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DIPLOMOVÁ PRÁCE

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Proces rizika s náhodným příjmem

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Prohlašuji, že jsem svou diplomovou práci napsala samostatně a výhradně s použitím citovaných pramenů. Souhlasím se zapůjčovaním práce a jejím zveřejňováním.

V Praze dne 10.8.2007 Anna Ringlerová

Název práce: *Proces rizika s náhodným příjmem*
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Abstrakt: *Táto práca sa zaoberá rizikovými procesmi. Popisuje klasický rizikový poces a zmieňuje pojem pravdepodobnosti ruinovania. Pre klasický rizikový proces sú odvodené vzťahy pre výpočet pravdepodobnosti ruinovania- konvolučná formula a Beekmanova konvolučná formula. V ďalšej časti sa práca zaoberá štúdiom Cramér-Lundbergovej, Beekman-Bowersovej a De Vylderovej aproximácie pravdepodobnosti ruinovania. Presnosť aproximácií je ilustrovaná v dvoch príkladoch. Následne je popísaný rizikový proces s náhodným príjmom a je odvodená konvolučná formula pre tento proces. V príklade uvažujeme o klasickom procese ako o špeciálnom prípade rizikového procesu s náhodným príjmom. Pre takýto proces je porovnaná pravdepodobnosť ruinovania vypočítaná pomocou konvolučnej formule a pravdepodobnosť ruinovania vypočítaná pomocou konvolučnej formule pre rizikový proces s náhodným príjmom. V práci je ukázané, že v prípade použitia konvolučnej formule pre klasický proces je pravdepodobnosť ruinovania niekedy podcenená.*
Klíčová slova: *klasický rizikový proces, pravdepodobnosť ruinovania, aproximácia, rizikový proces s náhodným príjmom*

Title: *Risk Process with Random Income*
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Abstract: *This diploma thesis deals with risk processes. It describes a classical risk process and mentions the ruin probability. A convolution formula and the Beekman convolution formula for calculating the ruin probability are deduced for the classical risk process. The following part of the thesis provides the investigation of the Cramér-Lundberg, the Beekman-Bowers and the De Vylder approximation to the ruin probability. The accuracy of approximations is illustrated in two examples. Afterwards, a risk process with random income is studied and a convolution formula for such a process is derived. In an example, the classical risk process is taken as a specific type of the risk process with random income. For such a process, the ruin probability computed by the convolution formula for classical risk process is compared to the ruin probability computed by the convolution formula for the risk process with random income. It is shown that sometimes the ruin probability is undervalued when computed by the convolution formula for classical risk process.*
Keywords: *classical risk process, ruin probability, approximation, risk process with random income*

Contents

Introduction

Insurance mathematics treats models of an insurance business. The main task of these should be describing the reality and serving as a tool for optimal managing of risk processes connected to insurance. Moreover, the models can be useful for planning actions to increase the value of business.

When discussing the management of risk processes, an insurance company should behave in the way to meet its liabilities towards insured persons. Therefore, falling of a reserve under zero, i.e. technical ruin, must be avoided. Hence, the probability of such an unpleasant event is calculated. Here arouses the question whether the model simplifications change the described reality or not. Another question is, if analysis made by a simplified model leads to the correct conclusion about the jeopardy of ruin. And this is the aim of our thesis. Specifically, we try to find out whether the simplified model of an insurance business provides the same conclusions about the probability of ruin as the one which is more complicated on one side, but more realistic, on the other side.

The thesis is divided into four chapters. First chapter provides a quick overview of the mathematical fundamentals that are applied further in the text.

In the second chapter we deal with the simplest model of the collective risk theory. In particular, we study a classical risk process. To specify such a process, premium is paid at constant rate per unit time, number of claims is described by a Poisson process and the claim sizes are considered to be independent and identically distributed. Later, the ruin probability is mentioned and this is deduced for the classical risk process. Such a probability can be obtained by a convolution formula and the Beekman convolution formula.

Due to the reason that treating any of the convolution formulas can be complicated, we investigate three approximations to the ruin probability in the third chapter. To be more specific, we study the Cramér-Lundberg, the Beekamn-Bowers and the De Vylder approximation respectively. To illustrate the accuracy of the approximations we provide two examples in which the exact ruin probability is calculated from the convolution formula and is compared to the approximations.

In the last, fourth, chapter we study a risk process with random income. In such a process we assume the flow of premiums to be directed by a Poisson process. Convolution formula for a risk process with random income is derived. Since we assume the classical risk process to act as the 'proper indicator' of the probability of ruin, we want to verify this. At first, we take the classical risk process as a specific type of the risk process with random income. Then, we compare the ruin probability computed by the convolution formula for classical risk process with the ruin probability computed by the convolution formula for the risk process with random income. It will be shown in the example that sometimes the ruin probability is undervalued when computed by the convolution formula for classical risk process. In the end, we mention two expressions which combined assist when computing the ruin probability in the random income process. These expressions are obtained by applying the ladder height distribution.

Chapter 1

Mathematical background

At the beginning we present some basic definitions and theorems which repeatedly show up in the following text. However, this chapter provides just a quick overview of the mathematical background. Further relevant aspects can be found in Dupač [5], Heilmann [7], Kalashnikov [8], Prášková [10] and Rolski et al. [11], which served as the sources for this part.

Random events and variables form the basis of the probability theory. To be able to study them we need to have an appropriate model. Let Ω be the *space of all elementary outcomes*. It is convenient to deal with subsets of Ω which form a *σ-algebra*. We just remark that a σ-algebra $\mathcal{A}$ is a non-empty system with the following properties

- if $A \in \mathcal{A}$, then $\Omega \backslash A \in \mathcal{A}$,
- if $A_1 \in \mathcal{A}, A_2 \in \mathcal{A}, \ldots$, then $\cup A_i \in \mathcal{A}$.

Let P be the *probability measure* defined on $\mathcal{A}$ fulfilling

- $P(A) \geq 0$ for $\forall A \in \mathcal{A}$,
- if $A_1, A_2, \ldots,$ are disjunct sets belonging to $\mathcal{A}$, then $P(\cup A_i) = \sum P(A_i)$,
- $P(\Omega) = 1$.

In insurance, Ω represents all the claim sizes, $\mathcal{A}$ is interpreted as the set of all possible events and P is a function assigning probability to an event.

Definition 1.1: A measurable real mapping $X : \Omega \to \mathcal{R}$ is a *random variable (r.v.)*.

Definition 1.2: Random variables X_1 and X_2 are *independent* if

$$P\left(X_1 \cap X_2\right) = P(X_1)\, P(X_2).$$

Theorem 1.1: Let $X_1, X_2 \in \mathcal{A}, P(X_2) > 0$. Then the conditional probability of an event X_1 given X_2 is

$$P(X_1|X_2) = \frac{P(X_1 \cap X_2)}{P(X_2)}.$$

Theorem 1.2: Total probability formula. If $P(\cup_n X_n) = 1$, where $\{X_n\}$ is finite or countably infinite sequence of events such that $X_i \cap X_j = \emptyset$ for $i \neq j$, if $P(X_n) > 0$ for all n and if $A \in \mathcal{A}$, then

$$P(A) = \sum_n P(A|X_n)\, P(X_n).$$

Total probability formula is sometimes also referred to as the law of total probability.

Definition 1.3: Let X be a random variable. The *distribution function (d.f.)* of X is defined as

$$F_X(x) = P(X \leq x), \quad x \in \mathcal{R}.$$

If there exists a derivative

$$f(x) = \frac{dF_X(x)}{dx},$$

then it is called the *density function* of X and the d.f. $F_X(x)$ is *absolutely continuous.* We write $F_X(x)$ if we want to emphasize the r.v. X to which the d.f. F is related, or simply $F(x)$ if the associated r.v. is unambiguous.

The distribution function F is *discrete* if there exists a finite or countably infinite sequence of mutually different real numbers $\{x_n\}_{n \in \mathcal{N}_0}$, where $\mathcal{N}_0 \subseteq \mathcal{N}$, and corresponding sequence of positive numbers $\{p_n\}_{n \in \mathcal{N}_0}$ such that $\sum_{n \in \mathcal{N}_0} p_n = 1$ and

$$F(x) = \sum_{n \in \mathcal{N}_0; x_n \leq x} p_n, \quad \forall x \in \mathcal{R}.$$

Nonnegative discrete random variable X is called a *lattice r.v.* in case that $\mathcal{N}_0 \subset h\mathcal{N}$ for some $h > 0$, i.e. $x_k = h\,k$. If X is a lattice r.v. we say that its distribution is lattice. Otherwise, the distribution is *non-lattice.*

Definition 1.4: The *mean value* or *expectation* of a r.v. X is defined as

$$EX = \int_\Omega X\, dP.$$

Let $r \in \mathcal{N}$. Then the *r-th power moment* of a r.v. X with d.f. F is

$$m_r = EX^r = \int_\Omega x^r\, dF(x),$$

if it exists.

In this moment we mention just the Poisson and exponential distribution more in detail, as these two are used further in the text.

Poisson distribution is an example of a discrete distribution of a r.v. X, which takes values $k = 0, 1, \ldots$ with probability $e^{-\lambda}\frac{\lambda^k}{k!}, 0 \leq k < \infty$. $\lambda > 0$ is a parameter of the Poisson distribution. The mean value and also the variance of this distribution is λ.

Exponential distribution is a type of absolutely continuous distribution and is defined by the succeeding density function

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0, \\ 0 & x \leq 0, \end{cases}$$

where $\lambda > 0$ is a parameter. For the distribution function holds the following

$$f(x) = \begin{cases} 1 - e^{-\lambda x} & x > 0, \\ 0 & x \leq 0. \end{cases}$$

The mean value and variance of an exponential distribution is $\frac{1}{\lambda}$ and $\frac{1}{\lambda^2}$ respectively.

Definition 1.5: Let X_1, X_2 be independent r.v's with distribution functions F_1, F_2 respectively. Then for a distribution function G of a random variable $Y = X_1 + X_2$ holds

$$G(y) = \int_{-\infty}^{\infty} F_1(y-x)dF_2(x) = \int_{-\infty}^{\infty} F_2(y-x)dF_1(x), \quad y \in \mathcal{R}.$$

Note that an operation assigning function G to functions F_1 and F_2 is called *(Stieltjes) convolution* and is denoted as $F_1 * F_2$.

Theorem 1.3: Duality property Let $Y_1, Y_2, \ldots$ be independent and identically distributed. Then,

$$(Y_1, Y_1 + Y_2, \ldots, Y_1 + \cdots + Y_n) \stackrel{d}{=} (Y_n, Y_{n-1} + Y_n, \ldots, Y_1 + \cdots + Y_n)$$

for all $n = 1, 2, \ldots$.

Theorem 1.4: Law of large numbers (strong) Let $X_n, n \geq 1$ be a sequence of independent, identically distributed r.v's with common distribution function F and power moments denoted by m_r. Let $S_n = X_1 + \cdots + X_n$. If $E|X_1| < \infty$, then

$$P\left(\lim_{n\to\infty} \frac{S_n}{n} = m_1\right) = 1.$$

Definition 1.6: Let $(\Omega, \mathcal{A}, P)$ be probability space and let $T \subset \mathcal{R}$. Family of real random variables $\{X_t, t \in T\}$ defined on $(\Omega, \mathcal{A}, P)$ is called a *stochastic process.* (S, ε), where S is a set of values of r.v's X_t and ε is a σ-algebra of subsets of S, is called the *state space* of the process $\{X_t, t \in T\}$. We say that a process $\{X_t, t \geq 0\}$ is a *Markov process* with state space (S, ε), if for any $t_0, t_1, \ldots, t_n, 0 \leq t_0 < t_1 < \cdots < t_n$, holds the following

$$P(X_{t_n} \leq x | X_{t_{n-1}}, \ldots, X_{t_0}) = P(X_{t_n} \leq x | X_{t_{n-1}}), \quad \text{for } \forall x \in \mathcal{R}. \tag{1.1}$$

1.1 is called the *markovian property.* The simplest examples of Markov processes are the ones with discrete states– *Markov chains with discrete and continuous time.*

To be able to solve gracefully some problems connected to the application of stochastic processes, we employ certain functions which are more easily manageable, but provide unmodified outcomes at the same time.

