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Povrchy asociované se sigma modely

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Abstrakt

Cílem práce bylo studium vlastností povrchů asociovaných s $CP^1$ sigma modelem na prostoru s Minkovského metrikou, založené na analýze konkrétních příkladů. Řešení rovnice $CP^1$ sigma modelu jsme nalezli metodou redukce diferenciální rovnice dle symetrií. Pomocí takto nalezených řešení $CP^1$ sigma modelu jsme určili parametrické rovnice odpovídajících ploch, spočítali jejich I. a II. fundamentální formu a jejich střední křivost. Protože takto konstruované povrchy mají zápornou konstantní Gaussovu křivost, dále jsme tímto způsobem k řešení $CP^1$ sigma modelu přidalí též odpovídající řešení sine–Gordonovy rovnice. Dalším úkolem práce bylo sumarizovat předchozí výsledky výzkumu na toto téma. Souhrn zahrnuje zejména zobecnění výše uvedených konstrukcí na $CP^{N-1}$ a obecněji na Grassmannovské sigma modely. Využili jsme při tom technik obvyklých při studiu povrchů vnořených do Lieových algeber. Pro popis Grassmannovských sigma modelů jsme zvolili výhodnější přístup pomocí ortogonálního projektoru. Práce je doplněna dodatky vysvětlujícími pojmy z diferenciální geometrie ploch, popis metody redukce dle symetrie a zavedení eliptických funkcí a theta funkcí.

Klíčová slova: Sigma modeley, strukturní rovnice povrchů, Lieovy algebry

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Abstract

The aim of this work was to study properties of surfaces related to the $CP^1$ sigma model on Minkowski space, based on analysis of specific examples. Solutions of equations of the $CP^1$ sigma model were found using the method of symmetry reduction of differential equations. When the solutions were found, we found parametrizations of the surfaces, their first and second fundamental forms and their mean curvatures. As the surfaces constructed in such
a manner have a constant negative Gaussian curvature, hence we assigned to a solution of the \( \mathbb{C}P^1 \) sigma model a corresponding solution of the sine–Gordon equation. Another aim of this work was to summarize results of previous researches on this topic including generalization of the above-mentioned constructions to \( \mathbb{C}P^{N-1} \) and, more generally, to Grassmanian sigma models. We followed the techniques used in the study of surfaces immersed in Lie algebras. We used an orthogonal projector approach to describe the Grassmannian sigma models, because it is more advantageous. The work is supplemented with several appendices explaining basic notions of the differential geometry of surfaces, a description of the symmetry reduction method and an introduction to the elliptic functions and the theta functions.

**Key words**: Sigma models, structural equations of surfaces, Lie algebras
Introduction

Many of the equations that are now called integrable were studied in differential geometry already in the 19th century. Probably the first was the famous sine–Gordon equation, which was derived to describe surfaces with constant negative Gaussian curvature. At that time many features of integrability of the sine-Gordon and other integrable equations were discovered, namely those which have clear geometrical interpretation (for example, the Bäcklund transformation). Next progress in this domain was done in the 1960’s when the inverse scattering method and Lax representation of the nonlinear equations were introduced. The Lax equation was recognized in differential geometry as the Gauss–Codazzi equation of the compatibility for a surface. The next important development in the study of the surfaces immersed in multidimensional spaces came with the use of harmonic maps, which greatly facilitated solution of the structural equations and construction of conformally parameterized surfaces in $\mathbb{R}^3$. The treatment of constant mean curvature surfaces in terms of harmonic maps can be found for example in [1]. These methods gained increased significance with the realization that the harmonic map equations correspond to what mathematical physicists used to call sigma models. An extensive research of surfaces immersed in $\mathbb{R}^n$ proceeded particularly in the last two decades and continues up to now. The motivation for this activity came largely from applications in various branches of physical, biological and chemical sciences as well as from engineering.

In this work we follow the techniques used in the study of surfaces immersed in Lie algebras. We apply the Weierstrass formula for the immersion to surfaces associated to the $\mathbb{C}P^{N-1}$ and more generally Grassmannian sigma models on Minkowski space. This line of investigation was initiated in [2], [6]. Before that research it was even shown that two–dimensional constant mean curvature surfaces in three– and eight–dimensional spaces are associated with the $\mathbb{C}P^1$ and $\mathbb{C}P^2$ sigma models defined on Euclidean spaces. The properties of surfaces obtained from the $\mathbb{C}P^{N-1}$ sigma models on Minkowski space turned out to be significantly different from the ones in the case of sigma models on Euclidean space.

Sigma models are of great interest in mathematical physics because many physical systems can be reduced to these models. One such example is the string theory in which sigma models on space-time and their supersymmetric extensions play a fundamental role. Classical configuration of strings can be described by common solutions of the Nambu–Goto–Polyakov action and a system of Dirac type equations intimately connected to $\mathbb{C}P^{N-1}$ models. Other relevant applications of recent interest are in the areas of statistical physics (for example reduction of self–dual Yang–Mills equations to the Ernst model), phase transitions (e.g. dynamics of vortex sheets, growth of crystals, surface waves etc.) and the theory of fluid membranes.

In this work, we use an orthogonal projector satisfying the Euler–Lagrange equations of the given sigma model to express fundamental forms, the Gauss curvature, the mean curvature and other quantities arising in the description of the associated surfaces. The advantage of the projector approach is that there are no gauge degrees of freedom in such description of the Grassmannian sigma models, compared to the usage of equivalence.
classes if Grassmannian manifold is viewed as the homogenous space.

Next we study the special case of the $\mathbb{C}P^1$ sigma model in more detail. The $\mathbb{C}P^1$ sigma model is probably the simplest model which is relativistically covariant and which admits the existence of localized soliton-like solutions. The exact solutions of this model were found using the group theoretical method of symmetry reduction [12]. This method exploits the symmetry of the equations to find solutions invariant under some subgroup. The method leads to equations whose solutions represent specific solutions of the full equations. In the case of the $\mathbb{C}P^1$ sigma model the symmetry reduction leads to coupled pair of ordinary differential equations.

Besides, in this simplest case of the $\mathbb{C}P^1$ sigma model, the resulting surface has constant negative Gaussian curvature. Therefore, we construct and investigate the corresponding solutions of the sine–Gordon equation. It turns out that the relation doesn’t necessarily involve the construction of surfaces, i.e. there is a direct reduction from the $\mathbb{C}P^1$ sigma model to the sine–Gordon equation, provided certain regularity conditions on the $\mathbb{C}P^1$ solution are met.

This work is organized as follows. In the first part of this work, we summarize the previous results on the subject of surfaces associated with sigma models, [2], [3], [4], [5], [6], [7]. We want to introduce basic constructions and clarify general formulas necessary for the calculations in the second part of this work. In the first chapter, we define actions of the $\mathbb{C}P^{N-1}$ and Grassmannian sigma model. Then we review the unified form of Euler–Lagrange equations of motion for the $\mathbb{C}P^{N-1}$ and Grassmannian sigma models in terms of a projector. In the second section we describe the Weierstrass immersion construction of associated surfaces and express the first and second fundamental forms and the Gaussian and mean curvature in terms of the projector. Finally we discuss the symmetries of Euler–Lagrange equations and its consequences. In the second chapter of the theoretical part, we consider the case of the $\mathbb{C}P^1$ sigma model. We pass through all the steps from the first chapter to show the concrete form of the general expressions. Then we derive the connection between the $\mathbb{C}P^1$ sigma model and the sine–Gordon equation, firstly in terms of the projector and secondly in terms of a complex field $w$. The second approach is useful for practical calculations, which we undertake in the second part of the work. For completeness, the results of the symmetry reduction for the $\mathbb{C}P^1$ sigma model finish the second chapter.

The second part of the work further develops the results from the previous papers [2], [3], [5], [6]. We describe rather technical but nontrivial calculations which yield particular own results. In the first section of the third chapter, the elliptic solutions obtained via symmetry reduction are defined in a dependence on several parameters. Then the constraints among the parameters are described there. We distinguish two types of solutions, which were not investigated before. The first original contribution is in the section 2, where concrete values of parameters satisfying complicated constraints are found for the both considered cases. The proof of the fact that the currents (which satisfy the equation of continuity), for every solution of the $\mathbb{C}P^1$ sigma model obtained by the symmetry reduction, are constant, can be found in the section 3. The proof is based on tedious but direct calculation. Solutions
of the sine–Gordon equation associated with the $\mathbb{CP}^1$ sigma model and their properties are studied in the last section of the third chapter. It is also a new part of the work. Finally, in the chapter 4 we solve the problem how to find the whole solution of the $\mathbb{CP}^1$ sigma model, if we know the solutions of one of two coupled ODE’s obtained by the symmetry reduction. It means that we integrate the second ordinary differential equation for the function $f$, which arises as a phase factor of the whole complex solution $w$. We found out that the problem of searching of the function $f$ can be converted to integration of the same elliptic integral for both of the considered cases. Evaluation of this integral involved advanced manipulations with the Jacobi’s elliptic functions and the theta functions. The closing chapter 5 concludes all of obtained results, compares them with proposed tasks in the submission of the work and suggests possible utilization and generalizations.

The basic facts from differential geometry of surfaces are summarized in the appendix A. The symmetry reduction method is explained in the appendix B. Definitions and some technical details of computations with theta functions can be found in the appendix C. The appendix D deals with the Jacobi’s elliptic functions. The proof of the fundamental identity used in the section (4.2) is relegated to the appendix E. For more information about the geometry of surfaces, see [8], or [9]. Recent developments in the subject of surfaces in connection with partial differential equations is broadly reviewed in [10]. The matrix approach to the treatment of surfaces is explained in [11]. The method of the symmetry reduction is readably described in [12].
Part I

Theoretical part
Chapter 1

Surfaces associated with $\mathbb{C}P^{N-1}$ and Grassmannian sigma models

1.1 Euler-Lagrange equations of motion for $\mathbb{C}P^{N-1}$ and Grassmannian sigma models

In the study of both models, i.e. the $\mathbb{C}P^{N-1}$ and the Grassmannian sigma models, we consider that the domain of the definition of the model is an open, connected and simply connected subset $\Omega$ in $\mathbb{R}^2$ with the Minkowski metric

$$(ds)^2 = d\xi_L d\xi_R. \quad (1.1.1)$$

From the form of the metric (1.1.1) it follows that $\xi_L, \xi_R$ are the light-cone coordinates in $\mathbb{R}^2$. They are connected to the standard Minkowski coordinates $\xi^0, \xi^1$ by the relations

$$\xi_L = \xi^0 + \xi^1, \xi_R = \xi^0 - \xi^1, \quad (ds)^2 = (d\xi^0)^2 - (d\xi^1)^2.$$ 

We denote $\partial_L \equiv \partial_{\xi_L}, \partial_R \equiv \partial_{\xi_R}$ and $\partial_\mu \equiv \partial_{\xi_\mu}, \mu = 0, 1$.

Naturally, in the case of the $\mathbb{C}P^{N-1}$ sigma models the target space is the complex projective space $\mathbb{C}P^{N-1}$, which is defined as a set of 1-dimensional subspaces in $\mathbb{C}^N$. The manifold structure on it is defined by an open covering

$$U_k = \{ [z] | z \in \mathbb{C}^N, z_k \neq 0 \}, \quad k = 1, \ldots, N,$$

where $[z] = \text{span}\{z\}$, and coordinate maps

$$\varphi_k : U_k \to \mathbb{C}^{N-1}, \quad \varphi_k(z) = \left( \frac{z_1}{z_k}, \ldots, \frac{z_{k-1}}{z_k}, \frac{z_{k+1}}{z_k}, \ldots, \frac{z_N}{z_k} \right).$$

The solution of the model is a map $[z] : \Omega \to \mathbb{C}P^{N-1}$, which is a stationary point of the action functional

$$S = \frac{1}{4} \int_\Omega (D_\mu z)^\dagger (D^n z) d\xi^0 d\xi^1, \quad z^\dagger z = 1. \quad (1.1.2)$$
$D_{\mu}$, $D^{\mu}$ are the covariant derivatives

$$D_{\mu}z = \partial_{\mu}z - (z^\dagger \partial_{\mu}z) z, \quad \mu = 0, 1.$$ 

One can verify that the action is defined correctly, i.e. that it depends only on $[z]$ and not on the choice of a representative of the class $[z]$.

The Grassmannian sigma models are a generalization of $\mathbb{C}P^{N-1}$ sigma models, i.e. $\mathbb{C}P^1 \simeq G(1, 1)$, $n = m = 1$, $N = n + m$, $\mathbb{C}P^{N-1} \simeq G(N - 1, 1)$, $m = N - 1, n = 1$. Notice that $G(m, n) \simeq G(n, m)$. The Grassmannian manifold

$$G(m, n) = \{ \text{m-dimensional subspaces of } \mathbb{C}^N, \; n + m = N \} \quad (1.1.3)$$

with standard open covering and coordinate maps, may be also expressed as a homogenous space

$$G(m, n) = \frac{SU(N)}{S(U(m) \times U(n))}, \; N = m + n. \quad (1.1.4)$$

The description of the Grassmannian sigma models using the equivalence classes of elements $g \in SU(N)$ has been done in [4]. However, it has been shown in [5] that the Grassmannian sigma models can be more naturally described by the projector acting on the corresponding subspace defining the element of (1.1.3). In this work we shall proceed in this second way.

Similarly as in the previous paragraph for a map $[z]$, here we consider a projector

$$P : \Omega \rightarrow \text{Aut}(\mathbb{C}^N), \quad P^3 = P, \quad P^2 = P, \quad \text{dim(rank} P) = m, \quad (1.1.5)$$

which is a stationary point of the action

$$S = \int_{\Omega} \text{tr} \{ \partial_L P \cdot \partial_R P \} \, d\xi_L d\xi_R. \quad (1.1.6)$$

Obviously the description by projectors can be reproduced also for the special case of the $\mathbb{C}P^{N-1}$ sigma models, namely the relation between $[z]$ and the projector $P$ is

$$z = \frac{f}{|f|}, \quad |f| = \sqrt{\mathcal{P}^\dagger f}, \quad P = 1 - \frac{1}{|f|^2} f \otimes f^\dagger. \quad (1.1.7)$$

The action (1.1.2) is then expressed as

$$S = \int_{\Omega} \mathcal{L} d\xi_L d\xi_R = \int_{\Omega} \frac{1}{4|f|^4} \left( \partial_L f^\dagger P \partial_R f + \partial_R f^\dagger P \partial_L f \right) d\xi_L d\xi_R. \quad (1.1.8)$$

One can verify that such defined action is a special case of (1.1.6).

The Euler–Lagrange equations are in the both cases obtained by a standard way by the variation of the corresponding actions, (1.1.2), (1.1.8), (1.1.6). For both of the cases, the equations of motion have clearly the same form in the formalism of projectors

$$[\partial_L \partial_R P, P] = 0. \quad (1.1.9)$$

In the form of a conservation law the equations of motion read

$$\partial_L [\partial_R P, P] + \partial_R [\partial_L P, P] = 0. \quad (1.1.10)$$
1.2 Surfaces associated with sigma models

We shall show that with the projectors $P$ given by (1.1.7), (1.1.5) one can associate two-dimensional surfaces $\mathcal{F}$ immersed in the $\mathfrak{su}(N)$ algebra. We follow the general description by projectors from [5]. The concrete form of the general formulas is illustrated on the special case of the $\mathbb{C}P^1$ model in the section (2.1).

