varphi(x,\bar{p})\rightarrow(\exists x_0)(\varphi(x_0,\bar{p})\wedge(\forall x<x_0)\neg\varphi(x,\bar{p}))].$$

  For a formula $\varphi(x,\bar{p})$, the expression $x_0=\mu x:\varphi(x,\bar{p})$ denotes the formula $\varphi(x_0,\bar{p})\wedge(\forall x<x_0)\neg\varphi(x,\bar{p})$ which reads 'x_0 is the least x such that $\varphi(x,\bar{p})$'.

- $\mathrm{L}\Delta_n$ denotes the theory obtained by adding to PA^- the scheme of axioms $(\forall\bar{x})(\varphi\leftrightarrow\neg\psi)\rightarrow\mathrm{L}\varphi$ for all $\varphi,\psi\in\Sigma_n$ (assuming all free variables of φ,ψ are among $\bar{x}$).

- $\mathrm{B}\Gamma$ is obtained by adding to $\mathrm{I}\Sigma_0$ the *collection axioms* $\mathrm{B}\varphi$ for all $\varphi\in\Gamma$, where $\mathrm{B}\varphi$ is the sentence

$$(\forall\bar{p})(\forall x_0)[(\forall x<x_0)(\exists y)\varphi(x,y,\bar{p})\rightarrow(\exists y_0)(\forall x<x_0)(\exists y<y_0)\varphi(x,y,\bar{p})].$$

- Coll_n is obtained by adding to PA^- the formula $\mathrm{B}\varphi$ for every $\varphi\in\Sigma_n$.

- $\mathrm{S}\Gamma$ is obtained by adding to $\mathrm{I}\Sigma_0$ the *strong collection axioms* $\mathrm{S}\varphi$ for all $\varphi\in\Gamma$, where $\mathrm{S}\varphi$ is the sentence

$$(\forall\bar{p})(\forall x_0)(\exists y_0)(\forall x<x_0)[(\exists y)\varphi(x,y,\bar{p})\rightarrow(\exists y<y_0)\varphi(x,y,\bar{p})].$$

1.2.2 Fact. *$\Sigma_n(\mathrm{Coll}_n)$, $\Pi_n(\mathrm{Coll}_n)$, and $\Delta_n(\mathrm{Coll}_n)$ are closed under bounded quantification for all $n\in\omega$. More specifically, if $\varphi(x,\bar{y})$ is a Σ_n formula, $\psi(x,\bar{y})$ a Π_n formula, and $t(\bar{y})$ an $\mathcal{L}^{Ar}$-term, then the formulae $(\forall x<t(\bar{y}))\varphi(x,\bar{y})$ and $(\forall x<t(\bar{y}))\psi(x,\bar{y})$ are $\Sigma_n(\mathrm{Coll}_n)$ and $\Pi_n(\mathrm{Coll}_n)$ respectively.* $\square$

1.2.3 Fact. *The following relations between arithmetic schemes are provable in* $\mathrm{I}\Sigma_0$*:*

$$\begin{array}{l}
\mathrm{I}\Sigma_{n+1} \Leftrightarrow \mathrm{L}\Sigma_{n+1} \Leftrightarrow \mathrm{S}\Pi_n \\
\Downarrow \\
\mathrm{B}\Sigma_{n+1} \Leftrightarrow \mathrm{B}\Pi_n \Leftrightarrow \mathrm{L}\Delta_{n+1} \Rightarrow \mathrm{I}\Delta_{n+1} \\
\Downarrow \\
\mathrm{I}\Sigma_n \Leftrightarrow \mathrm{I}\Pi_n \Leftrightarrow \mathrm{L}\Sigma_n \Leftrightarrow \mathrm{L}\Pi_n \Leftrightarrow \mathrm{S}\Sigma_n
\end{array}$$

($T \Rightarrow S$ *denotes* $T \vdash S$*, and* $T \Leftrightarrow S$ *denotes* $T \vdash S$ *and* $S \vdash T$*; the vertical arrows cannot be reverted).*

Proof. Most implications are folklore; the proofs can be found in [Kay91b], except for those concerning $\mathrm{S}\Gamma$, for which the reader may refer to [HP93]. □

1.2.4 Remark. Slaman [Sla04] proved recently $\mathrm{I}\Delta_{n+1} + \mathrm{Exp} \Rightarrow \mathrm{L}\Delta_{n+1}$, where Exp asserts that exponential is a total function.

1.3 Model-theoretic notation and terminology

We denote models of $\mathcal{L}^{Ar}$ theories by the letters $A, B, C, \ldots, M, N, \ldots$. If A is a model, then the domain (universe) of A will also be denoted by A. Unless stated otherwise, $A \subseteq B$ (or $A \subset B$) means that A is a (proper) submodel of B, rather than 'the domain of A is a (proper) subset of the domain of B'. If deleting or adding the symbols to the language or if the relaxed convention identifying a model with its domain leads to confusion, we specify structures by explicitly listing their domain and all their functions and relations, like $\langle A, F_1, F_2, \ldots, R_1, R_2, \ldots \rangle$.

For an $\mathcal{L}^{Ar}$-structure A, the symbols $+^A, \cdot^A, <^A, \ldots$ denote the realizations of $\mathcal{L}^{Ar}$-symbols $+, \cdot, <, \ldots$ in A. However, if it cannot lead to a confusion, we tend to drop the superscript. If A is a substructure of B, we use the same set of symbols for realizations in A and B with the understanding that the realizations in A are restrictions of those in B.

If $\varphi(x_1, \ldots, x_n)$ is a formula with all free variables among $x_1, \ldots, x_n$ and $a_1, \ldots, a_n \in A$, we write $A \models \varphi(a_1, \ldots, a_n)$, or briefly $A \models \varphi(\bar{a})$, asserting that the formula φ is true in A when each variable x_i is interpreted as a_i. In this notation we may sometimes mingle variables and elements of models in a formula, as in $A \models (\forall y)\psi(y, \bar{a})$ for $\bar{a} \in A$.

If A is an $\mathcal{L}^{Ar}$-structure and $X \subseteq A$ subset, then $\mathcal{L}^{Ar}_X$ denotes the language obtained by extending $\mathcal{L}^{Ar}$ by a new constant for every element of X. We naturally expand the model A to the language $\mathcal{L}^{Ar}_X$ by interpreting each of the constants by the corresponding element. Unless a confusion may arise, we denote the expanded model again by A.

1.3.1 Definition. Let A,B be $\mathcal{L}^{Ar}$-structures and $I \subseteq A$ a subset.

a) B is a *cofinal substructure* of A (written as $B \subseteq^{cf} A$) if B is a substructure of A and for every $a \in A$, there exists $b \in B$ such that $a < b$.

b) I is an *initial segment* of A if $b < a$ implies $b \in I$ for all $a \in I$. We use this term also if A is only a model for the language $\langle < \rangle$.

c) I is a *cut* of A if it is an initial segment of A closed under the successor. I is a *proper cut* of A if it is a cut and $I \neq A$.

d) I is an *an initial substructure* of A (written as $I \subseteq^e A$) if I is a cut of A closed under addition and multiplication. (In particular, I is a substructure of A).

e) If I is proper initial substructure of A (i.e. $A \neq I \subseteq^e A$, or shortly $I \subset^e A$), we also say that A is an *end-extension* of I.

Remark. In some literature, the term cut is used for what we call an initial segment, in other for what we call an initial substructure. Our use of the terms cut and initial segment is in agreement with the mainstream textbooks [Kay91b] and [KS06]. In [Kay91b], the definition of an initial segment is given for models, but the term appears to be used in the context of our definition. The term *initial substructure* is introduced to prevent a similar confusion.

Every model A of basic arithmetic includes a unique initial substructure isomorphic to the structure of natural numbers $\mathbb{N}$. We will always identify this unique initial substructure with $\mathbb{N}$ and assume $\mathbb{N} \subseteq^e A$. (We use the symbol $\mathbb{N}$ in the context of an $\mathcal{L}^{Ar}$-structure and ω as the first non-zero limit ordinal number; nevertheless, they are the same set).

1.3.2 Definition. Let A be an $\mathcal{L}^{Ar}$-structure.

a) For an $\mathcal{L}^{Ar}$-formula $\varphi(x,\bar{y})$ and parameters $\bar{p} \in A$, $\varphi(A,\bar{p})$ denotes the *subset of A defined by φ over $\bar{p}$*, i.e.

$$\varphi(A,\bar{p}) \stackrel{\text{df}}{=} \{a \in A \mid A \models \varphi(a,\bar{p})\}.$$

b) If $X \subseteq A$ is a subset and Γ a set of $\mathcal{L}^{Ar}$-formulae, then $\mathcal{D}_\Gamma(A,X)$ denotes the set of all Γ-*definable subsets* of A over parameters from X, i.e.

$$\mathcal{D}_\Gamma(A,X) \stackrel{\text{df}}{=} \{\varphi(A,\bar{p}) \mid \varphi \in \Gamma, \bar{p} \in X\}.$$

We denote $\mathcal{D}_{\mathrm{Fm}}(\mathcal{L}^{Ar})(A,X)$ by just $\mathcal{D}(A,X)$.

c) An element $a \in A$ is Γ-*definable in A over X* if $\{a\} \in \mathcal{D}_\Gamma(A,X)$, i.e. if $A \models (\exists! x)\varphi(x,\bar{p}) \wedge \varphi(a,\bar{p})$ for some $\varphi(x,\bar{y}) \in \Gamma$ and $\bar{p} \in X$.

d) The symbol $\mathrm{Dfe}_\Gamma(A,X)$ denotes the *set of all* Γ*-definable elements in* A *over* X. We let $\mathrm{Dfe}(A,X) \stackrel{\mathrm{df}}{=} \mathrm{Dfe}_{\mathrm{Fm}(\mathcal{L}^{Ar})}(A,X)$.

Note that $\mathcal{D}(A,X)$ is a Boolean algebra of sets.

1.3.3 Definition. For $\mathcal{L}^{Ar}$-structures A,B and an $\mathcal{L}^{Ar}$-formula $\varphi(\bar{x})$, $A \preccurlyeq_\varphi B$ denotes the fact that 'A is *φ-elementary substructure* of B', i.e. for all $\bar{a} \in A$, $A \models \varphi(\bar{a})$ iff $B \models \varphi(\bar{a})$. For a set of $\mathcal{L}^{Ar}$-formulae Γ, $A \preccurlyeq_\Gamma B$ denotes that $A \preccurlyeq_\varphi B$ for every $\varphi \in \Gamma$; $A \preccurlyeq_n B$ is an abbreviation for $A \preccurlyeq_{\Sigma_n} B$ (which we read 'A is an *n-elementary substructure* of B'), and $A \preccurlyeq B$ for $A \preccurlyeq_{\mathrm{Fm}(\mathcal{L}^{Ar})} B$ (i.e. the usual notion of 'A being an *elementary substructure* of B'). We write $A \preccurlyeq_n^e B$ if $A \preccurlyeq_n B$ and $A \subseteq^e B$, and $A \preccurlyeq_n^{cf} B$ if $A \preccurlyeq_n B$ and $A \subseteq^{cf} B$, etc.

Clearly, if $A \preccurlyeq B$, then $\mathcal{D}(A,A)$ and $\mathcal{D}(B,A)$ are isomorphic as Boolean algebras.

1.4 Second-order arithmetic

We now briefly introduce the basic second-order systems for arithmetic.

1.4.1 Definition. The language $\mathcal{L}^{\mathrm{II}}$ of second-order arithmetic is obtained by expanding the language $\mathcal{L}^{Ar}$ by a set of new variables (written in capital letters) to represent sets and the symbol $\in$ for set membership. Equality for set is defined by extensionality. A structure for $\mathcal{L}^{\mathrm{II}}$ is a pair $\langle N, \mathcal{X}\rangle$, where N is an $\mathcal{L}^{Ar}$-structure and $\mathcal{X}$ is the domain for the set variables, $\mathcal{X} \subseteq \mathcal{P}(N)$. We define Σ_0^0 as the smallest class of formulae containing all open $\mathcal{L}^{\mathrm{II}}$ formulae and closed under bounded quantification of the form $(\exists x < y)\ldots$ and $(\forall x < y)\ldots$, where x,y are distinct. Σ_n^0 and Π_n^0 formulae are defined just like Σ_n and Π_n but for the language $\mathcal{L}^{\mathrm{II}}$. Formulae occurring on some level of this hierarchy are called *arithmetic* $\mathcal{L}^{\mathrm{II}}$-formulae (that is to say, arithmetic are those $\mathcal{L}^{\mathrm{II}}$ formulae that do not quantifying over set variables).

If $\langle N, \mathcal{X}\rangle$ is as above and Γ is a class of $\mathcal{L}^{\mathrm{II}}$ formulae, then $\Gamma\mathcal{X}$ denotes the set of all subsets of N of the form

$$\{x \in N \mid \langle N, \mathcal{X}\rangle \models \varphi(x, Y_1, \ldots, Y_k)\},$$

with $\varphi(x, X_1, \ldots, X_k) \in \Gamma$ and $Y_1, \ldots, Y_k \in \mathcal{X}$. We put $\Delta_n^0\mathcal{X} \stackrel{\mathrm{df}}{=} \Sigma_n^0\mathcal{X} \cap \Pi_n^0\mathcal{X}$ as usual. The Γ*-comprehension* is a scheme of axioms postulating that every Γ formula defines some set, i.e. semantically, $\Gamma\mathcal{X} \subseteq \mathcal{X}$. (This semantic interpretation justifies use of the term Δ_n^0-comprehension, even though Δ_n^0 does not directly correspond to a class of formulae.) $\Gamma\mathrm{CA}_0$ denotes the theory consisting of first-order axioms of basic arithmetic PA^-, the Γ-comprehension scheme, and the following axiom of restricted induction:

$$(\forall X)[(0 \in X \wedge (\forall x)(x \in X \rightarrow x + 1 \in X)) \rightarrow (\forall x) x \in X]. \tag{1.1}$$

$A\mathrm{CA}_0$ denotes $\Gamma\mathrm{CA}_0$ where Γ are all arithmetic $\mathcal{L}^{\mathrm{II}}$-formulae, i.e. $\Gamma = \bigcup_{n\in\omega}\Sigma^0_n$. The theories $\mathrm{I}\Sigma^0_n$ and $\mathrm{B}\Sigma^0_n$ are defined analogously to their $\mathcal{L}^{Ar}$-counterparts (allowing set parameters). By (1.1), $\Sigma^0_n\mathrm{CA}_0$ implies $\mathrm{I}\Sigma^0_n$. In particular, if $\langle N, \mathcal{X}\rangle \models A\mathrm{CA}_0$, then $N \models \mathrm{PA}$.

1.4.2 Proposition. $\Sigma^0_1\mathrm{CA}_0$ *implies* $A\mathrm{CA}_0$.

Proof. Let $n \geq 1$ and let $\mathfrak{N} = \langle N, \mathcal{X}\rangle \models \Sigma^0_n\mathrm{CA}_0$. Then clearly $\mathfrak{N} \models \Pi^0_n\mathrm{CA}_0$, since $\mathcal{X}$ is closed under complement. Let φ be Π^0_n. If $Y = \{a \in N \mid \mathfrak{N} \models (\exists y)\varphi(a,y,\ldots)\}$, then $Y' \overset{\mathrm{df}}{=} \{b \in N \mid \mathfrak{N} \models \varphi((b)_0,(b)_1,\ldots)\} \in \mathcal{X}$ by $\Pi^0_n\mathrm{CA}_0$, so $Y = \{a \in N \mid \mathfrak{N} \models (\exists y)(y \in Y \wedge (y)_0 = a)\}$. Thus $Y \in \mathcal{X}$ by $\Sigma^0_1\mathrm{CA}_0$. This proves that $\mathfrak{N} \models \Sigma^0_{n+1}\mathrm{CA}_0$. □

Every $\mathcal{L}^{Ar}$ formula of the form $\varphi(x,\bar{p})$ can also be viewed as a *class* (more specifically, a *class with parameters* $\bar{p}$) of all individuals x satisfying $\varphi(x,\bar{p})$. We often enforce such interpretation, by denoting the class by a new set variable, say X, and writing $X = \{x \mid \varphi(x,\bar{p})\}$. Then φ is referred to as the *defining formula* of the class X. All arithmetical $\mathcal{L}^{\mathrm{II}}$ formulae in the variable X then translate to $\mathcal{L}^{Ar}$ formulae by replacing the atomic subformulae of the form $t \in X$ with $\varphi(t,\bar{p})$. We say that a class is Σ_n, Π_n,... etc., if its defining formula is. When we quantify over classes, we usually quantify over the defining formulae in metatheory. In 1.6 we introduce a method of quantifying over classes of bounded complexity in sufficiently strong theories. We then refer to such classes as sets.

For every first-order structure $A \models \mathcal{L}^{Ar}$, there is the canonical expansion of A into the second-order structure $\langle A, \mathfrak{D}(A,A)\rangle$, satisfying comprehension scheme for arithmetical formulae. We usually exploit this and denote the second-order structure just by A, writing $A \models \varphi(Y_1,Y_2,\ldots)$ for an arithmetical $\mathcal{L}^{\mathrm{II}}$ formula φ and $Y_1,Y_2,\ldots \in \mathfrak{D}(A,A)$. Sometimes we push this habit even further, writing $A \models \varphi(Y_1,Y_2,\ldots)$ even if $Y_1,Y_2,\ldots$ are just arbitrary subsets of A, not necessarily from $\mathfrak{D}(A,A)$. Such notation just abbreviates $\langle A,\{Y_1,Y_2,\ldots\}\rangle \models \varphi(Y_1,Y_2,\ldots)$. We only use these conventions for *arithmetical* formulae, so there is no real danger of confusion.

1.5 Coding and set-theoretic notation

It is well known that $\mathrm{I}\Sigma_1$ (as well some weaker theories) provides definable coding of bounded sequences of individuals (see e.g. [HP93] for details).

To avoid the tedious details, we take for granted that the predicates and functions described informally below have Σ_1-definitions and that $\mathrm{I}\Sigma_1$ proves all their usual properties:

$\mathrm{Seq}(u)$	$\overset{\mathrm{df}}{\iff}$	'u codes a sequence',
$\ell(u)$	$\overset{\mathrm{df}}{=}$	'the length of the sequence coded by u',
$\mathrm{Min}(u)$	$\overset{\mathrm{df}}{=}$	'the minimum of elements of the sequence u',
$\mathrm{Max}(u)$	$\overset{\mathrm{df}}{=}$	'the maximum of elements of the sequence u',
$(u)_y$	$\overset{\mathrm{df}}{=}$	'the $y+1$-st element of the sequence coded by u', provided $y < \ell(u)$ (otherwise we let $(u)_y \overset{\mathrm{df}}{=} 0$),
$u \smile v$	$\overset{\mathrm{df}}{=}$	'the code of the concatenation of the sequences coded by u and v',
$\langle x_0,\dots,x_{n-1}\rangle$	$\overset{\mathrm{df}}{=}$	'the code of the sequence $x_0,\dots,x_{n-1}$',
$x \,\varepsilon\, u$	$\overset{\mathrm{df}}{\iff}$	'x is an element of u', formally $(\exists y < \ell(u))x = (u)_y$,
$\mathrm{OSeq}(u)$	$\overset{\mathrm{df}}{\iff}$	'u codes an increasing sequence', formally $\mathrm{Seq}(u) \wedge (\forall x < y < \ell(u))(u)_x < (u)_y$.

We may further assume that all the functions above are in fact Δ_1-definable.

The subsequent definitions are meaningful at least in the following contexts: X,P,R,F are classes in some theory $T \supseteq \mathrm{I}\Sigma_1$, or X,P,R,F are subsets of some model A of $\mathrm{I}\Sigma_1$.

X is *bounded*, iff there is some x such that $(\forall y)(y \in X \to y < x)$. It is *unbounded* iff it is not bounded.

If our assumptions (e.g. induction) assure that X has a least element, we denote the element by $\min(X)$. If X has a greatest element, it is denoted by $\max(X)$.

X is said to be *codable* or *coded*, if there is an increasing sequence u that enumerates all elements of X, i.e. if

$$\mathrm{OSeq}(u) \wedge [x \in X \leftrightarrow (\exists y < \ell(u))x = (u)_y].$$

We the refer to u as the *code* of X and the length of $\ell(u)$ also as the *cardinality* of X, written as $|X|$.

In PA, a class is codable iff it is bounded; a subset $Y \subseteq A$ of a model $A \models \mathrm{PA}$ is codable (in A) iff it is bounded and definable.

In fragments of PA, the situation is more complicated. It is said that X has the *order-type of the universe* if for each y there is an increasing sequence u of length y that enumerates the first y elements of X (in such case we may write $|X| = \infty$). Now, $\mathrm{I}\Sigma_1$ proves that if a class X has the order-type of the universe, then X is unbounded. However, in general, $I\Sigma_n$ is required to prove that an unbounded Σ_n class has the order-type of the universe (cf. [HP93, Chapter I, 3(c)]).

X is said to be *relatively large*, if it is either unbounded or $\min(X) \leq |X|$.

We use Cantor's pairing function for coding pairs:

$$\langle x, y\rangle \overset{\mathrm{df}}{=} \frac{(x+y+1)(x+y)}{2} + x. \tag{1.2}$$

and define $X \times Y \stackrel{\text{df}}{=} \{\langle x, y\rangle \mid x \in X \wedge y \in Y\}$. For a pair u we denote $\langle u\rangle_0$ and $\langle u\rangle_1$ the elements such that $p \stackrel{\text{df}}{=} \langle\langle u\rangle_0, \langle u\rangle_1\rangle$.

With pairing at hand, we may introduce common set-theoretic notions to arithmetic in a natural way: A class R whose only elements are pairs is also called a *(binary) relation*. The *domain* of a relation R is the class $\mathrm{dom}(R) \stackrel{\text{df}}{=} \{x \mid (\exists y)\langle x, y\rangle \in R\}$. The *range* of R is the class $\mathrm{rng}(R) \stackrel{\text{df}}{=} \{y \mid (\exists x)\langle x, y\rangle \in R\}$. The image of X in R is $R''X \stackrel{\text{df}}{=} \{y \mid (\exists x \in X)\langle x, y\rangle \in R\}$. The class $R_x \stackrel{\text{df}}{=} R[x] \stackrel{\text{df}}{=} R''\{x\}$ is the *extension* of x in R. Other common set-theoretic notions and operations translate naturally.

Let P be an equivalence relation. Then $P[x]$ is the *P-equivalence class* of x. We write $x\,P\,y$ iff $\langle x, y\rangle \in P$. The family of P-equivalence classes forms a *partition* of $\mathrm{dom}(P)$. Since, conversely, every partition induces an equivalence on its domain whose equivalence classes are the blocks of the partition, we identify these two notions and represent partitions by the induced equivalence relations.

Another common way to represent partitions is by functions. If F is a function, then $P_F \stackrel{\text{df}}{=} \{\langle x, y\rangle \mid F(x) = F(y)\}$ is a partition of $\mathrm{dom}(F)$. Similarly, if P is a partition, we may put $F_P(x) \stackrel{\text{df}}{=} \min(P[x])$ for $x \in \mathrm{dom}(P)$; then $F = P_{F_P}$ and $F(x) \le x$ for all $x \in \mathrm{dom}(P)$.

Let P and F_P be as above and let $\mathrm{rng}(F_P)$ be coded or of the order-type of the universe. We then define $\|P\| \stackrel{\text{df}}{=} |\mathrm{rng}(F_P)|$ and let $P_{(i)} \stackrel{\text{df}}{=} P[x]$ if $P[x]$ the i-th block of P, i.e. if x satisfies $|\{y < x \mid y \in \mathrm{rng}(F_P)\}| = i$.

We say that $Y \subseteq \mathrm{dom}(P)$ is a *choice set* for a partition P, if $x\,P\,y$ implies $x = y$ whenever $x, y \in Y$; Y is a *total choice set* if $Y = \mathrm{dom}(P)$. Clearly, $\mathrm{rng}(F_P)$ with F_P defined as above is a total choice set for P.

Let $\langle X\rangle^d$ denote the class of codes of all increasing sequences with elements in X, i.e. such that $u \in \langle X\rangle^d$ iff $\mathrm{OSeq}(u) \wedge (\forall x)(x \,\varepsilon\, u \to x \in X) \wedge \ell(u) = d$.

Let P be a partition of $\langle X\rangle^y$ for some $y > 0$. We say that $Y \subseteq X$ is *homogeneous* for P, if $\langle Y\rangle^y \subseteq P[z]$ for some $z \in \langle X\rangle^y$.

If $\langle A, <^A\rangle$ is an ordered set and $X \subseteq A$, then $\mathsf{sup}_{\le^A}(X)$ denotes the smallest lower subset of $\langle A, <^A\rangle$ that includes X and $\mathsf{inf}_{\le^A}(X)$ denotes the largest lower subset of $\langle A, <^A\rangle$ disjoint from X; that is:

$$\mathsf{sup}_{\le^A}(X) \stackrel{\text{df}}{=} \{a \in A \mid (\exists b \in X)(a \le^A b)\},$$
$$\mathsf{inf}_{\le^A}(X) \stackrel{\text{df}}{=} \{a \in A \mid (\forall b \in X)(a <^A b)\}.$$

If $Y \subseteq X \subseteq A$ and $\mathsf{sup}_{\le^A}(Y) = \mathsf{sup}_{\le^A}(X)$, we say that Y is *cofinal* in X; if $\mathsf{inf}_{\le^A}(Y) = \mathsf{inf}_{\le^A}(X)$, we say that Y is *coinitial* in X. When A is known from the context, we abbreviate $\mathsf{sup}_{\le^A}(X)$ and $\mathsf{inf}_{\le^A}(X)$ as $\mathsf{sup}\,X$ and $\mathsf{inf}\,X$, respectively. We use (typographically distinct) symbols $\sup_{\le^A} X$ and $\inf_{\le^A} X$ to denote supremum and infimum of X in $\langle A, \le^A\rangle$ in the usual sense. If $X, Y \subseteq A$ are subsets of A, we write $a <^A X$ if $a \in \mathsf{inf}_{\le^A}(X)$ and write $X <^A a$ if $X \subseteq [0, a)$. We write $Y <^A X$ if $(\forall a \in Y) a <^A X$.

The following notation is used for intervals: $[x,y] \stackrel{\mathrm{df}}{=} \{z \mid x \le z \le y\}$, $[x,y) \stackrel{\mathrm{df}}{=} \{z \mid x \le z < y\}$, $(x,\rightarrow) \stackrel{\mathrm{df}}{=} \{z \mid z > x\}$; intervals $[x,y)$, (x,y), and $[x,\rightarrow)$ are defined analogously.

1.6 Satisfaction and Σ_n-sets

We now only roughly introduce the concepts of Σ_n and Π_n sets and their relativized counterparts. The details are tedious rather than particularly difficult. Hopefully, these things are generally well known; a reader unfamiliar with formalization may refer e.g. to [HP93].

It is well known that $\mathrm{I}\Sigma_1$ (and also some weaker theories) can formalize basic logical concepts, like terms, formulae, the language of arithmetic, classes of Σ_n and Π_n formulae, etc. and prove their basic properties; we refer to these formalized concepts as *formal*.

For $n \ge 1$, there are a $\Delta_1(\mathrm{I}\Sigma_1)$ formula $\mathrm{Sat}_{\Sigma_0}(x,y)$, a Π_n formula $\mathrm{Sat}_{\Pi_n}(x,y)$ and a Σ_n formula $\mathrm{Sat}_{\Sigma_n}(x,y)$ formalizing in $\mathrm{I}\Sigma_1$ satisfaction for formal Σ_0, Π_n and Σ_n formulae respectively, where x is the formula and y is a sequence representing an evaluation of variables that covers all free variables in x.

In other words, if Γ is one of Σ_0, Π_n, or Σ_n for $n \ge 1$, then $\mathrm{I}\Sigma_1$ proves:

> $\mathrm{Sat}_\Gamma(x,u) \leftrightarrow$ 'x is a Γ-formula whose all free variables are among the first $\ell(u)$ variables (in some fixed ordering of all formal variables $v_0, v_1, \ldots$) and that satisfies Tarski's truth conditions for the evaluation assigning $(u)_i$ to the variable v_i'.

(Notions appearing within '...' are meant in the appropriate formalization; this in particular applies the class of formulae Γ.) Moreover, for every (metamathematical) Γ-formula φ, there is a natural number $\ulcorner\varphi\urcorner$ called the Gödel number of φ such that

$$\mathrm{I}\Sigma_1 \vdash \varphi(\bar{x}) \leftrightarrow \mathrm{Sat}_\Gamma(\ulcorner\varphi(v_0)\urcorner, \langle\bar{x}\rangle).$$

This fact is often referred to as *It's snowing* $\leftrightarrow$ *'it's snowing'* lemma and says that a sentence does not change its meaning when formalized.

This justifies the following definitions which will allow us to quantify over (possibly infinite) Σ_n, Π_n, and Δ_n sets in $\mathrm{I}\Sigma_1$ using their codes:

1.6.1 Definition ($\mathrm{I}\Sigma_1$).

a) c is a (code of a) Σ_n *set* ($n \ge 0$) if c is a formal Σ_n formula whose only free variable is the 0-th formal variable v_0.

b) $x \in_{\Sigma_n} c$ if c is a Σ_n set and $\mathrm{Sat}_{\Sigma_n}(c, \langle x\rangle)$.

c) Π_n *sets* and the predicate $\in_{\Pi_n}$ ($n \ge 1$) are defined analogously.

d) For $n \geq 1$, a Δ_n *set* is a pair $\langle c, d \rangle$ where c is a Σ_n set, d is a Π_n set and for all x, $x \in_{\Sigma_n} c \leftrightarrow x \in_{\Pi_n} d$. We define $x \in_{\Delta_n} \langle c, d \rangle$ as $x \in_{\Sigma_n} c$ (which holds iff $x \in_{\Pi_n} d$).

Note that in $I\Sigma_1$, being a Σ_n (or Π_n) set is a Δ_1 predicate, $x \in_{\Sigma_0} c$ is Δ_1, $x \in_{\Sigma_n} c$ for $n \geq 1$ is Σ_n and $x \in_{\Pi_n} c$ for $n \geq 1$ is Π_n. The statement that e is a Δ_n set is Π_{n+1} in $I\Sigma_1$ but the predicate $x \in_{\Delta_n} e$ is Δ_n.

Instead of working with the codes explicitly, we shall normally use set (i.e. 2nd order) variables $X, Y, \ldots$ to denote Σ_n, Π_n and Δ_n sets and indicate to which of the classes each set belongs. We then write $\in$ instead of $\in_{\Sigma_n}$, $\in_{\Pi_n}$, and $\in_{\Delta_n}$.

We will also need relativized version of these concepts. This requires the following definition:

1.6.2 Definition.

a) In $I\Sigma_1$, we define a set X to be *piecewise coded* if for each u there is a sequence s of 0's and 1's of length u such that $(\forall i < u)((s)_i = 1 \leftrightarrow i \in X)$.

b) If Y is a set variable, then $\Sigma_n(Y)$ denotes the set of all Σ^0_n formulae that do not contain other set variables than Y. We refer to formulae from $\Sigma_n(Y)$ as Σ_n *in* Y.

1.6.3 Fact. *There are formulae* $\mathrm{Sat}_{\Sigma_n(Y)}(x, y, u)$ *($n \geq 0$) such that* $I\Sigma_1$ *proves (all notions formal): 'If y is a piecewise coded set, then* $\mathrm{Sat}_{\Sigma_n(Y)}$ *obeys Tarski's truth conditions for $\Sigma_n(Y)$ formulae x if Y is interpreted by y'. Moreover,* $\mathrm{Sat}_{\Sigma_n(Y)}$ *is $\Sigma_n(Y)$ for $n \geq 1$ and* $\mathrm{Sat}_{\Sigma_0(Y)}$ *is $\Delta_1(Y)$ in* $I\Sigma_1$ *under the assumption that 'Y is piecewise-coded'.*

The last sentence should be interpreted as follows: there are a $\Sigma_1(Y)$ formulae $\varphi(Y, \bar{z}), \psi(Y, \bar{z})$ such that

$$I\Sigma_1 \vdash (\forall x, y, \bar{z})(y \text{ is piecewise-coded} \to (\mathrm{Sat}_{\Sigma_0(Y)}(x, y, \langle \bar{z} \rangle) \leftrightarrow \varphi(y, \bar{z}) \leftrightarrow \psi(y, \bar{z})).$$

Under the assumption that X is piecewise coded we also have the corresponding version of the It's snowing $\leftrightarrow$ 'it's snowing' lemma.

The class of formulae $\Pi_n(Y)$ ($n > 1$) and related notions are introduced similarly. We further define:

1.6.4 Definition.

a) If H is a total Δ_1 function, we further define $\Sigma^H_0(X)$ as a class of formulae obtained from first-order atomic formulae and atomic formulae of the form $x \in X$ by bounded quantification of the forms $(\forall x \leq y)$ and $(\forall x \leq H(y))$.

b) A formula is $\Sigma^*_0(X)$ if it is $\Sigma^H_0(X)$ for some total Δ_1 function H.

The corresponding $\mathrm{Sat}_{\Sigma_0^H,X}$ formula for $\Sigma_0^H(X)$ formulae is introduced as before; under the assumption 'Y is piecewise-coded' it is $\Delta_1(Y)$ in $\mathrm{I}\Sigma_1$.

The satisfaction formulae allow us to define the notions of $\Sigma_n(X)$ $(n \geq 0)$, $\Pi_n(X)$ $(n > 0)$, and $\Sigma_0^*(X)$ sets and the corresponding membership relations, as we did in Definition 1.6.1 for Σ_n and Π_n sets. We refer to $\Sigma_n(X)$ sets as *sets* Σ_n *in* X; similarly for the other classes.

1.6.5 Definition. If each of Γ_1, Γ_1 is Σ_n $(n \geq 0)$, Π_n $(n > 0)$, Σ_0^H, then a set is said to be $\Gamma_1(\Gamma_2)$ if it is $\Gamma_1(X)$ for some Γ_2 set X.

To be able to quantify over $\Gamma_1(\Gamma_2)$ sets in a theory T, we must ensure that Γ_2 sets are piecewise coded in T. For this, we have

1.6.6 Lemma. [HP93, Lemma I.2.63]
For $n \geq 0$ (notions within '...' meant formally):

a) $\mathrm{I}\Sigma_n \vdash$ *'each Σ_n set is piecewise coded'.*

b) $\mathrm{B}\Sigma_{n+1} \vdash$ *'each Δ_{n+1} set is piecewise coded'.*

c) $\mathrm{I}\Sigma_n \vdash$ *'Δ_n total functions are closed under primitive recursion'.* □

1.7 Low sets and Low Basis Theorem

In Section 2.1, we rely on the following definitions and facts and on the Low basis theorem, and its corollary; all details can be found in [HP93], which is our fundamental reference on this topic:

1.7.1 Definition (*Low sets*). Let $n \geq 1$.

a) We define X to be a *low* Δ_{n+1} set if it is Δ_{n+1} and every $\Sigma_1(X)$ set Y is also Δ_{n+1}. The definition is meaningful in $\mathrm{B}\Sigma_{n+1}$, which has satisfaction for $\Sigma_1(\Delta_{n+1})$ formulae; in particular, we may quantify over *low* Δ_{n+1} sets in $\mathrm{B}\Sigma_{n+1}$.

b) Similarly, we define X to be a *low* $\Sigma_0^*(\Sigma_n)$ set (or LL_n set, briefly) if it is $\Sigma_0^*(\Sigma_n)$ and every $\Sigma_1(X)$ set Y is also $\Sigma^*(\Sigma_n)$. The definition is meaningful in $\mathrm{I}\Sigma_n$, which has satisfaction for $\Sigma_0^*(\Sigma_n)$ formulae; in particular, we may quantify over LL_n sets in $\mathrm{I}\Sigma_n$. C.f. [HP93, I.2.69,2.76, and 2.77] for details.

1.7.2 Fact. *Let $n \geq 1$. Then*

a) $\mathrm{B}\Sigma_n$ *proves:* $\Sigma_0^*(\Delta_n) \subseteq \Delta_n$ *(easy).*

b) $\mathrm{B}\Sigma_{n+1}$ *proves:* $\Delta_1(\mathit{low}\,\Delta_{n+1}) = \mathit{low}\,\Delta_{n+1}$ *and* $\Sigma_2(\mathit{low}\,\Delta_{n+1}) = \Sigma_{n+1}$ *(cf.* [HP93, I.2.71]*).*

c) $\mathrm{B}\Sigma_{n+1}$ *proves induction (and, trivially, collection) for low* Δ_{n+1} *formulae (since* $\mathrm{B}\Sigma_{n+1} \to \mathrm{I}\Delta_{n+1}$ *by 1.2.2).*

d) $\mathrm{I}\Sigma_n$ *proves:* $\Delta_1(LL_n) = LL_n$ *and* $\Delta_n \subseteq LL_n \subseteq \Delta_{n+1}$ *(cf.* [HP93, I.2.79 and I.2.72]*).*

e) $\mathrm{I}\Sigma_n$ *proves induction and collection for* LL_n *formulae.* □

1.7.3 Definition. A *tree* is a set T of sequences that is closed under initial subsequences, formally:

$$\mathrm{Tree}(T) \overset{\mathrm{df}}{\iff} (\forall s \in T)\mathrm{Seq}(s) \wedge (\forall s,t)(\mathrm{Seq}(s) \wedge s \subseteq t \wedge t \in T \to s \in T).$$

T is *finitely branching* if for each $s \in T$ the set of all sequences $t \in T$ such that $s \subseteq t$ and $\ell(t) = \ell(s) + 1$ is bounded. T is Γ-*estimated* if there is a Γ-definable function F such that $(\forall x)(\forall s \in T)(\ell(s) = x \to s \leq F(x))$. A subtree $B \subseteq T$ is a *branch* in T if it is linearly ordered by inclusion of sequences.

1.7.4 Theorem (*Low basis theorem*). $\mathrm{I}\Sigma_1 \vdash$ *'every unbounded* Δ_1 *tree that is* Δ_1*-estimated has a* $low\Sigma^*(\Sigma_1)$ *unbounded branch'.*

Proof. See [HP93, I.3.8] or [HK89]. □

1.7.5 Corollary. *Let* $k \geq 1$*.*

a) $\mathrm{I}\Sigma_{k+1} \vdash$ *'every* LL_k *unbounded finitely branching tree has an* LL_{k+1} *unbounded branch'.*

b) $\mathrm{B}\Sigma_{k+2} \vdash$ *'every* $low\Delta_{k+1}$ *unbounded finitely branching tree has a* $low\Delta_{k+2}$ *unbounded branch'.*

Proof. Follows from 1.7.4 by relativization and 1.7.2. See [HP93, I.3.10]. □

1.8 Useful facts

1.8.1 Fact (*Pigeon-hole principle for finite sets*).
$\mathrm{I}\Sigma_1$ *proves that if* $|x| < |y|$*, then there is no injection of* x *to* y*.* □

The following two facts have elementary proofs.

1.8.2 Fact. *Let* Γ *be a class of formulae and let* I *be a proper cut of* $M \models \mathrm{L}\Gamma$*.*

a) *(*Overspill*) If* $M \models \varphi(a)$ *for all* $a \in I$*, then there exists some* $b \in M \setminus I$ *such that* $M \models (\forall x < b)\varphi(x)$*.*

b) *(*Underspill*) If* $M \models \varphi(a)$ *for all* $a \in M \setminus I$*, then there exists some* $b \in I$ *such that* $M \models (\forall x)(x > b \to \varphi(a))$*.* □

1.8.3 Fact. *Let* $I \subseteq^e M \models \mathscr{L}^{Ar}$. *Then* $I \preccurlyeq_0 M$. *If* $M \models \mathrm{I}\Sigma_0$, *then* $I \models \mathrm{I}\Sigma_0$. □

1.8.4 Fact. *Let* $n \geq 0$, $M \models \mathrm{I}\Sigma_n$ *and* $I \prec_n^e M$ *(i.e.* I *is a proper* Σ_n*-elementary initial substructure of* M*). Then* $I \models \mathrm{B}\Sigma_{n+1}$.

Proof. First of all, $I \models \mathrm{I}\Sigma_0$ by Fact 1.8.3. Let $a \in I$ and $I \models (\forall x < a)(\exists y)\varphi(x,y)$ with $\varphi \in \Pi_n$. If $x < a$ and $y \in I$ such that $I \models \varphi(x,y)$, then $x, y \in M$ and $M \models \varphi(x,y)$, by n-elementarity. Therefore $M \models (\forall x < a)(\exists y < b)\varphi(x,y)$ for all $b \in M \setminus I$. By Fact 1.2.2, the last formula is $\Pi_n(\mathrm{Coll}_n)$, so by underspill, it holds also for some $b \in I$. If now $x < a$, $y < b$, and $M \models \varphi(x,y)$, then $x, y \in I$ and $I \models \varphi(x,y)$, by $I \preccurlyeq_n M$. Thus $I \models (\forall x < a)(\exists y < b)\varphi(x,y)$, as required. □

1.8.5 Fact (*MRDP Theorem*). *Every recursively enumerable set is Diophantine, i.e. of the form* $\{\bar{x} \mid (\exists \bar{y})p(\bar{x},\bar{y}) = q(\bar{x},\bar{y})\}$ *where* p, q *are multivariate polynomials with natural number coefficients.* □

The theorem is a result of combined efforts of Robinson, Davis, Putnam, and Matiyasevič; see e.g. [Dav73]. The proof can be carried out in $\mathrm{I}\Sigma_0 + \mathrm{Exp}$ (c.f. [GD82]), which yields the following important corollary: every Σ_1 formula is equivalent to an $\exists_1$ formula in $\mathrm{I}\Sigma_0 + \mathrm{Exp}$ (where $\exists_1$ formulae are formulae of the form $(\exists \bar{x})\varphi(\bar{x})$ with φ quantifier-free). In consequence, we obtain:

1.8.6 Corollary. *If* N, M *are models of* $\mathrm{I}\Sigma_0 + \mathrm{Exp}$ *and* $M \subseteq N$, *then* $M \preccurlyeq_0 N$.

Proof. [Kay91b, pp. 88–89] □

1.8.7 Fact (*Gaifman*).

a) *Let* $M \models \mathrm{PA}$, $M \subseteq^{cf} N \models \mathrm{PA}^-$ *and* $M \preccurlyeq_0 N$. *Then* $M \preccurlyeq N \models \mathrm{PA}$.

b) (Splitting theorem:) *If* $M \subseteq N$ *are both models of* PA, *then for* $I \stackrel{\mathrm{df}}{=} \sup_{\leq^N}(M)$, $M \subseteq^{cf} I \subseteq^e N$ *and* $M \preccurlyeq I \models \mathrm{PA}$.

Proof. See e.g. [Kay91b, pp. 87–89] for proof; in Example 2.3.8 on page 35 we give an alternative proof that illustrates our results of Section 2.3. □

The MRDP and Gaifman's theorems tell us that as long as we are interested in submodels of a given $M \models \mathrm{PA}$ that satisfy PA, we may narrow our concern to initial substructures. This does not mean, though, that we may also narrow to one particular model M. Firstly, one model of PA does not cover all non-isomorphic types: PA does not have the 0-joint embedding property, i.e. there a two (countable) models of PA that have no common extension satisfying PA [MS75]; similarly for fragments: [Ote92], [Ote93]. Secondly, no model of PA has submodels for all possible complete extensions of PA:

1.8.8 Fact. *For every model* $M \models \mathrm{PA}$, *there is a model* $N \models \mathrm{PA}$ *such that no submodel of* M *is elementary equivalent to* N.

Proof. Gaifman's theorem allows us to restrict ourselves to initial substructure, since if $N \equiv K \subseteq^{cf} I \subseteq^{e} M$ and $N \models \mathrm{PA}$, then $N \equiv I$.

If T is a consistent theory, we say that formulae φ,ψ are *mutually independent* over T iff each of $T \cup \{\varphi,\psi\}$, $T \cup \{\neg\varphi,\psi\}$, $T \cup \{\varphi,\neg\psi\}$, $T \cup \{\neg\varphi,\neg\psi\}$ is a consistent theory. The presented proof is based on the fact, shown below, that there is a pair of mutually independent Π_1 formulae over PA. The claim then follows in this way: assume φ,ψ is such a pair. If $M \models \varphi$, then φ (being Π_1), is satisfied in all initial substructures of M. Thus any $N \models \mathrm{PA} \cup \{\neg\varphi\}$ gives the claim. The case of $M \models \psi$ is analogous. Let, on the other hand, $M \models \neg\varphi \wedge \neg\psi$. Since $\neg\varphi$ and $\neg\psi$ are equivalent to some Σ_1 formulae, we can find the least $w_{\neg\varphi}$ and $w_{\neg\psi} \in M$ witnessing[†] $\neg\varphi$ and $\neg\psi$ in M, respectively. Assume for example $M \models w_{\neg\varphi} \leq w_{\neg\psi}$ and let $N \models \mathrm{PA} \cup \{\varphi,\neg\psi\}$. If $N \equiv I \subseteq^{e} M$, then $I \models \neg\psi$ and thus $w_{\neg\psi} \in I$, by $I \preccurlyeq_0 M$. Hence $w_{\neg\varphi} \in I$ and $I \models \neg\varphi$, by $I \preccurlyeq_0 M$; this contradicts $I \equiv N$.

It remains to find the mutually independent pair. This is analogous to the proof of the independence of Rosser's formula. The presented elementary form of the proof was suggested by E. Jeřábek.

For a pair of Σ_1 formulae φ,ψ of the form $(\exists x)\varphi'(x)$, $(\exists x)\psi'(x)$ with φ',ψ' bounded, we define the following Σ_1 *witness comparison sentence*:

$$\varphi \ll \psi \overset{\mathrm{df}}{\iff} (\exists x)(\varphi'(x) \wedge (\forall y < x)\neg\psi'(y)).$$

Let $\ulcorner\varphi\urcorner$ denote the numeral of the code of a formula φ in the Gödel numbering and let $\mathrm{Pr}(x)$ be a Σ_1 formula such that $\mathrm{PA} \vdash \varphi$ iff $\mathbb{N} \models \mathrm{Pr}(\ulcorner\varphi\urcorner)$. By a variant of Gödel's Diagonal lemma generalized for self-referential tuples of formulae (see e.g. [Hec]), there are Σ_1 sentences θ_i for $i \in \{1,\dots,4\}$ such that

$$\mathrm{PA} \vdash \quad \theta_i \leftrightarrow (\mathrm{Pr}(\ulcorner\neg\theta_i\urcorner) \ll \bigvee_{\substack{j\in\{1,\dots,4\}\\ i\neq j}} \mathrm{Pr}(\ulcorner\neg\theta_j\urcorner)) \tag{1.3}$$

Clearly, $\mathrm{PA} \vdash \theta_i \to \neg\theta_j$ for all $i \neq j$, $i,j \in \{1,\dots,4\}$. Let φ be the formula $\theta_1 \vee \theta_2$ and ψ the formula $\theta_1 \vee \theta_3$. If $\mathrm{PA} \cup \{\varphi,\psi\}$ were inconsistent, then $\mathrm{PA} \vdash \neg\theta_1$ (since $\theta_1 \to (\varphi \wedge \psi)$). Similarly, possible inconsistencies of $\mathrm{PA} \cup \{\neg\varphi,\psi\}$, $\mathrm{PA} \cup \{\varphi,\neg\psi\}$, and $\mathrm{PA} \cup \{\neg\varphi,\neg\psi\}$ imply $\mathrm{PA} \vdash \neg\theta_3$, $\mathrm{PA} \vdash \neg\theta_2$, and $\mathrm{PA} \vdash \neg\theta_4$, in order. Thus, it suffices to prove $\mathrm{PA} \nvdash \neg\theta_i$ for $i \in \{1,\dots,4\}$.

Suppose $\mathrm{PA} \vdash \neg\theta_i$. In particular, $\mathbb{N} \models \neg\theta_i$ and $\mathbb{N} \models \mathrm{Pr}(\ulcorner\neg\theta_i\urcorner)$. If there are more such i's, we take the one for which the witness of $\mathrm{Pr}(\ulcorner\neg\theta_i\urcorner)$ in $\mathbb{N}$ is minimal, i.e. such that $\mathbb{N} \models \mathrm{Pr}(\ulcorner\neg\theta_i\urcorner) \ll \mathrm{Pr}(\ulcorner\neg\theta_j\urcorner)$ whenever $i \neq j \in \{1,\dots,4\}$. Then $\mathbb{N} \models \theta_i$, by (1.3)—a contradiction. □

1.8.9 Fact (*Prime models*). *Let* $M \models \mathrm{PA}$*,* $X \subseteq M$*. Then* $X \subseteq \mathrm{Dfe}(M,X) \preccurlyeq M$*. In particular,* $\mathrm{Dfe}(M,X) \models \mathrm{PA}$*. Moreover, every complete type* $p(\bar{x})$ *over* X *in* M *that is realized in* $\mathrm{Dfe}(M,X)$*, is principal.*

[†] *$w \in M$ witnesses φ in M* iff φ is a sentence of the form $(\exists x)\psi(x)$ and $M \models \psi(a)$.

Consequently, if T is a complete extension of PA *and $M \models T$, then $K_T \stackrel{\text{df}}{=} \mathrm{Dfe}(M,\emptyset)$ is a minimal model of T (has no proper elementary substructures) and it is a prime model for T, i.e. for every $N \models T$, there is a unique elementary embedding of K_T into N (which sends K_T onto $\mathrm{Dfe}(N,\emptyset)$).* □

See e.g. [Kay91b, pp. 91-95] for proofs.

1.8.10 Definition. For a subset $X \subseteq M$ and a set of $\mathcal{L}^{Ar}$-formulae Γ, we let

$$I_\Gamma(M;X) \stackrel{\text{df}}{=} \sup_{\leq^M}(\mathrm{Dfe}_\Gamma(M;X))$$

be the cut determined by the set of all Γ-definable elements of M over X.

We say that an element $a \in M$ is Γ-*minimal* over X, if $a = \mu x : \varphi(x,\bar{p})$ for some $\varphi \in \Gamma$, $\bar{p} \in X$.

For any $n \geq 0$ and $X \subseteq M \models \mathrm{PA}$, $I_{\Sigma_{n+1}}(M;X) = I_{\Delta_{n+1}}(M;X)$. The right-to-left inclusion is trivial; conversely, if $a \in I_{\Sigma_{n+1}}(M;X)$ is defined by a formula $(\exists y)\varphi(x,y)$, where φ is Π_n over X, then $a \leq b = \mu x : \varphi((x)_0,(x)_1)$, where b is a Π_n-minimal element of M over X, hence a Δ_{n+1}-definable one. It also follows that Π_n-minimal elements of M over X are cofinal in $I_{\Sigma_{n+1}}(M;X)$. It can be shown that every Π_n-minimal element of M over X is less than some element from $\mathrm{Dfe}_{\Pi_n}(M;X)$, so we also have $I_{\Sigma_{n+1}}(M;X) = I_{\Pi_n}(M;X)$. Moreover, $\mathrm{Dfe}_{\Sigma_n}(M;X)$ is an initial substructure of $\mathrm{Dfe}_{\Pi_n}(M;X)$. Cf. [CFFMLM05].

The following well-known facts show among other things that the vertical arrows in 1.2.3 cannot be reverted (see e.g. [Kay91b] for proofs):

1.8.11 Fact. *Let $M \models \mathcal{L}^{Ar}$ and let $I \stackrel{\text{df}}{=} I_{\Sigma_{n+1}}(M;X)$ and $K \stackrel{\text{df}}{=} \mathrm{Dfe}_{\Sigma_{n+1}}(M;X)$ for a subset $X \subseteq M$ and $n \geq 0$.*

a) *If $M \models \mathrm{I}\Sigma_n$, then $K \preccurlyeq_{n+1} M$ and $I \preccurlyeq_n M$.*

b) *If $M \models \mathrm{PA}$, X is finite, and $K \neq \mathbb{N}$, then $I \models \mathrm{B}\Sigma_{n+1}$, but $I \not\models \mathrm{I}\Sigma_{n+1}$ and $K \not\models \mathrm{B}\Sigma_{n+1}$. In particular, $I \not\preccurlyeq_{n+1} M$.*

c) *If $M \models \mathrm{B}\Sigma_{n+1}$, and $\varphi(\bar{x})$ is Π_{n+2} and $\bar{a} \in X$, then $M \models \varphi(\bar{a})$ iff $I \models \varphi(\bar{a})$.*

d) *$\mathrm{B}\Sigma_{n+1}$ is Π_{n+2}-conservative over $\mathrm{I}\Sigma_n$ 'that is, $\mathrm{B}\Sigma_{n+1}$ and $\mathrm{I}\Sigma_n$ prove the same Π_{n+2} sentences'.* □

1.8.12 Corollary (*Ryll-Nardzewski* [RN52], *Rabin* [Rab61]).

a) *P has no finite axiomatization.*

b) *P is not implied by any consistent set of Σ_n sentences for any $n \in \mathbb{N}$.*

Proof. a) follows from b) since every finite set of $\mathcal{L}^{Ar}$-sentences in the prenex form is a set of Σ_n sentences for some n. Now, let Γ be a consistent set of

Σ_n-sentences, $n \geq 1$, such that Γ implies PA; let $M \models \Gamma$ be non-standard (Γ has non-standard models by compactness). Let $a \in M \setminus \mathbb{N}$ and $I \stackrel{\text{df}}{=} I_{\Sigma_{n+1}}(M,a)$. Then $I \preccurlyeq_n M$, so $I \models \Gamma$ and hence $I \models \text{PA}$. By 1.8.11, however, $I \not\models \text{I}\Sigma_{n+1}$. □

1.8.13 Definition. Let $M \subseteq N$ be models of $\mathfrak{L}^{Ar}$. We say that N is a *conservative extension* of M if for any $X \in \mathfrak{D}(N,N)$, $X \cap M \in \mathfrak{D}(M,M)$.