Definition 1.7: Let X be a random variable. The *moment generating function (m.g.f.)* of X is defined by $\hat{m}_X(s) = Ee^{sX}$.

Definition 1.8: Let $F(x)$ be a real function vanishing on $\mathcal{R}_-$ that can be expressed as $F(x) = F_+(x) + F_-(x)$, where $F_+(x)$ and $F_-(x)$ are nondecreasing nonnegative finite functions. Suppose that there exists $A \in \mathcal{R}$ and $s_0 \in \mathcal{R}$ such that

$$|F(x)| \leq A\, e^{s_0\, x}, \quad x \geq 0.$$

The *Laplace transform* of a function F is a function of a complex variable s defined as

$$F_L(s) = \int_0^\infty e^{-s\,x}\, F(x)\, dx, \quad \operatorname{Re} s > s_0.$$

Definition 1.9: Let $F(x)$ be a real function vanishing on $\mathcal{R}_-$ that can be expressed as $F(x) = F_+(x) + F_-(x)$, where $F_+(x)$ and $F_-(x)$ are nondecreasing nonnegative finite functions. Suppose that there exists $A \in \mathcal{R}$ and $s_0 \in \mathcal{R}$ such that

$$|F(x)| \leq A\, e^{s_0\, x}, \quad x \geq 0.$$

Then the *Laplace-Stieltjes transform* of the function F is a function of a complex variable s defined as

$$\mathcal{LS}(F)(s) = \int_0^\infty e^{-sx}\, dF(x), \quad \operatorname{Re} s > s_0.$$

In case of a r.v. X defined on $[0,\infty)$ with d.f. F that has the properties mentioned above, the Laplace transform can be expressed as

$$\mathcal{LS}(F)(s) = Ee^{-sX}.$$

Properties of the transforms

The Laplace-Stieltjes and Laplace transforms relate in the following way

$$\mathcal{LS}(F)(s) = s\,F_L(s).$$

At present we list the properties of Laplace transform.

1. If $H(x) = a\,F(x) + b\,G(x)$, where a and b are real, then $H_L(s) = a\ F_L(s) + b\ G_L(s)$.
2. If $G(x) = \int\limits_0^x F(u)\,du$, then $G_L(s) = \frac{F_L(s)}{s}$.
3. If $G(x) = e^{-ax}\,F(x)$, then $G_L(s) = F_L(s+a)$.
4. Let there exist a derivative $G(x) = \frac{dF(x)}{dx}$ for each $x \geq 0$ and let G be a function of a bounded variation. Then $G_L(s) = s\,F_L(s) - F(0)$.
5. Let $H(x) = \int\limits_0^x F(x-y)\,G(y)\,dy$ be the integral convolution of F and G. Then $H_L(s) = F_L(s)\,G_L(s)$.

Chapter 2

Classical Risk Process

The classical risk process is the simplest model of the collective risk theory. This chapter starts with formulating the classical risk process as it is stated in Grandell [6]. Afterwards, the ruin probability is expressed from the mathematical point of view. Finally, the ruin probability for the classical risk process is derived.

An insurance company is considered to have a certain *initial capital* $u \geq 0$. For the purpose of covering its liability, the insurance company receives premiums from clients. In the classical risk process, the premium rate is assumed to be constant. If $I(t)$ denotes the income process over time t, then

$$I(t) = c\,t, \tag{2.1}$$

where constant $c > 0$ is called the *gross premium rate.*

Let $\{T_k\}$ be the sequence of successive claim occurrence times, which is ordered $(T_0 = 0 < T_1 \leq T_2 \leq T_3 \leq \cdots)$ and usually random, and *N(t)* be the total number of claims occurring during the interval $[0,\ t]$. The occurrence of claims is described by a point process. In order to specify the classical risk process, the point process is a standard Poisson process with intensity λ, having the following distribution

$$P(N(t) = k) = \frac{(\lambda t)^k}{k!}\; exp(-\lambda t). \tag{2.2}$$

X_j, the amounts of money to be paid by the insurance company at each claim, are considered to represent a sequence of independent and identically distributed random variables (i.i.d.r.vs). Moreover, the point process of claim occurrence $N(t)$ and the sequence of claims $\{X_j\}$ are mutually independent. When $D(t)$ denotes the amount of money paid by the insurance company, we can express it in the following way:

$$D(t) = \sum_{j=1}^{N(t)} X_j, \tag{2.3}$$

where $\{X_j\}$ is a sequence of i.i.d.r.vs. Let $F(x)$ denote the common distribution function of the sequence $\{X_j\}$ and be defined as

$$F(x) = P(X_j \leq x).$$

In addition, let the power moments of claims, if they exist, be

$$m_r = EX_1^r = \int_0^\infty x^r \, dF(x), \qquad r > 0.$$

Hence, for instance, m_1 denotes the mean value of the claims.

When considering the amounts of income and claims together, we come to the *balance equation*, in which also initial capital occurs:

$$R(t) = u + I(t) - D(t). \tag{2.4}$$

Process $R(t)$ is called the *surplus process*. One of possible evolutions of a surplus process is depicted in Figure 3.1.

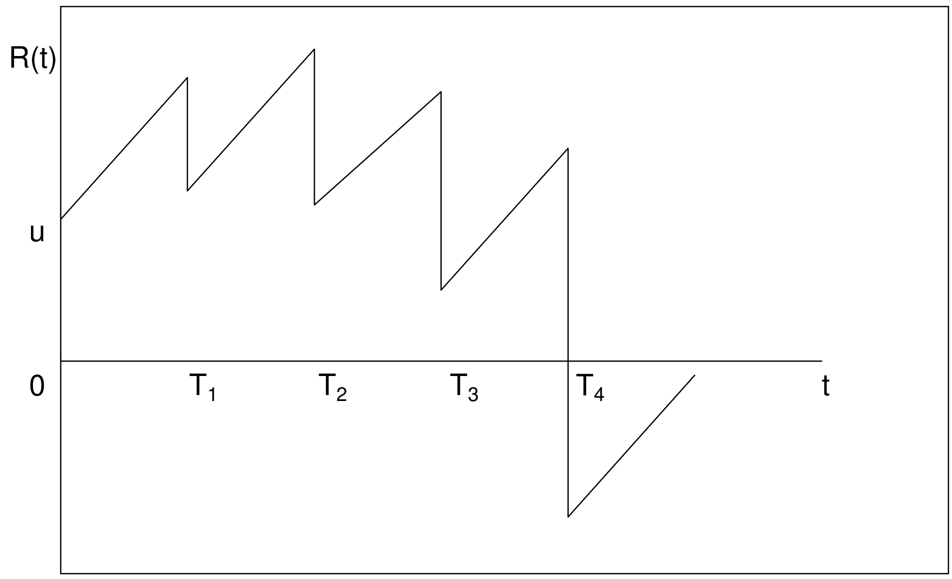

Figure 2.1: Surplus process

Since some quantities involved in 2.4 are random, surplus process $R(t)$ is random as well. Process $I(t) - D(t)$ is the *safety loading* of the company. If the average cost of claims is exceeded by the premium income, the safety loading is positive. Insurance company usually prefers to work with *relative safety loading* defined as

$$\rho = lim_{t\to\infty}\frac{E(I(t)-D(t))}{ED(t)}. \tag{2.5}$$

Relative safety loading is understood as the mean income per each claim. Due to Grandell [6], ρ can be considered as the indicator of allurement of an insurance company. In case that ρ is large enough, the insurance company is profitable. However, there is a danger of non-attractiveness for clients due to high premiums. On the other hand, if ρ is close to 0, insurance company is nearly unprofitable. Therefore, there is a big chance of getting ruined. In this work we consider only the cases in which the relative safety loading exists and is positive. Such a condition is called a PSL condition or net profit condition throughout the relevant literature, as for instance in Kalashnikov [8] or in Rolski et al. [11] respectively, and it is stated below.

PSL condition
If $\rho > 0$, we say that the risk process $R(t)$ satisfies the positive safety loading condition.

Example 2.1: Assume we have a classical risk process defined in 2.4. Let the sequence of i.i.d.r.vs $\{X_j\}$ have the mean value m_1. Due to the independence and identical distribution, the following holds

$$\begin{aligned} E\,N(t) &= \lambda\, t, \\ E(I(t)-D(t)) &= c\,t - E(N(t))\,E(X_j) = c\,t - \lambda\,t\,m_1, \\ \rho &= \frac{c}{\lambda\,m_1} - 1. \end{aligned}$$

With regard to the PSL condition, $c > \lambda\, m_1$. △

2.1 Ruin Probability

Risk theory is interested in the possibility that the surplus process of an insurance company ever becomes negative. Such a moment is called *technical ruin* and, from the practical point of view, it is connected to the solvency of insurance companies. Therefore much attention is paid to calculating and approximating the probability of ruin.

In the mathematical approach, the probability of ruin is the function of initial capital and it is defined in agreement with Grandell [6] in the following way

$$\Psi(u) = P(\inf_{t>0} R(t) < 0 | R(0) = u). \tag{2.6}$$

Ruining occurs when the surplus process is negative. This corresponds to the event when the safety loading is less than $-u$. It follows from the definition that $\Psi(u) = 1$ for $u < 0$. Let us introduce the *random time*

$$\tau = \inf\{t : R(t) < 0\}. \tag{2.7}$$

In terms of τ the ruin probability has a very simple form

$$\Psi(u) = P(\tau < \infty). \tag{2.8}$$

Naturally, the ruin probability is a function which is sensitive to the parameters of a risk model. Thus, it can be used for comparison of models having different parameters. Classical risk model does not reflect many real-life factors and the model is rather simplified. Therefore, in our work we attempt to recognize if for the insurance company it would be right to take $\Psi(u)$ in this model as the single indicator for the possibility of ruin.

2.2 Ruin Probability in the Classical Risk Process

In this section we are going to combine the preceding two sections and express the ruin probability in the classical risk process. According to Grandell [6] and Kalashnikov [8], it is possible to acquire it by at least two methods, in particular, by convolution formula and Beekman convolution formula. These two are deduced in the next sections by using the literature mentioned above.

