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BACHELOR THESIS

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Complementarity Problems in Economics

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Supervisor: Mgr. Michal Červinka
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Here, I would like to thank to my supervisor, Mgr. Michal Červinka, for his patient leadership, motivation and support throughout writing my bachelor thesis and also for encouraging me when I got lost. I appreciate his valuable advice and comments. Besides this, I would like to apologize for a short time giving him to read over my thesis and for all the trouble which I caused by this unhappy action.

I hereby proclaim that I wrote my bachelor thesis on my own under the leadership of my supervisor and that the references include all the materials I have used. I agree with lending of this thesis and its explore.

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Abstract: In the thesis presented we deal with the concept of complementarity problems. We focus on what complementarity problems are with a emphasis on the simplest and most widely studied linear complementarity problem. We also provide a brief introduction on the study of nonlinear complementarity problems. Subsequently, we describe some models, such as mathematical programming, game theory etc., leading to complementarity problems. Besides this, we mention different problems as variational inequalities, the fixed point problem etc. which are under some assumption equivalent to complementarity problem. Finally, we devote practical applications of complementarity problems in economic and engineering.

This thesis should serve the purpose of a gentle introduction desire to pursue people understanding of this interesting subject.

Keywords: *complementarity problems, linear complementarity problems, nonlinear complementarity problems, variational inequalities, optimization, competitive equilibrium, traffic equilibrium problem, network optimum*

Název práce: Úlohy komplementarity v ekonomii
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Abstrakt: V předložené práci se zabýváme úlohami komplementarity. Zaměříme se na definici úloh komplementarity s důrazem na nejjednodušší a neprostudovanější lineární úlohy komplementarity. Také krátce poskytneme úvod o nelineárních úlohách komplementarity. Následně popíšeme modely, jako je matematické modelování, teorie her atd., vedoucí k úlohám komplementarity. Kromě toho zmíníme různé problémy jako jsou variační nerovnice, problém pevného bodu atd., které za jistých předpokladů jsou ekvivalentní s úlohami komplementarity. Nakonec se věnujeme aplikacím úloh komplementarity v ekonomii a inženýrství.

Tato práce by měla být dobrým začátkem pro všechny co se rozhodnou dovědět o tomto zajímavém předmětu více.

Klíčová slova: *úlohy komplementarity, lineární úlohy komplementarity, nelineární úlohy komplementarity, variační nerovnice, optimalizace, nekooperativní rovnováha, rovnováha dopravního systému, optimalizace dopravní sítě*

Contents

1 Introduction

Given a mapping $f : \mathbb{R}^n \to \mathbb{R}^n$, the problem of finding an $x \in \mathbb{R}^+$ such that $f(x) \in \mathbb{R}^n_+$ and $x^T f(x) = 0$ is called the complementarity problem. The importance of the complementarity problem lies in the fact that its form is fairly general in the sense that several problems in different fields such as mathematical programming, game theory, mechanics, engineering, economics, etc. can, by appropriate choice of function f, be so posed.

Special instances of the complementarity problem can be found in the mathematical literature as early as 1940, but the problem received little attention until the mid 1960's at which time it became an object of study on its own right. W. S. Dorn was the first one who introduced the complementarity problem as an independent problem since then the theory of the complementarity problem has known a strong development [7]. During the past three decades (1961 – 1991), a number of important results have been established dealing with both the computational and theoretical aspects of the above problem. The study of the complementarity problem entered into a new phase when B.C.Eaves [8] and S.Karamardian [15] showed that the complementarity problem is closely related to two other problems: the solution of variational inequalities and the determination of the fixed point for a given map. Thus, the existence theorems and the methods used in the study of the last two problems are widely used in complementarity theory and conversely, the ideas and the methods developed specially for complementarity problems are used to solve variational inequalities or to solve fixed point problems. Nowadays, the broad field of complementarity problems is enormous, a lot of new material is presented. Some of sources are stated in the references in the end of this thesis.

The organization of this bachelor thesis is as follows. In the next section we give some notations and set up some definitions and theorems which needed throughout the thesis. It is strongly recommended that the reader peruse this section first at initial reading, and refer to it whenever there is a question about the meaning of some symbols or terms. In section 3 we give the definition of complementarity problem and we focus on the linear complementarity problem with short mention about the nonlinear complementarity problems. We also give a list of examples of problems from other fields such as mathematical programming, game theory, which have as mathematical model a specific complementarity problem. Section 4 is devoted to the study of the important mathematical problems such as variational inequalities over a locally convex space, the least element problem, fixed point theory etc. which are equivalent to the complementarity problems. This study helps in obtaining many new methods to solve complementarity problems. Some interesting applications are described in section 5 and through this, the reader will have a good insight into the applicability of the theory of the complementarity problem in many practical situation, such as economic and engineering models.

Theorems used in this thesis are introduced without the proofs, however each theorem include reference to a book where reader can find the proof. This thesis is meant to be an introductory book that covers shortly a variety of topics. It is thus inevitable that some sections have been treated in less details than others. The aim of the thesis is to explain the given subject of complementarity problems even for readers who are not thoroughly familiar with the optimization theory and other similar fields of interest.

2 Preliminaries

In this section we specify some notions and give the definitions and the fundamental properties to mathematical terms that are used throughout this bachelor thesis.

By $\mathbb{R}^n$ we denote n-dimensional Euclidean space with $||x||$ being the Euclidean norm of a vector $x \in \mathbb{R}^n$. The nonnegative orthant of $\mathbb{R}^n$ is denoted by $\mathbb{R}^n_+$, i.e., $y \in \mathbb{R}^n_+$ is equivalent to $y_i \geq 0,\ i = 1, \ldots, n$. The cardinality of a set M, i.e., a measure of the number of elements of the set M we denote by $card(M)$. All vectors used in this thesis are column vectors unless otherwise indicated. The only exception to this rule will be gradients. If $F : \mathbb{R}^n \to \mathbb{R}$ is any differentiable function, then

$$\nabla f := [\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}]$$

denotes a row vector.

Definition 2.1 *Let $A = (a_{ij})$ be $m \times n$ matrix, then the $n \times m$ matrix $A^T = (b_{ij})$, where $b_{ij} = a_{ji}$ is called the* ***transpose*** *of the matrix A.*

Properties of the transpose operation. If A and B are $m \times n$ matrices and C is an $n \times k$ matrix, then

(1) $(A + B)^T = A^T + B^T$

(2) $(AC)^T = C^T A^T$

(3) $(A^T)^T$.

Definition 2.2 *Let A be a $m \times m$ (square) matrix . If the matrix A is equal to its transpose, then is called* ***symmetric****.*

Definition 2.3

(1) A matrix A is called ***positive semidefinite*** *if the matrix is symmetric and $x^T Ax \geq 0$ for all vectors x.*

(2) A matrix A is called ***positive definite*** *if the matrix is symmetric and $x^T Ax > 0$ for all non-zero vectors x.*

Let us suppose the following problem. This problem, defined by a mapping $f : \mathbb{R}^n \to \mathbb{R}^n$, is to find a $x \in \mathbb{R}^n$ such that

$$0 \leq x \perp f(x) \geq 0,$$

where we use the notion " $\perp$ " to signify that in addition to the stated inequalities $0 \leq x$ and $f(x) \geq 0$, the equation $x^T f(x) = 0$ also holds. In other words we say that x and $f(x)$ are orthogonal.

Definition 2.4 ***Banach space*** *is a vector space V over the real or complex numbers with a norm $||.||$ such that every Cauchy sequence (with respect to the metric $d(x, y) = ||x - y||$) in V has a limit in V. Since the norm induces a topology on the vector space, a Banach space provides an example of a* ***topological vector space****.*

If V is a Banach space the **dual space** of V is the space V^* of all bounded linear functionals on V. **Hilbert space** is a special case of Banach space.

Definition 2.5 *Let V be a real vector space. A subset $K \subset V$ is said to be a* ***convex cone*** *if the following conditions are satisfied:*

(1) $K + K \subset K$

(2) $(\forall \lambda \in \mathbb{R}_+)(\lambda K \subset K)$.