Firstly, we introduce the matrix quantities

$$M_L = [\partial_L P, P], \quad M_R = [\partial_R P, P]. \quad (1.2.1)$$

Then we see from (1.1.10) that $M_L, M_R$ satisfy an equation

$$\partial_L M_R + \partial_R M_L = 0. \quad (1.2.2)$$

The key idea is to look at this equation in terms of $\mathfrak{su}(N)$–valued 1–form on $\Omega$. If we define

$$\mathcal{X} = M_L d\xi_L - M_R d\xi_R,$$

then the equation (1.2.2) implies

$$d\mathcal{X} = 0,$$

i.e. the equation (1.2.2) is a condition of a closedness of the 1–form $\mathcal{X}$. Now we refer to the Poincare’s lemma, which says that a closed 1–form on a connected and simply connected domain is also exact. It means that we can write $\mathcal{X} = dX$, where $X$ is well–defined $\mathfrak{su}(N)$–valued function on $\Omega$ and

$$\partial_L X = M_L, \quad \partial_R X = -M_R.$$

It follows, that integral of the 1–form $dX$ is independent on the choice of the path of integration. The sought surface $\mathcal{F}$ is defined by the formula (the so called Weierstrass coordinate representation)

$$\mathcal{F} : X(\xi_L, \xi_R) = \int_{\tilde{\gamma}(\xi_L, \xi_R)}^{\mathcal{X}}, \quad (1.2.3)$$

where $\tilde{\gamma}(\xi_L, \xi_R)$ is any curve in $\Omega$ connecting the end point $(\xi_L, \xi_R) \in \Omega$ with an arbitrary chosen initial point $(\xi^0_L, \xi^0_R) \in \Omega$. We obtain a surface in the $\mathfrak{su}(N)$ algebra. If we require to have a surface in an Euclidean space, we identify the $\mathfrak{su}(N)$ algebra supplemented with the scalar product

$$\langle A, B \rangle = -\frac{1}{2} \text{tr} (A, B)$$

and $\mathbb{R}^{(N^2-1)}$ with the standard scalar product, i.e

$$\{ \mathbb{R}^{N^2-1}, (A, B) \} \simeq \{ \mathfrak{su}(N), \langle A, B \rangle \}.$$
Further, in the quantities $\mathcal{X}_L = M_L$, $\mathcal{X}_R = -M_R$ we recognize tangent vectors to the surface $\mathcal{F}$. The coefficients of the first fundamental form are then computed by the standard way as scalar products, (see the appendix A, formula (A.5) for explication of the general formula)

$$I = \begin{pmatrix} \langle \mathcal{X}_L, \mathcal{X}_L \rangle & \langle \mathcal{X}_L, \mathcal{X}_R \rangle \\ \langle \mathcal{X}_R, \mathcal{X}_L \rangle & \langle \mathcal{X}_R, \mathcal{X}_R \rangle \end{pmatrix} = \frac{1}{2} \begin{pmatrix} p_{L|L} & -p_{L|R} \\ -p_{L|R} & p_{R|R} \end{pmatrix},$$

(1.2.4)

where

$$p_{B_1 \ldots B_k|D_1 \ldots D_l} = \text{tr} (\partial_{B_1 \ldots B_k} P \partial_{D_1 \ldots D_l} P),$$

(1.2.5)

and $k, l > 0$, $B_1, \ldots, B_k, D_1, \ldots, D_l = L, R$. Note that $p_{B_1 \ldots B_k|D_1 \ldots D_l} = p_{D_1 \ldots D_l|B_1 \ldots B_k}$ due to a cyclic property of the trace. This notation let us write the metric in the compact form

$$ds^2 = \left( \delta_{B,D} - \frac{1}{2} \right) p_{B|D} d\xi_B d\xi_D,$$

(1.2.6)

where we use the Einstein’s summation convention over repeated indices $B, D = L, R$. The signature of this form was investigated for example in [2]. We refer the reader to this source for a proof that $I$ is positive semi-definite. The proof is based on the Schwarz inequality. The conditions under which $I$ is positive definite, i.e. when the surface exists at least locally, are specified there. We shall suppose that these conditions are in our case satisfied.

We denote the diagonal terms of the matrix $I$ by

$$J_L = \frac{1}{2} p_{L|L}, \quad J_R = \frac{1}{2} p_{R|R}$$

(1.2.7)

and call them currents, because they satisfy equations of continuity

$$\partial_R J_L = 0, \quad \partial_L J_R = 0.$$  

(1.2.8)

To derive these equations, we use a differential consequence of $P^2 = P$, which gives

$$\partial_D P = \partial_D PP + P \partial_D P, \quad D = L, R.$$  

(1.2.9)

Multiplying this equation by $P$ from the left as well as from the right, we obtain

$$P \partial_D PP = 0, \quad D = L, R.$$  

(1.2.10)

Then we can write for example for $J_L$

$$\partial_R J_L = \frac{1}{2} \partial_R p_{L|L} = \frac{1}{2} \partial_R \text{tr} (\partial_L P \partial_L P) = \frac{1}{2} \cdot 2 \text{tr} (\partial_L \partial_R P \partial_L P) = p_{L|L} =$$

$$= \text{tr} (\partial_L \partial_R P (\partial_L PP + P \partial_L P)) = 2 \text{tr} (\partial_L \partial_R PP \partial_L P) = 2 \text{tr} (\partial_L \partial_R PPP \partial_L P) =$$

$$= 2 \text{tr} (\partial_L \partial_R PP \partial_L P) = 0,$$
where we used the cyclic property of the trace, the equation (1.2.9), the Euler–Lagrange equations (1.1.9) and finally (1.2.10). For $J_R$ is the derivation analogous. The equations of continuity are at the same time the conditions for the metric to be in the so called general Chebyshev form (A.12). We shall show in the next section (1.3) that due to a symmetries of (1.1.9), it is possible to make a transformation of the independent variables so that the metric is expressed in the Chebyshev form (A.11) and particularly we shall suppose such coordinates that $g_{11} = g_{22} = 1$, i.e. the metric is parametrized by

$$I = (d\tilde{\xi}_L)^2 - \tilde{p}_{L|R}d\tilde{\xi}_Ld\tilde{\xi}_R + (d\tilde{\xi}_R)^2. \quad (1.2.11)$$

From the other properties of the surface, we mention its scalar curvature $R$ or more precisely the Gaussian curvature $K$, where $R = 2K$. The general formula is given in terms of coefficients of $I$ by (A.15). To calculate $K$, we proceeded in the same manner as in the appendix A in the proof of the so-called ”Theorem Egregium”, proposition A.4, but we substituted the terms $p_{B|D}$ to the generals formulas. If we suppose the metric in Chebyshev coordinates and a suitable orientation of the axes (as in the proof of the above mentioned proposition 4), $K$ takes quite a simple form

$$K = \frac{2p_{LR|LR} - p_{LL|RR} - p_{LL|R}p_{L|R}p_{L|R}}{4 - (p_{L|R})^2} = \frac{2p_{LR|LR} - p_{LL|RR} - p_{LL|R}p_{R|R|L}}{4 - (p_{L|R})^2}. \quad (1.2.12)$$

We verified this result for the $\mathbb{C}P^1$ case, i.e. for the $\mathfrak{su}(2)$ algebra. It is also possible to find such coordinate system that, the second parenthesis in the numerator of the last equality vanishes.

The authors of the original papers [2], [3], [4], [5], [6] found the expressions for the moving frame of the surface, i.e. the set of $N^2 - 3$ normal vectors, which let to write the explicit form of Gauss-Weingarten equations of the surface. Then they computed the mean curvature and the second fundamental form. They found out that the Gauss–Codazzi–Ricci equations (GCR), which are the compatibility conditions for the Gauss–Weingarten equations, are satisfied for any solution $P$ of the Euler–Lagrange equations (1.1.9). The GCR are the necessary and sufficient conditions for the local existence of the corresponding surface $F$. This result finishes the proof that to every solution of (1.1.9) is associated a surface by (1.2.3). We refer the reader to the above mentioned sources for more information.

### 1.3 Consequences of symmetries of the Euler–Lagrange equations

The Euler–Lagrange equations (1.1.9) are invariant under the transformation

$$P \rightarrow UPU^\dagger, \quad U \in U(N). \quad (1.3.1)$$
At the level of surfaces, such a transformation amounts to a rotation in $\mathbb{R}^{N^2-1}$, so the geometry of the surface is unchanged. This transformation leaves also unchanged the currents $J_L$, $J_R$. Then there is the invariance of the equations (1.1.9) under the parity transformations

$$\xi_L \rightarrow \xi_R, \: \xi_R \rightarrow \xi_L.$$  

(1.3.2)

For the following calculations, an invariance under the conformal transformation of independent variables in the equations (1.1.9)

$$\xi_L \rightarrow \tilde{\xi}_L = f(\xi_L), \: \xi_R \rightarrow \tilde{\xi}_R = g(\xi_R).$$  

(1.3.3)

proves to be helpful. The effect of this transformation on the surface is a re-parametrization of its coordinates. This means that such a transformation can be used to bring the metric (1.2.6) to the Chebyshev’s form (A.11). Particularly, it correspondences to a requirement

$$\tilde{J}_L = \frac{1}{2} \tilde{\rho}_L|_{\tilde{L}} = \text{const.} > 0, \: \tilde{J}_R = \frac{1}{2} \tilde{\rho}_R|R = \text{const.} = \frac{1}{J_L}.$$  

(1.3.4)

It is even possible to require simply the metric of the form (1.2.11), which corresponds to the choice $\tilde{J}_L = \tilde{J}_R = 1$. To find $\tilde{\xi}_L(\xi_L), \: \tilde{\xi}_R(\xi_R)$ means to integrate the equations (1.3.4) with $P$ a solution of (1.1.9) given. In the second part of this work, we shall consider the solutions of $\mathbb{C}P^1$ sigma model obtained by the symmetry reduction. In that case, it will be shown that the currents are constants, so such a transformation amounts to a re-scaling only.

We conclude that the surface associated with a solution of (1.1.9) characterizes the symmetry equivalence class of solutions of (1.1.9).
Chapter 2

Special case: $\mathbb{C}P^1$ sigma model

2.1 Description of surfaces in $\mathfrak{su}(2)$ via $\mathbb{C}P^1$ sigma model

As an example of the general description from the previous section, we shall study the case of the $\mathbb{C}P^1$ sigma model in more detail. Firstly, note that $\mathbb{C}P^1 \simeq S^2$. We consider the map $[z]$ from (1.1.2). If we introduce a complex field $w(\xi_L, \xi_R)$ by an equation

$$ z = \frac{1}{\sqrt{1 + w\bar{w}}} (1, w), \quad (2.1.1) $$

then the action (1.1.2) can be rewritten as

$$ S = \frac{1}{4} \int_{\Omega} \frac{1}{(1 + w\bar{w})^2} (\partial_L w \partial_R \bar{w} + \partial_L \bar{w} \partial_R w) d\xi_L d\xi_R. \quad (2.1.2) $$

To such defined $w(\xi_L, \xi_R)$ corresponds the orthogonal projector (1.1.7) defined by

$$ P = 1 - \frac{1}{1 + w\bar{w}} \begin{pmatrix} 1 & \bar{w} \\ w & w\bar{w} \end{pmatrix}. \quad (2.1.3) $$

The Euler–Lagrange equations (1.1.9) then read

$$ \partial_L \partial_R w = \frac{2}{1 + w\bar{w}} \bar{w} \partial_L w \partial_R w. \quad (2.1.4) $$

The formulas for immersion of the corresponding surface in $\mathbb{R}^3$ are given by

$$ X_1 = \int_{\gamma} \frac{-i}{2(1 + w\bar{w})^2} \left[ (\partial_L w + w^2 \partial_L \bar{w} - \partial_L \bar{w} - \bar{w}^2 \partial_L w) d\xi_L + \right. $$

$$ \left. + (\partial_R \bar{w} + \bar{w}^2 \partial_R w - \partial_R w - w^2 \partial_R \bar{w}) d\xi_R \right], $$

$$ X_2 = \int_{\gamma} \frac{1}{2(1 + w\bar{w})^2} \left[ - (\partial_L w + w^2 \partial_L \bar{w} + \partial_L \bar{w} + \bar{w}^2 \partial_L w) d\xi_L + \right. $$

$$ \left. + (\partial_R w + w^2 \partial_R \bar{w} + \partial_R \bar{w} - \bar{w}^2 \partial_R w) d\xi_R \right], $$

$$ X_3 = \int_{\gamma} \frac{-i}{(1 + w\bar{w})^2} \left[ (w \partial_L \bar{w} - \bar{w} \partial_L w) d\xi_L + (\bar{w} \partial_R w - w \partial_R \bar{w}) d\xi_R \right]. \quad (2.1.5) $$
where $\gamma(\xi_L, \xi_R)$ is any curve in $\Omega$ connecting the point $(\xi_L, \xi_R) \in \Omega$ with an arbitrary chosen fixed point $(\xi^0_L, \xi^0_R) \in \Omega$. The first and second fundamental forms of the surface $\mathcal{F}$ take forms

$$
I = J_L d\xi^2_L - \frac{1}{(1 + w \bar{w})^2} (\partial_L w \partial_R \bar{w} + \partial_R w \partial_L \bar{w}) d\xi_L d\xi_R + J_R d\xi^2_R,
$$

$$
II = 2b_{LR} d\xi_L d\xi_R = \frac{2i}{(1 + w \bar{w})^2} (\partial_L w \partial_R \bar{w} - \partial_R w \partial_L \bar{w}) d\xi_L d\xi_R
$$

(2.1.6)

where $J_L, J_R$ are the currents (1.2.7)

$$
J_L = \frac{1}{(1 + w \bar{w})^2} \partial_L w \partial_L \bar{w}, \quad J_R = \frac{1}{(1 + w \bar{w})^2} \partial_R w \partial_R \bar{w}.
$$

(2.1.7)

We recall that $J_L, J_R$ satisfy the equation of continuity (1.2.8)

$$
\partial_R J_L = 0, \quad \partial_L J_R = 0.
$$

(2.1.8)

For the Gaussian and mean curvature of the surface $\mathcal{F}$ can be derived

$$
K = -4,
$$

$$
H = 2i \frac{1 + \text{tr} (\partial_L PP \partial_R P)^2}{1 - \text{tr} (\partial_L PP \partial_R P)^2} = -2i \frac{\partial_L w \partial_R \bar{w} + \partial_R w \partial_L \bar{w}}{\partial_L \bar{w} \partial_R \bar{w} - \partial_R \bar{w} \partial_L \bar{w}}.
$$

(2.1.9)

Since the Gaussian curvature is a negative constant, solutions of the equation of motion (2.1.4) are by the definition parametrization of pseudo-spherical surfaces in $\mathbb{R}^3$.

The normal to the surface $\mathcal{F}$ in $\mathbb{R}^3$ is given as a normalized vector product of the tangent vectors $\partial_L \mathcal{X} \times \partial_R \mathcal{X}$. A commutator of $[M_R, M_L]$ corresponds to the vector product in $\mathfrak{su}(2)$. It follows that the formula for the normal $n$ takes in $\mathfrak{su}(2)$ the form

$$
n = \frac{[M_R, M_L]}{|[M_R, M_L]|} = -i (1 - 2P), \quad n^\dagger = -n.
$$

(2.1.10)

This correspondence holds only for the case $N = 2$, in the higher dimensions the moving frame is not determined uniquely and one has to use a variant of Gramm-Schmidt procedure to make normal vectors orthogonal. Proofs of all of the statements in this section can be found in [6].

### 2.2 Relation between $\mathbb{C}P^1$ sigma model and the sine-Gordon equation

We know from the section (2.1) that any surface associated with a solution of the $\mathbb{C}P^1$ sigma model has a constant negative Gaussian curvature. From the differential geometry of surfaces arises 1-1 correspondence between pseudo-spherical surfaces and solutions of the
sine-Gordon equation. It follows that one can expect a connection between solutions of the $\mathbb{C}P^1$ sigma model and solutions of the sine-Gordon equation. This link can be effectively described in terms of projectors, which is done in [5]. We introduce this relation shortly here. Different approach yielding the same result is based on a special transformation of $w$ defined in the section (2.1), for more information on this approach see [6].

We consider the metric in Chebyshev coordinates, i.e. in the form

$$I = (d\xi_L)^2 - p_{LR} d\xi_L d\xi_R + (d\xi_R)^2.$$ 

If we specify the term $p_{LR}$, given by (1.2.5), and use the identity (1.2.9), we get

$$I = (d\xi_L)^2 - (\text{tr} \ (P \partial_L P \partial_R P) + \text{tr} \ (P \partial_R P \partial_L P)) \ d\xi_L d\xi_R + (d\xi_R)^2.$$ 

The advantage of Chebyshev coordinates is that the second fundamental form in such coordinates takes this simple form

$$II = 2 (\partial_L P, \partial_R P)) \ d\xi_L d\xi_R.$$ 

We recall the definition of the second fundamental form by scalar products of the normal vector with second derivatives of the tangent vectors (see the definition (A.6) in the appendix A)

$$II = \begin{pmatrix}
\langle X_{LL}, n \rangle & \langle X_{LR}, n \rangle \\
\langle X_{RL}, n \rangle & \langle X_{RR}, n \rangle 
\end{pmatrix}.$$ 

We know from the previous section that a unit normal to the surface is expressed as (2.1.10)

$$n = -i (1 - 2P).$$

Altogether we can calculate

$$II = \frac{2i}{2} \text{tr} \ (\partial_L P, \partial_R P) (1 - 2P)) \ d\xi_L d\xi_R = \frac{2i}{2} (-\text{tr} \ (P \partial_L P \partial_R P) + \text{tr} \ (P \partial_R P \partial_L P)) \ d\xi_L d\xi_R.$$ 

By deriving these relations, we used a cyclic property of the trace and the identities (1.2.9) again.