2.2.1 Convolution Formula

Since ruin can only happen at claim occurrence times T_k, we can rewrite the classical risk process in the succeeding recursive form

$$R(0) = u, \quad R(T_{k+1}) = R(T_k) + c(T_{k+1} - T_k) - X_{k+1}, \qquad k \geq 0.$$

Therefore, the ruin probability can also be rewritten as

$$\Psi(u) = R(\inf_{k>0} R(T_k) < 0 | R(0) = u).$$

Suppose that we are at the beginning of the surplus process. Initial capital is u. There are two possible sequels in time T_1. Firstly, the process ends with ruin. In this situation, $X_1 - c(T_1 - T_0) > u$ and, obviously, $\inf_k R(T_k) < 0$. Secondly, the surplus process continues with $R(T_1) = u + c(T_1 - T_0) - X_1$, $R(T_1) \geq 0$ because $X_1 - c(T_1 - T_0) \leq u$. As the inter-occurrence times and claim sizes are i.i.d.r.vs, the recursive sequence $R(T_k)$ comprises a homogeneous Markov chain. Thanks to the recursion and Markovian property, $\inf_k R(T_k)$ depends only on $R(T_1)$. Consequently,

$$P\left(\inf_{k>0} R(T_k) < 0 \mid R(T_1) = a\right) = \Psi(a).$$

By employing the total probability formula we obtain the following equations

$$\begin{aligned}
\Psi(u) &= EP(\inf_{k>0} R(T_k) < 0 \,|\, R(T_0) = u, T_1, X_1) \\
&= EP(\inf_{k>0} R(T_k) < 0 \,|\, R(T_0) = u, T_1, X_1, R(T_1) = u + c\,T_1 - X_1) \\
&= EP(\inf_{k>0} R(T_k) < 0 \mid R(T_1) = u + c\,T_1 - X_1) \\
&= E\Psi(u + c\,T_1 - X_1).
\end{aligned}$$

Sometimes it is advantageous to use expression for *survival probability* instead of ruin probability. It can be stated as $\phi(u) = 1 - \Psi(u)$. Considering that ruin can not occur until the first claim occurrence, this means in time interval $(0, T_1)$, we can write

$$\phi(u) = E\phi(u + cT_1 - X_1).$$

By using the characteristics of the classical risk process and taking the advantage of $\Psi(u) = 1$ for $u < 0$ we arrive at the relation

$$\phi(u) = E\phi(u + cT_1 - X_1) = \int_0^\infty \left(\int_0^{u+ct} \phi(u + ct - x)\lambda e^{-\lambda t} dF(x) \right) dt.$$

Plugging y instead of $u + ct$ we gain

$$\phi(u) = \int_u^\infty \frac{\lambda}{c} e^{\lambda(u-y)/c} \left(\int_0^y \phi(y - x)\, dF(x) \right) dy.$$

Immediately, we can see that function $\phi(u)$ is differentiable with respect to u. By differentiating the equation above we obtain

$$\phi'(u) = \frac{\lambda}{c} \left(\phi(u) - \int_0^u \phi(u - x)\, dF(x) \right). \tag{2.9}$$

Integrating 2.9 over time interval $(0, t)$ leads to

$$\phi(t) - \phi(0) =$$

$$
\begin{aligned}
&= \frac{\lambda}{c}\int_0^t \left(\phi(u) - \int_0^u \phi(u-x)dF(x)\right) du \\
&= \frac{\lambda}{c}\int_0^t \phi(u)\,du\, + \\
&+ \frac{\lambda}{c}\int_0^t \left(\phi(0)\,(1-F(u)) - \phi(u)\, + \int_0^u \phi'(u-x)(1-F(x))\,dx\right) du
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\lambda}{c}\int_0^t \phi(0)\,(1-F(u))\,du + \frac{\lambda}{c}\int_0^t\int_x^t \phi'(u-x)(1-F(x))\,dx\,du \\
&= \frac{\lambda}{c}\,\phi(0)\int_0^t (1-F(u))\,du + \frac{\lambda}{c}\int_0^t (1-F(x))\,\left(\phi(t-x)-\phi(0)\right)dx.
\end{aligned}
$$

This way we get

$$
\phi(u) = \phi(0) + \frac{\lambda}{c}\int_0^u (1-F(x))\,\phi(u-x)\,dx. \tag{2.10}
$$

In terms of monotone convergence and as $u \to \infty$, it follows from 2.10 that

$$
\phi(\infty) = \phi(0) + \frac{\lambda m_1}{c}\,\phi(\infty).
$$

According to the law of large numbers, $\phi(\infty) = 1$. Therefore, $1 \;-\; \phi(0) \;=\; \frac{\lambda m_1}{c}$. When the PSL condition is accomplished, returning to the ruin probability yields

$$
\Psi(0) \;=\; \frac{\lambda\, m_1}{c} \;=\; \frac{1}{1+\rho}. \tag{2.11}
$$

Combining the results 2.10 and 2.11 under the PSL condition we get

$$
1 - \Psi(u) =
$$

$$
\begin{aligned}
&= 1 - \frac{\lambda m_1}{c} + \frac{\lambda}{c}\int_0^u (1-\Psi(u-x))\,(1-F(x))\,dx \\
&= 1 - \frac{\lambda}{c}\left(m_1 - \int_0^u (1-F(x))dx + \int_0^u \Psi(u-x)\,(1-F(x))\,dx\right),
\end{aligned}
$$

which rearranged results in the following statement.

Theorem 2.1: Under the PSL condition, the ruin probability for the classical risk process satisfies the following convolution equation

$$\Psi(u) = \frac{\lambda}{c}\int_u^\infty (1-F(x))dx + \frac{\lambda}{c}\int_0^u \Psi(u-x)\,(1-F(x))\,dx. \tag{2.12}$$

One of the classical approaches to this probability equation is applying the Laplace transform. For functions $F(x)$ and $\Psi(u)$ the Laplace transforms are

$$F_L(s) = \int_0^\infty e^{-sx}\,F(x)\,dx,$$

$$\Psi_L(s) = \int_0^\infty e^{-su}\,\Psi(u)\,du,$$

respectively. First of all, the equation 2.12 is restated in the following way

$$\begin{aligned}\Psi(u) &= \frac{\lambda}{c}\int_0^\infty (1-F(x))dx - \frac{\lambda}{c}\int_0^u (1-F(x))dx\; + \\ &+ \frac{\lambda}{c}\int_0^u \Psi(u-x)\,dx - \frac{\lambda}{c}\int_0^u \Psi(u-x)\,F(x))\,dx.\end{aligned}$$

Using the 2 and 5 property of the Laplace transform and by transforming both sides of the equation above, we arrive at

$$\Psi_L(s) = \frac{\lambda m_1}{cs} - \frac{\lambda}{cs}\left(\frac{1}{s} - F_L(s)\right) + \frac{\lambda}{c}\,\Psi_L(s)\left(\frac{1}{s} - F_L(s)\right).$$

Rearranging leads to

$$\Psi_L(s) = \frac{\lambda}{s}\,\frac{m_1 - (1/s - F_L(s))}{c - \lambda\,(1/s - F_L(s))}. \tag{2.13}$$

Multiplying the right side of the equation 2.13 by s/s and subsequent addition and subtraction of cs in the nominator yields another expression for $\Psi_L(s)$

$$\Psi_L(s) = \frac{1}{s} - \frac{c - \lambda m_1}{cs - \lambda\,(1 - sF_L(s))}. \tag{2.14}$$

As noticed in Kalashnikov [8], by using the Laplace transform we obtain quite an exact result. Nevertheless, the problem of inverting the Laplace transform occurs and therefore it is rather difficult to study the ruin probability in detail from this point of view.

2.2.2 Beekman Convolution Formula

Naturally, it is possible to come to the Beekman convolution formula by several methods. One of them is by using the Laplace-Stieltjes transform (for further details see Kalashnikov [8]). However, we have chosen another approach. As is suggested in Dufresne and Gerber [3], suppose we have a classical risk process

$$R(t) = u + ct - D(t).$$

As we have already mentioned, m_1 is the mean value of claim size and $\Psi(0) = \frac{1}{1+\rho}$, where ρ is the relative safety loading. Let

$$L = \max_{t \geq 0}\{D(t) - I(t)\}$$

be the maximum total loss. This entire loss can be divided into some smaller losses and thus we can write

$$L = L_1 + L_2 + \cdots + L_N,$$

where $\{L_i\}$ are i.i.d.r.vs. L_i can be interpreted as the amount of the last exceeding of the total loss, which occurred as the i-th in the sequel. N denotes the number of excesses. If ruin has not occurred yet, the process will continue either by surpassing the value of the maximum loss, or not. Therefore, N has a geometric distribution

$$P(N = n) = \left(1 - \frac{1}{1+\rho}\right)\left(\frac{1}{1+\rho}\right)^n \quad n = 1,\ 2,\ 3, \ldots$$

Furthermore, N and $\{L_i\}$ are mutually independent. In this case, the ruin probability $\Psi(u)$ can be expressed as

$$\Psi(u) = P(L > u) = P(L_1 + \cdots + L_N > u).$$

Let $G(x)$ be the common d.f. of $L_i{}'s$

$$G(x) = P(L_i \leq x) = \frac{1}{m_1}\int_0^x (1 - F(y))\, dy,$$

$$G(x) = 0 \quad \text{if} \quad x < 0.$$

As a result,

$$\Psi(u) = \left(\frac{\rho}{1+\rho}\right)\sum_{n=1}^{\infty}\left(\frac{1}{1+\rho}\right)^n (1 - G^{*n}(u)) \tag{2.15}$$

$$= \left(\frac{\rho}{1+\rho}\right)\sum_{n=1}^{\infty}\left(\frac{1}{1+\rho}\right)^n \int_u^{\infty}\left(\frac{1-F(y)}{m_1}\right)^{*n} dy, \tag{2.16}$$

where $G^{*n}(u)$ means the n-fold Stieltjes convolution of the d.f. $G(x)$. In the risk theory, 2.15 is referred to as the Beekman convolution formula, while in queueing theory, it is called the Pollaczek-Khinchin formula, as for instance in Asmussen [1] and Rolski et al. [11].

Chapter 3

Approximation of the Ruin Probability

Although we have the Beekman convolution formula 2.15, in some cases it is still unlikely to provide the ruin probability explicitly. Therefore, it would be useful to have bounds and approximations for the ruin probability. In this chapter we study the Cramér-Lundberg approximation, the Beekman-Bowers approximation and the De Vylder approximation. The approximations are derived as in Rolski [11] and Kalashnikov [8]. At the end, we compare the approximations with the exact ruin probability in two examples.

3.1 The Cramér-Lundberg Approximation

To obtain the Cramér-Lundberg approximation we employ the moment generating function. Like the distribution function, the moment generating function determines the probability distribution uniquely. Process $D(t) - c\,t$ has the following m.g.f.

$$\hat{m}_{D(t)-ct}(s) = Ee^{s(D(t)-ct)}. \tag{3.1}$$

Suppose that the m.g.f. $\hat{m}_{D(t)-ct}(s) = Ee^{sD(t)-ct}$ exists over a specific interval of positive values $(0, z)$ (z can be also ∞) and let

$$lim_{s\to z\_}\hat{m}_{D(t)-ct}(s) = \infty. \tag{3.2}$$

Under the given assumptions, let us have a look at the existence of solutions to the equation

$$\hat{m}_{D(t)-ct}(s) = 1. \tag{3.3}$$

Apart from the trivial solution $s = 0$, the equation 3.3 has a unique positive solution. The moment generating function is infinitely times differentiable. Following the procedure from Mandl [9], thanks to the characteristics of $\hat{m}_{D(t)-ct}(s)$ we get

$$\begin{aligned}
\hat{m}_{D(t)-ct}(0) &= 1, \\
\hat{m}'_{D(t)-ct}(s) &= E\left(D(t)-ct\right)e^{s(D(t)-ct)}, \\
\hat{m}'_{D(t)-ct}(0) &= E\left(D(t)-ct\right) < 0, \\
\hat{m}''_{D(t)-ct}(s) &= E\left(D(t)-ct\right)^2 e^{s(D(t)-ct)} > 0.
\end{aligned}$$

Therefore, $\hat{m}_{D(t)-ct}(s)$ is a convex function and existence of a solution to 3.3 is guaranteed, as illustrates Figure 4.1.