Clearly, if N is a conservative extension of $M \models \text{PA}$, then $M \subseteq^e N$, since if $b \in N \setminus M$, then $Y \stackrel{\text{df}}{=} \{a \in M \mid N \models a < b\} \in \mathfrak{D}(M,M)$ and is unbounded in M, for otherwise $b = \max(Y) + 1 \in M$, contradicting $b \notin M$.

1.8.14 Theorem (*McDowell, Specker, Gaifman*). *Every model of* PA *has a proper conservative elementary extension of the same cardinality. Moreover, if* $M \models \text{PA}$ *is countable, there are* $2^{\aleph_0}$ *pairwise non-isomorphic countable conservative elementary extensions of* M.

Proof. A full proof is given in Appendix A as an application of a diagonal partition theorem discussed in Chapter 2. □

In Chapter 4, when proving basic properties of the Rudin-Keisler ordering, we recall Katětov's the lemma On Three Sets; for PA we prove it below.

1.8.15 Lemma. *Let* $M \models \text{PA}$*,* $Y \subseteq M$ *a subset and* $X, F \in \mathfrak{D}(M,Y)$*, where* $F \subseteq X \times X$ *is a partial function such that* $F(x) \neq x$ *for every* $x \in \text{dom}(F)$*. Then there are* $X_0, X_1, X_2 \in \mathfrak{D}(M,Y)$ *such that* $F[X_i] \cap X_i = \emptyset$ *for* $i \leq 2$*.*

Proof. For X bounded, the well-known proof by induction on $|X|$ formalizes in PA; here is a sketch: For $|X| \leq 2$, the claim is trivial. Let $|X| > 2$ and assume the claim holds for all $X' \subset X$. For some $x \in X$, $|F^{-1}[\{x\}]| \leq 1$ (F is a map). The induction hypothesis applied on $X' \stackrel{\text{df}}{=} X \setminus \{x\}$ and $F' \stackrel{\text{df}}{=} F \cap X' \times X'$ produces some X'_i, $i \leq 2$. Now, the set $Z \stackrel{\text{df}}{=} \{F(x)\} \cup F^{-1}[\{x\}]$ has at most two elements, so for some $j \leq 2$, $Z \cap X'_j = \emptyset$; take the least such j and put $X_j \stackrel{\text{df}}{=} X'_j \cup \{x\}$ and $X_i \stackrel{\text{df}}{=} X'_i$ for $i \neq j$. Clearly $F[X_i] \cap X_i = \emptyset$ for $i \leq 2$ and $\bigcup_{i \leq 2} X_i = X$.

We derive the unbounded case using McDowell-Specker's Theorem 1.8.14. We may assume $Y = M$ (otherwise we replace M with $M' \stackrel{\text{df}}{=} \text{Dfe}(M,Y)$ and use elementarity to translate between M' and M). Let $M \prec^e N$ be conservative and fix some $c \in N \setminus M$. For a formula $\varphi \in \text{Fm}(\mathfrak{L}^{Ar}_M)$ such that $X = \varphi(M)$, let $X' \stackrel{\text{df}}{=} [0,c] \cap \varphi(N) \in \mathfrak{D}(N,N)$; then $X = X' \cap M$. Now define F' from F similarly, so that $F' \restriction M = F$. But X' is bounded in N, so by the previous part of the proof, there are sets $X'_i \in \mathfrak{D}(N,N)$, $i \leq 2$ such that $\bigcup_{i \leq 2} X'_i = X'$ and $N \models F'[X'_i] \cap X_i = \emptyset$ for $i \leq 2$. Since N is a conservative extension of M, there are $X_i \in \mathfrak{D}(M,M)$, $i \leq 2$, such that $X_i = X'_i \cap M$. They have the required properties. □

CHAPTER 2

DIAGONALLY INDISCERNIBLE ELEMENTS

This chapter consists of three sections. In Section 2.1, we introduce and prove an infinite combinatorial principle in arithmetic that is a diagonal version of the Infinite Ramsey Theorem; the principle is tied closely to the notion of diagonally indiscernible elements. The results provide a detailed level-by-level analysis of the principle in terms of the arithmetical hierarchy and schemes of induction and collection. The proofs are based on the Low Basis Theorem and follow closely the pattern of a similar analysis for the usual Infinite Ramsey Theorem given in [HP93, II.1]. The principle is first formulated for PA in Theorem 2.1.2, which is in fact a corollary of the main results in Theorems 2.1.5 and 2.1.6, where restricted versions of the principle are proved in the theories $\mathrm{I}\Sigma_n$ and $\mathrm{B}\Sigma_{n+1}$. By finely balancing restrictions of the principle and the strengths of theories, we are able to derive further results, presented in Section 2.3. We conclude the first section by reformulating the combinatorial principle in terms of functions; in this light, the principle can be presented as an infinite version of the Kanamori-McAloon principle generalized to h-regressive functions.

Section 2.2 is a brief survey of finite partition principles in arithmetic. An iterated version of the Paris-Harrington principle is proved here using Theorem 2.1.3 in preparation for Chapter 4, where it is used to prove the existence of so-called Ramsey monads.

Section 2.3 provides a new theorem characterizing k-elementary cuts satisfying (fragments of) PA as cuts on certain systems of diagonal indiscernibles.

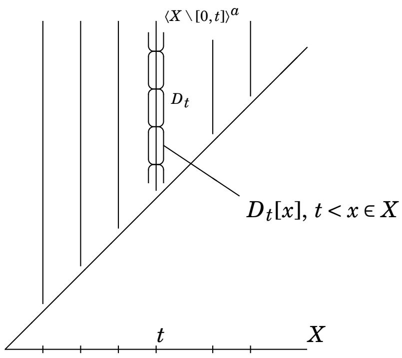
Remark. All following results are formulated for (fragments of) PA in the language $\mathcal{L}^{Ar}$. In spite of that, the same results apply without modification to any theory $T \supseteq \mathrm{PA}^-$ in any language $\mathcal{L}$ that extends $\mathcal{L}^{Ar}$ by adding some recursive set of new predicate symbols, provided that T has the amount of induction and collection for $\mathcal{L}$-formulae as required by the assumptions of the propositions.

2.1 Diagonal partition theorem in arithmetic

Ramsey-type theorems have been soon discovered to play an important role in the model theory of Peano arithmetic because they relate to the strength of the induction scheme. Many of the results in this area rely (often implicitly) on the notion of diagonal indiscernibles (in some variants also called *strong indiscernibles*). In this section, we present a form of the Infinite Ramsey theorem provable in PA (and partially in fragments) that explicates this notion. We also demonstrate its relationship to the induction and collection schemes.

2.1.1 Definition.

a) For $a \geq 1$, a *diagonal partition of* $\langle X\rangle^a$ is a system $D = \{D_t\}_{t\in X}$ such that each D_t is an equivalence on $\langle X \setminus [0,t]\rangle^a$ with a bounded number of equivalence classes ($\|D_t\| < \infty$). Formally, we represent the system D by a relation $D \subseteq X \times (\langle X\rangle^a \times \langle X\rangle^a)$ and define for $t \in X$, $D_t \stackrel{\text{df}}{=} D[t] = \{\langle x,y\rangle \mid \langle t,\langle x,y\rangle\rangle \in D\}$.



b) We say that $H \subseteq X$ is *diagonally homogeneous* for D (or simply D-*homogeneous*) if for every $t \in H$, the set $H \setminus [0,t]$ is homogeneous for the equivalence D_t.

With this, we may formulate our basic theorem, which will be proved and substantially refined in the course of this section.

2.1.2 Theorem (*Infinite Diagonal Partition Theorem*). *For every $n \geq 1$,* PA *proves: Let D be a diagonal partition of $\langle X\rangle^n$ with X unbounded. Then there exists an unbounded class $H \subseteq X$ diagonally homogeneous for D. (If the classes D,X are defined from parameters, then H is defined from the same parameters.)*

Of course, if we let $D_t \stackrel{\text{df}}{=} P$ for some partition P of X with $\|P\| < \infty$, we get just the Infinite Ramsey Theorem:

2.1.3 Corollary (*Infinite Ramsey Theorem*). *For each* $n \geq 1$, PA *proves: if* P *partitions an unbounded class* X *into boundedly many classes, there is an unbounded class* $H \subseteq X$ *homogeneous for* P.

In this formulation of the theorems, the existential quantification over H in the above propositions is meta-theoretic rather than 'inside PA'. We shall actually prove a more elaborate version of Theorem 2.1.2 that bounds the complexity of the homogeneous set; this will allow us to quantify inside the underlying theory. We first introduce some notation.

2.1.4 Definition. Let Γ_1 and Γ_2 be classes of sets. The following definitions are meaningful in every $T \supseteq \mathrm{I}\Sigma_1$ that allows for quantifying over Γ_1 and Γ_2 sets.

a) $\Gamma_1 \rightrightarrows (\Gamma_2)^a$ denotes the following assertion:

> For every unbounded Γ_1 set X and every Γ_1 diagonal partition D of $\langle X \rangle^a$, there is an unbounded Γ_2 set $Y \subseteq X$ diagonally homogeneous for D.

b) $\Gamma_1 \rightrightarrows (\Gamma_2)^a_{\Gamma_1}$ denotes the same assertion with the additional requirement that D is Γ_1-*estimated*, which means that for some Γ_1 function G with $\mathrm{dom}(G) \supseteq X$, $\|D_t\| \leq G(t)$ for all $t \in X$.

c) $\Gamma_1 \rightrightarrows (c)^a$ denotes the assertion:

> For every unbounded Γ_1 set X and every Γ_1 diagonal partition D of $\langle X \rangle^a$, there is a coded set $u \subseteq X$ diagonally homogeneous for D with $|u| = c$

The main theorems of this section are the following:

2.1.5 Theorem. *For every* $m, n \geq 1$,

a) $\mathrm{B}\Sigma_{n+m+1} \vdash \mathit{low}\,\Delta_{m+1} \rightrightarrows (\mathit{low}\,\Delta_{m+n+1})^{\underline{n}}$,

b) $\mathrm{I}\Sigma_{n+m} \vdash LL_m \rightrightarrows (LL_{m+n})^{\underline{n}}_{LL_m}$, *so in particular* $\mathrm{I}\Sigma_{n+m} \vdash \Delta_m \rightrightarrows (\Delta_{m+n+1})^{\underline{n}}_{\Delta_m}$.

Recall that $LL_k \stackrel{\text{df}}{=} \mathit{low}\,\Sigma^*_0(\Sigma_k)$ *(Definition 1.7.1 on page 16).*

2.1.6 Theorem. *For every* $m, n, k \geq 1$*:*

a) $\mathrm{B}\Sigma_{m+n} \vdash \mathit{low}\,\Delta_{m+1} \rightrightarrows (\underline{k})^{\underline{n}}$,

b) $\mathrm{I}\Sigma_{m+n-1} \vdash \Delta_m \rightrightarrows (\underline{k})^{\underline{n}}_{\Delta_m}$.

The Infinite Diagonal Partition Theorem 2.1.2 is an obvious consequence of 2.1.5 a) since every class in PA is Δ_m and hence *low* Δ_{m+1} for some m.

We now prove the main theorems 2.1.5 and 2.1.6. In the end of the section, we reformulate 2.1.5 b) and 2.1.6 b) in terms of functions.

The proofs are similar to [HP93, II.1]. We start with the following definition and lemma.

2.1.7 Definition. Let D be a diagonal partition of $\langle X\rangle^{\underline{n}}$, s an increasing sequence of elements from X (that is OSeq(s), $s \subseteq X$) and $x \in X$. We say that

- s is *D-pre-homogeneous* if either $\ell(s) \leq \underline{n}+1$ and s is an initial segment of the set X, or $\ell(s) > \underline{n}+1$ and $u\smile\langle x\rangle\, D_t\, u\smile\langle y\rangle$ whenever $u \in \langle s\rangle^{\underline{n}-1}$ and $t,x,y\,\varepsilon\, s$, $t < \mathrm{Min}(u)$, $\mathrm{Max}(u) < x,y$.

- $s\smile\langle x\rangle$ is a *minimal D-pre-homogeneous* extension of s if it is D-pre-homogeneous and for every $y \in (\mathrm{Max}(s),x)$ such that $s\smile\langle y\rangle$ is D-pre-homogeneous, there are $u \in \langle s\rangle^{\underline{n}}$ and $t\,\varepsilon\, s$ with $t < \mathrm{Min}(u)$, such that $\neg(u\smile\langle x\rangle\, D_t\, u\smile\langle y\rangle)$.

- s is *hereditarily minimal D-pre-homogeneous* (*D-h.m.p.h.*) if for all $i < \ell(s)$, $s\restriction i$ is D-minimal D-pre-homogeneous, where $s\restriction i$ denotes the unique initial subsequence s' of s with $\ell(s') = i$.

2.1.8 Lemma. *For every $m,n \geq 1$, $\mathrm{B}\Sigma_{m+1}$ proves: Let D be a diagonal partition of $\langle X\rangle^{\underline{n}}$, where D,X are low Δ_{m+1}, $n \geq 1$. Then $T_D \overset{\text{df}}{=} \{s \mid s \text{ is } D\text{-h.m.p.h.}\}$ ordered by inclusion (which on T_D coincides with the relation 'is an initial subsequence') is an unbounded finitely branching low Δ_{m+1} tree (see Definition 1.7.3 on page 17).*

Proof. First observe that the definition of a D-h.m.p.h. sequence can be expressed as a Δ_1-formula in D,X, so by 1.7.2 b), T_D is *low* Δ_{m+1}.

Clearly T_D is a tree. To show that T_D is finitely branching, let $s \in T_D$ and $Y \overset{\text{df}}{=} \{x \mid s\smile\langle x\rangle \in T_D\}$, and suppose Y is unbounded. We aim for a contradiction with the finite pigeon-hole principle in $\mathrm{I}\Sigma_1$ (1.8.1). Surely, Y is *low* Δ_{m+1}. Let F be a function on $\langle X\rangle^{1+\underline{n}}$ such that for $\langle t\rangle\smile u \in \langle X\rangle^{1+\underline{n}}$, $F(\langle t\rangle\smile u) = i$ iff i is the maximal cardinality of a choice set for D_t with elements less than $u_0 = \mu v \leq u : v\, D_t\, u$. This definition formalizes easily in $\mathrm{B}\Sigma_{m+1}$ as a Δ_1 formula in D,X, hence F is *low* Δ_{m+1}. It is evident that for every $\langle t\rangle\smile u, \langle t\rangle\smile v \in \langle X\rangle^{1+\underline{n}}$, $F(\langle t\rangle\smile u) = F(\langle t\rangle\smile v) \leftrightarrow u\, D_t\, v$. Since T_D is bounded,

$$(\forall t\,\varepsilon\, s)(\exists z)[(\forall u)(\langle t\rangle\smile u \in T \to F(\langle t\rangle\smile u) \leq z)]. \tag{2.1}$$

The subformula in square brackets is $\Pi_1(\textit{low}\,\Delta_{m+1})$, hence Δ_{m+1}. (Indeed, if $U \in \textit{low}\,\Delta_{m+1}$ and $W \in \Pi_1(U)$, then $-W \in \Sigma_1(U) \subseteq \Delta_{m+1}$, so $W \in \Delta_{m+1}$).

Thus, by Σ_{m+1}-collection, there exists z_0 such that

$$(\forall t\,\varepsilon\, s)(\forall u)(\langle t\rangle\smile u \in T \to F(\langle t\rangle\smile u) \leq z_0). \tag{2.2}$$

It follows that we may assign to every $x \in Y$ a finite function $f_x \subseteq \langle s\rangle^{\underline{n}} \times z_0$ such that $f_x(\langle t\rangle\smile u) = F(\langle t\rangle\smile t\smile\langle x\rangle)$. If $x < y$ are from Y, then by minimality there is some $\langle t\rangle\smile u \in \langle s\rangle^{\underline{n}}$ such that $u\smile\langle x\rangle$ and $u\smile\langle y\rangle$ fall into different equivalence classes of D_t, so $f_x(\langle t\rangle\smile u) \neq f_y(\langle t\rangle\smile u)$, hence $f_x \neq f_y$. Now $\langle s\rangle^{\underline{n}} \times z_0$ is clearly bounded, hence all $f \subseteq \langle s\rangle^{\underline{n}} \times z_0$ are less than some a. Then for every b, there is a finite subset s' of Y with $|s'| = b$ and a finite set f that is a 1-1 map of s' on a subset of $[0, a)$. (This is proved by induction on b where the induction formula is $\Sigma_1(Y)$). For $b = a+1$ this gives the desired contradiction.

In order to prove that T_D is unbounded, we show that for every $x \in X$, there is some s such that $s\smile\langle x\rangle \in T_D$. Assume it is not the case. Let s_0 consist of the first n elements of X. Then $s_0 \in T_D$, $x > \max(s_0)$ and $s_0\smile\langle x\rangle$ is clearly D-pre-homogeneous. Let $s \in T_D$ be such that $s\smile\langle x\rangle$ is D-pre-homogeneous and $\ell(s) \le x$ is maximal possible. By the assumption, $s\smile\langle x\rangle$ is not minimal (otherwise it would be D-h.m.p.h.), hence there is some $y < x$ such that $s_0\smile\langle y, x\rangle$ is pre-homogeneous. For the least such y, we have $s\smile\langle y\rangle \in T_D$. (Note that the argument only involves the Least number principle for Δ_{m+1} formulae, which is provable in $\mathrm{B}\Sigma_{m+1}$). □

2.1.9 Remark. By the same argument, $\mathrm{I}\Sigma_m$ proves that if D, X as above but LL_m and D is LL_m-estimated, then T_D is an unbounded finitely branching LL_m tree.

For the most part, it suffices to proceed exactly as above, only replacing everywhere $\mathrm{B}\Sigma_{m+1}$ with $\mathrm{I}\Sigma_m$ and *low* Δ_{m+1} with LL_m (since $\mathrm{I}\Sigma_m$ proves induction and collection for $\Sigma_1(LL_m)$ sets and $\Delta_1(LL_m) = LL_m$ under $\mathrm{I}\Sigma_m$). The only problematic spot is in Lemma 2.1.8, where $\Pi_1(LL_m)$-collection (which $\mathrm{I}\Sigma_m$ does *not* prove) is needed to derive (2.2) from (2.1). But if D is estimated by some LL_m function G, we may replace (2.2) with $(\forall t\ \varepsilon\ s)G(t) \le z_0$, where z_0 is obtained trivially from $(\forall t\ \varepsilon\ s)(\exists z)G(t) \le z$ by Σ_m-collection. The rest of the proof then goes as before.

Proof of 2.1.5 a). We prove for every $m, n \ge 1$ that

$$\mathrm{B}\Sigma_{n+m+1} \vdash low\,\Delta_{m+1} \rightrightarrows (low\,\Delta_{m+n+1})^{\underline{n}}.$$

Let D an diagonal partition of $\langle X\rangle^{\underline{n}}$, both D and X *low* Δ_{m+1}. By 2.1.8, T_D is an unbounded finitely branching *low* Δ_{m+1} tree, so by the corollary 1.7.5 to the Low Basis Theorem, $\mathrm{B}\Sigma_{m+2}$ proves that T_D has a *low* Δ_{m+2} unbounded branch Y.

We now proceed by induction on $n \ge 1$. If $n = 1$, Y is diagonally homogeneous for D and we are done. So, let $n \ge 2$ and suppose the theorem holds for $n-1$ and all m. Let T_D and Y be as above. We define a diagonal partition $\tilde{D}$ on $\langle Y\rangle^{n-1}$ by

$$u\, D_t\, v \overset{\mathrm{df}}{\iff} (\exists x > (u)_{n-2})(\exists y > (v)_{n-2}) u\smile\langle x\rangle\, D_t\, v\smile\langle y\rangle \tag{2.3}$$

for every $u,v \in \langle Y \setminus [0,t]\rangle^{n-1}$. The definition of $\tilde{D}$ is clearly Σ_1 in D,Y. Since Y is D-pre-homogeneous and unbounded, the $\exists$ quantifiers in (2.3) can be equivalently replaced with $\forall$, so $\tilde{D}$ is also $\Pi_1(D,Y)$; hence it is $\Delta_1(low\,\Delta_{m+2})$, which is $low\,\Delta_{m+2}$. By the induction hypothesis, $\mathrm{B}\Sigma_{(n-1)+(m+2)}(=\mathrm{B}\Sigma_{n+m+1})$ proves that $\tilde{D}$ has an unbounded diagonally homogeneous set $H \subseteq Y$ that is $low\,\Delta_{(n-1)+(m+2)} = low\,\Delta_{n+m+1}$. H is diagonally homogeneous also for D. □

Proof of 2.1.5 b). Repeat the previous proof replacing everywhere $\mathrm{B}\Sigma_{k+1}$ with $\mathrm{I}\Sigma_k$, $low\,\Delta_{k+1}$ with LL_k, and Lemma 2.1.8 by Remark 2.1.9. The sequel for Δ_m follows from the inclusions in 1.7.2 d). □

Proof of 2.1.6. a). For every $m,n,k \geq 1$, we prove

$$\mathrm{B}\Sigma_{m+n} \vdash low\,\Delta_{m+1} \rightrightarrows (\underline{k})^{\underline{n}}$$

by induction on $\underline{n}$. The induction step is as in the proof of Theorem 2.1.5, we only need to verify the case for $n=1$, i.e. $\mathrm{B}\Sigma_{m+1} \vdash low\,\Delta_{m+1} \rightrightarrows (\underline{k})^1$.

First observe that if T is a $low\,\Delta_{m+1}$ finitely branching unbounded tree, then for some $s \in T$, the subtree $T_s \stackrel{\mathrm{df}}{=} \{t \mid s \smile t \in T\}$ is a $low\,\Delta_{m+1}$ unbounded finitely branching tree. Indeed, since T is finitely branching, the set of $s \in T$ such that $\ell(s) = 1$ is bounded by some d. Suppose T_s is bounded for every $s \leq d$, i.e.

$$(\forall s \leq d)(\exists b)[(\forall t)(s \subseteq t \in T \to t < b)].$$

The subformula in square brackets is Π_1 in T, hence Δ_{m+1}. By Σ_{m+1} collection, there is some b_0 such that

$$(\forall s \leq d)(\forall t)(s \subseteq t \in T \to t < b_0).$$

Now, every $t \in T$ (except for $\emptyset$) prolongs some s with $\ell(s) = 1$, so it follows that $T \subseteq [0, b_0]$, which contradicts T being unbounded.

Now, D defines an unbounded finitely branching $low\,\Delta_{m+1}$ tree T of D-h.m.p.h. sequences. By iterating the observation in the last paragraph at most k-times, we arrive at some $s \in T$ with $\ell(s) \geq k$ (and T_s unbounded). In particular, s is pre-homogeneous for D; yet $n = 1$, so s is D-homogeneous.

b) We prove that for every $m,n,k \geq 1$

$$\mathrm{I}\Sigma_{m+n-1} \vdash \Delta_m \rightrightarrows (\underline{k})^{\underline{n}}_{\Delta_m}.$$

By a), the statement is provable in $\mathrm{B}\Sigma_{m+n}$. Because this theory is Π_{m+n+1}-conservative over $\mathrm{I}\Sigma_{n+m-1}$ by Fact 1.8.11, d), and $n \geq 1$, it suffices to check that the statement is Π_{m+2}. This is a straightforward task using the corresponding definitions and the facts from Section 1.6, namely that the statement 'X is Δ_m' can be expressed as a Π_{m+1} formula and '$x \in X$' as a Δ_m formula. □

The notions of diagonal partition and diagonal homogeneity can be rephrased it in terms of functions. Thus, our Infinite Diagonal Partition Theorem can be viewed as an infinite version of the Kanamori-McAloon principle (see 2.2.1) generalized to G-regressive functions.

2.1.10 Definition. Let F, G be functions, $\mathrm{dom}(F) = \langle X\rangle^a$, $a \geq 1$, $\mathrm{dom}(G) = X$.

a) F is said to be *G-regressive* on $\langle X\rangle^a$, if for all $u \in \langle X\rangle^a$, $F(u) = 0$ or $F(u) < G((u)_0)$. F is *regressive* if it is id-regressive.

b) A set $H \subseteq X$ is said to be *min-homogeneous* for F if for all $u, v \in \langle H\rangle^a$, $(u)_0 = (v)_0$ implies $F(u) = F(v)$.

A G-regressive function F with $\mathrm{dom}(F) = \langle X\rangle^{a+1}$ determines a diagonal partition D of $\langle X\rangle^a$ such that

$$u\, D_t\, v \overset{\mathrm{df}}{\iff} F(\langle t\rangle \smile u) = F(\langle t\rangle \smile v)$$

for $\langle t\rangle \smile u, \langle t\rangle \smile v \in \langle X\rangle^{a+1}$ and $\|D_t\| \leq G(t)$ for all $t \in X$. Clearly, if F is Δ_n, then D is Δ_n, too; similarly for *low* Δ_n.

Conversely, as in the proof of 2.1.8, every diagonal partition D of $\langle X\rangle^a$ determines a $\|D_t\|$-regressive function F with $\mathrm{dom}(F) = \langle X\rangle^{a+1}$.

The preceding results thus give the following simple corollary:

2.1.11 Corollary. *Let $m, n \geq 1$.*

a) $\mathrm{I}\Sigma_{n+m}$ *proves: For each $F, G, X \in \Delta_m$ where X is unbounded and F is a G-regressive function on $\langle X\rangle^{\underline{n}+1}$, there is an unbounded Δ_{m+n} subset $Y \subseteq X$, min-homogeneous for F.*

b) $\mathrm{I}\Sigma_{n+m-1}$ *proves: For every $k \geq 1$ and $F, G, X \in \Delta_m$ where X is unbounded and F is a G-regressive function on $\langle X\rangle^{\underline{n}+1}$, there is a subset $Y \subseteq X$ min-homogeneous for F with $|Y| = k$.* □

2.2 Variants of the Finite Ramsey Theorem

This section is a brief survey of variants of the Finite Ramsey theorem in Peano arithmetic and its fragments. In the end of the section we prove Lemma 2.2.7 which is an iterated version of the Paris-Harrington principle. We shall recall the lemma in Chapter 4.

2.2.1 Definition.

a) For a codable set u and $d, c > 0$, let $\langle u\rangle^d_c$ denote the set of all codable partitions p of $\langle u\rangle^d$ into at most c parts, i.e. such that $\|p\| \leq c$.

b) For a codable set u and $d,c,e>0$, the *Ramsey arrow* $u \to (e)^d_c$ denotes the assertion that for every $p \in \langle u\rangle^d_c$, there is a coded subset $v \subseteq u$ with $|v| \geq e$ such that v is homogeneous for p.

c) The *Paris-Harrington arrow* $u \xrightarrow{*} (e)^d_c$ is defined analogously, except that it further requires the homogeneous subset v to be relatively large, i.e. to satisfy $|v| \geq \min(v)$.

d) The *Kanamori-McAloon arrow* $u \to (e)^d_{reg}$ is the assertion that for every regressive function f on $\langle u\rangle^d$, there is a $v \subseteq u$ such that f is min-homogeneous on $\langle v\rangle^d$ and $|v| \geq e$.

The proofs of the following statements are well-known ([Kay91b]):

2.2.2 Fact. *For all* $1 \leq m \in \mathbb{N}$,

a) Paris-Harrington: $\mathrm{PA} \vdash (\forall a,c,e \geq 1)(\exists b)[a,b] \xrightarrow{*} (\underline{m}+1)^{\underline{m}}_c$

b) Kanamori-McAloon: $\mathrm{PA} \vdash (\forall a,e)(\exists b)[a,b] \to (e)^{\underline{m}}_{reg}$

c) Finite Ramsey Theorem: $\mathrm{I}\Sigma_1 \vdash (\forall d,c,e \geq 1)(\exists b)[0,b] \to (e)^d_c$ □

2.2.3 Remark. In c), $[0,b] \to (e)^d_c$ implies $u \to (e)^d_c$ for any u with $|u| > b$.

2.2.4 Remark. The Paris-Harrington (PH) and Kanamori-McAloon (KM) principles are provable in PA as schemes for m ranging over numerals. The full versions of PH and KM that quantify over all m are true in $\mathbb{N}$ but unprovable in PA ([PH77] and [KM87]).

A detailed discussion of these principles and their various restricted versions can be found in a recent paper [Bov05] by Andrey Bovykin.

The principles have refinements for fragments of Peano arithmetic:

2.2.5 Fact. *Let* $n \geq 1$.

a) $\mathrm{I}\Sigma_n \vdash (\forall a,c,e)(\exists b)[a,b] \xrightarrow{*} (e)^{\underline{n}}_c$

b) $\mathrm{I}\Sigma_n \nvdash (\forall a,c)(\exists b)[a,b] \xrightarrow{*} (\underline{n}+2)^{\underline{n}+1}_c$

c) $(\forall k \in \mathbb{N})\mathrm{I}\Sigma_n \vdash (\forall a,e)(\exists b)[a,b] \xrightarrow{*} (e)^{\underline{n}+1}_{\underline{k}}$

d) $\mathrm{I}\Sigma_1$ *proves: for a set* u, *if* $u \xrightarrow{*} (\underline{n}+2)^{\underline{n}+1}_e$ *and* $\min(u) \geq \max\{n^{2n}, 2e\}$, *then* $u \to (e-\underline{n})^{\underline{n}+1}_{reg}$. □

The first three results are due to Paris [Par80] who gave model-theoretic proofs; alternative proofs can be found e.g. in [HP93, Theorem II.1.9]. The last result is attributed to P. Clote and appears in [KM87]. Its proof relies substantially on earlier results by G. Mills [Mil80].

2.2.6 Definition. *The i-th iteration of the Paris-Harrington arrow* is defined as follows:

a) $u \xrightarrow{*}_0 (e)^d_c$ if u includes a relatively large subset v with $|v| > e$.

b) $u \xrightarrow{*}_{i+1} (e)^d_c$ if for every $p \in \langle u \rangle^d_c$, u includes a codable subset v homogeneous for p and satisfying $v \xrightarrow{*}_i (e)^d_c$.

Clearly, if $u \xrightarrow{*}_i (e)^d_c$ for $i \geq 0$, then u contains at least e elements and is relatively large.

2.2.7 Lemma. *For every unbounded class X and every $1 \leq m, i \in \mathbb{N}$,*

$$\mathrm{PA} \vdash (\forall c, e)(\exists j)(X \cap [0, j] \xrightarrow{*}_i (e)^m_c)$$

Proof. The claim is proved by induction on i. For $i = 0$, it suffices to take j such that $|X \cap [0, j]| > \mathrm{Max}(e, \min X)$, which is possible since X is unbounded.

Suppose the proposition holds for given m, i, and arbitrary unbounded class X. For the sake of obtaining a contradiction, suppose that for a certain X, $(\forall j)\neg(u_j \xrightarrow{*}_{i+1} (e)^m_c)$, where u_j denotes the codable set $X \cap [0, j]$. For convenience, we shall use functions $f : \langle u_j \rangle^m \to [1, c]$ instead of partitions from $\langle u_j \rangle^m_c$. Hence, for every j, there is a codable function $f_j : \langle u_j \rangle^m \to [1, c]$ such that for every $v \subseteq X \cap [0, j]$ satisfying $v \xrightarrow{*}_i (e)^m_c$, f_j is not constant on $\langle v \rangle^m$. For $x_1, \ldots, x_m, y \in X \setminus [0, c]$ such that $x_1 < \ldots < x_m < y$, let $F(‹x_1, \ldots, x_m, y›) \overset{\mathrm{df}}{=} f_y(‹x_1, \ldots, x_m›)$. Then $F : \langle X \rangle^{m+1} \to [1, c]$ is a definable function with bounded range, hence, by the Infinite Ramsey Theorem 2.1.3, there is an unbounded class H of $X \setminus [0, c]$ such that F is constant on $\langle H \rangle^{m+1}$. By induction hypothesis, there is some j_0 such that $H \cap [0, j_0] \xrightarrow{*}_i (e)^m_c$. Let $j_1 = \min(H \setminus [0, j_0])$. We show that for $v \overset{\mathrm{df}}{=} H \cap [0, j_0] \subseteq u_{j_1}$, f_{j_1} is constant on $\langle v \rangle^m$, which together with $v \xrightarrow{*}_i (e)^m_c$ gives the desired contradiction with the choice of the functions f_j. Let $‹\vec{x}›, ‹\vec{y}› \in \langle v \rangle^m$. Then $‹\vec{x}, j_1›, ‹\vec{y}, j_1› \in \langle H \rangle^{m+1}$, so $F(‹\vec{x}, j_1›) = F(‹\vec{y}, j_1›)$, hence $f_{j_1}(‹\vec{x}›) = f_{j_1}(‹\vec{y}›)$. Indeed, f_{j_1} is constant on $\langle v \rangle^m$. □

2.3 Systems of diagonal indiscernibles

In this section, we characterize Σ_n-elementary cuts satisfying $\mathrm{I}\Sigma_{k+n}$ by the existence of certain systems of diagonally indiscernible elements. The main results are formulated after the following definition.

2.3.1 Definition.

a) An *overlay* in a model M is a sequence $\mathcal{O} = \{X_i \mid i \in \omega\}$ of non-empty subsets of M with no last element such that $\sup X_i = \sup X_j$ and $i < j \to X_j \subseteq X_i$ for every $i, j \in \omega$. In particular, if $\emptyset \neq X \subseteq M$ and X has no last element, then $\{X\}$ is an overlay in M. Let I be a cut of M. We write $\mathcal{O}_I \overset{\mathrm{df}}{=} \{X \cap I; X \in \mathcal{O}\}$ and say that $\mathcal{O}$ is *unbounded* in I, if for every $X \in \mathcal{O}$, $X \cap I$ is cofinal in I.

Clearly, if $\mathcal{O}$ is unbounded in I, then $\mathcal{O}_I$ is an overlay in I (but note that not all overlays in I have to be unbounded in it and that an overlay can be unbounded in many different cuts).

b) Let φ be a Σ_k formula ($k \geq 1$) of the form $(\exists x_0)(\forall x_1)\ldots\psi(\bar{y},\bar{x})$ with $\psi \in \Delta_0$. A set $X \subseteq M$ is said to be *φ-bounding in M* if for every $e < e_0 < \cdots < e_{k-1}$ from X,

$$M \models (\forall \bar{y} < e)(\varphi(\bar{y}) \leftrightarrow (\exists x_0 < e_0)(\forall x_1 < e_1)\ldots\psi(\bar{y},\bar{x})). \tag{2.4}$$

An overlay $\mathcal{O}$ is *Σ_k-bounding in M* if for every $\varphi \in \Sigma_k$ (of the considered form), there exists $X \in \mathcal{O}$ such that X is φ-bounding in M. Bounding for Π_k formulae is defined by duality.

c) The symbol $\varphi(\bar{y};\bar{x})$ denotes a formula φ with two *designated lists of variables* $\bar{y}$ and $\bar{x}$ (not necessarily of the same length) such that φ has all free variables among those in $\bar{y},\bar{x}$ and each two variables from these lists are distinct. If Γ is a class of formulae, then $\Gamma(\bar{y};\bar{x})$ denotes a class of all $\varphi(\bar{y};\bar{x})$ where $\varphi \in \Gamma$; $\Gamma(l;k)$ further denotes a class of all $\varphi(\bar{y};\bar{x})$ with arbitrary designated lists $\bar{y},\bar{x}$ of lengths $l,k \geq 0$, in order. We identify Γ with $\bigcup_{l,k\in\omega}\Gamma(l;k)$.

d) We say that $X \subseteq M$ is a set of *$\varphi(\bar{y};\bar{x})$-diagonally indiscernible elements* in M if for each $e \in X$ and every two increasing tuples of elements $\vec{a},\vec{b}$ from $X \setminus [0,e]$ (of the length of $\bar{x}$)

$$M \models (\forall \bar{y} < e)(\varphi(\bar{y},\vec{a}) \leftrightarrow \varphi(\bar{y},\vec{b})).$$

X is a *set of Γ-diagonally indiscernible elements* in M if X is a set of $\varphi(\bar{y};\bar{x})$-diagonally indiscernible elements for every $\varphi(\bar{y};\bar{x})$ from Γ. An overlay $\mathcal{O}$ in M is *Γ-diagonally indiscernible* in M, if for every $\varphi(\bar{y};\bar{x})$ from Γ, $\mathcal{O}$ contains a set of $\varphi(\bar{y};\bar{x})$-diagonally indiscernible elements in M.

The main results of this section are the following theorems (note that for $n = 0$, 2.3.2 b) fully characterizes cuts satisfying PA in terms of overlays):

2.3.2 Theorem. *Let $M \models \mathrm{I}\Sigma_0$, I a cut of M and $n \geq 0, k \geq 1$.*

a) *If $\mathcal{O}$ is a $\Sigma_n(1;k)$-diagonally indiscernible Σ_n-bounding overlay in M, unbounded in I, then $I \preccurlyeq_n M$ and $I \models \mathrm{I}\Sigma_{n+k}$. If, moreover, $J \subset I$ is a cut and $\mathcal{O}$ is unbounded also in J, then $J \preccurlyeq_{n+k} I$ and $J \models \mathrm{B}\Sigma_{n+k+1}$.*

b) *$I \models$ PA and $I \preccurlyeq_n M$ iff there is a Σ_n-bounding Δ_0-diagonally indiscernible overlay $\mathcal{O}$ in M that is unbounded in I. In left-to-right, $\mathcal{O}$ can be formed so as to consist only of $\emptyset$-definable subsets of I.*

c) *If $\mathcal{O}$ is a Δ_0-diagonally indiscernible overlay in M, $J \subset I$ are cuts, and $\mathcal{O}$ is unbounded in both I,J, then $J \preccurlyeq I$.*

We additionally prove that models of $\mathrm{I}\Sigma_{n+2}$ can capture all its $(n+1)$-elementary cuts by a single unbounded overlay:

2.3.3 Theorem. *If $M \models \mathrm{I}\Sigma_{n+2}$ $(n \geq 0)$, then there exists an unbounded $\Sigma_n(1;1)$-diagonally indiscernible overlay $\mathcal{O}$ in M consisting of Δ_{n+3}-definable subsets of M such that for every cut I of M, $I \preccurlyeq_{n+1} M$ iff $\mathcal{O}$ is unbounded in I.*

The rest of this section is devoted to proving these results; we first make some easy observations.

2.3.4 Proposition. *Let $I \preccurlyeq^e_k M$, $M \models \mathcal{L}^{Ar}$ and let $\mathcal{O}$ be an overlay in I. Then $\mathcal{O}$ is $\Sigma_k(\bar{y};\bar{x})$-diagonally indiscernible in I iff it is $\Sigma_k(\bar{y};\bar{x})$-diagonally indiscernible in M.*

Proof. Clearly $\mathcal{O}$ is an overlay in M, too. Let $\varphi(\bar{x};\bar{y})$ be Σ_k and let $\chi(z,\bar{u},\bar{v})$ be the formula $(\forall \bar{y} < z)(\varphi(\bar{y},\bar{u}) \leftrightarrow \varphi(\bar{y},\bar{v}))$. Now, $I \preccurlyeq^e_\varphi M$ implies $I \preccurlyeq_\chi M$, so in particular, $I \models \chi(e,\bar{a},\bar{b})$ iff $M \models \chi(e,\bar{a},\bar{b})$ whenever $e,\bar{a},\bar{b} \in X \in \mathcal{O}$ and the proposition follows. □

Trivially, the property of being φ-bounding is preserved downwards, i.e. if $X \subseteq M$ is φ bounding in M, then so is every $X' \subseteq X$.

2.3.5 Proposition. *Let $M \models \mathrm{I}\Sigma_0$, $k \geq 0$, and let $\mathcal{O}$ be an overlay in M. Then $\mathcal{O}$ is Σ_{k+1}-bounding in M iff for every $\varphi(y,x) \in \Pi_k$ there is $X \in \mathcal{O}$ such that*

(2.5) *for every $a < b$ in X, $M \models (\forall y < a)[(\exists x)\varphi \to (\exists x < b)\varphi]$.*

Proof. Left to right: Let $\varphi(y,x)$ be a Π_k formula of the form $(\forall x_0)\ldots\psi(y,x,\bar{x})$ with $\psi \in \Delta_0$. There is some $X \in \mathcal{O}$ that is both φ- and $(\exists x)\varphi$-bounded in M; we prove that X has the required property. Take $a < b$ from X, some $p < a$, and assume $M \models (\exists x)\varphi(p,x)$. Since X has no last element, we may take some $b_0 < \cdots < b_{k-1}$ from $X \setminus [0,b]$; applying $(\exists y)\varphi$-bounding we get $M \models (\exists x < b)(\forall x_0 < b_0)\ldots\psi(p,x,\bar{x})$. Let $d < b$ be a witness of the first existential quantifier in the last formula; using φ-bounding, we obtain $M \models \varphi(p,d)$; in particular $M \models (\exists x < b)\varphi(p,x)$ as required.

Right to left is proved by induction on $k \geq 0$. First observe that the condition on the right implies a stronger one that allows a tuple $\bar{y}$ in place of y. To see that, first find $X' \in \mathcal{O}$ satisfying (2.5) for the formula $(y+1)^m = x$, $m \geq 1$. This ensures that $a^m < b$ whenever $a < b$ are from X'. Then translate $\varphi(\bar{y},x)$ to some $\varphi'(y,x) \in \Pi_k$ employing a suitable Δ_0-coding of m-tuples from a given interval such that, for a given a, the code of each $\bar{p} < a$ is less than a^m, take $X \subseteq X'$ in $\mathcal{O}$ satisfying (2.5) for φ'. The set X then satisfies (2.5) for φ; With this observation the equivalence for $k = 0$ is trivial.

For the induction step, assume $\mathcal{O}$ is Σ_k-bounding (hence, by duality, also Π_k-bounding) and let φ be a Σ_{k+1} formula of the form $(\exists x)(\forall x_0)\ldots\psi(\bar{y},x,\bar{x})$ with $\psi \in \Delta_0$. We prove that $\mathcal{O}$ contains some φ-bounding set. By induction hypothesis, $\mathcal{O}$ contains a $(\forall x_0)\ldots\psi(\bar{y},x,\bar{x})$-bounding set X'. Using (2.5) (generalized for tuples), we may find some $X \in \mathcal{O}$ such that for every $a < b$ from

X, $M \models (\forall \bar{y} < a)[\varphi(\bar{y}) \leftrightarrow (\exists x < b)(\forall x_0)\ldots\psi(\bar{y},x,\bar{x})]$. We may assume $X \subseteq X'$ (otherwise $X' \subseteq X$ and we take X' instead), and thus by the choice of X', the subformula $(\forall x_0)\ldots\psi$ on the right side of the equivalence can be equivalently replaced by $(\forall x_0 < b_0)\ldots\psi$, where $b_0 < \cdots < b_{k-1}$ are arbitrary elements from X with $b < b_0$; this finishes the proof of the induction step. □

2.3.6 Corollary. *If $M \models \mathrm{I}\Sigma_0$, $k \geq 1$, and $\mathcal{O}$ is an unbounded Σ_k-bounding overlay in M, then $M \models \mathrm{I}\Sigma_k$.*

Proof. For every $\varphi(y,x) \in \Pi_{k-1}$, there is an unbounded $X \in \mathcal{O}$ such that (2.5) holds. Thus, $M \models \mathrm{S}\Pi_{k-1}$, which equivalent to $\mathrm{I}\Sigma_k$ in $\mathrm{I}\Sigma_0$ by Fact 1.2.3. □

2.3.7 Proposition. *Let $M \models \mathrm{I}\Sigma_0$, I a cut of M and $n,l \geq 0$, $k \geq 1$.*

a) *If $\mathcal{O}$ is a Σ_k-bounding overlay in M unbounded in I, then $I \preccurlyeq_k M$.*

b) *If $\mathcal{O}$ is a $\Sigma_n(1;1)$-diagonally indiscernible overlay unbounded in M, then $\mathcal{O}$ is Σ_{n+1}-bounding in M and $M \models \mathrm{I}\Sigma_{n+1}$.*

c) *Let $\mathcal{O}$ be a $\Sigma_n(l;k+1)$-diagonally indiscernible overlay unbounded in M. Then it is $\Sigma_{n+1}(l;k)$-diagonally indiscernible.*

Proof. a) Applying (2.5) on the formulae $y+y=x$ and $y\cdot y=x$ and considering the fact that $\mathcal{O}$ is unbounded in I, observe that I is closed under operations, so $I \subseteq^e M$. The rest follows from (2.5) by Vaught-Tarski's test.

b) It suffices to prove that $\mathcal{O}$ is Σ_{n+1}-bounding in M since then $M \models \mathrm{I}\Sigma_{n+1}$, by 2.3.6. Let $\varphi(y,x)$ be a Π_n. It suffices to verify that X satisfies the condition (2.5) for $\varphi(y,x)$. We proceed by induction on $n \geq 0$, so for a given n, we may assume $M \models \mathrm{I}\Sigma_n$ by induction hypothesis. By $\mathrm{B}\Sigma_n$, $(\exists x < z)\varphi(y,x)$ is equivalent in M to some Π_n formula $\theta(y,z)$; since $\mathcal{O}$ is $\Sigma_n(1;1)$- and hence $\Pi_n(1;1)$-diagonally indiscernible, there exists a set of $\theta(y;z)$-diagonally indiscernible elements $X \in \mathcal{O}$. Let $a < b$ be from X, $p < a$ and $d \in M$ such that $M \models \varphi(p,d)$. Since X is unbounded in M, there is some $c \in X$, $c > \max\{a,d\}$. Then $M \models (\exists x < c)\varphi(p,x)$ and by the choice of X, $M \models (\exists x < b)\varphi(p,x)$. This is exactly what was required for (2.5).

c) This proof follows the same pattern as the one above, only involving extra parameters. Let $\varphi(\bar{y};\bar{x}) \in \Sigma_{n+1}(l;k)$ be of the form $(\exists z)\psi(\bar{y},\bar{x},z)$ with $\psi \in \Pi_n$ and let $X \in \mathcal{O}$ be a set of $((\exists z \leq x)\psi)(\bar{y};\bar{x},x)$-diagonally indiscernible elements. Such X exists because $\mathcal{O}$ is $\Sigma_n(l;k+1)$-diagonally indiscernible and the formula in question is equivalent to a Π_n formula in Coll_n, which is satisfied in M due to b). Take $e \in X$, $\bar{p} < e$, and two increasing k-tuples $\vec{a}$, $\vec{b}$ from $X \setminus [0,e]$ and assume $M \models \varphi(\bar{p},\vec{a})$. We must prove $M \models \varphi(\bar{p},\vec{b})$. Let $d \in M$ be such that $M \models \psi(\bar{p},\vec{a},d)$; since X is unbounded in M, there is some $c \in X$ with $c > \max\{d,\bar{a},\bar{b}\}$. But then $M \models (\exists x < c)\psi(\vec{p},\vec{a},x)$ and also $M \models (\exists x < c)\psi(\bar{p},\vec{b},x)$ due to the choice of X and c, hence $M \models \varphi(\bar{p},\vec{b})$ as required. □

Proof of 2.3.2.

a) Let $\mathcal{O}$ be a $\Sigma_n(1;k)$-diagonally indiscernible Σ_n-bounding overlay in M and let I be a cut of M such that $\mathcal{O}$ is unbounded in I.

If $n \geq 1$, we have $I \preccurlyeq_n M$ by 2.3.7 a). For $n = 0$ it suffices to prove that I is closed under addition and multiplication. Let $X \in \mathcal{O}$ be a set of diagonally indiscernible elements w.r.t. the formulae $y + y \leq x$ and $y \cdot y \leq x$ with designated variables $(y;x)$. Suppose $c \in I$ but $c + c \in M \setminus I$. Then for some $a < b$ from X, $c < a$ but $b < c + c$. Fix these a, b and the least c with this property (using $\mathrm{I}\Sigma_0$ in M). Obviously $c > 0$, so $c = d + 1$ for some $d \in I$. Then $d < a$ and $d + d \leq b$. By the choice of X, however, this last inequality holds for every $b' \in X \setminus [0, b]$. Since X is cofinal in I, we thus have $d + d \in I$ and hence $c + c = d + d + 2 \in I$, since I is a cut. The proof for multiplication is similar.

Now, since $I \preccurlyeq_n^e M$, $\mathcal{O}_I$ is a $\Sigma_n(1;k)$-diagonally indiscernible overlay in I, by 2.3.4; hence by $k - 1$ applications of 2.3.7 c) and by 2.3.7 b), $I \models \mathrm{I}\Sigma_{n+k}$ and $\mathcal{O}_I$ is Σ_{n+k}-bounding in I. If $J \subset I$ is a cut and $\mathcal{O}_I$ is unbounded in J, then by the same argument as above, J is closed under addition and multiplication, so $J \subset^e I$. By 2.3.7 a) (where $J, I, \mathcal{O}_I, k + n$ are to be substituted for $I, M, \mathcal{O}, k$), $J \preccurlyeq_{n+k} I$. Finally, $J \models \mathrm{B}\Sigma_{n+k+1}$, by Fact 1.8.4.

b) The implication from right to left follows from a) taking $n = 0$ and every $k \geq 1$. To prove left to right, it suffices to show that if $I \models \mathrm{PA}$, there is a Δ_0-diagonally indiscernible overlay $\mathcal{O}$ in I unbounded in I, since then $\mathcal{O}$ is a Δ_0-diagonally indiscernible overlay in M by 2.3.4; moreover $\mathcal{O}$ is Σ_n-bounding in I for all n, by 2.3.7 c), b); thus, if $I \preccurlyeq_n M$, $\mathcal{O}$ is Σ_n-bounding in M, using 2.3.5.

We construct $\mathcal{O} = \{X_i \mid i \in \omega\}$ in stages along some fixed enumeration of Δ_0 formulae. We start with $X_0 = I$. At $(i+1)$-th stage, let $\varphi_i(\bar{y};\bar{x})$ be the i-th formula considered and assume X_i is $\emptyset$-definable and unbounded in I. Let m be the length of $\bar{x}$ in $\varphi_i(\bar{y};\bar{x})$. By Infinite Diagonal Partition Theorem (2.1.2) in I, there is an $\emptyset$-definable and unbounded subset $X_{i+1} \subseteq X_i$ that is homogeneous for the diagonal partition D^i of $\langle X_i \rangle^m$, defined by

$$u\, D^i_t\, v \overset{\mathrm{df}}{\iff} (\forall \bar{y} < t)(\varphi_i(\bar{y}, (u)_0, \ldots, (u)_{m-1}) \leftrightarrow \varphi_i(\bar{y}, (v)_0, \ldots, (v)_{m-1}), \quad t \in X_i.$$

Clearly $\|D^i_t\|$ is bounded in I for each t, so the definition of X_{i+1} is correct. Obviously, X_{i+1} is a set of $\varphi_i(\bar{y};\bar{x})$-diagonally indiscernible elements in I.

c) First, $I \models \mathrm{PA}$, by b). Applying 2.3.7 c) and b), $\mathcal{O}_I$ is Σ_k-bounding for every $k \geq 1$, therefore by 2.3.7 a), $J \preccurlyeq_k I$ for every $k \geq 1$. $\square$

Proof of 2.3.3. First, if $\mathcal{O}$ is an unbounded $\Sigma_n(1;1)$-diagonally indiscernible overlay in M, then by 2.3.7 b), $\mathcal{O}$ is Σ_{n+1}-bounding, so if $\mathcal{O}$ is unbounded in a cut I of M, then $I \preccurlyeq_{n+1} M$, by 2.3.7 a). It remains to prove the existence.