To finish the proof, it remains to introduce a new function $Q$, or more precisely a real-valued function $u_{sG}(\xi_L, \xi_R)$, by

$$Q = e^{iu_{sG}} = -\text{tr} \ (\partial_L P \partial_P \partial_R P).$$ 

(2.2.1)

Comparing the form of the metric and the second fundamental form in terms of $u_{sG}$

$$I = (d\xi_L)^2 + 2 \cos (u_{sG}) \ d\xi_L d\xi_R + (d\xi_R)^2,$n

$$II = 4 \sin (u_{sG}) \ d\xi_L d\xi_R.$$ 

(2.2.2)
with the classical theorem of the differential geometry of surfaces, we conclude that \( u_{sG} \) satisfies the re-scaled sine–Gordon equation
\[
\partial_L \partial_R u_{sG} = 4 \sin u_{sG}.
\] (2.2.3)

In practical calculations, which we undertake in the section (3.4), we have to rewrite the defining relation (2.2.1) involving the projector \( P \) (2.1.3) in terms of the variable \( w \). We present here only the result:
\[
Q = \sqrt{\frac{\partial_L \bar{w} \partial_R w}{\partial_L w \partial_R \bar{w}}},
\] (2.2.4)

because it is a rather lengthy and straightforward calculation. By inspection we find that \(|Q| = 1\). The equation (2.2.3) is in terms of \( Q \) equivalent to
\[
\partial_L \partial_R \ln Q = 2 \left( \frac{1}{Q} - Q \right).
\] (2.2.5)

If the currents were not constant (but we shall show in the section (3.3) that they are), a term \( J = 2\sqrt{J_L J_R} \) occurs as a multiplication factor on the right-hand side of this equation. To transform away the scaling factor and to obtain the standard form of the sine–Gordon equation, we make a change of variables
\[
\eta_L = 2\xi_L, \quad \eta_R = -2\xi_R.
\] (2.2.6)

Then we have for \( Q \) the equation
\[
\partial_{\eta_L} \partial_{\eta_R} \ln Q = \frac{1}{2} \left( Q - \frac{1}{Q} \right) = i \Im Q.
\] (2.2.7)

The last equation will be used in the section (3.4) to verify the correctness of searched solution of the sine–Gordon equation, because this form is more convenient for numerical calculations.

### 2.3 Symmetry reduction of the \( CP^1 \) sigma model

This part is an adaptation of the corresponding chapter from [6]. For the purposes of the investigation of symmetries it appears to be useful to rewrite the complex-valued function \( w \) (2.1.1) to its real and imaginary part \( w = u + iv \). We then obtain
\[
\begin{align*}
\partial_L \partial_R u &= \frac{2}{1 + u^2 + v^2} \left( u (\partial_L u \partial_R u - \partial_L v \partial_R v) + v (\partial_L v \partial_R u + \partial_L u \partial_R v) \right), \\
\partial_L \partial_R v &= \frac{2}{1 + u^2 + v^2} \left( u (\partial_L v \partial_R u + \partial_L u \partial_R v) - v (\partial_L u \partial_R u - \partial_L v \partial_R v) \right),
\end{align*}
\] (2.3.1)
which is another form of the Euler–Lagrange equations of motion. Then we used the method described in the appendix (B) to find the algebra of symmetry generators. The calculation was done by the standard procedure of the programme Maple 10. We found out that the algebra is infinite dimensional and can be decoupled to the direct sum

$$\mathcal{G} = \mathcal{C}_{\xi_L} \oplus \mathcal{C}_{\xi_R} \oplus \mathfrak{su}(2),$$

(2.3.2)

where $\mathcal{C}_{\xi_D}, D = L, R$ denote infinite dimensional algebras of conformal transformations

$$\mathcal{C}_{\xi_D} = \{ f_D(\xi_D) \partial_{\xi_D} \mid f_D \in C^\infty(\mathbb{R}) \}$$

and $\mathfrak{su}(2)$ is generated by the following transformations involving only dependent coordinates

$$L_1 = u \partial_v - v \partial_u,$$

$$L_2 = \frac{1}{2} (1 + u^2 - v^2) \partial_u + uv \partial_v,$$

$$L_3 = -uv \partial_u + \frac{1}{2} (-1 + u^2 - v^2) \partial_v.$$  

(2.3.3)

Our aim is to obtain solutions of (2.3.1), which are invariant under subgroups of this symmetry group $\mathcal{G}$. This will reduce the original partial differential equations (2.3.1) to a system of ordinary differential equations.

Specially, for the construction of solutions invariant under some 1–parametric subgroup, the conformal factors $f_D(\xi_D)$ can be absorbed into a suitable choice of independent variables, so that we are free to consider only one generator in each $\mathcal{C}_{\xi_D}$,

$$\Xi_L = \partial_{\xi_L}, \quad \Xi_R = \partial_{\xi_R}. $$

By the standard method one finds that all solutions invariant under

$$a \Xi_L + b \Xi_R$$

are singular, i.e. not usable for the construction of the associated surface.

Therefore we have to consider a solution invariant under a vector field which is a combination of $\Xi_L, \Xi_R, L_k$. A vector from $\mathfrak{su}(2)$ itself cannot be used for symmetry reduction, because its orbits are not of codimension 1 in the space of independent variables. We can fix the $\mathfrak{su}(2)$ part of a general vector using the $SU(2)$ symmetry to be $L_1$ and consider only the following vector field

$$Y = L_1 + a \Xi_L + b \Xi_R, \quad a, b \in \mathbb{R}. $$

(2.3.4)

Then the method of characteristics gives a solution invariant under (2.3.4) with a constriction to be of the form

$$w = R(\chi) e^{\frac{i}{2}(\xi_L - f(\chi))}, \quad \chi = \frac{\xi_L}{a} - \frac{\xi_R}{b}, $$

(2.3.5)
where \( R, f : \mathbb{R} \to \mathbb{R} \). Substituting this form of \( w \) into the Euler–Lagrange equation (2.1.4) one finds two coupled ordinary differential equations

\[
R'' - \frac{2R}{1 + R^2} R^2 + \frac{R(1 - R^2)}{1 + R^2} \left( f' - f'^2 \right) = 0, \tag{2.3.6}
\]
\[
f'' + \frac{1 - R^2}{R(1 + R^2)} \left( 2R' f' - R' \right) = 0, \tag{2.3.7}
\]

where \( R', f' \) etc. denote derivatives with respect to \( \chi \). The system (2.3.6),(2.3.7) has a form similar to the one obtained by the symmetry reduction of the equations of the \( \mathbb{C}P^1 \) sigma model in (1+2)–dimensions in [13], which we follow now.

By integrating we rewrite (2.3.7) in an equivalent form

\[
f' = A \left( 1 + \frac{R^2}{R^2} \right) + \frac{1}{2}, \tag{2.3.8}
\]

where \( A \) is a constant of integration. Substituting (2.3.8) into (2.3.6) we get a single second order ODE

\[
R'' - \frac{2R}{1 + R^2} R^2 - A^2 \frac{(1 - R^2)(1 + R^2)^3}{R^3} + \frac{R(1 - R^2)}{4(1 + R^2)} = 0. \tag{2.3.9}
\]

Investigation of the singularity structure of the equation (2.3.9), i.e. after the so called Painlevé analysis, we find that we can transform it into one of the standard Painlevé forms listed in [15]. Performing the change of the dependent variable

\[
R(\chi) = \sqrt{-U(\chi)} \tag{2.3.10}
\]

we find that the function \( U \) obeys the Painlevé equation \( P_{XXXVIII} \) from the list in the standard handbook of ordinary differential equations [15]

\[
U'' = \left( \frac{1}{2U} + \frac{1}{U - 1} \right) U'^2 + 2A^2 \frac{(1 + U)(1 - U)^3}{U} + \frac{U(1 + U)}{2(U - 1)}. \tag{2.3.11}
\]

The order of (2.3.11) can be reduced by integration

\[
U'^2 = -4A^2 U^4 + 4KU^3 + (8A^2 - 8K - 1)U^2 + 4KU - 4A^2, \quad K \in \mathbb{C}. \tag{2.3.12}
\]

A considerable number of solution of (2.3.12) exists, but most of them leads to \( R \) being complex of \( f \) not expressible in terms of elementary functions. Special solutions of this equation will be discussed in the following section.
Part II

Computational part:
Searching for new examples of surfaces
Chapter 3

Solutions of the $\mathbb{CP}^1$ model obtained by the symmetry reduction

3.1 Elliptic solutions

Our first intention is to specify cases that will lead to new examples of surfaces associated with the $\mathbb{CP}^1$ sigma model. As we saw in the section (2.3), all the solutions of the $\mathbb{CP}^1$ model (2.1.4) that are obtained by the symmetry reduction and allow associated surfaces are of the form

$$
\chi = \frac{\xi_L}{a} - \frac{\xi_R}{b}, \quad w(\xi_L, \xi_R) = R(\chi) \ e^{i\left(\xi_L - f(\chi)\right)},
$$

(3.1.1)

where the functions $R(\chi), f(\chi) : \mathbb{R} \rightarrow \mathbb{R}$ are solutions of the following differential equations (an explanation is in the section (2.3), for more details, see [2], [13])

$$
R(\chi) = \sqrt{-U(\chi)}, \quad f(\chi)' = A \frac{(1 + R(\chi)^2)^2}{R(\chi)^2} + \frac{1}{2},
$$

(3.1.2)

$$(U''(\chi))^2 = -4A^2U(\chi)^4 + 4KU(\chi)^3 + (8A^2 - 1 - 8K)U(\chi)^2 + 4KU(\chi) - 4A^2,
$$

(3.1.3)

and $A, K$ are arbitrary real constants. Many solutions of the problem (3.1.3) were found in [13]. In that work, the authors considered system a more general than (3.1.2), (3.1.3):

$$
f(\chi)' = A \frac{(1 + R(\chi)^2)^2}{R(\chi)^2} \cdot g(\chi) - h(\chi), \quad A = \text{const.},
$$

$$
R'' \left(\frac{2R}{1 + R^2} \cdot (R')^2 - \frac{g'}{g} R' - A^2 g^2 \frac{(1 - R^2) \cdot (1 + R^2)^3}{R^3} + \frac{R(1 - R^2)}{1 + R^2} \cdot (h^2 - l) = 0 \right).
$$

(3.1.4)

In the last equation we shorten a notation, i.e. $R = R(\chi), g = g(\chi), h = h(\chi)$ and $l = l(\chi)$. They introduced a function $B(\chi) = 2 \frac{h^2(\chi) - l(\chi)}{g^2(\chi)}$. For $B$ constant and $R(\chi) = \sqrt{-U(\chi)}$, the second equation of the general system (3.1.4) becomes

$$
(U'(\chi))^2 = -4A^2U(\chi)^4 + 4KU(\chi)^3 + (8A^2 - 2B - 8K)U(\chi)^2 + 4KU(\chi) - 4A^2.
$$

(3.1.5)
We denote $U_1, U_2, U_3, U_4$ the roots of the polynomial of the fourth order in a variable $U$

$$-4A^2U^4 + 4KU^2 + (8A^2 - 2B - 8K)U^2 + 4KU - 4A^2 = 0. \quad (3.1.6)$$

We see that the coefficients of this polynomial are given by $A, B, K$. In [13] there are listed over 13 types of particular solutions of the equation (3.1.5). Every type of solutions is determined by the constraints for $A, B, K$ and the four roots $U_i, \ i = 1, \ldots, 4$. Since these roots are expressed by the coefficients of the polynomial (3.1.6), the relations among $U_i$ yield additional constraints for $A, B, K$.

In the two-dimensional $CP^1$ model the function $B$ is fixed to 1/2. Therefore in some of the variants of the solutions, the constraints for the constants $A, B, K$ are inconsistent with each other. We excluded such cases. Some of the remaining variants were investigated in detail in [2], [3], [5], etc. The surfaces associated with the tanh solution, the exponential well solution, the elliptic sine solution, and several solutions obtained via other methods were described there. Here we shall study the solutions of the elliptic type. We recall the known elliptic sine solution (e.g. [3]). It is defined by the conditions

$$A = 0, \ K < 0, \ B > -8K, \ U_3 = U_4 = 0, \ U_2 < U_1 \leq U(\chi) \leq 0,$$

and the relation for $R(\chi) = \sqrt{-U(\chi)}$

$$R(\chi) = \sqrt{-U_1}\sinh\sqrt{\frac{U_2K}{2}\beta}\chi, \quad k = \sqrt{\frac{U_1}{U_2}}.$$

We are going to study the last two unexplored possibilities from the list in [13].

1. In the first case, two roots of the polynomial of the fourth order (3.1.6) are complex conjugate and the other two are real:

$$A \neq 0, \ U_4 < U(\chi) < U_1, \ U_{2,3} = p \pm iq, \ q > 0.$$

Moreover, neither of the real roots $U_1, U_4$ cannot be greater than zero. The reason is, that $R$ is defined as $R(\chi) = \sqrt{-U(\chi)}$ and it is required to be real. Since $U'(\chi)$ is given by the polynomial (3.1.5) that does not have any other real roots than $U_1, U_4$, it is clear that $U$ grows or falls monotonously from $U_1$ to $U_4$ or vice versa. Its behaviour can change only in the points where $U = U_1$ or $U = U_4$ (see Fig. 3.3).

The solution $U(\chi)$ in this case has the form

$$U(\chi) = \frac{CU_4 + DU_1 + (CU_4 - DU_1)\text{cn} [\beta\chi, k]}{C + D + (C - D)\text{cn} [\beta\chi, k]}, \quad (3.1.7)$$

where the constants $C, D, \beta, k$ are expressed in terms of the roots $U_i$ as follows

$$C = \sqrt{(U_1 - p)^2 + q^2}, \quad D = \sqrt{(U_4 - p)^2 + q^2}, \quad (3.1.8)$$

$$k = \sqrt{\frac{(U_1 - U_4)^2 - (C - D)^2}{4CD}}, \quad \beta = 2A\sqrt{CD}. \quad (3.1.9)$$
2. In the second case, all the roots of the polynomial of the fourth order (3.1.6) are real and mutually distinct:

\[ A \neq 0, \ U_4 \leq U(\chi) \leq U_3 < U_2 < U_1. \]

For the same reason as in the case (3.1.7), the roots \( U_3, U_4 \) can not be greater than zero. If \( B \) is an arbitrary constant, there can occur two alternatives. The first one is \( U_1 < 0 \), then we must have \( B > 4(A^2 + (-K)) > 0 \). In the second alternative is \( U_4 < U_3 < 0 < U_2 < U_1 \), with no other restrictions.

The solution \( U(\chi) \) in this case is given by (see Fig. 3.4)

\[ U(\chi) = \frac{U_4(U_1 - U_3) + U_1(U_3 - U_4) \text{sn}[\beta \chi, k]^2}{U_1 - U_3 + (U_3 - U_4) \text{sn}[\beta \chi, k]^2}, \quad (3.1.10) \]

where the constants \( k, \beta \) are again defined by the roots \( U_i \)

\[ k = \sqrt{\frac{(U_1 - U_2)(U_3 - U_4)}{(U_1 - U_3)(U_2 - U_4)}}, \quad \beta = A\sqrt{(U_1 - U_3)(U_2 - U_4)}. \quad (3.1.11) \]

We also mention a modification of this solution, which has the form

\[ U(\chi) = \frac{U_1(U_2 - U_4) + U_4(U_1 - U_2) \text{sn}[\beta \chi, k]^2}{U_2 - U_4 + (U_1 - U_2) \text{sn}[\beta \chi, k]^2}, \quad (3.1.12) \]

with \( k, \beta \) as in the equation (3.1.11). To this solution we add the constraints

\[ A \neq 0, \ U_4 < U_3 < U_2 \leq U(\chi) \leq U_1 < 0, \ B > 4(A^2 + (-K)) > 0. \]

In all of these formulas, the parameter of the elliptic sine and cosine \( k \) is required to lie in the closed interval \([0, 1]\), otherwise the elliptic functions are not real-valued. In all of the cases (3.1.7), (3.1.10) and (3.1.12) the parameter \( k \) is defined as a function of the roots \( U_i \), thus is a function of the parameters \( A, K \). The condition \( 0 \leq k \leq 1 \) leads to additional constraints for \( A, K \).

We verified analytically, that \( U \) according to (3.1.7), (3.1.10) and (3.1.12) are solutions of the equation (3.1.3). There were some misprints in the original source [13], which we had to correct.