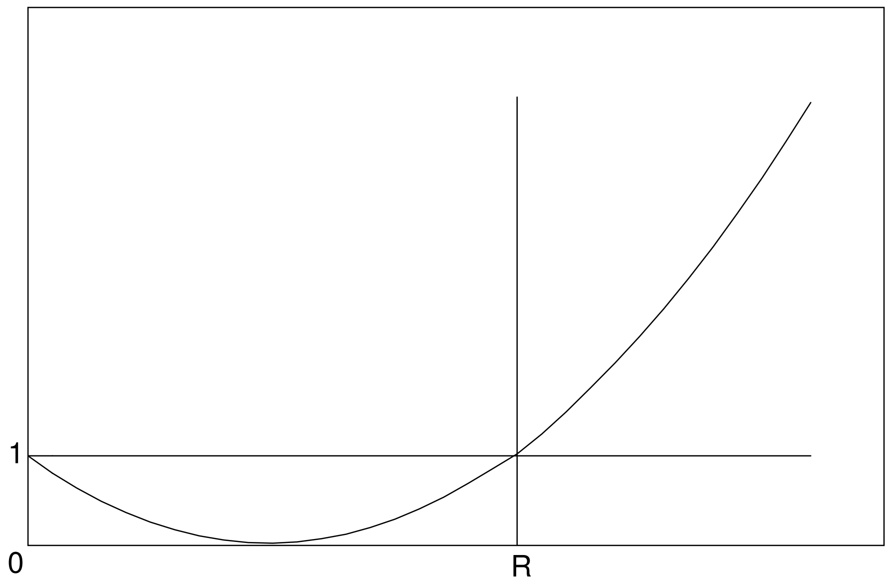

Figure 3.1: Typical shape of a moment generating function $\hat{m}_{D(t)-ct}(s)$ guarantees the existence of an adjustment coefficient R

The strictly positive solution to 3.3 is called the *adjustment coefficient*, which will be denoted by R in the subsequent text. Another name for R is *Lundberg constant* or *coefficient.* The adjustment coefficient is a measure of risk, as it indicates how quickly the ruin probability decreases depending on the amount of risk reserve.

Theorem 3.1: Assume that the adjustment coefficient $R > 0$ exists. Let $\hat{m}_X(s)$ be the m.g.f. of a single claim size X. Suppose that $\hat{m}'_X(R) < \infty$. Then the following Cramér-Lundberg approximation of the ruin probability is true

$$\Psi_{app}(u) = \frac{c - \lambda m_1}{\lambda \hat{m}'_X(R) - c}\, e^{-Ru}. \tag{3.4}$$

To be able to prove the Cramér-Lundberg approximation, we need the succeeding definition and auxiliary result from the renewal theory.

Definition 3.1: Let $z : R \to R_+$ be a locally bounded function and F be a distribution on R_+. Then

$$g(x) = z(x) + \int_0^x g(x-y)\, dF(y) \tag{3.5}$$

is the renewal equation.

Lemma 3.1: Let function $z_1 : R_+ \to (0, \infty)$ be increasing and let function $z_2 : R_+ \to R_+$ be decreasing, such that

$$\int_0^\infty z_1(x)\, z_2(x) dx < \infty$$

and

$$\lim_{h \to 0} \sup\{z_1(x+y)/z_1(x) : x \le 0, 0 \le y \le h\} = 1.$$

Then for $z(x) = z_1(x)\, z_2(x)$ and for each non-lattice distribution F on R_+

$$g(u) = z(u) + \int_0^u g(u-y)\, dF(y), \;\; u \ge 0$$

admits a unique locally bounded solution g(u)

$$\lim_{u \to \infty} g(u) = \begin{cases} \frac{1}{m_{1;F}} \int\limits_0^\infty z(u)\, du & m_{1;F} \le \infty, \\ 0 & m_{1;F} = \infty, \end{cases}$$

where $m_{1;F}$ is the mean value of the distribution F.

The proof of lemma 3.1 is omitted and can be found, for instance, in Rolski et al. [11]. In this moment we have the tools necessary to prove the theorem with Cramér-Lundberg approximation. The proof is from Rolski et al. [11].

Proof: Multiplying 2.12 by e^{Ru} leads to

$$\Psi(u) e^{Ru} =$$

$$= \frac{\lambda}{c}\int_u^\infty e^{Ru}\,(1-F(x))\,dx + \frac{\lambda}{c}\int_0^u \Psi(u-x)\,(1-F(x))\,e^{Ru}\,dx$$
$$= \frac{\lambda}{c}\int_u^\infty e^{Ru}\,(1-F(x))\,dx + \frac{\lambda}{c}\int_0^u \Psi(u-x)\,e^{R(u-x)}\,(1-F(x))\,e^{Rx}\,dx.$$

From the definition of the adjustment coefficient R it follows that

$$\frac{\lambda}{c}\int_0^\infty (1-F(x))\,e^{Rx}\,dx = 1. \tag{3.6}$$

Thus, 3.6 is a renewal equation. Applying the lemma 3.1, the integrating distribution is

$$dF(x) = \frac{\lambda}{c}\,(1-F(x))\;e^{Rx}\,dx.$$

The mean value of the distribution is

$$m_{1;F} = \int_0^\infty x\,\frac{\lambda}{c}\,(1-F(x))\;e^{Rx}\,dx. \tag{3.7}$$

Integral 3.7 is solved by the method per-partes where $1-F(x)$ is the differentiated part and xe^{Rx} is the integrated part

$$m_{1;F} = \frac{\lambda}{c}\left(\frac{1}{R^2} + \int_0^\infty \left(\frac{x}{R} - \frac{1}{R^2}\right)\,e^{Rx}\,dF(x)\right).$$

At present, we use the facts that $\hat{m}'_X(R) = \int_0^\infty xe^{Rx}\,dF(x)$ and 3.6 to find the solution

$$\begin{aligned} m_{1;F} &= \frac{\lambda}{c}\left(\frac{1}{R^2} + \frac{\hat{m}'_X(R)}{R} - \left(\frac{Rc/\lambda+1}{R^2}\right)\right) \\ &= \frac{\lambda}{c}\,\frac{\hat{m}'_X(R)}{R} - \frac{1}{R} = \frac{\lambda\hat{m}'_X(R)-c}{Rc}. \end{aligned}$$

As the function $z(u)$ from the lemma 3.1 will be taken function

$$z(u) = \frac{\lambda}{c}\,e^{Ru}\int_u^\infty (1-F(x))\,dx$$

Using lemma 3.1 we get

$$\begin{aligned} \int_0^\infty z(u)\,du &= \int_0^\infty \frac{\lambda}{c}\,e^{Ru}\int_u^\infty (1-F(x))\,dx \\ &= \frac{c-\lambda m_1}{cR}. \end{aligned}$$

Consequently,

$$\lim_{u\to\infty} \Psi(u)\, e^{Ru} = \frac{c-\lambda m_1}{\lambda \hat{m}'_X(R) - c},$$

and finally

$$\Psi_{app}(u) = \frac{c-\lambda m_1}{\lambda \hat{m}'_X(R) - c}\, e^{-Ru}. \tag{3.8}$$

□

The other two described approximations are based on moment matching.

3.2 The Beekman-Bowers Approximation

As the ruin probability function $\Psi(u)$ is decreasing to 0 and non-negative, it follows from 2.11 that $H(u) = 1-(1+\rho)\Psi(u)$ is a distribution function, which is vanishing on $(-\infty, 0)$ and $H(0) = 0$. Let $H(u)$ be the distribution function of a r.v. U. The idea of the Beekamn-Bowers approximation is to replace function $H(u)$ by a gamma distribution $W(u, a, b)$ by fitting their first two moments. For detecting the moments of H(u) we use the moment generating function

$$\begin{aligned} \hat{m}_U(s) &= Ee^{sU} = \int_0^\infty (1+\rho)(-1)\, e^{su}\, d\Psi(u) \\ &= (-1)\, \frac{c}{\lambda m_1} \int_0^\infty e^{su}\, d\Psi(u). \end{aligned}$$

To work out the integral $\int_0^\infty e^{su}\, d\Psi(u)$ we apply the Laplace-Stieltjes transform. Remarking that the formal relationship between moment generating function and Laplace-Stieltjes transform is

$$\hat{m}_X(s) = \mathcal{LS}(F)(-s), \tag{3.9}$$

we obtained the integral $\int_0^\infty e^{-su}\, d\Psi(u)$, which is the same as the Laplace transform of the function $\Psi'(u)$

$$\Psi'_L(u) = \int_0^\infty e^{-su}\, \Psi'(u)\, du.$$

Since $\Psi(u)$ is absolutely continuous, we use the 4-th property of the Laplace transform to calculate the integral. As a result of this property and 2.14, it can be easily seen that

$$\int_0^\infty e^{-su}\,\Psi'(u)\,du = s\left(\frac{1}{s} - \frac{c-\lambda m_1}{cs-\lambda(1-sF_L(s))}\right) - \frac{\lambda m_1}{c}.$$

Due to 3.9, returning to the m.g.f. $\hat{m}_U(s)$ gives

$$\begin{aligned}\hat{m}_U(s) &= \frac{(-1)c}{\lambda m_1}\left(1 - \frac{\lambda m_1}{c} + \frac{s\,(c-\lambda m_1)}{cs-\lambda(\hat{m}_X(s)-1)}\right)\\ &= 1 - \frac{c}{\lambda m_1} + \frac{c\,(c-\lambda m_1)}{\lambda m_1}\,\frac{s}{cs-\lambda(\hat{m}_X(s)-1)}.\end{aligned}$$

If the n-th moment of the r.v. U is finite, then the n-th derivative of $\hat{m}_U(s)$ exists and $EU^n = \hat{m}_U^{(n)}(0)$ holds. Hence, to obtain the first two moments of r.v. U we derive m.g.f. $\hat{m}_U(s)$

$$\hat{m}'_U(s) = \frac{c\,(c-\lambda m_1)}{m_1}\,\frac{s\hat{m}'_X(s)-(\hat{m}_X(s)-1)}{(cs-\lambda(\hat{m}_X(s)-1))^2}. \tag{3.10}$$

To calculate $\hat{m}'_U(s)$ as s approaches 0, we use

$$\lim_{s\to 0}\frac{c\,(c-\lambda m_1)}{m_1}\,\frac{s\hat{m}'_X(s)-(\hat{m}_X(s)-1)}{(cs-\lambda(\hat{m}_X(s)-1))^2}\,\frac{s^2}{s^2}.$$

Since

$$\lim_{s\to 0}\frac{s\hat{m}'_X(s)-(\hat{m}_X(s)-1)}{s^2} \overset{L'H}{=} \lim_{s\to 0}\frac{s\hat{m}_X^{(2)}(s)}{2s} = \frac{m_2}{2}$$

and

$$\lim_{s\to 0}\frac{s}{s-\lambda(\hat{m}_X(s)-1)} \overset{L'H}{=} \lim_{s\to 0}\frac{1}{c-\lambda\,\hat{m}'_X(s)} = \frac{1}{c-\lambda m_1}$$

the mean value of r.v. U is

$$EU = \hat{m}'_U(0) = \frac{c\,(c-\lambda m_1)}{m_1}\,\frac{1}{(c-\lambda m_1)^2}\,\frac{m_2}{2} = \frac{cm_2}{2\,m_1\,(c-\lambda m_1)}. \tag{3.11}$$

To get the second derivative of $\hat{m}_U(s)$, we differentiate both sides of 3.10

$$\begin{aligned}\hat{m}_U^{(2)}(s) &= \frac{c\,(c-\lambda m_1)}{m_1}\,\frac{s\,\hat{m}_X^{(2)}(s)\,(cs-\lambda(\hat{m}_X(s)-1))}{(cs-\lambda(\hat{m}_X(s)-1))^3} - \\ &- \frac{c\,(c-\lambda m_1)}{m_1}\,\frac{2\,(c-\lambda\hat{m}'_X(s))(s\,\hat{m}'_X(s)-(\hat{m}_X(s)-1))}{(cs-\lambda(\hat{m}_X(s)-1))^3}.\end{aligned}$$

Therefrom the second moment of r.v. U is calculated as the limit of $\hat{m}_U^{(2)}(s)$ when s approaches 0

$$\lim_{s\to 0} \hat{m}_U^{(2)}(s)\,\frac{s^3}{s^3} =$$

$$\begin{aligned}
&= \frac{c(c-\lambda m_1)}{m_1}\,\frac{1}{(c-\lambda m_1)^3}\left(\frac{m_3}{3}(c-\lambda m_1) + \lim_{s\to 0}\frac{3\lambda\,\hat{m}_X^{(2)}(s)(s\,\hat{m}'_X(s)+1)}{3s^2}\right)\\
&= \frac{c}{m_1(c-\lambda m_1)^2}\left(\frac{m_3}{3}(c-\lambda m_1) + \frac{\lambda(m_2)^2}{2}\right)\\
&= \frac{c}{m_1}\left(\frac{m_3}{3(c-\lambda m_1)} + \frac{\lambda(m_2)^2}{2(c-\lambda m_1)^2}\right). \qquad (3.12)
\end{aligned}$$