If $K \subset V$ is a **pointed convex cone**, that is, K is a convex cone and satisfies in addition, $K \cap (-K) = \{0\}$ then the "$\leq$" is **reflexive**.
If for the vector space V is defined a pointed convex cone $K \subset V$ then we say that (V, K) is an **ordered vector space**.
An ordered vector space (V, K) is said to be a **vector lattice** if in addition every non-empty finite subset of V has greatest lower bound.

3 Models Leading to Complementarity Problems

The complementarity theory is a new domain in applied mathematics and is concerned with the study of complementarity problems. The distinguishing feature of a complementarity problem is the set of *complementarity conditions.* Each of these conditions requires that the product of nonnegative quantities should be zero, where each quantity is either decision variable, or a function of the decisions variables.

These problems represent a wide class of mathematical models. They appear prominently in the study of equilibrium problems and naturally in numerous applications from economics, engineering, biology etc.

Our purpose is to describe what complementarity problems are and try to give a sense of which classes of problems can be fit into this framework. Proofs of theorems mentioned in this chapter are left out since they are either obvious or the reader can find them in most optimization books, e.g. [3], [16], [17].

Definition 3.1 *Given a function $f : \mathbb{R}^n \to \mathbb{R}^n$, problem to find a vector $x \in \mathbb{R}^n_+$ such that, $f(x) \in \mathbb{R}^n_+$ and $x \perp f(x)$, is called the* ***complementarity problem****.*

The simplest and most widely studied of the complementarity problems is the *Linear Complementarity Problem* (LCP). The LCP is complementarity problem with a linear function f. The LCP unifies linear and quadratic programming program and bimatrix games. Iterative methods developed for solving LCPs hold great promise for handling very large scale linear programs which cannot be solved with the simplex method because of their large size and the consequent numerical difficulties. For these reasons the study of LCPs offers rich rewards for people learning or doing research in optimization or engaged in practical applications of optimization.

Given $M \in \mathbb{R}^{n\times n}, q \in \mathbb{R}^n$ the LCP can be formulated in the following form:

$$\begin{aligned} &\text{find } w \in \mathbb{R}^n, \ z \in \mathbb{R}^n \text{ satisfying} \\ &w - Mz = q, \\ &0 \leq w \perp z \geq 0. \end{aligned} \tag{1}$$

We observe that there is no objective function in an LCP to be optimized. The only data in the problem is the column vector q and the square matrix M. We will denote the LCP by the symbol (q, M). It is said to be an LCP of *order* n.

In an LCP of order n there are $2n$ variables. The LCP (q, M) is said to be *monotone* if the matrix M is positive semidefinite.

Example 3.1

Let $n = 2$, $M = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, $q = \begin{bmatrix} -5 \\ -6 \end{bmatrix}$. This leads to the LCP

$$\begin{aligned} w_1 - 2z_1 - z_2 &= -5, \\ w_2 - z_1 - 2z_2 &= -6, \\ w_1 z_1 &= 0, \\ w_2 z_2 &= 0, \\ w_1, \ w_2, \ z_1, \ z_2 &\geq 0. \end{aligned} \tag{2}$$

The problem (??) can be expressed in the form of a vector equation as

$$w_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + w_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + z_1 \begin{bmatrix} -2 \\ -1 \end{bmatrix} + z_2 \begin{bmatrix} -1 \\ -2 \end{bmatrix} = \begin{bmatrix} -5 \\ -6 \end{bmatrix} \tag{3}$$

$$w_1 z_1 = w_2 z_2 = 0 \text{ and } w_1,\ w_2,\ z_1,\ z_2 \geq 0 \tag{4}$$

In any solution satisfying (??), at least one of the variables in each pair (w_j, z_j), $j = 1, 2$, has to equal zero. One approach for solving this problem is to pick one variable from each of the pairs $(w_1, z_1), (w_2, z_2)$ and to fix them at zero value in (??). The remaining variables in the system may be called *usable variables.* After eliminating the zero variables from (??), if the remaining system has a solution in which the usable variables are nonnegative, that would provide a solution to (??) and (??). Pick w_1, w_2 as the zero-valued variables. After setting w_1, w_2 equal to 0 in (??), the remaining system is

$$z_1 \begin{bmatrix} -2 \\ -1 \end{bmatrix} + z_2 \begin{bmatrix} -1 \\ -2 \end{bmatrix} = \begin{bmatrix} -5 \\ -6 \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = q \tag{5}$$

$$z_1 \geq 0,\ z_2 \geq 0$$

It means that equation (??) has a solution if the vector q can be expressed as a nonnegative linear combination of the vectors $(-2, -1)^T$ and $(-1, -2)^T$. The set of all nonnegative linear combinations of $(-2, -1)^T$ and $(-1, -2)^T$ is a cone. Only if the given vector $q = (-5, -6)^T$ lies in this cone, does the LCP have a solution in which the usable variables are z_1, z_2. We verify that the point $(-5, -6)^T$ does lie in the cone, that the solution of (??) is $(z_1, z_2) = (\frac{4}{3}, \frac{7}{3})$ and hence a solution for (??) is $(w_1, w_2, z_1, z_2) = (0, 0, \frac{4}{3}, \frac{7}{3})$.

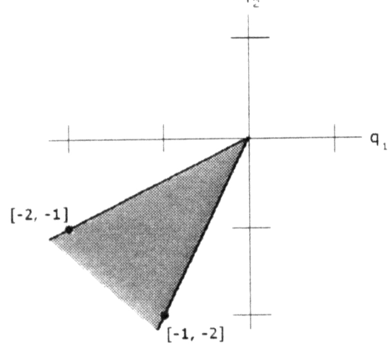

Figure 1: A complementary cone associated with the LCP (??)

Problem to find $x \in \mathbb{R}^n$ such that for a given nonlinear function $F : \mathbb{R}^n \to \mathbb{R}^n$ hold $0 \leq x \perp F(x) \geq 0$, is called the *nonlinear complementarity problem* (NLCP). If F is a linear function then the NLCP becomes the LCP. Thus, the LCP is a special case of the NCP. Other than this, we will not discuss any detailed results on NLCP, but the interested reader can find them in [14].

3.1 Linear Programming

In this subsection that is based on [16] and [17] we deal with the linear programming program (LPP) and its important properties. A linear programming problem may be defined as the problem of minimizing or maximizing a linear function subject to linear constraints. The constraints can be equalities or inequalities.

Let $b=(b_i) \in \mathbb{R}^I, c=(c_j) \in \mathbb{R}^J$ be vectors and let $A=(a_{ij}) \in \mathbb{R}^{I \times J}$ be a matrix. For $I_1, I_2, I_3 \subset I$ let $I_1 \cup I_2 \cup I_3 = I; I_1 \cap I_2 \cap I_3 = \emptyset$; $card(I) = m$ and for $J_1, J_2, J_3 \subset J$ let $J_1 \cup J_2 \cup J_3 = J; J_1 \cap J_2 \cap J_3 = \emptyset$; $card(J) = n$. Let us consider the following LPP:

$$\begin{array}{rl} \text{minimize} & c^T x \\ \text{subject to} & A_{I_1 \times J}\, x \geq b_{I_1}, \\ & A_{I_2 \times J}\, x \leq b_{I_2}, \\ & A_{I_3 \times J}\, x = b_{I_3}, \\ & x_{J_1} \geq 0;\ x_{J_2} \leq 0;\ x_{j_3} \in \mathbb{R}. \end{array}$$

Definition 3.2

(1) *For $I_1=\emptyset$, $I_2=\emptyset$, $J_2=\emptyset$, $J_3=\emptyset$ the LPP is called the* ***standart LPP****.*

(2) *For $I_2=\emptyset$, $I_3=\emptyset$, $J_2=\emptyset$, $J_3=\emptyset$ or $I_1=\emptyset$, $I_3=\emptyset$, $J_2=\emptyset$, $J_3=\emptyset$ the LPP is called the LPP in the* ***form of inequalities****.*

(3) *For the all non considered cases the LPP is called the* ***mixed LPP****.*

Definition 3.3

(1) *The function to be minimized or maximized is called the* ***objective function****.*

(2) *A vector x is said to be* ***feasible*** *if it satisfies the corresponding constraints. The set of feasible vectors is called the* ***set of feasible solutions****. A LPP is said to be* ***feasible*** *if the set of feasible solutions in not empty, otherwise it is said to be* ***infeasible****.*

(3) *A feasible minimum (maximum) is said to be* ***unbounded*** *if the objective function can assume arbitrarily large negative (positive) values at feasible vectors, otherwise, it is said to be* ***bounded****. The value of a bounded feasible minimum (maximum) is the minimum (maximum) value of the objective function as the variables range over the set of feasible solutions. A feasible vector at which the objective function achieves the value is called* ***optimal****.*

LPPs in standard form can be rewritten into an equivalent problem in form of inequalities or mixed form. LPPs in form of inequalities can be rewritten into an equivalent problem in standard form or mixed form. Finally, LPPs in mixed form can be rewritten into an equivalent problem in standard form or form of inequalities. The changes can be achieved by the following set of rules.