### 3.2 Allowed values of parameters

The first task, which we shall undertake, is to find parameters \( A, K \) such that, together with \( B \) fixed to \( \frac{1}{2} \), all of the above mentioned conditions are satisfied. Some of the inequalities for \( A, K \) are derived from the properties of the roots of the polynomial (3.1.6). Therefore these inequalities are also of the fourth order. It is clear that it would be too complicated...
to solve them in general. For our purposes, it is enough to determine sufficient conditions to fulfil all of the conditions. We begin with the general formula for the four roots of the polynomial (3.1.6)

\[
U_{1,4} = \frac{K - \sqrt{16A^4 + s}}{4A^2} \pm \frac{\sqrt{K^2 - 2K \sqrt{16A^4 + s} + s}}{4A^2}, \\
U_{2,3} = \frac{K + \sqrt{16A^4 + s}}{4A^2} \pm \frac{\sqrt{K^2 + 2K \sqrt{16A^4 + s} + s}}{4A^2},
\]

(3.2.1)

where \(s = K^2 - (1 + 8K)A^2\). We notice, that, due to a symmetry of the coefficients of the polynomial (3.1.6), its non-zero roots are always in pairs reciprocal, i.e.

\[U_1 = U_4^{-1}, \quad U_2 = U_3^{-1}.\]

The Viét formulas reduce to

\[
U_1 + \frac{1}{U_1} + U_2 + \frac{1}{U_2} = \frac{K}{A^2}, \\
2 + U_1U_2 + \frac{1}{U_1U_2} + \frac{U_1}{U_2} + \frac{U_2}{U_1} = -2 + \frac{1}{4A^2} + \frac{2K}{A^2}.
\]

The roots of the following biquadratic equation for \(A\)

\[16A^4 + s = 16A^4 - (1 + 8K)A^2 + K^2 = 0\]

are \(\pm \frac{1}{8} (1 \pm \sqrt{1 + 16K})\). We conclude that the sufficient (but not necessary) condition for \(16A^4 + s > 0\) is the choice \(K < -\frac{1}{16}\). Now we shall distinguish three possible cases:

- The choice \(K = -\frac{1}{16}\) gives all of the four roots equal.
- The choice \(K < -\frac{1}{16}\) will lead to the first case (3.1.7) as we shall show below. For such \(K\), we distinguish two further cases,

  \(K^2 + s \geq 0\): The inequality \(K^2 - 2K \sqrt{16A^4 + s} + s > 0\) is satisfied trivially, whereas the inequality \(K^2 + 2K \sqrt{16A^4 + s} + s > 0\) leads to a contradiction. (The s-roots of \(K^2 \pm 2K \sqrt{16A^4 + s} + s = 0\) are \(K^2 \pm 8KA^2\). A comparison with \(K^2 - (1 + 8K)A^2 < K^2 + 8KA^2\) shows, that it is possible to fulfil the above inequality for \(K < 0\) only if \(-\frac{1}{16} < K\).

  \(K^2 + s < 0\): The inequality \(K^2 + 2K \sqrt{16A^4 + s} + s > 0\) is never satisfied for \(K < -\frac{1}{16}\), whereas the inequality \(K^2 - 2K \sqrt{16A^4 + s} + s > 0\) holds for every \(K < -\frac{1}{16}\) (for the same reason as in the parenthesis above).

Thus we have two real roots \(U_1, U_4\) and two complex conjugate roots \(U_2, U_3\) for all \(K < -\frac{1}{16}\). If we want to consider the case (3.1.7), we need also to fulfil the inequality for the elliptic parameter \(0 \leq k \leq 1\). The expression for \(k\) from (3.1.9) as a function of \(A, K\) is too complicated just to write down explicitly, let alone to solve the above
mentioned inequalities. Therefore, we consider the graph of the dependence (see Fig. 3.1). We can easily read a broad range of allowed parameters from that graph. For the following computations, we have chosen the sets of $A, K$ according to the Table 3.1. It is interesting to compare the changing periods $2\pi(\theta_3[0, q(k)])^2$ of the elliptic cosine in the dependence on $k$, see Fig. 3.2. We notice that $q(k)$ is the elliptic nome. We also plotted the corresponding graph of the function $U$ for the case (3.1.7), Fig. 3.3.

![Graph of the dependence of the elliptic parameter $k$ as a function of $A, K$ for the case (3.1.7). The dependence is even in $A$.](image)

**Figure 3.1:** The graph of the dependence of the elliptic parameter $k$ as a function of $A, K$ for the case (3.1.7). The dependence is even in $A$.

<table>
<thead>
<tr>
<th>$K$</th>
<th>$A$</th>
<th>$U_1$</th>
<th>$U_2$</th>
<th>$k$</th>
<th>$2\pi(\theta_3[0, q(k)])^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>1</td>
<td>-0.39083</td>
<td>0.97474+0.22332i</td>
<td>0.04947</td>
<td>2.00123\pi</td>
</tr>
<tr>
<td>-1/10</td>
<td>1/10</td>
<td>-0.10207</td>
<td>-0.05051+0.99872i</td>
<td>0.65073</td>
<td>2.28438\pi</td>
</tr>
<tr>
<td>-10/159</td>
<td>1/10</td>
<td>-0.24313</td>
<td>-0.96663+0.25617i</td>
<td>0.97787</td>
<td>3.78476\pi</td>
</tr>
</tbody>
</table>

**Table 3.1:** Testing sets of allowed parameters for the case (3.1.7)

- (The case $K > -\frac{1}{16}$). In further analysis of the formulas (3.2.1), we want to fulfil sufficient conditions for the second solution, i.e. for (3.1.10). Therefore we limit ourselves to the case

$$K > -\frac{1}{16} \quad \text{and} \quad s = K^2 - (1 + 8K)A^2 > 0.$$  

The inequality $s > 0$ gives a condition for $A$

$$-\frac{|K|}{\sqrt{8K+1}} < A < \frac{|K|}{\sqrt{8K+1}}.$$  

(3.2.2)
Figure 3.2: The dependence of the period of elliptic functions on the elliptic parameter $k$ (in the multiples of $\pi$)

Figure 3.3: The graph of the elliptic solution $U(\chi)$ of (3.1.3) given by the formula (3.1.7) (for the choice of the parameters $A = 1$, $K = -1$)

Figure 3.4: The graph of the elliptic solution $U(\chi)$ of (3.1.3) given by the formula (3.1.10) (for the choice of the parameters $K = -\frac{1}{32}$, $A = K/(2\sqrt{8K + 1})$)
Figure 3.5: The graph of the dependence of the elliptic parameter $k$ as a function of $A, K$ for the case (3.1.10). The dependence is even in $A$.

Consequently, the term under the square root $16A^4 + s$ is also positive and we can proceed similarly as in the preceding paragraph. The inequalities

$$K^2 \pm 2K\sqrt{16A^4 + s} + s > 0$$

hold for every $K > -\frac{1}{16}$ and $A$ in the interval given by (3.2.2). These special conditions are examples of the situation in which we obtain four real roots. For $K < 0$ their ordering is in accordance with the numbering in (3.2.1). In the case $K > 0$, we must change the pairs to get $U_4 < U_3 < U_2 < U_1$.

<table>
<thead>
<tr>
<th>$K$</th>
<th>$A = K/(2\sqrt{8K + 1})$</th>
<th>$U_1$</th>
<th>$U_2$</th>
<th>$k$</th>
<th>$2\pi(\theta_3[0, q(k)])^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1/32</td>
<td>-0.01804</td>
<td>-0.01116</td>
<td>-0.16072</td>
<td>0.14983</td>
<td>2.01137π</td>
</tr>
<tr>
<td>-1/17</td>
<td>-0.04042</td>
<td>-0.02969</td>
<td>-0.59061</td>
<td>0.57094</td>
<td>2.20194π</td>
</tr>
<tr>
<td>-500/8001</td>
<td>-0.04419</td>
<td>-0.03336</td>
<td>-0.97638</td>
<td>0.97477</td>
<td>3.70519π</td>
</tr>
</tbody>
</table>

Table 3.2: Testing sets of allowed parameters for the case (3.1.10)

We notice that from the formulas (3.2.1) it furthermore follows, that the general variant (for $B$ arbitrary constant) $U_1 < U_3 < 0 < U_2 < U_1$ is now excluded. It is clear that if $K > 0$ then $U_1$ is (after the above mentioned reordering) always greater than 0. Consequently $U_4 = U_1^{-1} > 0$. Analogous argument can be made for $K < 0$, because then $U_4$ is surely negative. To sum up, using the condition $U(\chi) < 0$, we have only one allowed case $K < 0$, and $U_4 < U_3 < U_2 < U_1 < 0$. 

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The condition $B = \frac{1}{2} > 4(A^2 + (-K)) > 0$ is satisfied for all $-\frac{1}{16} < K < 0$ and $A$ such that (3.2.2) holds. To fit $A, K$ so that we have $0 \leq k \leq 1$, we made a graph of the dependence of $k$ from (3.1.11) as a function of $A, K$ (see Fig. 3.5). Our choice of suitable parameters is in the Table 3.2. The function $U$ for the case (3.1.10) is plotted in the Figure 3.4.

An interesting problem is to find such parameters that in the both cases (3.1.7), (3.1.10) the elliptic parameter $k$ equals $k = 0$, or $k = 1$. The elliptic functions then reduce to goniometric or hyperbolic functions. The case $k = 0$ for both of the considered cases reduces to a trivial solution. However, if we found such parameters to have $k = 1$, we could obtain simpler but nontrivial examples of surfaces, which we are searching for. The problem for $A, K$ to have $k = 1$ was postponed to future research, because the constraints for $A, K$ are deeply non-trivial. We hope that we will report on these tasks soon. Analogous possibly fruitful question is to discuss the case

$$A = \sqrt{\frac{8K + 1}{8}},$$

when the coefficient of $U(\chi)^2$ in the polynomial (3.1.3) disappears. This will also be undertaken in our future work.

### 3.3 Computing of conserved currents

We have shown in the section (1.2) that the currents $J_L, J_R$ defined by the equation (2.1.7) satisfy the equations of continuity in the form (1.2.8). Now, we shall prove that in all the solutions of the $\mathbb{CP}^1$ sigma model constructed by the symmetry reduction (2.3), the currents are constant. This fact proves to be very helpful in the section (3.4) for a reduction of the $\mathbb{CP}^1$ sigma model to the sine-Gordon equation. If we want to determine $J_L, J_R$, it is necessary to substitute the reduced solution $\bar{w}$ from (3.1.1) into the general equation for currents (2.1.7):

$$J_L(\xi_L, \xi_R) = \frac{1}{(1 + w(\xi_L, \xi_R)\bar{w}(\xi_L, \xi_R))^2} \partial_L w(\xi_L, \xi_R) \partial_L \bar{w}(\xi_L, \xi_R) =$$

$$= \frac{1}{(1 + R(\chi)^2)^2} \left( \partial_L R(\chi) e^{i(\frac{\xi_L}{a} - f(\chi))} + R(\chi) \frac{i}{a} (1 - f'(\chi)) e^{i(\frac{\xi_R}{b} - f(\chi))} \right) \cdot$$

$$\cdot \left( \partial_L R(\chi) e^{-i(\frac{\xi_L}{a} - f(\chi))} - R(\chi) \frac{i}{a} (1 + f'(\chi)) e^{-i(\frac{\xi_R}{b} - f(\chi))} \right) =$$

$$= \frac{1}{(1 + R(\chi)^2)^2} \left( (\partial_L R(\chi))^2 + R(\chi)^2 \frac{(-1)^2}{a^2} (1 - f'(\chi))^2 \right).$$

We recall that $\chi = \frac{\xi_L}{a} - \frac{\xi_R}{b}$, and $f'(\chi)$ denotes the derivative with respect to $\chi$. 

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Similarly for $J_R$

$$J_R(\xi_L, \xi_R) = \frac{1}{(1 + w(\xi_L, \xi_R) \bar{w}(\xi_L, \xi_R))^2} \frac{1}{(1 + R(\chi))^2} \left( (\partial_R R(\chi))^2 + R(\chi)^2 \frac{1}{b^2} (f'(\chi))^2 \right).$$

Using the differential equation (3.1.2) for $f(\chi)$ we find

$$J_L(\chi) = \frac{1}{(1 - U(\chi))^2} \left( \left( \frac{1}{2a} \frac{U'(\chi)}{\sqrt{-U(\chi)}} \right)^2 - U(\chi) \frac{1}{a^2} \left( -A \frac{1 + R(\chi)^2}{R(\chi)^2} + \frac{1}{2} \right)^2 \right) =$$

$$= \frac{1}{(1 - U(\chi))^2} \left( \frac{1}{4a^2} \frac{U'(\chi)^2}{U(\chi)} - U(\chi) \frac{1}{a^2} \left( A \frac{1 - U(\chi)^2}{U(\chi)} + \frac{1}{2} \right)^2 \right) =$$

$$= \frac{1}{4a^2 U(\chi)(1 - U(\chi))^2} \left( (U'(\chi))^2 + (2A(1 - U(\chi))^2 + U(\chi))^2 \right).$$

For $J_R$ we get an analogous expression

$$J_R(\chi) = \frac{1}{(1 - U(\chi))^2} \left( \frac{1}{4b^2} \frac{U(\chi)^2}{U(\chi)} - U(\chi) \frac{1}{b^2} \left( -A \frac{1 - U(\chi)^2}{U(\chi)} + \frac{1}{2} \right)^2 \right) =$$

$$= \frac{1}{4b^2 U(\chi)(1 - U(\chi))^2} \left( (U'(\chi))^2 + (2A(1 - U(\chi))^2 - U(\chi))^2 \right).$$

Recalling the differential equation for $U(\chi)$ (3.1.3) we find that the currents have a constant value, namely

$$J_L = \frac{4A^2 - K - A}{a^2}, \quad J_R = \frac{4A^2 - K + A}{b^2}. \quad (3.3.1)$$

To make some of the following expressions simpler, we choose the constants $J_L, J_R$ to be equal to $1/4$. That leads to the requirement

$$a = 2\sqrt{4A^2 - K - A}, \quad b = -2\sqrt{4A^2 - K + A}. \quad (3.3.2)$$

Such choice amounts to a suitable choice of the parametrization of the corresponding surfaces due to a conformal invariance, or more precisely to an invariance under a re-scaling.

### 3.4 Associated solutions of the sine-Gordon equation

We want to find a solution of the sine-Gordon equation which is associated with the solution of the $\mathbb{C}P^1$ model (3.1.1) given by the reduced solution $U$ of (3.1.3) for the cases (3.1.7), (3.1.10). The calculation of the function $Q$ given by (2.2.1) proceeds in the same way as the calculation of the currents $J_L, J_R$. From the defining relation for
\[ Q(\xi_L, \xi_R)^2 = \left( \frac{\partial_L \bar{w}(\xi_L, \xi_R) \partial_R \bar{w}(\xi_L, \xi_R)}{\partial_L w(\xi_L, \xi_R) \partial_R \bar{w}(\xi_L, \xi_R)} \right)^2 = \]
\[ = \left\{ \frac{1}{a} R'(\chi) - \frac{i}{a} R(\chi) (1 - f'(\chi)) \right\} \cdot \left\{ -\frac{i}{b} R'(\chi) + \frac{i}{b} R(\chi) f'(\chi) \right\} = \]
\[ = -(R'(\chi))^2 + R(\chi)^2 (1 - f'(\chi))^2 f'(\chi) + iR(\chi) R'(\chi) = \]
\[ = -(R'(\chi))^2 + R(\chi)^2 (1 - f'(\chi))^2 f'(\chi) - iR(\chi) R'(\chi) = \]
\[ = \frac{(U''(\chi))^2 - U(\chi) \left( \frac{1}{2} + A \frac{(1-U(\chi))^2}{U'(\chi)} \right)}{(U'(\chi))^2 - U(\chi) \left( \frac{1}{2} + A \frac{(1-U(\chi))^2}{U'(\chi)} \right)} - \frac{i}{2} U'(\chi) = \]
\[ = \frac{(U''(\chi))^2 - U(\chi) \left( \frac{1}{2} + A \frac{(1-U(\chi))^2}{U'(\chi)} \right)}{(U'(\chi))^2 - U(\chi) \left( \frac{1}{2} + A \frac{(1-U(\chi))^2}{U'(\chi)} \right)} + \frac{i}{2} U'(\chi) \]
\[ = \frac{(U'(\chi))^2 - U(\chi)^2 + 4A^2 (1 - U(\chi))^2 - 2iU(\chi) U'(\chi)}{(U'(\chi))^2 - U(\chi)^2 + 4A^2 (1 - U(\chi))^2 + 2iU(\chi) U'(\chi)}. \]

We used the relations (3.1.1), (3.1.2). The prime denotes the derivative with respect to \( \chi \). After substitution of \((U')^2\) from (3.1.3), and elimination of the imaginary part from the denominator, we get
\[ Q(\chi) = \frac{\sqrt{(8A^2 - 2K)(U(\chi) - 1)^2 + U(\chi) + iU'(\chi)}}{\sqrt{(8A^2 - 2K)(U(\chi) - 1)^2 + U(\chi) - iU'(\chi)}} \]
\[ = \frac{1}{2 \sqrt{(4A^2 - K)^2 - A^2}} \left( \frac{8A^2 - 2K + U(\chi) + iU'(\chi)}{(U(\chi) - 1)^2} \right). \]

The logarithm of \( Q \) is purely imaginary, because \(|Q| = 1 \), and reads
\[ \ln Q(\chi) = i \arctan \frac{U'(\chi)}{U(\chi) + (8A^2 - 2K)(U(\chi) - 1)^2}. \quad (3.4.1) \]

We verified analytically, that (after appropriate changes of variables (2.2.6)) such \( Q \) is a solution of the equation (2.2.7) for any \( U \) that is a solution of (3.1.3) and \( a, b \) chosen according to (3.3.2).