Gamma distribution with d.f. $W(x,a,b)$ has the density function in the form

$$w(x,a,b) = \frac{x^{a-1}\,e^{-x/b}}{b^a\,\Gamma(a)}, \quad x \geq 0,\ a > 0,\ b > 0,$$

where gamma function $\Gamma(a)$ is defined as

$$\Gamma(a) = \int_0^\infty e^{-x}\,x^{a-1}\,dx, \quad a > 0$$

with the following properties

$$\begin{aligned}
\Gamma(1) &= 1,\\
\Gamma(a+1) &= a\,\Gamma(a).
\end{aligned}$$

Obviously, we also need to know the moments of Γ-distribution to be able to compare them with 3.11 and 3.12

$$\begin{aligned}
\int_0^\infty x\ w(x,a,b)\,dx &= \int_0^\infty \frac{x^a e^{-x}}{b^a\ \Gamma(a)}\,dx = a\,b,\\
\int_0^\infty x^2\ w(x,a,b)\,dx &= \int_0^\infty \frac{x^{a+1} e^{-x}}{b^a\ \Gamma(a)}\,dx = a\,(a+1)\,b^2.
\end{aligned}$$

Both integrals above are calculated by integration by parts. According to the main idea of the Beekman-Bowers approximation, we fit the moments of both gamma distribution and function $H(u)$

$$\begin{aligned}
a\,b &= \frac{cm_2}{2\,m_1\,(c-\lambda m_1)},\\
a\,(a+1)\,b^2 &= \frac{c}{m_1}\left(\frac{m_3}{3(c-\lambda m_1)} + \frac{\lambda(m_2)^2}{2(c-\lambda m_1)^2}\right).
\end{aligned}$$

By solving the system of equations above we find out the parameters of gamma distribution that approximates function $H(u)$ superbly

$$\begin{aligned} a &= \frac{3\,c\,m_2^2}{4\,m_1\,m_3\,(c-\lambda m_1)+3\,m_2^2\,(2\lambda m_1 - c)}, \\ b &= \frac{2\,m_3}{3\,m_2}+\frac{m_2\,(2\lambda m_1 - c)}{2\,m_1\,(c-\lambda m_1)}. \end{aligned}$$

The gamma function $W(u,a,b)$ with parameters a,b specified above best approximates function $H(u)$. As a consequence, the Beekman-Bowers approximation of the ruin probability is

$$\Psi_{app}(u) = \frac{1}{1+\rho}(1 - W(u,a,b)). \tag{3.13}$$

3.3 The De Vylder Approximation

De Vylder suggested to approximate the surplus process $R(t)$ by another risk process $\tilde{R}(t)$ with exponentially distributed claim sizes, as described in Asmussen [1] for instance. Of course, it is desirable to make the approximation as accurate as possible. Therefore it is required that the first three power moments of processes $R(t)$ and $\tilde{R}(t)$ coincide

$$ER^i(t) = E\tilde{R}^i(t), \quad i = 1,2,3, \quad t \geq 0.$$

Let us denote the parameter of exponential distribution, the arrival intensity and premium rate of the process $\tilde{R}(t)$ by $\tilde{\delta}, \tilde{\lambda}, \tilde{c}$ respectively. In this case, the PSL condition takes the form $\tilde{c} > \frac{\tilde{\lambda}}{\tilde{\delta}}$ as the mean value of exponential distribution with parameter $\tilde{\delta}$ is $\frac{1}{\tilde{\delta}}$. Naturally, the initial capital should be the same for both of the processes. Thus it can be eliminated when calculating the mean value of the processes

$$\begin{aligned} E(R(t)-u) &= (c - \lambda\, m_1)\, t, \\ E(\tilde{R}(t)-u) &= (\tilde{c} - \frac{\tilde{\lambda}}{\tilde{\delta}})\, t. \end{aligned}$$

Since we require

$$ER^2(t) = varR(t) + (ER(t))^2 = var\tilde{R}(t) + (E\tilde{R}(t))^2 = E\tilde{R}^2(t),$$

and the equality of $ER(t)$ and $E\tilde{R}(t)$ is ensured, it is sufficient to work with variance instead of working with the second moment of the processes

$$\begin{aligned} varR(t) = var(R(t)-u-c\,t) &= \lambda\, m_2\, t, \\ var\tilde{R}(t) = var(\tilde{R}(t)-u-\tilde{c}\,t) &= \frac{2\,\tilde{\lambda}\,t}{\tilde{\delta}^2}, \end{aligned}$$

because for the variance of a compound distribution of a r.v. $S = \sum_{i=1}^{M(t)} Z_i$ holds the following

$$varS = EM\, varZ + varM\,(EZ)^2.$$

And last but not least,

$$\begin{aligned} E(R(t) - u - c\,t)^3 &= \lambda\, m_3\, t, \\ E(\tilde{R}(t) - u - \tilde{c}\,t)^3 &= \frac{6\,\tilde{\lambda}\,t}{\tilde{\delta}^3}, \end{aligned}$$

by reason that for a compound distribution of a r.v. S defined above

$$ES^3 = E(E(S|M)) = EM(EZ)^3.$$

Resultantly, the main concept of the De Vylder approximation is fulfilled if and only if

$$c - \lambda\, m_1 = \tilde{c} - \frac{\tilde{\lambda}}{\tilde{\delta}}, \quad \lambda\, m_2 = \frac{2\,\tilde{\lambda}}{\tilde{\delta}^2}, \quad \lambda\, m_3 = \frac{6\,\tilde{\lambda}}{\tilde{\delta}^3}.$$

This leads to the parameters specified below

$$\tilde{\delta} = \frac{3\,m_2}{m_3}, \quad \tilde{\lambda} = \frac{9\,\lambda\, m_2^3}{2\,m_3^2}, \quad \tilde{c} = c - \lambda\, m_1 + \frac{3\,m_2^2\,\lambda}{2\,m_3}.$$

At present, it still remains to express the function with given parameters, which is involved as the ruin probability approximation. To attain the ruin probability $\Psi(u)$, we use its Laplace transform 2.14. For exponentially distributed claims, the Laplace transform of its distribution function takes the form

$$F_{L(exp)}(s) = \int_0^\infty e^{-sx}\,(1 - e^{-\tilde{\delta}x})\,dx = \frac{1}{s} - \frac{1}{s+\tilde{\delta}} = \frac{\tilde{\delta}}{s(s+\tilde{\delta})},$$

where *exp* in the subscript stresses the fact that we deal with exponentially distributed claims.

Plugging into 2.14, we arrive at

$$\begin{aligned} \Psi_{L(exp)}(s) &= \frac{1}{s} - \frac{\tilde{c} - \tilde{\lambda}/\tilde{\delta}}{\tilde{c}\,s - \tilde{\lambda}\left(1 - s\frac{\tilde{\delta}}{s(s+\tilde{\delta})}\right)} = \frac{\tilde{c}\,(s+\tilde{\delta}) - \tilde{\lambda} - (\tilde{c} - \tilde{\lambda}/\tilde{\delta})(s+\tilde{\delta})}{s\,(\tilde{c}\,(s+\tilde{\delta}) - \tilde{\lambda})} \\ &= \frac{\tilde{\lambda}}{\tilde{\delta}\,\tilde{c}}\,\frac{1}{s+\tilde{\delta} - \tilde{\lambda}/\tilde{c}}. \end{aligned}$$

Fortunately, in this case it is easy to invert the Laplace transform. As a result, we obtain the De Vylder approximation for the ruin probability

$$\Psi_{app}(u) = \frac{\tilde{\lambda}}{\tilde{\delta}\,\tilde{c}}\, e^{-(\tilde{\delta} - \frac{\tilde{\lambda}}{\tilde{c}})\,u}. \tag{3.14}$$

As we can see, De Vylder chose the risk process $\tilde{R}(t)$ in such a way that resolving the ruin probability was straightforward. Obviously, De Vylder's $\Psi_{app}(u)$ is valid for any classical risk process. Moreover, $\Psi_{app}(u)$ corresponds with $\Psi(u)$ in the case of exponentially distributed claim sizes.

To illustrate the accuracy of the approximations studied in this chapter, we provide the following examples.

3.4 Examples

Example 3.1: Firstly, consider $F(x) = 1 - \frac{1}{2}e^{-x} - \frac{1}{2}e^{-2x}$ as the distribution function of claims. By applying the expression 2.14 and by adjusting we obtain $\Psi_L(s)$ in the form

$$\Psi_L(s) = \frac{1}{s} - \frac{A_1}{s} - \frac{A_2}{s - s_1} - \frac{A_3}{s - s_2}, \tag{3.15}$$

where

$$s_1 = \frac{\lambda - 3c + \sqrt{c^2 + \lambda^2}}{2c}, \qquad s_2 = \frac{\lambda - 3c - \sqrt{c^2 + \lambda^2}}{2c},$$

$$A_1 = \frac{4(c - \lambda m_1)}{4c - 3\lambda}, \quad A_2 = \frac{\lambda(c - \lambda m_1)(\lambda + \sqrt{c^2 + \lambda^2})}{c\sqrt{c^2 + \lambda^2}(\lambda - 3c + \sqrt{c^2 + \lambda^2})}$$

and

$$A_3 = \frac{\lambda(c - \lambda m_1)(\lambda - \sqrt{c^2 + \lambda^2})}{c\sqrt{c^2 + \lambda^2}(3c - \lambda + \sqrt{c^2 + \lambda^2})}.$$

Fortunately, expression 3.15 is invertible and we arrive at the exact formula for the ruin probability

$$\Psi(u) = 1 - A_1 - A_2\, e^{s_1 u} - A_3\, e^{s_2 u}.$$

Secondly, the approximations need to be figured out for our special case of d.f. $F(x)$. The first, second and third moment of claim sizes is

$$m_1 = \frac{3}{4}, \quad m_2 = \frac{5}{4}, \quad m_3 = \frac{27}{8}$$

respectively. Let assume the inter-occurrence time of claims as time unit, i.e. $\lambda = 1$. In addition, let the premium rate be unitary, i.e. $c = 1$. The value of adjustment coefficient is obtained as the solution to the equation 3.6. This yields

$$R = \frac{3c - \lambda - \sqrt{c^2 + \lambda^2}}{2c}.$$

Last but not least, $m'_X(R)$ is gained from the integration $\int_0^\infty x\, e^{Rx}\, dF(x)$ as

$$m'_X(R) = \frac{1}{2\,(1 - R)^2} + \frac{1}{(2 - R)^2}.$$

To sum it up, the parameters for approximations 3.8, 3.13 and 3.14 are

$$\begin{aligned} R &= \frac{2-\sqrt{2}}{2}, \\ a &= \frac{25}{26}, \quad b = \frac{52}{15}, \\ \tilde{\delta} &= \frac{10}{9}, \quad \tilde{\lambda} = \frac{125}{162}, \quad \tilde{c} = \frac{17}{18}. \end{aligned}$$

Thus, we can compare the exact ruin probability with the approximations. The results are provided in tables 3.1, 3.2, 3.3 below. Each table contains exact ruin probability $\Psi(u)$, its approximation (Cramér-Lundberg's, Beekman-Bowers' and De Vylder's respectively) and relative error calculated as $(\Psi(u) - \Psi_{app}(u))/\Psi(u)$ multiplied by 100, which is given in percent.