(1) A minimization problem can be converted into a maximization problem by multiplying the objective function by -1. Similarly, constraints of the form $\sum_{j\in J} a_{ij}x_j \geq b_i$ can be converted into the form $\sum_{j\in J}(-a_{ij})\, x_j \leq -b_i$.

(2) An unrestricted variable x_j may be replaced by the difference of two nonnegative variables, $x_j = u_j - v_j$, where $u_j \geq 0$ and $v_j \geq 0$. This adds one variable and two nonnegativity constraints to the problem.

(3) An equality constraint $\sum_{j\in J} a_{ij}x_j = b_i$ may be replaced by two inequality constraints $\sum_{j\in J} a_{ij}x_j \geq b_i$ and $\sum_{j\in J} a_{ij}x_j \leq b_i$.

(4) Inequality constraints $\sum_{j\in J} a_{ij}x_j \geq b_i$, $\sum_{j\in J} a_{ij}x_j \leq b_i$ may be replaced by $\sum_{j\in J} a_{ij}x_j - v = b_i$, $\sum_{j\in J} a_{ij}x_j + v = b_i$, respectively, where the slack variable v is nonnegative.

To every linear program there is a dual linear program with which it is intimately connected. Duality is one of the most important concepts in linear programming, it allows to provide a proof of optimality.

Let us state this duality for the LPP in the form of inequalities. Any theory derived for the LPP in the form of inequalities is applicable to the LPP in standard form or mixed form.

Definition 3.4 *For $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, $A \in \mathbb{R}^{m\times n}$ and the primal LPP*

$$\begin{array}{rl} \textit{minimize} & c^T x \\ \textit{subject to} & Ax \geq b \\ & x \geq 0, \end{array} \tag{6}$$

the ***dual problem*** *is defined to be the LPP*

$$\begin{array}{rl} \textit{maximize} & b^T y \\ \textit{subject to} & A^T y \leq c \\ & y \geq 0. \end{array} \tag{7}$$

The LPP and the corresponding dual LPP may be simultaneously exhibited in the following table.

		x_1	x_2	$\dots$	x_n	
		$\geq$	$\geq$	$\dots$	$\geq$	
y_1	$\geq$	a_{11}	a_{12}	$\dots$	a_{1n}	$\geq b_1$
y_2	$\geq$	a_{21}	a_{22}	$\dots$	a_{2n}	$\geq b_2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$ $\vdots$
y_m	$\geq$	a_{m1}	a_{m2}	$\dots$	a_{mn}	$\geq b_m$
		$\leq$	$\leq$	$\dots$	$\leq$	
		c_1	c_2	$\dots$	c_n	

If the table is written in rows (columns), we obtain the primal LPP (the dual LPP). For details, see [16].

The relation between the LPP and its dual is seen from the following theorems.

Theorem 3.1 ***(Weak Duality)****[16, 2.29]*
If x is feasible for the LPP (??) and if y is feasible for its dual (??), then $c^T x \geq b^T y$.

Theorem 3.2 ***(Duality)****[16, 2.30]*
If there exists a feasible x and a feasible y for the LPP (??) and its dual (??) such that $c^T x = b^T y$, then x is optimal solution of (??) and y is optimal solution of (??).

Theorem 3.3 ***(Strong Duality)****[16, 2.31]*
If the LPP (??) is bounded feasible, then so is its dual, their values are equal, and there exist optimal vectors for both problems.

Applying the Theorem ?? we can arrive at a linear complementarity problem (LCP), which is equivalent to the couple primal-dual of the LPP (??) and (??).

Let us add slack variables $u \in \mathbb{R}^n$ and $v \in \mathbb{R}^m$ such that, $Ax - v = b$ and $A^T y + u = c$. The vectors u and v together satisfying

$$\begin{bmatrix} u \\ v \end{bmatrix} - \begin{bmatrix} 0 & -A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c \\ -b \end{bmatrix}$$

$$\begin{bmatrix} u \\ v \end{bmatrix} \geq 0, \begin{bmatrix} x \\ y \end{bmatrix} \geq 0 \text{ and } \begin{bmatrix} u \\ v \end{bmatrix}^T \begin{bmatrix} x \\ y \end{bmatrix} = 0.$$

If we denote $z := \begin{bmatrix} x \\ y \end{bmatrix}$, $w := \begin{bmatrix} u \\ v \end{bmatrix}$, $d := \begin{bmatrix} c \\ -b \end{bmatrix}$ and $M := \begin{bmatrix} 0 & -A^T \\ A & 0 \end{bmatrix}$, then we obtain the linear complementarity problem:

$$\begin{aligned} &\text{find } z \in \mathbb{R}^{n+m} \text{ such that,} \\ &0 \leq z \perp w = Mz + d \geq 0. \end{aligned}$$

It is natural, after modelling of linear programming, to consider problems which do not fit into the framework of this technique, because either the constraints, or objective function, cannot be formulated in linear terms. We will deal with this type of problems in the next section.

3.2 Nonlinear Programming

The nonlinear programming problem that will concern us has three fundamental ingredients: a finite number of real variables, a finite number of constraints which the variables must satisfy, and a function of the variables which must be minimized or maximized. Mathematically speaking we can state the problem as follows:

$$\begin{aligned} \text{minimize} \quad & f(x) \\ \text{subject to} \quad & g_i(x) \leq 0, \ i = 1, \dots, m \\ & h_j(x) = 0, j = 1, \dots, k \\ & x \in \mathbb{R}^n, \end{aligned} \tag{8}$$

where f, g_i, $i = 1, \dots m$, h_j, $j = 1, \dots k$, are generally nonlinear continuously differentiable functions from $\mathbb{R}^n$ to $\mathbb{R}$.

An important question is related to the existence of optimal solutions for this problem. We already know that even linear programming programs may not have optimal solutions. This is also true for nonlinear programming program. A typical result ensuring the existence of optimal solutions is based on the continuity of the functions involved in such a problem. In many practical settings the following theorem is enough to ensure the existence of optimal solution.

Theorem 3.4 (3, 3.3.1) *Assume that f, g_i, $i = 1, \dots m$, h_j, $j = 1, \dots k$, are continuous functions and one of the following situations hold:*

(1) the set of feasible vectors x is a bounded set in $\mathbb{R}^n$,

(2) the set of feasible vectors x is not bounded, but function f is ***coercive****, e.i.,*

$$\lim_{|x| \to \infty,\ g(x) \leq 0,\ h(x) = 0} f(x) = \infty.$$

Then the associated minimization problem admits at least one solution.

To find an optimal solution of the problem (**??**) is one of the most important problem in optimization. Further, we will write the conditions that a vector must satisfy so that it can possibly be an optimal solution for the problem (**??**). Now, we need to write some definitions. Let $h : \mathbb{R}^n \to \mathbb{R}^k$ and $g : \mathbb{R}^n \to \mathbb{R}^m$ denote function such that $h(x) = (h_1(x), \dots, h_k(x))^T$ and $g(x) = (g_1(x), \dots, g_m(x))^T$.