The associated solution of the sine-Gordon equation is then
\[ u_{SG}(\chi) = \arctan \frac{U'(\chi)}{U(\chi) + (8A^2 - 2K)(U(\chi) - 1)^2}. \]

It can be rewritten using the identity \( \arctan x = \arcsin \frac{x}{\sqrt{1+x^2}} \) in a form which is more convenient for us:
\[ u_{SG}(\chi) = \arcsin \frac{U'(\chi)}{\sqrt{(U'(\chi))^2 + (U(\chi) + (8A^2 - 2K)(U(\chi) - 1)^2)^2}}. \]

This gets further simplified after substitution of \((U'(\chi))^2\) from (3.1.3) into the denominator to
\[ u_{SG}(\chi) = \arcsin \left( \frac{1}{2 \sqrt{(4A^2 - K)^2 - A^2}} \right) \frac{U'(\chi)}{U(\chi) - 1)^2} = \arcsin \left( \frac{-2U'(\chi)}{ab(U(\chi) - 1)^2} \right). \]

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Now we shall construct a concrete form of the solution of the sine–Gordon equation for the first case (3.1.7). If we want to find these terms using the solution (3.1.7), we must calculate \( U'(\chi), (U(\chi) − 1)^2 \)

\[
U'(\chi) = \frac{2\beta CD(U_1 - U_4) \text{dn}[\beta \chi, k] \cdot \text{sn}[\beta \chi, k]}{(C + D + (C - D) \text{cn}[\beta \chi, k])^2},
\]

\[
(U(\chi) − 1)^2 = \left(\frac{C(U_4 - 1) + D(U_1 - 1) + (C(U_4 - 1) - D(U_1 - 1)) \text{cn}[\beta \chi, k]}{C + D + (C - D) \text{cn}[\beta \chi, k]}\right)^2.
\]

To express the required quantities, let us denote

\[
c = 8A^2 - 2K, \quad V = C(U_4 - 1) + D(U_1 - 1), \quad W = C(U_4 - 1) - D(U_1 - 1), \quad X = C + D, \quad Y = C - D, \quad Z = 2\beta CD(U_1 - U_4).
\]

We obtain \( Q \) and \( u_{sG} \)

\[
Q(\chi) = \frac{-2}{ab} \left( c + \frac{(V + X + (W + Y) \text{cn}[\beta \chi, k]) \cdot (X + Y \text{cn}[\beta \chi, k]) + iZ \text{dn}[\beta \chi, k] \cdot \text{sn}[\beta \chi, k]}{(V + W \text{cn}[\beta \chi, k])^2} \right),
\]

\[
u_{sG}(\chi) = \frac{Z \text{dn}[\beta \chi, k] \cdot \text{sn}[\beta \chi, k]}{(V + X + (W + Y) \text{cn}[\beta \chi, k]) \cdot (X + Y \text{cn}[\beta \chi, k]) + c(V + W \text{cn}[\beta \chi, k])^2}.
\]

The form using arcsin is

\[
u_{sG}(\chi) = \frac{-4\beta CD(U_1 - U_4) \text{dn}[\beta \chi, k] \cdot \text{sn}[\beta \chi, k]}{ab(D(U_1 - 1) + C(U_4 - 1) + (C(U_4 - 1) - D(U_1 - 1)) \text{cn}[\beta \chi, k])^2} = \frac{-Z \text{dn}[\beta \chi, k] \cdot \text{sn}[\beta \chi, k]}{ab(V + W \text{cn}[\beta \chi, k])^2}.
\]

We verified numerically for the parameters \( A, K \) from the Table 3.1, that it is a solution of the sine-Gordon equation (2.2.3).

We will investigate this solution in more detail. In order to transform away the uniform motion of the wave, we make the following transformations. Firstly, we transform the light-cone coordinates \( \eta_L, \eta_R \) to standard Minkowski coordinates \( X_0, T_0 \). We get the usual hyperbolic form of the sine-Gordon equation

\[
X_0 = \eta_L + \eta_R, \quad T_0 = \eta_L - \eta_R, \quad \partial_{T_0} u_{sG} - \partial_{X_0} u_{sG} + \sin u_{sG} = 0.
\]

Secondly, we use the fact that the sine-Gordon equation is invariant with respect to the Lorentz boost transformation

\[
X = \frac{X_0 - VT_0}{\sqrt{1 - V^2}}, \quad T = \frac{T_0 - VX_0}{\sqrt{1 - V^2}}.
\]
The original variables \( \eta_L, \eta_R \) are then given by \( X = \alpha \eta_L + \frac{1}{\alpha} \eta_R, \ T = -\alpha \eta_L + \frac{1}{\alpha} \eta_R \), where \( \alpha = \sqrt{\frac{1+V}{1-V}} \) and \( V \) is a suitable velocity. Consequently, these two transformations correspond to a re-scaling of variables \( \xi_L = \eta_L, \ \xi_R = -\eta_R \):

\[
\tilde{\xi}_L = \frac{X - T}{2} = \alpha \xi_L, \quad \tilde{\xi}_R = \frac{X + T}{2} = \frac{1}{\alpha} \xi_R.
\]

The choice \( \alpha = \sqrt{-\frac{b}{a}} \) leads to the equation \( \frac{d^2}{dX^2} \tilde{u}_{sG}(\frac{1}{\sqrt{-ab}} X) = \sin \tilde{u}_{sG}(\frac{1}{\sqrt{-ab}} X) \). Then \( \tilde{u}_{sG}(X) = u_{sG}(\frac{1}{\sqrt{-ab}} X) \) is a stationary solution of the sine-Gordon equation

\[
\frac{d^2}{dX^2} \tilde{u}_{sG}(X) = \sin \tilde{u}_{sG}(X).
\]

This is an oscillating solution a little resembling sine (see Fig. 3.6). In the case (3.1.7) the factor in the argument of elliptic functions \( \sqrt{-ab} \) equals one, therefore the period of the function \( u_{sG}(X) \) is the same as for the elliptic sine (see Tab. 3.1 for numeric values).

![Figure 3.6: The solution of the sine-Gordon equation associated with (3.1.7) for the choice of the parameters \( A = 1, K = -1 \)](image)

The solution of the sine–Gordon equation for the second case (3.1.10) can be obtained by a similar procedure and we postponed its calculation, because we expect that it will be of the same type as already found one. For future research there are some open questions like the finiteness of the obtained solution and its physical interpretation, e.g. its energy etc.
Chapter 4

Searching for corresponding solutions of the $\mathbb{C}P^1$ sigma model

4.1 Formulation of the problem for $f$

Our intention is to find the formulas for the corresponding solutions of the $\mathbb{C}P^1$ sigma model $w$. We know from the previous chapters that $w$ is given by

$$w(\chi) = R(\chi) \, e^{i \frac{4\lambda}{\alpha} - f(\chi)}$$  \hfill (4.1.1)

where $\chi = \frac{\xi_L}{a} - \frac{\xi_R}{b}$, $R(\chi) = \sqrt{-\frac{1}{U(\chi)}}$ and $U(\chi)$ is one of the two considered solutions of (3.1.3). To get $w(\chi)$, we have to solve the differential equation (3.1.2) for $f$

$$f'(\chi) = A \frac{(1 + R(\chi)^2)^2}{R(\chi)^2} + \frac{1}{2}.$$  

We found out that in both of the investigated cases, $f(\chi)$ can be obtained by integration of the same expression. To prove this, we firstly consider the case (3.1.10). When we express the derivative of $f(\chi)$ in terms of the solution $U(\chi)$ from (3.1.10), it is useful to decompose it to partial fractions:

$$f'(\chi) = \frac{1}{2} - \frac{A}{U_1} \left( \frac{(U_1 - 1)^2}{1 - kU_1^{-1} \text{sn}[\beta_X, k]^2} + \frac{U_1^2 - 1}{1 - kU_1 \text{sn}[\beta_X, k]^2} \right),$$  \hfill (4.1.2)

where we used the fact that the roots $U_i$ are in pairs reciprocal, i.e. $U_1 = U_1^{-1}, U_2 = U_3^{-1}$. Because of that the expression for the elliptic modulus $k$ from (3.1.11) simplifies to

$$k = \sqrt{\frac{(U_1 - U_2) \cdot (U_3 - U_4)}{(U_1 - U_3) \cdot (U_2 - U_4)}} = \frac{U_1 - U_2}{1 - U_1 U_2}.$$  

The terms arising in the denominators of (4.1.2) share the common subexpression

$$\frac{U_3 - U_4}{U_1 - U_3} = -\frac{k}{U_1}.$$  

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We denote this term
\[(\alpha_1)^2 = -k U_1^{-1}.\]  
Similarly we get
\[(\alpha_2)^2 = -k U_1.\]

Notice that both of them are positive. This follows from the previous chapter, where we showed \(U_1 < 0\) and \(0 \leq k \leq 1\).

We conclude that we transformed the problem for \(f(\chi)\) to the integration of \((4.1.2)\) and consequently of
\[
\Lambda(u, \alpha, k) = \int_0^u \frac{dv}{1 - \alpha^2 \text{sn}^2[v, k]}, \quad \text{or} \quad \Lambda(u, i\alpha, k) = \int_0^u \frac{dv}{1 + \alpha^2 \text{sn}^2[v, k]}, \quad \alpha > 0. 
\]

For the case \((3.1.7)\) the treatment is analogous. Decomposition to partial fractions gives
\[
f'(\chi) = \frac{1}{2} - \frac{(CU_4 - DU_1 - C + D)^2}{(C - D) \cdot (CU_4 - DU_1)} \frac{2CD(U_1 - U_4)}{2CD(U_1 - U_4)} + \frac{(C - D) \cdot (C + D + (C - D) \text{cn}[\beta \chi, k])}{(CU_4 - DU_1) \cdot (CU_4 + DU_1 + (CU_4 - DU_1) \text{cn}[\beta \chi, k])}.
\]

We recall that \(C, D\) are given by the roots \(U_i\), see \((3.1.8)\). Since the roots \(U_i\) are in pairs reciprocal, some terms can be simplified: \(C - D = CU_4 - DU_1\) and \(C + D = -(CU_4 + DU_1)\). One term of the constant part of \(f'(\chi)\) vanishes. We also write the remaining terms of \(f'(\chi)\) in the form of common factors in the numerators and the denominators.

\[
\alpha = \frac{C - D}{C + D}; \quad \gamma = \frac{2CD(U_1 - U_4)}{(C + D) \cdot (C - D)}; 
\]

\[
f'(\chi) = \frac{1}{2} - \frac{\gamma}{1 + \alpha \text{cn}[\beta \chi, k]} - \frac{\gamma}{1 - \alpha \text{cn}[\beta \chi, k]}.
\]

We shall show that the integrals of the last fractions can be transformed to the integral \((4.1.5)\) of \(\Lambda(u, \tilde{\alpha}, k)\). Let us firstly suppose \(\alpha^2 < 1\). An easy calculation shows
\[
\int \frac{du}{1 + \alpha \text{cn}[u, k]} = \int \frac{(1 - \alpha \text{cn}[u, k])du}{1 - \alpha^2 + \alpha^2 \text{sn}^2[u, k]} = \frac{1}{1 - \alpha^2} \left( \Lambda(u, \tilde{\alpha}, k) - \alpha \int \frac{\text{cn}[u, k]du}{1 + \alpha^2 \text{sn}^2[u, k]} \right),
\]
where \(\tilde{\alpha} = \frac{|\alpha|}{\sqrt{1 - \alpha^2}}\). The last integral is then evaluated by a sequence of substitutions
\[
v = \frac{\text{cn}[u, k]}{\text{sn}[u, k]}, \quad dv = -\frac{\text{dn}[u, k]}{\text{sn}^2[u, k]} du,
\]
\[\text{sn}^2[u, k] = \frac{1}{1 + v^2}, \quad \text{cn}[u, k] = \frac{v}{\sqrt{1 + v^2}}, \quad \text{dn}[u, k] = \frac{\sqrt{v^2 + k^2}}{\sqrt{1 + v^2}},
\]
\[w = v^2 + k^2, \quad dw = 2vdv,
\]
\[t = \sqrt{w}, \quad dt = \frac{1}{2\sqrt{w}} dw.
\]
\[
\int \frac{\text{cn}[u, k]}{1 + \alpha^2 \text{sn}^2[u, k]} \, du = \int \frac{-v \, dv}{\sqrt{k^2 + v^2(1 + \alpha^2 + v^2)}} = \int \frac{-dw}{2 \sqrt{w(k^2 + \alpha^2 + w)}} = \int \frac{-dt}{k^2 + \alpha^2 + t^2}
\]

Finally, after a few standard manipulations, we conclude that if \(\alpha^2 < 1\), we have
\[
\Sigma(u, \alpha, k) = \int_0^u \frac{dv}{1 + \alpha \text{cn}[v, k]} = \frac{1}{1 - \alpha^2} \Lambda \left( u, \frac{|\alpha|}{\sqrt{1 - \alpha^2}}, k \right) - \frac{\alpha}{\sqrt{(1 - \alpha^2)(k^2 + k^2\alpha^2)}} \cdot \arctan \left( \frac{k^2 + k^2\alpha^2 \text{sn}[u, k]}{\sqrt{1 - \alpha^2} \text{dn}[u, k]} \right).
\]

By analogous procedure one can derive a similar formula in the second case \(\alpha^2 > 1\)
\[
\Sigma(u, \alpha, k) = \int_0^u \frac{dv}{1 + \alpha \text{cn}[v, k]} = -\frac{1}{\alpha^2 - 1} \Lambda \left( u, \frac{|\alpha|}{\sqrt{\alpha^2 - 1}}, k \right) - \frac{\alpha}{\sqrt{(\alpha^2 - 1)(k^2 + k^2\alpha^2)}} \cdot \text{arctanh} \left( \frac{k^2 + k^2\alpha^2 \text{sn}[u, k]}{\alpha^2 - 1 \text{dn}[u, k]} \right).
\]

### 4.2 Evaluation of the elliptic integral \(\Pi(u, a, k)\)

The starting point in the integration of \(\Lambda(u, \alpha, k)\), given by (4.1.5), is to introduce the integral
\[
\Pi(u, a, k) = \int_0^u k^2 \text{sn}[a, k] \cdot \text{cn}[a, k] \cdot \text{dn}[a, k] \frac{\text{sn}^2[v, k]}{1 - k^2 \text{sn}^2[a, k] \cdot \text{sn}^2[v, k]} \, dv,
\]
which can be found for example in the handbook of the elliptic functions [14] as a form of an elliptic integral of the third kind. In that book the integral of \(\Lambda(u, \alpha, k)\) is given as an exercise with instructions and results. We shall compute it in detail in the section (4.3).

From the following paragraph further, we will shorten the notation and suppose that the second argument of all of the elliptic functions is equal to \(k\). We shall write \(k\) only, where it is necessary and also in the expressions \(\Pi(u, a, k)\), \(\Lambda(u, \alpha, k)\).

To express \(\Lambda(u, \alpha, k)\) in terms of \(\Pi(u, a, k)\), it is sufficient to choose \(a\) such that
\[
\text{sn} a = \frac{\alpha}{k}.
\]

Then, after a simple calculation, we find the relation
\[
\Lambda(u, \alpha, k) = u + \frac{\text{sn} a}{\text{cn} a \cdot \text{dn} a} \cdot \Pi(u, a, k).
\]

The evaluation of the integral \(\Pi(u, a, k)\) will be performed according to [14]. We want to rewrite the integrand of \(\Pi(u, a, k)\) so that it decomposes to parts, whose integrals are easy to compute. For this purpose, we use the addition formula for the elliptic sine
\[
\text{sn}(u + v) + \text{sn}(u - v) = \frac{2 \text{sn} u \cdot \text{cn} v \cdot \text{dn} v}{1 - k^2 \text{sn}^2 u \cdot \text{sn}^2 v}.
\]

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For a derivation of this formula see [14], page 34. This identity shows that the integrand of $\Pi(u, a, k)$ can be modified to the form

$$\frac{k^2 \text{sn} a. \text{cn} a. \text{dn} a. \text{sn}^2 v}{1 - k^2 \text{sn}^2 a. \text{sn}^2 v} = \frac{1}{2} k^2 \text{sn} a. (\text{sn}(v + a) + \text{sn}(v - a)). \text{sn} v. \quad (4.2.4)$$

Further, we use the fact that the terms on the right-hand side of (4.2.4) are similar to the terms, which occur in the superposition formula for an incomplete elliptic integral of the second kind. It is the integral

$$E[u, k] = \int_0^u \text{dn}^2 v \, dv = \int_0^\tau \sqrt{\frac{1 - k^2 t^2}{1 - t^2}} \, dt, \quad (4.2.5)$$

after the substitution $t = \text{sn} v, \tau = \text{sn} u$. We shall simply write $E(u)$ instead of $E[u, k]$, where it could not lead to any confusion.