Let D be a diagonal partition of M such that

$$x\, D_t\, y \text{ iff } \mathrm{Sat}_{\Sigma_n}(f, \langle p, x \rangle) \leftrightarrow \mathrm{Sat}_{\Sigma_n}(f, \langle p, y \rangle),$$

whenever $f, p < t$ and f codes a Σ_n formula in variables $(v_0;v_1)$. It straightforward to check that D is Δ_{n+1} and that $\mathrm{I}\Sigma_{n+2}$ proves $\|D_t\| \leq 2^{t^2}$ for every

t, hence D is Δ_1 estimated. In particular, D is a LL_{n+1}-estimated LL_{n+1}-diagonal partition of M. Let T_D be the tree of D-h.m.p.h. sequences ordered by inclusion (c.f. Definition 2.1.7). Then by 2.1.9, T_D is LL_{n+1}; by the Corollary 1.7.5 to the Low Basis Theorem, T_D has an LL_{n+2} unbounded branch B in M. Let $X \stackrel{\text{df}}{=} \{a \in M \mid (\exists s \in B) a \in s\}$; then X is D-homogeneous and LL_{n+2}, hence Δ_{n+3}. Now, the set $\mathcal{O} \stackrel{\text{df}}{=} \{X \setminus [0,m] \mid m \in \mathbb{N}\}$ is an unbounded $\Sigma_n(1;1)$-diagonally indiscernible overlay in M, since for a $\Sigma_n(v_0;v_1)$-formula φ with $m = \ulcorner\varphi\urcorner$, $X \setminus [0,m]$ is a set of φ-diagonally indiscernible elements in M. Let $I \preccurlyeq_{n+1} M$; it remains to be proved that $X \cap I$ is cofinal in I. Assume $X \cap I < a$ for some $a \in I$ and let $b \stackrel{\text{df}}{=} \min(X \setminus [0,a])$. Then there is a sequence $s \smile \langle b \rangle \in B$ such that $\mathrm{Max}(s) < a$. There is a Δ_{n+1} formula $\theta(s,y,w)$ expressing that '$s \smile \langle y \rangle$ is D-homogeneous and w codes the set $\{\langle f,p\rangle \in [0,\mathrm{Max}(s)]^2 \mid \mathrm{Sat}_{\Sigma_n}(f,\langle p,y\rangle)\}$'. Let w_b be such that $M \models \theta(s,b,w_b)$; since $\mathrm{Max}(s) \in I$, $w_b \in I$, too. Then $M \models (\exists y)\theta(s,y,w_b)$ and by $I \preccurlyeq_{n+1} M$, $I \models \theta(s,c,w_b)$ for some $c \in I$. But then $s \smile \langle c,b \rangle$ is D-homogeneous; this, however, contradicts the minimality condition of $s \smile \langle b \rangle$ in the definition of T_D. This contradiction completes the proof. □

2.3.8 Example. Our results on overlays reveal somewhat hidden structure of models of arithmetic. As an example of this, we give an illuminating proof of Gaifman's Theorem 1.8.7, a) based on overlays. With them at hand, the whole thing becomes obvious:

Proof of Gaifman's theorem. Let $M \models \mathrm{PA}$ and $M \preccurlyeq_0^{cf} N$. Surely all Π_1 sentences true in M hold in N as well, hence $N \models \mathrm{I}\Sigma_0$. Now, M has an unbounded Δ_0-diagonally indiscernible overlay $\mathcal{O}$ which, due to $M \preccurlyeq_0 N$ and $M \subseteq^{cf} N$, is also an unbounded Δ_0-diagonally indiscernible overlay in N. Thus $N \models \mathrm{PA}$. Also, if φ is an $\mathcal{L}^{Ar}$-formula, then some $X \in \mathcal{O}$ is φ-bounding in both M and N; thus $M \preccurlyeq_\varphi N$. Thus M is elementary in N. □

2.3.9 Remark. We may finally note that there is no unbounded set X of Δ_0-diagonally indiscernible elements in $M \models \mathrm{PA}$ definable in M from parameters; i.e. a definable Δ_0-diagonally indiscernible overly cannot be singleton or finite. Indeed, if M is non-standard (otherwise pass to some elementary end-extension) and X is as above with $X = \varphi(M,\bar{p})$ for some $\varphi \in \Sigma_n$ and $\bar{p} \in M$, then for $K \stackrel{\text{df}}{=} \mathrm{Dfe}_{\Sigma_{n+1}}(M;a,\bar{p})$ with $a \in M \setminus \mathbb{N}$ we have $K \preccurlyeq_{n+1} M$, hence $\varphi(K,\bar{p}) = X \cap K$ is unbounded in K. It follows that $\{X \cap K\}$ is an unbounded Δ_0-diagonally indiscernible overlay in K and consequently $K \models \mathrm{PA}$. This contradicts 1.8.11, b).

CHAPTER 3

FAMILIES OF CUTS

Given a countable model $M \models \mathrm{I}\Sigma_1$, we refer to any set of its cuts that have a certain common property as a family. Each family is linearly ordered by inclusion. Under this order, the family of all cuts of M is topologically isomorphic to the Cantor space. Given a particular family, say $\mathcal{P}_0 \stackrel{\mathrm{df}}{=} \{I \subseteq^e M \mid I \models \mathrm{PA}\}$, there are many natural questions to ask, such as: What is the order type of $\mathcal{P}_0$? What is the order type of its topological closure $\overline{\mathcal{P}}_0$? What are the relationships between $\mathcal{P}_0$ and other similar families of cuts of M?

In Theorem 3.2.1, we gather general topological consequences obtained for any family $\mathcal{R}$ that has an Δ_n-definable indicator (a notion introduced by Kirby and Paris [KP76]) with reasonable properties. The results apply to majority of those families of cuts that could be described as 'interesting'. We cannot say that the information provided by the theorem is entirely new, because for certain particular families, some of the properties have been observed and pointed out by others ([Kot83], [Kot84b],[Ign86], etc.). But to our knowledge, the scattered ideas were never collected in a single theorem of a similar general form.

The rest of the chapter is devoted to some concrete subfamilies of the family $\mathcal{E}_n$ of Σ_n-elementary cuts. Among others they include: family $\mathcal{P}_n$ of the cuts satisfying PA, family $\mathcal{I}_n$ of the cuts that are isomorphic to the model M, and family $\mathcal{D}_n$ of the cuts determined by Σ_n-definable elements. For each of the families we provide additional results that cannot be derived from the general theorem and conclude the study by proving, with some assumptions, the following inclusions $\overline{\mathcal{P}_n} \supsetneq \overline{\mathcal{I}_n} \supseteq \overline{\mathcal{D}_{n+1}} \supsetneq \mathcal{E}_{n+1} \supsetneq \overline{\mathcal{P}_{n+1}}$, where the overline denotes topological closure in the set of all cuts. We apply the results on overlays from Section 2.3 in Theorem 3.4.1.

We also correct a false claim made by R. Kaye in [Kay91b] (Remark 3.5.13).

3.1 Definitions and preliminaries

Assumption. Throughout this chapter, M is an arbitrary but fixed countable non-standard model of $\mathrm{I}\Sigma_1$.

We study a variety of families of cuts and initial substructures of M.

3.1.1 Definition. If $a,b \in M$ and I is a cut of M, we say that the interval (a,b) *contains* I, or that I *lies between* a *and* b, if $a \in I < b$.

For a family of cuts $\mathfrak{R}$, let $\sim_{\mathfrak{R}}$ denote the relation on M such that $x \sim_{\mathfrak{R}} y$ iff no cut from $\mathfrak{R}$ lies between x and y.

Two families of cuts of M, $\mathfrak{R}$ and $\mathfrak{R}'$, are said to be *symbiotic*, if every interval (a,b) of M containing a cut from $\mathfrak{R}$ contains a cut from $\mathfrak{R}'$ and vice versa.

Clearly $\sim_{\mathfrak{R}}$ is an equivalence on M, partitioning M into contiguous blocks w.r.t. the order $<$. Hence, the order $<$ of M induces a total order on $M/{\sim_{\mathfrak{R}}}$.

3.1.2 Proposition. *Two families of cuts of* M*,* $\mathfrak{R}$ *and* $\mathfrak{R}'$*, are symbiotic if* $\sim_{\mathfrak{R}} = \sim_{\mathfrak{R}'}$*, that is, if* $M/{\sim_{\mathfrak{R}}} = M/{\sim_{\mathfrak{R}'}}$ $\square$

3.1.3 Definition. Let $\mathfrak{R}$ be a non-empty family cuts of M. We define:

$$I^-_{a,\mathfrak{R}} \stackrel{\text{df}}{=} \bigcup\{J \subseteq^e M \mid J \in \mathfrak{R} \text{ and } a \in M \setminus J\},$$
$$I^+_{a,\mathfrak{R}} \stackrel{\text{df}}{=} \bigcap\{J \subseteq^e M \mid a \in J \in \mathfrak{R}\}.^{\dagger}$$

For $I \subseteq^e M$, we write

$$\mathfrak{R} \nearrow I \quad \text{iff} \quad I = \bigcup\{J \subset^e I \mid J \in \mathfrak{R}\},$$
$$I \swarrow \mathfrak{R} \quad \text{iff} \quad I = \bigcap\{J \subseteq^e M \mid I \subset^e J \in \mathfrak{R}\},$$

If $\mathfrak{R}'$ is also a family of cuts of M, we write $\mathfrak{R} \nearrow \mathfrak{R}'$ iff $\mathfrak{R} \nearrow I$ for every $I \in \mathfrak{R}'$, and we write $\mathfrak{R}' \swarrow \mathfrak{R}$ iff $I \swarrow \mathfrak{R}$ for every $I \in \mathfrak{R}'$, $I \neq M$.

We further use the symbols $\not\nearrow$ and $\not\swarrow$ to denote respectively the negations of the relations $\nearrow$ and $\swarrow$.

(Since M is fixed, we do not decorate the symbols $I^+_{a,\mathfrak{R}}, I^-_{a,\mathfrak{R}}$ and $\swarrow, \nearrow$ with an additional superscript indicating their reference to the model M.)

Clearly, $I^-_{a,\mathfrak{R}} \not\swarrow \mathfrak{R}$ and $\mathfrak{R} \not\nearrow I^+_{a,\mathfrak{R}}$. Also, if $\emptyset \neq I^-_{a,\mathfrak{R}} \notin \mathfrak{R}$, then $\mathfrak{R} \nearrow I^-_{a,\mathfrak{R}}$, and if $I^+_{a,\mathfrak{R}}$ is defined and $I^+_{a,\mathfrak{R}} \notin \mathfrak{R}$, then $I^+_{a,\mathfrak{R}} \swarrow \mathfrak{R}$. Moreover, for $I \subset^e M$, $I \not\swarrow \mathfrak{R}$ iff $I^-_{a,\mathfrak{R}} \subseteq I < a$ for some $a \in M$, and $I \not\nearrow \mathfrak{R}$ iff either $I^+_{a,\mathfrak{R}}$ is undefined or $I \subseteq I^+_{a,\mathfrak{R}}$ for some $a \in I$.

† $I^+_{a,\mathfrak{R}}$ is undefined if $\bigcap\emptyset$ stands on the right .

3.1.4 Definition. For a family $\mathfrak{R}$ of cuts of M, let

$$\overline{\mathfrak{R}} \stackrel{\mathrm{df}}{=} \{I \subseteq^e M \mid I \in \mathfrak{R} \text{ or } \mathfrak{R} \nearrow I \text{ or } I \swarrow \mathfrak{R}\}$$

Clearly $\langle \overline{\mathfrak{R}}, \subseteq \rangle$ is the unique (up to isomorphism) order-completion of $\langle \mathfrak{R}, \subseteq \rangle$; it is the topological closure of $\mathfrak{R}$ in the space of all cuts of M whose topology is induced by the ordering $\subseteq$. We further define

$$\begin{aligned} \mathfrak{R}^+ &\stackrel{\mathrm{df}}{=} \{I^+_{a,\mathfrak{R}} \mid a \in M\}, \text{ and} \\ \mathfrak{R}^- &\stackrel{\mathrm{df}}{=} \{I^-_{a,\mathfrak{R}} \mid a \in M\}. \end{aligned}$$

Finally, let for $a, b \in M$, $a < b$, $(a,b)^{\mathfrak{R}}$ denote the subfamily $\{I \in \mathfrak{R} \mid a \in I < b\}$.

Note that $M/\sim_{\mathfrak{R}} = \{I^+_{a,\mathfrak{R}} \setminus I^-_{a,\mathfrak{R}} \mid a \in M\}$; this set with its natural ordering induced by $\leq^M$ it is order-isomorphic to $\langle \mathfrak{R}^+, \subseteq \rangle$.

3.1.5 Proposition. *Families of cuts $\mathfrak{R}$ and $\mathfrak{R}'$ are symbiotic iff $\overline{\mathfrak{R}} = \overline{\mathfrak{R}}'$.*

Proof. First let $\mathfrak{R}$ and $\mathfrak{R}'$ be symbiotic and let $I \in \overline{\mathfrak{R}}$. If $\mathfrak{R} \nearrow I$, then for every $a \in I$ there exists $b \in I$, $a < b$ such that $(a,b)^{\mathfrak{R}}$ is non-empty. Hence $(a,b)^{\mathfrak{R}'}$ is non-empty and $\mathfrak{R}' \nearrow I$. The remaining cases when $I \swarrow \mathfrak{R}$ or when $I \in \mathfrak{R}$ but neither $\mathfrak{R} \nearrow I$ nor $I \swarrow \mathfrak{R}$ are similar. We thus have $\overline{\mathfrak{R}} \subseteq \overline{\mathfrak{R}}'$. The other inclusion follows from symmetry.

Conversely, if $\overline{\mathfrak{R}} = \overline{\mathfrak{R}}'$ and the interval (a,b) contains a cut $I \in \mathfrak{R}$, then $I \in \overline{\mathfrak{R}}'$, hence either $I \in \mathfrak{R}'$ in which case we are done, or one of $\mathfrak{R}' \nearrow I$ or $I \swarrow \mathfrak{R}'$ holds, and we can therefore easily find a cut $J \in \mathfrak{R}'$ such that either $a \in J \subset^e I$ or one satisfying $I \subset^e J < b$. □

The following definitions extend Kirby and Paris' notion of a well-behaved indicator:

3.1.6 Definition (*Indicators*).

a) Let $M \models \mathrm{PA}^-$ be non-standard and let $\mathfrak{R}$ be a family of cuts of M. A function $Y : M^2 \to M$ is said to be an *indicator* for $\mathfrak{R}$ in M if the following properties hold for every $a, b \in M$:

(3.1) $\quad Y(a,b) \geq \mathbb{N}$ iff there exists $I \in \mathfrak{R}$ with $a \in I < b$

(3.2) $\quad$ if $M \in \mathfrak{R}$, then $M \models (\forall x)(\exists y) Y(x,y) > n$ for every $n \in \mathbb{N}$.

If Γ is a class of $\mathfrak{L}^{Ar}$ formulae and Y is defined in M (without parameters) by a Γ-formula, we say that Y is a Γ-*indicator*. We say that $\mathfrak{R}$ *has a* Γ-*indicator in* M, if there exists a Γ-indicator Y for $\mathfrak{R}$ in M.

b) More generally, if T is a theory extending PA^-, and $\mathfrak{R}$ is defined by some property $\Phi(I,M)$ of cuts of M, i.e. $\mathfrak{R} = \{I \subseteq^e M \mid \Phi(I,M)\}$, then a Γ-formula $\xi(x,y,z)$ (also written as $Y(x,y) = z$) is said to be a Γ-*indicator* *for* $\mathcal{R}$ *in* T if in every countable $N \models T$ the formula $\xi(x,y,z)$ defines a Γ-indicator for $\mathcal{R}^N \stackrel{\text{df}}{=} \{I \subseteq^e N \mid \Phi(I,N)\}$ in N. We then say that $\mathcal{R}$ *has a* Γ*-indicator in* T.

3.1.7 Remark. Indicators were introduced by Paris and Kirby in [KP76]. Indicators provide a method for finding independent results ([Par78], [PH77]) and reveal symbiosis as well as the distribution of initial substructures of various types in the base model (cf. for example [Kir77], [Par80], and [Kot84b]). In this chapter we pursue the latter application.

We only deal with Γ-indicators for some $\Gamma \subseteq \mathrm{Fm}(\mathcal{L}^{Ar})$. Note that if Y is a Σ_n-indicator in M for a family $\mathcal{R}$, then there exists a Σ_n-indicator Y' for $\mathcal{R}$ in M with the following 'monotony' property:

$$(y \le x \to Y'(x,y) = 0) \wedge (x_1 \le x_2 \le y_2 \le y_1 \to Y'(x_2,y_2) \le Y'(x_1,y_1)). \tag{3.3}$$

Assumption. From now on, we assume all considered indicators satisfy (3.3).

3.1.8 Remark. For every Σ_{n+1}-indicator Y the relation $Y(x,y) = z$ is in fact Δ_{n+1} in M, since $M \models (\forall x,y)(\exists z)Y(x,y) = z$ and hence $Y(x,y) = z$ is equivalent in M to the Π_{n+1} formula $(\forall z' \neq z)\neg Y(x,y) = z'$.

We use the following lemma from [Kot84b], which characterizes the order-type of the Cantor set.

3.1.9 Lemma. *A complete linear ordering* $\langle X, \le \rangle$ *is isomorphic to the Cantor set (that is to* $\langle {}^{\omega}2, <_{Le} \rangle$ *where* $<_{Le}$ *is the lexicographic ordering) iff there is a subset* $Y \subseteq X$ *such that* $\langle Y, \le \rangle$ *has the order type* $1+\eta^{\dagger}$*, and*

a) $x = \sup_{\langle X,\le\rangle}\{y \in Y \mid y < x\}$*, for all* $x \in X \setminus Y$*, and*

b) $x > \sup_{\langle X,\le\rangle}\{y \in Y \mid y < x\}$ *(and the supremum exists), for all* $x \in Y$.

Sketch of the proof. Since Y has order-type $1+\eta$, there is an isomorphism between Y and the set of functions in ${}^{\omega}2$ that are eventually constantly 0, and this isomorphism extends uniquely (via suprema) to an isomorphism of $\langle X, \le \rangle$ and $\langle {}^{\omega}2, <_{Le} \rangle$. □

3.2 The general theorem

Let I be a cut of M and Y an indicator of some family in M. For future reference, we recognize the following situations:

† That is, $\langle Y, \le \rangle$ is a countable dense linear order with a least but no last element.

(A) $(\forall a \in I)(\exists c \in I \setminus \mathbb{N})(\exists b \in I) M \models Y(a,b) > c$,

(B) $(\exists c \in I \setminus \mathbb{N})(\forall a \in I)(\exists b \in I) M \models Y(a,b) > c$.

3.2.1 Theorem. *If $M \models \mathrm{I}\Sigma_{n+1}$ and $\mathfrak{R} \neq \emptyset$ has a Σ_{n+1}-indicator Y in M, then*

a) *$\langle \overline{\mathfrak{R}}, \subseteq \rangle$ is order-isomorphic to the Cantor set. In particular, $\overline{\mathfrak{R}}$ has the cardinality $2^{\aleph_0}$; for every $I \in \overline{\mathfrak{R}}$ either $\mathfrak{R} \nearrow I$ or $I \swarrow \mathfrak{R}$.*

b) *Under this isomorphism, $\mathfrak{R}^-$ and $\mathfrak{R}^+$ correspond to the sets of the right- and left-isolated points of the Cantor set, respectively.*

c) *There is a densely ordered countable subset $\mathfrak{Q} \subseteq \mathfrak{R}$ such that $\overline{\mathfrak{Q}} = \overline{\mathfrak{R}}$.*

d) *If I is any cut of M satisfying (A), then $\mathfrak{R} \nearrow I$.*

Assume additionally that every $I \in \mathfrak{R}$ satisfies (B). Then:

e) *$\mathfrak{R} \nearrow \mathfrak{R} \swarrow \mathfrak{R}$. In other words, $\mathfrak{R}$ has no left- or right- isolated points (except possibly for end-points $\mathbb{N}$ and M, which, if in $\mathfrak{R}$, may be left- and right-isolated, respectively), and, in particular, $I^-_{a,\mathfrak{R}} \notin \mathfrak{R}$, $I^+_{a,\mathfrak{R}} \notin \mathfrak{R}$ for all $a \in M$ non-standard.*

f) *$\mathfrak{R}$ is meager in $\overline{\mathfrak{R}}$.*

g) *Every non-empty interval $(a,b)^{\overline{\mathfrak{R}}}$ includes a subset $X \subseteq \overline{\mathfrak{R}} \setminus \mathfrak{R}$ with $|X| = 2^{\aleph_0}$ and $\langle X, \subseteq \rangle$ ordered as reals.*

In the proof and also later in this chapter we refer to the following definition:

3.2.2 Definition. A sequence $\{c_k\}_{k\in\omega}$ of elements of M is called *downward cofinal in M* if for every $a \in M$, a is non-standard iff $a > c_k$ for some $k \in \omega$, that is, if $\{c_k\}_{k\in\omega}$ is coinitial in $M \setminus \mathbb{N}$.

Since M is countable, there is a downward cofinal sequence in M.

Proof of 3.2.1. Let Y be a Δ_{n+1}-indicator for $\mathfrak{R}$. First observe that if $a < b$, then $I^+_{a,\mathfrak{R}} \neq I^-_{b,\mathfrak{R}}$ (that is $\mathfrak{R}^+ \cap \mathfrak{R}^- = \emptyset$). Otherwise we have $(a,b)^{\mathfrak{R}} = \{I\}$ where $I^+_{a,\mathfrak{R}} = I^-_{b,\mathfrak{R}} = I \in \mathfrak{R}$. But then $Y(a,b) = c$ is nonstandard. For every $n \in \mathbb{N}$, considering the element $\mu x : Y(a,x) > n$, we obtain

$$M \models (\exists x)(a < x < b \wedge Y(a,y) > n \wedge Y(y,b) > n)$$

and thus by overspill, there are $c \in M \setminus \mathbb{N}$ and $d \in M$ such that $a < d < b \wedge Y(a,d) > c \wedge Y(d,b) > c$. Then $(a,d)^{\mathfrak{R}}$ and $(d,b)^{\mathfrak{R}}$ are disjoint, yet both non-empty, in contradiction to $(a,b)^{\mathfrak{R}}$ being a singleton.

We prove a) and b) using Lemma 3.1.9, in which we let $\mathfrak{R}^+$ take the role of the set Y. First we must show that the order-type of $\mathfrak{R}^+$ is $1+\eta$. Countability of $\mathfrak{R}^+$ follows from that of M. Since $\mathfrak{R} \neq \emptyset$, $I^+_{0,\mathfrak{R}}$ is defined and is clearly the least element of $\mathfrak{R}^+$. Suppose $I \stackrel{\text{df}}{=} I^+_{a,\mathfrak{R}}$ for some $a \in M$ is the greatest element of $\mathfrak{R}^+$. Then $I \in \mathfrak{R}$ since if $I \subset^e J \in \mathfrak{R}$, we would have some $b \in J \setminus I$ and $I^+_{b,\mathfrak{R}}$ greater than I. If $I = M$, then $Y(a,b) > \mathbb{N}$ for some $b \in M$ by overspill and (3.2) in the definition of an indicator; this produces a cut from $\mathfrak{R}$ between a and I, which contradicts $I = I^+_{a,\mathfrak{R}}$. Hence $I \neq M$. But then again, there is some $b \in M$ with $a \in I < b$ and since we observed, $I \notin \mathfrak{R}^-$, so $I \neq I^-_{b,\mathfrak{R}}$, hence there must be a cut from $\mathfrak{R}$ between I and b, contradicting maximality of I.

The proof that $\mathfrak{R}^+$ is densely ordered goes similarly: if $I \subset^e J$ for $I \stackrel{\text{df}}{=} I^+_{a,\mathfrak{R}}$, $J \stackrel{\text{df}}{=} I^+_{b,\mathfrak{R}}$ and some $a, b \in M$, then $I \subset^e I^-_{b,\mathfrak{R}}$. Hence, for arbitrary $d \in I^-_{b,\mathfrak{R}} \setminus I$ we have $I < d \in I^+_{d,\mathfrak{R}} < b \in J$.

It remains to verify the conditions 3.1.9 a), b), which means to prove that $\mathfrak{R}^+ \nearrow I$ when $I \in \overline{\mathfrak{R}} \setminus \mathfrak{R}^+$ and $\mathfrak{R}^+ \not\nearrow I$ when $I \in \mathfrak{R}^+$. The latter is obvious. Let $a \in I \in \overline{\mathfrak{R}} \setminus \mathfrak{R}^+$. We show that $I^+_{a,\mathfrak{R}} \subset^e I$. If $I \in \mathfrak{R}$, then clearly $I^+_{a,\mathfrak{R}} \subseteq^e I$ and the inequality follows from $I \notin \mathfrak{R}^+$. If $\mathfrak{R} \nearrow I$, we may apply the same argument on some $J \in \mathfrak{R}$ with $a \in J \subset^e I$. Finally, $I \swarrow \mathfrak{R}$ implies $\mathfrak{R} \nearrow I$ in this case, since otherwise $I^+_{b,\mathfrak{R}} = I$ for some large enough $b \in I$, contradicting $I \notin \mathfrak{R}^+$.

c) For every $a, b \in M$ such that $(a,b)^{\mathfrak{R}} \neq \emptyset$ pick $I_{a,b} \in (a,b)^{\mathfrak{R}}$ and put

$$\mathfrak{Q} \stackrel{\text{df}}{=} \{I_{a,b} \mid a, b \in M\}.$$

$\mathfrak{Q}$ is countable and it is easy to verify that $\overline{\mathfrak{Q}} = \overline{\mathfrak{R}}$.

d) is trivial.

Assume from now that every $I \in \mathfrak{R}$ satisfies (B).

e) Let $I \in \mathfrak{R}$. Then immediately $\mathfrak{R} \nearrow I$ for $I \neq \mathbb{N}$, by from (B). It remains to show $I \swarrow \mathfrak{R}$ for $I \neq M$. We first deal with the case $I \neq \mathbb{N}$. Let $c \in I \setminus \mathbb{N}$ be as in (B) and fix arbitrary $b > I$. We have $M \models (\exists y < b) Y(a,y) > c$ for all $a \in I$. Applying overspill to this formula, we obtain some a_0 and b_0 such that $I < a_0 < b_0 < b$ and $M \models Y(a_0, b_0) > c$. It follows that some $I \subset^e J < b$ for some $J \in \mathfrak{R}$. The case of $I = \mathbb{N} \in \mathfrak{R}$ goes analogously: we first note that $M \models Y(m,b) > m$ for all $m \in \mathbb{N}$, $b \in M \setminus \mathbb{N}$, then apply overspill to obtain a non-standard $a < b$ satisfying $M \models Y(a,b) > a$.

f) Let $\{c_k\}_{k \in \omega}$ be a downward cofinal sequence in M. Let

$$\mathfrak{R}_k \stackrel{\text{df}}{=} \{I \in \mathfrak{R} \mid c_k \in I \text{ and } (\forall a \in I)(\exists b \in I) M \models Y(a,b) > c_k\}.$$

Clearly, $\mathfrak{R} \setminus \{\mathbb{N}\} = \bigcup_{k \in \omega} \mathfrak{R}_k$, since for every $\mathbb{N} \neq I \in \mathfrak{R}$ we have some $c \in I \setminus \mathbb{N}$ as in (B). Then $c > c_k$ for some $k \in \omega$, so $I \in \mathfrak{R}_k$. It remains to show that all $\mathfrak{R}_k$ are nowhere dense in $\overline{\mathfrak{R}}$.

Fix $k \in \omega$. Every non-empty open subset of $\overline{\mathfrak{R}}$ includes a non-empty open subset of the form $(a,b)^{\overline{\mathfrak{R}}}$, so it suffices to show that if $(a,b)^{\overline{\mathfrak{R}}}$ is non-empty, then for some $d \in (a,b)$, $(a,d)^{\overline{\mathfrak{R}}} \neq \emptyset$ but $(a,d)^{\mathfrak{R}_k} = \emptyset$. Assume $(a,b)^{\overline{\mathfrak{R}}} \neq \emptyset$, $a,b \in M$. Then clearly $(a,b)^{\mathfrak{R}} \neq \emptyset$, hence $M \models Y(a,b) > \mathbb{N}$. Let, in M,

$$d \stackrel{\text{df}}{=} \max\{x < b \mid Y(a,x) \leq c_k\}.$$

It is easy to see that d is correctly defined and that $M \models Y(a,d) > \mathbb{N}$. Let I be a cut from $(a,d)^{\mathfrak{R}}$ and assume $c_k \in I$. Then for all $e \in I$, $M \models Y(a,e) \leq Y(a,d) \leq c_k$, which means that $I \notin \mathfrak{R}_k$.

g) First observe that whenever $u,v \in M$ satisfy $M \models \mathbb{N} < Y(u,v)$, we can find $e \in (u,v)$ so that $M \models \mathbb{N} < Y(u,e)$ and $M \models \mathbb{N} < Y(e,v)$, and that we may further require $M \models \mathbb{N} < Y(u,e) < c$ for a given non-standard $c \in M$. Indeed, we may e.g. put

$$e \stackrel{\text{df}}{=} \max\{x < v \mid Y(u,x) \leq c \wedge Y(u,x) \leq Y(x,v)\} \tag{3.4}$$

and observe that if $Y(u,e) \in \mathbb{N}$, then $Y(u,e+1)$ would be in $\mathbb{N}$, too, and thus less than both c and $Y(e,v)$ (which must be non-standard). This contradicts maximality of e.

Let $(a,b)^{\overline{\mathfrak{R}}}$ be non-empty, $a,b \in M$, and let $\{c_k\}_{k\in\omega}$ be a downward cofinal sequence in M as before. We use the fact just proved to construct a family $\{a_\sigma\}_{\sigma\in{}^{<\omega}2}$ of points from the interval (a,b) indexed by finite sequences of 0's and 1's such that the following conditions hold for every $\tau \in {}^{<\omega}2$:

1) $a < a_\tau < a_{\tau 0} < a_{\tau 00} < a_{\tau 01} < a_{\tau 1} < a_{\tau 10} < a_{\tau 11} < b$

2) $M \models \mathbb{N} < Y(a_\tau, a_{\tau 0})$, $M \models \mathbb{N} < Y(a_{\tau 0}, a_{\tau 1}) < c_{|\tau|}$, and $M \models \mathbb{N} < Y(a_{\tau 1}, b)$,

where τi denotes the function $\tau \cup \{\langle |\tau|, i\rangle\}$. We put $a_\emptyset \stackrel{\text{df}}{=} a+1$ and proceed by induction on the length of τ. Let $\sigma \in {}^{<\omega}2$ and assume a_σ is defined together with all a_τ such that $|\tau| \leq |\sigma|$. We obtain $a_{\sigma 0}$ and $a_{\sigma 1}$ as follows: let $b' \stackrel{\text{df}}{=} \tau 1$ if $|\sigma| \geq 1$ and σ is of the form $\tau 0$ for some $\tau \in {}^{|\sigma|-1}2$, and $b' \stackrel{\text{df}}{=} b$ otherwise. In both cases we assume $M \models \mathbb{N} < Y(a_\sigma, b')$ based on the property 2). Now, we first apply (3.4) with $u = a_\sigma$, $v = b'$ and $c = c_0$ to find $a_{\sigma 0}$ and then again with $u = a_{\sigma 0}$, $v = b'$, and $c = c_{|\sigma|}$ to find $a_{\sigma 1}$. Clearly, 1) holds and 2) holds with σ substituted for τ, so the induction may proceed.

For any $f \in {}^\omega 2$ we now let

$$I_f \stackrel{\text{df}}{=} \{x \in M \mid x \leq a_{\tau 0} \text{ for some } \tau = f{\restriction}k, k \in \omega\}.$$

If $f,g \in {}^\omega 2$ and $f \neq g$, let $k \in \omega$ be the least such that $f(k) \neq g(k)$. If, say, $f(k) = 0$, then $a_{\sigma 0} \in I_f < a_{\sigma 1} \in I_g$, where $\sigma = f{\restriction}k = g{\restriction}k$, so $I_f \neq I_g$. Note also that $\mathfrak{R} \nearrow I_f$ for every $f \in {}^\omega 2$, since $\{a_{f\restriction k} \mid k \in \omega\}$ is cofinal in I_f and every $(a_{f\restriction k}, a_{f\restriction(k+1)})^{\mathfrak{R}}$ is non-empty by 2). Finally, note that perhaps with the countably many exceptions when f is eventually constantly 1, $I_f \notin \mathfrak{R}$. Indeed, assume $I_f \in \mathfrak{R}$ and let $c > \mathbb{N}$ be as in (B) for $I \stackrel{\text{df}}{=} I_f$; since f is not eventually constantly 1, there exists $k \in \omega$ such that $f(k) = 0$ and $c_k < c$. Then for $\tau = f{\restriction}k$

we have $a_{\tau 0} < a_{\tau 00} \in I_f < a_{\tau 1}$ and $M \models Y(a_{\tau 0}, a_{\tau 1}) < c_k$. For all $e \in I$ thus $M \models Y(a_{\tau 0}, e) < c$, which contradicts the choice of c.

We have shown that $X' = \{I_f \mid f \in {}^{\omega}2 \wedge f$ is not eventually constantly 1$\} \subseteq \overline{\mathfrak{R}} \setminus \mathfrak{R}$. By removing from X' the least element $I_{\overline{0}}$ where $\overline{0} = \omega \times \{0\}$, we obtain a set X of cuts from $(a,b)^{\overline{\mathfrak{R}}} \setminus \mathfrak{R}$ ordered as reals. □

There are two typical examples for the conditions (A), (B).

3.2.3 Proposition. *Let $n \geq 0$, $M \models \mathrm{I}\Sigma_{n+1}$ and let Y be an indicator for a family $\mathfrak{R}$ in M that is Δ_{n+1} in $\mathrm{I}\Sigma_{n+1}$. Then, for every $I \in \mathfrak{R}$,*

a) *If $\mathbb{N} \neq I \preccurlyeq_n M$ and $I \models \mathrm{I}\Sigma_{n+1}$, then I satisfies* (A).

b) *If $\mathbb{N} \neq I \preccurlyeq_n M$ and $I \models \mathrm{I}\Sigma_{n+2}$, then I satisfies* (B).

Proof. If $M \in \mathfrak{R}$, then (3.4) with $I = M$ follows immediately by overspill from the condition (3.2) in the definition of indicator.

Let $I \in \mathfrak{R}$, $\mathbb{N} \neq I \prec_n^e M$, $I \models \mathrm{I}\Sigma_{n+1}$. Then for a fixed $m \in \mathbb{N}$ and every $a \in I$ and $b \in M \setminus I$, $M \models Y(a,b) > m$. Let us fix $a \in I$. Since $I \preccurlyeq_n M$ I satisfies all Π_{n+1} formulae true in M with parameters from I. Now, the assumption ensure that $Y(a,b) > m$ is equivalent to a Π_{n+1} formula in both M and I. So, using these facts and underspill in M, there exists some $b \in I$ such that $I \models Y(a,b) > m$. Now a) follows by Σ_{n+1}-overspill of $I \models (\exists y) Y(a,y) > m$ in I and n-elementarity and b) follows by Π_{n+2}-overspill of $I \models (\forall x)(\exists y) Y(a,y) > m$ in I and n-elementarity. □

In [Kot84b], Kotlarski investigated the family of cuts satisfying PA and obtained a theorem which generalizes easily as follows:

3.2.4 Theorem (*Kotlarski*). *Let $M \models \mathrm{I}\Sigma_{n+1}$ and assume $\mathfrak{R} \neq \emptyset$ has a Δ_{n+1}-indicator in M. Then there exists a countably infinite family $\{q_k\}_{k \in \omega}$, of recursive Σ_{n+1}-types such that if a realizes q_i and $j \neq i$, then q_j is not realized by any element of $I^+_{a,\mathfrak{R}} \setminus I^-_{a,\mathfrak{R}}$. It follows that there is a countable family $\{a_k\}_{k \in \omega}$ of elements of M such that the models $I^+_{a_k,\mathfrak{R}}$ are pairwise non-isomorphic.*

Proof. C.f. [Kot84b, Theorem 4] and note that the proof there translates directly from the family of initial substructure satisfying PA with $Y \in \Sigma_1$ to our general setting. The existence of the family $\{a_k\}_{k \in \omega} \subseteq M$ follows from the bounded complexity of the types $\{q_k\}_{k \in \omega}$. □

3.2.5 Remark. Indicators for models of theories yield unprovable theorems of the form $(\forall x)(\exists y) Y(x,y) > x$ (see esp. [Par78], [Kir77]). We may approximate the distribution of initial substructures satisfying this formula. Let Y be a Δ_{n+1}-indicator for a family $\mathfrak{R}$ of cuts of $M \models \mathrm{I}\Sigma_{n+1}$ and let

$$\mathfrak{R}_Y \overset{\mathrm{df}}{=} \{I \subseteq^e M \mid (\forall a \in I)(\exists b \in I) M \models Y(a,b) > a\}. \tag{3.5}$$

Then $\mathcal{R}_Y = \overline{\mathcal{R}_Y} \subseteq \overline{\mathcal{R}}$, yet $\mathcal{R} \not\subseteq \mathcal{R}_Y$.

The closeness of $\mathcal{R}_Y$ is trivial and clearly, if $\mathbb{N} \neq I \in \mathcal{R}_Y$, then $\mathcal{R} \nearrow I$. If $\mathbb{N} \in \mathcal{R}_Y$, then for every standard n and non-standard b, $M \models (\exists y < b) Y(n,y) > n$. By overspill, $M \models (\exists y < b) Y(a,y) > a$ for some non-standard a; in particular, $(a,b)^{\mathcal{R}}$ is non-empty. Since we can start with arbitrarily small $b > \mathbb{N}$, we have $\mathbb{N} \swarrow \mathcal{R}$. Finally, in order to show that $\mathcal{R}$ is not a subset of $\mathcal{R}_Y$, take any non-standard $I \in \mathcal{R}$ and $a \in I \setminus \mathbb{N}$ and let $b \stackrel{\text{df}}{=} \mu y : Y(a,y) > a$. Note that $(a,b)^{\mathcal{R}_Y}$ is empty, while $(a,b)^{\mathcal{R}}$ is not. $\square$

3.3 Specific families, n-elementary cuts

In the rest of the chapter, we focus on some prominent families of cuts.

3.3.1 Definition. Let $n \geq 0$, $a \in M$. We define the following families of cuts of M:

$\mathcal{E}_n \stackrel{\text{df}}{=} \{I \subseteq^e M \mid I \preccurlyeq_n M\}$ (n-elementary cuts)

$\mathcal{R}_T \stackrel{\text{df}}{=} \{I \subseteq^e M \mid I \models T\}$ (cuts satisfying an $\mathcal{L}^{Ar}$-theory T)

$\mathcal{P}_n \stackrel{\text{df}}{=} \mathcal{E}_n \cap \mathcal{R}_{\mathrm{PA}}$ (n-elementary cuts satisfying PA)

$\mathcal{I}_n \stackrel{\text{df}}{=} \{I \in \mathcal{E}_n \mid M \cong I\}$ (n-elementary cuts isomorphic to M)

$\mathcal{D}_n \stackrel{\text{df}}{=} \{I_{\Sigma_n}(M;a) \mid a \in M\}$ (cuts determined by elements Σ_n-defined from a parameter)

3.3.2 Remark.

a) $\mathcal{E}_0$ is the family of all initial substructures of M.

b) $\overline{\mathcal{E}_n} = \mathcal{E}_n$ for all $n \geq 0$.

c) It is easy to see that if the family $\mathcal{E} = \bigcap_{n\in\omega} \mathcal{E}_n$ of all elementary initial substructures of $M \models \mathrm{PA}$ is non-empty, it does not have a $\emptyset$-defined indicator in M (for consider some non-standard a for which $b \stackrel{\text{df}}{=} \mu y : Y(a,y) > a$ exists in M, take $I \in (a,b)^{\mathcal{E}}$, and arrive at a contradiction by proving that $b \in I$). The same argument applies to any non-empty subfamily of $\mathcal{E}$.

d) No non-empty subfamily of $\mathcal{E}_n$ has a Σ_n- or Π_n-indicator (by the same argument as above).

e) The order type and related properties of the family $\mathcal{E}$ heavily depend on the model M. The saturated and recursively saturated cases have been investigated by Kotlarski in [Kot83], [Kot84a].

If $\mathcal{R}$ is either of the above families, $\mathcal{R}^N$ is defined by replacing our model M with N in the definition of $\mathcal{R}$, as in item b) in the Definition 3.1.6.

3.3.3 Fact. *Let T be a recursively axiomatized $\mathcal{L}^{Ar}$-theory. Then*

a) $\mathcal{R}_T$ *has a* $\Delta_1(\mathrm{I}\Sigma_1)$*-indicator in* PA.

b) $\mathcal{R}_T \cap \mathcal{E}_n$ *has a* $\Delta_{n+1}(\mathrm{I}\Sigma_{n+1})$*-indicator in* PA.

c) $\mathcal{R}_T$ *is symbiotic with its subfamily consisting of recursively saturated initial substructures satisfying* T. *Moreover, if* $(a,b)^{\mathcal{R}_T}$ *is non-empty, then there are* $2^{\aleph_0}$ *initial substructures* $I \models T$ *such that* $a \in I < b$ *and* I *is recursively saturated.*

d) *If* T' *is a recursively axiomatized* $\mathcal{L}^{Ar}$*-theory that proves the same* Π_2 *sentences as* T*, then* $\mathcal{R}_T$ *and* $\mathcal{R}_{T'}$ *are symbiotic.*

Proof. Cf. e.g. [Kay91b], pp. 198-206. □

3.3.4 Remark. Alternative proofs can be found in [KMM81] (and references), where it is further shown that if T is a recursively axiomatized second-order arithmetical theory then the family of initial substructures of M that are expandable to models of T (meaning there exists $\exists\mathcal{X} \subseteq \mathcal{P}(M)$ such that $\langle I, \mathcal{X}\rangle \models T$) is symbiotic with the family of initial substructures satisfying the first-order Π_2-consequences of T.

Applying the general Theorem 3.2.1, we obtain:

3.3.5 Corollary.

- $\mathcal{E}_n$ *and* $\mathcal{E}_n \cap \mathcal{R}_{\mathrm{I}\Sigma_k}$ *with* $k \le n$ *satisfy 3.2.1* a)–c).
- $\mathcal{E}_n \cap \mathcal{R}_{\mathrm{I}\Sigma_{n+1}}$ *satisfies 3.2.1* a)–d).
- $\mathcal{P}_n$ *and* $\mathcal{E}_n \cap \mathcal{R}_{\mathrm{I}\Sigma_k}$ *with* $k \ge n+2$ *satisfy 3.2.1* a)–g).
- *If* $\mathcal{R} \subseteq \mathcal{E}_n \cap \mathcal{R}_{\mathrm{I}\Sigma_{n+1}}$ *has a* $\Delta_{n+1}(\mathrm{I}\Sigma_{n+1})$ *indicator, then* $I^+_{a,\mathcal{R}} \not\models \mathrm{I}\Sigma_{n+1}$ *for very* $a \in M \setminus \mathbb{N}$*, so* $\mathcal{R}^+ \cap \mathcal{R} = \emptyset$.

Proof. Follows from 3.3.3, b) and 3.2.3. The last item follows by 3.2.1 d). □

3.3.6 Fact. *For every* $n \in \mathbb{N}$*, if* $\mathcal{R} \subseteq \mathcal{P}_0$ *is a family of initial substructures of* $M \models \mathrm{PA}$ *closed under isomorphism, then for every* $a, b \in M$*,* $(a,b)^{\mathcal{R}\cap\mathcal{E}_n}$ *is either empty or of cardinality* $2^{\aleph_0}$.

Proof. Cf. e.g. [Kay91b, Theorem 12.7. on page 167]. □

3.3.7 Proposition. *Let* $M \models \mathrm{I}\Sigma_{n+1}$ *and* $a \in M \setminus \mathbb{N}$*. Then* $I^+_{a,\mathcal{E}_n} \subset^e I^+_{a,\mathcal{E}_n\cap\mathcal{R}_{\mathrm{I}\Sigma_{n+1}}}$*. In particular,* $\overline{\mathcal{E}_n \cap \mathcal{R}_{\mathrm{I}\Sigma_{n+1}}} \subsetneq \mathcal{E}_n$ *and* $\overline{\mathcal{P}_n} \subsetneq \mathcal{E}_n$.

Proof. Inclusion $\subseteq^e$ is clear, we show that it is strict. We define

$$\xi(x,y,z,u) \overset{\mathrm{df}}{\iff} \begin{cases} x+z \le y \wedge u \in \langle [x,y]\rangle^{z+1} \wedge (\forall i < z-1)(\forall f < i)(\forall p < (u)_i) \\ \quad [(\exists s)\mathrm{Sat}_{\Pi_{n-1}}(f, \langle p, s\rangle) \to (\exists s < (u)_{i+1})\mathrm{Sat}_{\Pi_{n-1}}(f, \langle p, s\rangle)]. \end{cases}$$

The formula in square brackets is Π_n, so ξ is Δ_{n+1} in $\mathrm{I}\Sigma_1$. For z non-standard, the formula ensures that u is a sequence of elements from the interval $[x,y]$ such that $\mathcal{O} \stackrel{\mathrm{df}}{=} \{\{(u)_i \mid i \geq j\} \mid j \in \omega\}$ is a Σ_n-bounding overlay in M (c.f. 2.3.5). It follows that a formula $Y(x,y)=z$ of the form $z=\max\{z' \leq y \mid (\exists u)\xi(x,y,z',u)\}$ is an indicator for the family $\mathcal{E}_n$ in models of $\mathrm{I}\Sigma_{n+1}$. Observe that $Y(x,y)=z$ is equivalent to a Σ_{n+1} formula (since we may find some w such that all sequences of elements from (x,y) are below w and then replace the unbounded quantifier $(\exists u)$ occurring within the maximization with $(\exists u < w)$).

We now show that $\mathrm{I}\Sigma_{n+1} \vdash (\forall x,z)(\exists y)(\exists u)\xi(x,y,z,u)$. By induction on z: For $z=0$, we have trivially $\xi(x,x,0,\langle x\rangle)$. Let $z \geq 1$ and let $\xi(x,y',z,v)$ for some v,y'. By $\mathrm{S}\Pi_{n-1}$ (provable in $\mathrm{I}\Sigma_n$) we have

$$(\exists y)(\forall f \leq z)(\forall p \leq (v)_{z-1})[(\exists s)\mathrm{Sat}_{\Pi_{n-1}}(f,\langle p,s\rangle) \rightarrow (\exists s<y)\mathrm{Sat}_{\Pi_{n-1}}(f,\langle p,s\rangle)].$$

This is a Σ_{n+1}-formula in $\mathrm{B}\Sigma_{n+1}$, Let y be as above; we may assume $y > y'+1$. Then clearly $\xi(x,y,z+1,v\smile\langle y\rangle)$. This completes the induction. In particular, we have

$$\mathrm{I}\Sigma_{n+1} \vdash (\forall x,z)(\exists y)Y(x,y) \geq z. \tag{3.6}$$

Let $a \in M$ be non-standard and let, in M, $b = \mu y : Y(a,y) \geq a$ whose existence is ensured by (3.6) and $M \models \mathrm{I}\Sigma_{n+1}$. Then $Y(a,b) > \mathbb{N}$, hence there exists $I \in (a,b)^{\mathcal{E}_n}$ and so $I^+_{a,\mathcal{E}_n} \subset I < b$. We show that $I \not\models \mathrm{I}\Sigma_{n+1}$, i.e. that $(a,b)^{\mathcal{R}_{\mathrm{I}\Sigma_{n+1}}} = \emptyset$, which gives $b < I^+_{a,\mathcal{E}_n \cap \mathcal{R}_{\mathrm{I}\Sigma_{n+1}}}$ and completes the proof. Indeed, if $I \models \mathrm{I}\Sigma_{n+1}$, then by (3.6) $I \models Y(a,d) \geq a$ for some $d \in I$. But since this is a Σ_{n+1} formula, we have by n-elementarity $M \models Y(a,d) \geq a$ with $d < b$, contradicting the minimality of b. □

3.4 Cuts satisfying $\mathrm{I}\Sigma_n$ or PA

We now turn our attention to the families $\mathcal{P}_n$ and $\mathcal{E}_n \cap \mathcal{R}_{\mathrm{I}\Sigma_{n+k}}$. We already know a lot about them from Corollary 3.3.5. To squeeze a bit more, we applying our previous results on overlays, studied in Section 2.3:

3.4.1 Theorem. *Let $M \models \mathrm{I}\Sigma_{n+1}$, $n \geq 0$, $k \geq 1$, and let $\mathcal{R}$ be either $\mathcal{E}_n \cap \mathcal{R}_{\mathrm{I}\Sigma_{n+k}}$ or $\mathcal{P}_n$. Then $(a,b)^{\mathcal{R}}$ is non-empty iff $(a,b)^{\mathcal{R}}$ contains a closed subset $\mathcal{R}'$ isomorphic to the Cantor set; moreover, if $I \subseteq^e J$ are from $\mathcal{R}'$, then $I \preccurlyeq_{n+k} J$ (and $I \preccurlyeq J$ if $\mathcal{R} = \mathcal{P}_n$).*

Proof. Let $(a,b)^{\mathcal{R}} \neq \emptyset$. We will define a certain increasing sequence u of elements from $[a,b]$ with $\ell(u) > \mathbb{N}$ and define $\mathcal{R}'$ as the family of all cuts I such that $I \cap u$ is unbounded in I. Then, clearly, $\mathcal{R}'$ is closed under union and intersection and order-isomorphic to the set of all cuts of M in the interval $(0,\ell(u))$, hence also to the Cantor set.

Let $\Sigma_m^{<z}$ denote the set of all formal Σ_m formulae f with $f < z$. We may define (formal) notions such as $\Sigma_m^{<z}$*-bounding* and $\Sigma_m^{<z}$*-diagonally indiscernible* as in Definition 2.3.1, using $\mathrm{Sat}_{\Sigma_n}(f, \langle\ldots\rangle)$.

Depending on the choice of $\mathcal{R}$, let $\xi(u,a,b)$ be a Σ_{n+1}-formula expressing:

1) for $\mathcal{R} = \mathcal{E}_n \cap \mathcal{R}_{\mathrm{I}\Sigma_{n+k}}$: $\xi(u,a,b) \overset{\mathrm{df}}{\iff}$ 'u is an increasing sequence of elements from $[a,b]$ coding a $\Sigma_n^{<\ell(u)}$-bounding set of $\Sigma_n^{<\ell(u)}(v_0;v_1,\ldots,v_k)$-diagonally indiscernible elements',

2) for $\mathcal{R} = \mathcal{P}_n$: $\xi(u,a,b) \overset{\mathrm{df}}{\iff}$ 'u is an increasing sequence of elements from $[a,b]$ coding a $\Sigma_n^{<\ell(u)}$-bounding set of $\Delta_0^{<\ell(u)}$-diagonally indiscernible elements'.

If $M \models \xi(u,a,b)$ for u with $\ell(u) > \mathbb{N}$, then $\mathcal{R}' \overset{\mathrm{df}}{=} \{I \mid u \cap I \text{ is unbounded in } I\}$ has the required properties by Theorem 2.3.2, a) (or b) and c) for $\mathcal{P}_n$) applied on the overlay $\mathcal{O} \overset{\mathrm{df}}{=} \{U\}$ where $U \overset{\mathrm{df}}{=} \{(u)_i \mid \mathbb{N} < i \text{ and } \mathbb{N} < \ell(u)-i\}$. It suffices to show that for every $l \in \mathbb{N}$, $M \models (\exists u)(\ell(u) = l \wedge \xi(u,a,b))$ and apply Σ_{n+1}-overspill in M to obtain u of non-standard length.

Fix some $I \in (a,b)^{\mathcal{R}}$ and l standard. We discuss the two cases separately:

1) $\mathcal{R} = \mathcal{P}_n$: By Theorem 2.3.2 b), there is a Δ_0-diagonally indiscernible Σ_n-bounding overlay $\mathcal{O}$ in M, unbounded in I. Since l is finite, there is some $X \in \mathcal{O}$ that is a $\Sigma_n^{<l}$-bounding $\Delta_0^{<l}$-diagonally indiscernible elements. X is unbounded in I and we may take u as the first l elements from X (since l is truly finite, we may code u in I).