Definition 3.5 *A vector $x_0 \in \mathbb{R}^n$ is a* ***local minimum*** *of function f subject to $g(x) \leq 0$, $h(x) = 0$ if $g(x_0) \leq 0$, $h(x_0) = 0$, and $f(x_0) \leq f(x)$ for all x such that $g(x) \leq 0$, $h(x) = 0$, $|x - x_0| < \epsilon$, for some $\epsilon < 0$. A vector x_0 is a* ***global minimum*** *of function f subject to $g(x) \leq 0$, $h(x) = 0$ if $g(x_0) \leq 0$, $h(x_0) = 0$, and $f(x_0) \leq f(x)$ for all x such that $g(x) \leq 0$, $h(x) = 0$.*

Definition 3.6 *A feasible vector x of the problem (??) is said to be* ***linear independence constraint qualification*** *(LICQ) if the equality constraint gradients $\nabla h_j(x)$, $j = 1, \dots, k$, and the inequality constraint gradients $\nabla g_i(x)$, $i \in A(x)$, where $A(x) = \{i | \ g_i(x) = 0\}$ are linearly independent.*

Theorem 3.5 *(Karush-Kuhn-Tucker necesary conditions)[3, 3.3.2]*
Let $\hat{x}$ be a local minimum of the problem (??) and assume that $\hat{x}$ is LICQ. Then there exist unique vectors $\hat{\lambda} \in \mathbb{R}^k$, $\hat{\mu} \in \mathbb{R}^m$ such that

$$\begin{aligned} &\nabla_x L(\hat{x}, \hat{\lambda}, \hat{\mu}) = 0 \\ &\hat{\mu}_i \geq 0,\ i = 1, \ldots, m \\ &\hat{\mu}_i = 0,\ \forall i \notin A(\hat{x}). \end{aligned}$$

If in addition f, g_i, $i = 1, \ldots m$, h_j, $j = 1, \ldots k$, are twice continuously differentiable, there holds

$$y^T \nabla^2_{xx} L(\hat{x}, \hat{\lambda}, \hat{\mu}) y \geq 0, \quad \forall y \in \mathbb{R}^n$$

such that $\quad \nabla h_j(\hat{x})^T y = 0,\ \forall j = 1, \ldots k,\ \nabla g_i(\hat{x})^T y = 0,\ \forall i \in A(\hat{x}).$

Definition 3.7 *The vectors $\hat{\lambda}$, $\hat{\mu}$ in (??) are called* ***Lagrange multiplicators*** *and the function $L(\hat{x}, \hat{\lambda}, \hat{\mu}) := f(x) + \sum_{j=1}^{k} \lambda_j h_j(x) + \sum_{i=1}^{m} \mu_i g_i(x)$ is called the* ***Lagrange function***.

Let us now consider the following nonlinear programming program:

$$\begin{aligned} \text{minimize} \quad & f(x) \\ \text{subject to} \quad & g_i(x) \leq 0,\ i = 1, \ldots, m \\ & x \in \mathbb{R}^n_+. \end{aligned}$$

We suppose that functions f, g_i, $i = 1, \ldots, m$, are convex and differentiable. The Lagrange function $L(x, \lambda)$ can be written as

$$L(x, \lambda) = f(x) + \sum_{i=1}^{m} \lambda_i g_i(x).$$

The Karush-Kuhn-Tucker necessary conditions can be written as

$$\begin{aligned} \frac{\partial L(x, \lambda)}{\partial x_j} &= h_j(x, \lambda) \geq 0,\ j = 1, \ldots, n, \\ -\frac{\partial L(x, \lambda)}{\partial \lambda_i} &= h_{n+i}(x, \lambda) \geq 0,\ i = 1, \ldots, m, \\ \sum_{j=1}^{n} x_j h_j(x, \lambda) = 0,\ & \sum_{i=1}^{m} \lambda_i h_{n+i}(x, \lambda) = 0, \\ x \geq 0,\ & \lambda \geq 0. \end{aligned} \tag{9}$$

If we denote $z := \begin{bmatrix} x \\ \lambda \end{bmatrix}$, $h(z) := [h_1(z), \ldots, h_n(z), \ldots, h_{n+m}(z)]^T$, then the Kuhn Tucker condition (??) may be stated as the following complementarity problem:

$$\begin{aligned} &\text{find } z \in \mathbb{R}^{n+m} \text{ such that,} \\ &0 \leq z \perp h(z) \geq 0. \end{aligned}$$

3.3 Quadratic Programming

In this section we turn our focus to the class of quadratic programming program (QPP), which is a special subclass of mathematical optimization problem. It involves optimization of a quadratic objective function subject to linear equality and inequality constraints. The QPP is a very useful tool in optimization since it forms the basis of several general nonlinear programming algorithms.

The QPP can be formulated as:

$$\begin{aligned} \text{minimize} \quad & c^T x + \tfrac{1}{2} x^T Q\, x \\ \text{subject to} \quad & Ax \leq b, \\ & x \geq 0, \end{aligned}$$

where $c \in \mathbb{R}^n, b \in \mathbb{R}^m$ are vectors, $A \in \mathbb{R}^{m \times n}$ is a matrix and $Q \in \mathbb{R}^{n \times n}$ is a symmetric matrix. There is no loss of generality in assuming that Q is a symmetric matrix, because if it is not symmetric replacing Q by $(Q + Q^T)/2$ (which is a symmetric matrix) leaves $f(x)$ unchanged.

Let $\lambda \in \mathbb{R}^m$ and $\mu \in \mathbb{R}^n$ denote *Lagrange multipliers* of the constraints $Ax \leq b$ and $x \geq 0$ respectively. Then the *Lagrange function* for the QPP is given by

$$L(x, \lambda, \mu) = c^T x + \frac{1}{2} x^T Q x + \lambda^T (ax - b) - \mu^T x.$$

The *Karush-Kuhn-Tucker optimality conditions* (KKT) for the QPP are given by

$$\begin{aligned} & c + Qx + A^T \lambda - \mu = 0, \\ & \lambda^T (Ax - b) = 0, \ \mu^T x = 0, \\ & Ax - b \leq 0, \ x \geq 0, \\ & \lambda \geq 0, \ \mu \geq 0. \end{aligned}$$

Let us denote $v := b - Ax$, then the KKT conditions can be written as:

$$\begin{bmatrix} \mu \\ v \end{bmatrix} - \begin{bmatrix} Q & A^T \\ -A & 0 \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} = \begin{bmatrix} c \\ b \end{bmatrix}$$

$$\begin{bmatrix} \mu \\ v \end{bmatrix} \geq 0, \begin{bmatrix} x \\ \lambda \end{bmatrix} \geq 0 \ \text{and} \ \begin{bmatrix} \mu \\ v \end{bmatrix}^T \begin{bmatrix} x \\ \lambda \end{bmatrix} = 0.$$

If we denote $z := \begin{bmatrix} x \\ \lambda \end{bmatrix}$, $h(z) := \begin{bmatrix} \mu \\ v \end{bmatrix}$, $g := \begin{bmatrix} c \\ b \end{bmatrix}$ and $M := \begin{bmatrix} Q & A^T \\ -A & 0 \end{bmatrix}$, we obtain the linear complementarity problem:

$$\begin{aligned} & \text{find } z \in \mathbb{R}^{n+m} \text{ such that,} \\ & 0 \leq z \perp Mz + q \geq 0. \end{aligned}$$

If Q is a positive semidefinite matrix, then the quadratic function is a convex function. Moreover, the problem has a local minimum which is also the global minimum. If the matrix Q is positive definite then this global minimum is even unique.

If Q is zero, then the QPP becomes a linear program. The quadratic minimization problem with a convex objective function is said to be a *convex quadratic problem.* If there are only equality constraints, then the QPP can be solved by a linear system. Otherwise, a variety methods for solving the QPP are commonly used, including interior point and conjugate gradient method.

A feasible solution x for the QPP, is said to be a *Karush-Kuhn-Tucker point* (KKT point) if there exist Lagrange multiplier vectors λ, μ, such that x, λ, μ together satisfy the KKT conditions. In other words, the LCP corresponding to the QPP is the problem of finding a KKT point for the QPP.

We now have the following results.