The complete integral of the second kind is denoted by $\mathcal{E}$ and is defined by the equation

$$\mathcal{E} = E[K, k] = \int_0^1 \sqrt{\frac{1 - k^2 t^2}{1 - t^2}} \, dt,$$

where $K$ is the value of the complete elliptic integral of the first kind

$$K = \text{sn}^{-1}(1) = \int_0^1 \left( (1 - t^2)(1 - k^2 t^2) \right)^{-1/2} \, dt. \quad (4.2.6)$$

$K$ is also connected to the real period of the elliptic sine, which is $4K$.

The addition formula for $E(u)$ is given by

$$E(u + v) = E(u) + E(v) - k^2 \text{sn} u. \text{sn} v. \text{sn}(u + v).$$

For a derivation of this formula, see [14], page 64.

Now we can anticipate that it will be necessary to integrate the function $E(u)$. The clue for this problem is that $E(u)$ can also be expressed as a logarithmic derivative of the standard Jacobi’s theta functions. The theta functions are defined by infinite sums, see Appendix C for the definitions and a description of their properties. The following identity is derived in Appendix E

$$E(u) = \frac{1}{\theta_2^2(0)} \cdot \frac{\theta_4'(x)}{\theta_4(x)} + \frac{2E x}{\pi} = \frac{d}{du} \ln \theta_4 \left( \frac{\pi u}{2K} \right) + \mathcal{E} \quad (4.2.7)$$

The variable $u$ is a multiple of the variable $x, \ u = \theta_3^2(0)x$. We used the fact that the complete integral $K$ in terms of the theta functions is given by $K = \frac{1}{2} \pi \theta_3^2(0)$.

For the following computations, it is convenient to introduce a new notation for the logarithmic derivative of $\theta_4$. It is the so-called Jacobi’s zeta function

$$Z(u) = \frac{d}{du} \ln \theta_4 \left( \frac{\pi u}{2K} \right) = E(u) - \frac{\mathcal{E}}{K}. \quad (4.2.8)$$
$Z(u)$ is odd, because $\theta_4(u)$ is even. From the equation (4.2.8) follows that there is the same superposition rule for $Z(u)$ as for $E(u)$

$$Z(u + v) = Z(u) + Z(v) - k^2 \text{sn} u. \text{sn} v. \text{sn}(u + v).$$

This identity allows us to write the integrand (4.2.1) of $\Pi(u, a, k)$ as

$$\frac{1}{2} k^2 \text{sn} a. (\text{sn}(v + a) + \text{sn}(v - a)). \text{sn} v = \frac{1}{2} (Z(v - a) - Z(v + a) + 2Z(a)).$$

Using the definition of the zeta function $Z(u)$, we get

$$\Pi(u, a, k) = \frac{1}{2} \int_a^u (Z(v - a) - Z(v + a) + 2Z(a))dv =$$

$$= \frac{1}{2} \int_a^u Z(w)dw - \frac{1}{2} \int_a^{u-a} Z(w)dw + uZ(a) =$$

$$= \frac{1}{2} (\ln(\Theta(u - a)) - \ln(\Theta(-a)) - \ln(\Theta(u + a)) + \ln(\Theta(a))) + uZ(a),$$

where $\Theta(u) = \theta_4 \left( \frac{u}{2k} \right)$. This leads to the final result

$$\Pi(u, a, k) = \frac{1}{2} \ln \frac{\Theta(u - a)}{\Theta(u + a)} + uZ(a) = \frac{1}{2} \ln \theta_4 \left( \frac{\pi}{2k} (u-a) \right) + uZ(a). \quad (4.2.9)$$

### 4.3 Evaluation of the elliptic integral $\Lambda(u, \alpha, k)$

Now we want to express $\Lambda(u, \alpha, k)$ using the equation (4.2.9). We shall distinguish two cases:

(i) $\alpha > 1$,

(ii) $\alpha = i\beta$, $\beta > 0$.

We have shown that the case (ii) holds for every solution of (3.1.10), because the coefficients $(\alpha_1)^2 > 0$, $(\alpha_2)^2 > 0$. For the solutions of (3.1.7) the second argument of $\Lambda$ in (4.1.7), (4.1.8) is either pure imaginary or greater than 1, because $\frac{\ln |\alpha|}{\sqrt{\alpha^2 - 1}} > 1$ for every $\alpha > 1$. Therefore, the case $0 < \alpha < 1$ is not needed in this work. The case $\alpha = 1$ reduces directly to an elliptic integral of the second kind, but we did not find values of the parameters $A, K$ to have $\alpha = 1$. Therefore, we do not consider this case here either.

**Case (i) $\alpha > 1$:** The equation (4.2.2) cannot be fulfilled by any real $a$, because for $\alpha > 1$ and $0 \leq k \leq 1$ the right-hand side of the equation is greater than one, whilst the left-hand side is never greater than one. To obtain $\frac{1}{\text{sn} b}$ on the left-hand side, we will use the identity (see [14], page 28)

$$\text{sn}(b + iK') = \frac{1}{k \text{sn}(b)}, \quad (4.3.1)$$

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where $\mathcal{K}'$ is the complementary complete elliptic integral of the first kind, $\mathcal{K}'(k) = \mathcal{K}(k')$ and $k^2 + k'^2 = 1$. Like $\mathcal{K}$ for the real period, $\mathcal{K}'$ is also connected to the imaginary period of the elliptic sine, which is $2i\mathcal{K}'$. If we set $a = b + i\mathcal{K}'$, the equation (4.2.2) is equivalent to the requirement

$$\frac{1}{k \text{ sn}(b)} = \frac{\alpha}{k}.$$

Thus, (we choose $b \in < 0, \mathcal{K} >$ to take an injective part of the elliptic sine)

$$b = \text{sn}^{-1} \left( \frac{1}{\alpha} \right) = F(\psi), \quad \text{where} \quad \sin \psi = \frac{1}{\alpha} \quad (4.3.2)$$

and $F$ is the Legendre’s form of the elliptic integral of the first kind

$$F(\psi) = \text{sn}^{-1}(\sin \psi) = \int_{0}^{\psi} \left(1 - k^2 \sin^2 \theta \right)^{-\frac{1}{2}} d\theta.$$

We use the Legendre’s form of this integral, because $F(\psi)$ is usually evaluated in mathematical tables. Then $a$ is given by

$$a = F \left( \arcsin \left( \frac{1}{\alpha} \right) \right) + i\mathcal{K}'(k). \quad (4.3.3)$$

To obtain $\Lambda(u, \alpha, k)$, we have to express $\text{sn} a$, $\text{cn} a$ and $\text{dn} a$ in terms of $\alpha$ and $k$. We use the identity (4.3.1) and its modifications for $\text{cn} a, \text{dn} a$ (see [14], page 28):

$$\text{cn}(b + i\mathcal{K}') = \frac{1}{ik} \frac{dn b}{\text{sn} b}, \quad \text{dn}(b + i\mathcal{K}') = -i \frac{cn b}{\text{sn} b}. \quad (4.3.4)$$

From (4.3.2) and the equations $\text{sn}^2 u + \text{cn}^2 u = 1$, $\text{dn}^2 u + k^2 \text{sn}^2 u = 1$ then follows

$$\frac{\text{sn} a}{\text{cn} a. \text{dn} a} = \frac{i^2 \alpha}{\sqrt{(\alpha^2 - k^2)(\alpha^2 - 1)}}. \quad (4.3.5)$$

To compute $\Pi(u, a, k)$ for this case, it is necessary to adjust $Z(a)$. For this purpose, we use the identity for $E(a)$ (see [14], page 66)

$$E(a) = E(b + i\mathcal{K}') = E(b) + i(\mathcal{K}' - \mathcal{E}') + \frac{\text{cn} b. \text{dn} b}{\text{sn} b}, \quad (4.3.6)$$

where $\mathcal{E}'$ is the complementary complete elliptic integral of the second kind, $\mathcal{E}'(k) = \mathcal{E}(k')$. To express $E(b) = E \left( \text{sn}^{-1} \left( \frac{1}{\alpha} \right) \right)$, we introduce the Legendre’s form of the elliptic integral of the second kind

$$D(\phi) = \int_{0}^{\phi} \sqrt{1 - k^2 \sin^2 \theta} \, d\theta. \quad (4.3.7)$$

This integral can be obtained from (4.2.5) by the substitution $t = \sin \theta$, $\tau = \sin \phi$. Then we have $\phi = \arcsin \text{sn} u$. We can verify by differentiation that $\int \text{dn} u du = \arcsin(\text{sn} u)$,
because \((\text{sn } u)' = \text{cn } u \cdot \text{dn } u\). If we now define \(am u = \int_0^u \text{dn } v \, dv\) (see [14], chapter 2.7), it is clear that \(\phi = am u\). This means \(E[u, k] = D[am u, k]\). Now we can write

\[
E(a) = D \left( \arcsin \left( \frac{1}{\alpha} \right) \right) + i(\mathcal{K}' + \mathcal{E}') + \frac{1}{\alpha} \sqrt{(\alpha^2 - k^2)(\alpha^2-1)} \tag{4.3.8}
\]

and similarly for \(Z(a) = E(a) - \frac{\mathcal{E}}{\mathcal{K}} a\). The terms \(\Theta(u \pm a)\) are transformed according to relations (C.11) as follows

\[
\frac{1}{2} \ln \frac{\Theta(u - a)}{\Theta(u + a)} = \frac{1}{2} \ln \frac{\theta_4 \left( \frac{\pi}{2 \mathcal{K}} (u - F(\psi) - i\frac{\mathcal{E}}{2 \mathcal{K}}) \right)}{\theta_4 \left( \frac{\pi}{2 \mathcal{K}} (u + F(\psi) + i\frac{\mathcal{E}}{2 \mathcal{K}}) \right)} = i\frac{\pi u}{2 \mathcal{K}} + \frac{1}{2} \ln \frac{\theta_4 \left( \frac{\pi}{2 \mathcal{K}} (F(\psi) - u) \right)}{\theta_4 \left( \frac{\pi}{2 \mathcal{K}} (F(\psi) + u) \right)}, \tag{4.3.9}
\]

where \(\theta_4\) is defined in (C.1), resp. (C.2). Collecting the appropriate terms together gives, with the usage of the following identity (see [14], page 76)

\[
\mathcal{E}\mathcal{K}' + \mathcal{E}'\mathcal{K} - \mathcal{K}\mathcal{K}' = \frac{1}{2} \pi, \tag{4.3.10}
\]

the final result for \(\Lambda(u, \alpha, k)\):

\[
\Lambda(u, \alpha, k) = \frac{\alpha}{\sqrt{(\alpha^2 - k^2)(\alpha^2-1)}} \left( \frac{1}{2} \ln \frac{\theta_4 \left( \frac{\pi}{2 \mathcal{K}} (F[\psi, k] + u) \right)}{\theta_4 \left( \frac{\pi}{2 \mathcal{K}} (F[\psi, k] - u) \right)} + u \left( \frac{\mathcal{E}}{\mathcal{K}} F[\psi, k] - D[\psi, k] \right) \right), \tag{4.3.11}
\]

where \(\psi = \arcsin \left( \frac{1}{\alpha} \right)\).

**Case** (ii) \(\alpha = i\beta\): In this case, the equation (4.2.2) can be solved by a substitution \(a = id\). Using the transformation of the elliptic sine from an imaginary argument \(iu\) to a real argument \(u\), this yields

\[
\text{sn}[id, k] = i \frac{\text{sn}[d, k']}{\text{cn}[d, k']} = \frac{i \beta}{k}.
\]

The requirement \(\text{sn}[d, k'] = \frac{\beta \text{cn}[d, k']}{k}\) is equivalent to

\[
d = \text{sn}^{-1} \left[ \frac{\beta}{\sqrt{\beta^2 + k^2}}, k' \right] = F[\omega, k'], \tag{4.3.12}
\]

where

\[
\sin \omega = \frac{\beta}{\sqrt{\beta^2 + k^2}}
\]

and \(F\) was defined by (4.3.2). For the following computations, we will need these equations

\[
\begin{align*}
\text{sn}[d, k'] &= \frac{\beta}{\sqrt{\beta^2 + k^2}}, \\
\text{cn}[d, k'] &= \frac{k}{\sqrt{\beta^2 + k^2}}, \\
\text{dn}[d, k'] &= \frac{k\sqrt{\beta^2 + 1}}{\sqrt{\beta^2 + k^2}},
\end{align*} \tag{4.3.13}
\]

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which we obtained from (4.3.12) and the identities \( sn^2 u + cn^2 u = 1, \ dn^2 u + k'^2 \ sn^2 u = 1. \)

The rule for a change from an imaginary argument \( iu \) to a real argument \( u \) for \( E[iu, k] \) takes the form

\[
E[iu, k] = \frac{i \text{sn}[u, k'], \text{dn}[u, k']}{\text{cn}[u, k']} - iE[u, k'] + iu.
\]

For a derivation see [14], page 66. This identity and the transformation equations (4.3.13), give the value of \( E[a, k] \)

\[
E[a, k] = E[id, k] = \frac{i^3 \sqrt{\beta^2 + 1}}{\sqrt{\beta^2 + k'^2}} - iE[d, k'] + id = \frac{i^3 \sqrt{\beta^2 + 1}}{\sqrt{\beta^2 + k'^2}} - iD[\omega, k'] + iF[\omega, k'],
\]

where \( D \) was defined in the case (4.3.7). From (4.2.8), we can easily write

\[
Z[id, k] = \frac{i^3 \sqrt{\beta^2 + 1}}{\sqrt{\beta^2 + k'^2}} - iD[\omega, k'] + iF[\omega, k'] - i \frac{\mathcal{E}}{K} F[\omega, k']. \tag{4.3.14}
\]

After substituting of these results into the equation (4.2.9), we proceed by the equation

\[
\Pi(u, id, k) = \frac{1}{2} \ln \frac{\Theta(u-a)}{\Theta(u+a)} + uZ(id) = \frac{1}{2} \ln \frac{\Theta(u-a)}{\Theta(u+a)} + u \left( \frac{i^3 \sqrt{\beta^2 + 1}}{\sqrt{\beta^2 + k'^2}} - iD[\omega, k'] + iF[\omega, k'] - i \frac{\mathcal{E}}{K} F[\omega, k'] \right),
\]

where we have for the arguments of theta functions

\[
\Theta(u-a) = \theta_4 \left( \frac{\pi}{2K} (u-a) \right), \quad \Theta_4 \left( \frac{\pi}{2K} (u-a) \right) = \theta_4 \left( \frac{\pi}{2K} \right) \cdot \left( u - iF[\omega, k'] \right),
\]

\[
\Theta(u+a) = \theta_4 \left( \frac{\pi}{2K} (u+a) \right), \quad \Theta_4 \left( \frac{\pi}{2K} (u+a) \right) = \theta_4 \left( \frac{\pi}{2K} \right) \cdot \left( u + iF[\omega, k'] \right).
\]

Transformation rules for an imaginary argument of the elliptic functions and the identities (4.3.13) yield for \( \Lambda(u, i\beta, k) \), given by the equation (4.2.3),

\[
\Lambda(u, i\beta, k) = u + \frac{\text{sn} \text{id}}{\text{cn} \text{id} \cdot \text{dn} \text{id}} \Pi(u, id, k) = u + i \frac{\text{sn}[d, k'] \cdot \text{cn}[d, k'] \cdot \text{dn}[d, k']}{\text{dn}[d, k']} \Pi(u, id, k) = u + i \frac{\beta}{\sqrt{\beta^2 + 1} \sqrt{\beta^2 + k'^2}} \Pi(u, id, k).
\]

It remains to substitute for \( \Pi(u, id, k) \) computed above.

\[
\Lambda(u, i\beta, k) = \frac{k'^2 u + \frac{\beta}{\sqrt{\beta^2 + 1} \sqrt{\beta^2 + k'^2}} \left( u \left( D[\omega, k'] - \left( 1 - \frac{\mathcal{E}}{K} \right) F[\omega, k'] \right) + \frac{1}{2} \frac{\ln \Theta(u - iF[\omega, k'])}{\Theta(u + iF[\omega, k'])} \right), \tag{4.3.15}
\]

where \( \Theta(u) = \theta_4 \left( \frac{\pi u}{2K} \right) \) and \( \omega = \arcsin \left( \frac{\beta}{\sqrt{\beta^2 + k'^2}} \right). \)
4.4 Solution of the problem for $f$

To conclude, we derived that $f(\chi)$ for the case (3.1.10) is given by

$$f(\chi) = \left( \frac{1}{2} - \frac{A}{U_1}(U_1 - 1)^2 \right) \cdot \chi + \frac{A(U_1^2 - 1)}{U_1} \cdot (\Lambda(\beta \chi, i \alpha_1, k) - \Lambda(\beta \chi, i \alpha_2, k)),$$

where $\alpha_1, \alpha_2$ are defined by (4.1.3) and $\Lambda(\beta \chi, i \alpha_1, k), \Lambda(\beta \chi, i \alpha_2, k)$ are given by (4.3.15). We verified numerically the validity of this result for the choice of the testing parameters $K = -1/17$ and corresponding $A$ from the Table 3.2, page 30.