2) $\mathcal{R} = \mathcal{E}_n \cap \mathcal{R}_{\mathrm{I}\Sigma_{n+k}}$: In I, using IΣ_{n+1} we may find an unbounded Δ_{n+1}-set X that is $\Sigma_n^{<l}$-bounding in I. As $I \preccurlyeq_n M$, X (as a subset of M) is $\Sigma_n^{<l}$-bounding in M. Let D be a diagonal partition of $\langle X\rangle^k$ defined by

$$u\, D_t\, v \overset{\mathrm{df}}{\iff} (\forall y < t) \bigwedge_{f \in \Sigma_n^{<l}} [\mathrm{Sat}_{\Sigma_n}(f, \langle y\rangle ^\frown u) \leftrightarrow \mathrm{Sat}_{\Sigma_n}(f, \langle y\rangle ^\frown v)], \quad t \in X,\, u,v \in \langle X\rangle^k.$$

D is clearly Δ_{n+1} in I and Δ_1-estimated ($\|D_t\| \le 2^{lt}$). By 2.1.6 b), X has a D-homogeneous (in I) subset $u \subseteq X$ with $|u| = l$; u is $\Delta_0^{<l}(1;k)$-diagonally indiscernible in I (and hence in M).

□

3.4.2 Remarks.

a) If $M \models \mathrm{I}\Sigma_{n+1}$, then $\mathcal{E}_n \subseteq \mathcal{R}_{\mathrm{I}\Sigma_n}$, so in the last theorem, the case of $k = 0$ is trivial.

b) If $\xi(u,a,b)$ is the formula from the last proof (for either $\mathcal{R}$), then

$$Y(x,y) = \max\{z \le b \mid (\exists u)(\xi(u,x,y) \wedge \ell(u) = z)\}$$

is a Σ_{n+1}-indicator for $\mathcal{R}$ (the quantifier $(\exists u)$ can be bounded by a Δ_1-definable function of y).

c) $\mathcal{E}_n \cap \mathcal{R}_{\mathrm{B}\Sigma_{n+k+1}}$ and $\mathcal{E}_n \cap \mathcal{R}_{\mathrm{I}\Sigma_{n+k}}$ are symbiotic families in models of IΣ_{n+1}. This is a consequence of the last theorem and 1.8.4.

3.5 Isomorphic cuts

In this section, we study the family $\mathcal{I}_n$. First, we abbreviate the notation:

3.5.1 Definition. If A is a substructure of B, $a \in A$ and $b \in B$, we write $\langle A, a\rangle \cong_n \langle B, a\rangle$ if $A \preccurlyeq_n B$ and $\langle A, a\rangle \cong \langle B, b\rangle$ (i.e. there is an isomorphism of A and B sending a on b).

The main tool in this part will be the following well-known fundamental lemma:

3.5.2 Lemma (*Friedman*). *Let M be a countable model of* PA*, $n \geq 0$ and $a, b, c \in M$. Then the following conditions are equivalent:*

a) *there exists $I \subset^e M$ such that $I < b$ and $\langle I, a\rangle \cong_n \langle M, c\rangle$*

b) *$M \models (\exists \bar{x})\theta(\bar{x}, c) \to (\exists \bar{x} < b)\theta(\bar{x}, a)$ for any Π_n formula $\theta(\bar{x}, y)$.*

Proof. See [Kay91b, Theorem 12.3.]. □

3.5.3 Corollary. *Let M be a countable model of* PA*, let a, b, $n \geq 0$. Then the following are equivalent:*

a) *there exists an initial substructure $I < b$ of M such that $\langle M, a\rangle \cong_n \langle I, a\rangle$,*

b) *$M \models (\exists \bar{x})\theta(\bar{x}, a) \to (\exists \bar{x} < b)\theta(\bar{x}, a)$ for any Π_n formula $\theta(\bar{x}, y)$.* □

3.5.4 Proposition. *Let $\mathbb{N} \neq I \in \mathcal{P}_n$ and $I \swarrow \mathcal{I}_n$. Then $\mathcal{I}_n \nearrow I$.*

Proof. It follows from 3.5.3 that there is a Π_{n+1} formula $\xi(x, y, z)$ such that for every non-standard model $N \models$ PA, $N \models \xi(a, b, c)$ for $a, b, c \in N$ with c, b non-standard if and only if there is an n-elementary initial substructure I of N such that $I < b$ and $\langle N, a\rangle \cong \langle I, a\rangle$. By Friedman's theorem for any $N \models$ PA and arbitrary large $a \in N$ there exists an initial substructure $I \preccurlyeq_n N$ satisfying $\langle N, a\rangle \cong \langle I, a\rangle$.

Let $I \subset^e M$ satisfy the hypothesis of the claim and fix some $a \in I$. We may assume the formula $\xi(x, y, z)$ to be of the form $(\forall u)\theta(x, y, z, u)$ with $\theta \in \Sigma_n$. Since $I \models$ PA, there is an initial substructure $I' \preccurlyeq_n I$ with $\langle I', a\rangle \cong \langle I, a\rangle$. Hence, for some non-standard $b, c \in I$, $I \models (\forall u)\theta(a, b, c, u)$. Thus, using n-elementarity of I in M and overspill in M, we have $M \models (\forall u < d)\theta(a, b, c, u)$ for some $d \in M \setminus I$. Since $I \swarrow \mathcal{I}_n$, there exists some $J \in \mathcal{I}_n$ with $I \subset^e J < d$. By n-elementarity of J, $J \models (\forall u)\theta(a, b, c, u)$. It follows that there exists an initial substructure $J' \preccurlyeq_n J$ with $J' < b$ and $\langle J', a\rangle \cong \langle J, a\rangle$; we thus have $M \cong J \cong J' \preccurlyeq_n J \preccurlyeq_n M$ so $J' \in \mathcal{I}_n$ and $a \in J' \subset^e I$, which finishes the proof. □

We now also immediately obtain the following provisional information about the set $\bigcap \mathcal{I}_n$, which we fully characterize later.

3.5.5 Corollary. *If $\bigcap \mathcal{I}_n \neq \mathbb{N}$, then $\bigcap \mathcal{I}_n \in \mathcal{E}_n \setminus \mathcal{P}_n$.* □

Recall from 1.8.10 that for $X \subseteq M$ and a set of $\mathcal{L}^{Ar}$-formulae Γ,

$$I_\Gamma(M;X) \stackrel{\text{df}}{=} \{x \in M \mid M \models x < a \text{ for some } a \in \mathrm{Dfe}_\Gamma(M;X)\}.$$

3.5.6 Proposition. *For every $I \subseteq^e M$ and $n \geq 0$, $I \preccurlyeq_{n+1} M$ iff for every $a \in I$, $I_{\Sigma_{n+1}}(M;a) \subseteq I$.*

Proof. Left-to-right: Let $a \in I \preccurlyeq_{n+1} M$, $b \in I_{\Sigma_{n+1}}(M;a)$, and let $\varphi(x,y)$ be a Σ_{n+1}-formula such that $M \models \varphi(b,a) \wedge (\forall x)(\varphi(x,a) \to x = b)$. Then $M \models (\exists x)\varphi(x,a)$, hence by $(n+1)$-elementarity of I, $I \models \varphi(b',a)$ for some $b' \in I$. But then again $M \models \varphi(b',a)$, so $b' = b$ and $b \in I$ as required.

Conversely, assume $I_{\Sigma_{n+1}}(M;a) \subseteq I$ for all $a \in I$. Clearly, I is closed under operations. Let $\bar{a} \in I$ and let $\varphi(x,\bar{y})$ be a Π_n-formula such that $M \models (\exists x)\varphi(x,\bar{a})$. Let $b \stackrel{\text{df}}{=} \mu x : \varphi(x,\bar{a})$. Then $b \in I_{\Sigma_{n+1}}(M;\bar{a})$. Since I is closed under operations, we may use some trivial coding to represent $\bar{a}$ as an element $c \in I$ and see that $b \in I_{\Sigma_{n+1}}(M;c)$. Then, by the premise, $b \in I$. □

3.5.7 Proposition. *Let $n \geq 0$, $a,b \in M \models \mathrm{B}\Pi_n$. Then $b \in \mathrm{Dfe}_{\Delta_{n+1}}(M;a)$ iff $h(b) = b$ for every every isomorphism $h : M \to I$ with $I \preccurlyeq_n^e M$ and $h(a) = a$.*

Proof. Let $b \in \mathrm{Dfe}_{\Delta_{n+1}}(M;a)$. Then some Σ_{n+1}-formula $\varphi(x,y)$ defines b in M over a. In particular, we have

$$M \models \varphi(b,a) \wedge (\forall y)(\varphi(y,a) \to y = b).$$

Let $I \in \mathcal{I}_n^a$ be arbitrary and let $h : M \to I$ be an arbitrary isomorphism of $\langle M,a\rangle$ and $\langle I,a\rangle$. Let $b' = h(b)$. Then $I \models \varphi(b',a)$. Since φ is Σ_{n+1} and $I \preccurlyeq_n M$, we have $M \models \varphi(b',a)$ and hence $b' = b$.

Conversely, suppose $b \notin \mathrm{Dfe}_{\Delta_{n+1}}(M;a)$. We find h and I so that $h(b) \neq b$. Let

$$p(x) \stackrel{\text{df}}{=} \{\varphi(x,a) \mid \varphi \in \Sigma_{n+1} \text{ and } M \models \varphi(b,a)\} \cup \{x \neq b\}.$$

Clearly $p(x)$ is a type over M of bounded complexity, codable in M, and finitely satisfied in M (since otherwise some $\varphi_1(x,a) \wedge \ldots \varphi_k(x,a)$ with $\varphi_i(x,a) \in p(x)$ Σ_{n+1}-defines b in M over a). It follows that $p(x)$ is realized in M by some b' (c.f. e.g. [Kay91b, 12.1 and 12.2]). Since $M \models \mathrm{B}\Pi_n$, there exists $c \in M$ such that the following holds for every Π_n formula $\psi(\bar{u},x,y)$:

$$M \models (\exists \bar{u})\psi(\bar{u},a,b') \to (\exists \bar{u} < c)\psi(\bar{u},a,b').$$

Since b and b' satisfy in M the same Σ_{n+1} formulae over a, we have for every Π_n-formula

$$M \models (\exists \bar{u})\psi(\bar{u},a,b) \to (\exists \bar{u} < c)\psi(\bar{u},a,b').$$

By Corollary 3.5.3 (and an obvious use of coding on the pairs $\langle a,b\rangle$ and $\langle a,b'\rangle$), there exists $I \prec_n^e M$ such that $a,b' \in I < c$ and $\langle M,a,b\rangle \cong \langle I,a,b'\rangle$, which completes the proof. □

We now investigate a localized version of the family $\mathcal{I}_n$, namely the family of initial substructures of M satisfying $\langle I, a\rangle \cong_n \langle M, a\rangle$ for some $a \in M$. For this we introduce:

3.5.8 Definition. Let $n \geq 0$ and $a \in M$. Define

$$\mathcal{I}_n^a \stackrel{\text{df}}{=} \{I \subseteq^e M \mid \langle I, a\rangle \cong_n \langle M, a\rangle\}.$$

In particular, $\mathcal{I}_n^0 = \mathcal{I}_n$.

3.5.9 Proposition. *Let $n \geq 0$, $a \in M$. Then $\bigcap \mathcal{I}_n^a = I_{\Sigma_{n+1}}(M; a)$.*

Proof. For $\supseteq$, let $e \in \mathrm{Dfe}_{\Sigma_{n+1}}(M; a)$ and $I \in \mathcal{I}_n^a$. We show that $e \in I$. Let $(\exists y)\varphi(x, y, a)$ with $\varphi \in \Pi_n$ be a Σ_{n+1}-formula defining e in M over $\{a\}$. Since $\langle I, a\rangle \cong_n \langle M, a\rangle$, $I \models (\exists x, y)\varphi(x, y, a)$. Let $c, d \in I$ be such that $I \models \varphi(c, d, a)$. Since $I \preccurlyeq_n M$, we have $M \models \varphi(c, d, a)$, hence $M \models (\exists y)\varphi(c, y, a)$ which yields $c = e$, so $e \in I$.

Conversely, let $b > I_{\Sigma_{n+1}}(M; a)$. We show that there is an $I \in \mathcal{I}_n^a$ with $b > I$. By Corollary 3.5.3, we only need to check that for every Π_n-formula $\varphi(x, y)$ satisfies $M \models (\exists x)\varphi(x, a) \rightarrow (\exists x < b)\varphi(x, a)$. Assume otherwise; that is, let us for some Π_n-formula $\varphi(x, y)$ have

$$M \models (\exists x)\varphi(x, a) \wedge (\forall x < b)\neg\varphi(x, a). \tag{3.7}$$

By Π_n-induction in M, we can take $b_0 \stackrel{\text{df}}{=} \mu x : \varphi(x, a)$. Then $b_0 \in \mathrm{Dfe}_{\Sigma_{n+1}}(M; a)$, so $b_0 < b$ in contradiction with (3.7). □

3.5.10 Corollary.

a) *$\mathbb{N} \neq I \preccurlyeq_{n+1} M$ iff for all $a \in I$, there exists $J \subset^e I$ with $\langle J, a\rangle \cong_n \langle M, a\rangle$.*

b) *$\mathbb{N} \preccurlyeq_{n+1} M$ iff $\mathbb{N} \swarrow \mathcal{I}_n$.*

Proof. a) If a is trivial, the claim is trivial; for $a > \mathbb{N}$ it follows immediately from 3.5.6, 3.5.9 and 1.8.11 b).

b), apply 3.5.6 on $I = \mathbb{N}$ and 3.5.9 on $a \in \mathbb{N}$. □

The following result indicates that there is a substantial difference between the families $\mathcal{I}_n^a$ and $\{I \in \mathcal{I}_n \mid a \in I\}$.

3.5.11 Proposition. *Let $M \models \mathrm{PA}$ be non-standard. For every $c \in M$, there are $a, b \in M$ such that $c < a < b$ and $(a, b)^{\mathcal{I}_n} \neq \emptyset$, but $(a, b)^{\mathcal{I}_n^a} = \emptyset$.*

Proof. Let $c \in M \setminus \mathbb{N}$. Using $\mathrm{Sat}_{\Sigma(n+1)}$, it is easy to verify that the cut $I_c \stackrel{\text{df}}{=} \bigcup_{x \leq c} I_{\Sigma_{n+1}}(M; x)$ is bounded in M. Let $I_c < d \in M$. Then $J \stackrel{\text{df}}{=} I^+_{d, \mathcal{I}_n}$ is the intersection of all $I \in \mathcal{I}_n$ containing d. We show that for some $a \in J$,

$I_{\Sigma_{n+1}}(M;a) \not\subseteq J$, which gives both $c < a$ (by our choice of d) and using 3.5.9 also

$$\bigcap\{I \in \mathcal{I}_n \mid a \in I\} = I^+_{a,\mathcal{I}_n} = I^+_{d,\mathcal{I}_n} = J \subsetneq I_{\Sigma_{n+1}}(M;a) = \bigcap \mathcal{I}^a_n. \tag{3.8}$$

Indeed, suppose $I_{\Sigma_{n+1}}(M;a) \subseteq J$ for all $a \in J$; then $J \preccurlyeq_{n+1} M$, by 3.5.6, and $J = I_{\Sigma_{n+1}}(M;d)$, since by the assumption, $I_{\Sigma_{n+1}}(M;d) \subseteq J$ and $J = I^+_{d,\mathcal{I}_n} \subseteq \bigcap \mathcal{I}^d_n = I_{\Sigma_{n+1}}(M;d)$, by 3.5.9. We thus have $I_{\Sigma_{n+1}}(M;d) \models \mathrm{I}\Sigma_{n+1}$, contradicting 1.8.11. This proves (3.8). Now, any $b \in I_{\Sigma_{n+1}}(M;a) \setminus J$ has the required property. □

3.5.12 Corollary. $\mathcal{E}_{n+1} \subsetneq \overline{\mathcal{I}}_n$. □

3.5.13 Remark. R. Kaye claims in the Exercise 12.8 in his book [Kay91b] that '*if $M \models$ PA is countable and non-standard, $a, b \in M$, $n \in N$, and there is $I \preccurlyeq_n M$ with $a \in I < b$ and $M \cong I \subseteq^e M$, then there exists $J \preccurlyeq_n M$ with $a \in J < b$, $J \subseteq^e M$, and an isomorphism $h : M \to J$ such that $h(a) = a$*'. Our last Proposition clearly falsifies his claim. In fact, it can be refuted in an even more straightforward way, just by using 3.5.3 (or Theorem 12.3. of [Kay91b]) and Vaught-Tarski's test to show that if the claim were true, then every non-standard initial substructure I of M such that $M \cong I$ would satisfy $I \preccurlyeq M$, which is of course contradictory.

3.5.14 Proposition. *Let $M \models \mathrm{I}\Sigma_{n+1}$. Let Y be a Δ_{n+1}-indicator Y in M for a family $\mathcal{R}$ such that $\mathcal{R} \nearrow M$. Let $p \in M$ be such that for some non-standard $c \in \mathrm{Dfe}_{\Sigma_{n+1}}(M;p)$, $M \models (\forall x)(\exists y)Y(x,y) > c$. Then $\mathcal{R} \nearrow I_{\Sigma_{n+1}}(M;p)$.*

(The last assumption is satisfied for example, if $M \models \mathrm{I}\Sigma_{n+2}$ and M has a downward cofinal sequence of Π_n-minimal elements in M. Then the required c can be found for arbitrary p.)

Proof. Let $\mathcal{R}$, Y be as above. First, if $M \models \mathrm{I}\Sigma_{n+2}$, then $M \models (\forall x)(\exists y)Y(x,y) > c$ for some non-standard c by overspill; thus if M has a downward cofinal sequence of Π_n-minimal elements, we may find such a non-standard c in $\mathrm{Dfe}_{\Sigma_{n+1}}(M;\emptyset)$. This proves the sequel. Let $p, c \in M$ meet the assumptions and let $a \in I_{\Sigma_{n+1}}(M;p)$. We must find some $I \in \mathcal{R}$ with $a \in I \subset^e I_{\Sigma_{n+1}}(M;p)$. Let $b = \mu y : Y(a,y) > c$. Note that $b \in \mathrm{Dfe}_{\Sigma_{n+1}}(M;c)$ and since $c \in \mathrm{Dfe}_{\Sigma_{n+1}}(M;p)$, we have $b \in \mathrm{Dfe}_{\Sigma_{n+1}}(M;p)$. Now, $Y(a,b) > \mathbb{N}$, so $a \in I < b$ for some $I \in \mathcal{R}$ as required. □

3.5.15 Remark. Models of PA with downward cofinal sequences of Π_n-minimal elements exist by [McA78, Theorem 4.4]; McAloon proved that every first-order theory T extending PA consistent with the set of true Π_{n+2} sentences and represented in $\mathbb{N}$ by a Π_n formula has such a model and every recursively axiomatized extension of PA has a model with a downward cofinal sequence of Δ_1-definable elements.

3.5.16 Corollary. *$\mathcal{P}_n \nearrow I$ does not necessarily imply $\mathcal{I}_n \nearrow I$.*

Proof. Assume $M \models \mathrm{I}\Sigma_{n+2}$ has a downward cofinal sequence of Π_n-minimal elements. Then by 3.5.14, we have $\mathcal{P}_n \nearrow I_{\Sigma_{n+1}}(M;\emptyset) = \bigcap \mathcal{I}_n^0 = \bigcap \mathcal{I}_n \neq \mathbb{N}$. □

3.6 Cuts determined by Σ_n-definable elements

We have seen that elements of the family $\mathcal{D}_n \stackrel{\mathrm{df}}{=} \{I_{\Sigma_{n+1}}(M;a) \mid a \in M\}$ are exactly intersections of the form $\bigcap \mathcal{I}_n^a$ with $a \in M$; in particular $\mathcal{D}_{n+1} \subseteq \overline{\mathcal{I}_n}$.

3.6.1 Proposition. *For every $n \geq 0$ the family $\mathcal{D}_{n+1}$ has a Δ_{n+2}-indicator in models of $\mathrm{I}\Sigma_{n+1}$.*

Proof. Let $Y(x,y) = z$ be the following formula:

$$z = \max\{z' \leq y \mid (\exists w \in (x,y))[(\forall f < z')((\exists v)\mathrm{Sat}_{\Pi_n}(f,\langle w,v\rangle) \rightarrow (\exists v < y)\mathrm{Sat}_{\Pi_n}(f,\langle w,v\rangle))]\} \tag{3.9}$$

Note that this formula is Δ_{n+2} in $\mathrm{B}\Sigma_{n+1}$. We claim that $Y(x,y) = z$ is an indicator for the family $\mathcal{D}_{n+1}$. Indeed, if $Y(a,b) = c \geq \mathbb{N}$, then according to (3.9), there exists $d \in (a,b)$ such that for every Π_n formula $\varphi(w,v)$ $(\exists v)\varphi(v,d)$ implies $(\exists v < b)\varphi(v,d)$. In particular, if $e \in \mathrm{Dfe}_{\Pi_n}(M;d)$, then there is a Π_n formula $\varphi(w,v)$ such that $M \models \varphi(d,e) \wedge (\forall v)(\varphi(d,v) \rightarrow v = e)$. It follows that $e < b$. Since $\mathrm{Dfe}_{\Pi_n}(M;d)$ is cofinal in $I_{\Sigma_{n+1}}(M;d)$, we have $a \in I_{\Sigma_{n+1}}(M;d) < b$.

Conversely, let $a \in I_{\Sigma_{n+1}}(M;d) < b$ for some $d \in M$. Then there exists $e \in (a,b) \cap \mathrm{Dfe}_{\Pi_n}(M;d)$. Clearly, $\mathrm{Dfe}_{\Pi_n}(M;e) \subseteq \mathrm{Dfe}_{\Pi_n}(M;d)$, so $a \in I_{\Sigma_{n+1}}(M;e) < b$. Let $\varphi(w,v)$ be a Π_n formula such that $M \models (\exists v)\varphi(e,v)$ and let $v_0 \stackrel{\mathrm{df}}{=} \mu v : \varphi(e,v)$. Then $v_0 \in I_{\Sigma_{n+1}}(M;e) = I_{\Pi_n}(M;e)$. In particular, $v_0 < b$. Hence the subformula in square brackets in (3.9) holds for every z' standard and w,x,y equal to e,a,b respectively. In particular, $M \models Y(a,b) > \mathbb{N}$ (the subformula in square brackets is Π_{n+1}, so $\mathrm{I}\Sigma_{n+1}$ suffices to define $Y(a,b)$). □

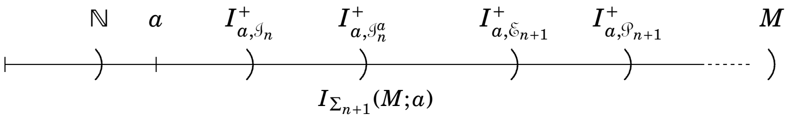
3.6.2 Remark. There can be no Σ_{n+1} or Π_{n+1} indicator for $\mathcal{D}_{n+1}$ in models of PA since otherwise, as we show below, the families $\mathcal{D}_{n+1}$ and $\mathcal{P}_n$ would be symbiotic. This, however, is not the case since $\overline{\mathcal{P}_n} \neq \overline{\mathcal{D}_{n+1}}$ (this will be discussed in the concluding Section 3.7 that follows). Suppose Y is either a Σ_{n+1}- or Π_{n+1}-indicator for $\mathcal{D}_{n+1}$ in models of PA. Let $a \in I \in \mathcal{P}_n$ be non-standard. Then there are some $b,c \in I$ non-standard such that $I \models Y(a,b) = c$. If Y is Σ_{n+1}, then, by Σ_n-elementarity, $M \models Y(a,b) = c$, and consequently $\mathcal{D}_{n+1} \nearrow I$, so $I \in \overline{\mathcal{D}_{n+1}}$. For a Π_{n+1} indicator, we arrive at the same conclusion because $I \models Y(a,b) = c$ implies $M \models Y(a,b) \neq n$ for every $n \in \mathbb{N}$.

3.6.3 Remark. We do not know any Δ_{n+2} indicator for the family $\mathcal{I}_n$. Of course, it would exist if $\mathcal{D}_{n+1}$ and $\mathcal{I}_n$ turned out to be symbiotic, but as we remark later, this remains unclear. A Δ_{n+3} indicator can be constructed using Friedman's theorem:

$$Y(x,y) \stackrel{\text{df}}{=} \max\{z' \le y \mid (\exists w_0)(\exists w \in (x,y))[(\forall f < z')((\exists v)\mathrm{Sat}_{\Pi_n}(f,\langle w_0,v\rangle) \to (\exists v < y)\mathrm{Sat}_{\Pi_n}(f,\langle w,v\rangle))]\}$$

3.7 Conclusion to the Chapter

For a given non-standard element a, the cuts $I^+_{a,\mathcal{R}}$ for the families studied in this chapter are ordered according to the following figure:



For some a, the first two cuts on the picture may actually equal (for example, when a is Σ_{n+1}-definable in M), however, by 3.5.11, $I^+_{a,\mathcal{I}^a_n} \ne I^+_{a,\mathcal{I}_n}$ for unboundedly many a's. All other cuts on the picture are distinct.

In terms of the closure of the families of cuts in question, we have

$$\overline{\mathcal{P}_n} \supsetneq \overline{\mathcal{I}_n} \supseteq \overline{\mathcal{D}_{n+1}} \supsetneq \mathcal{E}_{n+1} \supsetneq \overline{\mathcal{P}_{n+1}}.$$

The inclusions are clear, only their strictness deserves some explanation:

- $\overline{\mathcal{P}_n} \supsetneq \overline{\mathcal{I}_n}$: for example, if $\mathbb{N} \preccurlyeq_n M$ but $\mathbb{N} \not\preccurlyeq_{n+1} M$, then $\mathbb{N} = I^+_{0,\mathcal{P}_n} \ne I^+_{0,\mathcal{I}_n}$. C.f. also Corollary 3.5.16.
- $\overline{\mathcal{D}_{n+1}} \supsetneq \mathcal{E}_{n+1}$: $I_{\Sigma_{n+1}}(M;a)$ for a non-standard belongs to the family on the left by definition, but not to the family on the right (Fact 1.8.11).
- $\mathcal{E}_{n+1} \supsetneq \overline{\mathcal{P}_{n+1}}$: see 3.3.7.
- $\overline{\mathcal{I}_n} \supseteq \overline{\mathcal{D}_{n+1}}$: although we have proved that for some $a \in M$ there are cuts from $\mathcal{I}_n$ below $I_{\Sigma_{n+1}}(M;a)$, we did not actually prove that there is no $b \in I_{\Sigma_{n+1}}(M;a)$ with $a < b$ and $I_{\Sigma_{n+1}}(M;b) \subset I_{\Sigma_{n+1}}(M;a)$. Whether such a b exists for every a we leave as an open problem (although the solution might turn out to be easy). To that we remark, that if M includes a cofinal subset X such that $\mathrm{Dfe}_{\Sigma_1}(N;a) \subseteq M$ whenever $a \in X$ and $M \subset^e N \models \mathrm{PA}$, then $\mathcal{I}_0$ and $\mathcal{D}_1$ are symbiotic.

3.7.1 Remark. The inclusion $\overline{\mathcal{I}_n} \supseteq \overline{\mathcal{P}_{n+1}}$ was proved by Ignjatović's in [Ign86]; we have presented a finer result slicing $\mathcal{E}_{n+1}$ strictly in between.

3.7.2 Remark. Let $M \models \mathrm{PA}$ be non-standard, $I \subseteq^e M \cong_n I$, $a,b \in M$, and $a \in I < b$. Do these assumption imply either of the following?

a) There exists exist some $c < b$ and $J \subseteq^e M$ such that $a \le c \in J < b$ and $\langle J,c\rangle \cong_n \langle M,c\rangle$.

b) There exist some $c < b$ and $J \subseteq^e M$ such that $\langle J,a\rangle \cong_n \langle M,c\rangle$?

Chapter 4

The Stone space of the algebra of definable subsets

In this chapter, we study non-principal ultrafilters on the algebra $\mathcal{D}(M,M)$ of definable subsets of some countable model $M \models \mathrm{PA}$, confronting properties of the ultrafilters with properties of the corresponding cuts.

There is a natural analogy with a classical topic, the study of $\beta\omega$, the Stone space of $\mathcal{P}(\omega)$; here ω is replaced by the model M and the full algebra $\mathcal{P}(\omega)$ by a countable subalgebra with arithmetical comprehension; the notion of finiteness is replaced with M-finiteness or I-finiteness for some cut I of M. This analogy brings both similarities and differences (for example, while in $\beta\omega$ selective ultrafilters are exactly the Ramsey ultrafilters [Boo71], here these two notions do not coincide with a prominent counterexample where $\mathbb{N}$ is not a strong cut of M).

This field has been studied before, most notably by Kirby in [Kir82] and esp. [Kir84], where he introduced and analyzed the notions of definable and weakly definable ultrafilters (see also [KP86], [Sch93]), giving a partial solution to a related 2-3-problem.

The topic of this chapter relates closely to the study of types of a complete extension T of PA, pioneered by Gaifman [Gai76]. If M is a minimal model of T and $p(x)$ is a complete type in T, then $p(x)$ determines uniquely an ultrafilter on the algebra of definable subsets of M. Conversely, every such ultrafilter determines uniquely a type in T. These connection have been further pursued by Kirby in [Kir84]. For arithmetic the situation is simplified by the fact that due to coding, it suffices to consider 1-types instead of n-types for all $n \in \omega$ as usual in general model theory.

This chapter describes of the topic from a non-standard viewpoint: we situate the countable base model M into some $\aleph_1$-saturated elementary extension C, which can be seen as a 'big model', a tool widely exploited in modern model theory allowing useful descriptions of different situations. In particular, this allows us to replace the algebra $\mathfrak{D}(M,M)$ with its isomorphic copy $\mathfrak{D}(C,M)$ and identify ultrafilters on the latter algebra with their non-empty intersections, called *monads*.

This description establishes a close relationship with the so-called basic equivalences studied in the context of Vopěnka's Alternative Set Theory (AST) [Vop79], see e.g. [ČK83], [ČV87], [ČV89]; specifically, we use here a certain description of minimal monads as stated in Theorem 4.3.40.

In Section 4.2, we introduce the basic notions, namely monads and gaps, and develop some essential tools for their study, namely an overspill principle for certain Π-like properties of monads that covers most combinatorial notions studied in this Chapter and then standard tools like the Rudin-Keisler ordering on monads and model extensions that go hand in hand with each other as expected.

In Section 4.3 we first briefly survey semi-regular, regular and strong cuts as introduced by Kirby and Paris, and then characterize these cuts in terms of monads and gaps.

The chief new results are contained in Section 4.4, where we return to the topic of diagonal indiscernibility explored previously in Chapter 2, this time in a new context of monads. We analyze in detail the combinatorial strength of diagonally indiscernible monads in Theorems 4.4.6, 4.4.11, and 4.4.16. In the second part of the section, we investigate relationships between relatively large RK-minimal monads (corresponding to selective ultrafilters), relatively large p-monads, and regular monads, proving that, in this order, the properties are of strictly decreasing strength. In particular, we show that p-monads that are not RK-minimal form a dense subset of every gap that contains a regular monad. This result contrasts with the classical situation in $\beta\omega$, where the proof of the existence of p-point ultrafilters that are not selective relies on assumptions beyond ZFC, such as Martin's axiom ([Boo71]).

To make the text to a large extent self-contained, this Chapter is accompanied by Appendix B, where we reprove the well-known fact that a cut I is strong iff $I^* \models A\mathrm{CA}_0$ and a theorem derived by Kirby, which complements our investigation of diagonal partition properties of monads.

4.1 Definitions and preliminaries

Recall that by *C is $\aleph_1$-saturated* it is meant that every 1-type $p(x)$ in C over a countable set of parameters $X \subseteq C$ is realized in C, i.e. $C \models p(a)$ for some $a \in C$. Equivalently, every countable system of definable subsets of C (from parameters) with the finite intersection property has non-empty intersection.

Assumption. Throughout this chapter, M is a fixed countable model of PA and C is a fixed $\aleph_1$-saturated elementary extension of M.

4.1.1 Definition. Let B be a boolean algebra. The *Stone space of B*, $\mathfrak{S}(B)$, is a set of all ultrafilters on B augmented with the topology whose (cl)open basis consists of all sets of the form $\{\mathfrak{U} \in \mathfrak{S}(B) \mid b \in \mathfrak{U}\}$ with $b \in B$.

4.1.2 Fact. *The topology of $\mathfrak{S}(B)$ is Hausdorff, compact, and totally disconnected.* □

4.1.3 Definition. For a model $N \models \mathfrak{L}^{Ar}$, a class Γ of $\mathfrak{L}^{Ar}$-formulae closed under propositional combinations, and a set $X \subseteq N$,

$$\mathfrak{D}_\Gamma(N,X) \stackrel{\mathrm{df}}{=} \{\varphi(N,\bar{p}) \mid \bar{p} \in X \text{ and } \varphi(x,\bar{y}) \in \Gamma\},$$

is the *set of Γ-definable subsets of N over parameters from X*; $\mathfrak{D}_\Gamma(N,X)$ forms a Boolean algebra of sets. We denote the Stone space of this algebra by $\mathfrak{S}_\Gamma(N,X)$.

Notation. In the most typical setting for this chapter, $N = C$. In that case we simplify the notation by omitting the reference to C, writing just $\mathfrak{D}_\Gamma(X)$ and $\mathfrak{S}_\Gamma(X)$ instead of $\mathfrak{D}_\Gamma(C,X)$ and $\mathfrak{S}_\Gamma(C,X)$; if further $X = M$, we reduce the notation to just $\mathfrak{D}_\Gamma$, $\mathfrak{S}_\Gamma$. If Γ consists of all $\mathfrak{L}^{Ar}$-formulae, we omit the subscript Γ. In particular,

$$\mathfrak{D} \stackrel{\mathrm{df}}{=} \mathfrak{D}_{\mathrm{Fm}(\mathfrak{L}^{Ar})}(C,M) \text{ and } \mathfrak{S} \stackrel{\mathrm{df}}{=} \mathfrak{S}_{\mathrm{Fm}(\mathfrak{L}^{Ar})}(M,C).$$

Since $M \preccurlyeq C$, a function sending each $X \in \mathfrak{D}$ to $X \cap M \in \mathfrak{D}(M,M)$ is an isomorphism of $\mathfrak{D}$ and $\mathfrak{D}(M,M)$:

$$\mathfrak{D} \cong \mathfrak{D}(M,M).$$

4.1.4 Proposition. *Every infinite set from $\mathfrak{D}$ can be divided into two infinite sets from $\mathfrak{D}$. Consequently, $\mathfrak{D}$ is the up to isomorphism unique countable atomic saturated Boolean algebra.*

Proof. The following is known: there is an up to isomorphism unique Boolean algebra B that is countable, atomic and satisfies the following *splitting property*: 'if the set of atoms below an $a \in B$ is infinite, then there are disjoint $a_1, a_2 \in B$ whose supremum is a such that the sets of atoms below a_1 and

a_2 are also infinite'. There exists a countable saturated model for the theory of atomic Boolean algebras; by saturation, this model has the splitting property. Consequently, every countable atomic Boolean algebra with the splitting property is saturated.

Now, $\mathfrak{D}$ is countable and atomic ($\{\min X\}$ being an atom under $X \in \mathfrak{D}$). If $X \in \mathfrak{D}$, let $X^{(0)} \stackrel{\text{df}}{=} \{\alpha \in X \mid |X \cap [0,\alpha]| \text{ is odd}\}$ and $X^{(1)} \stackrel{\text{df}}{=} \{\alpha \in X \mid |X \cap [0,\alpha]| \text{ is even}\}$. Then $X^{(0)} \cup X^{(1)} = X$, $X^{(0)} \cap X^{(1)} = \emptyset$, and if X is infinite, then so are $X^{(0)}, X^{(1)}$. Thus $\mathfrak{D}$ has the splitting property. □

4.2 The enlarged setting

We now have the basic situation $\langle M, \mathfrak{D}(M,M)\rangle$ naturally enlarged by $\langle C, \mathfrak{D}(C,M)\rangle$. In this section, we introduce basic notions related to this enlarged setting, such as monads and gaps, and develop some necessary tools and methods of description, namely an overspill principle for properties of monads, Rudin-Keisler pre-order, and model extensions. We show that the Rudin-Keisler pre-order is closely related to model extensions in a way usually expected in similar contexts.

Equivalences, monads, and gaps

4.2.1 Definition. Let Γ be a class of $\mathfrak{L}^{Ar}$-formulae and $X \subseteq C$ countable.

a) We define an equivalence $\sim_{\Gamma,X}$ on C by

$$\alpha \sim_{\Gamma,X} \beta \stackrel{\text{df}}{\Longleftrightarrow} (\forall Y \in \mathfrak{D}_\Gamma(X))(\alpha \in Y \leftrightarrow \beta \in Y).$$

b) Equivalence classes of $\sim_{\Gamma,X}$ are called *monads of* $\sim_{\Gamma,X}$ or just $\sim_{\Gamma,X}$-*monads*. (The notion is borrowed from AST, c.f. [ČK83]).

c) A $\sim_{\Gamma,X}$-monad is said to be *trivial*, if it is an atom of the algebra $\mathfrak{D}_\Gamma(X)$; otherwise we say that the monad is *non-trivial*.

d) The equivalence $\sim_{\Gamma,X}$ induces a natural topology on C whose basic open sets are the sets from $\mathfrak{D}_\Gamma(X)$. We refer to this topology as $\sim_{\Gamma,X}$-*topology* on C; thus we say, for example, that $Y \subseteq C$ is $\sim_{\Gamma,X}$-*closed* if it is a complement of an open set in the $\sim_{\Gamma,X}$-topology.

e) $Z \subseteq C$ is $\sim_{\Gamma,X}$-*figure* if $\alpha \sim_{\Gamma,X} \beta \in Z$ implies $\alpha \in Z$.

If $X \subseteq C$ and X is countable, there is a one-to-one correspondence between monads of $\sim_{\Gamma,X}$ and ultrafilters from $\mathfrak{S}_\Gamma(X)$: if $\mathfrak{U} \in \mathfrak{S}_\Gamma(X)$, then, by $\aleph_1$-saturation of C, $\bigcap \mathfrak{U}$ is non-empty and clearly a $\sim_{\Gamma,X}$-monad; conversely, every $\sim_{\Gamma,X}$-monad Z determines an ultrafilter

$$\mathfrak{U}(Z) \stackrel{\text{df}}{=} \{Y \mid Y \in \mathfrak{D}_\Gamma(X) \wedge Z \subseteq Y\}$$

Then $\mathfrak{U}(Z) \in \mathfrak{S}_\Gamma(X)$ and $\bigcap \mathfrak{U}(Z) = Z$. Moreover, $\mathfrak{U}(Z)$ is principal iff Z is a trivial $\sim_{\Gamma,X}$-monad.

Observe that the quotient $C/{\sim_{\Gamma,X}}$ with the topology induced by $\sim_{\Gamma,X}$-topology on C is homeomorphic to $\mathfrak{S}_\Gamma(X)$ via the correspondence between monads and ultrafilters outlined above. We therefore refer to this topology as the *Stone topology* of $C/{\sim_{\Gamma,X}}$.

$\sim_{\Gamma,X}$-figures are just unions of arbitrary subsets of the factor $C/{\sim_{\Gamma,X}}$. In particular, every $Y \in \mathfrak{D}_\Gamma(X)$ is a $\sim_{\Gamma,X}$-figure, as is every $\sim_{\Gamma,X}$-open or $\sim_{\Gamma,X}$-closed set.

Notation. From now on, we reserve the letters $a, b, \ldots$ (possibly with subscripts or primes) for elements of M and the symbols $\alpha, \beta, \ldots$ (possibly with subscripts or primes) for elements of C. When comparing elements from C, the symbol $<$ always refers to the ordering $<^C$. Since the model C underlines most our work, we shall from now implicitly consider all intervals of the form $[\alpha, \beta]$, (α, β) with $\alpha, \beta \in C$ as intervals in $\langle C, <^C \rangle$, even if $\alpha, \beta \in M$. We further abbreviate the symbols $\sup_{\leq^C}(Z)$ and $\inf_{\leq^C}(Z)$ for $Z \subseteq C$ (see the definitions on page 13) as $\sup Z$ and $\inf Z$.

Let $\Gamma_<$ be the class of all formulae in the language $\langle < \rangle$. The algebra $\mathfrak{D}_{\Gamma_<}$ is isomorphic to the interval algebra of $\langle M, <^M \rangle$; it coincides with the subalgebra of $\mathfrak{D}$ generated by intervals of the form $[a, \rightarrow)$ with $a \in M$.

4.2.2 Definition. We denote $\sim_{\Gamma_<}$ as $\Bumpeq$ and refer to non-trivial $\Bumpeq$-monads as *gaps*.† We reserve the symbol $\mathfrak{g}$ for denoting gaps.

For $\alpha < \beta$ from C, $\alpha \Bumpeq \beta$ iff $[\alpha, \beta] \cap M = \emptyset$; thus the trivial $\Bumpeq$-monads are of the form $\{a\}$ with $a \in M$, while non-trivial $\Bumpeq$-monads (gaps) are maximal convex subsets of $\langle C, < \rangle$ that are disjoint from M.

The factor $C/{\Bumpeq}$ is naturally ordered by representatives. Under this ordering, the set of all gaps, $C/{\Bumpeq} \setminus \{\{a\} \mid a \in M\}$, is order-isomorphic to the Cantor's set.

To each cut I of M naturally correspond two cuts of C: $\sup I$ and $\inf(M \setminus I)$. Their difference is the *gap of I*:

$$\mathfrak{g}_I \stackrel{\text{df}}{=} \inf(M \setminus I) \setminus \sup I = \bigcap \{[a, b] \mid a \in I \wedge b \in M \setminus I\}.$$

Correspondingly, each gap $\mathfrak{g}$ determines a unique cut of M:

$$I_\mathfrak{g} \stackrel{\text{df}}{=} \inf(\mathfrak{g}) \cap M = \sup(\mathfrak{g}) \cap M.$$

The gap $\mathfrak{g}_M$ is called the *unbounded gap* and is also denoted by $\mathfrak{g}_\infty$; all other gaps are *bounded*.

†In some literature the term *gap* is used for a different concept, namely for *interstical gaps*, also called *skies*, c.f. [KS06, p. 17].

For a gap $\mathfrak{g}$, we put

$$\mathfrak{D}(\mathfrak{g}) \stackrel{\text{df}}{=} \{X \in \mathfrak{D} \mid \mathfrak{g} \cap X \neq \emptyset\}.$$

(This slight abuse of notation is hopefully redeemed by the fact that the symbols for gaps are reserved so there should be no confusion with other usage of the symbol $\mathfrak{D}(\ldots)$. Observe that $\mathfrak{D}(\mathfrak{g})$ is not a Boolean algebra!)

4.2.3 Proposition. *$X \in \mathfrak{D}(\mathfrak{g})$ iff $X \in \mathfrak{D}$ and $X \cap I_{\mathfrak{g}}$ is cofinal in $I_{\mathfrak{g}}$.*

Proof. Immediate from $M \preccurlyeq C$ and $\aleph_1$-saturation of C. □

The larger the class Γ, the finer the equivalence $\sim_\Gamma$. For example: if $\Gamma_+ = \mathrm{Fm}((\langle 0, S, +, <\rangle))$, then applying the well known quantifier elimination theorem for Presburger arithmetic, we obtain

$$\alpha \sim_{\Gamma_+} \beta \text{ iff } \alpha \doteqdot \beta \wedge (\forall k \in \mathbb{N})(\alpha \equiv \beta \pmod{k}).$$

Our fundamental equivalence will be $\sim_{\Gamma,M}$ for $\Gamma \stackrel{\text{df}}{=} \mathrm{Fm}(\mathfrak{L}^{Ar})$. This equivalence represents the basic form of indiscernibility in the model C with respect to M, sometimes referred to as *1-indiscernibility*—if $\alpha \sim_{\Gamma,M} \beta$, then α cannot be distinguished from β by any $\mathfrak{L}^{Ar}$-formula with parameters from M. The equivalence refines both $\doteqdot$ and $\sim_{\Gamma_+}$; in particular, non-trivial $\sim_{\Gamma,M}$-monads are included in gaps.

4.2.4 Definition. We shall denote $\sim_{\Gamma,M}$ with $\Gamma \stackrel{\text{df}}{=} \mathrm{Fm}(\mathfrak{L}^{Ar})$ by just $\sim$ and refer to $\sim$-monads simply as *monads*. We reserve symbols $\mathfrak{m}$ and $\mathfrak{n}$ (possibly with subscripts or primes) for non-trivial monads. For a non-trivial monad $\mathfrak{m}$, the symbol $I_{\mathfrak{m}}$ will denote the cut $\inf(\mathfrak{m}) \cap M \subseteq^e M$. For $\alpha \in C \setminus M$, the symbol $\mathfrak{m}(\alpha)$ will denote the non-trivial monad containing α.

4.2.5 Proposition.

a) *Every infinite $X \in \mathfrak{D}$ intersects a gap. For every gap $\mathfrak{g}$ and every $X \in \mathfrak{D}(\mathfrak{g})$, $X \cap \mathfrak{g}$ decomposes into $2^{\aleph_0}$ monads.*

b) *Every infinite interval $[a,b]$ with $a,b \in M$ includes $2^{\aleph_0}$ gaps.*

c) *Every monad is both cofinal and coinitial in its gap.*

d) *Every monad is densely ordered by $<$.*

Proof. a) If $X \in \mathfrak{D}$ is infinite, let $Y \stackrel{\text{df}}{=} \{\alpha \in X \mid |X \cap [0,\alpha]| \in \mathbb{N}\}$. Then Y consists of the first ω elements from X in the order $<$ and $I = M \cap \sup Y$ is a cut of M. By saturation, $X \cap \mathfrak{g}_I \neq \emptyset$.

The proof that every gap decomposes into $2^{\aleph_0}$ monads is basically iteration of the splitting process described in the proof of 4.1.4 above. We postpone the details to 4.2.15 where we present a generalized version of the construction. Its application yielding our present claim is described in Example 4.2.16.

b) This has an elementary proof, but we may reuse our previous general results: By a), an infinite interval (a,b) with $a,b \in M$ intersects a gap. Thus, for the family $\mathfrak{R} \stackrel{\text{df}}{=} \{I \mid I \subseteq^e M\}$ of all cuts of M, $(a,b)^{\mathfrak{R}} \neq \emptyset$. The function $Y(x,y) = y - x$ is clearly an indicator for $\mathfrak{R}$. Thus, by 3.2.1, $(a,b)^{\mathfrak{R}}$ is of cardinality $2^{\aleph_0}$.

c) First, for every $X \in \mathfrak{D}(\mathfrak{g})$, $X \cap \mathfrak{g}$ is both cofinal and coinitial in $\mathfrak{g}$. Indeed, for every $a \in I$, $C \models (\exists x \in X)x > a$. The same holds in M and for $b \stackrel{\text{df}}{=} \min(X \cap (a,\rightarrow))$ we thus have $b \in M$, $b \leq \alpha$, so $b \in I$. This proves that $X \cap I$ is unbounded in I. Then $\beta \stackrel{\text{df}}{=} \max X \cap [a,\alpha)$ gives $\beta \in \mathfrak{g} \cap X$ with $\beta < \alpha$, proving coinitiality of $X \cap \mathfrak{g}$ in $\mathfrak{g}$. Cofinality is proved similarly.

For monads, the claim follows by $\aleph_1$-saturation: Let $\mathfrak{m} = \bigcap_{n \in \omega} X_n$ with $X_n \in \mathfrak{D}$ for $n \in \omega$ and let $\alpha \in \mathfrak{g}$. Then $[0,\alpha) \cap X_n \neq \emptyset$ for all $n \in \omega$. Hence by saturation, $[0,\alpha) \cap \mathfrak{m} \neq \emptyset$, so $\mathfrak{m}$ is coinitial in $\mathfrak{g}$. The proof of cofinality is similar.

d) To prove that $<$ is a dense order on $\mathfrak{m}$, it suffices to show that $X \cap (\alpha,\beta) \neq \emptyset$ for every $X \in \mathfrak{U}(\mathfrak{m})$ and $\alpha < \beta$ from $\mathfrak{m}$, and apply $\aleph_1$-saturation. Indeed, if $X \cap (\alpha,\beta)$ were empty, then α,β would each fall in a different block of the partition $\{X^{(0)}, X^{(1)}\}$ of X defined above. But as both blocks are in $\mathfrak{D}$, $\mathfrak{m}$ must be included in one of them. □

The following lemma is a variant of the overspill principle.

4.2.6 Lemma. *Let $\mathfrak{g}$ be a gap, $\gamma \in \mathfrak{g}$, $X \in \mathfrak{D}$. If $\mathfrak{g} \cap [0,\gamma] \subseteq X$ or $\mathfrak{g} \cap [\gamma,\rightarrow) \subseteq X$, then $\mathfrak{g} \subseteq X$.*

Proof. Assume $\mathfrak{g} \cap [0,\gamma] \subseteq X$ with $\gamma \in \mathfrak{g}$ and let $\delta \stackrel{\text{df}}{=} \mu x < \gamma : [x,\gamma] \subseteq X$. Then $\delta \in \inf \mathfrak{g}$, so there is some $d \in I_{\mathfrak{g}}$ such that $\delta < d$; then $[d,\gamma] \subseteq X$. If $\mathfrak{g} = \mathfrak{g}_\infty$, then $[d,\rightarrow) \cap M \subseteq X \cap M$, so $[d,\rightarrow) \subseteq X$ by elementarity. Otherwise let $c \in M \setminus \mathbb{N}$ and $e \stackrel{\text{df}}{=} \max\{y < c \mid [d,y] \subseteq X\}$. Then $e \in M$ and $e > \gamma$, so $e > \mathfrak{g}$ and we are done. The case with $[\gamma,\rightarrow) \subseteq X$ is proved similarly. □

Functions and images of monads and gaps

We apply the basic set-theoretic notation and terminology described in Section 1.5 to the model C. In particular, functions and relations on C are sets of pairs obtained using Cantor's pairing function.

4.2.7 Definition. We define

$$\mathfrak{F} \stackrel{\text{df}}{=} \{F \in \mathfrak{D} \mid F \text{ is a function}\}.$$

As usual, if $F \in \mathfrak{F}$, we write $F(\alpha) = \beta$ if $\langle \alpha, \beta \rangle \in F$. For a cut I, we further denote
$$\mathfrak{F}(I) \stackrel{\mathrm{df}}{=} \{F \in \mathfrak{F} \mid F''I \subseteq I \wedge \mathrm{dom}(F) \in \mathfrak{D}(\mathfrak{g}_I)\}$$
and refer to functions from $\mathfrak{F}(I)$ as *I-functions*.