Theorem 3.6 (17, 1.13) *If x is an optimum solution of QPP, then x must be a KKT point of this QPP, regardless of convexity of the quadratic objective function.*

Theorem 3.7 (17, 1.14) *If Q is a positive definite matrix and x is a KKT point of QPP, then x is an optimum solution of QPP.*

Now we consider a more general case of the quadratic program in the form:

$$\begin{array}{ll} \text{minimize} & c^T x + \frac{1}{2} x^T Q\, x \\ \text{subject to} & Ax \geq b, \\ & \quad x \in \mathbb{R}^n, \end{array} \tag{10}$$

where $c \in \mathbb{R}^n, b \in \mathbb{R}^m, A \in \mathbb{R}^{m\times n}$ and $Q \in \mathbb{R}^{n\times n}$ is a symmetric, positive definite matrix. The dual of (??) is given by:

$$\begin{array}{ll} \text{maximize} & b^T u - \frac{1}{2} x^T Q\, x \\ \text{subject to} & Qx - A^T u + c = 0, \\ & \quad u \geq 0,\ x \in \mathbb{R}^n. \end{array} \tag{11}$$

If we substitute x in (??) by expression $x = Q^{-1}(A^T u - c)$, we obtain the following equivalent problem:

$$\begin{array}{ll} \text{maximize} & -\frac{1}{2} u^T (AQ^{-1} A^T u) + (b + AQ^{-1} c)^T u \\ \text{subject to} & \quad u \geq 0. \end{array} \tag{12}$$

The matrix $AQ^{-1}A^T$ is a positive semidefinite so the problem (??) can be rewritten as a symmetric linear complementarity problem:

$$\begin{array}{l} \text{find } u \in \mathbb{R}^m \text{ such that,} \\ 0 \leq u \perp AQ^{-1} A^T u - (b + AQ^{-1}) \geq 0. \end{array}$$

3.4 Variational Inequalities

The subject of variational inequalities has its origin in the calculus of variations associated with the minimization of infinite dimensional functionals. The theory of variational inequalities was introduced by Hartman and Stampacchia (1966) in [12] as a tool for the study of partial differential equations with applications principally drawn from mechanics. To date variational inequalities provides a broad unifying setting for optimization and equilibrium problems in the study of traffic network, oligopolistic market, migration etc. For a comprehensive treatment of variational inequalities we refer the reader to [?].

Definition 3.8 *Given a continuous mapping $f : K \rightarrow \mathbb{R}^n$, where $K \subset \mathbb{R}^n$ be a closed convex set, the* ***variational inequality problem****, denoted $VI(f, K)$, is to find a vector $x \in K$ such that $f(x)^T(y - x) \geq 0,\ \forall y \in K$.*

In geometric terms, the variational inequalities state that a point x in the set K is a solution of the variable inequalities if and only if $f(x)$ forms a non-obtuse angle with every vector of the form $y - x$ for all y in K.

Many mathematical problems can be formulated as variational inequality problems, and several examples applicable to equilibrium analysis are mentioned below.

System of Equations

Economic equilibrium problems have been formulated as systems of equations, since market clearing conditions necessarily equate the total supply with the total demand. In terms of a variational inequality problem, the formulation of a system of equalities is as follows.

Theorem 3.8 (18,1) *Let $K = \mathbb{R}^n$ and let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a given function. A vector $x \in \mathbb{R}$ solves $VI(f, \mathbb{R}^n)$ if and only if $f(x) = 0$.*

Note that systems of equations preclude the introduction of inequalities, which may be needed, for example, in the case of nonnegativity assumptions on certain variables such as prices.

Optimization Problems

Both unconstrained and constrained optimization problems can be formulated as variational inequality problem. The following theorem identify the relationship between an optimization problem and a variational inequality problem.

Theorem 3.9 (18,2) *Let x_0 be a solution to the optimization problem:*

$$\begin{aligned} &\textit{minimize}\ \ f(x) \\ &\textit{subject to}\ \ x \in K, \end{aligned} \tag{13}$$

where f is continuously differentiable and K is closed and convex. Then x_0 is a solution of the variational inequality problem:

$$\nabla f(x_0)^T(x - x_0) \geq 0,\ \forall x \in K.$$

Theorem 3.10 (18, 3) *If $f(x)$ is a convex function and x_0 is a solution to $VI(\nabla f, K)$, then x_0 is a solution to the optimization problem (??).*

If the feasible set $K = \mathbb{R}^n$, then the unconstrained optimization problem is also a variational inequality problem. The variational inequality problem also contains the complementarity problem as a special case. In case that the variational inequality formulation is characterized by a function with a symmetric Jacobian, then the variational inequality problem can be reformulated as an optimization problem.

3.5 Game Theory

Game theory provides mathematical models for situations where the interests of several individuals play a role. It was the book of von Neumann and Morgenstern (1944) that started game theory. Nowadays game theory has applications in economics, biology, social and political science.

The several branches of game theory can roughly be divided into the theories of *cooperative* and *noncooperative* games. In both theories a modelled situation is called a *game* and the involved individuals are called *players*. In general the players are understood to be perfectly rational, i.e. each player is concerned with doing as well for himself as possible, subject to rules and possibilities imposed by the model. It is the aim of game theory to search for *solutions* of games, enforced by the self-interest of the players.

Cooperative games theory focuses on possible cooperation of the players. In noncooperative games no cooperation is allowed. Each player must choose a strategy from a set of possible strategies. This must happen independent of the other players.

The next part of this section is denoted to bimatrix games. We show that bimatrix games are equivalent to the linear complementarity problems.

Bimatrix Games

A bimatrix game is a strategic confrontation of two players I and II. A bimatrix game can be, for example, chess, checkers, etc. Each player has a finite number of strategies, commonly called *pure strategies*. Player I has to choose between n pure strategies, while player II has to choose between m pure strategies. A bimatrix game is described through a pair of $n \times m$ payoff matrices A_1 and A_2. The elements a_{ij1} and a_{ij2} of the matrices A_1 and A_2 are respectively the immediate payoffs of player I and player II when the first plays his ith strategy while the second simultaneously plays his jth strategy. Each player attempts to maximize his own payoff by selecting a probability vector over his set of pure strategies. These vectors are combinations of pure strategies, called *mixed strategies* and represented by probability vector $x_1 \in \mathbb{R}^n$ and $x_2 \in \mathbb{R}^m$ such that $x_1 \geq 0$, $x_2 \geq 0$ and $e^T x_1 = 1$, $e^T x_2 = 1$, where e denotes the unit vector (each component of e is equal to 1). Hence, player I's payoff is $x_1^T A_1 x_2$ and player II's payoff is ${x_1}^T A_2 x_2$.

Mathematically described, the first player solves the program:

$$\begin{aligned} \text{maximize} \quad & {x_1}^T A_1 x_2 \\ \text{subject to} \quad & \sum_{i=1}^{n} x_1^i = 1, \\ & x_1 \geq 0 \end{aligned} \tag{14}$$

and the second player solves the program:

$$\begin{aligned} \text{maximize} \quad & {x_1}^T A_2 x_2 \\ \text{subject to} \quad & \sum_{j-1}^{m} x_2^j = 1, \\ & x_2 \geq 0. \end{aligned} \tag{15}$$

Let (A_1, A_2) stands for this game and X^1 and X^2 denotes the set of mixed strategies for player I and player II respectively.

Definition 3.9 *A pair of mixed strategies $x_1^0, x_2^0 \in X^1 \times X^2$ is said to be* ***Nash equilibrium point*** *for game (A_1, A_2) if and only if*

(1) ${x_1^0}^T A_1 x_2^0 \geq {x_1}^T A_1 x_2^0 \quad \forall x_1 \in X^1$, and

(2) ${x_1^0}^T A_2 x_2^0 \geq {x_1^0}^T A_2 x_2 \quad \forall x_2 \in X^2$.

In other words, a Nash equilibrium point is a pair of strategies that do not motivate any one of the players to change his/her strategy as long as the other stay with his/her strategy. In addition Nash proved that every bimatrix game admits a Nash equilibrium point.

We show that the Definition ?? is equivalent to the following pair of inequalities:

$$\begin{aligned} {x_1^0}^T A_1 x_2^0 e &\geq A_1 x_2^0, \\ {x_1^0}^T A_2 x_2^0 e &\geq A_2^T x_1^0, \end{aligned} \tag{16}$$

where e denotes the unit vector (each component of e is equal to 1). For the vector e_j with all zeros except for 1 in the jth element from the Definition ?? we have,

$$\begin{aligned} {x_1^0}^T A_1 x_2^0 &\geq {e_i}^T A_1 x_2^0, \quad && \forall i = 1, \ldots, n, \\ {x_1^0}^T A_2 x_2^0 &\geq x_1^0 A_2 e_j, \quad && \forall j = 1, \ldots, m, \end{aligned} \tag{17}$$

and we observe that (??) implies (??).
Conversely, if (??) holds and x_1, x_2 are mixed strategies then we deduce,

$$\begin{aligned} {x_1^0}^T A_1 x_2^0 \, {x_1}^T e &\geq {x_1}^T A_1 x_2^0, \\ {x_1^0}^T A_2 x_2^0 \, e^T x_2 &\geq {x_1^0}^T A_2 x_2 \end{aligned} \tag{18}$$

and since ${x_1}^T e = 1$, ${x_2}^T e = 1$, we observe that the Definition ?? holds.