For the case (3.1.7), we obtained the formula

$$f(\chi) = \frac{\chi}{2} - \gamma \cdot (\Sigma(\beta \chi, \alpha, k) + \Sigma(\beta \chi, -\alpha, k)),$$

where $\alpha, \gamma$ are defined by (4.1.6) and $\Sigma(\beta \chi, \alpha, k), \Sigma(\beta \chi, -\alpha, k)$ are given by (4.1.7), (4.1.8). We verified numerically the validity of this result for the choice of the testing parameters $A = 1, K = -1$ from the Table 3.1, page 28. For this case it should be noted that if the integrand of $\Lambda(u, \tilde{\alpha}, k)$ is computed according to (4.3.11), i.e. $\alpha^2 > 1$ in the formula (4.1.8), it becomes infinite when the variable in the integrand reaches

$$\text{sn} \, v = 1/\tilde{\alpha} = \frac{\sqrt{\alpha^2 - 1}}{|\alpha|}.$$

The integral $\Lambda(\beta \chi, |\alpha|/\sqrt{\alpha^2 - 1}, k)$ diverges for the upper bound

$$\beta \chi = \text{sn}^{-1} \left( \frac{\sqrt{\alpha^2 - 1}}{|\alpha|} \right) = F[\arcsin \left( \frac{\sqrt{\alpha^2 - 1}}{|\alpha|} \right), k].$$

Therefore, we require

$$0 \leq \chi < \frac{F[\psi, k]}{\beta}, \quad \text{where} \quad \psi = \arcsin \left( \frac{\sqrt{\alpha^2 - 1}}{|\alpha|} \right),$$

in the formula (4.4.2).

These results allow us construct the complete solution $w(\chi)$ of the Euler–Lagrange equations (2.1.4) for the $\mathbb{C}P^1$ sigma model

$$\partial_L \partial_R w = 2 \frac{w}{1 + \bar{w}w} \partial_L w \partial_R w.$$

(4.4.3)

We recall that for solutions obtained by the symmetry reduction the function $w$ from (4.1.1) is of the following form

$$w(\chi) = R(\chi) \, e^{i \frac{\xi_R}{a} - f(\chi)}$$

(4.4.4)

where $\chi = \frac{\xi_L}{a} - \frac{\xi_R}{b}$, $R(\chi) = \sqrt{-U(\chi)}$ and $U(\chi)$ is one of the two considered solutions of (3.1.3).
Particularly, $U(\chi)$ for the case (3.1.10) is given by

$$U(\chi) = \frac{U_4(U_1 - U_3) + U_1(U_3 - U_4) \text{sn}[\beta \chi, k]^2}{U_1 - U_3 + (U_3 - U_4) \text{sn}[\beta \chi, k]^2}, \quad (4.4.5)$$

where the constants $k$, $\beta$ are again defined by (3.1.11). The phase factor $f(\chi)$ is determined by the formula (4.4.1), which completes the searched result for $w(\chi) = R(\chi) e^{i\left(\frac{\beta \chi}{\alpha} - f(\chi)\right)}$.

For the case (3.1.7) the defining equation (3.1.10) for $U(\chi)$ reads

$$U(\chi) = \frac{CU_4 + DU_1 + (CU_4 - DU_1) \text{cn}[\beta \chi, k]}{C + D + (C - D) \text{cn}[\beta \chi, k]}, \quad (4.4.6)$$

where the constants $C$, $D$, $\beta$, $k$ are defined by (3.1.8), (3.1.9). The function $f(\chi)$ is now computed from (4.4.2). We verified numerically for the choice of the testing parameters $A = 1$, $K = -1$ that such constructed $w(\chi) = R(\chi) e^{i\left(\frac{\beta \chi}{\alpha} - f(\chi)\right)}$ satisfies the Euler-Lagrange’s equations of motion (4.4.3).
Chapter 5

Conclusion and final remarks

The main goal of this work was to find some new non-trivial examples of surfaces related to the $\mathbb{C}P^1$ sigma model and to investigate their main properties. The research was based on an analysis of specific examples of solutions of the $\mathbb{C}P^1$ sigma model, which were found using the method of the symmetry reduction for partial differential equations.

In the first theoretical part there is a comprehensive summary of most of the results in this subject achieved up to date. We verified many of the original results from the papers [2], [3], [5], [6] and we corrected some misprints, particularly in formulas (3.1.8), (3.1.9), (3.1.11) and we improved the formula (1.2.12). We are aware of the danger of mixing too many different approaches from various sources in one work. We tried to unify the nomenclature and the notation. In the description of the Grassmannian sigma models we decided to use a formulation using projectors, which are in $1 \to 1$ correspondence with elements of $G(m, n)$. This allows to avoid from the beginning the superfluous gauge degrees of freedom involved in the use of representatives of elements of $G(m, n)$ as equivalence classes in $su(N)$, as was employed in [4]. This way is advantageous in the theoretical approach. For practical calculations we had to express the general formulas in terms of explicit functions, which made the results less transparent.

While composing this review we pursued the aim to create a work which is readable even for a beginner without searching for references. That is why the work is supplemented by several appendices. In the first appendix we begin by recalling various important notions and theorems from differential geometry of surfaces, which are not parts of the standard university courses. Since I had to become familiar with many of new notions from the Lie groups and Lie algebras theory and their applications to geometry and mathematical physics, we decided to add an appendix concerning the method of the symmetry reduction. The remaining appendices deal with a technical background for the second part of the work.

The focal point of the work is the second computational part. Firstly, we have found concrete values of parameters satisfying complicated constraints for the both considered cases, that are necessary for next computations. Then we derived by a direct calculation that the currents (which satisfy the equation of continuity), for every solution of the $\mathbb{C}P^1$
sigma model obtained by the symmetry reduction, are constant. This fact let us construct
an example of a solution of the sine–Gordon equation associated with one of the considered
solutions of the $\mathbb{C}P^1$ sigma model. This solution shows to be of a stationary oscillating
type.

The second relevant contribution to the research in this topic was finding complete ana-
lytic forms of solutions of the $\mathbb{C}P^1$ sigma model for both of the considered cases obtained by
the symmetry reduction. From these formulas, it is possible to find then a parametrization
of the surface, its first and second fundamental form and its mean curvature. These results
showed to be so complicated that it was not possible to write them in a compact form
in this work. Particularly, the relations for the tangent and normal vectors (2.1.5) to the
 corresponding surfaces were so complicated, that they clearly seem to be integrable only
numerically. A straightforward integration by programmes like Maple or Mathematica was
extremely time-consuming and does not yield any applicable result. It means that such
procedure would require a knowledge of some optimization methods (perhaps an usage
of parallel programming) and also more sophisticated numerical methods than those that
were available to us. That is the reason why we cannot present any pictures of searched
 surfaces.

Regrettably, we had to postpone searching for such special choice of parameters that our
considered cases (3.1.10), (3.1.7) would reduce to simpler problems. For a lack of the time
and due to the complexity of this problem this was not done, because the computations
with elliptic functions were too tedious by themselves. We hope that this problem can lead
to interesting results and that we will be able to report on this in the near future. We
conclude that the majority of the proposed tasks was fulfilled.

For me, the work on this topic was very interesting and fruitful, so I hope to use gained
experience during my further work on my diploma thesis, in which an investigation of the
linear problem for the $\mathbb{C}P^1$ sigma model and its relation to integrable systems or possible
generalizations to sigma models in (2+1) dimensions can be treated. For more possible
utilizations of gained results, see the original papers [2], [5].
Appendix A
Elementary review of differential geometry of surfaces

Surfaces in a 3–dimensional Euclidean space can be described by two different approaches:

(i) as the graph of a function of two variables or more generally, by the implicit equation

\[ F(x, y, z) = 0; \quad (A.1) \]

(ii) by parametric equations

\[ r = r(u, v), \quad r = (x, y, z), \quad x = x(u, v), \quad y = y(u, v), \quad z = z(u, v). \quad (A.2) \]

Definition 1 We say that the surface given by the equation (A.1) is non-singular at the point \( p = (x_0, y_0, z_0) \) on it (i.e. satisfying \( F(x_0, y_0, z_0) = 0 \)), if the gradient of \( F \) at \( p \) is non-zero:

\[ \frac{\partial F}{\partial x} e_1 + \frac{\partial F}{\partial y} e_2 + \frac{\partial F}{\partial z} e_3 \neq 0 \quad \text{at} \quad x = x_0, \ y = y_0, \ z = z_0, \]

where \( e_i, \ i = 1, \ldots, 3 \) form an orthonormal basis in the point \( p \). Then we can suppose for example \( \frac{\partial F}{\partial x} \big|_{x_0,y_0,z_0} \neq 0 \). By the Implicit Function Theorem, near the point \( p = (x_0, y_0, z_0) \) on the surface, the equation \( F(x, y, z) = 0 \) can be solved for \( z \) as a continuously differentiable function \( z = f(x, y) \), such that \( z_0 = f(x_0, y_0) \), and in a neighbourhood of the point \( (x_0, y_0) \) holds \( F(x, y, f(x, y)) = 0 \).

From the above Implicit Function Theorem and the definition (1) it follows that around each non-singular point of a surface given by (A.1), there is a neighbourhood such that the piece of the surface enclosed by that neighbourhood is the graph of a function (where if \( z \) is not a dependent variable, then we rename dependent and independent variables). Hence locally, i.e. in a neighbourhood of a non-singular point, the surface is given by the parametric equations \( z = f(u, v), \ x = u, \ y = v \) around the point \( x_0 = u_0, \ y = v_0 \). This is often expressed that in a neighbourhood of a non-singular point of a surface there local coordinates \( u, v \) exist.

Suppose now that a surface is given parametrically by (A.2).
Definition 2 A point \( p = (x_0, y_0, z_0) = (x(u_0, v_0), y(u_0, v_0), z(u_0, v_0)) \) on the surface given by (A.2) is said to be non-singular if the rank of the following matrix is equal to 2

\[
\text{rank} \begin{pmatrix}
\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\
\frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\
\frac{\partial z}{\partial u} & \frac{\partial z}{\partial v}
\end{pmatrix}_{(u_0, v_0)} = 2.
\]

The next theorem tells that definitions (1) and (2) are locally equivalent.

Proposition 1 If a surface is given parametrically as in (A.2) and the point \( p = (x_0, y_0, z_0) \) corresponding to \((u_0, v_0)\) is non-singular according to the definition (2), then there is an equation \( F(x, y, z) = 0 \) which defines the surface in a neighbourhood of \( p \) and has the property that \( (\nabla F)(x_0, y_0, z_0) \neq 0 \), i.e. \( p \) is a non-singular point (also according to the definition (1)).

Furthermore, we can conclude from this that locally, i.e. in a neighbourhood of a non-singular point of a surface, the above two ways of presenting a surface are equivalent.

The geometric significance of the property of being non-singular lies in the fact that the tangent plane to a surface at a non-singular point has the same dimension as that of the surface, i.e. the vectors \( r_u = (x_u, y_u, z_u), r_v = (x_v, y_v, z_v) \) are linearly independent and every vector tangent to the surface is their linear combination.

Now we shall define the metric on a surface in Euclidean space. We consider the surface as a space parametrized by \( u, v \). Then it is natural to ask how to measure lengths in this space in terms of the coordinates \( u, v \). More exactly, suppose we have a curve \( r = r(t) = (x(t), y(t), z(t)) \) on the surface. Since this curve lies in the Euclidean space, by its length we shall mean its length in that space, given by

\[
l = \int_a^b \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} \, dt. \tag{A.3}
\]

We can find a pair of functions \( u = u(t), v = v(t) \) determining the curve on the surface, i.e.

\[
x(t) = x(u(t), v(t)), \quad y(t) = y(u(t), v(t)), \quad z(t) = z(u(t), v(t)).
\]

Now it is natural to ask which metric \( g_{ij} = g_{ij}(u, v) \) leads to the same result for the length of the curve (given by \((u(t), v(t))\)) as the formula (A.3). It means we want to find \( g_{ij} \) such that

\[
l = \int_a^b \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} \, dt = \int_a^b \sqrt{g_{ij} \dot{x}^i \dot{x}^j} \, dt, \tag{A.4}
\]

where \( i, j \in 1, 2, \quad x^1 = u, x^2 = v \) and a dot denotes a derivative with respect to \( t \).

The answer is that the coefficients of the Riemannian metric \( g_{ij} \) are given by

\[
g_{ij} = \langle r_x^i, r_x^j \rangle, \quad x^1 = u, \quad x^2 = v.
\]

We shall say that the metric is induced on the surface.
Definition 3 We then define the first fundamental form on the surface by the matrix

\[ I = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} \langle \mathbf{r}_u, \mathbf{r}_u \rangle & \langle \mathbf{r}_u, \mathbf{r}_v \rangle \\ \langle \mathbf{r}_v, \mathbf{r}_u \rangle & \langle \mathbf{r}_v, \mathbf{r}_v \rangle \end{pmatrix} . \tag{A.5} \]

We shall also define the second fundamental form on the surface. Firstly, denote \( n \) a normal to the surface, given as a normalized vector product of the tangent vectors \( \mathbf{r}_u, \mathbf{r}_v \). An orientation of the normal is not relevant for our purposes. Let \( r(t) = r(u(t), v(t)) \) be a curve on the surface. Then it can be deduced that

\[ \langle \mathbf{r}, n \rangle = \langle \mathbf{r}_{uu}, n \rangle \mathbf{u}^2 + 2\langle \mathbf{r}_{uv}, n \rangle \mathbf{u} \mathbf{v} + \langle \mathbf{r}_{vv}, n \rangle \mathbf{v}^2 . \]

Definition 4 We then define the second fundamental form on the surface by the matrix

\[ II = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} = \begin{pmatrix} \langle \mathbf{r}_{uu}, n \rangle & \langle \mathbf{r}_{uv}, n \rangle \\ \langle \mathbf{r}_{vu}, n \rangle & \langle \mathbf{r}_{vv}, n \rangle \end{pmatrix} . \tag{A.6} \]

Definition 5 The principal curvatures of the surface at the point are the eigenvalues of the pair of quadratic forms (A.5), (A.6), see [9] for explanation. The product of the principal curvatures is called the Gaussian curvature \( K \) of the surface at the point and their sum (or sometimes their arithmetic average sum) is the mean curvature \( H \).

Proposition 2 The Gaussian curvature of a surface is equal to the ratio of the determinants of the second and the first fundamental forms

\[ K = \frac{h_{11}h_{22} - (h_{12})^2}{g_{11}g_{22} - (g_{12})^2} . \tag{A.7} \]

Particularly, if the surface is defined as a graph of a function \( z = f(x, y) \), then the Gaussian curvature is given by

\[ K = \frac{f_{xx}f_{yy} - f_{xy}^2}{(1 + f_x^2 + f_y^2)^2} . \tag{A.8} \]

Now we recall one of the fundamental theorems of the differential geometry of surfaces.

Proposition 3 Let \( r(u, v) \) be a parametrization (A.2) of a surface \( M \) in \( \mathbb{R}^3 \). If \( p \in M \) is a non-singular point, then the so called Gauss-Weingarten equations are satisfied in \( p \).

i.e.:

\[ n_u = \frac{h_{12}g_{12} - h_{11}g_{22}}{g} r_u + \frac{h_{11}g_{12} - h_{12}g_{11}}{g} r_v , \]

\[ n_v = \frac{h_{22}g_{12} - h_{12}g_{22}}{g} r_u + \frac{h_{12}g_{12} - h_{22}g_{11}}{g} r_v . \tag{A.9} \]
where $g = g_{11}g_{22} - (g_{12})^2$ and $\Gamma^i_{jk}$ are the standard Christoffel symbols given by formula
\begin{equation}
\Gamma^k_{ij} = \frac{1}{2} \delta^{kl} \left( \frac{\partial g_{lj}}{\partial z^i} + \frac{\partial g_{li}}{\partial z^j} - \frac{\partial g_{ij}}{\partial z^l} \right).
\end{equation}
(A.10)

The compatibility conditions for the system (A.9) yield
\begin{equation}
(r_{uu})_v = (r_{uv})_u, \quad (r_{uu})_v = (r_{uv})_u, \quad n_{uu} = n_{vu},
\end{equation}
which are called the Gauss-Codazzi equations. These equations are necessary and sufficient conditions for existence of a surface.