We now describe how functions operate on monads and gaps:

4.2.8 Theorem. *Let $\mathfrak{g}$ be a gap, $\mathfrak{m} \subseteq \mathfrak{g}$ a monad and $F \in \mathfrak{F}$. Then:*

a) *If $\mathrm{dom}(F) \cap \mathfrak{m} \neq \emptyset$, then $\mathrm{dom}(F) \subseteq \mathfrak{m}$ and $F''\mathfrak{m}$ is a monad (trivial or non-trivial).*

b) *If $F''\mathfrak{m} \cap \mathfrak{m} \neq \emptyset$, then $F \restriction \mathfrak{m}$ is the identity on $\mathfrak{m}$.*

c) *If $F''I_\mathfrak{g} \subseteq I_\mathfrak{g}$, then $F'' \sup I_\mathfrak{g} \subseteq \sup I_\mathfrak{g}$ and $F'' \sup \mathfrak{g} \subseteq \sup \mathfrak{g}$. If, additionally, $x \leq F(x)$ for all $x \in \mathrm{dom}(F)$, then $F''\mathfrak{g} \subseteq \mathfrak{g}$.*

d) *Let $\alpha, \beta \in \mathfrak{g}$ be such that $G(\alpha) < \beta$ for every $G \in \mathfrak{F}(I_\mathfrak{g})$ with $\alpha \in \mathrm{dom}(G)$. Then $\mathfrak{m} \cap [\alpha, \beta] \neq \emptyset$.*

Proof. a) Let $F \in \mathfrak{F}$ be a function with $\mathrm{dom}(F) \cap \mathfrak{m} \neq \emptyset$; then $\mathfrak{m} \subseteq \mathrm{dom}(F)$ because $\mathrm{dom}(F) \in \mathfrak{D}$. Let $\alpha, \beta \in \mathfrak{m}$ and $X \in \mathfrak{D}$. If $F(\alpha) \in X \in \mathfrak{D}$, then $\alpha \in F^{-1}[X] \in \mathfrak{D}$, thus $\mathfrak{m} \subseteq F^{-1}[X]$, hence $F(\beta) \in X$. Thus $F(\alpha) \sim F(\beta)$, so $F''\mathfrak{m}$ is included in some monad; if it is a trivial monad, we are done. Otherwise assume $F''\mathfrak{m} \subseteq \mathfrak{n}$ where $\mathfrak{n}$ is a non-trivial monad. Let $\gamma \in \mathfrak{n}$, $X \in \mathfrak{U}(\mathfrak{m})$, and $\alpha \in \mathfrak{m}$. Then $\gamma \sim F(\alpha) \in F''X \in \mathfrak{D}$, so $\gamma \in F''X$. By $\aleph_1$-saturation, $\gamma \in F''\mathfrak{m}$; altogether, $F''\mathfrak{m} = \mathfrak{n}$.

b) Let $F''\mathfrak{m} \cap \mathfrak{m} \neq \emptyset$. By Item a), $F''\mathfrak{m}$ is a monad, so $F''\mathfrak{m} = \mathfrak{m}$. Suppose $F(\alpha) \neq \alpha$ for some $\alpha \in \mathfrak{m}$. Then $\beta \neq F(\beta)$ for all $\beta \in \mathfrak{m}$ since $\alpha \in Y \stackrel{\mathrm{df}}{=} \{\beta \in \mathrm{dom}(F) \mid \beta \neq F(\beta)\} \in \mathfrak{D}$ and hence $\mathfrak{m} \subseteq Y$. Let $X \stackrel{\mathrm{df}}{=} Y \cap F^{-1}[Y]$. Clearly, $\mathfrak{m} \subseteq X$. By the 3-set Lemma 1.8.15, there is an M-definable partition P of X with $\|P\| = 3$ such that $F''P_{(i)} \cap P_{(i)} = \emptyset$ for all $i < 3$. For some $i_0 \in \{0,1,2\}$ thus $\alpha \in P_{(i_0)}$ and hence $\mathfrak{m} \subseteq P_{(i_0)}$. Now $F(\alpha) \notin P_{(i_0)}$ contradicts the fact that $F(\alpha) \in \mathfrak{m} \subseteq P_{(i_0)}$.

c) Let $I \stackrel{\mathrm{df}}{=} I_\mathfrak{g}$ and $F \in \mathfrak{F}$ such that $F''I \subseteq I$. Let $a \in I$. Now, $F''[0,a] \subseteq \sup I$, for otherwise $F''[0,a] \cap [b, \to) \neq \emptyset$ for all $b \in I$ and by overspill in M, this holds also for some $b \in M \setminus I$. Thus there exist $c \in M \cap [0,a]$ such that $F(c) > b > I$, contradicting $F''I \subseteq I$. This proves the inclusion $F'' \sup I \subseteq \sup I$. Similarly, if $\alpha \in \mathfrak{g}$ and $F(\alpha) > \mathfrak{g}$, then $F(\alpha) > b$ for some $b \in M \setminus I$. Let $c \stackrel{\mathrm{df}}{=} \mu x : F(x) > b$. Then $c \in M$ and $c < \alpha$, contradicting $F''I \subseteq I$. This proves $F'' \sup \mathfrak{g} \subseteq \sup \mathfrak{g}$. The sequel is an immediate consequence.

d) Let $\alpha, \beta \in \mathfrak{g}$ be as in the claim. It suffices to show that $X \cap [\alpha, \beta] \neq \emptyset$ for every $X \in \mathfrak{U}(\mathfrak{m})$ and apply $\aleph_1$-saturation. For a given X, let $G_X \in \mathfrak{F}$ be defined by $G_X(x) \stackrel{\mathrm{df}}{=} \min(X \cap [x, \to))$. Now, $X \cap I$ is cofinal in I, so we have $G_X \in \mathfrak{F}(I)$. Also, since $X \cap \mathfrak{g}$ is cofinal in $\mathfrak{g}$, we have $\mathfrak{g} \subseteq \mathrm{dom}(G_X)$. In particular, $G_X(\alpha)$ is defined and by our assumption, $G_X(\alpha) < \beta$. Now, $G_X(\alpha) \in X$, so $G_X(\alpha) \in [\alpha, \beta] \cap X$. $\square$

Monadic overspill and distribution of monads in gaps

For properties of monads studied in this work, occurrence of one monad with such property in a gap often yields a great plenitude of monads of the kind. In this paragraph, we identify some common features of such properties and give a general theorem about this.

4.2.9 Definition. Let $\varphi(X,\bar{x})$ be an arithmetical $\mathfrak{L}^{\mathrm{II}}$-formula in one set variable X. We say that the formula $\varphi(X,\bar{x})$ *has monadic overspill* over X for parameters $\bar{a} \in C$, if for every non-trivial monad $\mathfrak{m}$,

$$C \models \varphi(\mathfrak{m},\bar{a}) \quad \text{iff} \quad (\exists Y_0 \in \mathfrak{U}(\mathfrak{m}))(\forall Y \in \mathfrak{U}(\mathfrak{m}))C \models (Y \subseteq Y_0 \to \varphi(Y,\bar{a})). \tag{4.1}$$

(This definition is meaningful according to the notation established in Section 1.5; $C \models \varphi(\mathfrak{m},\bar{a})$ just means that $\varphi(X,\bar{a})$ holds in the expansion of C in which X is interpreted by $\mathfrak{m}$.)

4.2.10 Definition. Let $X \subseteq C$ be any subset (not necessarily M-definable). Then for $\alpha \in C$, $\langle X\rangle^{\alpha}$ denotes the set of all increasing sequences (coded in C) of length α with elements in X.

Note that for $X \in \mathfrak{D}$ and $a \in M$, $\langle X\rangle^{a} \in \mathfrak{D}$.

4.2.11 Lemma (*on monadic overspill*).
Let $\varphi(X,z,\bar{y})$ be an arithmetical $\mathfrak{L}^{II}$-formula of the form

$$(\forall x_1,\dots,x_n \in \langle X\rangle^{z})\psi(x_1,\dots,x_n,\bar{y}),$$

where ψ is an $\mathfrak{L}^{Ar}$-formula. Then $\varphi(X,d,\bar{a})$ has monadic overspill over X for parameters $d \in M$, $\bar{a} \in C$.

Proof. The implication from right to left in (4.1) is trivial in this case. For the converse implication, assume $C \models \varphi(\mathfrak{m},\bar{a})$, but suppose that for every $Y_0 \in \mathfrak{U}(\mathfrak{m})$ there is $Y \subseteq Y_0$, $Y \in \mathfrak{U}(\mathfrak{m})$, such that $C \models \neg\varphi(Y,\bar{a})$; then for some $\bar{\delta} \in \langle Y\rangle^{d} \subseteq \langle Y_0\rangle^{d}$, $C \models \neg\psi(\bar{\delta},\bar{a})$. This makes $p(\bar{y}) \stackrel{\text{df}}{=} \{\neg\psi(\bar{y},\bar{a}) \wedge \bar{y} \in \langle Y_0\rangle^{a} \mid Y_0 \in \mathfrak{U}(\mathfrak{m})\}$ an n-type in C. By $\aleph_1$-saturation, $p(\bar{y})$ is satisfied by some $\bar{\gamma} \in C$. Then $\bar{\gamma} \in \langle\mathfrak{m}\rangle^{a} = \bigcap\{\langle Y_0\rangle^{a} \mid Y_0 \in \mathfrak{U}(\mathfrak{m})\}$ and $C \models \neg\psi(\bar{\gamma},\bar{a})$. This contradicts the assumption $C \models \varphi(\mathfrak{m},\bar{a})$. $\square$

4.2.12 Example. For a fixed $F \in \mathfrak{F}$, the formulae 'F is 1-1 on X' and 'F is constant on X' have monadic overspill over X (applying the lemma for $z = 2$ and replacing F with its $\mathfrak{L}^{Ar}_{M}$ definition to meet the requirements on ψ).

4.2.13 Remark. The restriction on the form (or rather complexity) of φ in the Lemma on monadic overspill is important. For example the formula $(\forall x \in X)(\exists y \in X)y < x$ holds if X is a monad but not if $X \in \mathfrak{D}$.

4.2.14 Definition.

a) A *simple property* in C is a sequence $\Phi(X) = \{\varphi(X)_n\}_{n\in\omega}$ of arithmetical $\mathfrak{L}^{\mathrm{II}}$-formulae with parameters from M and a single set variable X. We say that a subset $Y \subseteq C$ satisfies (or has) the simple property $\Phi(X)$ in C, writing $C \models \Phi(Y)$ or just $\Phi(Y)$, if $C \models \varphi(Y)$ for each formula $\varphi(X)$ from $\Phi(X)$.

b) We say that a simple property $\Phi(X)$ *has monadic overspill* (over X) if every $\varphi(X)$ from $\Phi(X)$ has monadic overspill over X (for the parameters from M occurring in φ).

c) A simple property $\Phi(X)$ is *dense in a gap* $\mathfrak{g}$ if

$$(\forall Y \in \mathfrak{D}(\mathfrak{g}))(\exists \mathfrak{m} \subseteq Y) C \models \Phi(\mathfrak{m}).$$

(This just means that the set of monads $\{\mathfrak{m} \mid C \models \Phi(\mathfrak{m})\}$ is dense in $\mathfrak{g}/\!\sim$ with the Stone topology).

d) A simple property $\Phi(X)$ is *distributive over a gap* $\mathfrak{g}$ if for each $Y \in \mathfrak{D}(\mathfrak{g})$, $Y \cap \mathfrak{g}$ includes a closed $\sim$-figure Z consisting of $2^{\aleph_0}$ monads that satisfy $\Phi(X)$.

4.2.15 Theorem. *Let $\Phi(X)$ be a simple property with monadic overspill. Then $\Phi(X)$ is dense in a gap $\mathfrak{g}$ if and only if it is distributive over $\mathfrak{g}$.*

Proof. Right-to-left, it is trivial. Let $\Phi(X)$ be as assumed and dense in $\mathfrak{g}$ and let $Y \in \mathfrak{D}(\mathfrak{g})$. Let $\{D_i\}_{i\in\omega}$ be a complete enumeration of $\mathfrak{D}$ and $L_0 \supset L_1 \supset \ldots$ a sequence of intervals with endpoints in M such that $\mathfrak{g} = \bigcap_{n\in\omega} L_n$.

For $X \in \mathfrak{D}$, let $X^{(0)}, X^{(1)}$ be the partition of X such that $X^{(0)}$ contains just every other element of X in the canonical order and $X^{(1)} = X \setminus X^{(0)}$ (as in the proof of 4.1.4). Clearly for $X \in \mathfrak{D}(\mathfrak{g})$, both $X^{(0)}, X^{(0)} \in \mathfrak{D}(\mathfrak{g})$.

We shall construct an ω-high binary tree with elements from $\mathfrak{D}(\mathfrak{g})$ indexed by functions from ${}^n2$. For $f : n \to 2$, let $f\smile i$ denote the function $f \cup \{\langle n, i\rangle\}$. The construction starts with $X_\emptyset = Y$. At the n-th stage, given $X_f \in \mathfrak{D}(\mathfrak{g})$ with $f \in {}^n2$, we find $X_{f\smile 0}$ and $X_{f\smile 1}$ in $\mathfrak{D}(\mathfrak{g})$ such that for $i \in \{0,1\}$:

1) $X_{f\smile 0} \cap X_{f\smile 1} = \emptyset$,

2) either $X_{f\smile i} \subseteq D_n$ or $X_{f\smile i} \subseteq C \setminus D_n$,

3) $X_{f\smile i} \subseteq X_f \cap L_n$,

4) $\varphi_k(X_{f\smile i})$ for all $k < n$.

We first show that such $X_{f\smile 0}$ and $X_{f\smile 1}$ can indeed be found. Fix $i \in \{0,1\}$, and using the fact that the property $\Phi(X)$ is dense in $\mathfrak{g}$, let $\mathfrak{m}$ be arbitrary monad satisfying $\Phi(\mathfrak{m})$ and included in $\mathfrak{g} \cap X_f^{(i)}$. One of the sets $D_n, C \setminus D_n$ includes $\mathfrak{m}$; let us denote that set by D. Using monadic overspill for $\mathfrak{m}$, we may now find $X_{f\smile i} \subseteq X_f^{(i)} \cap L_n \cap D$ such that $\varphi_k(X_{f\smile i})$ for all $k < n$. Thus, all requirements of 3)–4) have been met.

For every $f \in {}^\omega 2$ now put $X_f \stackrel{\mathrm{df}}{=} \bigcap_{n\in\omega} X_{f\restriction n}$ and let $Z \stackrel{\mathrm{df}}{=} \bigcup_{f\in{}^\omega 2} X_f$.

Let $f \in {}^{\omega}2$. The condition 1) ensures that if $g \in {}^{\omega}2$ and $g \neq f$, then $X_f \cap X_g = \emptyset$, 2) ensures that X_f is a monad, 3) ensures that $X_f \subseteq Y \cap \mathfrak{g}$, and 4) ensures that $\Phi(X_f)$ by monadic overspill. Thus Z is a figure included in $Y \cap \mathfrak{g}$ and consists of $2^{\aleph_0}$ monads that all satisfy $\Phi(X)$.

It thus remains to show that Z is closed in the $\sim$-topology. For that, observe that $\bigcup_{f \in {}^{\omega}2} X_f = \bigcap_{n \in \omega} \bigcup_{g \in {}^{n}2} X_g$. The inclusion $\supseteq$ follows from 1). The set on the right is closed because it is an intersection of sets from $\mathcal{D}$, which are clopen. □

4.2.16 Example. Every gap includes $2^{\aleph_0}$ monads. This follows from Theorem 4.2.15 by the fact that $\{(\forall x \in Y) x = x\}$ is a simple property that (trivially) has monadic overspill and is dense in every gap.

Rudin-Keisler pre-order on monads

4.2.17 Definition. The *Rudin-Keisler (RK) pre-order* on non-trivial monads is defined as follows:

$$\mathfrak{m} \preceq_{\mathrm{RK}} \mathfrak{n} \overset{\mathrm{df}}{\iff} (\exists F \in \mathcal{F}) F''\mathfrak{n} = \mathfrak{m}.$$

It is accompanied by the *Rudin-Keisler equivalence* on non-trivial monads:

$$\mathfrak{m} \asymp_{\mathrm{RK}} \mathfrak{n} \overset{\mathrm{df}}{\iff} (\exists F \in \mathcal{F})(F''\mathfrak{m} = \mathfrak{n} \wedge F \text{ is one-to-one}).$$

For $F, \mathfrak{m}, \mathfrak{n}$ as above, we write $F : \mathfrak{m} \asymp_{\mathrm{RK}} \mathfrak{n}$.

4.2.18 Theorem. *Let $\mathfrak{m}, \mathfrak{n}$ be non-trivial monads and $F, G \in \mathcal{F}$.*

a) *If $\mathfrak{m} \asymp_{\mathrm{RK}} \mathfrak{n}$ and $F''\mathfrak{m} = \mathfrak{n}$, then F is one-to-one on $\mathfrak{m}$.*

b) $\mathfrak{m} \asymp_{\mathrm{RK}} \mathfrak{n}$ *iff* $\mathfrak{m} \preceq_{\mathrm{RK}} \mathfrak{n}$ *and* $\mathfrak{n} \preceq_{\mathrm{RK}} \mathfrak{m}$.

c) *If $F''\mathfrak{m} = G''\mathfrak{m} = \mathfrak{n}$ and $\mathfrak{m} \asymp_{\mathrm{RK}} \mathfrak{n}$, then $F \restriction \mathfrak{m} = G \restriction \mathfrak{m}$.*

d) *$\preceq_{\mathrm{RK}}$ is a pre-order and induces a partial order $\sqsubseteq_{\mathrm{RK}}$ on the equivalence classes of $\asymp_{\mathrm{RK}}$. The partial order $\sqsubseteq_{\mathrm{RK}}$ is upward directed (thus for every $\mathfrak{m}_0, \mathfrak{m}_1$ there is $\mathfrak{m}$ such that both $\mathfrak{m}_0 \preceq_{\mathrm{RK}} \mathfrak{m}$ and $\mathfrak{m}_1 \preceq_{\mathrm{RK}} \mathfrak{m}$) and has no maximal elements (thus for every $\mathfrak{m}$ there is $\mathfrak{n}$ such that $\mathfrak{m} \preceq_{\mathrm{RK}} \mathfrak{n} \not\asymp_{\mathrm{RK}} \mathfrak{m}$).*

e) *Every equivalence class $[\mathfrak{m}]_{\asymp_{\mathrm{RK}}}$ is countable and totally ordered by the relation $\le_{\mathcal{F}}$ defined for $\mathfrak{m}_1, \mathfrak{m}_2 \in [\mathfrak{m}]_{\asymp_{\mathrm{RK}}}$ as follows:*

$$\mathfrak{m}_1 \le_{\mathcal{F}} \mathfrak{m}_2 \overset{\mathrm{df}}{\iff} \alpha \le F(\alpha), \text{ whenever } \alpha \in \mathfrak{m}_1 \text{ and } F''\mathfrak{m}_1 \asymp_{\mathrm{RK}} \mathfrak{m}_2.$$

Moreover, if $\inf(\mathfrak{m}) \neq \mathbb{N}$, then the order type of $\langle [\mathfrak{m}]_{\asymp_{\mathrm{RK}}}, \le_{\mathcal{F}} \rangle$ is $\eta \cdot ({}^{*}\omega + \omega)$, i.e. that of the lexicographical ordering on $\mathbb{Q} \times \mathbb{Z}$.

Proof. a) Let F, $\mathfrak{m}$, and $\mathfrak{n}$ be as assumed in the claim. The relation $\asymp_{\mathrm{RK}}$ is clearly an equivalence, so $G : \mathfrak{n} \asymp_{\mathrm{RK}} \mathfrak{m}$ for some $G \in \mathfrak{F}$. Then $F \circ G$ maps $\mathfrak{m}$ onto $\mathfrak{m}$, hence $(F \circ G)\restriction\mathfrak{m} = \mathrm{id}\restriction\mathfrak{m}$ by 4.2.8 b). It follows that F is one-to-one on $\mathfrak{m}$.

b) Left-to-right is immediate. Assume $F, G \in \mathfrak{F}$, $F''\mathfrak{m} = \mathfrak{n}$ and $G''\mathfrak{n} = \mathfrak{m}$. Then, just as above, $F \circ G$ is identical on $\mathfrak{m}$, so F is 1-1 on $\mathfrak{m}$. Using monadic overspill, F is 1-1 on some X. The restriction $F\restriction X$ witnesses $\mathfrak{m} \asymp_{\mathrm{RK}} \mathfrak{n}$.

c) By item a), F is 1-1 on $\mathfrak{m}$ so by monadic overspill, we may assume F to be 1-1 on its domain. Then $F^{-1} \in \mathfrak{F}$ and maps $\mathfrak{n}$ onto $\mathfrak{m}$. Thus $G \circ F^{-1}$ is identical on $\mathfrak{m}$. Hence, restricting everything to $\mathfrak{m}$, $F = \mathrm{id} \circ F = G \circ F^{-1} \circ F = G$.

d) Clearly $\preceq_{\mathrm{RK}}$ is reflexive and transitive, so it is a pre-order on monads. It induces a partial order $\sqsubseteq_{\mathrm{RK}}$ on the set of $\asymp_{\mathrm{RK}}$-equivalence classes as follows:

$$[\mathfrak{m}]_{\asymp_{\mathrm{RK}}} \sqsubseteq_{\mathrm{RK}} [\mathfrak{n}]_{\asymp_{\mathrm{RK}}} \overset{\mathrm{df}}{\Longleftrightarrow} \mathfrak{m} \preceq_{\mathrm{RK}} \mathfrak{n}.$$

The correctness of the definition is obvious. The anti-symmetry of $\sqsubseteq_{\mathrm{RK}}$ follows from b).

Let $\mathfrak{m}_0, \mathfrak{m}_1$ be non-trivial monads and take $\alpha_0 \in \mathfrak{m}_1, \alpha_0 \in \mathfrak{m}_1$ and let $\mathfrak{m} \overset{\mathrm{df}}{=} \mathfrak{m}(\langle \alpha_0, \alpha_1 \rangle)$. For $i \in \{0,1\}$, the i-th projections of the Cantor's pairing function, $\pi_i(x) = \langle x \rangle_i$, maps $\mathfrak{m}$ onto $\mathfrak{m}_i$, so $\mathfrak{m}_i \preceq_{\mathrm{RK}} \mathfrak{m}$.

Let $\mathfrak{m}$ be a non-trivial monad. Take $\alpha, \beta \in \mathfrak{m}$ such that $\alpha \neq \beta$ and for $\gamma = \langle \alpha, \beta \rangle$, let $\mathfrak{n} \overset{\mathrm{df}}{=} \mathfrak{m}(\gamma)$. For π_i as above, we have ${\pi_i}''\mathfrak{n} = \mathfrak{n}$ for both $i = 0, 1$. Thus $\mathfrak{m} \preceq_{\mathrm{RK}} \mathfrak{n}$. But $\mathfrak{m} \not\asymp_{\mathrm{RK}} \mathfrak{n}$, since otherwise $\pi_0\restriction\mathfrak{n} = \pi_1\restriction\mathfrak{n}$ by item c) which is not possible, since $\pi_1(\gamma) = \alpha \neq \beta = \pi_2(\gamma)$, $\gamma \in \mathfrak{n}$. For any $\mathfrak{m}$, we have found $\mathfrak{n}$ strictly

e) Clearly, if $\alpha, \beta \in \mathfrak{m}_1$, then $F(\alpha) \leq \alpha \leftrightarrow F(\beta) \leq \beta$. This, and item c) establish correctness of the definition of the order $\leq_{\mathfrak{F}}$. The weak anti-symmetry of the relation $\leq_{\mathfrak{F}}$ is proved as follows: let $\mathfrak{m}_1 \leq_{\mathfrak{F}} \mathfrak{m}_2$ and $\mathfrak{m}_2 \leq_{\mathfrak{F}} \mathfrak{m}_1$. Then for $F : \mathfrak{m}_1 \asymp_{\mathrm{RK}} \mathfrak{m}_2$ and $G : \mathfrak{m}_2 \asymp_{\mathrm{RK}} \mathfrak{m}_1$, $G(F(\alpha)) \leq F(\alpha) \leq \alpha$. Since $G(F(\alpha)) \in \mathfrak{m}_1$, $F \circ G$ coincides with the identity map on $\mathfrak{m}_1$ by 4.2.8 b). It thus follows that $G(F(\alpha)) = F(\alpha) = \alpha$, hence $\mathfrak{m}_1 = \mathfrak{m}_2$. Verifying that $\leq_{\mathfrak{F}}$ is reflexive, transitive, and total is trivial.

Since $\mathfrak{F}$ is countable, $[\mathfrak{m}]_{\asymp_{\mathrm{RK}}}$ is countable, too. If $\mathfrak{m}$ is a monad, then $\mathfrak{m} + 1 \overset{\mathrm{df}}{=} \{\alpha + 1 \mid \alpha \in \mathfrak{m}\}$ and $\mathfrak{m} - 1 \overset{\mathrm{df}}{=} \{\alpha - 1 \mid \alpha \in \mathfrak{m}\}$ are its immediate successor and predecessor in the order of $\leq_{\mathfrak{F}}$. Hence the $[\mathfrak{m}]_{\asymp_{\mathrm{RK}}}$ decomposes into $\mathbb{Z}$-blocks of the ordering $\leq_{\mathfrak{F}}$. Let $\mathfrak{m}_1 \leq_{\mathfrak{F}} \mathfrak{m}_2$ be monads from different $\mathbb{Z}$-blocks of $[\mathfrak{m}]_{\asymp_{\mathrm{RK}}}$ and $F : \mathfrak{m}_1 \asymp_{\mathrm{RK}} \mathfrak{m}_2$. Then $x < F(x)$ on $\mathfrak{m}_1$. Since $\mathbb{N} < F(x) - x$, there exists $a \in M \setminus \mathbb{N}$ such that $x < x + 2a < F(x)$. It is obvious that the $\mathbb{Z}$-block of the monad $\mathfrak{m}_1 + a \overset{\mathrm{df}}{=} \{\alpha + a \mid \alpha \in \mathfrak{m}\} \in [\mathfrak{m}]_{\asymp_{\mathrm{RK}}}$ lies strictly between the $\mathbb{Z}$-blocks of $\mathfrak{m}_1$ and $\mathfrak{m}_2$. $\square$

RK-minimal monads

By the last theorem, there may be no $\preceq_{\mathrm{RK}}$-maximal monad. We shall see, however, that there are (in fact many) minimal non-trivial monads, which contrasts with the situation of $\langle\omega,\mathcal{P}(\omega)\rangle$, where the existence of minimal ultrafilters cannot be established using the axioms of ZFC.

4.2.19 Definition. A non-trivial monad $\mathfrak{m}$ is said to be *RK-minimal* or just *minimal*, if it is minimal with respect to the pre-order $\preceq_{\mathrm{RK}}$ on non-trivial monads, i.e. if $\mathfrak{n} \preceq_{\mathrm{RK}} \mathfrak{m}$ implies $\mathfrak{n} \asymp_{\mathrm{RK}} \mathfrak{m}$ for every non-trivial monad $\mathfrak{n}$.

4.2.20 Theorem. *Let $\mathfrak{m}$ be a non-trivial monad. The following statements are equivalent:*

a) $\mathfrak{m}$ *is minimal.*

b) *For every $F \in \mathcal{F}$ with $\mathfrak{m} \subseteq \mathrm{dom}(F)$, $F\restriction\mathfrak{m}$ is constant or one-to-one.*

c) *For every $F \in \mathcal{F}(I_{\mathfrak{m}})$ with $\mathfrak{m} \subseteq \mathrm{dom}(F)$, $F\restriction\mathfrak{m}$ is constant or one-to-one.*

d) $\mathcal{U}(\mathfrak{m})$ *is a* selective ultrafilter*, i.e. for every $X \in \mathcal{U}(\mathfrak{m})$ and every partition $P \in \mathcal{D}$ of X, there is $Y \in \mathcal{U}(\mathfrak{m})$ such that Y is either homogeneous or anti-homogeneous for P.*

Proof. a)→b) Let $\mathfrak{m}$ be minimal, $F \in \mathcal{F}$, and $\mathfrak{m} \subseteq \mathrm{dom}(F)$, and assume F is not constant on $\mathfrak{m}$. Then $F''\mathfrak{m}$ is not a singleton, so by 4.2.8 a), it is a non-trivial monad. Let $\mathfrak{n} = F''\mathfrak{m}$. Then $\mathfrak{n} \preceq_{\mathrm{RK}} \mathfrak{m}$ and by minimality $G : \mathfrak{n} \asymp_{\mathrm{RK}} \mathfrak{m}$ for some 1-1 function $G \in \mathcal{F}$. By 4.2.18 c), $F\restriction\mathfrak{m} = G\restriction\mathfrak{m}$, hence $F\restriction\mathfrak{m}$ is 1-1.

b)→c) is trivial. c)→d) Let P be a partition on X, $P \in \mathcal{D}$, $X \in \mathcal{U}(\mathfrak{m})$. Let $G(x) \stackrel{\mathrm{df}}{=} \min(P[x])$ for every $x \in X$; then $G(x) \le x$. Thus $G(x) \in \mathcal{F}(I_{\mathfrak{m}})$. By the assumption, one of the following is true: 1) G is constant on $\mathfrak{m}$; then $\mathfrak{m}$ is homogeneous for P. 2) G is 1-1 on $\mathfrak{m}$; then $\mathfrak{m}$ is anti-homogeneous for P. Both homogeneity and anti-homogeneity for P are properties with monadic overspill (4.2.11), so the claim follows.

d)→a) Let $F''\mathfrak{m} = \mathfrak{n}$ be a non-trivial monad, $F \in \mathcal{F}$, $X = \mathrm{dom}(F) \in \mathcal{U}(\mathfrak{m})$. We must show that $\mathfrak{m} \asymp_{\mathrm{RK}} \mathfrak{n}$. F naturally determines a partition P on X such that $x\, P\, y$ iff $F(x) = F(y)$. Let $Y \subseteq X$, $Y \in \mathcal{U}(\mathfrak{m})$ be homogeneous for P or anti-homogeneous for the partition; in the first case, $F''Y$ is a singleton, so $\mathfrak{n}$ is trivial—a contradiction. So Y must be anti-homogeneous for P, so $F\restriction Y$ is 1-1. Thus $F\restriction X : \mathfrak{m} \asymp_{\mathrm{RK}} \mathfrak{n}$ as required. □

4.2.21 Proposition. *If $\mathfrak{m}$ is a minimal monad included in a gap $\mathfrak{g}$, then for every $X \in \mathcal{D}(\mathfrak{g})$, there exists a (minimal) monad $\mathfrak{n} \subseteq X \cap \mathfrak{g}$ such that $\mathfrak{m} \asymp_{\mathrm{RK}} \mathfrak{n}$. In other words, $[\mathfrak{m}]_{\asymp_{\mathrm{RK}}}$ is dense in the Stone topology of $\mathfrak{g}/\!\sim$.*

Proof. Let $\mathfrak{m}$ be a minimal monad from the gap $\mathfrak{g}$ and let $X \in \mathcal{D}(\mathfrak{g})$. Fix arbitrary $Y \in \mathcal{U}(\mathfrak{m})$ and define $F : Y \to X$ by $F(\alpha) = \mu\gamma : (\gamma \in X \wedge \alpha \le \gamma)$. Since $X \cap \mathfrak{g}$ is cofinal in $\mathfrak{g}$, $Y \cap \mathfrak{g} \subseteq \mathrm{dom}(F)$ and $F''\mathfrak{m} \subseteq \mathfrak{g}$. Thus $F''\mathfrak{m} = \mathfrak{n}$ is a non-trivial monad by 4.2.8 a); it is minimal since $\mathfrak{n} \preceq_{\mathrm{RK}} \mathfrak{m}$. □

By 4.2.12, $\{$'*F is one-to-one or constant on X*'$\}_{F\in\mathfrak{F}}$ is a simple property with monadic overspill and by 4.2.20 and 4.2.21 it is dense in every gap that includes a minimal monad. Thus by Theorem 4.2.15, minimal monads are distributive over every gap that includes one.

Relatively large monads

4.2.22 Definition.

a) A set $X \in \mathfrak{D}$ *relatively large* if it is unbounded or $\min(X) \leq |X|$.

b) A monad $\mathfrak{m}$ is *relatively large* if every $X \in \mathfrak{U}(\mathfrak{m})$ is relatively large.

4.2.23 Proposition. *Let I be a proper cut and $\mathfrak{m} \subseteq \mathfrak{g}_I$ a non-trivial monad. Then the following conditions are equivalent:*

a) $\mathfrak{m}$ *is relatively large*

b) $(\forall a \in I)\langle\mathfrak{m}\rangle^a \neq \emptyset$,

c) *If $X \in \mathfrak{U}(\mathfrak{m})$ is bounded, then $a \leq |X \setminus [0,a]|$ for some $a \in M \setminus I$,*

d) $(\forall \alpha,\beta \in \mathfrak{g}_I)\langle\mathfrak{m} \cap [\alpha,\rightarrow)\rangle^\beta \neq \emptyset$.

e) $\mathfrak{m} \not\preceq_{\mathrm{RK}} \mathfrak{n}$ *for every non-trivial monad* $\mathfrak{n} \subseteq \inf\mathfrak{m}$

f) $\mathfrak{m} \not\asymp_{\mathrm{RK}} \mathfrak{n}$ *for every non-trivial monad* $\mathfrak{n} \subseteq \inf\mathfrak{m}$

Proof. a)→b) Let $a \in I$ be arbitrary. For every $X \in \mathfrak{U}(\mathfrak{m})$, $|X \setminus [0,a]| \geq a$ by the assumption, so $\langle X\rangle^a \neq \emptyset$. Since $\langle\mathfrak{m}\rangle^a = \bigcap\{\langle X\rangle^a \mid X \in \mathfrak{U}(\mathfrak{m})\}$, the claim follows by $\aleph_1$-saturation.

b)→c) Let $X \in \mathfrak{U}(\mathfrak{m})$ be bounded. Then the set $Y \overset{\mathrm{df}}{=} \{a \mid a \leq |X\setminus[0,a]|\} \in \mathfrak{D}$ is bounded too. Now, $I \subseteq Y$ because for each $b \in I$ we have some $u \subseteq \mathfrak{m} \subseteq X\setminus[0,b]$ with $|u| = b$, as $\langle\mathfrak{m}\rangle^b \neq \emptyset$. Let $a \overset{\mathrm{df}}{=} \max Y \in M$. Then a has the desired property.

c)→d) Let $\alpha,\beta \in \mathfrak{g}_I$ and $\gamma = \max\{\alpha,\beta\}$. Let $X \in \mathfrak{U}(\mathfrak{m})$. By the assumption, either $|X\setminus[0,\gamma]|$ is unbounded or $|X\setminus[0,\gamma]| \geq \gamma$. In either case, $\langle X\setminus[0,\gamma]\rangle^\gamma \neq \emptyset$. The claim now follows by $\aleph_1$-saturation.

d)→e) Assume $F : \mathfrak{n} \succeq_{\mathrm{RK}} \mathfrak{m}$ for some $F \in \mathfrak{F}$ and $\mathfrak{n} \subseteq \inf\mathfrak{m}$. The latter gives $\mathfrak{n} \subseteq [0,a)$ for some $a \in I$. By the assumption, there is a coded subset $u \subseteq \mathfrak{m}$, $u \in C$, such that $|u| \in \mathfrak{g}_I$. Then F maps some coded subset $v \subseteq [0,a)$ onto u; we have $a \geq |v| \geq |u| \in \mathfrak{g}_I$—a contradiction.

e)→f) is trivial. f)→a) Let $X \in \mathfrak{U}(\mathfrak{m})$ and assume it is not relatively large. Then $a = |X| < \min X$ for some $a \in M$, so in fact, $a \in I$. There exists a 1-1 function $F \in \mathfrak{F}$ such that $F''X = [0,a)$. But then F maps $\mathfrak{m}$ on a non-trivial monad $F''\mathfrak{n} = \mathfrak{n} \subseteq [0,a) \subseteq \inf\mathfrak{m}$ and $F : \mathfrak{m} \asymp_{\mathrm{RK}} \mathfrak{n}$—a contradiction. □

Original gaps

4.2.24 Definition. A gap $\mathfrak{g}$ is *original* if

$$(\forall \mathfrak{m} \subseteq \mathfrak{g})(\forall \mathfrak{n} \subseteq \inf \mathfrak{g})(\mathfrak{m} \not\asymp_{\mathrm{RK}} \mathfrak{n}).$$

An original gap consists of monads that are new, i.e. inequivalent to the monads in the preceding part of the model C; hence the term.

It is easy to see that the unbounded gap $\mathfrak{g}_\infty$ is original; since every $X \in \mathfrak{D}(\mathfrak{g}_\infty)$ is unbounded, all monads from $\mathfrak{g}_\infty$ are relatively large.

4.2.25 Proposition. *The following statements are equivalent:*

a) *The gap $\mathfrak{g}$ is original.*

b) *Every monad from $\mathfrak{g}$ is relatively large.*

Proof. Follows from 4.2.23, a)↔e). □

4.2.26 Proposition. *Let $\mathfrak{g}$ be an original gap, $\mathfrak{m} \subseteq \mathfrak{g}$ a monad and $X \in \mathfrak{D}(\mathfrak{g})$. Then there is a monad $\mathfrak{n} \subseteq \mathfrak{g} \cap X$ such that $\mathfrak{m} \asymp_{\mathrm{RK}} \mathfrak{n}$.*

Proof. For $\mathfrak{m}$, $\mathfrak{g}$, and X as above, let $F \in \mathfrak{F}$ be the unique enumeration of X, i.e. an order-preserving map such that $\mathrm{rng}(F) = X$ and $\mathrm{dom}(F)$ is some interval in C starting at 0. Then F is 1-1 so $\mathfrak{n} = F''\mathfrak{m} \asymp_{\mathrm{RK}} \mathfrak{m}$. Moreover, $F(x) \leq x$ for all $x \in X$, so $\mathfrak{n} \subseteq \sup \mathfrak{g}$. Originality of $\mathfrak{g}$ gives $\mathfrak{n} \not\subseteq \inf \mathfrak{g}$, hence $\mathfrak{n} \subseteq \mathfrak{g}$, □

4.2.27 Remark. The proposition says that for an original gap $\mathfrak{g}$ and a monad $\mathfrak{m} \subseteq \mathfrak{g}$, $[\mathfrak{m}]_{\asymp_{\mathrm{RK}}}$ is dense in $\mathfrak{g}/\sim$. Still, Theorem 4.2.15 fails, since $[\mathfrak{m}]_{\asymp_{\mathrm{RK}}}$ is only countable and thus not distributive over $\mathfrak{g}$. This means that for a given monad $\mathfrak{m}$, $X \in [\mathfrak{m}]_{\asymp_{\mathrm{RK}}}$ cannot be expressed by a simple property with monadic overspill over X.

4.2.28 Remark. Proposition 4.2.26 cannot be reversed in the following sense: Let $\mathfrak{g}$ be a bounded original gap and $a \in M \setminus I_{\mathfrak{g}}$. Then $a + \mathfrak{g} \stackrel{\mathrm{df}}{=} \{a + \alpha \mid \alpha \in \mathfrak{g}\}$ is clearly a gap that is not original but every $[\mathfrak{m}]_{\asymp_{\mathrm{RK}}}$ with $\mathfrak{m} \subseteq a + \mathfrak{g}$ is still dense in $(a + \mathfrak{g})/\sim$. Indeed, if we let $F(\alpha) \stackrel{\mathrm{df}}{=} \alpha - a$ for $\alpha \geq a$, then for $\mathfrak{m} \subseteq a + \mathfrak{g}$ and $X \in \mathfrak{D}(a + \mathfrak{g})$ we have $F''\mathfrak{m} \subseteq \mathfrak{g}$ and $F''X \in \mathfrak{D}(\mathfrak{g})$. Applying 4.2.26, there is a monad $\mathfrak{n} \subseteq \mathfrak{g} \cap F''X$ such that $F''\mathfrak{m} \asymp_{\mathrm{RK}} \mathfrak{n}$. Then $\mathfrak{m} \asymp_{\mathrm{RK}} F''\mathfrak{m} \asymp_{\mathrm{RK}} \mathfrak{n} \asymp_{\mathrm{RK}} a + \mathfrak{n} \subseteq a + \mathfrak{g} \cap X$, where $a + \mathfrak{n} \stackrel{\mathrm{df}}{=} \{a + \alpha \mid \alpha \in \mathfrak{n}\}$.

Model extensions

We now introduce a natural way of obtaining elementary extensions of the model M. This will allow us to give model-theoretic interpretations to various properties of monads studied in this chapter.

4.2.29 Definition. Let $Z \subseteq C$ be a set. Then $M[Z]$ denotes the smallest elementary submodel of C that includes $M \cup Z$. If Z is finite, say $\{\alpha_1, \ldots, \alpha_n\}$, we often write $M[Z]$ as $M[\alpha_1, \ldots, \alpha_n]$.

Clearly $M \preccurlyeq M[Z] \preccurlyeq C$. Then 1.8.9 gives

$$M[Z] = \mathrm{Dfe}(C, M \cup Z),$$

which can be in our present context expressed as

$$M[Z] = \{F(\langle \bar{\alpha} \rangle) \mid F \in \mathfrak{F} \wedge \bar{\alpha} \in Z\}. \tag{4.2}$$

Indeed, if $\gamma \in \mathrm{Dfe}(C, M \cup Z)$, there is an $\mathcal{L}_M^{Ar}$-formula $\varphi(x, \bar{y})$ and $\bar{\alpha} \in Z$ such that $C \models \gamma = \mu x : \varphi(x, \bar{\alpha})$. Then $F(y) = \mu x : \varphi(x, (y)_0, \ldots)$ defines a function from $\mathfrak{F}$ such that $F(\langle \bar{\alpha} \rangle) = \gamma$. This yields $\subseteq$ in the equation (4.2). The reverse inclusion is similar.

We shall often use the representation of $M[Z]$ given in (4.2).

Let us remark that for $\alpha \in C$, $M[\alpha]$ is isomorphic to the so-called *definable ultrapower* of M over the ultrafilter $\mathcal{U} = \mathcal{U}(\mathfrak{m}(\alpha))$ on $\mathfrak{D}$:

$$M^{\mathcal{U}} \overset{\mathrm{df}}{=} \{[F]_{\mathcal{U}} \mid F \in \mathfrak{F} \wedge \mathrm{dom}(F) \in \mathcal{U}\}, \text{ where } [F]_{\mathcal{U}} \overset{\mathrm{df}}{=} \{G \in \mathfrak{F} \mid \mathrm{dom}(F \cap G) \in \mathcal{U}\},$$

with operations defined on $M^{\mathcal{U}}$ in the usual way.

4.2.30 Theorem. *Let $\alpha, \beta \in C$. Then*

a) $\mathfrak{m}(\alpha) \preceq_{\mathrm{RK}} \mathfrak{m}(\beta)$ *iff there is an elementary embedding $f : M[\alpha] \to M[\beta]$ that is identical on M.*

b) $\mathfrak{m}(\alpha) \asymp_{\mathrm{RK}} \mathfrak{m}(\beta)$ *iff there is an isomorphism $f : M[\alpha] \to M[\beta]$ that is identical on M.*

c) $\alpha \sim \beta$ *iff there is an isomorphism $f : M[\alpha] \to M[\beta]$ such that $f(\alpha) = \beta$ and f is identical on M.*

Proof. We start by proving c). If $\alpha \sim \beta$, they satisfy the same complete type in C over M and hence the map $\mathrm{id}{\restriction} M \cup \langle \alpha, \beta \rangle$ extends uniquely to an isomorphism of $M[\alpha]$ and $M[\beta]$. Conversely, if f is an isomorphism of $M[\alpha]$ and $M[\beta]$ identical on M and $f(\alpha) = \beta$, then α and β satisfy the same complete type in C over M and hence $\alpha \sim \beta$.

a) Let $\mathfrak{m}(\alpha) \preceq_{\mathrm{RK}} \mathfrak{m}(\beta)$. This means that for some $F \in \mathfrak{F}$, $F(\beta) \sim \alpha$. By c), $M[\alpha] \cong M[F(\beta)]$ via some isomorphism f identical on M. Since $F(\beta) \in M[\beta]$, we have $M[F(\beta)] \preccurlyeq M[\beta]$ and f is the required embedding. Conversely, if $f : M[\alpha] \to M[\beta]$ is an elementary embedding identical on M, then $M[\alpha] \cong M[f(\alpha)] \preccurlyeq M[\beta]$. But then $\alpha \sim f(\alpha)$ by c) and since $f(\alpha) \in M[\beta]$, $f(\alpha) = F(\beta)$ for some $F \in \mathfrak{F}$, hence $\mathfrak{m}(\alpha) = \mathfrak{m}(f(\alpha)) \preceq_{\mathrm{RK}} \mathfrak{m}(\beta)$.

b) Let $F : \mathfrak{m}(\alpha) \asymp_{\mathrm{RK}} \mathfrak{m}(\beta)$, $F \in \mathfrak{F}$. Since F is 1-1, $F^{-1} \in \mathfrak{F}$, so $\alpha = F^{-1}(F(\alpha)) \in M[F(\alpha)]$. Thus $M[\alpha] = M[F(\alpha)]$. Now, $F(\alpha) \sim \beta$, so by c), $M[F(\alpha)] \cong M[\beta]$ via an isomorphism identical on M. The converse implication follows by a). □

We now give two characterizations of minimal monads based on extensions:

4.2.31 Proposition. *The following are equivalent for a non-trivial monad* $\mathfrak{m}$*:*

a) $\mathfrak{m}$ *is minimal*

b) $M[\alpha] \cap M[\beta] = M$ *for all* $\alpha, \beta \in \mathfrak{m}$, $\alpha \neq \beta$.

c) *for some (all)* $\alpha \in \mathfrak{m}$, $M[\alpha]$ *is a minimal elementary extension of* M *(i.e. if* $M \preccurlyeq N \preccurlyeq M[\alpha]$*, then either* $N = M$ *or* $N = M[\alpha]$*).*

Proof. a)→b) Let $\mathfrak{m}$ be minimal, $\alpha, \beta \in \mathfrak{m}$, $\alpha \neq \beta$. If $\gamma \in M[\alpha] \cap M[\beta]$, then there are functions $F, G \in \mathfrak{F}$ such that $F(\alpha) = G(\beta) = \gamma$. By minimality, F is either one-to-one or constant on $\mathfrak{m}$. In the later case, $F''\mathfrak{m} = \{\gamma\}$ and $\gamma \in M$. If F is one-to-one on $\mathfrak{m}$, we may assume it is one-to-one on its whole domain. Then $F^{-1}(G(\beta)) = \alpha$, hence $(F^{-1} \circ G''\mathfrak{m}) \cap \mathfrak{m} \neq \emptyset$ and by item b) of Theorem 4.2.8, $F^{-1} \circ G$ is identical on $\mathfrak{m}$, hence $\alpha = \beta$. This contradiction finishes the proof of the left-to-right implication.

b)→c) Let $\alpha \in \mathfrak{m}$ and $M \prec N \preccurlyeq M[\alpha]$. Let $\gamma \in N \setminus M$. Since $\gamma \in M[\alpha]$, there is a function $F \in \mathfrak{F}$ such that $F(\alpha) = \gamma$. Now, for every $\beta \in \mathfrak{m}$, $F(\beta) \in M[\beta]$; but $M[\beta]$ coincides with $M[\alpha]$ only on M, hence $F(\beta) \neq \gamma$. It follows that F is 1-1 on $\mathfrak{m}$. But then there is $G \in \mathfrak{F}$ such that $G(\gamma) = \alpha$. Thus $G(\gamma) \in N$, hence $\alpha \in N$, so $M[\alpha] \subseteq N$.

c)→a) Let $M[\alpha]$ with $\alpha \in \mathfrak{m}$ be a minimal elementary extension of M and $\mathfrak{n} \leq_{\mathrm{RK}} \mathfrak{m}$. Let $F \in \mathfrak{F}$ be such that $F(\alpha) = \beta \in \mathfrak{n}$. Then $\beta \in M[\alpha]$. Since $\beta \notin M$, we have $M \prec M[\beta] \preccurlyeq M[\alpha]$, hence $M[\alpha] = M[\beta]$ by minimality of the extension. In particular, $\alpha \in M[\beta]$, thus for some function $G \in \mathfrak{F}$, $G(\beta) = \alpha$. It follows that $\mathfrak{m} \leq_{\mathrm{RK}} \mathfrak{n}$. □

4.3 Cuts in the enlarged setting

This section studies how properties of monads and gaps relate to the properties of the corresponding cuts. For this, we start with a brief survey of basic types of cuts of models of arithmetic identified by Kirby and Paris.

Basic types of cuts

Kirby and Paris, [Kir77], defined several types of cuts of models of arithmetic that proved to play prominent roles in the model theory of PA; these are *semi-regular*, *regular*, and *strong* cuts. We briefly introduce these notions and recall some well known results about them. The reader may refer to Kossak's and Schmerl's book [KS06] or Kirby's thesis [Kir77] for details.

4.3.1 Definition. A cut I is said to be *semi-regular* (in M) if for every $F \in \mathfrak{F}$

$$(\forall a \in I)(F''[0,a] \cap I \text{ is bounded in } I).$$

Note that if $F''[0,a]$ is bounded in I, then $F''[0,a]\cap\mathfrak{g}_I=\emptyset$ and hence by overspill, there are $b,c\in M$ such that $b\in I<c$ and $F''[0,a]\cap[b,c]=\emptyset$. Put in yet another way, I is semi-regular iff $\mathfrak{g}_I\cap F''\sup I=\emptyset$ for every $F\in\mathfrak{F}$.

Semi-regularity is a direct analogy of regularity for cardinal numbers in set theory; this analogy becomes explicit when the notion of cofinality for cuts is introduced: For $I\subseteq^e M$, the *cofinality* of I in M, $\mathrm{cf}^M(I)$, is the intersection of all $J\subseteq^e M$ for which there is a function $F\in\mathfrak{F}$ such that $J\subseteq\mathrm{dom}(F)$ and $F''J$ is cofinal in I, i.e. $\sup_{\le^M}(F''J)=I$. Then I is semi-regular in M iff $\mathrm{cf}^M(I)=I$ [Kir77, 2.7]. Thus, I is semi-regular iff for every $X\in\mathfrak{D}$ such that $X\cap I$ is cofinal in I there exists an order-preserving function $F\in\mathfrak{F}$ 'compressing' $X\cap I$ onto I.

In analogy with regular filters rather than regular cardinals, the established terminology in the model theory of arithmetic uses the attribute *regular* for cuts with the following property:

4.3.2 Definition. A cut $I\subseteq^e M$ is *regular* if for every $F\in\mathfrak{F}$ such that $I\subseteq\mathrm{dom}(F)$ and $F''I\subseteq[0,a]$ for some $a\in I$, there exists $b\in[0,a]$ such that $F^{-1}[\{b\}]\cap I$ is cofinal in I.

4.3.3 Remark. Regularity implies semi-regularity, for let I be a regular cut and let $F\in\mathfrak{F}$, $\mathrm{dom}(F)=[0,a]$, $a\in I$; aiming for a contradiction, assume $\mathrm{rng}(F)\in\mathfrak{D}(\mathfrak{g}_I)$. The elements of $\mathrm{rng}(F)$ naturally partition I into intervals: let G be a function with $\mathrm{dom}(G)=\sup(\mathrm{rng}(F))$ such that $G(x)\stackrel{\mathrm{df}}{=}\min\{y\mid x\le F(y)\wedge[x,F(y))\cap\mathrm{rng}(F)=\emptyset\}$. Then $G\in\mathfrak{F}$, $I\subseteq\mathrm{dom}(G)$, $\mathrm{rng}(G)\subseteq[0,a]$, and each $G^{-1}[\{y\}]$ with $y\in\mathrm{rng}(G)$ is an interval, whose only intersection with $\mathrm{rng}(F)$ is its end point, $F(y)$. By regularity, there is $b\in[0,a]$ such that $G^{-1}[\{b\}]\in\mathfrak{D}(\mathfrak{g}_I)$. But then $\mathfrak{g}_I\subseteq G^{-1}[\{b\}]$ and there is some $b'\in\mathrm{rng}(F)$ such that $(F(b'),F(b))\cap\mathrm{rng}(F)=\emptyset$. Then $F(b')$ is the last element of $\mathrm{rng}(F)$ in I—a contradiction.

Using semi-regularity, we may now replace the condition $I\subseteq\mathrm{dom}(F)$ in the definition of regularity with the condition that $\mathrm{dom}(F)\cap I$ is cofinal in I.

4.3.4 Definition. Let $X\subseteq C$ be any subset (not necessarily M-definable).

- Recall that for an equivalence $P\in\mathfrak{D}$, $\|P\|$ is the number of equivalence classes of P (assuming there are only boundedly many of them).
- Let $a,b\in M$, $a,b\ge1$. Let $I\subseteq^e M$. We define:

$$\langle X\rangle^a_b \stackrel{\mathrm{df}}{=} \{P\in\mathfrak{D}\mid P\text{ is an equivalence, }\|P\|\le b,\text{ and }\langle X\rangle^a\subseteq\mathrm{dom}(P)\},$$
$$\langle X\rangle^a_{<I} \stackrel{\mathrm{df}}{=} \bigcup_{b\in I}\langle X\rangle^a_b.$$

We refer to equivalences from $\langle X\rangle^a_b$ as *partitions* and to their equivalence classes as their *blocks* or *parts*. Note that if $a,b\in M$, then by our definition $\langle X\rangle^a_b\subseteq\mathfrak{D}$ for any subset $X\subseteq C$ and $\langle X\rangle^a\in\mathfrak{D}$ for $X\in\mathfrak{D}$. Moreover, if X is bounded, it is coded in M, and hence $\langle X\rangle^a$, every element of $\langle X\rangle^a_b$, as well as $\langle X\rangle^a_b$ itself are coded in M, too.

The combinatorial property of cuts introduced with regularity naturally generalizes as follows:

4.3.5 Definition. Let $I \subseteq^e M$ and $X \subseteq C$. Then

$$X \to (I)^a_b$$

asserts that for every $P \in \langle X\rangle^a_b$ there exists $Y \subseteq X$ such that $Y \in \mathfrak{D}(\mathfrak{g}_I)$ and Y is homogeneous for P. The symbol $X \to (I)^a_{<I}$ is defined similarly for partitions from $\langle X\rangle^a_{<I}$.

An equivalent definition in terms of functions would be: $X \to (I)^a_b$ iff for every $F : \langle X\rangle^a \to [0,b)$ there is $Y \subseteq X$ with $Y \in \mathfrak{D}(\mathfrak{g}_I)$ such that $F\restriction\langle Y\rangle^a$ is constant.