Now, we prove that the bimatrix game (A_1, A_2) is equivalent to the following linear complementarity problem:

$$\begin{aligned} &\text{find } X \in \mathbb{R}^{n+m} \text{ such that,} \\ &0 \leq X \perp q + MX \geq 0, \end{aligned} \tag{19}$$

where $X = \begin{bmatrix} x^1 \\ x^2 \end{bmatrix}$, $q = \begin{bmatrix} -e \\ -e \end{bmatrix}$ and $M = \begin{bmatrix} 0 & -A_1 \\ -A_2{}^T & 0 \end{bmatrix}$.

Let (x_1, x_2) be a solution of (??). Then $x_1 \in \mathbb{R}^n_+$, $x_2 \in \mathbb{R}^m_+$,

$$A_1 x_2 \leq -e, \ A_2^T x_1 \leq -e, \tag{20}$$

$$x_1{}^T A_1 x_2 = -e^T x_1, \ x_1{}^T A_2 x_2 = -e^T x_2. \tag{21}$$

Since $x_i \neq 0, \ i = 1, 2$ then $e^T x_i > 0, \ i = 1, 2$ and we can denote

$$x_i^0 := \frac{x_i}{e^T x_i} \quad i = 1, 2. \tag{22}$$

Now, we prove that x_i^0, $i = 1, 2$ satisfy (??).
Using, (??) from (??) and (??) we deduce,

$$\begin{aligned} &A_1 x_2^0 \leq \frac{-e}{e^T x_2}, \ A_2{}^T x_1^0 \leq \frac{-e}{e^T x_1} \\ &x_1^{0T} A_1 x_2^0 = -\frac{1}{e^T x_2}, \ x_1^{0T} A_2 x_2^0 = -\frac{1}{e^T x_1} \end{aligned}$$

which imply,

$$A_1 x_2^0 \leq x_1^{0T} A_1 x_2^0 e \ \text{ and } \ A_2{}^T x_1^0 \leq x_1^{0T} A_2 x_2^0 e \tag{23}$$

that (??) holds.
Conversely, we already know if x_1^0, x_2^0 satisfy (??) then Definition ?? holds. We can show that

$$x_1 = -\frac{x_1^0}{x_1^{0T} A_2 x_2^0}, \ x_2 = -\frac{x_2^0}{x_1^{0T} A_1 x_2^0}$$

is a solution of (??).

4 Equivalences

4.1 Basic Theorems

Complementarity problems have under certain conditions close relationship with several problems such as least element problems, variational inequality problems, fixed point problems, nonlinear programming problems. The purpose of this section is to study the equivalences in a Banach space. We will show that the complementarity problem can be considered as equivalent form of the variational inequality problem and we will prove that the complementarity problem is equivalent to a fixed point problem.

Let us start with the *Hartman-Stampacchia variational inequality*, which was first study by Hartman and Stampacchia in finite dimensional spaces and then it was generalized by Browder to the case of infinite dimensional space and was called the *Browder-Hartman-Stampacchia variational inequality* and the results concerning this variational inequality have been applied to many important problems, i.e., mechanics, control theory, game theory, mathematical economics. Recently, the Browder-Hartman-Stampacchia variational inequality was extended to the case of set-valued monotone mappings in reflexive Banach spaces.

If it is not said otherwise, in whole this section E stands for a Banach space and E^* stands for the topological dual of E.

Theorem 4.1 *(Hartman Stampacchia)[13, 3.1]*
Let C be a compact convex subset of a locally convex space $E(\tau)$ and let $f : C \to E^*$ be continuous. Then there exist $y \in C$ such that for every $x \in C$, $(x - y)^T f(x) \geq 0$.

Theorem 4.2 *(Browder-Hartman-Stampacchia)[13,3.2]*
Consider a reflexive Banach space E, $f : E \to E^*$ a monotone hemicontinuous operator and $K \subset E$ a closed convex cone. If there exist $r > 0$ such that, $x^T f(x) > 0$, for every $x \in K$ such that, $||x|| = r$, then there exist $y \in K$, such that, $||y|| \leq r$ and $(x - y)^T f(y) \geq 0$ for every $x \in K$. Moreover, if f is strictly monotone then y is unique.

4.2 Existence Theorems

Given a closed convex cone $K \subset E$ and two mappings, $f : E \to E^*$ and $h : E \to R$. Let $F = \{x \in E |\, x \in K \text{ and } f(x) \in K^*\}$ denote the *feasible set* of f with respect to K. Let us look at several problems.

$$\textit{Nonlinear program:} \quad \begin{array}{l} \text{for a given } u \in E^* \\ \text{find } y \in F \text{ such that,} \\ u(y) = \min_{x \in F} u(x). \end{array} \tag{24}$$

$$\textit{The least element problem:} \quad \begin{array}{l} \text{find } y \in F \text{ such that,} \\ y \leq x, \forall x \in F. \end{array} \tag{25}$$

$$\textit{Variational inequality:} \quad \begin{array}{l} \text{find } y \in K \text{ such that,} \\ (x - y)^T f(y) \geq 0, \ \forall x \in K. \end{array} \tag{26}$$

$$\textit{Complementarity problem:} \quad \begin{array}{l} \text{find } y \in F \text{ such that,} \\ y \perp f(y). \end{array} \tag{27}$$

If E is a Hilbert space and $f : K \to E$ has the form $f(x) = x$, where $g : K \to K$, then we consider

$$\textit{Fixed point problem:} \quad \begin{array}{l} \text{find } y \in K \text{ such that,} \\ g(y) = y. \end{array} \tag{28}$$

Considering now the problems (1) – (5) we state several fundamental equivalence theorems.

Theorem 4.3 (13, 3.5) *Let $E(\tau)$ be a locally convex space, $K \subset E$ a closed convex cone and $f : E \to E^*$ a mapping. An element $y \in K$ is a solution of (**??**) if and only if y is a solution of (**??**).*

Definition 4.1 *A linear mapping $f : E \to E^*$ is a **Z-mapping** with respect to K, if and only if, for every $x, y \in K$ such that $\inf(y, x) = 0$ we have $y^T f(x) \leq 0$.*

Theorem 4.4 (13, 3.6) *Let $E(\tau)$ be a locally convex space, which is also a vector lattice, $K = \{x \in E|\ x \geq 0\}$ and consider $F : E \to E^*$ to be a Z-mapping strictly monotone. If y is a solution of (**??**), then y is a solution of (**??**).*

Theorem 4.5 (13, 3.7) *Let $E(\tau)$ be a locally convex space ordered by a closed convex cone $K \subset E$ and let $f : E \to E^*$ be a mapping. for an arbitrary element $u \in K^*$, if y is a solution of (**??**) then y is a solution of (**??**).*

Now, we are interested to know when the complementarity problem (**??**) is equivalent to a variational inequality (**??**).
We observe that if y satisfy (**??**), then it also satisfies the complementarity problem (**??**). Substituting $x := y + e_i$ into (**??**), where e_i denotes the vector with 1 in the ith element and 0 elsewhere, one concludes that

$$f_i(y) \geq 0 \to f(y) \geq 0.$$

Substituting now $x := 2y$ into (**??**), one obtains

$$y^T f(y) \geq 0. \tag{29}$$

Substituting then $x := 0$ into the (**??**), one obtains

$$y^T f(y) \leq 0. \tag{30}$$

(**??**) and (**??**) together imply that $y^T f(y) = 0$.
Conversely, if y satisfies the complementarity problem, then since $y \in F$ and $f(y) \geq 0$ holds $(x - y)^T f(y) \geq 0$.

We showed that the variational inequality (**??**) problem also contains the complementarity problem (**??**)as a special case and the following theorem holds.