u	$\Psi(u)$	$C - L\,app.$	$ER(\%)$
0,00	0,750000000	0,728553390	2,859547920
0,10	0,725604922	0,707524027	2,491837495
0,25	0,691108873	0,677112617	2,025188299
0,50	0,638437995	0,629303908	1,430692793
0,75	0,590831806	0,584870817	1,008914779
1,00	0,547465197	0,543575000	0,710583421
1,50	0,471181613	0,469524782	0,351633080
2,00	0,406267931	0,405562289	0,173688869
5,00	0,168446774	0,168442562	0,002500118
7,50	0,080992981	0,080992922	0,000072863
10,00	0,038944156	0,038944156	0,000002123

Table 3.1: Cramér-Lundberg approximation to the ruin probability for classical risk process with claim distribution $F(x) = 1 - \frac{1}{2}e^{-x} - \frac{1}{2}e^{-2x}$

u	$\Psi(u)$	$B-B\,app.$	$ER(\%)$
0,00	0,750000000	0,750000000	0
0,10	0,725604922	0,725162724	0,060941977
0,25	0,691108873	0,691304198	-0,028262636
0,50	0,638437995	0,639594169	-0,181094203
0,75	0,590831806	0,592444455	-0,272945597
1,00	0,547465197	0,549146238	-0,307058885
1,50	0,471181613	0,472417955	-0,262391877
2,00	0,406267931	0,406861505	-0,146104026
5,00	0,168446774	0,167768648	0,402575303
7,50	0,080992981	0,080677881	0,389045873
10,00	0,038944156	0,038896375	0,122692800

Table 3.2: Beekman-Bowers approximation to the ruin probability for classical risk process with claim distribution $F(x) = 1 - \frac{1}{2}e^{-x} - \frac{1}{2}e^{-2x}$

u	$\Psi(u)$	$DV\,app.$	$ER(\%)$
0,00	0,750000000	0,735294117	1,960784313
0,10	0,725604922	0,713982758	1,601720655
0,25	0,691108873	0,683168249	1,148968654
0,50	0,638437995	0,634737644	0,579594334
0,75	0,590831806	0,589740343	0,184733305
1,00	0,547465197	0,547932953	-0,085440354
1,50	0,471181613	0,472999394	-0,385792059
2,00	0,406267931	0,408313509	-0,503504602
5,00	0,168446774	0,168963437	-0,306721948
7,50	0,080992981	0,080995064	-0,002572775
10,00	0,038944156	0,038826154	0,303003594

Table 3.3: De Vylder approximation to the ruin probability for classical risk process with claim distribution $F(x) = 1 - \frac{1}{2}e^{-x} - \frac{1}{2}e^{-2x}$

One can see that the differences between exact values and approximations are quite low. Although the relative errors in the case of Beekman-Bowers approximation are all less than 1%, they behave irregularly. Likewise, the unpredictable behavior of relative errors can be observed in the De Vylder approximation. From the regularity point of view, the Cramér-Lundberg approximation seems to be the best, as the relative error decreases with increasing u. Moreover, from some value of u the approximation equals the exact ruin probability. To summarize it up, for our d.f. $F(x)$ the most accurate outcomes for small u (approximately $u < 5$) provides the Beekman-Bowers approximation while for big u acts the Cramér-Lundberg approximation as the best one. △

Example 3.2: Let standard exponential distribution $F(x) = 1 - e^{-\alpha x}$ be the distribution function of claims. In this case, $EX = \frac{1}{\alpha}, varX = \frac{1}{\alpha^2}$ and $m_i = \frac{i!}{\alpha^i}$. Similarly to the example 3.1 we calculate the exact ruin probability. Following the method for calculating the convolution formula 2.14, $\Psi_L(s)$ results in

$$\Psi_L(s) = \frac{1}{s} - \frac{A_1}{s} - \frac{A_2}{s - s_1},$$

where

$$A_1 = \frac{\alpha\,(-1)\,(c - \lambda m_1)}{\lambda - c\,\alpha} \qquad A_2 = \frac{\lambda\,(c - \lambda m_1)}{c\,(\lambda - c\,\alpha)}$$

and

$$s_1 = \frac{\lambda - c\,\alpha}{c}.$$

Consequently, $\Psi(u) = 1 - A_1 - A_2\,e^{s_1 u}$. As the next step of calculation, we determine the value of variables. Let assign α to 1. In contrast to the previous example, we assume that $\lambda = 2$ and set the premium rate c to 50. By repeating the procedure of example 3.1 we obtain also the other variables. In particular,

$$\begin{aligned} R &= \frac{24}{25}, \quad m'_X(R) = 25, \\ a &= 1, \quad b = \frac{25}{24}, \\ \tilde{\delta} &= 1, \quad \tilde{\lambda} = 2, \quad \tilde{c} = 50. \end{aligned}$$

Since the claim distribution is exponential, the De Vylder approximation matches with the exact ruin probability. Cramér-Lundberg approximation is nearly identical to the exact value of ruin probability, these two are equal to 17th significant digit. Table below provides the values of exact ruin probability, Beekman-Bowers approximation to it and difference between these two.

u	$\Psi(u)$	$B - B\,app.$	$ER(\%)$
0,00	0,04000000000	0,04000000000	0,0
0,10	0,03633856064	0,03633856064	-0,0001088 $.10^{-5}$
0,25	0,03146511444	0,03146511445	-0,0026914 $.10^{-5}$
0,50	0,02475133567	0,02475133576	-0,0388876 $.10^{-5}$
0,75	0,01947009023	0,01947009045	-0,1099140 $.10^{-5}$
1,00	0,01531571543	0,01531571571	-0,1730252 $.10^{-5}$
1,50	0,00947711034	0,00947711034	-0,0004171 $.10^{-5}$
2,00	0,00586427848	0,00586427848	-0,0004171 $.10^{-5}$
5,00	0,00032918988	0,00032918988	-0,0004171 $.10^{-5}$
7,50	0,00002986343	0,00002986343	-0,0004169 $.10^{-5}$
10,00	0,00000270914	0,00000270914	-0,0004103 $.10^{-5}$

Table 3.4: Beekman-Bowers approximation to the ruin probability for classical risk process with claim distribution $F(x) = 1 - e^{-x}$

It can be seen that the Beekman-Bowers approximation behaves irregularly. Although the relative error between exact ruin probability and approximation is biggest in case of Beekman-Bowers approximation, as a whole, it is quite small. Therefore we can conclude that all three approximations work well for this type of claim distribution, even for small probabilities. $\triangle$

Chapter 4

Risk Process with Random Income

It can be said that several assumptions made in chapter 3 are rather unrealistic. Obviously, the classical risk process can be generalized in many ways to become a suitable description of the cash flows of an insurance company. Some of the possibilities are mentioned below:

- the occurrence of the claims may be described by a more general process than the Poisson process,
- inflation may be included in the model,
- the premium rate does not have to be constant.

Our task in this chapter is to focus on the last-mentioned generalization. To be more concrete, we study the risk process with random income as it is described in Temnov [12].

In case of an insurance company, income can no more be treated as a linear function as in the classical risk process. The reason is in the changing number of clients, different gross premium rates and frequency of payments. Therefore we will assume that the income process is a stochastic process represented as a random sum

$$I(t) = \sum_{i=1}^{N_2(t)} Y_i, \quad t \geq 0, \tag{4.1}$$

where $N_2(t)$ is a standard Poisson process with intensity λ_2, $\{Y_i\}$ is a sequence of i.i.d.r.vs and $N_2(t)$ is independent of $\{Y_i\}$. Similarly to the claims, $N_2(t)$ indicates the number of payments and the sequence $\{Y_i\}$ determines the size of premium payments. Altogether with process describing the claims, which is defined as in chapter 2, we come to the balance equation which defines *a risk process with* *random income*

$$R(t) = \sum_{i=1}^{N_2(t)} Y_i - \sum_{j=1}^{N(t)} X_j, \quad t \geq 0, \tag{4.2}$$

where sequence of i.i.d.r.vs $\{Y_i\}$ is independent of $\{X_j\}$ and $N(t)$; $N_2(t)$ is independent of $N(t)$ and r.vs. $\{X_j\}$. Like in the classical process, let $F(x)$ be the common distribution function of the sequence $\{X_j\}$, and m_r be its power moments, if they exist. In addition, let $G(y)$ be the common distribution function of the premium sequence $\{Y_i\}$ defined as

$$G(y) = P(Y_i \leq y),$$

and n_r be its power moments

$$n_r = EY_1^r = \int_0^\infty y^r \, dG(y), \quad r > 0.$$

In this case, the relative safety loading takes the form

$$\rho = \frac{\lambda_2 \, t \, n_1 - \lambda \, t \, m_1}{\lambda \, t \, m_1} = \frac{\lambda_2 \, n_1}{\lambda \, m_1} - 1 > 0.$$

Like in the classical risk process, ruin appears only at claim occurrence time T_k. Let $\{I_k\}$ be the sequence of incomes that come into the process between $(n-1)$-th and n-th claim. For the classical risk process, $I_k = c\,(T_k - T_{k-1})$. Let T_k^I, T_k^C be the occurrence times of incomes and claims respectively, provided that $T_0^I = 0$, $T_0^C = 0$. Then I_k can be expressed as

$$I_k = \sum_{i=0}^{N_2(T_k^C)} Y_i - \sum_{i=0}^{N_2(T_{k-1}^C)} Y_i. \tag{4.3}$$

In addition, let U be the distribution function of claim inter-occurrence times.

Since the inter-occurrence times $\{T_k^C - T_{k-1}^C\}$ are i.i.d.r.vs, for the distribution of I_k holds the following

$$P(I_k \leq x) =$$

$$= \sum_{i=0}^{\infty} \left(P(N_2(T_k^C - T_{k-1}^C) = i) \, P(\sum_{l:1 \leq l \leq i} Y_l \leq x | N_2(T_k^C - T_{k-1}^C) = i) \right)$$

$$= \sum_{i=0}^{\infty} P\left(N_2(T_k^C - T_{k-1}^C) = i\right) P(\sum_{l:1 \leq l \leq i} Y_l < \infty).$$

Hence, the sequence $\{I_k\}$ is also a sequence of i.i.d.r.vs.

As in the classical risk process, we shall derive a formula for ruin probability because it is one of the most significant problems in the investigation of risk processes. To accomplish such a task, we apply a random walk process.

4.1 Random Walk

In case of computing ruin probabilities for a process with random income, random walk is considered to be helpful. First of all, we need to define some terms. The definition is taken from Rolski et al. [11].

Definition 4.1: Let $Z_1, Z_2, \dots$ be a sequence of i.i.d.r.vs with distribution D, which can take positive as well as negative values. The sequence $\{S_n, n \in \mathcal{N}\}$ defined as

$$S_0 = 0, \quad S_n = Z_1 + \cdots + Z_n$$

is called a *random walk*.

We assume that EZ exists and that $P(Z = 0) < 1$. Depending on the value of EZ, we distinguish between three types of random walk

- random walk with positive drift, if $EZ > 0$,
- random walk with negative drift, if $EZ < 0$,
- random walk without drift, if $EZ = 0$.

Another aspect relying on EZ is the evolution of random walk, as states the following lemma.

Lemma 4.1:
If $EZ > 0$, then $lim_{n\to\infty} S_n = \infty$.
If $EZ < 0$, then $lim_{n\to\infty} S_n = -\infty$.
If $EZ = 0$, then $\limsup_{n\to\infty} S_n = \infty$ and $\liminf_{n\to\infty} S_n = -\infty$.

The proof of lemma is omitted and can be found, for instance, in Rolski et al. [11]

Let $\nu^+ = inf\{n : S_n > 0\}$ be a random variable denoting the time of the first entrance of the random walk into the positive half-line. ν^+ is called the *first ascending ladder epoch*. In case that $S_n \leq 0$ for all $n \in \mathcal{N}$, let $\nu^+ = \infty$. Similarly, let $\nu^- = inf\{n : S_n \leq 0\}$ be the *first descending ladder epoch*. ν^- denotes the time of the first entrance into the non-positive half-line. Let $\nu^- = \infty$, if $S_n > 0$ for all $n \in \mathcal{N}$. Depending on the type of random walk, we can say whether ν^+ and ν^- are proper random variables.