Under this notation, an cut I is regular in M iff $I \to (I)^1_{<I}$.

4.3.6 Proposition. *Let I be a semi-regular cut, $a \in M$, and $b \in I$. Then the following statements are equivalent:*

a) $I \to (I)^a_b$,

b) $(\forall X \in \mathfrak{D}(\mathfrak{g}_I))X \to (I)^a_b$.

Proof. b)→a) is trivial. Conversely: Let $X \in \mathfrak{D}(\mathfrak{g}_I)$ and $P \in \langle X\rangle^a_b$. Let $F \in \mathfrak{F}$ be the unique order-preserving map of X onto a lower subset of C (i.e. $x \in X$ is the $F(x)$-th element in X in the ordering by $\leq$) and let $X' \stackrel{\mathrm{df}}{=} \mathrm{rng}(F)$. It follows easily from semi-regularity, that $F^{-1} \in \mathfrak{D}(I)$. In particular, $X' \cap I$ is unbounded in I, so $I \subseteq X'$. We transfer the partition P to X' as follows: For $\alpha \in \langle X\rangle^a$, let $F\alpha$ denote the unique element of $\langle X'\rangle^a$ such that $(F\alpha)_i = F((\alpha)_i)$ for all $i < a$. We may now define an equivalence Q on $\langle X'\rangle^a$ by

$$\langle \alpha,\beta\rangle \in Q \stackrel{\mathrm{df}}{\iff} \langle F\alpha, F\beta\rangle \in P, \tag{4.3}$$

for all $\alpha,\beta \in \langle X'\rangle^a$. Clearly $\|P\| = \|Q\| \leq b$, so we have $Q \in \langle X'\rangle^a_b$. Now, $I \to (I)^a_b$ gives some $Y' \in \mathfrak{D}(\mathfrak{g}_I)$, $Y' \subseteq X'$, homogeneous for Q. Then $Y = F^{-1}[Y'] \subseteq X$ is clearly homogeneous for P; since $F^{-1} \in \mathfrak{F}(I)$ and $x \leq F^{-1}(x)$ for all $x \in X'$, $Y \cap I$ is unbounded in I. □

4.3.7 Definition. A cut $I \subseteq^e M$ is *strong* in M, if $I \to (I)^n_{<I}$, for every $n \in \mathbb{N}$.

There is a nice equivalent of strength:

4.3.8 Lemma. *A proper cut I of M is strong iff for every $F \in \mathfrak{F}$ there exists $c \in M \setminus I$ such that $F''I \subseteq I \cup [c,\to)$.*

Proof. We shall sometimes use the implication from left-to-right, which we prove now. We may assume dom(F) is C. Let $X \stackrel{\text{df}}{=} \{\langle a,b,c\rangle \in \langle C\rangle^3 \mid F''[0,a] \cap [b,c] = \emptyset\}$ and $X' = \langle C\rangle^3 \setminus X$. The partition of $\langle C\rangle^3$ to X, X' has a homogeneous set $Y \in \mathfrak{D}(\mathfrak{g}_I)$. The result follows immediately if $\langle Y\rangle^3 \subseteq X$ by taking arbitrary $c \in X \cap (M \setminus I)$. Now, semi-regularity eliminates the other case as it gives for every $a \in Y$ some $b,c \in M$ such that $\mathfrak{g}_I \subseteq [b,c]$ and $F''[0,a] \cap [b,c] = \emptyset$; since $Y \in \mathfrak{D}(\mathfrak{g}_I)$, we may take $b,c \in Y$ and $a < b$; then $\langle a,b,c\rangle \in X$.

For proof of the converse implication, see [KS06, 7.3] or [Kir77, 7.5]. □

We are now about to summarize some fundamental facts concerning the types of cuts we have just introduced. Some of them are best formulated in terms of second-order arithmetic, which we introduced in 1.4.

4.3.9 Definition. For an initial substructure $I \subseteq^e M$, we define

$$\mathrm{Cod}(M/I) \stackrel{\text{df}}{=} \{I \cap X \mid X \in \mathfrak{D}\}.$$

We shall denote the $\mathcal{L}^{\mathrm{II}}$ structure $\langle I, \mathrm{Cod}(M/I)\rangle$ by I^*.

$\mathrm{Cod}(M/I)$ is a countable Boolean subalgebra of $\mathcal{P}(I)$; obviously, $\mathrm{Cod}(M/M)$ is just an isomorphic copy of $\mathfrak{D}$. For $I = \mathbb{N}$, $\mathrm{Cod}(M/\mathbb{N})$ is the *standard system of M*, $\mathrm{SSy}(M)$, introduced by H. Friedman ([Fri73]).

Note that for $I \subset^e M$, $\mathrm{Cod}(M/I) = \{I \cap X \mid X \in \mathfrak{D}_{\Delta_1}\}$, since every set from $\mathrm{Cod}(M/I)$ is of the form $A = \{(a)_i \in I \mid i < \ell(a)\}$ for some $a \in M$. Moreover, if $I \models \mathrm{I}\Sigma_1$, then for a given A, the coding element a can be taken from $M \setminus I$ arbitrarily close to I, since every bounded part of A is coded in I. Thus if $I \subset^e J \subset^e M$ and $I, J \models \mathrm{I}\Sigma_1$, then $\mathrm{Cod}(M/I) = \mathrm{Cod}(J/I)$.

If $I \subset^e M$ and $X, Y \in \mathfrak{D}$ determine the same set from $\mathrm{Cod}(M/I)$, i.e. $I \cap X = I \cap Y$, then the least element of $X \dot{-} Y$ is in $M \setminus I$; in particular $X \cap \sup \mathfrak{g}_I = Y \cap \sup \mathfrak{g}_I$.

4.3.10 Lemma. *For any $I \subset^e M \models \mathrm{I}\Sigma_0$, $I^* \models \Delta^0_1\mathrm{CA}_0 + \mathrm{B}\Sigma^0_1$.*

Proof. We prove this well-known lemma in B.1. □

Now we are ready to formulate

4.3.11 Fact. *Let I be an initial substructure of M.*

a) *I is semi-regular in M iff $I^* \models \mathrm{I}\Sigma^0_1$.*

b) *I is regular in M iff $I^* \models \mathrm{B}\Sigma^0_2$.*

c) *I is strong in M iff $I^* \models \mathrm{ACA}_0$.*

Proof. C.f. for example [KS06, Chapter 7]. We prove c) in B.2. □

4.3.12 Corollary.

a) *If I is semi-regular, then $I \models \mathrm{I}\Sigma_1$.*

b) *If I is regular, then $I \models \mathrm{B}\Sigma_2$.*

c) *If I is strong, then $I \models$ PA.*

The converse implications do not hold. More specifically, for every $n \geq 0$ there are n-elementary initial substructures of M satisfying PA that are not even semi-regular. Consider the indicator Y from 3.4.2 b) for the family $\mathcal{P}_n$ of n-elementary initial substructures of M satisfying PA. If $Y(a,b) = c > \mathbb{N}$ for $a,b \in M$, then the interval (a,b) includes a coded sequence u of length $\ell(u) = c$, such that every cut in which the elements of u are unbounded, belongs to $\mathcal{P}_n$. Thus $I = \sup_{\leq M} \{(u)_n \mid n \in \mathbb{N}\} \in \mathcal{P}_n$. Yet, if a is non-standard, then $\{(u)_i \mid i < \min(a,c)\} \in \mathcal{D}(\mathfrak{g}_I)$ while $\min(a,c) \in I$, hence I is not semi-regular. It even follows that the families $\mathcal{P}_n$ and $\{I \in \mathcal{P}_n \mid I \text{ is not semi-regular in } M\}$ are symbiotic.

The following well known results are due Kirby and Paris. The proofs can be found in [Kir77, namely 2.16, 2.18,4.13, 4.19].

4.3.13 Theorem (*Kirby and Paris*).

a) *There is a Σ_1 indicator for semi-regular cuts.*

b) *Semi-regular cuts of M and cuts from $\mathcal{R}_{\mathrm{I}\Sigma_1}$ are symbiotic.*

c) *Regular cuts and semi-regular cuts are symbiotic. In particular, there is a Σ_1 indicator for regular cuts.*

d) *For every $a \in M$, there is a semi-regular cut I of M such that $a \in I$ and I is not regular; i.e., semi-regular $\neq$ regular.*

e) *For every $a \in M$, there is a regular cut I such that $a \in I$ and $I \not\models \mathrm{I}\Sigma_2$ (and in fact $\mathcal{R}_{\mathrm{I}\Sigma_2} \not\nearrow I$). Hence $\mathrm{B}\Sigma_2$ in 4.3.12 b) is optimal.*

f) *A cut I of M is strong in M iff I is semi-regular and $I \to (I)^3_2$.*

g) *There is a Σ_1 indicator for the family of strong cuts.*

h) *There is an elementary end extension N of M such that M is a strong cut of N.*

i) *Strong cuts in M and cuts from $\mathcal{P}_0$ are symbiotic.* □

In particular, since each of the families of cuts has a Σ_1 (and thus Δ_1) indicator, all items from our Theorem 3.2.1 apply. We now start relating the basic properties of cuts with properties of monads and gaps.

4.3.14 Proposition. *A cut I is semi-regular iff $\mathfrak{g}_I$ is an original gap.*

Proof. Let $\mathfrak{g}$ denote $\mathfrak{g}_I$. Left-to-right: suppose $F : \mathfrak{m} \asymp_{\mathrm{RK}} \mathfrak{n}$, with $\mathfrak{m} \subseteq \mathfrak{g}$ and $\mathfrak{n} \subseteq \inf \mathfrak{g}$. Then $\mathfrak{n} \subseteq [0, a]$ for some $a \in I$; we have $F^{-1} \in \mathfrak{F}$, $\mathfrak{m} \subseteq F^{-1\prime\prime}[0,a] \in \mathfrak{D}(\mathfrak{g})$), which contradicts semi-regularity of I. Conversely: let $a \in I$, $F \in \mathfrak{F}$, and suppose $F''[0,a] \in \mathfrak{D}(\mathfrak{g})$. Let $G(x) \stackrel{\mathrm{df}}{=} \min(F^{-1}[\{x\}])$ for $x \in \mathrm{dom}(F)$. Then G is a 1-1 map with $\mathrm{dom}(G) = \mathrm{rng}(F)$ and $\mathrm{rng}(F) \subseteq [0,a]$. In particular, it maps some monad $\mathfrak{m} \subseteq \mathfrak{g}$ onto a non-trivial monad $G''\mathfrak{m} \subseteq \inf \mathfrak{g}$, contradicting originality. □

Regular monads

In analogy with regular cuts we define:

4.3.15 Definition. Let I be a cut and $\mathfrak{m}$ a monad in the gap of I. We call $\mathfrak{m}$ a *regular monad* if every function $F \in \mathfrak{F}$ such that $\mathfrak{m} \subseteq \mathrm{dom}(F)$ and $\mathrm{rng}(F) \subseteq [0,a]$ for some $a \in I$ is constant on $\mathfrak{m}$.

In terms of partitions, $\mathfrak{m}$ is regular iff it is homogeneous for every partition $P \in \langle \mathfrak{m} \rangle^1_{<I}$, symbolically $\mathfrak{m} \to (\mathfrak{m})^1_{<I}$.

4.3.16 Remark. The property of the ultrafilter $\mathfrak{U}(\mathfrak{m})$ corresponding to regularity is called *additivity* by Kirby [Kir84].

4.3.17 Proposition. *Every relatively large minimal monad is regular.*

Proof. Let $\mathfrak{m}$ be relatively large and minimal, $F : X \to [0,a]$, $X \in \mathfrak{U}(\mathfrak{m})$, $F \in \mathfrak{F}$, and $a \in \inf(\mathfrak{m})$. By minimality, F is either constant on $\mathfrak{m}$ (in which case we are done), or 1-1. In the latter case, F is 1-1 on some $Y \subseteq X$, $Y \in \mathfrak{U}(\mathfrak{m})$. Since $\mathfrak{m}$ is relatively large, $|Y| \geq |Y \setminus [0,a]| > a + 1$, a contradiction. □

4.3.18 Remark.

a) Every regular monad $\mathfrak{m}$ is relatively large. Otherwise we could enumerate some Y from $\mathfrak{U}(\mathfrak{m})$ by elements less than $a = \min Y < \mathfrak{m}$ and thus partition $\mathfrak{m}$ into less than a singleton parts.

b) Not every minimal monad is relatively large; in particular, there are minimal monads that are not regular. Indeed, if $\mathfrak{m}$ is a minimal monad and $\mathfrak{m} < a \in M$, then $a + \mathfrak{m} \stackrel{\mathrm{df}}{=} \{a + \alpha \mid \alpha \in \mathfrak{m}\}$ is minimal, but not relatively large.

c) Not every regular monad is minimal; in particular, 4.3.17 cannot be reversed. As a simple counterexample, consider the gap $\mathfrak{g}_{\mathbb{N}}$ in which all monads are regular for trivial reasons, yet not all of them are minimal. To find a non-minimal one, take e.g. the first projection of Cantor's pairing function: $\pi_0 : x \mapsto \langle x \rangle_0$. This $\emptyset$-definable function partitions $\mathbb{N}$ into infinitely many infinite blocks. Since $\mathfrak{D}$ is countable, we may enumerate (in meta-theory) all M-definable choice sets for this partition. We may thus construct a non-increasing sequence $\{X_n\}_{n\in\omega}$ of subsets of $\mathbb{N}$ codable in M such that X_n avoids both the n-th choice set and the n-th block of the considered partition, while ensuring that π_0 still partitions X_n into infinitely many infinite blocks. The intersection of $\{X_n\}_{n\in\omega}$ yields a non-minimal monad. We have a stronger result that generalizes of this construction in Theorem 4.4.25.

4.3.19 Theorem (*On regularity*). *Let I be a cut. The following conditions are equivalent:*

a) *I is regular.*

b) *I is semi-regular and and $\mathfrak{g}_I$ includes a minimal monad.*

c) *$\mathfrak{g}_I$ includes a relatively large minimal monad.*

d) *$\mathfrak{g}_I$ includes a regular monad.*

Proof. Let $\mathfrak{g}$ denote the gap $\mathfrak{g}_I$ for brevity.

a)→b) I is semi-regular by 4.3.3. We shall construct a non-increasing sequence $\{X_n\}_{n\in\omega}$ of sets from $\mathfrak{D}(\mathfrak{g})$ whose intersection will be a minimal monad included in $\mathfrak{g}$. For that, let $\{F_n\}_{n\in\omega}$ be some enumeration of $\mathfrak{F}$ and let $\{L_n\}_{n\in\omega}$ be a sequence of intervals with endpoints in M such that $\bigcap_{n\in\omega} L_n = \mathfrak{g}$. We define $\{X_n\}_{n\in\omega}$ inductively, starting from arbitrary $X_0 \in \mathfrak{D}(\mathfrak{g})$. At $(n+1)$-st stage we ensure that $X_{n+1} \subseteq L_n$ and that F_n is either constant of 1-1 on X_{n+1}. Let $Y \overset{\text{df}}{=} \operatorname{dom}(F_n) \cap X_n \cap L_n$. If $Y \cap \mathfrak{g} = \emptyset$, we put $X_{n+1} \overset{\text{df}}{=} X_n \cap L_n$. Otherwise $Y \in \mathfrak{D}(\mathfrak{g})$. Let $F \overset{\text{df}}{=} F_n \restriction Y$ and $Z \overset{\text{df}}{=} \{\alpha \in Y \mid (\forall \beta < \alpha)(\beta \in Y \to F(\alpha) \neq F(\beta))\}$. Clearly, F is 1-1 on Z, so if $Z \in \mathfrak{D}(\mathfrak{g})$, we may put $X_{n+1} \overset{\text{df}}{=} Z$. Otherwise Z is bounded in I, so for some $a \in I$, $Z \subseteq [0,a]$. Let $G(x) = \min(F^{-1}[\{F(x)\}])$ for every $x \in Y$. Then $G \in \mathfrak{F}$, $\operatorname{dom}(G) = Y$ and $\operatorname{rng}(G) = Z \subseteq [0,a]$. Applying regularity of I (and the sequel in Remark 4.3.3), we have $G^{-1}[\{b\}] \in \mathfrak{D}(\mathfrak{g})$ for some $b \in Z$; we put $X_{n+1} \overset{\text{df}}{=} G^{-1}[\{b\}]$. Now, for every $x \in X_{n+1}$, $G(x) = b$, thus $F(x) = F(b)$, hence F is constant on X_{n+1}. This completes the induction step. By $\aleph_1$-saturation, there exists $\alpha \in \bigcap_{n\in\omega} X_n$. The construction ensures that $\mathfrak{m}(\alpha) \subseteq \mathfrak{g}$ and that every $F \in \mathfrak{F}$ with $\alpha \in \operatorname{dom}(F)$ is either constant or 1-1 on $\mathfrak{m}(\alpha)$, so the monad is minimal by 4.2.20.

b)→c) Trivial, since if I is semi-regular, then every monad included in $\mathfrak{g}$ is relatively large by 4.2.25,

c)→d) is a consequence of 4.3.17 .

d)→a) Let $\mathfrak{m} \subseteq \mathfrak{g}$ be a regular monad. If $F \in \mathfrak{F}$ satisfies $I \subseteq \operatorname{dom}(F)$ and $F''I \subseteq [0,a]$ for some $a \in I$, then $\mathfrak{m} \subseteq \operatorname{dom}(F)$, so by definition of regularity for monads, F is constant on $\mathfrak{m}$. By monadic overspill, F is constant on some $X \in \mathfrak{D}(\mathfrak{m})$. In particular, $X \in \mathfrak{D}(\mathfrak{g})$, witnessing regularity of $\mathfrak{g}$. □

The construction in the proof of the implication a)→b) started with arbitrary $X_0 \in \mathfrak{D}(\mathfrak{g})$ and yielded a relatively large minimal monad $\mathfrak{m} \subseteq X_0 \cap \mathfrak{D}(\mathfrak{g})$. This means that the set of relatively large minimal monads is dense in the Stone topology of $\mathfrak{g}/\sim$. Applying 4.2.15, we have

4.3.20 Corollary. *Let $\mathfrak{g}$ be a gap of a regular cut. Then the set of relatively large minimal monads is distributive over $\mathfrak{g}$.* □

We now give two characterizations of regular monads, one based on the notion of the Rudin-Keisler pre-order, the other of a model-theoretic nature. For the latter we need a definition.

4.3.21 Definition. Let $N \subseteq C$ be a submodel such that $M \subseteq N$ and let I be a cut of M. We write $M \preccurlyeq_I N$ if N is *an elementary I-end extension of M*, i.e. if $M \preccurlyeq N$ and $N \cap \operatorname{sup}_{\leq^N}(I) = I$.

4.3.22 Proposition. *Let $M \preccurlyeq_I N$. If $X \in \mathfrak{D}(C,N)$ and $Y \in \mathfrak{D}$ are such that $X \cap I = Y \cap I$, then $X \cap \operatorname{sup} I = Y \cap \operatorname{sup} I$.*

Proof. Let X, Y be as assumed and let $Z = X \dot{-} Y$. Then $Z \in \mathfrak{D}(C,N)$. Let $z \in Z$ be the least element of Z. Then clearly $z \in N$; suppose $z \in \operatorname{sup} I$. Then $z \in N \cap \operatorname{sup}_{\leq^N}(I) = I$, which contradicts $X \cap I = Y \cap I$. □

4.3.23 Theorem. *Let $\mathfrak{m}$ be a non-trivial monad, $I = I_{\mathfrak{m}}$, and $\alpha \in \mathfrak{m}$. The following properties are equivalent:*

a) $\mathfrak{m}$ *is regular,*

b) $M \preccurlyeq_I M[\alpha]$,

c) $\mathfrak{n} \not\leq_{\mathrm{RK}} \mathfrak{m}$ *for every non-trivial monad* $\mathfrak{n} \subseteq \operatorname{inf} \mathfrak{m}$.

Proof. a)→b) Let $\mathfrak{m}$ be a regular monad, $\alpha \in \mathfrak{m}$. We only have to prove that $M[\alpha] \cap \operatorname{sup} I = I$. Recall that $M[\alpha] = \{F(\alpha) \mid \alpha \in C \wedge F \in \mathfrak{F}\}$. Let $F \in \mathfrak{F}$ satisfy $F(\alpha) \in \operatorname{sup} I$. There is some $a \in I$ such that $F(\alpha) < a$. Let $X \overset{\mathrm{df}}{=} F^{-1\prime\prime}[0,a)$. Then $X \in \mathfrak{U}(\mathfrak{m})$ and $F \restriction X$ induces a partition $P \in \langle X \rangle_a^1$ such that $P_{(y)} = F^{-1\prime\prime}\{y\}$ for $y \in [0,a)$. Being regular, $\mathfrak{m}$ is homogeneous for P, hence $F''\mathfrak{m}$ is a singleton. It follows that $F(\alpha) \in I$.

b)→c) Let $M \preccurlyeq_I M[\alpha]$ and suppose there is some $\mathfrak{n} \subseteq \operatorname{sup} I$ such that $\mathfrak{n} \leq_{\mathrm{RK}} \mathfrak{m}$. Then for a certain $F \in \mathfrak{F}$, $F''\mathfrak{m} = \mathfrak{n}$, so $F(\alpha) \in M[\alpha] \cap \operatorname{sup} I \setminus I$—a contradiction.

c)→a) To see that $\mathfrak{m}$ is regular, we only have to verify that every $F \in \mathfrak{F}$ with $\operatorname{dom}(F) \in \mathfrak{U}(\mathfrak{m})$ and $\operatorname{rng}(F) \supseteq [0,a]$ for some $a \in I$ is constant on $\mathfrak{m}$. But if, for a certain F this was not the case, then $\mathfrak{n} = F''\mathfrak{m}$ would be a non-trivial monad from $\operatorname{sup} I$, less or equal to $\mathfrak{m}$ in the Rudin-Keisler pre-order, which contradicts our assumption. □

4.3.24 Remark. Item c) above with 4.2.23 e) give that a regular monad $\mathfrak{m}$ is RK-incomparable with any <-less non-trivial monad.

From Remark 4.3.18 we know there are regular monads that are not minimal in the Rudin-Keisler pre-order. At least, regularity is preserved downwards within a gap:

4.3.25 Proposition. *Let* $\mathfrak{m}, \mathfrak{n} \subseteq \mathfrak{g}$ *be monads such that* $\mathfrak{n} \preceq_{\mathrm{RK}} \mathfrak{m}$. *If* $\mathfrak{m}$ *is regular, then* $\mathfrak{n}$ *is regular, too.*

Proof. Let $F \in \mathfrak{F}$ be such that $F''\mathfrak{m} = \mathfrak{n}$. It suffices to prove that $M[\beta]$ is a I-end extension of M for all $\beta \in \mathfrak{n}$. Indeed, if $\beta \in \mathfrak{n}$, there is some $\alpha \in \mathfrak{m}$ with $F(\alpha) = \beta$. Clearly $M[\beta] \subseteq M[\alpha]$, since $\beta = F(\alpha) \in M[\alpha]$, and $M[\alpha]$ is a I-end extension of M by regularity of $\mathfrak{m}$. Hence $M[\beta]$ must be a I-end extension of M as well. □

Ramsey monads

We have already expressed regularity of a monad $\mathfrak{m} \subseteq \mathfrak{g}_I$ as $\mathfrak{m} \to (\mathfrak{m})^1_{<I}$. The first item in the following definition generalizes the associated combinatorial property in the usual way.

4.3.26 Definition. Let $k \in \mathbb{N}$, $a, b \in M$, $a, b, k \geq 1$.

- Let $\mathfrak{m}$ be a non-trivial monad. We write $\mathfrak{m} \to (\mathfrak{m})^a_b$ if $\mathfrak{m}$ is homogeneous for every partition $P \in \langle \mathfrak{m} \rangle^a_b$. If $I \subseteq^e M$, we write $\mathfrak{m} \to (\mathfrak{m})^a_{<I}$ meaning that $\mathfrak{m} \to (\mathfrak{m})^a_b$ holds for every $b \in I$.
- In accordance with the terminology for cuts, we say that $\mathfrak{m}$ is a *strong monad*, if $\mathfrak{m} \to (\mathfrak{m})^n_2$ for every $n \in \mathbb{N}$.
- A non-trivial monad $\mathfrak{m}$ is an *a-monad* if $\langle \mathfrak{m} \rangle^a$ is a monad.
- A subset $Y \subseteq C$ is a set of *k-indiscernible elements* (over M) if for any $\mathfrak{L}^{Ar}_M$-formula $\varphi(x_1, \ldots, x_k)$ and any two increasing k-tuples $\vec{\alpha}$, $\vec{\beta}$ of elements from Y, $C \models \varphi(\vec{\alpha}) \leftrightarrow \varphi(\vec{\beta})$.

Of course, every non-trivial monad $\mathfrak{m}$ is a 1-monad and satisfies $\mathfrak{m} \to (\mathfrak{m})^1_2$.

Note that by 4.2.11, homogeneity for a partition P with $P \in \mathfrak{D}$ is a property with monadic overspill; thus, if $\mathfrak{m}$ is homogeneous for $P \in \langle \mathfrak{m} \rangle^a_b$, then so is some $X \in \mathfrak{U}(\mathfrak{m})$.

The following two propositions are easy.

4.3.27 Proposition. *Let* $a, b \in M$, $a \geq 1$, $b \geq 2$.

a) *If* $\mathfrak{m} \to (\mathfrak{m})^a_2$ *then* $\mathfrak{m} \to (\mathfrak{m})^a_{<\mathbb{N}}$.

b) *If* $\mathfrak{m} \to (\mathfrak{m})^a_b$ *and* $\langle \mathfrak{m} \rangle^a \neq \emptyset$, *then* $(\forall c \leq a)\mathfrak{m} \to (\mathfrak{m})^c_b$.

Proof. a) Let $\mathfrak{m} \to (\mathfrak{m})^a_n$ with $n \geq 2$ and $P \in \langle\mathfrak{m}\rangle^a_{n+1}$. By joining the first two blocks of P together (the blocks being naturally ordered by their first elements), we obtain some $P' \in \langle\mathfrak{m}\rangle^a_n$. The hypothesis gives homogeneity for P'. If $\langle\mathfrak{m}\rangle^a$ falls into the joint pair of blocks of P, we apply $\mathfrak{m} \to (\mathfrak{m})^a_2$ on a partition consisting just of these two blocks.

b) For $c < a$, $P \in \langle X\rangle^c_b$, and $X \in \mathfrak{U}(\mathfrak{m})$, let $P' \stackrel{\mathrm{df}}{=} \{\langle\alpha,\beta\rangle \mid \alpha,\beta \in \langle X\rangle^a \wedge \langle\alpha\restriction c, \beta\restriction c\rangle \in P\}$, where $\alpha\restriction c$ denotes the initial subsequence of the sequence α with $\ell(\alpha\restriction c) = c$. Then $P' \in \langle\mathfrak{m}\rangle^a_b$. By the assumption, $\mathfrak{m}$ is homogeneous for P'. Let $\gamma,\delta \in \langle\mathfrak{m}\rangle^c$. We shall show that there is some $\varepsilon \in \langle\mathfrak{m}\rangle^{a-c}$ such that $(\varepsilon)_0 > \max((\delta)_{c-1},(\gamma)_{c-1}) \in \mathfrak{m}$. Then $\langle\gamma\smile\varepsilon, \delta\smile\varepsilon\rangle \in P'$, so $\langle\gamma,\delta\rangle \in P$. The existence of ε follows from the fact that $\langle\mathfrak{m}\rangle^a \neq \emptyset$; by this assumption, for every $X \in \mathfrak{U}(\mathfrak{m})$ and $e \in I$, $|X \setminus [0,e]| \geq a$. For every $\varepsilon_0 \in \mathfrak{m}$, $|X \setminus [0,\varepsilon_0]| \geq a$, i.e. $\langle X \setminus [0,\varepsilon_0]\rangle^a \neq \emptyset$, by overspill. Thus $\langle\mathfrak{m} \setminus [0,\varepsilon_0]\rangle^a \neq \emptyset$, by $\aleph_1$-saturation. □

4.3.28 Proposition. *Let $\mathfrak{m}$ be a non-trivial monad and $k \geq 2$ natural. Then the following properties are equivalent:*

a) $\mathfrak{m} \to (\mathfrak{m})^k_2$

b) $\mathfrak{m}$ *is a monad of k-indiscernible elements.*

c) $\mathfrak{m}$ *is a k-monad.*

Proof. a)→b) Let $\varphi(x_1,\dots,x_k)$ be an $\mathfrak{L}^{Ar}_M$-formula and let

$$P_\varphi \stackrel{\mathrm{df}}{=} \{\langle u,v\rangle \mid u,v \in \langle C\rangle^k \wedge C \models \varphi((u)_0,\dots,(u)_{k-1}) \leftrightarrow \varphi((v)_0,\dots,(v)_{k-1})\}.$$

Then $P_\varphi \in \langle C\rangle^k_2$. Hence $\mathfrak{m}$ is homogeneous for P_φ. For any two increasing k-tuples $\vec{\alpha},\vec{\beta} \in \mathfrak{m}$ thus $\langle\langle\vec{\alpha}\rangle,\langle\vec{\beta}\rangle\rangle \in P_\varphi$, so $C \models \varphi(\vec{\alpha}) \leftrightarrow \varphi(\vec{\beta})$.

b)→c) We have to show that $\langle\mathfrak{m}\rangle^k$ is a monad. Let $\alpha,\beta \in \langle\mathfrak{m}\rangle^k$ and $X \in \mathfrak{D}$. The set X is defined in C by an $\mathfrak{L}^{Ar}_M$, so there is an $\mathfrak{L}^{Ar}_M$ formula $\varphi(x_0,\dots,x_{k-1})$ such that $\gamma \in X$ iff $C \models \varphi((\gamma)_0,\dots,(\gamma)_{k-1})$, whenever $\gamma \in \langle C\rangle^k$. Applying this equivalence and k-indiscernibility of $\mathfrak{m}$ for φ, we have $\alpha \in X$ iff $\beta \in X$. Since $X \in \mathfrak{D}$ was arbitrary, we have $\alpha \sim \beta$.

c)→a) Let $P \in \langle X\rangle^k_2$ with $X \in \mathfrak{U}(\mathfrak{m})$. Since there are only two blocks of the partition P, they are both in $\mathfrak{D}$. By the assumption, $\langle\mathfrak{m}\rangle^k$ is a monad, so it falls into exactly one of them. □

4.3.29 Proposition.

a) *If* $\mathfrak{m} \to (\mathfrak{m})^2_2$, *then* $\mathfrak{m}$ *is minimal.*

b) *Let $a,b \in M$, $a,b \geq 2$. If* $\mathfrak{m} \to (\mathfrak{m})^a_b$ *and* $\mathfrak{n} \asymp_{\mathrm{RK}} \mathfrak{m}$, *then* $\mathfrak{n} \to (\mathfrak{n})^a_b$.

Proof. a) By 4.3.28, $\mathfrak{m}$ be a 2-monad. Assume $F \in \mathfrak{F}$ with $\mathfrak{m} \subseteq \mathrm{dom}(F)$ is not 1-1 on $\mathfrak{m}$. Then $F(\alpha) = F(\beta)$ for some $\alpha < \beta$ from $\mathfrak{m}$; since $\langle\mathfrak{m}\rangle^2$ is a monad, the equality holds for every $\langle\alpha,\beta\rangle \in \langle\mathfrak{m}\rangle^2$. Thus F is constant on $\mathfrak{m}$.

b) Let $\mathfrak{m}, a, b$ be as assumed and $F : \mathfrak{m} \asymp_{\mathrm{RK}} \mathfrak{n}$. Let $P \in \langle X\rangle^a_b$ with $X \in \mathfrak{U}(\mathfrak{n})$ and let $Y \overset{\mathrm{df}}{=} F^{-1}[X]$. Then $Y \in \mathfrak{U}(\mathfrak{m})$. For $\alpha \in \langle Y\rangle^a$, let α_F denote the element from $\langle X\rangle^a$ such that $\{(\alpha_F)_i \mid i < a\} = \{F((\alpha)_j) \mid j < a\}$, i.e. α_F is a pointwise image of α, with elements arranged to form an increasing sequence; since F is 1-1, this rearrangement is possible and is unique. The map $\alpha \mapsto \alpha_F$ is clearly definable. We may now define a partition $P' \in \langle Y\rangle^a_b$ as follows: $P' \overset{\mathrm{df}}{=} \{\langle\alpha,\beta\rangle \in \langle Y\rangle^a \mid \langle\alpha_F,\beta_F\rangle \in P\}$. Since $\mathfrak{m}$ is P'-homogeneous each pair $\gamma,\delta \in \langle\mathfrak{n}\rangle^a$ is of the form α_F, β_F for some $\alpha,\beta \in \langle\mathfrak{m}\rangle^a$, $\mathfrak{n}$ is P-homogeneous. □

4.3.30 Remark.

1. Item b) also works for $a = 1$ and $\mathfrak{n} \preceq_{\mathrm{RK}} \mathfrak{m}$.
2. By b), monads satisfying $\mathfrak{m} \to (\mathfrak{m})^a_b$ do not have to be relatively large. For instance, if $c > I_{\mathfrak{m}}$, then the monad $c + \mathfrak{m}$ defined as in 4.3.18 is not relatively large, but if $\mathfrak{m}$ is as above we still have $c + \mathfrak{m} \to (c + \mathfrak{m})^a_b$.

4.3.31 Proposition. *Let $I \subseteq^e M$ and $a, b \in M$, $a, b \geq 2$. The following are equivalent:*

a) *I is semi-regular and $I \to (I)^a_b$.*

b) *There exists a relatively large monad $\mathfrak{m} \subseteq \mathfrak{g}_I$ such that $\mathfrak{m} \to (\mathfrak{m})^a_b$.*

Proof. a)→b). Let $\{P_n\}_{n\in\omega}$ be an enumeration of $\langle C\rangle^a_b$, and assume $\mathfrak{g}_I = \bigcap_{n\in\omega} L_n$ where $\{L_n\}_{n\in\omega}$ are intervals with endpoints in M. We define a non-increasing sequence $\{X_n\}_{n\in\omega}$ of sets from $\mathfrak{D}(\mathfrak{g}_I)$ in stages, starting with arbitrary $X_0 \in \mathfrak{D}(\mathfrak{g}_I)$. If X_n is defined, then by 4.3.6, there exists some $X_{n+1} \subseteq X_n \cap L_n$ homogeneous for P_n such that $X_{n+1} \in \mathfrak{D}(\mathfrak{g}_I)$. By $\aleph_1$-saturation, there exists $\alpha \in \bigcap_{n\in\omega} X_n \subseteq \mathfrak{g}_I \neq \emptyset$. Then $\mathfrak{m}(\alpha) \subseteq X_n$ for all $n \in \omega$, so $\mathfrak{m}(\alpha)$ is homogeneous for every P_n $n \in \omega$, i.e. for every $P \in \langle C\rangle^a_b$. This gives the arrow $\mathfrak{m}(\alpha) \to (\mathfrak{m}(\alpha))^a_b$.

b)→a) By 4.3.29, $\mathfrak{m}$ is minimal, so I is even regular, by 4.3.19. The implication from $\mathfrak{m} \to (\mathfrak{m})^a_b$ to $I \to (I)^a_b$ is trivial. □

4.3.32 Corollary. *I is a strong cut iff $\mathfrak{g}_I$ includes a relatively large strong monad.*

Note that the initial choice of $X_0 \in \mathfrak{D}(\mathfrak{g}_I)$ in the proof of 4.3.31 was arbitrary. This means that if $\mathfrak{g}$ contains a relatively large monad satisfying $\mathfrak{m} \to (\mathfrak{m})^a_b$ $(a, b \geq 2)$, then such monads form a dense subset of $\mathfrak{g}/\sim$ with Stone topology. Now, $\mathfrak{m} \to (\mathfrak{m})^a_b$ can be expressed using a simple property $\Phi(\mathfrak{m})$, where

$$\Phi(X) \overset{\mathrm{df}}{=} \{(\forall x, y \in \langle X\rangle^a)\langle x, y\rangle \in P \mid X \in \mathfrak{D}, P \in \langle X\rangle^a_b\}.$$

By 4.2.11, $\Phi(X)$ has monadic overspill, so applying 4.2.15 we have our usual corollary on distributivity.

4.3.33 Corollary. *Relatively large monads satisfying the arrow* $\mathfrak{m} \to (\mathfrak{m})^a_b$ *with* $2 \le a, b \in M$ *are distributive in every gap that includes one.* □

Infinite Ramsey Theorem in PA 2.1.3 gives $M \to (M)^n_{<M}$ for every $n \in \mathbb{N}$; in particular, M is strong in M. Consequently, $\mathfrak{g}_\infty$ includes a strong monad. The following will show that bounded gaps containing strong monads are frequent, too. The results concerning strong cuts are known [Kir77], the proofs are ours.

Recall from Definition 2.2.6 on page 30 that for a set u codable in M and for $i, e, d, c \in M$, $u \xrightarrow{*}_i (e)^d_c$ denotes the i-th iteration of the Paris-Harrington arrow in C.

4.3.34 Proposition. *Let* $a, b, c, d, i \in M$ *and let* $d \ge 1$, $c \ge 2$, *and* $i \ge \mathbb{N}$. *If* $[a, b] \xrightarrow{*}_i (d+1)^d_c$, *then the interval* $[a, b]$ *includes a relatively large monad* $\mathfrak{m}$ *such that* $\mathfrak{m} \to (\mathfrak{m})^d_c$.

Proof. Let $\{P_n \mid n \in \omega\}$ be some fixed enumeration of $\langle C \rangle^d_c$. Put $X_0 = [a, b]$. If X_n is defined and $X_n \xrightarrow{*}_{i-n} (d+1)^d_c$, then X_n includes a relatively large subset X_{n+1} coded in M such that, X_{n+1} is homogeneous for P_n and satisfies $X_{n+1} \xrightarrow{*}_{i-n-1} (d+1)^d_c$. Let $\alpha \in \bigcap_{n \in \omega} X_n$, by $\aleph_1$-saturation. Then $\mathfrak{m}(\alpha)$ is a relatively large monad that is homogeneous for every $P \in \langle C \rangle^d_c$. □

4.3.35 Corollary.

a) *For every* $a \in M$, *the interval* $[a, \to)$ *includes a bounded relatively large strong monad.*

b) *For every* $a \in M$, *there exists a strong cut* $I \subset^e M$ *such that* $a \in I$.

c) $Y(a, b) \overset{\text{df}}{=} \max\{c \le b \mid [a, b] \xrightarrow{*}_c (c+1)^c_2\}$ *is a* Δ_1*-indicator for strong cuts. Thus all items of Theorem 3.2.1 for* $n = 1$ *apply to the family* $\mathcal{R}_{strong}$ *of strong cuts in M. In fact, Y is also an indicator for* $\mathcal{P}_0$, *so* $\overline{\mathcal{R}_{strong}} = \overline{\mathcal{P}_0}$.

Proof. a) Let $a, c \in M$ be fixed. By Lemma 2.2.7, for every $k \in \mathbb{N}$,

$$M \models (\exists b)[a, b] \xrightarrow{*}_k (k+1)^k_c. \tag{4.4}$$

By overspill, there is some $d \in M \setminus \mathbb{N}$ such that (4.4) with k replaced by d holds. Then 4.3.34 produces a relatively large bounded monad $\mathfrak{m} \subseteq [a, \to)$ satisfying $\mathfrak{m} \to (\mathfrak{m})^d_c$. By 4.3.27, $\mathfrak{m} \to (\mathfrak{m})^k_2$ for every $k \in \mathbb{N}$, so $\mathfrak{m}$ is strong.

b) By a), there is a bounded, relatively large strong monad $\mathfrak{m}$ such that $a \in I_\mathfrak{m}$. By 4.3.32, $I_\mathfrak{m}$ is a strong cut.

c) It is a straightforward task to verify that Y is Σ_1 and thus Δ_1. If $Y(a, b) = c > \mathbb{N}$, then $[a, b]$ contains a strong cut I by the arguments used in the proofs of a) and b). By 4.3.12, $I \models$ PA, so $I \in \mathcal{P}_0$. It now suffices to prove that, conversely, if $a \in I < b$ and $I \models$ PA, then $Y(a, b) > \mathbb{N}$. We may assume $I \ne \mathbb{N}$, since $\mathbb{N} \not\in \mathcal{P}_0$ by 3.2.1, e). Then by 2.2.2, $I \models (\exists b)[a, b] \xrightarrow{*}_k (k+1)^k_2$, i.e. $I \models (\exists b)Y(a,b) \geq k$, for every $k \in \mathbb{N}$. Applying overspill in I, we get $I \models Y(a,b) \geq c$ for some $b,c \in I$ non-standard. Since Y is Σ_1, we have $M \models Y(a,b) \geq c$ by Δ_0-elementarity of I. □

2-monads and regular monads of pairs

For a non-trivial monad $\mathfrak{m}$, we now have:

- $\mathfrak{m} \to (\mathfrak{m})^1_{<\mathbb{N}}$.
- $\mathfrak{m} \to (\mathfrak{m})^1_{<I_\mathfrak{m}}$ iff $\mathfrak{m}$ is regular.
- $\mathfrak{m} \to (\mathfrak{m})^2_2$ iff $\mathfrak{m} \to (\mathfrak{m})^2_{<\mathbb{N}}$ iff $\langle\mathfrak{m}\rangle^2$ is a monad; also, $\mathfrak{m}$ is minimal.
- $\mathfrak{m} \to (\mathfrak{m})^2_{<I_\mathfrak{m}}$ iff $\langle\mathfrak{m}\rangle^2$ is a regular monad.

4.3.36 Proposition.

a) *Let $\mathfrak{m}$ be a relatively large 2-monad. Then for every Δ_0 formula $\theta(x,y,\bar{z})$, $\bar{a} \in M$, and $‹\alpha_0,\alpha_1› \in \langle\mathfrak{m}\rangle^2$,*

$$(4.5)\quad I_\mathfrak{m} \models (\forall x_0)(\exists x_1)\theta(x_0,x_1,\bar{a}) \textit{ iff } C \models (\forall x_0 < \alpha_0)(\exists x_1 < \alpha_1)\theta(x_0,x_1,\bar{a}).$$

b) *If $\mathfrak{m} \to (\mathfrak{m})^2_{<I_\mathfrak{m}}$, then $I_\mathfrak{m} \models \mathrm{I}\Sigma_2$.*

Proof. Let I denote the cut $I_\mathfrak{m}$.

a) The assumption ensures that $\mathfrak{m}$ is relatively large and minimal; in particular, I is semi-regular, and $I \preccurlyeq_0 M$. Let $\alpha_0 < \alpha_1$ be elements of $\mathfrak{m}$. Let θ be a bounded $\mathfrak{L}^{Ar}_I$-formula. If $I \models (\forall x_0)(\exists x_1)\theta(x_0,x_1)$, then Δ_0-elementarity gives $C \models (\exists x_1 < \alpha_1)\theta(c,x_1)$ for every $c \in I$. By overspill, there is $\gamma \in \mathfrak{g}_I$ such that $C \models (\forall x_0 < \gamma)(\exists x_1 < \alpha_1)\theta(x_0,x_1)$. Since $\mathfrak{m}$ is coinitial in $\mathfrak{g}_I$, we may assume $\gamma \in \mathfrak{m}$ and $\gamma < \alpha_1$. Thus $‹\gamma,\alpha_1› \in \langle\mathfrak{m}\rangle^2$. Now, $\mathfrak{m}$ is a 2-monad, so we may replace γ with α_0, concluding the left-to-right implication. The converse follows easily by underspill and does not even require $\mathfrak{m}$ to be a 2-monad.

b) As above, $I \preccurlyeq_0 M$; we prove $I \models \mathrm{L}\Pi_2$. Let $\varphi(y)$ be an $\mathfrak{L}^{Ar}_I$-formula of the form $(\forall x_0)(\exists x_1)\theta(x_0,x_1,y)$ with θ bounded; assume $I \models \varphi(c)$ for some $c \in I$. For $‹\alpha_0,\alpha_1› \in \langle\mathfrak{m}\rangle^2$, let $\gamma \stackrel{\text{df}}{=} \mu x : (\forall x_0 < \alpha_0)(\exists x_1 < \alpha_1)\theta(x_0,x_1,y)$. Then $\gamma \in M[‹\alpha_0,\alpha_1›]$. By (4.5), $\gamma \leq c$. Now, $\langle\mathfrak{m}\rangle^2$ is a regular monad ($\subseteq \mathfrak{g}_I$), therefore $M \preccurlyeq_I M[‹\alpha_0,\alpha_1›]$, by 4.3.23. Thus $\gamma \in I$. By (4.5), $I \models \varphi(\gamma) \wedge (\forall y < \gamma)\neg\varphi(y)$. □

4.3.37 Corollary. *There is a gap that includes relatively large minimal monads, but no monad satisfying $\mathfrak{m} \to (\mathfrak{m})^2_{<I_\mathfrak{m}}$.*

Proof. By 4.3.13, there is a regular cut I such that $I \not\models \mathrm{I}\Sigma_2$. Its gap includes relatively large minimal monads and 4.3.36 b) gives the result. □

4.3.38 Remark. This corollary only partially answers the following natural questions: Are there I,M such that $I \subset^e M \models \mathrm{PA}$ and

- I is regular but fails to satisfy $I \to (I)^2_2$?
- $I \to (I)^2_2$ but not $I \to (I)^2_{<I}$?
- $I \to (I)^2_{<I}$ but not $I \to (I)^3_2$?

At least to our knowledge, these particular questions remain open, although there is a strong suggestion of positive answers to the latter two of them: Cholak, Jockusch, and Theodore ([CJS01]) proved recently using forcing that over $\mathrm{I}\Sigma^0_1 + \Delta_1\mathrm{CA}_0$, Ramsey Theorem for pairs with 2 colors (RT^2_2) is strictly weaker than Ramsey Theorem for pairs for any number of colors ($\mathrm{RT}^2_{<\infty}$) and that the latter is strictly stronger than RT^3_2. This improved earlier Seetapun and Slaman's [SS95] negative answer to Jockuch's well-known 2-3-problem.

In contrast, it follows from a result by Kirby [Kir84, Theorem 5.7] on ultrafilters on models of $A\mathrm{CA}_0$ that if I is a strong cut (i.e. its gap already contains some 3-monad), then every 2-monad from $\mathfrak{g}_I$ is a 3-monad (see B.3 for proof). In light of this Kirby's result, a strong cut with $\mathrm{Cod}(M/I)$ resemble a bit closer ω with $\mathcal{P}(()\omega)$, where every non-trivial selective ultrafilter is Ramsey.

We now return briefly to minimal monads to see how M-definable functions operate on them locally. A theorem given below has a sequel about 2-monads, which is why we had inserted it here. The theorem will be useful in the subsequent study of diagonal partition properties.

4.3.39 Definition. For $\alpha \in C$, we write

$$[\![\alpha]\!] \stackrel{\mathrm{df}}{=} \{\gamma \in C \mid \neg(\exists \beta \sim \alpha)(\gamma \le \beta < \alpha \vee \alpha < \beta \le \gamma\}.$$

Thus, $[\![\alpha]\!]$ is the maximal convex subset around α that does not contain any other element from $\mathfrak{m}(\alpha)$. Since every non-trivial monad $\mathfrak{m}$ spans across its gap and is densely ordered by $<$, $\{[\![\gamma]\!] \mid \gamma \in \mathfrak{m}\}$ is a partition of the gap containing $\mathfrak{m}$.

The following theorem is a variant of [ČV87, Theorem 8] in AST. It says that for a minimal monad $\mathfrak{m}$, $\alpha \in \mathfrak{m}$, and $F \in \mathfrak{F}(I_\mathfrak{m})$, the image $F(\alpha)$ cannot leave $[\![\alpha]\!]$ (that is to say, α cannot jump over another element from $\mathfrak{m}$) unless it leaves the entire gap.

4.3.40 Theorem (Čuda-Vojtášková). *Let $\mathfrak{m}$ be a minimal monad, $F \in \mathfrak{F}$, $\mathfrak{m} \subseteq \mathrm{dom}(F)$. If F is regressive on $\mathfrak{m}$ and $I_\mathfrak{m} \ne M$, then either*

a) $F(\alpha) \in [\![\alpha]\!]$ *for all* $\alpha \in \mathfrak{m}$, *or*

b) $(\exists d \in M \setminus I_\mathfrak{m}) F(d) \in I_\mathfrak{m}$.

If $\mathfrak{m}$ is a relatively large 2-monad or if $I_\mathfrak{m} = M$, then b) *can be replaced by the condition*

c) *F is constant on* $\mathfrak{m}$.

Similarly, if $F(\alpha) \geq \alpha$ *for some (and hence all)* $\alpha \in \mathfrak{m}$ *and* $F''I_{\mathfrak{m}} \subseteq I_{\mathfrak{m}}$*, then* $F(\alpha) \in [\![\alpha]\!]$ *for all* $\alpha \in \mathfrak{m}$.

Proof. Let I denote $I_{\mathfrak{m}}$. The sequel about functions with $F(\alpha) \geq \alpha$ for $\alpha \in \mathfrak{m}$, is obtained from the first part by passing from F to F^{-1} and from $\mathfrak{m}$ to the monad $F''\mathfrak{m}$, which is also minimal. The assumption of $F''I \subseteq I$ eliminates the alternative b) for F^{-1}. It thus suffices to prove the case for F regressive.

Now, F is either constant or one-to-one on $\mathfrak{m}$. In the first case, c) holds, which for $I \neq M$ gives b). This reduces the problem to the case when F is 1-1 on $\mathfrak{m}$; we may assume F is 1-1 on its whole domain.

We first deal with the case when $\mathfrak{m}$ is a relatively large 2-monad. Assume $\neg$a). Then $F(\alpha) \geq \beta$ for some $\alpha, \beta \in \mathfrak{m}$, $\beta < \alpha$; by 2-indiscernibility of $\mathfrak{m}$, the inequality holds for arbitrarily small $\beta \in \mathfrak{m}$, hence, by underspill, $F(\alpha) < b$ for some $b \in I$. Thus $F''\mathfrak{m} \subseteq [0, b]$, in contradiction with $\mathfrak{m}$ being relatively large.

Now we resolve the general case, for $\mathfrak{m}$ minimal. Aiming towards a contradiction, assume neither a) nor b) holds. In particular, $F^{-1\,''}I \subseteq I$ by $\neg$b). Let $b \overset{\mathrm{df}}{=} \min(\mathrm{rng}(F))$. For each $\alpha \in [b, \rightarrow)$ define $H(\alpha) \overset{\mathrm{df}}{=} \max((F^{-1\,''}[0, \alpha]) \cup \{\alpha + 1\})$. Clearly $H \in \mathfrak{F}$, H is non-decreasing, $H(\alpha) > \alpha$ for $\alpha \in [b, \rightarrow)$, and, by $\neg$b), $H''I \subseteq I$. Let $G \in \mathfrak{F}$ be defined as follows: $G(0) \overset{\mathrm{df}}{=} b$, $G(\gamma+1) \overset{\mathrm{df}}{=} H(G(\gamma))$, i.e. $G(\gamma)$ is the γ's iteration of H. Thus $F(\alpha) \leq G(\gamma) \rightarrow \alpha \leq G(\gamma+1)$ for every $\alpha \in \mathrm{dom}(F)$. Finally, we put $g(\alpha) \overset{\mathrm{df}}{=} \max\{\gamma \mid G(\gamma) < \alpha\}$ for every $\alpha \in (b, \rightarrow)$. Then $g \in \mathfrak{F}$, g is non-decreasing, and $G(g(\alpha)) < \alpha \leq G(g(\alpha)+1)$ for all $\alpha \in (b, \rightarrow)$.