Theorem 4.6 (18, 4) *(**??**) and (**??**) have precisely the same solution, if any.*

Finally, let us deal with the equivalence between complementarity problems (??) and fixed points problem (??).

Theorem 4.7 (13, 3.10) *Let H be a Hilbert space and let $K \subset H$ be a convex cone. If $f : K \to H$ has the form, $f(x) = x - g(x)$, where $g : K \to K$, then y is a solution of (??), if and only if, y is a solution of (??).*

[13, 3.11] The case when f does not have the special form $f(x) = x - g(x)$, where $G : K \to K$, is more complicated. However, the following theorem holds.

Theorem 4.8 *Let H be a Hilbert space and let $K \subset H$ be a closed convex cone. If $f : K \to H$ is an arbitrary mapping we consider the complementarity problem*

$$\begin{aligned} &\text{find } y \in K \text{ such that,} \\ &f(y) \in K^* \text{ and } y \perp f(y). \end{aligned} \tag{31}$$

Also, for an arbitrary $\tau \in \mathbb{R}_+ \setminus \{0\}$ we consider the mapping $T : H \to K$ defined by $T(x) = P_K(x - \tau f(x))$. The problem (??) is equivalent to the following fixed point problem

$$\begin{aligned} &\text{find } y \in K \text{ such that,} \\ &T(y) = y. \end{aligned}$$

5 Economic Applications of the Complementarity Problems

Complementarity problems arise naturally in the study of equilibrium of many phenomena in economics. A comprehensive treatment of applications of complementarity problems is provided in [10]. Additionally, a large collection of problems from a variety of application areas can be found in the MCPLIB library of test problems [6].

Most models characterized in this chapter has the nature of the *excess demand functions* that determines the properties of equilibria. Different assumptions on the behavior of the agents in the economy determine different forms of equilibria.

We consider several economic situations which imply diverse complementarity problems. For the situations, we describe the problem briefly, state the defining equations of the models, and give functional expressions for the complementarity formulations.

5.1 Equilibrium in an Exchange Economy

Let us consider a market with no production. It means that traders simply exchange goods with each other. We suppose a system with n different commodities and m traders buying and selling these commodities. We also presume competitive behavior, that is, traders do not perceive that they can have any influence over the market prices. Each trader wants to maximize utility, but is constrained by the budget.

Let trader i has a concave utility function $u_i : \mathbb{R}^n_+ \to \mathbb{R}_+, i = 1, \dots, m$, which represents her preferences for the different commodities. We also assume that trader i is initially endowed with w^i_j of commodities j and $w^i := (w^i_1, w^i_2, \dots, w^i_n)^T \in \mathbb{R}^n_+$ denote the corresponding vector of endowments of the ith trader. Let $p_j, j = 1, \dots, n$ denote the price of commodity j and $p := (p_1, p_2, \dots, p_n)^T \in \mathbb{R}^n_+$ is the corresponding price vector. At given prices $p_j, j = 1, \dots, n$, the ith trader will sell his endowment and get the bundle of commodities $x^i := (x^i_1, x^i_2, \dots, x^i_n)^T \in \mathbb{R}^n_+$ which maximizes $u_i(x^i)$ subject to her budget constraint $\sum_{j=1}^{n} p_j x^i_j \le \sum_{j=1}^{n} p_j w^i_j, i = 1, \dots, m$.

An equilibrium is a vector of prices p at which, for each trader i, there is a bundle x^i of goods such that the following two conditions hold:

(1) For each trader i, the vector x^i maximizes $u_i(x^i)$ subject to the constraints $\sum_{j=1}^{n} p_j x^i_j \le \sum_{j=1}^{n} p_j w^i_j, i = 1, \dots, m$.

(2) $\sum_{i=1}^{m} x^i_j \le \sum_{i=1}^{m} w^i_j, j = 1, \dots, n$.

The condition (1) says that the bundle of commodities x^i should cost no more than the income of trader i. The conditions (2) says that the total consumption of commodity j cannot exceed the total amount of commodity j in the market.

For any price vector p, the vector $x^i(p)$ satisfying (1) is called the *demand* of trader i at prices p. The *excess demand* of trader i is given by $f^i(p) = x^i(p) - w^i$. Then $f_j(p) = \sum_{i=1}^{m}(x^i_j(p) - w^i_j)$ denotes the *excess demand* of good j at prices p and $f(p) = (f_1(p), \dots, f_n(p))^T$ is called the *excess demand*.

The excess demand represents the result of consumer optimization. At the equilibrium prices it must be possible for all consumers to optimize at the same time. This means that in market for each good demand to buy cannot exceed zero (demand may be less than supply), since there is no production or outside agent to provide supply to the market. Mathematically, it means that for the equilibrium price p we have $f(p) \le 0$.

According to the Walras's law we have $\sum_{j=1}^{n} p_j f_j(p) = 0$. However, in equilibrium each term $p_j f_j(p) \ge 0$. It follows this and the Walras's law that each term must actually equal zero. This leaves two possibilities: in market j either supply equals demand or the price is equal zero. The latter is not typically the case, although it is possible if individuals have saturated preferences.

To find the economic equilibrium in this case we in fact have to solve the following complementarity problem:

$$\begin{aligned} &\text{find } p \in \mathbb{R}^n \text{ such that,} \\ &0 \le p \perp f(p) \ge 0. \end{aligned}$$

5.2 Equilibrium of an Economy with Production

The model of a sector with production is the basis of much of mathematical economics and general equilibrium theory and has proven very useful in macroeconomics, particularly in analyzing tax policy, international trade, issues in energy economics, to name a few applications.

Let us simplify the process by considering an economy with competitive behavior without any price distortions and assume that our economy involves n commodities indexed by $j = 1, \ldots, n$, and m activities indexed by $i = 1, \ldots, m$, with constant returns to scale production. In particular, this means that if the technology can output a certain quantity of a good using as input certain quantities of other goods, then scaling all these quantities by a common nonnegative number also results in a technologically feasible plan.

Let the unit cost of operating the ith activity is c_i and the initial endowment of the jth commodity is b_j. The unknown level of the ith activity is denoted y_i and the price of the jth commodity is denoted p_j. The demand function for the jth commodity is $d_j(p)$, where $p := (p_j) \in \mathbb{R}^n$ is the price vector of all commodities. We suppose that the demand function is single-valued and continuously differentiable. The technology input-output matrix of the economy is given by the $m \times n$ matrix $A = (a_{ij})$, where $a_{ij} > 0$ ($a_{ij} < 0$) stands for an output (input). The transpose of this matrix converts levels of activities into vectors of commodities. Specially, for a vector $y := (y_i) \in \mathbb{R}^m$ of activities, $A^T y$ is the vector of commodities resulting from these activities, for a vector p, Ap is the vector of per unit activity returns.

Because of the generality of the theory of economic equilibrium, there are several ways to characterize an equilibrium. We shall use here the Scarf's definition.

Definition 5.1 *A price vector p and a vector of activity levels y constitute a* ***competitive equilibrium*** *if :*

1) *all prices and activity levels are nonnegative,*
   $p \geq 0, \ y \geq 0,$

2) *all activities yield a nonpositive profits,*
   $c - Ap \geq 0,$

3) *no commodity is in excess demand,*
   $b + A^T y - d(p) \geq 0,$

4) *an activity incurring a loss is not performed and an operated activity runs at a zero loss,*
   $y^T(c - A^T p) = 0,$

5) *a commodity in excess supply has zero price and a positive price implies market clearance (demand equals supply),*
   $p^T(b + A^T y - d(p)) = 0.$

We can associate to this model the following nonlinear complementarity problem:

$$\begin{aligned} &\text{find } z \in \mathbb{R}^{n+m} \text{ such that,} \\ &0 \leq z \perp F(z) \geq 0, \end{aligned}$$

where $z = \begin{bmatrix} y \\ p \end{bmatrix}$, $F(z) = \begin{bmatrix} c - A^T p \\ b + Ay - d(p) \end{bmatrix}$.

We observe that if the demand functions are linear in prices, then our definition of an equilibrium will be a linear complementarity problem.