At the time of the first ascending ladder epoch, let S_{ν^+} be the value of random walk above the zero level and call it *first ascending ladder high*. If the first ascending ladder epoch does not occur, then the first ascending ladder high equals infinity. Let $M = \sup_{n\geq 0} S_n$ be the *maximum* of a random walk. Obviously, $M \geq 0$.

To be able to use the duality property from theorem 1.3 later in this chapter, we also introduce the first descending ladder high. Let S_{ν^-} be the value below the zero level of random walk at the time of the first descending ladder epoch ν^-, and call it first descending ladder high.

Henceforth we assume that the random walk has a negative drift. According to lemma 4.1, this is the only case when M is finite with probability 1. Since $S_n \to -\infty$ with probability 1, it is possible that $q = P(\nu^+ = \infty) > 0$. This means that ν^+ is a *non-proper random variable* and

$$F_\nu(x) = P(S_{\nu^+} \leq x | \nu^+ < \infty)$$

denotes the conditional distribution of S_{ν^+} under the condition $\nu^+ < \infty$.

In case of negatively drifted random walk the ruin probability has the form

$$\Psi(u) = P(M > u).$$

To state the ruin probability for a random walk we use the following lemma. The proof is from Kalashnikov [8].

Lemma 4.2: For a random walk with negative drift, the survival probability fulfils the following formula

$$\phi(x) = q + (1-q)F_\nu(x - M') * \phi(M'). \tag{4.4}$$

Proof: If $\nu^+ = \infty$, then $M = 0$. Otherwise, $\nu^+ < \infty$ and then M can be expressed as $M = S_{\nu^+} + M'$, where

$$M' = \sup_{k \geq 0} S'_k$$

and

$$S'_k = S_{\nu^+ + k} - S_{\nu^+}, \quad k \geq 0.$$

Since the sequence $\{S'_k\}$ does not depend on $(\nu^+, S_{\nu^+}), M'$ also does not depend on (ν^+, S_{ν^+}). Moreover, the distributions of $\{S'_k\}$ and M' are matched by the distributions of $\{S_k\}$ and M respectively. Therefore we can write M as

$$M = \mathbf{1}(\nu^+ < \infty)(\mathbf{S}_{\nu^+} + \mathbf{M}'),$$

where $\mathbf{1}(\nu^+ < \infty)$ is an *indicator function.* In this moment we can use the total probability formula to calculate $P(M \leq x)$

$$\begin{aligned} P(M \leq x) &= P\left(\mathbf{1}(\nu^+ < \infty)(\mathbf{S}_{\nu^+} + \mathbf{M}') \leq \mathbf{x}\right) \\ &= P(\nu^+ = \infty)P(0 \leq x) + (\nu^+ < \infty)P(S_{\nu^+} + M' \leq x | \nu^+ < \infty). \end{aligned}$$

Using the independence of M' and S_{ν^+}, we obtain the formula 4.4. □

Following the procedure from Kalashnikov [8] we calculate the survival probability more precisely. Since 4.4 is a convolution equation, we use the Laplace-Stieltjes transform to solve it

$$\mathcal{LS}(\phi)(s) = q + (1-q)\mathcal{LS}(F_\nu)(s) * \mathcal{LS}(\phi)(s).$$

From the equation above, we express $\mathcal{LS}(\phi)(s)$ as

$$\mathcal{LS}(\phi)(s) = \frac{q}{1-(1-q)\,\mathcal{LS}(F_\nu)(s)}. \tag{4.5}$$

By applying relation for the sum of geometric series we obtain the result

$$\mathcal{LS}(\phi)(s) = q\sum_{n=0}^{\infty}(1-q)^n\,(\mathcal{LS}(F_\nu)(s))^n\,. \tag{4.6}$$

Inverting 4.6 leads to

$$\phi(x) = q\sum_{n=0}^{\infty}(1-q)^n\,F_\nu^{*\,n}(x). \tag{4.7}$$

It is possible to come to 4.7 also by applying Fourier transform or renewal process, as it is done in Temnov [12] and Rolski et al. [11] respectively.

Theorem 4.1: The following convolution equation holds for a random walk with negative drift

$$\Psi(u) = q\sum_{n=0}^{\infty}(1-q)^n\,\left(1-F_\nu^{*\,n}(u)\right), \quad u \geq 0. \tag{4.8}$$

PROOF: Since $\Psi(u) = 1-\phi(u)$, 4.7 directly infers 4.8. □

4.2 Convolution Formula in the Random Income Process

In this section we apply the results obtained with the help of random walk to the risk process with random income. To start with, we redefine the random income process by using the terms from random walk. Consider the following random walk

$$S_0 = 0, \quad S_k = Z_1 + .. + Z_k,$$

where

$$Z_k = X_k - I_k, \quad k \in \mathcal{N}. \tag{4.9}$$

From the PSL condition we have $EZ < 0$. Therefore, the process defined in 4.9 is a random walk with negative drift. In addition, the maximum of such a random walk is finite. Since the assumptions of lemma ozn are fulfilled, the relation 4.4

holds. Furthermore, from this relation we get theorem 4.1 and. Consequently, the following convolution formula holds for a risk process with random income

$$\Psi(u) = q \sum_{k=0}^{\infty} (1-q)^k \left(1 - F_\nu^{*k}(u)\right), \quad u \geq 0, \tag{4.10}$$

where F_ν is the conditional distribution of the first ascending ladder height of the random walk defined in 4.9.

Since the classical risk process is a special case of a random income risk process, we can apply the results obtained in this section to the classical risk process. This way we are able to compare the convolution formulas for the ruin probability.
As the 0−th summand in 4.10 is 0,

$$\Psi(0) = q \sum_{k=1}^{\infty} (1-q)^k \left(1 - F_\nu^{*k}(0)\right) = 1 - q. \tag{4.11}$$

It follows from 4.11 and 2.11 that

$$q_c = 1 - \frac{\lambda m_1}{c}, \tag{4.12}$$

where q_c is the probability that the first ascending ladder epoch in the classical risk process equals infinity. Plugging q_c into 4.10 and subsequent confronting it with 2.15 yields

$$F_{\nu;c}(u) = \frac{1}{m_1} \int_0^u (1 - F(x))\, dx, \tag{4.13}$$

where $F_{\nu;c}(u)$ denotes $F_\nu(u)$ for the classical risk process. Consequently, 4.10 can be considered as the extension of the Beekman convolution formula to the processes with random income.

In the following part we are going to compare the convolution formulas for the classical risk process and random income process. However, to be able to deal with the convolution of the ladder heights more in detail, we need the characteristic function. It is defined as

$$\varphi(t) = E(e^{itX}) = \int_{-\infty}^{\infty} e^{itx} dF_X(x),$$

where $t \in \mathcal{R}$ and i is an imaginary unit. Such a function always exists and uniquely characterizes the distribution function from which it is calculated. We denote the characteristic function by $\varphi(t)$ in case it is clear to which function is the characteristic function calculated. Otherwise, we will specify it in the subscript. Let us only mention, that the relation between the characteristic function and the distribution function is one-to-one and it is possible to use either of them to describe the probability distribution, as is written in [2].

4.3 Examples

Example 4.1: Let us turn back to the example 3.1 from the previous chapter. Since we appointed the distribution function of claims and thus we know it, we can express the ruin probability as if for the risk process with random income. Consequently, we can compare obtained results with the exact ruin probability. By substituting the distribution function of claims $F(x) = 1 - \frac{1}{2}e^{-x} - \frac{1}{2}e^{-2x}$ and $m_1 = \frac{3}{4}$ into 4.13 we get the following distribution of the ladder height

$$F_{\nu;c}(u) = \frac{4}{3}\int_0^u \left(1 - (1 - \frac{1}{2}\,e^{-x} - \frac{1}{2}e^{-2x})\right) dx.$$

After integration we gain

$$F_{\nu;c}(u) = \begin{cases} 1 - \frac{2}{3}e^{-u} - \frac{1}{3}e^{-2u} & \text{for } u \geq 0, \\ 0 & \text{for } u < 0. \end{cases}$$

As $\lambda = c = 1$, the probability of infinite first ascending ladder epoch is $q_c = \frac{1}{4}$. Altogether, the convolution formula 4.10 takes the form

$$\Psi(u) = \frac{1}{4}\sum_{k=1}^{\infty}\left(\frac{3}{4}\right)^k \left(1 - \left(1 - \frac{2}{3}\,e^{-u} - \frac{e^{-2u}}{3}\right)^{*\,k}\right).$$

To calculate the convolution of the distribution function we use the characteristic function

$$\begin{aligned} \varphi(t) &= \int_{-\infty}^{\infty} e^{itu}\,dF(u) \\ &= \int_0^{\infty} \frac{2}{3}\,(e^{-u} + e^{-2u})\,e^{itu}\,du \\ &= \frac{2}{3}\left(\frac{1}{1 - i\,t} + \frac{1}{2 - i\,t}\right). \end{aligned}$$

Further we use one of the properties of the characteristic function. Particularly, the characteristic function of a convolution of probability densities g and h equals multiplication of characteristic functions of probability densities g and h. Written in symbols,

$$\varphi_{g*h}(t) = \varphi_g(t)\,.\,\varphi_h(t).$$

Thus the k-th convolution is expressed as

$$\begin{aligned} \varphi^k(t) &= \left(\frac{2}{3}\right)^k \cdot \left(\frac{1}{1 - i\,t} + \frac{1}{2 - i\,t}\right)^k \\ &= \left(\frac{2}{3}\right)^k \cdot \sum_{j=0}^{k}\binom{k}{j}\left(\frac{1}{1 - i\,t}\right)^j \left(\frac{1}{2 - i\,t}\right)^{k-j}. \end{aligned}$$

At this point we need to get back to the distribution function. Therefore we apply the inverse to the characteristic function. However, inverting the characteristic function is rather laborious for some bigger k. Due to this reason we use the software Mathematica 5.2 to calculate it. Nevertheless, calculating infinite sum in 4.10 would be complicated even for this software. Hence, we decided to calculate the ruin probability using finite sum. Difference between exact value of ruin probability and ruin probability calculated by the convolution formula for random income process is less than 1% for majority of initial capitals in case when maximum k equals 20. Thus we consider 20 to be large enough for the calculation. Resulting ruin probabilities and relative errors are provided in table 4.1.

u	Classical process	Random income process	Δ (%)
0,00	0,750000000	0,750000000	0,0
0,10	0,725604922	0,723226513	0,327782913
0,25	0,691108873	0,688730464	0,344143889
0,50	0,638437995	0,636059586	0,372535621
0,75	0,590831806	0,588453397	0,402552626
1,00	0,547465197	0,545086788	0,434440210
1,50	0,471181613	0,468803204	0,504775417
2,00	0,406267931	0,403889522	0,585428671
5,00	0,168446774	0,166068368	1,411962798
7,50	0,080992981	0,078615313	2,935646655
10,00	0,038944156	0,036582712	6,063666904

Table 4.1: Classical process vs. Random income process. Ruin probability for risk process with claim distribution $F(x) = 1 - \frac{1}{2}e^{-x} - \frac{1}{2}e^{-2x}$

△

Example 4.2: Now we return to the example 3.2 and again calculate the ruin probability as if for the random income process. This time we plug $F(x) = 1 - e^{-x}$ and $m_1 = 1$ into 4.13. Following the procedure from the example 4.1 we obtain the succeeding distribution function for ladder height

$$F_{\nu;c}(u) = \begin{cases} 1 - e^{-u} & \text{for } u \geq 0, \\ 0 & \text{for } u < 0. \end{cases}$$