Now, fix $\alpha \in \mathfrak{m}$ such that $F(\alpha) \neq [\![\alpha]\!]$ and let $\gamma \overset{\mathrm{df}}{=} g(\alpha)$. Since $F(\alpha) < \alpha$ and $F(\alpha) \neq [\![\alpha]\!]$, there is some $\beta \in \mathfrak{m}$ such that $F(\alpha) \leq \beta < \alpha$. We first aim for proving that $\gamma \in M$ (and thus $\gamma \in I$). For that, it suffices to show that $g(\alpha) = g(\beta)$. Indeed, as g is non-decreasing, we have $g(\beta) \leq g(\alpha)$. For the converse inequality, we have $\beta < \alpha \in (G(\gamma), G(\gamma+1)]$, so we only need to show that $G(\gamma) < \beta$. Assume otherwise and let $\delta \overset{\mathrm{df}}{=} g(\beta)$, that is, $\beta \in (G(\delta), G(\delta+1)]$ with $\delta < \gamma$. Now, $F(\alpha) \leq \beta$, so $F(\alpha) \in [0, G(\delta+1)]$, so we have $\alpha \in [0, G(\delta+2)]$ and hence $\gamma = \delta + 1$. Thus $g(\alpha) = g(\beta) + 1$, but then parity of the g-image discerns α from β, contradicting $\alpha \sim \beta$. This proves that $g(\alpha) = g(\beta) = \gamma$. Thus by minimality, g is constant on $\mathfrak{m}$, so $\gamma \in M$ and also $G(\gamma), G(\gamma+1) \in M$. Since $G(\gamma) < \alpha \leq G(\gamma+1)$, $G(\gamma) \in I$, but $H(G(\gamma)) = G(\gamma+1) \in M \setminus I$. This contradicts $H''I \subseteq I$. □

4.4 Diagonal partitions of monads and p-monads

In the first part of this section, we return to the topic of diagonal indiscernibility started in Chapter 2 by studying diagonal partition properties of monads. In the second part, we introduce p-monads (in analogy with p-points in $\beta\omega$) and prove that being a p-monad is a property of monads that stands strictly between RK-minimality and regularity (in any gap that contains a regular monad).

On overlays in the enlarged setting

First of all, we briefly note what our previous results concerning diagonal homogeneity, namely the results on overlays from Section 2.3.1, tell us in the present context.

4.4.1 Proposition.

a) *There is an unbounded $\sim$-closed subset $O \subseteq C$ such that $\mathfrak{g}_\infty \cap O$ is a monad and for every $I \subseteq^e M$,*

$$I \preccurlyeq M \text{ iff } \mathfrak{g}_I \cap O \neq \emptyset.$$

(If $\mathfrak{g}_I \cap O$ is non-empty, then it is a $\sim$-figure and a $\sim_{\mathrm{Fm}(\mathcal{L}^{Ar}),I}$-monad.)

b) *There is an unbounded $\sim$-closed subset $O_n \subseteq C$ such that for $I \subseteq^e M$,*

$$I \preccurlyeq_n M \text{ iff } \mathfrak{g}_I \cap O_n \neq \emptyset.$$

c) *If $I \subseteq^e M$, then $I \models \mathrm{PA}$ iff there exists a subset $Z \subseteq C$ of Δ_0-diagonally indiscernible elements such that $\sup I = \inf Z$. If $I \models \mathrm{PA}$ then Z can be taken as an intersection of countably many coded subsets of C.*

Sketch of the proof. a) Let $\mathcal{O}$ be an unbounded Δ_0-diagonally indiscernible overlay in C consisting of $\emptyset$-definable sets; such an overlay exists by 2.3.2 b). By $\aleph_1$-saturation, $O = \bigcap \mathcal{O}$ is non-empty; since it is an intersection of clopen sets, it is $\sim$-closed; it is unbounded in C by $\aleph_1$-saturation. Clearly O intersects the unbounded gap $\mathfrak{g}_\infty$. By 2.3.7, O is a set of $\mathrm{Fm}(\mathcal{L}^{Ar})$-diagonally indiscernible elements from which it follows easily that $O \cap \mathfrak{g}_\infty$ is a monad. Moreover, if $I \preccurlyeq M$, $I \subseteq^e M$, then by elementarity, every $X \cap I$ with $X \in \mathcal{O}$ is cofinal in I, so $O \cap \mathfrak{g}_I \neq \emptyset$. This time $O \cap \mathfrak{g}_I$ is a $\sim$-closed figure, but not necessarily a single $\sim$-monad; nevertheless, diagonal indiscernibility gives that elements from $O \cap \mathfrak{g}_I$ are indiscernible over parameters from I, so it is a $\sim_{\mathrm{Fm}(\mathcal{L}^{Ar}),I}$-monad. Conversely, if $O \cap \mathfrak{g}_I \neq \emptyset$, then 2.3.2 b) gives $I \preccurlyeq M$.

b) is proved similarly, using Theorem 2.3.3 for a $\Sigma_n(1;1)$-diagonally indiscernible overlay and cuts with $I \preccurlyeq_{n+1} M$.

c) If $I \models \mathrm{PA}$, $I \subseteq^e M$, then there is a Δ_0-diagonally indiscernible overlay $\mathcal{O}$ unbounded in I. A set $X \in \mathcal{O}$ may not be definable in M, but, due to countability of I, we may define (outside C) an increasing sequence $\{a_n\}_{n\in\omega}$ cofinal in I such that for each n the subsequence $\{a_k\}_{k\geq n}$ is a set of $\Delta_0^{<k}$-diagonally indiscernible elements (where $\Delta_0^{<k}$ denotes the set of the first k formal Δ_0 formulae, c.f. the proof of 3.4.1). By $\aleph_1$-saturation in C, there is an element $\gamma \in C$ coding an increasing sequence of some non-standard length such that $(\gamma)_n = a_n$ for $n \in \mathbb{N}$ and with the property that for every $\alpha < \ell(\gamma)$, the set

$u_\alpha \stackrel{\text{df}}{=} \{(\gamma)_\beta \mid \alpha \le \beta < \ell(\gamma)\}$ is a set of $\Delta_0^{<\alpha}$-diagonally indiscernible elements. Then $Z = \bigcap_{n\in\omega} u_n$ has the required property. Conversely, let $Z \subseteq C$ be a set of Δ_0-diagonally indiscernible elements in C such that $\sup I = \inf Z$ for $I \subseteq^e M$. We have $I \preccurlyeq \sup I$, by $I \preccurlyeq_0 M \preccurlyeq C$, $\sup I \preccurlyeq_0 C$, and 1.8.7 a). It is easy to verify by induction on number of quantifiers (using Δ_0-diagonal indiscernibility and overspill) that for every bounded $\mathfrak{L}^{Ar}$-formula ψ, $\bar{a} \in I$, and $\vec{e} \in Z$,

$$\sup I \models (\exists x_0)(\forall x_1)\ldots\psi(\bar{a},\bar{x}) \text{ iff } C \models (\exists x_0 < e_0)(\forall x_1 < e_1)\ldots\psi(\bar{a},\bar{x})).$$

From this, it follows easily using $\sup I = \inf Z$ that $\sup I \models \mathrm{PA}$, hence $I \models \mathrm{PA}$ by elementarity. □

Diagonal partitions of monads

We now turn our attention to M-definable diagonal partitions on monads.

4.4.2 Definition.

a) Let D be a diagonal partition of $\langle X\rangle^a$ (see Definition 2.1.1), $X \in \mathfrak{D}$, $a \in M$, and let $h \in \mathfrak{F}$. Say that D is *h-estimated*, if for every $t \in X$, $\|D_t\| \le h(t)$.

b) Let $\mathfrak{m}$ be a non-trivial monad and $h \in \mathfrak{F}(I_\mathfrak{m})$. Then $\mathfrak{m} \rightrightarrows (\mathfrak{m})^a_h$ denotes the following assertion:

> For every $X \in \mathfrak{U}(\mathfrak{m})$ and every h-estimated diagonal partition D of $\langle X\rangle^a$, there exists a set $Y \in \mathfrak{U}(\mathfrak{m})$ diagonally homogeneous for D.

If $\mathfrak{m} \rightrightarrows (\mathfrak{m})^a_h$ holds for every $h \in \mathfrak{F}(I_\mathfrak{m})$, we write $\mathfrak{m} \rightrightarrows (\mathfrak{m})^a_{\mathfrak{F}(I_\mathfrak{m})}$. If $\mathfrak{m} \rightrightarrows (\mathfrak{m})^a_h$ for a constant function h with the value $b \in I_\mathfrak{m}$, we write $\mathfrak{m} \rightrightarrows (\mathfrak{m})^a_b$. Finally, we write, $\mathfrak{m} \rightrightarrows (\mathfrak{m})^a_{<I_\mathfrak{m}}$ if $(\forall b \in I_\mathfrak{m})\mathfrak{m} \rightrightarrows (\mathfrak{m})^a_b$.

4.4.3 Remark. By Lemma 4.2.11, D-homogeneity for diagonal partition $D \in \mathfrak{D}$ is a property with monadic overspill.

Notice that h-estimated diagonal homogeneity can equivalently be expressed as min-homogeneity for h-regressive functions. Indeed, if D is a h-estimated diagonal partition of $\langle X\rangle^n$, let for each $t \in X$ and $u \in \langle X \setminus [0,t]\rangle^n$, $F(\langle t\rangle \smile u) = y$ if u falls into the y-th block of D_t, i.e. $u \in (D_t)_{(y)}$. Then $F(v) < h(\mathrm{Min}(v))$ for every $v \in \langle X\rangle^{n+1}$ (which is referred to as F being *h-regressive*). Clearly, a subset $H \subseteq X$ is diagonally homogeneous for D if and only if it is *min-homogeneous for F* i.e. if $F(v) = F(w)$ for every $v,w \in \langle H\rangle^{n+1}$ with $(v)_0 = (w)_0$.

Similarly, every h-regressive $F : \langle X\rangle^{n+1} \to C$ determines a h-estimated diagonal partition of $\langle X\rangle^n$ such that for all $t \in X$ and $u,v \in \langle X \setminus [0,t]\rangle^n$, $u\, D_t\, v$ iff $F(\langle t\rangle \smile u) = F(\langle t\rangle \smile v)$. Then min-homogeneity for F causes diagonal homogeneity for D. From this, we have:

4.4.4 Proposition. $\mathfrak{m} \rightrightarrows (\mathfrak{m})^a_h$ *iff* $\mathfrak{m}$ *is min-homogeneous for every h-regressive function* $F \in \mathfrak{F}$ *with* $\langle\mathfrak{m}\rangle^{a+1} \subseteq \mathrm{dom}(F)$. □

For its brevity, we prefer this functional description, namely in proofs.

4.4.5 Lemma. *If* $\mathfrak{m} \rightrightarrows (\mathfrak{m})^1_2$, *then* $\mathfrak{m}$ *is minimal.*

Proof. Let $F \in \mathfrak{F}$ with $\mathrm{dom}(F) = X \in \mathfrak{U}(\mathfrak{m})$ be not 1-1 on $\mathfrak{m}$. For $\langle\alpha,\beta\rangle \in \langle X\rangle^2$, we put $G(\langle\alpha,\beta\rangle) \stackrel{\mathrm{df}}{=} 0$ if $F(\alpha) = F(\beta)$ and $G(\langle\alpha,\beta\rangle) \stackrel{\mathrm{df}}{=} 1$ otherwise. Then some $Y \in \mathfrak{U}(\mathfrak{m})$ is min-homogeneous for G. Let $a \stackrel{\mathrm{df}}{=} \mu x \in Y : (\forall y \in Y)(x < y \rightarrow G(\langle x,y\rangle) = 0)$. If $a \in I_{\mathfrak{m}}$, then clearly $F''\mathfrak{m} = \{F(a)\}$, so F is constant on $\mathfrak{m}$. Otherwise $\mathfrak{m} < a$, but in that case for every $\alpha \in \mathfrak{m}$, if $\alpha < \beta \in Y$, then $G(\langle\alpha,\beta\rangle) = 1$, so $F(\alpha) \neq F(\beta)$. In particular, F is 1-1 on $\mathfrak{m}$. □

4.4.6 Theorem. *Let I be a semi-regular cut, $\mathfrak{m} \subseteq \mathfrak{g}_I$, $a, b \in I$, $a \geq 1$, $b \geq 2$, and $h \in \mathfrak{F}(I)$:*

a) $\mathfrak{m} \rightrightarrows (\mathfrak{m})^a_h$ *iff* $(\forall k \in \mathbb{N})\mathfrak{m} \rightrightarrows (\mathfrak{m})^a_{h^k}$, *where* $h^k(x) \stackrel{\mathrm{df}}{=} (h(x))^k$.

b) $g''\mathfrak{m} \rightrightarrows (g''\mathfrak{m})^a_h$ *implies* $\mathfrak{m} \rightrightarrows (\mathfrak{m})^a_{g\circ h}$, *whenever* $\mathfrak{m} \subseteq \mathrm{dom}(g \circ h)$, $g \in \mathfrak{F}(I)$ *is increasing, and* $h \in \mathfrak{F}$.

c) $\mathfrak{m} \rightrightarrows (\mathfrak{m})^a_b$ *iff* $\mathfrak{m} \to (\mathfrak{m})^{a+1}_b$. *Consequently,* $\mathfrak{m} \rightrightarrows (\mathfrak{m})^a_{<I}$ *iff* $\mathfrak{m} \to (\mathfrak{m})^{a+1}_{<I}$.

d) $\mathfrak{m} \to (\mathfrak{m})^{a+2}_2$ *implies* $\mathfrak{m} \rightrightarrows (\mathfrak{m})^a_{\mathfrak{F}(I)}$

e) $\mathfrak{m} \rightrightarrows (\mathfrak{m})^{a+1}_2$ *implies* $\mathfrak{m} \rightrightarrows (\mathfrak{m})^a_{\mathfrak{F}(I)}$

Proof. a) The right-to-left is trivial. Suppose $\mathfrak{m} \rightrightarrows (\mathfrak{m})^a_h$ for $h \in \mathfrak{F}(I)$. We proceed by induction on $k \in \mathbb{N}$. For $k \leq 1$ the claim is trivial. Let $X \in \mathfrak{U}(\mathfrak{m})$ and $F : \langle X\rangle^{a+1} \to C$, and let F be h^{k+1}-regressive. We define $G : \langle X\rangle^{a+1}$ by $G(\alpha) \stackrel{\mathrm{df}}{=} (F(\alpha) \bmod h((\alpha)_0))$. Then $G(\alpha) < h((\alpha)_0)$, so G is h-regressive (and thus also h^k-regressive). The induction hypothesis gives some $Y \in \mathfrak{U}(\mathfrak{m})$ that is min-homogeneous for G. We now put $F'(\alpha) \stackrel{\mathrm{df}}{=} \lfloor F(\alpha)/h((\alpha)_0)\rfloor$ for every $\alpha \in \langle Y\rangle^{a+1}$; clearly, $F(\alpha) = F'(\alpha)\cdot h((\alpha)_0) + G(\alpha)$. Since $F(\alpha) < h^{k+1}((\alpha)_0)$, $F'(\alpha) < h^k((\alpha)_0)$. Applying again the induction hypothesis, we have that $\mathfrak{m}$ is min-homogeneous for F' (and of course also for G). Thus, if $\alpha, \beta \in \langle\mathfrak{m}\rangle^a$ and $(\alpha)_0 = (\beta)_0$, we have

$$F(\alpha) = F'(\alpha)\cdot h((\alpha)_0) + G(\alpha) = F'(\beta)\cdot h((\beta)_0) + G(\beta) = F(\beta).$$

b) Let g be increasing on $\mathfrak{m}$. Then $g''\mathfrak{m}$ is a non-trivial monad and $g''\mathfrak{m} \asymp_{\mathrm{RK}} \mathfrak{m}$. Moreover, I is semi-regular, so $\mathfrak{g}_I$ is original and thus $g''\mathfrak{m} \not\subseteq \sup I$. Yet $g''I \subseteq I$, so $g''\mathfrak{m}$, like $\mathfrak{m}$, belongs to the gap $\mathfrak{g}_I$. Let $F \in \mathfrak{F}$ be a $(g\circ h)$-regressive function defined on $\langle X\rangle^{a+1}$ for some $X \in \mathfrak{U}(\mathfrak{m})$. We may assume that g is increasing on X. For $\alpha \in \langle X\rangle^a$, let $G(\alpha)$ denote the pointwise image of α over g, i.e. $(\forall i < a)(G(\alpha))_i = g((\alpha)_i$. Then $G \in \mathfrak{F}$ and since g is increasing, $G : \langle X\rangle^a \to \langle g''X\rangle^a$. Let $\beta \in \langle g''X\rangle^a$ and let $\beta \overset{\text{df}}{=} G(\alpha)$ for $\alpha \in \langle X\rangle^a$. Then $F(G^{-1}(\beta)) = F(\alpha) < h(g((\alpha)_0)) = h((G(\alpha))_0) = h((\beta)_0)$, so $G^{-1} \circ F$ is h-regressive on $\langle g''X\rangle^a$. According to the premise, $g''\mathfrak{m}$ is min-homogeneous for $G^{-1} \circ F$. It now follows easily that $\mathfrak{m}$ is min-homogeneous for F.

c) First, let $\mathfrak{m} \rightrightarrows (\mathfrak{m})^a_b$, $a \geq 1$, $b \geq 2$, $a, b \in I$, and let $F : \langle X\rangle^{a+1} \to [0,b)$ for some $X \in \mathfrak{U}(\mathfrak{m})$. We must prove that F is constant on $\langle \mathfrak{m}\rangle^{a+1}$. By the assumption, there exists $Y \in \mathfrak{U}(\mathfrak{m})$ that is min-homogeneous for F. For $t \in Y$ such that $|Y \cap [t, \to)| \geq a$, let $\gamma_t \in \langle Y\rangle^a$ consist of the first a elements following t in Y and let $G(t) \overset{\text{df}}{=} F(\langle t\rangle \smile \gamma_t)$. Since $\mathfrak{m}$ is relatively large, $\mathfrak{m} \subseteq \mathrm{dom}(G)$. Now, $\mathfrak{m}$ is minimal by 4.4.5, so by 4.3.17 it is regular. Thus G is constant on $\mathfrak{m}$ with some value $d \in [0,b) \cap M$. We show that F is constantly d on $\langle \mathfrak{m}\rangle^{a+1}$. Indeed, if $\langle t\rangle \smile \gamma \in \langle \mathfrak{m}\rangle^{a+1}$, then by min-homogeneity for F, $F(\langle t\rangle \smile \gamma) = F(\langle t\rangle \smile \gamma_t) = G(t) = d$.

As for the reversed implication, we are supposed to prove that if $X \in \mathfrak{U}(\mathfrak{m})$ and $F : \langle X\rangle^{a+1} \to [0,b)$, then $\mathfrak{m}$ is min-homogeneous for F. But this is trivial, since $\mathfrak{m} \to (\mathfrak{m})^{a+1}_b$ establishes that F is actually constant on $\langle \mathfrak{m}\rangle^{a+1}$.

d) (The method of this proof is borrowed from [KM87, Theorem 1.3]). Let $F : \langle X\rangle^{a+1} \to C$ with $X \in \mathfrak{D}$ and F h-regressive for some $h \in \mathfrak{F}(I)$. Assume $\mathfrak{m} \to (\mathfrak{m})^{a+2}_2$. By 4.3.27, $\mathfrak{m} \to (\mathfrak{m})^{a+2}_3$. We define $G : \langle X\rangle^{a+2} \to [0,2]$ as follows: for $u \in \langle X\rangle^{a-1}$ (for $a = 1$ let u be an empty sequence) and $\alpha, \beta, \gamma \in X$ such that $\alpha < \beta < (u)_0$ and $(u)_{a-2} < \gamma$, put

$$G(\langle \alpha,\beta\rangle \smile u \smile \langle\gamma\rangle) \overset{\text{df}}{=} \begin{cases} 0 & \text{if } F(\langle \alpha,\beta\rangle \smile u) = F(\langle\alpha\rangle \smile u \smile \langle\gamma\rangle), \\ 1 & \text{if } F(\langle \alpha,\beta\rangle \smile u) < F(\langle\alpha\rangle \smile u \smile \langle\gamma\rangle), \\ 2 & \text{if } F(\langle \alpha,\beta\rangle \smile u) > F(\langle\alpha\rangle \smile u \smile \langle\gamma\rangle). \end{cases} \tag{4.6}$$

By the assumption, there exists $Y \in \mathfrak{U}(\mathfrak{m})$ such that G is constant on $\langle Y\rangle^{a+2}$. We show that the value of G on this set is 0. Otherwise, let $c \overset{\text{df}}{=} \min Y$ and for $\alpha \in Y \cap (c, \to)$ let $f(\alpha) \overset{\text{df}}{=} F(\langle c, \alpha\rangle \smile u_\alpha)$ where $u_\alpha \in \langle Y\rangle^{a-1}$ consists of the first $a-1$ elements following α in Y. We have $c \in I$, and thus $h(c) \in I$. Since $\mathfrak{m}$ is relatively large, $|Y \cap (c, \to)| \geq h(c) + a$. If G were 1 or 2 on $\langle Y\rangle^{a+2}$, then f would be increasing or decreasing, respectively, producing at least $h(c)+1$ different values on $Y \cap (c, \to)$. But this contradicts the assumption that F is h-bounded.

Thus G is constantly 0 on $\langle \mathfrak{m}\rangle^{a+2}$. We show that $\mathfrak{m}$ is min-homogeneous for F. For $\alpha \in \mathfrak{m}$, $u, v \in \langle \mathfrak{m}\rangle^a$, $\alpha < (u)_0, (v)_0$, we must show that $F(\langle\alpha\rangle \smile u) = F(\langle\alpha\rangle \smile v)$. Since $\mathfrak{m}$ is relatively large, there is a $w \in \langle \mathfrak{m}\rangle^a$ such that $\max\{(u)_{a-1}, (v)_{a-1}\} < (w)_0$. Now consider the sequence $\sigma \overset{\text{df}}{=} u \smile w$. Let $\sigma^{(i)}$ be a contiguous subsequence of σ of length a whose first element is $(\sigma)_i$ and whose last element is $(\sigma)_{i+a-1}$. Then $u = \sigma^{(0)}$ and $w = \sigma^{(a)}$, and by (4.6) (the case for 0), $F(\langle\alpha\rangle \smile \sigma^{(i)}) = F(\langle\alpha\rangle \smile \sigma^{(i+1)})$ for every $i < a$. Thus (using induction in C), $F(\langle\alpha\rangle \smile u) = F(\langle\alpha\rangle \smile v)$. The same argument gives $F(\langle\alpha\rangle \smile w) = F(\langle\alpha\rangle \smile v)$ and we are done.

e) By c), $\mathfrak{m} \rightrightarrows (\mathfrak{m})^{a+1}_2$ implies $\mathfrak{m} \to (\mathfrak{m})^{a+2}_2$, the rest follows by d). $\square$

4.4.7 Corollary. *Let* $\mathfrak{g}$ *be a gap. Let* $f \in \mathcal{F}(I_{\mathfrak{g}})$ *be increasing. If* $\mathfrak{g} \cap \mathrm{rng}(f)$ *includes a monad* $\mathfrak{m}$ *satisfying* $\mathfrak{m} \rightrightarrows (\mathfrak{m})^1_{\mathrm{id}}$*, then* $\mathfrak{g}$ *includes a monad satisfying* $\mathfrak{n} \rightrightarrows (\mathfrak{n})^1_f$*.*

Proof. Since $\mathfrak{m}$ is clearly regular, $\mathfrak{g}$ is original. Thus $f''\mathfrak{g} \subseteq \mathfrak{g}$ and also $f^{-1}[\mathfrak{g}] \subseteq \mathfrak{g}$ using the fact that f is increasing. Now $\mathfrak{n} \stackrel{\mathrm{df}}{=} f^{-1}[\mathfrak{m}]$ witnesses the claim by 4.4.6 b). □

We shall now apply the notions from Definition 2.3.1 c) on subsets of C and formulae with parameters from M.

Notation. For brevity, $(k;n)$ denotes the class of formulae $\mathrm{Fm}(\mathcal{L}^{Ar}_M)(k;n)$.

4.4.8 Remark. By Lemma 4.2.11, the property 'X is a set of $(k;n)$-diagonally indiscernible elements' has monadic overspill.

4.4.9 Definition. Let $\varphi(\bar{x};\bar{y})$ be a $(k;n)$ formula and let $X \in \mathcal{D}$. Then $D^{X,\varphi(\bar{z};\bar{x})}$ denotes the diagonal partition of $\langle X\rangle^n$ such that for every $t \in X$ and $\langle\vec{\alpha}\rangle, \langle\vec{\beta}\rangle \in \langle X \setminus [0,t]\rangle^n$,

$$\langle\vec{\alpha}\rangle\, D^{X,\varphi(\bar{z};\bar{x})}_t\, \langle\vec{\beta}\rangle \stackrel{\mathrm{df}}{\Longleftrightarrow} (\forall \bar{z} < t)[\varphi(\bar{z},\vec{\alpha}) \leftrightarrow \varphi(\bar{z},\vec{\beta})]. \tag{4.7}$$

It follows from definitions that $Y \subseteq C$ is a set of $\varphi(\bar{z};\bar{x})$-diagonally indiscernible elements iff it is diagonally homogeneous for $D^{C,\varphi(\bar{z};\bar{x})}$. The following lemma is easy:

4.4.10 Lemma.

a) $D^{X,\varphi(\bar{z};\bar{x})}$ *is* $\exp_k$*-estimated, where* k *is the length of* $\bar{z}$ *and* $\exp_k(x) \stackrel{\mathrm{df}}{=} 2^{x^k}$.

b) *If* $\mathfrak{m} \rightrightarrows (\mathfrak{m})^n_{\exp_k}$*, then* $\mathfrak{m}$ *is a monad of* $(k;n)$*-diagonally indiscernible elements.* □

4.4.11 Theorem. *The following assertions are equivalent for every non-trivial relatively large monad* $\mathfrak{m}$*:*

a) $\mathfrak{m}$ *is a set of* $(1;1)$*-diagonally indiscernible elements,*

b) $\mathfrak{m}$ *is a set of* $(n;1)$*-diagonally indiscernible elements for all* $n \in \omega$*,*

c) $\mathfrak{m} \to (\mathfrak{m})^n_2$ *for all* $n \in \mathbb{N}$*, i.e.* $\mathfrak{m}$ *is strong,*

d) $\mathfrak{m} \rightrightarrows (\mathfrak{m})^n_{\mathcal{F}(I_{\mathfrak{m}})}$ *for all* $n \in \mathbb{N}$*.*

e) $\mathfrak{m} \to (\mathfrak{m})^3_2$,

f) $\mathfrak{m} \rightrightarrows (\mathfrak{m})^1_{\exp}$, *Proof.* Let I denote the cut $I_{\mathfrak{m}}$. By 4.4.5 and either of the premises, $\mathfrak{m}$ is minimal, so I is regular, by 4.3.19.

a)→b) Let $G_n : C^n \to C$ be defined by $G(\bar{x}) \stackrel{\text{df}}{=} 2^{x_0} \cdot 3^{x_1} \cdots p_n^{x_{n-1}}$, where p_n is the n-th prime. G is 1-1 and provides a coding of n-tuples such that $G''J^n \subseteq J$ whenever J is closed under operations. Let further $g_n \in \mathfrak{F}$ be defined by $g_n(x) \stackrel{\text{df}}{=} G_n(x,\dots,x)$. Then $g_n \in \mathfrak{D}(I)$, $x \le g_n(x)$ for all $x \in C$, and $G_n(x_1,\dots,x_n) \le g_n(y)$ whenever $y \ge \max\{x_1,\dots,x_n\}$. Hence, if $\bar{\alpha} < \gamma < \beta$, with $\beta,\gamma \in \mathfrak{m}$, then by Theorem 4.3.40, $g_n(\gamma) \in [\![\gamma]\!]$, hence $G_n(\bar{\alpha}) \le g_n(\gamma) < \beta$. Now, let $\varphi(z_1,\dots,z_n,x)$ be an $\mathfrak{L}^{Ar}_M$-formula. We prove that $\mathfrak{m}$ is a set of $\varphi(\bar{z};x)$-diagonally indiscernible elements. Let $\psi(z,x)$ denote the formula $(\exists z_1,\dots,z_n)(z = G_n(\bar{z}) \wedge \varphi(\bar{z},x))$. Then $C \models (\forall \bar{z})(\forall x)(\varphi(\bar{z},x) \leftrightarrow \psi(G_n(\bar{z}),x))$. Let $\gamma,\beta_1,\beta_2 \in \mathfrak{m}$ be such that $\gamma < \beta_1 < \beta_2$ and let $\bar{p} < \gamma$. Since $\mathfrak{m}$ is densely ordered, there is some $\gamma' \in \mathfrak{m}$ such that $G_n(\bar{p}) \le g_n(\gamma) < \gamma' < \beta_1$, hence $C \models \psi(G_n(\bar{p}),\beta_1) \leftrightarrow \psi(G_n(\bar{p}),\beta_2)$ by (1;1)-diagonal indiscernibility of $\mathfrak{m}$. But then $C \models \varphi(\bar{p},\beta_1) \leftrightarrow \varphi(\bar{p},\beta_2)$. Since γ,β_1,β_2 were arbitrary, $\mathfrak{m}$ is a monad of $\varphi(\bar{z},x)$-diagonally indiscernible elements.

b)→c) We show by induction on $n \ge 0$ that if $\mathfrak{m}$ is a monad of $(n;1)$-diagonally indiscernible elements, then $\mathfrak{m}$ is a monad of $(n+1)$-indiscernible elements (and thus $\mathfrak{m} \to (\mathfrak{m})^{n+1}_2$, by 4.3.28). The argument is based on trading between the number of parameters and the dimension. There is nothing to prove if $n = 0$. Assume the implication to hold for $n-1$, $n \ge 1$, i.e. in particular, $\mathfrak{m} \to (\mathfrak{m})^n_2$. Let $\varphi(x_1,\dots,x_{n+1})$ be an $\mathfrak{L}^{Ar}_M$-formula. By $(n;1)$-diagonal indiscernibility of $\mathfrak{m}$, there exists a set X of $\varphi(x_1,\dots,x_n;x_{n+1})$-diagonally indiscernible elements, such that $X \in \mathfrak{U}(\mathfrak{m})$. If $I \ne M$, let $a \in X \cap (M \setminus I)$ and put $\psi(x_1,\dots,x_n) \stackrel{\text{df}}{\iff} \varphi(x_1,\dots,x_n,a)$; if $I = M$, let $\psi(x_1,\dots,x_n) \stackrel{\text{df}}{\iff} (\exists a_0 \in X)(\forall a \in X)(a_0 < a \to \varphi(x_1,\dots,x_n,a))$. Now, let $\langle\bar{\alpha}\rangle,\langle\bar{\beta}\rangle \in \langle\mathfrak{m}\rangle^{n+1}$. There exists $\gamma \in \mathfrak{m}$ such that $\alpha_n < \gamma < \alpha_{n+1}$, hence by $\varphi(x_1,\dots,x_n;x_{n+1})$-diagonal indiscernibility of X, $\varphi(\alpha_1,\dots,\alpha_{n+1}) \leftrightarrow \psi(\alpha_1,\dots,\alpha_n)$. Similarly, $\varphi(\beta_1,\dots,\beta_{n+1}) \leftrightarrow \psi(\beta_1,\dots,\beta_n)$. By our induction hypothesis, $\mathfrak{m}$ is a monad of n-indiscernible elements, hence $\psi(\alpha_1,\dots,\alpha_n) \leftrightarrow \psi(\beta_1,\dots,\beta_n)$, that is $\varphi(\alpha_1,\dots,\alpha_{n+1}) \leftrightarrow \varphi(\beta_1,\dots,\beta_{n+1})$. Since φ was arbitrary, $\mathfrak{m}$ is a monad of $(n+1)$-indiscernible elements.

c)→d) follows from 4.4.6 d). d)→e) is trivial. e)→f) follows again from 4.4.6 d). f)→a) follows from Lemma 4.4.10. □

4.4.12 Remark. The equivalence of c) and e) was first proved by Kirby [Kir84] in terms of ultrafilters.

The weakest diagonal partition property, $\mathfrak{m} \rightrightarrows (\mathfrak{m})^1_2$, is equivalent to $\mathfrak{m} \to (\mathfrak{m})^2_2$. Let D be a 2-estimated diagonal partition. Diagonal homogeneity for D ensures D_t-homogeneity for a 'large' set of indeces t. To ensure D_t-homogeneity for *all* indeces t in a general case, one must replace D_t with a common refinement of the equivalences D_s for all $s < t$. The resulting partition is exp-estimated, so $\mathfrak{m} \rightrightarrows (\mathfrak{m})^1_{\exp}$ is required. This is exactly how a monad of (1;1)-diagonally indiscernible elements is obtained. A seemingly weaker partition property than $\mathfrak{m} \rightrightarrows (\mathfrak{m})^1_{\mathrm{exp}}$ would be $\mathfrak{m} \rightrightarrows (\mathfrak{m})^1_{\mathrm{id}}$. We shall see that, considering relatively large monads, it is no weaker at all. This will need some preparation.

4.4.13 Definition. We say that a model $N \subseteq C$ is a *I-conservative extension* of M, written as $M \preccurlyeq^c_I N$, if N is an elementary extension of M, and

$$(\forall X \in \mathfrak{D}(N))(\exists Y \in \mathfrak{D}(M))(X \cap I = Y \cap I).$$

4.4.14 Lemma. *$M \preccurlyeq^c_I N$ implies $M \preccurlyeq_I N$.*

Proof. Assume, for the sake of obtaining a contradiction, that there are $\alpha \in N \setminus M$ and $a \in I$ such that $\alpha < a$. For $X \stackrel{\mathrm{df}}{=} \{\beta \in N \mid \alpha \leq \beta\}$, there is some $Y \in \mathfrak{D}(M)$ with $X \cap I = Y \cap I$. Clearly, $a \in Y \cap I \neq \emptyset$. Let $b \stackrel{\mathrm{df}}{=} \min(Y)$. Then $b > \alpha$ since $\alpha \notin M$. Consequently $b - 1 \in X \cap I = Y \cap I$. This contradicts the choice of b. □

4.4.15 Lemma. *Let I be a cut and $\alpha \in \mathfrak{g}_I$ such that $M \preccurlyeq^c_I M[\alpha]$. Then*

a) *$M \preccurlyeq^c_I M[\beta]$ for all $\beta \sim \alpha$.*

b) *If $\mathfrak{n} \leq_{\mathrm{RK}} \mathfrak{m}(\alpha)$ and $\beta \in \mathfrak{n}$, then $M \preccurlyeq^c_I M[\beta]$.*

Proof. a) To prove that the condition $M \preccurlyeq^c_I M[\alpha]$ does not depend on the choice of $\alpha \in \mathfrak{m}$, assume that $\alpha, \beta \in \mathfrak{m}$ and $Z \in \mathfrak{D}(M[\beta])$. Then $Z = \varphi(C, \beta)$ for some $\mathfrak{L}^{Ar}_M$-formula $\varphi(x, y)$ (recall that $\varphi(C, \beta) \stackrel{\mathrm{df}}{=} \{\gamma \in C \mid C \models \varphi(\gamma, \beta)\}$). Since $\alpha \sim \beta$, we have $\varphi(C, \beta) \cap M = \varphi(C, \alpha) \cap M$. In particular, if $Z' = \varphi(C, \alpha)$ and for some $X \in \mathfrak{D}$ $X \cap I = Z' \cap I$, then $X \cap I = Z \cap I$, too.

b) By 4.4.14 and 4.3.23, $\mathfrak{n} \subseteq \mathfrak{g}_I$. Let $F \in \mathfrak{F}$ be such that $F''\mathfrak{m} = \mathfrak{n}$ and $\alpha \in \mathfrak{m}$. Then $F(\alpha) \in M[\alpha]$, so $M[F(\alpha)] \preccurlyeq_M [\alpha]$. Hence, if $M \preccurlyeq^c_I M[\alpha]$, then $M \preccurlyeq^c_I M[F(\alpha)]$. The rest follows from a). □

4.4.16 Theorem. *The following are equivalent for a monad $\mathfrak{m}$:*

a) *$\mathfrak{m}$ is a relatively large strong monad.*

b) *$\mathfrak{m}$ is relatively large and $\mathfrak{m} \rightrightarrows (\mathfrak{m})^1_{\mathrm{id}}$.*

c) *$\mathfrak{m}$ is minimal and $M \preccurlyeq^c_{I_\mathfrak{m}} M[\alpha]$ for some (and hence all) $\alpha \in \mathfrak{m}$.*

Proof. Let I denote the cut $I_\mathfrak{m}$.

a)→b) From 4.4.11 we even know $\mathfrak{m} \rightrightarrows (\mathfrak{m})^1_{\mathrm{exp}}$, which is stronger.

b)→c) Let $X \in \mathfrak{D}(M[\alpha])$ for $\alpha \in \mathfrak{m}$ be of the form $\varphi(C, \alpha)$ for some $\mathfrak{L}^{Ar}_M$-formula $\varphi(x, y)$. Define $F \in \mathfrak{F}$ so that $F(\langle t, u\rangle)$ codes the set $\{z \mid 2^z < t \wedge \varphi(z, u)\}$. By our choice of coding of bounded sets, we may assume $F(\langle t, u\rangle) < t$. Now let D be a diagonal partition of C such that $u\, D_t\, v$ iff $F(\langle t, u\rangle) = F(\langle t, v\rangle)$. Clearly

$\|D_t\| \le t$, so there is some $Y \in \mathfrak{U}(\mathfrak{m})$ homogeneous for D. Let $I \stackrel{\mathrm{df}}{=} I_{\mathfrak{m}}$ and $a \in I$. By 4.4.5, $\mathfrak{m}$ is minimal and relatively large, so I is regular (in particular, I is closed under exponentiation). Since Y is unbounded in I, there is a $t \in Y \cap I$ such that $2^a < t$ and some $c \in Y \cap I$ such that $t < c$. By D-homogeneity of Y and the definition of F we now have:

$$\varphi(a,\alpha) \text{ iff } \varphi(a,c) \text{ iff } (\forall x,y \in Y)(2^a < x < y \to \varphi(a,y)).$$

In particular, if $\psi(a)$ denotes the formula on the right, then $\varphi(C,\alpha)\cap I = \psi(C)\cap I$ and $\psi(C) \in \mathfrak{D}$ as required.

c)→a) Let $M \preccurlyeq^c_{I_{\mathfrak{m}}} M[\alpha]$ for all $\alpha \in \mathfrak{m}$ and, aiming towards contradiction, suppose that $\mathfrak{m}$ is not strong, in particular not a set of (1;1)-diagonally indiscernible elements. Thus, there are an $\mathfrak{L}^{Ar}_M$-formula $\varphi(x,y)$, $\gamma, \alpha_1, \alpha_2 \in \mathfrak{m}$, and $\delta \in C$ such that $\delta \le \gamma < \alpha_1 < \alpha_2$ but $C \models \varphi(\delta,\alpha_1) \wedge \neg\varphi(\delta,\alpha_2)$.

Let $Y_x = \varphi(C,x)$. Then $\delta \in Y_{\alpha_1} \setminus Y_{\alpha_2}$. Let $Y \in \mathfrak{D}$ be such that $Y \cap I = Y_{\alpha_1} \cap I$. Then $Y \cap I = Y_{\alpha_2} \cap I$ and by 4.3.22, $Y \cap \sup I = Y_{\alpha_i} \cap \sup I$ for $i = 1,2$. It follows that $\delta \in \mathfrak{g}_I$. Let $X \stackrel{\mathrm{df}}{=} \{x \in C \mid Y \neq Y_x\}$ and let $F(x) \stackrel{\mathrm{df}}{=} \min(Y \dot{-} Y_x)$ for all $x \in X$. Then $X \in \mathfrak{D}$ and $F \in \mathfrak{F}$. If $\delta \in Y$, then $\alpha_2 \in X$, if $\delta \notin Y$, then $\alpha_1 \in X$; so, either way, $\mathfrak{m} \subseteq X$. For at least one $i \in \{1,2\}$, $F(\alpha_i) \le \delta$. Since, $\delta < \alpha_1 < \alpha_2$, this among other means that F is regressive on $\mathfrak{m}$. It further follows that F is not constant on $\mathfrak{m}$ since otherwise we would have $F''\mathfrak{m} = \{b\}$ for some $b \in I$, and thus $b \in Y \dot{-} Y_{\alpha_1}$, contradicting our choice of Y. Thus F is 1-1 on $\mathfrak{m}$. By restricting F if necessary, we may assume F is 1-1 on its domain. For the last step of the proof we need to ensure that

$$F''(M \setminus I) \cap I = \emptyset.$$

If F has this property, let $\tilde{F} \stackrel{\mathrm{df}}{=} F$. Otherwise we proceed as follows: Let $G \stackrel{\mathrm{df}}{=} F^{-1}$ and $Z \stackrel{\mathrm{df}}{=} \{y \in \mathrm{dom}(G) \mid y < \alpha_1 < G(y)\}$. By the assumption, $Z \cap I \neq \emptyset$. There exists some $Z_0 \in \mathfrak{D}$ such that $Z \cap I = Z_0 \cap I$ and we may assume $Z_0 \subseteq \mathrm{dom}(G)$. Let $c \stackrel{\mathrm{df}}{=} \min(G''Z_o)$. Clearly $G''Z_0 \subseteq [c,\to)$. Moreover, $c \in M \setminus I$ (otherwise $G^{-1}(c) < c$ since $G^{-1} = F$ and F is regressive; hence $G^{-1}(c) \in Z_0 \cap I = Z \cap I$ and by definition of Z, $\alpha_1 < G(G^{-1}(c)) = c$, a contradiction with $c \in I$). If $a \in I$ is such that $G(a) \in M \setminus I$; then $G(a) > \alpha_1$, so $a \in Z$ and also $a \in Z_0$, hence $G(a) > c$. This proves $G''I \subseteq I \cup [c,\to)$. Now let $\tilde{F} \stackrel{\mathrm{df}}{=} F \restriction [0,c)$. We have $\tilde{F}''(M \setminus I) \cap I = \emptyset$ and $\tilde{F} \restriction \mathfrak{m} = F \restriction \mathfrak{m}$.

Now, $\tilde{F}$ is regressive on a minimal monad $\mathfrak{m}$ and $\tilde{F}''(M \setminus I) \cap I = \emptyset$, so by 4.3.40, for every $\alpha \in \mathfrak{m}$, $\tilde{F}(\alpha) = F(\alpha) \in [\![\alpha]\!]$. But for some $i \in \{1,2\}$ we have $F(\alpha_i) \le \delta < \gamma < \alpha_i$ with $\gamma \in \mathfrak{m}$, so $F(\alpha_i) \notin [\![\alpha_i]\!]$—a contradiction. □

4.4.17 Remark. The requirement of minimality in item c) of the preceding theorem is necessary since there exist non-minimal monads $\mathfrak{m}$ satisfying the condition $M \preccurlyeq^c_I M[\alpha]$ for all $\alpha \in \mathfrak{m}$. Indeed, if $\mathfrak{n}$ is a strong monad, then $\mathfrak{n}$ is a monad of (1;2)-diagonally indiscernible elements and also a 2-monad, so $\mathfrak{m} \stackrel{\mathrm{df}}{=} \langle\mathfrak{n}\rangle^2$ is a monad, but clearly not minimal ($\mathfrak{m} \prec_{\mathrm{RK}} \mathfrak{n}$). We show that $M \preccurlyeq^c_I M[\alpha]$.

For simplicity, we shall further suppose that $I_{\mathfrak{m}} \neq M$ (although the argument could be easily modified for $I_{\mathfrak{m}} = M$, too). Let $\alpha \in \mathfrak{m}$, $X \in \mathfrak{D}(M[\alpha])$. Then $\alpha = \langle \alpha_1, \alpha_2 \rangle$ for some $\alpha_1, \alpha_2 \in \mathfrak{n}$, hence $X \in \mathfrak{D}(M[\alpha_1, \alpha_2])$. Assume for example $\alpha_1 < \alpha_2$ and let $\varphi(x, y_1, y_2) \in \mathrm{Fm}(\mathfrak{L}^{Ar}_M)$ be such that $X = \varphi(C, \alpha_1, \alpha_2)$. Let $X \in \mathfrak{U}(\mathfrak{n})$ be a set of $\varphi(x; y_1, y_2)$-diagonally indiscernible elements and $a_1 < a_2$ arbitrary elements from $X \cap (M \setminus I)$. Then $X \cap I = \varphi(C, a_1, a_2) \cap I$ and $\varphi(C, a_1, a_2) \in \mathfrak{D}$ as required.

Nevertheless, we have

4.4.18 Proposition. *A cut I is strong iff for some $\alpha \in \mathfrak{g}_I$, $M \preccurlyeq^c_I M[\alpha]$.*

Proof. The left-to-right implication follows from 4.4.16 by the fact that every strong cut includes a relatively large strong monad (Corollary 4.3.32). Conversely, assume $\alpha \in \mathfrak{g}_I$ satisfies $M \preccurlyeq^c_I M[\alpha]$. Then I is regular, by 4.4.14 and 4.3.23. Thus every monad from $\mathfrak{g}_I$ is relatively large; by 4.4.16, it suffices to show that at least one of them satisfies $\mathfrak{m} \rightrightarrows (\mathfrak{m})^1_{\mathrm{id}}$. For that again, it suffices to show that if $X \in \mathfrak{D}(\mathfrak{g}_I)$ and a diagonal partition D of X with $\|D_t\| \leq t$ for every $t \in X$, there is a D-homogeneous $H \subseteq X$ with $H \in \mathfrak{D}(\mathfrak{g}_I)$. Then $\mathfrak{m}$ is obtained by a construction analogous to that in the proof of 4.3.31.

Let D, X be as above and, in C, let σ be a D-h.m.p.h. sequence (recall Definition 2.1.7) such that $\sigma \subseteq [0, \alpha]$, $\sigma \smile \langle\alpha\rangle$ is D-homogeneous and σ is of maximal length. Clearly σ exists, is D-homogeneous, and $\sigma \in M[\alpha]$. Let $Y \stackrel{\mathrm{df}}{=} \{(\sigma)_i \mid i < \ell(\sigma)\}$. Then $Y \in \mathfrak{D}(M[\alpha])$. We first show that $Y \cap I$ is unbounded in I. It is non-empty, since $\min Y = \min X$. Let $(\sigma)_i \in I$, $i < \ell(\sigma)$. Then the initial subsequence of σ up to $(\sigma)_i$, i.e. $\sigma \restriction (i+1)$, is in $\sup I$ and hence in I, too. For a given $x \in X \cap [(\sigma)_i, \rightarrow)$, let $h(x)$ denote the unique sequence τ with $\ell(\tau) = i+1$ such that and for each $j \leq i$, x belongs to the $(\tau)_j$-th block of the partition $D_{(\sigma)_j}$ (counting from 0). Thus $(\tau)_j < \|D_{(\sigma)_j}\| \leq i$. Then $h \in \mathfrak{F}$, so $h(\alpha) \in M[\alpha]$. Moreover, since $h(\alpha)$ is a sequence of length $i+1$ with each element less than $i \in I$, we have that $h(\alpha) \in \sup I$ by semi-regularity, and in fact $h(\alpha) \in I$, by 4.4.14. Clearly $(\sigma)_{i+1} = \mu x : h(x) = h(\alpha)$, so $(\sigma)_{i+1} \in I$. Hence $Y \cap I$ is unbounded. Let $H' \in \mathfrak{D}$ be such that $Y \cap I = H' \cap I$. For every $a \in I$, $H' \cap [0, a]$ is D-homogeneous. Thus, if $I = M$, we may take $H \stackrel{\mathrm{df}}{=} H'$. For the bounded case, there is by overspill some $a \in M \setminus I$ such that $H' \cap [0, a]$ is D-homogeneous. Then $H \stackrel{\mathrm{df}}{=} H' \cap [0, a]$ is as required. □

4.4.19 Remarks.

a) The ultrafilter over α satisfying $M \preccurlyeq^c_I M[\alpha]$ would be called *definable* in Kirby's terminology [Kir84]. We do not adopt this terminology for monads to avoid confusion with definable sets.

b) Diagonal partition properties of singletons are closely related to diagonal intersections and normal ultrafilters, which are some of the properties studied in a similar context by Kirby in [Kir84]. In analogy with set theory, he defines an ultrafilter $\mathcal{U}$ on M-coded subsets of $I \subseteq^e M$ to be *weakly normal* if for every M-coded system of sets $\{A_i\}_{i\in I}$ such that $(\forall i \in I) A_i \in \mathcal{U}$, there is $X \in \mathcal{U}$ such that $\langle i, j\rangle \in \langle X\rangle^2 \to j \in A_i$ An ultrafilter $\mathcal{U}$ is *weakly definable* if for any $\{A_i\}_{i\in I}$, there is $X \in \mathcal{U}(\mathfrak{m})$ such that either $(\forall i \in X) A_i \in \mathcal{U}$, or $(\forall i \in X) A_i \notin \mathcal{U}$. It follows easily from the definitions that $\mathfrak{m} \to (\mathfrak{m})^1_2$ iff $\mathcal{U}(\mathfrak{m})$ is weakly normal and weakly definable. From left to right, it works as follows: given a 2-estimated diagonal partition, weak definability uniformly selects the large one of the two classes of D_t on a large set of indeces t. Once this is done, weak normality produces a large subset from the diagonal intersection. The notion of *normal ultrafilter* is obtained by replacing $j \in A_i$ with $j \in \bigcap_{k\le j} A_i$ in the definition of weak normality; this produces the full diagonal intersection. As easily observed, for a relatively large monad $\mathfrak{m}$, $\mathcal{U}(\mathfrak{m})$ is normal and weakly definable iff $\mathfrak{m} \to (\mathfrak{m})^2_2$. Now, $\mathfrak{m} \rightrightarrows (\mathfrak{m})^1_{\mathrm{exp}}$ clearly implies that $\mathcal{U}(\mathfrak{m})$ is normal; and if $\mathfrak{m}$ is further relatively large, we have that $\mathcal{U}(\mathfrak{m})$ is definable from 4.4.16. On the other hand, Kirby proved that either of 'normal+definable' and 'weakly normal+definable' give $\mathfrak{m} \to (\mathfrak{m})^3_2$.

c) Our so far 'weakest' equivalent of strength in terms of diagonal partition properties for monads has been $\mathfrak{m} \rightrightarrows (\mathfrak{m})^1_{\mathrm{id}}$. In [Kir84, Theorem 5.7], Kirby gave a result which translated to our terminology says that every 2-monad included in the gap of a strong cut is a 3-monad. For completeness, we provide a proof of his result in Appendix B as B.3. Kirby's result gives us the following corollary, via 4.4.6 and 4.4.11.

4.4.20 Corollary. *Let I be a strong cut and $\mathfrak{m} \subseteq \mathfrak{g}_I$. Then $\mathfrak{m} \rightrightarrows (\mathfrak{m})^1_2$ iff $\mathfrak{m}$ is strong.* □

P-monads

Recall that a non-trivial ultrafilter $\mathcal{U}$ on ω is a *p-point* iff for every $f : \omega \to \omega$, f is either constant on finite-to-one on some set from $\mathcal{U}$. A *q-point* is a non-principal ultrafilter $p \in \beta\omega$ such that every finite-to-one function $f : \omega \to \omega$ is one-to-one on some set from p. Clearly, an ultrafilter $p \in \beta\omega$ is selective iff p is both a p-point and q-point.

The definition will apply to our situation if we translate the notion finite-to-one in the right way:

4.4.21 Definition.

a) Say that $F \in \mathcal{F}(I)$ is *I-finite-to-one* on $X \in \mathfrak{D}$ with $X \subseteq \mathrm{dom}(F)$ if for every $x \in X \cap I$ the set

$$\{y \in X \mid F(x) = F(y)\} = F^{-1''}\{F(x)\} \cap X$$

is bounded in I. In other words, F partitions the set $X \cap I$ into blocks that are bounded in I. Say further that F is *I-finite-to-one* on a monad $\mathfrak{m}$ if it is I-finite-to-one on some $X \in \mathcal{U}(\mathfrak{m})$.

b) A non-trivial monad $\mathfrak{m}$ is a *p-monad* if for every $F \in \mathcal{F}(I_{\mathfrak{m}})$, F is either constant on $\mathfrak{m}$ or I-finite-to-one on $\mathfrak{m}$.

c) A non-trivial monad $\mathfrak{m}$ is a *q-monad* if every function which is $I_{\mathfrak{m}}$-finite-to-one on $\mathfrak{m}$ is one-to-one on $\mathfrak{m}$

In other words, $F \in \mathcal{F}(I)$ is I-finite-to-one on X if F is non-constant on all monads included in $X \cap \mathfrak{g}_I$. Hence a monad $\mathfrak{m}$ is a p-monad iff every $F \in \mathcal{F}(I_{\mathfrak{m}})$ is either constant on $\mathfrak{m}$, or F is non-constant on every monad from a certain neighborhood of $\mathfrak{m}$, i.e. $\mathfrak{m}$ is not an accumulation point of monads on which F is constant. Similarly, $\mathfrak{m}$ is a q-monad iff for every $F \in \mathcal{F}(I_{\mathfrak{m}})$ with $\mathfrak{m} \subseteq \mathrm{dom}(F)$, F is 1-1 on $\mathfrak{m}$ or $\mathfrak{m}$ is an accumulation point of monads on which F is constant.