5.3 Equilibrium of International or Interregional Trade

In this section we concentrate our attention on the model of grouping of traders into regions (states). We consider n regions, which are engaged in trade with each other and for simplicity assume there is a homogenous commodity in the market. The homogenous commodities only differs by region and are otherwise physically identical. Each region has a linear supply and demand function, depending on price at the region. The intersection of these functions determine the equilibrium price and amounts of the commodity produced and consumed in the region with no imports and exports included. Adding imports to this model enables consumption to outstrip production but at a lower equilibrium price.

To make the situation easier, we assume that supply and demand of each region are irrelevant. Only the net import and the equilibrium price in each region are of any

relevance. Moreover, we suppose that for every region there is a linear relation between the equilibrium price and the net imports given by

$$p_i = a_i - b_i y_i,$$

where p_i denotes the equilibrium price in the ith region, a_i denotes the equilibrium price in the absence of imports and exports, b_i is related to elasticity of supply and demand of the ith region, and y_i denotes the net import of the ith region.

We note that if p_i exceeds a_i, then supply in the ith region exceeds demand and the surplus can be passed on as export. Thus, y_i can reach a negative value which can be interpreted as export.

We also assume that the shipments take place on the most cost effective routes and hence the unit cost of shipping from region i to region j described by c_{ij} will satisfy the triangle inequality

$$c_{ij} \leq c_{ik} + c_{kj} \qquad \forall i, j, k = 1, 2, \ldots, n.$$

Let x_{ij} denotes exports from region i to region j, to find the trade equilibrium is needed $\forall i, j = 1, 2, \ldots, n$ solve the following problem:

$$\begin{aligned}
&\text{find } p_i, y_i, x_{ij} \text{ such that,} \\
&\sum_{j=1}^{n} x_{ji} - \sum_{j=1}^{n} x_{ij} = y_i, && (32) \\
&a_i - b_i y_i = p_i, && (33) \\
&x_{ij}(p_i + c_{ij} - p_j) = 0, \\
&p_i + c_{ij} - p_j \geq 0, \\
&x_{ij} \geq 0.
\end{aligned}$$

If we eliminate p_i and y_i using relations (??) and (??), we obtain the linear complementarity problem if p_i is linear for every i.

Economics is not just the social science that studies the production, distribution, and consumption of goods and services. Economics is close to some other sciences, for example, such as engineering. Last few years economics investigate road accident and traffic flows. There ask themselves many questions. How should society go about expanding its road system? Who would decide where to prove more road capacity, and how much more? Where would funds for expansion come from?

Complementarity problems in engineering have received wide attention in recent years. An important reason why complementarity problems are so pervasive in engineering is because the concept of complementarity is synonymous with the notion of system equilibrium.

Optimization is a recurring theme in numerous engineering applications. On the other hand, many engineering systems involve the notion of equilibrium without an objective being optimized. For instance, the Wardrop principle of user equilibrium in traffic theory has a natural formulation as a nonlinear complementarity system.

Complementarity problems are widely used in contact mechanic problems, structural mechanic problems, elastohydrodynamic lubrication problems etc. Since we are not experts in the disciplines described herein, especially in view of the fact that several of them are based on related physical and mechanical principles, we mention just one of them in this section.

5.4 Equilibrium of Traffic Flows

There are a number of ways how to model the flow of traffic in networks. Here we use a popular formulation of the traffic equilibrium problem with fixed demand in terms of path flows. The purpose of a traffic equilibrium model is to predict steady-state traffic flows in a congested traffic network.

Let us suppose that drivers drive along paths which minimize their cost (or time) with no cooperative among drivers and there is a transportation network given by a set of nodes $\mathcal{N}$ and a set of arcs $\mathcal{A}$. Nodes represents districts, street intersections etc and arcs model streets and arteries or might be introduced to model connection between streets at an intersection. There are two subsets of $\mathcal{N}$ that represent the set of origin nodes $\mathcal{O}$ and destination nodes $\mathcal{D}$, respectively. The set of origin-destination (O, D) pairs is a given subset of $\mathcal{O} \times \mathcal{D}$, associated with each such pair is a travel demand that represents the required flow from the origin node to the destination node.

To describe this situation mathematically it is necessary to make some denotations. Let I denotes the suitable index set of (O, D) pairs. Let p_i denotes the set of paths connecting the (O, D) pair i for $i \in I$ and P stands for the set of all paths joining all (O, D) pairs of the network. We denote the flow (volume) of the commodity transported on path p for $p \in P$ by h_p and let the vector h with dimension n_1 contain all path flows in the network. We also assume that u_i is a variable that depict the minimum transport cost (or time) between (O, D) pair i and n_2 dimensional vector u contain all components u_i. The function $D_i : \mathbb{R}_+^{n2} \to \mathbb{R}_+$ stands for the demand function of the vector u in the travel demand between (O, D) pair i measured in number of drivers per time unit. The function $T_p : \mathbb{R}_+^{n_1} \to \mathbb{R}$ stands for the cost of flow on the path p, which is a function of the path flow vector h.

We suppose that the network is strongly connected, i.e. for any (O, D) pair $i \in I$ there is at least one path joining the origin to the destination. The model is called a *fixed-demand model* if each $D_i(u)$ is a constant function. The general model is often called the *elastic demand model.*

The problem to find the equilibrium on this transportation network is to solve the following problem:

$$\text{find } h \in \mathbb{R}_+^{n_1} \text{ and } u \in \mathbb{R}_+^{n_2} \text{ such that,} \tag{34}$$
$$(T_p(h) - u_i)h_p = 0, \ \forall p \in P_i, \ \forall i \in I, \tag{35}$$
$$T_p(h) - u_i \geq 0, \ \forall p \in P_i, \ \forall i \in I, \tag{36}$$

$$\sum_{p \in P_i} \big(h_p - D_i(u)\big)u_i = 0, \ \forall i \in I, \tag{37}$$
$$\sum_{p \in P_i} \Big(h_p - D_i(u)\Big) \geq 0, \ \forall i \in I. \tag{38}$$

The equations (??) and (??) states that each driver will choose the minimum cost path between every origin destination pair. Paths with cost higher than the minimum will have no flow. The equilibrium condition of zero excess demand can be stated as (??) and (??). The demand is satisfied if relations (??) hold.

Let $x := (h, u) \in \mathbb{R}_+^n$, where $n = n_1 + n_2$ and we denote,

$$f_p(x){:=}T_p(h) - u_i, \forall p \in P_i, \forall i \in I,$$

$$g_i(x){:=}\sum_{p \in P_i} h_p - D_i(u), \forall i \in I,$$

then we can transform the complementarity problem into the following nonlinear complementarity problem:

$$\begin{aligned} &\text{find } x \in \mathbb{R}_+^n \text{ such that,} \\ &f_p(x)h_p = 0, \ \forall p \in P_i, \ \forall i \in I, \\ &f_p(x) \geq 0, \ \forall p \in P_i, \ \forall i \in I, \\ &g_i(x)u_i = 0, \ \forall i \in I, \\ &g_i(x) \geq 0, \ \forall i \in I. \end{aligned} \tag{39}$$

Theorem 5.1 (4, 2.3) *If the function $T_p : \mathbb{R}_+^n \to \mathbb{R}$ is a positive function $\forall p \in P$, then problem (??)-(??) is equivalent to the nonlinear complementarity problem (??).*

For networks of reasonable size, the enumeration of all paths connecting elements of I is prohibitive, if there are many (O, D) pairs. Thus the above path-flow formulation is not suitable for direct computational use. Nevertheless, there are path-generation schemes that utilize this formulation and generate the paths only if they are needed. More details in [4].

6 Conclusion

In this bachelor thesis we have presented an introductory view on the theory of complementarity problems. After defining complementarity problems we investigated various models of mathematical programming and game theory which can be converted into forms of complementarity problems and subsequently we dealt with models which are equivalent to complementarity problems, if certain assumptions are satisfied. So, by these connection we obtained both, new methods to study complementarity problems and the possible use of the complementarity theory in the study of these problems.

At the end of this thesis we gave a summary of some applications of complementarity problems in the field of economics and engineering. We restricted our study to linear models, namely a model of exchange, a model of production and a model of international trade. Besides, we focused on traffic equilibrium problem since network models are used extensively in practise, in an ever expanding spectrum of applications. For each of the models mentioned above we found out the corresponding problem in complementarity theory.

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