Since $\lambda = 2$ and $c = 50$, the probability that the first ascending ladder epoch is infinite equals $\frac{24}{25}$, i.e. $q_c = \frac{24}{25}$. Therefore the entire convolution formula takes the form

$$\Psi(u) = \frac{24}{25} \sum_{k=1}^{\infty} \left(\frac{1}{25}\right)^k \left(1 - \left(1 - e^{-u}\right)^{*k}\right).$$

As in the example 4.1, we use the characteristic function of the ladder height distribution function

$$\varphi(t) = \int_{-\infty}^{\infty} e^{itu}\, dF(u) = \int_{0}^{\infty} e^{-u}\,.\, e^{itu}\, du = \frac{i}{t + a\, i}.$$

At present we state the k-th convolution from the expression above by using the characteristic function

$$\varphi^k(t) = \left(\frac{i}{t + a\, i}\right)^k .$$

Final calculation, again for maximum $k = 20$, is executed by the software Mathematica 5.2. and the results are given in table 4.2.

u	Classical process	Random income process	Δ (%)
0,00	0,040000000	0,040000000	0,0
0,10	0,036338560	0,037761099	-3,9146808
0,25	0,031465114	0,034635510	-10,0759063
0,50	0,024751335	0,029990464	-21,1670516
0,75	0,019470090	0,025968375	-33,3757304
1,00	0,015315715	0,022485698	-46,8145441
1,50	0,009477110	0,016858913	-77,8908546
2,00	0,005864278	0,012640165	-115,5451037
5,00	0,000329189	0,002245391	-582,0958469
7,50	0,000029863	0,000531995	-1681,4273179
10,00	0,000002709	0,000126044	-4552,5474439

Table 4.2: Classical process vs. Random income process. Ruin probability for risk process with claim distribution $F(x) = 1 - e^{-x}$

△

As we see from the examples, the probabilities calculated by using the classical risk process, on one side, and random income risk process, on the other side, differ. Let us review the results from the insurance company's point of view. Suppose that the insurance company takes the classical risk process as a risk indicator.

In the first example, the classical risk convolution formula supplies higher probabilities of ruin. Therefore the insurance company perceives the risk and takes some actions necessary to improve its position or, at least, to keep it unchanged. This way it minimizes the danger of ruin. On the other hand, difference between ruin probabilities is not very big, and thus random income risk process would also be appropriate as the risk indicator for the insurance company.

In the second example, the ruin probabilities obtained by the two mentioned methods vary much more. Classical risk process provided undervalued ruin probability when comparing it to the ruin probability calculated by the random income process. In this case, the insurance company does not expect any jeopardy, does not behaves overcautiously and this may lead to an inability of meeting its liabilities towards insured persons. To analyze this case more in detail, the relative safety loading equals 24. As mentioned in the second chapter, such a high relative safety loading does not have to attract clients due to high amount of premiums.

To summarize it up, the examples show that the ruin probability is sometimes undervalued when calculated by the convolution formula for classical risk process. Even though the classical risk process works, it is more risky. Consequently, it is important to explore the risk processes with random income as they describe the reality more sensibly.

4.4 Ladder Height Distribution

A definite expression for the conditional distribution of the first ascending ladder high in 4.10 can be found awkwardly. Therefore, if we want to compute the ruin probability, we have to determine the ladder high distribution $F_\nu(u)$. In this section we provide one possible way how to calculate the ruin probability for risk process with random income. However, this method is still quite complicated.

At the beginning, we just note that throughout this section $EZ < 0$. Let us define the *ladder high distribution* $H^-(x)$, concentrated on $\mathcal{R}_-$ as

$$H^-(x) = P(S_{\nu^-} \leq x), \quad x \in \mathcal{R}.$$

Similarly, let the *ladder high distribution* $H^+(x)$, concentrated on $(0, \infty)$ be

$$H^+(x) = P(S_{\nu^+} \leq x), \quad x \in \mathcal{R}.$$

As proposed in Rolski et al. [11], ladder high distributions $H^+(x)$ and $H^-(x)$ can be used to represent the distribution $D(x)$ of the sequence $Z_1, Z_2, \ldots$ Such an expression is given by the so-called Wiener-Hopf factorization. As the Wiener-Hopf factorization of D is frequently applied for working out the distribution of the maximum M of the random walk 4.9, we state it in the following theorem.

Theorem 4.2: The distribution function D can be expressed in terms of the ladder high distributions $H^+(x)$ and $H^-(x)$ as

$$D(x) = H^+(x) + H^-(x) - H^-(x) * H^+(x).$$

The Wiener-Hopf factorization leads to the following important lemma, as it is stated in Rolski et al. [11].

Lemma 4.3: If for some $z \in \mathcal{C}$ all the moment generating functions $\hat{m}_D(z)$, $\hat{m}_{H^+}(z)$ and $\hat{m}_{H^-}(z)$ exist, in particular if $Re(z) = 0$, then

$$1 - \hat{m}_D(z) = (1 - \hat{m}_{H^+}(z))(1 - \hat{m}_{H^-}(z)). \tag{4.14}$$

The expression 4.14 favors us to obtain the ruin probability for the risk process with random income.

Following the procedure from Temnov [12], but by using the moment generating function, we can state the relation between $\hat{m}_{H^+}(s)$ and $\hat{m}_{F_\nu}(s)$ as

$$\begin{aligned}\hat{m}_{H^+}(s) &= E\left(e^{s\,S_\nu}, \nu^+ < \infty\right) = P(\nu^+ < \infty) E\left(e^{s\,S_\nu} | \nu^+ < \infty\right) \\ &= (1-q)\hat{m}_{F_\nu}(s).\end{aligned}$$

At present we attempt to express the m.g.f. of Z_1, i.e. $\hat{m}_{Z_1}(s)$, by applying the moment generating functions of claims and premiums, i.e. $\hat{m}_{X_1}(s)$ and $\hat{m}_{Y_1}(s)$. Due to the independence of X_1 and I_1

$$\begin{aligned}\hat{m}_{Z_1}(s) &= Ee^{s\,Z_1} = Ee^{s(X_1 - I_1)} = E(e^{s\,X_1})\, E(e^{-s\,I_1}) \\ &= \hat{m}_{X_1}(s)\, \hat{m}_{I_1}(-s),\end{aligned}$$

where $\hat{m}_{I_1}(s)$ is the m.g.f of I_1. By using 4.3, we state $\hat{m}_I(-s)$ as

$$\hat{m}_I(-s) = Ee^{-s\,I_k} = E\left(-s \sum_{i=1}^{N_2(T_k^C)} Y_i + s \sum_{i=1}^{N_2(T_{k-1}^C)} Y_i\right)$$

Independence of $\{I_k\}$ leads to

$$Ee^{-s\,I_1} = E\left(e^{-s\,\sum_{i=1}^{N_2(T_1^C)} Y_i}\right) = \int_0^\infty E\left(e^{-s\,\sum_{i=1}^{N_2(t)} Y_i} \,|\, t\right) dU_1(t). \tag{4.15}$$

For fixed time t,

$$\begin{aligned}E\left(e^{-s\,\sum_{i=1}^{N_2(t)} Y_i} \,|\, t\right) &= \sum_{k=0}^{\infty} P\left(N_2(t) = k\right) E\left(e^{-s\,\sum_{1\le j\le k} Y_j} \,|\, N_2(t) = k\right) \\ &= \sum_{k=0}^{\infty} P\left(N_2(t) = k\right) E\left(e^{-s\,\sum_{1\le j\le k} Y_j}\right),\end{aligned}$$

because $N_2(t)$ is independent of Y_j. Due to the Poisson distribution of N_2,

$$\begin{aligned}E\left(e^{-s\,\sum_{i=1}^{N_2(t)} Y_i} \,|\, t\right) &= \sum_{k=0}^{\infty} e^{-\lambda_2\,t} \frac{(\lambda_2\,t)^k}{k!} \prod_{j=1}^{k} E(e^{-s\,Y_j}) \\ &= e^{-\lambda_2\,t} \sum_{k=0}^{\infty} \frac{(\lambda_2\,t)^k (E(e^{-s\,Y_1})^k)}{k!} \\ &= e^{-\lambda_2\,t(1-\hat{m}_Y(-s))}\end{aligned}$$

Therefore, for 4.15

$$\hat{m}_{I_1}(-s) = \int_0^\infty e^{-\lambda_2\, t(1-\hat{m}_Y(-s))}\, \lambda\, e^{-\lambda\, t}.$$

This integral converges if $(-\lambda_2\, t(1-\hat{m}_Y(-s)) - \lambda\, t)$ is negative or equals zero. Assuming that this condition holds, by calculating the integral above we get

$$\hat{m}_{I_1}(-s) = \frac{\lambda}{\lambda_2(1-\hat{m}_Y(-s)+\lambda)} = \frac{1}{1+\lambda_2/\lambda(1-\hat{m}_Y(-s))}.$$

Thus

$$\hat{m}_{Z_1}(s) = \frac{\hat{m}_{X_1}(s)}{1+\lambda_2/\lambda(1-\hat{m}_Y(-s))}.$$

By applying the geometric expansion,

$$\hat{m}_{Z_1}(s) = \frac{\hat{m}_X(s)}{1+\lambda_2/\lambda}\sum_{k=0}^{\infty}\left(\frac{\lambda_2/\lambda}{1+\lambda_2/\lambda}\right)^k \hat{m}_Y^k(-s). \tag{4.16}$$

Further we use the characteristics of the moment generating function. Specifically, the m.g.f. of a convolution of mutually independent r.vs X and Y equals multiplication of m.g.fs. of random variables X and Y. Written in symbols,

$$\hat{m}_{X+Y}(s) = \hat{m}_X(s)\ \hat{m}_Y(s).$$

Due to the above mention property of the moment generating function, when turning back to the distribution functions, 4.16 yields

$$D(x) = \frac{1}{1+\lambda_2/\lambda}\sum_{k=0}^{\infty}\left(\frac{\lambda_2/\lambda}{1+\lambda_2/\lambda}\right)^k F_X(x) * \overline{G_Y}(x), \tag{4.17}$$

where $\overline{G_Y}(x) = 1 - G_Y(-x-0)$.

In this moment, connection between 4.17 and 4.10 is needed. As shown in Feller [4], the following representation is valid for the characteristic function $\hat{f}_\nu(s)$ of the ladder heights distribution F_ν

$$\ln\frac{1}{1-(1-q)\hat{f}_\nu(s)} = \sum_{n=1}^{\infty}\frac{1}{n}\int_{0+}^{\infty} e^{i\,s\,x}\, dD^{*n}(x). \tag{4.18}$$

To summarize it up, 4.10, 4.17 and 4.18 must be combined to obtain the ruin probability in the risk process with random income.

Conclusion

The diploma thesis dealt with risk processes. At the beginning, a classical risk process was described. For such a process, the ruin probability was derived. Probability of ruin is connected to the solvency of an insurance company and this way also with its ability to fulfil its liabilities. Therefore evaluation of the ruin probability is one of the crucial tasks of a risk theory.

Naturally, if we do not have an exact formula for the ruin probability, we want to approximate it. Therefore, in the other part of the thesis, three approximations were studied. Their accuracy was illustrated in examples. As a whole, all the approximations served good results. However, they sometimes behaved irregularly and unpredictable.

Last, but not least, we studied a risk process with random income. Classical process was assumed to be a good approximation of the risk process with random income. The main aim of our thesis was to show, that investigating a more complicated model is worth it. We managed to show it in examples. To be more concrete, we demonstrated that ruin probability had been undervalued when calculated by using the classical risk process. Such a situation can be dangerous for an insurance company. Hence, it can ce concluded that the model simplifications changed the described reality.

Finally, we mentioned two expressions which assist when computing the ruin probability in the random income process. These expressions were obtained by applying the ladder height distribution. However, an explicit expression for ladder height distribution can be found only for few types of distributions of claims and payments. In general case it may be quite a challenging task.

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