4.4.22 Remark. In topological terminology, a point is called a p-point if the intersection of any countable system of its neighborhoods is also its neighborhood. Since our algebra is countable, we cannot apply the topological definition; for every non-trivial monad $\mathfrak{m}$, $\{\mathfrak{m}\}$ is not a neighborhood of $\mathfrak{m}$, but is an intersection of the countable system of its (clopen) neighborhoods of $\mathfrak{m}$ in the Stone topology of $C/\sim$.

4.4.23 Proposition. *A non-trivial monad $\mathfrak{m}$ is minimal iff it is both a p-monad and a q-monad.*

Proof. Trivially from 4.2.20. □

4.4.24 Proposition. *A non-trivial monad $\mathfrak{m} \subseteq \mathfrak{g}_I$ is a p-monad iff every function $F \in \mathcal{F}(\mathfrak{m})$ such that $F(x) \leq x$ for all $x \in \mathrm{dom}(F)$, is either constant or I-finite-to-one on some set from $\mathcal{U}(\mathfrak{m})$.*

Proof. The implication from left to right is trivial. Suppose $\mathfrak{m}$ has the later property. Let $F \in \mathcal{F}(I)$ and put $\tilde{F}(x) = \min(F(x), x)$. Clearly, $\tilde{F}(x) \leq x$ for all $x \in \mathrm{dom}(\tilde{F})$, so $\tilde{F}$ is either constant or I-finite-to-one on some $X \in \mathcal{U}(\mathfrak{m})$ by the premise. Yet, for every y, the sets $F^{-1}[\{y\}]$ and $\tilde{F}^{-1}[\{x\}]$ may only differ by the point y. Hence, if $\tilde{F}$ is constant on X, then so is F. Let $\tilde{F}$ be I-finite-to-one on $X \in \mathcal{U}(\mathfrak{m})$ and let $x \in X$. We show that the set $Z \stackrel{\text{df}}{=} \{z \in X \mid F(z) = F(x)\}$ is bounded in I. Assume otherwise. Then there is some $z_0 \in Z \cap I$ such that $z_0 \geq F(x)$ (note that $F(x) \in I$ since otherwise $Z \cap I$ would be empty as F is a I-function). Then $F(z_0) = \tilde{F}(z_0) = F(x) \in X$. Since F equals $\tilde{F}$ on $Z \setminus [0, z_0)$, the set $\{z \in X \mid \tilde{F}(z) = \tilde{F}(z_0)\}$ is unbounded in I, in contradiction with $\tilde{F}$ being I-finite-to-one on X. □

In the rest of the paragraph, we investigate the relationship between regular monads, (relatively large) p-monads, and (relatively large) minimal monads.

As we already remarked in 4.3.18, there are regular non-minimal monads in the gap of $\mathbb{N}$. The following proposition is a generalization.

4.4.25 Proposition. *If $I \to (I)^2_{<I}$, then $\mathfrak{g}_I$ includes a regular monad that is not minimal and, in fact, not even a p-monad.*

Proof. By 4.3.31, the gap $\mathfrak{g}_I$ includes a monad $\mathfrak{m}$ such that $\mathfrak{m} \to (\mathfrak{m})^2_{<I}$. In particular, $\mathfrak{n} = \langle \mathfrak{m} \rangle^2$ is a regular monad. Notice that $\mathfrak{n} \subseteq \mathfrak{g}_I$. Let $\pi_1 \in \mathfrak{F}$ be the mapping $\pi_1 : \langle C \rangle^2 \to C$ defined by $\pi_1(\langle \alpha, \beta \rangle) = \alpha$. Then $\pi_1'' \mathfrak{n} = \mathfrak{m}$ and π_1 is clearly neither constant nor one-to-one on $\mathfrak{n}$ (we also see that $\mathfrak{m} \prec_{\mathrm{RK}} \mathfrak{n}$). Hence, $\mathfrak{n}$ is a non-minimal regular monad. Clearly, $\pi_1(x) \le x$ and, in particular, $\pi_1 \in \mathfrak{F}(I)$. Suppose π_1 was I-finite-to-one on $X \in \mathfrak{U}(\mathfrak{n})$. It follows easily by saturation that there is some $Y \in \mathfrak{U}(\mathfrak{m})$ such that $\langle Y \rangle^2 \subseteq X$. Now, if $y \in \langle Y \rangle^2 \cap M$ and $y_1 = \pi_1(y)$, then $y_1 \in Y \cap M$ and $Y \setminus [0, y_1] \subseteq \pi_1^{-1}[\{y_1\}] \cap X$, hence the latter set belongs to $\mathfrak{U}(\mathfrak{m})$ and is therefore unbounded in I. □

We may improve the result slightly for strong cuts:

4.4.26 Corollary. *If I is strong, then regular monads that are not p-monads are dense in $\mathfrak{g}_I/\sim$.*

Proof. It is straightforward to verify that if $F : \mathfrak{m} \asymp_{\mathrm{RK}} \mathfrak{n}$, where $\mathfrak{m}$ and $\mathfrak{n}$ both belong to the gap $\mathfrak{g}_I$ and $F \in \mathfrak{F}(I)$, then $\mathfrak{m}$ is a p-monad if and only if $\mathfrak{n}$ is a p-monad. For $I = M$, $\mathfrak{F}(I) = \mathfrak{F}$, so the result is clear.

Otherwise, let I be a proper strong cut of M, $F \in \mathfrak{F}$, and $F : \mathfrak{m} \asymp_{\mathrm{RK}} \mathfrak{n}$. Then by 4.3.8, for some $c \in M \setminus I$, $F''I \cap [0, c]$. Now, for $G \overset{\mathrm{df}}{=} F \restriction (F^{-1}{}''[0, c])$, $G \in \mathfrak{F}(I)$ and $G \restriction \sup I = F \restriction \sup I$. Thus $G : \mathfrak{m} \asymp_{\mathrm{RK}} \mathfrak{n}$. □

4.4.27 Proposition. *Let I be a regular cut. Then every p-monad in $\mathfrak{g}_I$ is regular.*

Proof. Let $\mathfrak{m} \subseteq \mathfrak{g}_I$ be a p-monad and $F : C \to [0, a]$ for some $a \in I$. We need to show that F is constant on some $X \in \mathfrak{U}(\mathfrak{m})$. Since $\mathfrak{m}$ is a p-monad, F is either I-finite-to-one or constant on some $X \in \mathfrak{U}(\mathfrak{m})$. In the later case, we are done, so suppose that F was I-finite-to-one on X. By 4.3.19, X includes some regular monad (even a relatively large minimal one), say $\mathfrak{n}$. But then F is constant on $\mathfrak{n}$, hence $F''\mathfrak{n} = \{b\}$ for some $b \in [0, a]$. Thus $X \cap F^{-1}[\{b\}]$ is unbounded in I, in contradiction with F being I-finite-to-one on X. □

We have shown that the property of being a p-monad is strictly stronger than being regular. We now show that it is strictly weaker than being minimal.

4.4.28 Remark. An equivalent of the following theorem for $\langle \mathbb{N}, \mathcal{P}(\mathbb{N}) \rangle$ requires assumptions beyond ZFC, such as Martin's axiom (see [Boo71, 4.12]).

4.4.29 Theorem. *If I is a regular cut, then non-minimal p-monads form a dense subset of $\mathfrak{g}_I$.*

Proof. Let $\mathfrak{g}$ denote the gap of I and let $X \in \mathfrak{D}(\mathfrak{g})$. Let P be a partition of X into a sequence of adjacent intervals of $\langle X, \leq \rangle$ of increasing lengths so that the set of the start points of the intervals is unbounded in I. Specifically, we shall assume that:

- for each $a \in X$, $P[a]$ is an interval of $\langle X, \leq \rangle$,
- for each $a \in X \cap I$ there is some $b > a$, $b \in X$, such that $b \notin P[a]$,
- $|P[a]| = \min(P[a]) + 1$ for all $a \in X$. (We may need to extend or truncate X to meet this condition on the last block $P[a]$).

Note that such a partition P can easily be obtained, say by letting $P_{(\alpha)} = [F(\alpha), F(\alpha+1))$ where F is a function defined by $F(0) \stackrel{\text{df}}{=} \min(X)$ and

$$F(\alpha+1) \stackrel{\text{df}}{=} \mu\gamma \in X : (\gamma = \max(X) \vee |X \cap [F(\alpha), \gamma)| = F(\alpha) + 1).$$

We will show that there is a p-monad $\mathfrak{m} \subseteq X \cap \mathfrak{g}$ that is not anti-homogeneous for P. (Clearly, $\mathfrak{m}$ cannot be homogeneous for P, since otherwise some $P[a]$ with $a \in I \cap X$ would intersect $\mathfrak{g}$, which is not possible due to our assumptions on P and semi-regularity of I.)

Let $\{H_n\}_{n\in\omega}$ be an enumeration of all M-definable P-homogeneous sets, $\{F_n\}_{n\in\omega}$ an enumeration of all functions $F \in \mathfrak{F}$ satisfying $F(x) \leq x$ for all $x \in \operatorname{dom}(F)$, and $\{I_n\}_{n\in\omega}$ a non-increasing sequence of closed intervals of $\langle C, \leq \rangle$ with endpoints in M such that $\mathfrak{g} = \bigcap_{n\in\omega} L_n$. We shall define a non-increasing sequence $\{X_n\}_{n\in\omega}$ of sets from $\mathfrak{D}$ that will satisfy the following:

1) $L_n \supseteq X_n \in \mathfrak{D}(\mathfrak{g})$,
2) F_n is either constant or I-finite-to-one on $\operatorname{dom}(F_n) \cap X_{n+1}$,
3) $X_{n+1} \cap H_n = \emptyset$,
4) the partition $Q = P \cap (X_n \times X_n)$ of X_n has the following properties:

    (Q-1) For every $r \in I$ there exists $a \in \operatorname{dom}(Q) \cap I$ such that $r < |Q[a]|$

    (Q-2) $|Q[x]| \leq |Q[y]|$ for all $x < y$ from $\operatorname{dom}(Q)$.

If such a sequence $\{X_n\}_{n\in\omega}$ is given, there exists some $\alpha \in \bigcap_{n\in\omega} X_n \subseteq \mathfrak{g}$ by saturation and the monad $\mathfrak{m}(\alpha)$ is a p-monad due to 2), but not a minimal monad due to 3) and 4) (Q-1). In the rest of the proof we construct the family $\{X_n\}_{n\in\omega}$.

First observe that if the property (Q-1) is assured for a partition $Q \subseteq P$ of $Y \in \mathfrak{D}(\mathfrak{g})$, we may find $Y' \subseteq Y$, $Y' \in \mathfrak{D}(\mathfrak{g})$ so that $Q' \stackrel{\text{df}}{=} Q \cap (Y' \times Y')$ satisfies both (Q-1) and (Q-2). We obtain Y' by simply throwing away from Y

those equivalence classes $Q[x]$, $x \in Y$, for which there is some $y < x$ in Y with $|Q[y]| > |Q[x]|$. Then $Y' \in \mathfrak{D}(\mathfrak{g})$ by the fact that for all $a \in I$, $Q[a] \subseteq P[a]$ and the latter is bounded in I.

Now, we may put $X_0 \overset{\text{df}}{=} L_0$. If X_n is given, let $Y \overset{\text{df}}{=} (L_n \cap X_n) \setminus H_n$. The intersection with L_n clearly affects neither of the conditions (Q-1) and (Q-2).

In the following, we let $P_n \overset{\text{df}}{=} P \cap (X_n \times X_n)$. Since H_n takes out at most one point from each block of P_n and P_n satisfies the condition (Q-1), the partition $P' = P \cap (Y \times Y)$ satisfies (Q-1) too. As noted above, we may assume without loss of generality that P' satisfies (Q-2) as well (replacing Y with a suitable subset, if needed).

Without loss of generality, we may further assume $Y \subseteq \mathrm{dom}(F_n)$ (otherwise we take $F_n' = F_n \cup \{\langle x, 0\rangle \mid x \in Y \setminus \mathrm{dom}(F_n)\}$ instead). In order to assure 2) while retaining (Q-1), we consider the following cases:

CASE I. Assume $\mathrm{rng}(F_n) \cap I \subseteq [0,d]$ for some $d \in I$. Let $Y' \overset{\text{df}}{=} Y \cap F_n^{-1\prime\prime}[0,d]$. Since $F_n(x) \le x$ on the domain of F_n, $F_n^{\prime\prime} I \subseteq I$, hence $Y \cap I \subseteq Y'$ and $Y' \in \mathfrak{D}(\mathfrak{g})$. Let $P' \overset{\text{df}}{=} P \cap (Y' \times Y')$. Now, on each block of $P'[\alpha]$ with $\alpha \in Y' \cap I$, F_n induces a partition with at most $d+1$ sub-blocks; we keep from each $P'[\alpha]$ one sub-block whose size is at least average. Formally: for each $x \in \{\min(P'[\alpha]) \mid \alpha \in Y'\}$, let

$$G(x) \overset{\text{df}}{=} \mu z \le d : (d+1) \cdot |\{y \in P'[x] \mid F_n(y) = z\}| \ge |P'[x]|. \tag{4.8}$$

Then $\mathrm{rng}(G) \subseteq [0,d]$ and $\mathrm{dom}(G) \in \mathfrak{D}(\mathfrak{g})$. Since I is regular, there is some $c \in [0,d] \cap M$ such that $Z \overset{\text{df}}{=} G^{-1\prime\prime}[c]$ is unbounded in I. We now let $X_{n+1} \overset{\text{df}}{=} \bigcup_{x \in Z} P'[x] \cap F_n^{-1\prime\prime}[c]$. Clearly $X_{n+1} \subseteq Y' \subseteq X_n$. Also, since for every $x \in Z$, $X_{n+1} \cap P'[x] \neq \emptyset$, X_{n+1} is unbounded in I. If $\alpha \in X_{n+1}$, then $\alpha \in P'[x]$ for precisely one $x \in \mathrm{dom}(G)$ and $F_n(\alpha) = c$, so F_n is constant on X_{n+1}, hence the condition 2) is fulfilled. It remains to check that the property (Q-1) was preserved. Let $r \in I$. We know that (Q-1) holds for P'. Hence, there is some $x \in \mathrm{dom}(G) \cap I$ such that $r \cdot (d+1) < |P'[x]|$. Using (Q-2) for P' and the fact that Z is unbounded in I, we can conclude that $r \cdot (d+1) < |P'[x]|$ for some $x \in Z$ as well. But then for $x' \in P'[x] \cap F_n^{-1\prime\prime}[c]$, $|P_{n+1}[x']| = |P'[x] \cap F_n^{-1\prime\prime}[c]| > r$ by (4.8). As observed above, now that (Q-1) is established for P_{n+1}, we may assume the condition (Q-2) fulfilled as well (correcting X_{n+1} slightly, if needed).

CASE II. Assume $\mathrm{rng}(F_n) \cap I$ is unbounded in I, but for some $d \in I$, the partition $P' \overset{\text{df}}{=} P \cap (Y' \times Y')$, where $Y' \overset{\text{df}}{=} Y \cap F_n^{-1\prime\prime}[0,d]$ has the property (Q-1). In that case, we prune Y' in the usual way to satisfy (Q-2) and proceed with the result as in Case I.

CASE III. Assume $\mathrm{rng}(F_n)$ is unbounded in I, yet for every $d \in I$, there is some $r_d \in I$ such that for all $x \in I$, $|P'[x] \cap F_n^{-1\prime\prime}[0,d]| \le r_d$. Let $Z \overset{\text{df}}{=} \{\min(P'[\alpha]) \mid \alpha \in Y'\}$. We now define a function h with $\mathrm{rng}(h) \subseteq Z$ in the following way: $h(0) \overset{\text{df}}{=} \min(Y')$. If $h(\alpha)$ is defined, let $h(\alpha+1)$ be the least element of $Z \setminus [0, h(\alpha)]$ such that

$$|P'[h(\alpha+1)] \setminus F_n^{-1}[0, h(\alpha)]| > h(\alpha). \tag{4.9}$$

In other words, $h(\alpha+1) = x$ if x is the least element of the first P'-block such that after removing the first $h(\alpha)$ equivalence classes of the partition induced by F_n from $P'[x]$, there are still at least $h(\alpha)+1$ elements left (provided such x exists).
We shall now prove that $\mathrm{rng}(h)$ is unbounded in I. Suppose otherwise and let $d \stackrel{\mathrm{df}}{=} h(\alpha)$ be the last element of $\mathrm{rng}(h) \cap I$. By our assumption, there is some $r_d \in I$ such that $|P'[x] \cap F_n^{-1''}[0,d]| \le r_d$ for all $x \in I$. Using the property (Q-1) of P', let $a \in I \setminus [0,d]$ be such that $|P''[a]| > r_d + d$. Then clearly $|P'[a] \setminus F_n^{-1''}[0,d]| > d$. It follows that $h(\alpha+1)$ is defined and belongs to I in contradiction with our choice of d and α as the last element of $\mathrm{rng}(h)$ in I.
Now that the unboundedness of $\mathrm{rng}(h)$ in I is established, we let

$$(X_{n+1} \stackrel{\mathrm{df}}{=} \bigcup\{P'[h(\alpha+1)] \setminus F_n^{-1''}[0,h(\alpha)] \mid \alpha \in \mathrm{dom}(h)\}.$$

Since the condition (4.9) guarantees that P_{n+1} has the property (Q-1), we are done (the property (Q-2) can be established as usual). It remains to show that F_n is I-finite-to-one on X_{n+1}. Indeed, if $y \in I$, take some $\alpha \in I$ with $h(\alpha) > y$ and note that according to the definition of X_{n+1}, $F_n^{-1''}\{y\} \cap [h(\alpha+1), \rightarrow) = \emptyset$.
□

More extensions

4.4.30 Definition. For $I \subseteq^e M$ and $\alpha \in \sup \mathfrak{g}_I$, let $I[\alpha] \stackrel{\mathrm{df}}{=} \{F(\alpha) \mid F \in \mathfrak{F}(I) \wedge \alpha \in \mathrm{dom}(F)\}$.

Note that our notation suppresses M, which (represented in the definition by the set $\mathfrak{F}$) is an important parameter of the definition.

Clearly, $I[\alpha]$ is closed under every function from $\mathfrak{F}(I)$; in particular, it is closed under operations, so $I[\alpha]$ is a substructure if $M[\alpha]$.

4.4.31 Proposition. *For every $I \subseteq^e M$ and $\alpha \in \sup \mathfrak{g}_I$, $I \preccurlyeq_1 I[\alpha] \preccurlyeq_0 M[\alpha]$. Moreover, if I is a strong cut, $I \preccurlyeq I[\alpha]$.*

Proof. For an open $\mathfrak{L}^{Ar}$ formula φ, we clearly have

$$I \preccurlyeq_\varphi I[\alpha] \preccurlyeq_\varphi M[\alpha] \tag{4.10}$$

Assume $\varphi(x,\bar{y})$ is bounded and satisfies (4.10). We first show that $I[\alpha] \preccurlyeq_{(\exists x \le u)\varphi} M[\alpha]$. Let $\beta, \bar{\gamma} \in I[\alpha]$ and $M[\alpha] \models (\exists x \le \beta)\varphi(x,\bar{\gamma})$. Let $F(\langle u,\bar{y}\rangle) \stackrel{\mathrm{df}}{=} \mu x \le u : \varphi(x,\bar{y})$. Clearly F is regressive on its domain, so $F \in \mathfrak{F}(I)$, and as $\langle \beta,\bar{\gamma}\rangle \in I[\alpha] \cap \mathrm{dom}(F)$, we have $F(\langle\beta,\bar{\gamma}\rangle) \in I[\alpha]$. This proves $I[\alpha] \preccurlyeq_0 M[\alpha]$.

Next we prove $I \preccurlyeq_{(\exists x)\varphi} I[\alpha]$ for $\varphi(x,\bar{y})$ a bounded φ satisfying (4.10). (Then, by induction on complexity of formulae, we first obtain $I \preccurlyeq_o I[\alpha]$, and subsequently $I \preccurlyeq_1 I[\alpha]$). For that, let $\beta \in I[\alpha]$, $\bar{a} \in I$, and $I[\alpha] \models \varphi(\beta,\bar{a})$. Then $C \models \varphi(\beta,\bar{a})$ (by the first part of the proof). There exists $F \in \mathfrak{F}(I)$ such that $F(\alpha) = \beta$. Let $X \stackrel{\text{df}}{=} \{\gamma \in C \mid C \models \varphi(F(\gamma), \bar{a})\}$. Clearly $X \in \mathfrak{D}$ and $\alpha \in X$, so $I \cap X \neq \emptyset$, since $\alpha \in \sup \mathfrak{g}_I$. For $c \in I \cap X$ we thus have $F(c) \in I$ and $C \models \varphi(F(c), \bar{a})$, so $I \models \varphi(F(c), \bar{a})$, by $I \preccurlyeq_0 M$.

Finally, assume I is strong. By 4.3.11, a), $I^* \models ACA_0$. Let for each $X \in \mathfrak{D}$, $X^I = X \cap I$. Then $X^I \in \mathrm{Cod}(I/M)$. Conversely, for each $A \in \mathrm{Cod}(I/M)$, let $\tilde{A} \in \mathfrak{D}$ be such that $A = \tilde{A}^I$. Note that $\tilde{A} \cap \sup \mathfrak{g}_I$ is uniquely determined, since if $X, Y \in \mathfrak{D}$ and $X^I = Y^I$, $X \neq Y$, then $\min X \dot{-} Y > I$. Let $\mathfrak{U}_I(\alpha)$ denote the ultrafilter on $\mathrm{Cod}(I/M)$ defined by $\mathfrak{U}_I(\alpha) \stackrel{\text{df}}{=} \{X^I \mid \alpha \in X \in \mathfrak{D}\}$. Clearly $A \in \mathfrak{U}_I(\alpha)$ iff $\tilde{A} \in \mathfrak{U}(\mathfrak{m}(\alpha))$. Clearly for all $F \in \mathfrak{F}(I)$, F^I is a function with $\mathrm{dom}(F) = (\mathrm{dom}(F))^I$ and $\mathrm{rng}(F) = (\mathrm{rng}(F))^I$.

We first prove, the following version of Łoś's lemma: for any $\mathfrak{L}^{Ar}$-formula $\varphi(x_1, \ldots, x_k)$ and $F_i \in \mathfrak{F}(I)$, $i = 1, \ldots, k$:

(4.11) $\quad I[\alpha] \models \varphi(F_1(\alpha), \ldots, F_k(\alpha))$ iff $\{a \in I \mid I^* \models \varphi(F_1^I(a), \ldots, F_k^I(a))\} \in \mathfrak{U}_I(\alpha)$,

where on the left side, we evaluate $F_i(\alpha)$ in C and the set on the right is in $\mathrm{Cod}(I/M)$ by ACA_0, so the equivalence makes sense. For bounded φ this follows from $I[\alpha] \preccurlyeq_0 C$. If the equivalence holds for $\varphi(x, \bar{y})$, $A \stackrel{\text{df}}{=} \{a \in I \mid I^* \models (\exists x)\varphi(x, F_1^I(a), \ldots)\}$, and $A \in \mathfrak{U}_I(\alpha)$, then $A \neq \emptyset$. In I^*, we may define a function $f(y) = \mu x : \varphi(x, F_1^I(y), \ldots)$. Then $f \in \mathrm{Cod}(I/M)$ by ACA_0, so $f = F^I$ for some $F \in \mathfrak{F}$. We may clearly assume $\mathrm{dom}(F) \cap I = A$, so $F \in \mathfrak{F}(I)$. But then $A = \{a \in I^* \mid I \models \varphi(F^I(a), F_1^I(a), \ldots)\}$, so by induction hypothesis $I[\alpha] \models \varphi(F(\alpha), F_1(\alpha), \ldots)$. The proof of the converse implication in the induction step is simple.

Now, if $\bar{c} \in I$ and $k_c \stackrel{\text{df}}{=} C \times \{c\}$, we have $I[\alpha] \models \varphi(c_1, \ldots)$ iff $I[\alpha] \models \varphi(k_{c_1}(\alpha), \ldots)$ iff $A \stackrel{\text{df}}{=} \{a \in I \mid I \models \varphi(k_{c_1}(a), \ldots)\} = \{a \in I \mid I \models \varphi(\bar{c})\} \in \mathfrak{U}(\alpha)$ iff $A \neq \emptyset$ iff $I \models \varphi(\bar{c})$. $\square$

4.4.32 Lemma. *If I is strong, $\alpha, \beta \in \mathfrak{g}_I$, and $\mathfrak{m}(\alpha) \leq_{\mathrm{RK}} \mathfrak{m}(\beta)$, then $I[\beta] \preccurlyeq I[\alpha]$.*

Proof. Let $\beta = G(\alpha)$ for some $G \in \mathfrak{F}$. Using 4.3.8, we may assume $G \in \mathfrak{F}(I)$. Then using the notation of the previous proof, $B \in \mathfrak{U}_I(\beta)$ iff $\beta \in \tilde{B}$ iff $G(\alpha) \in \tilde{B}$ iff $\alpha \in G^{-1}[\tilde{B}]$ iff $(G^I)^{-1}[B] \in \mathfrak{U}_I(\alpha)$. Thus by (4.11), we have for $F_1, \ldots, F_k \in \mathfrak{F}(I)$ and a $\mathfrak{L}^{Ar}$ formula $\varphi(x_1, \ldots, x_k)$: $I[\alpha] \models \varphi(F_1(\beta), \ldots)$ iff $I[\alpha] \models \varphi(F_1(G(\alpha)), \ldots)$ iff $\{a \in I \mid I^* \models \varphi((G \circ F_1)^I(a), \ldots)\} \in \mathfrak{U}_I(\alpha)$ iff $\{a \in I \mid I^* \models \varphi(F_1^I(G^I(a)), \ldots)\} \in \mathfrak{U}_I(\alpha)$ iff $\{G^I(a) \mid a \in I$ and $I^* \models \varphi(F_1^I(G^I(a)), \ldots)\} \in \mathfrak{U}_I(\beta)$ iff $\{b \in I \mid I^* \models \varphi(F_1^I(b), \ldots)\} \in \mathfrak{U}_I(\beta)$ iff $I[\beta] \models \varphi(F_1^I(\beta), \ldots)$. $\square$

4.4.33 Proposition. *Let I be a strong cut, $\alpha \in \mathfrak{g}_I$. Then $\mathfrak{m}(\alpha)$ is a p-monad iff $I[\alpha]$ is a minimal elementary end-extension of I, i.e., if $I \preccurlyeq^e I[\alpha]$, and $I \preccurlyeq^e N \preccurlyeq^e I[\alpha]$ implies $I = N$ or $I[\alpha] = N$.*

Proof. For left-to-right, assume $I \prec^e N \prec^e I[\alpha]$ and let $\beta \in N \setminus I$. Then $I[\beta] \prec^e I[\alpha]$. For some $F \in \mathfrak{F}(I)$, $F(\alpha) = \beta$. It follows that F is not constant on $\mathfrak{m}(\alpha)$, so by the property of p-monad, there exists $X \in \mathfrak{U}(\mathfrak{m}(\alpha))$ such that F is I-finite-to-one on X. For $\gamma \in F''X$, let $G(\gamma) \stackrel{\text{df}}{=} \max(F^{-1}[\{\gamma\}])$; then $G''I \subseteq I$, so $G \in \mathfrak{F}(I)$.

Since clearly $\beta \in \mathrm{dom}(G)$, $\alpha \le G(\beta)$. But $G(\beta)$ is a I-function, so $G(\beta) \in I[\beta]$. This contradicts with $I[\beta] \prec^e I[\alpha]$.

Conversely, let $I[\alpha]$ be a minimal elementary end-extension of I; in particular, $M \preccurlyeq_I M[\alpha]$, so $\mathfrak{m}(\alpha)$ is regular. Let $F \in \mathfrak{F}(I)$ be such that $\mathfrak{m}(\alpha) \subseteq \mathrm{dom}(F)$ and let $F(x) \le x$ holds on the domain of F. If F is not constant on $\mathfrak{m}(\alpha)$, then $F(\alpha) \in \mathfrak{g}_I$ by regularity. Let $\beta \stackrel{\mathrm{df}}{=} F(\alpha)$ and let $N \stackrel{\mathrm{df}}{=} I[\beta]$. Then $I \preccurlyeq^e N \preccurlyeq I[\alpha]$. But $I \neq N$, since $\beta \notin I$. Suppose $I[\alpha] = N$; then $\alpha \in I[\beta]$, so some function from $\mathfrak{F}(I)$ sends β to α. In that case, F is 1-1 on $\mathfrak{m}$. Assume $I[\alpha] \neq N$. Then $N \not\preccurlyeq^e I[\alpha]$. But then for some $\gamma \in N$, $\alpha \le \gamma$. There is a function $G \in \mathfrak{F}(I)$ such that $\gamma = G(\beta)$. Then $\alpha \le G(F(\alpha))$. The same inequality holds on some $X \in \mathfrak{U}(\mathfrak{m}(\alpha))$. Let $a, b \in X$ be such that $F(a) = F(b) = c$. Then $\max\{a,b\} \le G(F(a)) = G(F(b)) \in I$. Thus $F^{-1}(\{c\})$ cannot be unbounded in I, so F is I-finite-to-one. □

4.4.34 Proposition. *If* $\mathfrak{m}$ *is a relatively large strong monad, then for any* $Z \subseteq \mathfrak{m}$*,* $M \preccurlyeq^c_{I_\mathfrak{m}} M[Z]$ *and* $I_\mathfrak{m}$ *is a strong cut of* $M[Z]$*.*

Proof. Let $\mathfrak{m}$ and Z be as above. The proof that $M \preccurlyeq^c_{I_\mathfrak{m}} M[Z]$ is a generalization of Remark 4.4.17. Let $\varphi(x,\bar{y}) \in \mathrm{Fm}(\mathfrak{L}^{Ar}_M)$ and let $X \stackrel{\mathrm{df}}{=} \varphi(C,\vec{\alpha})$ for some parameters $\alpha_1 < \cdots < \alpha_k$, or briefly $\vec{\alpha}$, from Z. We must show that $X \cap I = Y \cap I$ for some $Y \in \mathfrak{D}$. But $\mathfrak{m}$ is a set of $\varphi(x;\bar{y})$-diagonally indiscernible elements. Thus if $c_1 < \cdots < c_k$ are arbitrary elements from $X \cap (M \setminus I)$, we have $\varphi(C,\vec{c}) \in \mathfrak{D}$ and $X \cap I = \varphi(C,\vec{c}) \cap I$, as required. (If $I = M$, we may not take such $\vec{c}$, but we may replace $\varphi(C,\vec{c})$ with $(\exists y)(\forall \bar{y} \in X)(y < y_1 < \cdots < y_k \to \varphi(C,\bar{y}))$.)

To see that $I_\mathfrak{m}$ is strong in $M[Z]$, simply deduce from $M \preccurlyeq^c_{I_\mathfrak{m}} M[Z]$, that for any partition $Q \in \mathfrak{D}(M[Z])$ of $\langle M[Z]\rangle^n$ $(n \in \mathbb{N})$ into two sets coincides on $I_\mathfrak{m}$ with some partition $P \in \mathfrak{D}$ of $\langle M\rangle^n$ into two sets. Since $I_\mathfrak{m}$ is strong in M, there is an P-homogeneous subset $X \in \mathfrak{D}$ unbounded in $I_\mathfrak{m}$. Let $Y \in \mathfrak{D}(M[Z])$ be the largest Q-homogeneous initial segment of the set X. Then $Y \cap I = X \cap I$. This gives $(\forall n \in \mathbb{N}) I_\mathfrak{m} \to (I_\mathfrak{m})^n_2$ with respect to partitions from $\mathfrak{D}(M[Z])$. □

APPENDICES

APPENDIX A

CONSERVATIVE EXTENSIONS

This appendix contains a proof of the well-known McDowell, Specker theorem with an addendum attributed to Gaifman. The crucial step of the proof, Lemma A.3, is an immediate consequence of our Infinite Diagonal Partition Theorem 2.1.2, and thus serves also as an example of its usage.

A.1 Theorem (*McDowell, Specker, Gaifman*). *Every model of* PA *has a proper conservative elementary extension of the same cardinality. Moreover, if* $M \models \mathrm{PA}$ *is countable, there are* $2^{\aleph_0}$ *pairwise non-isomorphic countable conservative elementary extensions of* M.

A.2 Definition. *For the rest of this section, say that a formula* $\psi(x)$ decides $\theta(x,\bar{a})$ *in* M*, where* $\bar{a} \in M$*, iff* $\psi(M)$ *is unbounded in* M *and* $M \models (\exists x_0)(\forall x > x_0)(\psi(x) \to \theta(x,\bar{a}))$*. Say that* $\psi(x)$ forces $\theta(x,\bar{y})$ *in* M *iff for every* $\bar{a} \in M$*,* $\psi(x)$ *either decides* $\theta(x,\bar{a})$ *or decides* $\neg\theta(x,\bar{a})$ *in* M.

A.3 Lemma. *Let* M *be a model of* P *and* $\varphi(x)$ *an* $\mathcal{L}^{Ar}$*-formula such that* $\varphi(M)$ *is unbounded in* M*. Then for every* $\mathcal{L}^{Ar}$*-formula* $\theta(x,\bar{y})$ *there is an* $\mathcal{L}^{Ar}$*-formula* $\psi(x)$ *such that* $\psi(M) \subseteq \varphi(M)$ *and* $\psi(x)$ *forces* $\theta(x,\bar{y})$ *in* M.

Proof. Given $\varphi(x), \theta(x,\bar{y})$, ψ is found using 2.1.2 so that $\psi(M)$ is an unbounded diagonally homogeneous set for the diagonal partition D of $\varphi(M)$ defined by:

$$\langle t, \langle u,v\rangle\rangle \in D \overset{\mathrm{df}}{\iff} [\varphi(u) \wedge \varphi(v) \wedge (\forall t_0,\ldots,t_{n-1} < t)(\theta(u,\bar{t}) \leftrightarrow \theta(v,\bar{t}))].$$

D_t is clearly an equivalence on $\varphi(M)$ and $\|D_t\| \le 2^{t^n}$, so ψ exists. Let $\bar{a} \in M$. There is some $d \in \psi(M)$ such that $\bar{a} < d$; by D-homogeneity of $\psi(M)$, either $(\forall x > d)(\psi(x) \to \theta(x,\bar{a}))$ or $(\forall x > d)(\psi(x) \to \neg\theta(x,\bar{a}))$, as required. □

Proof of McDowell, Specker, Gaifman. We shall prove both claims at the same time. However, in order to derive the second part for countable M, we need to fix arbitrary $F \in {}^{\omega}2$, as a parameter of our construction. Now, let $\{\xi_j(x)\}_{j\in\omega}$ be some enumeration of all $\mathscr{L}^{Ar}$-formulae with the only free variable x and let $\{\theta_i(x,\bar{y})\}_{i\in\omega}$ enumerate all $\mathscr{L}^{Ar}$-formulae with all free variables among $x,\bar{y}$, where $\bar{y}$ stands for tuples of arbitrary lengths. For a formula $\xi(x)$, let $\xi^{[0]}(x)$ and $\xi^{[1]}(x)$ denote $\mathscr{L}^{Ar}$-formulae equivalent in PA to $\xi(x)\wedge$'$|\{y \leq x \mid \xi(y)\}|$ is even' and $\xi(x)\wedge$'$|\{y \leq x \mid \xi(y)\}|$ is odd', respectively. Then $\{\xi^{[0]}(M),\xi^{[1]}(M)\}$ is a partition of $\xi(M)$ and if $\xi(M)$ is unbounded in M, then so are both $\xi^{[0]}(M)$ and $\xi^{[1]}(M)$.

The sequence of $\mathscr{L}^{Ar}$-formulae $\{\varphi_i^F(x)\}_{i\in\omega} = \{\varphi_i(x)\}_{i\in\omega}$ is defined by induction on i as follows: $\varphi_0(x)$ is the formula $(x=x)^{[F(0)]}$. Clearly, $\varphi_0(M)$ is unbounded in M. If φ_i is defined, let $\varphi_{i+1}(x)$ be the formula $\xi_j^{[F(i)]}$ where j is least such that $\xi_j(M) \subseteq \varphi_i(M)$ and $\xi_j(x)$ forces $\theta_i(x,\bar{a})$ in M. Since $\{\xi_j\}_{j\in\omega}$ enumerates all $\mathscr{L}^{Ar}$-formulae with one free variable, such j exists due to the preceding lemma. Especially, all $\varphi_i(M)$ are unbounded.

Let T denote the following theory in the language $\mathscr{L}^{Ar}_{M\cup\{c\}}$ with a new constant symbol c:

$$\mathrm{Thm}(M) \cup \{\varphi_{i+1}(c) \wedge \theta_i(c,\bar{a}) \mid \bar{a} \in M \text{ and } \varphi_{i+1}(x) \text{ decides } \theta_i(x,\bar{a}) \text{ in } M\}$$

Surely, T is consistent, since every finite subset of T is satisfied in some expansion of M where c is realized by a sufficiently large element.

T is complete. Indeed, let θ be an $\mathscr{L}^{Ar}_{M\cup\{c\}}$-formula. There are $\bar{a} \in M$ and $i,k \in \omega$ such that θ and $\neg\theta$ are of the form (or equivalent to in predicate calculus) $\theta_i(c,\bar{a})$ and $\theta_k(c,\bar{a})$, respectively. Without loss of generality, we may assume that $i<k$. By the definition of φ_{i+1}, $\theta_i(x,\bar{y})$ is forced in M by some $\xi_j(x)$. Now, if $\xi_j(x)$ decides $\theta_i(x,\bar{a})$, then so does φ_{i+1} and $T \vdash \theta$ because $\varphi_{i+1}(c)\wedge\theta_i(c,\bar{a}) \in T$. Otherwise $\xi_j(x)$ decides $\theta_k(x,\bar{a})$ in M and since $\varphi_{k+1}(M) \subseteq \varphi_{i+1}(M) \subseteq \xi_j(M)$, so does $\varphi_{k+1}(x)$. It follows that $\varphi_{k+1}(c)\wedge\theta_k(c,\bar{a}) \in T$ and therefore and $T \vdash \neg\theta$. We also see that $T \vdash \theta_i(c,\bar{a})$ iff $\varphi_{i+1}(x)$ decides $\theta_i(c,\bar{a})$ in M.

Now, let $N \models T$ and $K \overset{\mathrm{df}}{=} K(F) = \mathrm{Dfe}(N, M\cup\{c^N\})$. Then $M \preccurlyeq K$. We shall show that $c \in K\setminus M$ and that K is a conservative extension of M. Let $c^N = a$ for some $a \in M$. Then the formula $c = y$ is $\theta_i(c,y)$ for some $i \in \omega$ and $T \vdash \theta_i(c,a)$. Thus $\varphi_{i+1}(x,y)$ decides $\theta_i(x,a)$ in M and since $\varphi_{i+1}(M)$ is unbounded, $x=a$ must hold in M for unboundedly many x, which, of course, is not possible—a contradiction.

Let $\varphi(z,\bar{y},x)$ be an $\mathscr{L}^{Ar}$-formula and $b_1,\dots,b_n$ elements defined in K respectively by $\mathscr{L}^{Ar}$-formulae $\eta_k(u,c,\bar{a})$, $1\leq k\leq n$, where $\bar{a}\in M$. There exists $i\in\omega$ such that $\theta_i(c,z,\bar{y})$ is the formula

$$(\exists\bar{u})\Big(\bigwedge_{j=1}^{n}\eta_j(u_j,c,\bar{y})\wedge\varphi(z,\bar{y},c)\Big).$$

Then for any $d \in M$,

$$K \models \varphi(d,\bar{b},c) \text{ iff } K \models \theta_i(c,d,\bar{a}) \text{ iff}$$
$$T \vdash \theta_i(c,d,\bar{a}) \text{ iff } M \models (\exists x_0)(\forall x > x_0)(\varphi_{i+1}(x) \to \theta_i(x,d,\bar{a})).$$

It follows that for any choice of φ and $\bar{b}$, the set $\{d \in M \mid K \models \varphi(d,\bar{b},c)\}$ is definable in M, hence K is a proper conservative extension of M.

We now proceed to show the remaining part of the claim. Assume M is countable. Then $K(F)$ is also countable and $c^{K(F)}$ realizes the complete 1-type $p_F(x) \stackrel{\text{df}}{=} \{\varphi_i^F(x) \mid i \in \omega\}$. Let $F_1, F_2 \in {}^{\omega}2$, $F_1 \neq F_2$ and let $i_0 \in \omega$ be the least i satisfying $F_1(i) \neq F_2(i)$. Then for all $i < i_0$, $\varphi_i^{F_1} = \varphi_i^{F_2}$. Hence $\varphi_{i_0}^{F_1} = \xi_j^{[F_1(i_0)]}$, $\varphi_{i_0}^{F_2} = \xi_j^{[F_2(i_0)]}$ for some $j \in \omega$. Now $\varphi_{i_0}^{F_1}(x) \in p_1^F(x)$, $\varphi_{i_0}^{F_2}(x) \in p_2^F(x)$, but $F_1(i_0) \neq F_2(i_0)$ and hence $\xi_j^{[F_1(i_0)]}(M) \cap \xi_j^{[F_1(i_0)]}(M) = \emptyset$, so $p_{F_1}(x) \neq p_{F_2}(x)$.

It follows that there are 2^{ω}-many complete 1-types in the language $\mathfrak{L}_M^{Ar}$ each of which can be realized in some proper countable conservative elementary extension of M. If there were, up to isomorphism, only κ-many such extensions for some cardinal $\kappa < 2^{\omega}$, then they would altogether realize at most $\omega \cdot \kappa < 2^{\omega}$ complete 1-types. Hence, to realize all the 2^{ω} different complete 1-types, the number of such non-isomorphic models must be also at least (and obviously at most, too) 2^{ω}. $\square$

APPENDIX B

STRONG CUTS AND ACA_0

In this appendix, we provide a proof of the well-known fact that a cut I is strong iff $I^* \models ACA_0$ and a theorem derived by Kirby. The proofs use the background of Chapter 4.

Recall that for $I \subseteq^e M$, I^* denotes the model $\langle I, \mathrm{Cod}(I/M)\rangle$ of $\mathcal{L}^{\mathrm{II}}$.

B.1 Lemma. *If $I \subset^e M \models \mathrm{I}\Sigma_1$, then $I^* \models \Delta^0_0\mathrm{CA}_0 + \mathrm{B}\Sigma^0_1$ and*

$$I^* \models (\forall X)[(0 \in X \wedge (\forall x)(x \in X \to x+1 \in X)) \to (\forall x) x \in X]. \tag{B.1}$$

In particular, if $I^* \models ACA_0$, then $I \models \mathrm{PA}$.

Proof. $\mathrm{B}\Sigma^0_1$ follows as in 1.8.4. For every $A \in \mathrm{Cod}(I/M)$, let $\tilde{A}$ denote some (arbitrary but fixed) set from $\mathfrak{D}$ such that $A = \tilde{A} \cap I$. In particular, we may further assume that $\tilde{A}$ is coded by some element in M. The induction formula (B.1) for I^* thus follows easily from $\mathrm{L}\Sigma_1$ in M. To prove $I^* \models \Delta^0_0\mathrm{CA}_0$, first observe that if $t(\bar{x})$ is an $\mathcal{L}^{Ar}$-term and $\bar{a} \in I$, then $t^M(\bar{a}) = t^I(\bar{a}) \in I$, so $I \models t(\bar{a}) \in A$ iff $M \models t(\bar{a}) \in \tilde{A}$. Since $\mathrm{Cod}(I/M)$ is a Boolean algebra, we have for every open $\mathcal{L}^{\mathrm{II}}$-formula φ and any $\bar{a} \in I$, $A_1, \dots, A_k \in \mathrm{Cod}(I/M)$

$$I^* \models \varphi(\bar{a}, A_1, \dots, A_k) \text{ iff } M \models \varphi(\bar{a}, \tilde{A}_1, \dots, \tilde{A}_k). \tag{B.2}$$

If (B.2) holds for a formula $\varphi(\bar{x}, y, X_1, \dots, X_k)$ (with all second-order variables among $X_1, \dots, X_k$ and all first-order variables among $\bar{x}, y$) and ψ is of the form $(\exists y \le t(\bar{x}))\varphi$, then (B.2) holds for ψ too since I is a cut. Consequently, (B.2) holds for all Δ^0_0 formulae, so $I^* \models \Delta^0_0\mathrm{CA}_0$. The sequel about ACA_0 is obvious. □

B.2 Theorem. *If I is strong iff $\langle I, \mathrm{Cod}(I/M)\rangle \models ACA_0$.*

Proof. From left to right:

Let I be strong, $\alpha \in \mathfrak{g}_I$, and $M \preccurlyeq^c_I M[\alpha]$. By 1.4.2, it suffices to prove $I^* \models \Sigma^0_1\mathrm{CA}_0$. For a Δ^0_0 formula $\varphi(x, y, \bar{x}, X_1, \dots, X_k)$ and parameters $\bar{a} \in I$,

$A_1,\dots,A_k \in \mathrm{Cod}(I/M)$, let $A \overset{\mathrm{df}}{=} \{a \in I \mid I^* \models (\exists y)\varphi(a,y,\bar{a},A_1,\dots,A_k)\}$. We must show that $A \in \mathrm{Cod}(I/M)$. Let $Y \overset{\mathrm{df}}{=} \{\gamma \in C \mid C \models (\exists y < \alpha)\varphi(\gamma,y,\bar{a},\tilde{A}_1,\dots,\tilde{A}_k)\}$. Then $Y \in \mathfrak{D}(M[\alpha])$ and hence for some $\tilde{A} \in \mathfrak{D}$ we have $Y \cap I = \tilde{A} \cap I$. It now suffices to show that $\tilde{A} \cap I = A$. Let $a \in A$. Then there exists $b \in I$ such that $I^* \models \varphi(a,b,\bar{a},A_1,\dots,A_k)$ hence by (B.2), $C \models \varphi(a,b,\bar{a},\tilde{A}_1,\dots,\tilde{A}_k)$, $b < \alpha$, so $a \in \tilde{A} \cap I$. Conversely, if $a \in \tilde{A} \cap I$, then $C \models (\exists y < \alpha)\varphi(a,y,\bar{a},\tilde{A}_1,\dots,\tilde{A}_k)$. Let $Y \overset{\mathrm{df}}{=} \{\beta \in C \mid (\exists y < \beta)\varphi(a,y,\bar{a},\tilde{A}_1,\dots,\tilde{A}_k)\}$. Then $\alpha \in Y \in \mathfrak{D}$, so $Y \cap I$ is non-empty (and cofinal in I), hence for some $b \in I$, $M \models \varphi(a,b,\bar{a},\tilde{A}_1,\dots,\tilde{A}_k)$ and thus $a \in A$ by (B.2).

For the reversed implication, it suffices to observe that the Infinite Ramsey Theorem holds in I^* in the following form: for every unbounded $A \in \mathrm{Cod}(I/M)$ and a partition $P \in \mathrm{Cod}(I/M)$ of $\langle A\rangle^n$ $(n \geq 1)$ into a parts with $a \in I$, there is $B \in \mathrm{Cod}(I/M)$, homogeneous for P. Indeed, by B.1, $\langle I,A,P\rangle$ is a model of Peano arithmetic with induction scheme extended to all formulae of the language $\mathcal{L}' = \mathcal{L}^{Ar} \cup \{\dot{A},\dot{P}\}$ with two new predicate symbols $\dot{A},\dot{P}$. Hence Theorem 2.1.3 applies and provides an unbounded P-homogeneous subset $B \subseteq I$ defined in I by some $\mathcal{L}'$-formula. In particular, B is definable in I^* by some $\mathcal{L}^{\mathrm{II}}$ formula with parameters A,P; by ACA_0, $B \in \mathrm{Cod}(I/M)$. Thus $I \to (I)^n_{<I}$, so I is strong. □

For completeness, we now provide is a short version of Kirby's proof of [Kir84, Theorem 5.7] that every 2-monad in a gap of a strong cut is strong, which gave us Corollary 4.4.20.

B.3 Theorem (*Kirby*). *Let I be strong. Then every 2-monad from $\mathfrak{g}_I$ is strong.*

Proof. Let $\mathfrak{m}$ be a 2-monad, $\alpha \in \mathfrak{m}$. Then $\mathfrak{m}$ is minimal and by semi-regularity of I, relatively large; it follows that $\mathfrak{m}$ is regular, so $M \preccurlyeq_I M[\alpha]$. By 4.4.16, it suffices to prove that $M \preccurlyeq_I^c M[\alpha]$. For a set $X \in \mathfrak{D}(C,C)$, let X^I denote $X \cap I$. For that, let $\varphi(x,y)$ be an $\mathcal{L}^{Ar}_M$-formula; we must show that $\varphi(C,\alpha)^I \in \mathrm{Cod}(M/I)$. Let $A_x \overset{\mathrm{df}}{=} (\varphi(C,x) \cap [0,x]) \cup \{x\}$; then $\varphi(C,\alpha)^I = A^I_\alpha$. Note that for $x \neq y$, $A_x \dot{-} A_y$ is non-empty, so we may define:

$$\begin{aligned} P_0 &\overset{\mathrm{df}}{=} \{\langle x,y\rangle \in \langle C\rangle^2 \mid \min(A_x \dot{-} A_y) \in A_y\}, \\ P_1 &\overset{\mathrm{df}}{=} \{\langle x,y\rangle \in \langle C\rangle^2 \mid \min(A_x \dot{-} A_y) \in A_x\} \quad = \quad \langle C\rangle^2 \setminus P_0. \end{aligned}$$

Now, $\mathfrak{m}$ is homogeneous for the partition $P \in \langle C\rangle^2_2$ with blocks P_0,P_1, so for some $H \in \mathfrak{U}(\mathfrak{m})$, either $\langle H\rangle^2 \subseteq P_0$ or $\langle H\rangle^2 \subseteq P_1$. Let

$$\begin{aligned} B &\overset{\mathrm{df}}{=} \{b \in I \mid I^* \models (\exists x_0)(\forall x \geq x_0)(x \in H^I \to b \in A^I_x)\}, \\ B' &\overset{\mathrm{df}}{=} \{b \in I \mid I^* \models (\exists x_0)(\forall x \geq x_0)(x \in H^I \to b \notin A^I_x)\}. \end{aligned}$$

These definitions are correct because $\{A^I_x\}_{x\in I}, H^I \in \mathrm{Cod}(M/I)$. Since $I^* \models ACA_0$, $B,B' \in \mathrm{Cod}(M/I)$. We show that $B = A^I_\alpha$. Observe that for $b \in I$, $b \in B$

iff $b \in A_\gamma$ for every $\gamma \in H \cap \mathfrak{g}_I$; similarly, $b \in B'$ iff $b \notin A_\gamma$ for every $\gamma \in H \cap \mathfrak{g}_I$. In particular, $B \subseteq A^I_\alpha$ and $B' \subseteq I \setminus A_\alpha$, since $\alpha \in H \cap \mathfrak{g}_I$. To complete the proof, it suffices to show that $I = B \cup B'$. Aiming for a contradiction, suppose the set $I \setminus B \cup B'$ is non-empty and let b be its least element (which exists by restricted induction in I^*). Then for all $\gamma, \delta \in H \cap \mathfrak{g}_I$, $A^I_\gamma \cap [0, b) = A^I_\delta \cap [0, b)$. Since $b \notin B'$, there is some $\gamma \in H \cap \mathfrak{g}_I$ such that $b \in A_\gamma$. Let $X \overset{\text{df}}{=} \{x \mid b \in A_x\}$. If $\langle H \rangle^2 \subseteq P_0$, then $b \in A_\delta$ for every $\delta > \gamma$ from $H \cap \mathfrak{g}_I$, so $H \cap \mathfrak{g}_I \cap (\gamma, \rightarrow) \subseteq X$. Similarly, if $\langle H \rangle^2 \subseteq P_1$, then $b \in A_\delta$ for every $\delta < \gamma$ from $H \cap \mathfrak{g}_I$, so $H \cap \mathfrak{g}_I \cap [0, \gamma) \subseteq X$. In either case we obtain the full equality $X \cap \mathfrak{g}_I = H \cap \mathfrak{g}_I$ as an easy application of 4.2.6. Thus $b \in A_\delta$ for all $\delta \in H \cap \mathfrak{g}_I$, i.e. $b \in B$—a contradiction. □

B.4 Remark. The use of ACA_0 (i.e. strength of the cut) was essential for the proof. Thus, the theorem provides only little guidance towards answering the general question whether all (relatively large) 2-monads are strong and is in no conflict with Seetapun and Slaman's [SS95] result that suggests the answer to that question is more likely to be negative (see Remark 4.3.38 for more details).

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Bibliography

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