**Definition 6** We say that the metric is in the Chebyshev form if the coefficients $g_{ij}$ are of the form
\begin{equation}
I = \lambda(dx)^2 - 2g_{12}dx dy + \lambda^{-1}(dy)^2
\end{equation}
We say that the metric is in the general Chebyshev form if the coefficients $g_{11}, g_{22}$ satisfy
\begin{equation}
\partial_x g_{11} = 0, \quad \partial_y g_{22} = 0.
\end{equation}
(A.12)

We conclude our short review by a version of the standard "Theorema Egregium":

**Proposition 4** For a 2–dimensional surface in the 3–dimensional space endowed with a Riemannian metric, the scalar curvature $R$ is twice the Gaussian curvature, $R = 2K$. It follows that the Gaussian curvature of a surface $K$ is, in contrast with the mean curvature, an intrinsic invariant of the surface, because it is expressible in terms of the induced metric on the surface.

**Proof:** Suppose $p$, any non–singular point on a surface $M$. Let us choose coordinates $x, y, z$ on the surface such that $p$ is the origin, the $z$–axis is normal to the surface at $p$, and the $x$– and $y$–axes are tangent to the surface at $p$.

Then locally the surface is given as a graph of a function $z = f(x, y)$, where $\text{grad } f|_p = 0$. The induced metric in a neighbourhood of $p$ is then given by
\begin{equation}
g_{ij} = \delta_{ij} + \frac{\partial f}{\partial z^i} \frac{\partial f}{\partial z^j}, \quad i, j \in 1, 2, \quad z^1 = x, z^2 = y.
\end{equation}
(A.13)

From this and due to $\text{grad } f|_p = 0$ we have that the derivatives $\frac{\partial g_{ij}}{\partial z^k}$ are zero for $i, j, k = 1, 2$, at $p$. Since the unique symmetric connexion on the surface compatible with the induced metric is given by the Christoffel formula (A.10), it follows that $\Gamma^k_{ij}$ are also zero at $p$.

From the general formula for components of the Riemann curvature tensor
\begin{equation}
-R^i_{qkl} = \frac{\partial \Gamma^i_{ql}}{\partial z^k} - \frac{\partial \Gamma^i_{qk}}{\partial z^l} + \Gamma^i_{pl} \Gamma^p_{ql} - \Gamma^i_{pq} \Gamma^p_{ql}
\end{equation}
now remains
\begin{equation}
-R^i_{qkl} = \frac{\partial \Gamma^i_{ql}}{\partial z^k} - \frac{\partial \Gamma^i_{qk}}{\partial z^l}.
\end{equation}

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If we express the Christoffel symbols in terms of the coefficients of the induced metric, we get
\[ R_{ijkl} = \frac{1}{2} \left( \frac{\partial^2 g_{ij}}{\partial z^k \partial z^l} + \frac{\partial^2 g_{jk}}{\partial z^i \partial z^l} - \frac{\partial^2 g_{ik}}{\partial z^j \partial z^l} - \frac{\partial^2 g_{il}}{\partial z^j \partial z^k} \right). \]

Concretely for \( R_{1212} \) it leads to
\[ R_{1212} = \frac{1}{2} \left( \frac{\partial^2 g_{12}}{\partial x \partial y} - \frac{\partial^2 g_{11}}{\partial y^2} - \frac{\partial^2 g_{22}}{\partial x^2} \right). \quad (A.14) \]

Further, substitution of the special form of the metric (A.13) and the relation \( \text{grad} f |_p = 0 \) yield
\[ R_{1212} = f_{xx} f_{yy} - f_{xy}^2, \]
in which we recognize the numerator of the formula (A.8) for the Gaussian curvature \( K \). The scalar curvature \( R \) is defined by \( R = g^{kl} R_{kl} \). Due to symmetries of the Riemann curvature tensor, this formula reduces to
\[ R = \frac{2}{g} R_{1212}. \]

Finally we use that the determinant \( g \) is in the point \( p \) equal to 1. This gives \( R = 2K \) at the point \( p \). Since \( R \) and so \( K \) are scalars, i.e. they are coordinate-independent, we conclude that
\[ R = 2K = \frac{R_{1212}}{g} \quad (A.15) \]
at every point, which finishes the proof. \( \square \)
Appendix B

Basic concepts of the method of the symmetry reduction

We shall shortly explain the method of the symmetry reduction for partial differential equations. We follow the chapter 6 from [16] and also [17]. The starting point is to realize how group actions act on jet bundles. We explicate these important notions and basic ideas on an ordinary differential equation $f(x, u, u') = 0$ firstly. Then the set $f(x, u, p) = 0$ is a smooth submanifold in $\mathbb{R}^3$. A solution of the differential equation is a curve $(x, u(x), p(x))$ on the surface $f = 0$ such that $p(x) = u'(x)$. Now a question arises: What transformations of the $x-u$ plane leave the set of solutions invariant?

Consider a one-parameter group of transformations in the $x-u$ plane be given by

$$\tilde{x} = X(x, u, \varepsilon), \quad \tilde{u} = U(x, u, \varepsilon).$$

This transformation converts the graph of a function $u(x)$, given by $\Gamma = \{(x, u)|u = u(x)\}$ into a new graph $\tilde{\Gamma}$. Such a transformation thus carries the original function $u(x)$ into a new one $\tilde{u}(\tilde{x})$. Such transformations we call point transformations, because transformations change dependent and independent variables simultaneously, but the new variables depend only on the old ones. Parametrically, the new function $\tilde{u}(\tilde{x})$ is expressed by

$$\tilde{x} = X(x, u(x), \varepsilon), \quad \tilde{u} = U(x, u(x), \varepsilon). \quad (B.1)$$

For $\varepsilon = 0$ the transformation is the identity and the family of such transformations form a local Lie-group. Therefore (B.1) can be locally inverted and $x$ may be expressed as a function of $\tilde{x}$ for sufficiently small $\varepsilon$. The result is then substituted into the equation for $\tilde{u}$ to obtain the desired result. But such a computation is in general quite complicated. Lie showed that all relevant information may be obtained from an infinitesimal approximation, i.e. from infinitesimal generators of the transformations.

We shall directly extend the suggested concepts to the case of several variables, but we shall not do it in detail. This can be found for example in [16] and a yet more extended description is in [12]. We only recall the notation: let us denote $x = (x_1, \ldots, x_p)$, $u = (u^1, \ldots, u^n)$, the standard multi-index $J = (j_1, \ldots, j_p)$, where $j_i$ are non-negative integers.
and $|J| = \sum_{i=1}^p j_i$. Let $\mathcal{J}_k$ be the vector space $X \times U \times \mathbb{R}^k$ with coordinates labelled $(x, u, u^1, ..., u^k)$. We call $\mathcal{J}_k$ the $k$-th order jet bundle. Then

$$\partial_J = \frac{\partial^{|J|}}{\partial x_1^{j_1} \cdots \partial x_p^{j_p}}.$$ 

By $(x_i, u^i(x), u^i_J(x))$, where $u^i_J(x) = \partial_J u^i(x)$, we mean a section in $\mathcal{J}_k$. On the spaces $\mathcal{J}_k$ we define differential operators

$$D_i = \frac{\partial}{\partial x_i} + \sum_{i,j} u^j_{i,j} \frac{\partial}{\partial u^j},$$

where $J_i = (j_1, ..., j_i + 1, ..., j_n)$. The meaning of $D_i$ is that it acts as the total derivative on sections.

Now we shall accept the fact, that for a general one-parameter transformation group

$$\tilde{x}_i = X_i(x, u, \varepsilon), \quad \tilde{u}^j = U^j(x, u, \varepsilon), \quad 1 \leq i \leq p, \quad 1 \leq j \leq n,$$

the infinitesimal generator is expressed as

$$\alpha = \xi^i \frac{\partial}{\partial x_i} + \varphi^j \frac{\partial}{\partial u^j}.$$ 

More precisely the components $\xi^i$, $\varphi^j$ of the vector field $\alpha$ are functions $\xi^i = \xi^i(x, u)$ and $\varphi^j = \varphi^j(x, u)$. The vector fields $\alpha$ act as differential operators and form a Lie-algebra. Every field separately generates one-parametric subgroup of the symmetry group. These subgroups may be obtained by integration of the defining relations for the vector field

$$\begin{align*}
\frac{d\tilde{x}_i(\varepsilon)}{d\varepsilon} &= \xi_i(\tilde{x}, \tilde{u}), \quad \tilde{x}_i(0) = x_i, \\
\frac{d\tilde{u}^j(\varepsilon)}{d\varepsilon} &= \varphi^j(\tilde{x}, \tilde{u}), \quad \tilde{u}^j(0) = u^j.
\end{align*}$$

We summarize that these vector fields are associated with independent infinitesimal symmetries as follows

$$\begin{align*}
\frac{\partial}{\partial x^i} &\quad \text{translations}, \\
x_j \frac{\partial}{\partial x^i} - x_i \frac{\partial}{\partial x^j} &\quad \text{rotations}, \\
x^j \frac{\partial}{\partial x^i} &\quad \text{dilatations}, \\
\frac{\partial}{\partial u} &\quad u \to u + c, \\
\frac{\partial}{\partial u} &\quad u \to \lambda u \text{ scalar multiplication}, \\
\xi(x) \frac{\partial}{\partial x} + \eta(y) \frac{\partial}{\partial y} &\quad \text{conformal transformations}.
\end{align*}$$

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Let us consider firstly the set of m algebraic equations

\[ f^i(x, u) = 0, \text{ where } x \in \mathbb{R}^p, \ u \in \mathbb{R}^n, \ i = 1..m \]

as before. Then we can calculate the group of its symmetries by solving a system

\[ \alpha(f^i(x, u))|_{f^i(x, u) = 0} = 0, \ i = 1..m, \tag{B.4} \]

viewed as a system for \( \xi_i(x, u), \varphi^i(x, u) \) unknown. The condition (B.4) is necessary and also sufficient for the system \( f \) to be invariant with respect to a Lie group \( \mathfrak{G} \) iff \( \alpha(f) = 0|_{f(x, u) = 0} \) for all \( \alpha \) in the Lie algebra \( \mathfrak{g} \) of the group \( \mathfrak{G} \). Note that we require \( \alpha(f) = 0|_{f(x, u) = 0} \) only on the manifold \( \mathcal{M} \) and not everywhere, so we are not requiring \( F \) to be invariant. This subtle point will be important in our calculation of symmetry groups of differential equations.

However, for a differential equation, we need to a point \((x, u(x), (du/dx))\) be carried to \((\tilde{x}, \tilde{u}(x), (d\tilde{u}/d\tilde{x}))\), i.e. we have to take into account also transformations of the derivatives. Therefore consider an action of a group of transformations \( \Phi_g \) acting in the \( x - u \) space. Since \( \Phi_{g_1g_2} = \Phi_{g_1} \circ \Phi_{g_2} \), we have \( \tilde{\Gamma}_{g_1g_2} = \Phi_{g_1g_2}(\Gamma) = \Phi_{g_1}(\Phi_{g_2}(\Gamma)) \). To take into account the derivatives, we can determine the action induced on \((x, u, p)\) space. We denote this action on \( \mathbb{R}^3 \) by \( \Phi^{(1)}_g \) and call it the first prolongation. For the action of the one-parameter transformation group \( \Phi_\varepsilon \) on \( x - u \) plane and a prolonged action \( \Phi^{(1)}_\varepsilon \), which is also an one-parameter transformation group, we can compute the first prolongation of the infinitesimal generator \( \alpha \), which we denote \( pr^{(1)} \alpha \).

Therefore there is a natural prolongation of any group action on \( \mathbb{R}^2 \) to \( \mathbb{R}^3 \). Moreover, we shall extend this argument to derivatives of all orders of the initial function \( u(x) \). The \( k \)-th prolongation of the action \( \Phi_\varepsilon \) on \( \mathbb{R}^2 \) is the action induced on \( k \)-th order Taylor polynomials as follows. Given a point \((\bar{x}, \bar{u}, \bar{u}^1, ..., \bar{u}^k)\) in the \( k \)-th order jet bundle \( \mathcal{J}_k \), we construct the polynomial

\[ f(x) = \sum_{j=0}^{k} \frac{\bar{u}^j(x - \bar{x})^j}{j!}, \]

where \( \bar{u}^0 = \bar{u} \). A smooth transformation \( \Phi_g \) on the base space naturally leads to a transformation \( x \to \tilde{x}, \ u \to \tilde{u} \) on the space \( \mathcal{J}_k \). An action \( \Phi^{(k)}_g \) on \( \mathcal{J}_k \) induced by the action \( \Phi_g \) is called the \( k \)-th prolongation.

The important step, which brings us further, is to realize that if a Lie group \( \mathfrak{G} \) acts on \( X \times U \), the prolongations of the infinitesimal generators \( pr^{(k)} \alpha \) of the \( k \)-th prolongation form also a Lie algebra. We denote it by \( \mathfrak{g} \), because it corresponds to the group \( \mathfrak{G} \). Particularly we have

\[ [pr^{(k)} \alpha, pr^{(k)} \beta] = pr^{(k)}[\alpha, \beta]. \]

The crucial result may be summarized in the following proposition:

**Proposition 5** The first prolongation of \( \alpha \) is given by

\[ pr^{(1)} \alpha = \alpha + \sum_{i,t} \varphi^i_t \frac{\partial}{\partial u^i_t}, \tag{B.5} \]
where
\[ \varphi_i^J = D_i \varphi^J - u_m^J D_i \xi^m. \]  \hspace{1cm} (B.6)

The prolongations of higher orders are calculated by recursion method. Denoting \( \alpha_k = pr^{(k)} \alpha \), we obtain \( \alpha_{k+1} \) by
\[ \alpha_{(k+1)} = pr^{(1)} \alpha_k. \]

Formally, after replacement of \( i \) by \( J \) in the formula (B.5), \( k \)-th prolongation \( pr^{(k)} \alpha \) is given by
\[ pr^{(k)} \alpha = \alpha + \sum_{i,J} \varphi^J_i \frac{\partial}{\partial u^J_i}. \]  \hspace{1cm} (B.7)

Similarly, generalization of the formula (B.6) is
\[ \varphi^J_{J_i} = D_i \varphi^J_j - u_m^J D_i \xi^m, \]  \hspace{1cm} (B.8)

where \( J_i = (j_1, \ldots, j_i + 1, \ldots, j_n) \).

**Proposition 6** The functions \( \varphi^J_j \) in the formula (B.7) for the \( k \)-th prolongation of \( \alpha \) are given by
\[ \varphi^J_j = D_j (\varphi^J - \xi^m u_m^J) - (D_m u_m^J) \xi^m. \]  \hspace{1cm} (B.9)

Let us now have a system of \( m \) partial differential equations of the \( n \)-th order given by \( \Delta(x, u, u_J) = 0 \), where \( \Delta(x, u, u_J) \) is smooth mapping from \( J_k \) to \( \mathbb{R}^q \) such that the set of \( x, u, u_J \) in \( J_k \) for which \( \Delta(x, u, u_J) = 0 \) forms a smooth submanifold of \( J_k \). If the Jacobian
\[ \frac{\partial (\Delta^1, \ldots, \Delta^q)}{\partial (x, u^J, u_m^J)} \]

is of rank \( q \) everywhere on \( \Delta = 0 \), the set \( \Delta = 0 \) is a smooth manifold of a codimension \( q \). By a solution we mean a section \( (x, u(x), u_J(x)) \) such that \( \Delta((x, u(x), u_J(x)) = 0 \) in some open domain of \( X \).

**Proposition 7** Let \( G \) be a local group acting on \( X \times U \) by
\[ \tilde{x} = X(x, u, g), \tilde{u} = U(x, u, g), \]  \hspace{1cm} (B.10)

and let \( \Delta(x, u, u_J) \) be a system of partial differential equations which is invariant under the prolongation of the action \( \Phi_g^{(k)} \) to \( J_k \). Then this action (B.10) preserves solutions of \( \Delta = 0 \). It means that if \( u(x) \) is a solution so is \( \tilde{u}(\tilde{x}) \). A necessary and sufficient condition that the equations be invariant with respect to \( \Phi_g^{(k)} \) is that
\[ pr^{(k)}(\alpha) \Delta = 0 \quad \text{whenever} \quad \Delta = 0, \]  \hspace{1cm} (B.11)

for every infinitesimal generator of the group action (B.10).
The conditions (B.6), (B.9) give a way to calculate the generators $\alpha$ of the symmetry group of a system of differential equations.

During the calculations of symmetries, three alternatives can occur:

1. While evaluating (B.11) the components of the vector fields we get only trivially zero results;

2. A general solution of (B.11) depends on $N$ constants of integration. The Lie-algebra of the symmetry group is finite-dimensional and its dimension is $N$;

3. A general solution of (B.11) depends on arbitrary functions. The Lie-algebra of the symmetry group is infinite-dimensional. This case occurs always in the case of linear differential equations as a consequence of the linear superposition rule.
Appendix C

Theta functions

Here we summarize the results of the chapter 1 of [14]. For a detailed information about the theta functions, see this source. We have chosen the approach to use the four theta functions $\theta_1$, $\theta_2$, $\theta_3$, $\theta_4$, defined by the following series for all complex values of $z$ and $q$ such that $|q| < 1$:

\[
\begin{align*}
\theta_1[z, q] &= 2 \sum_{n=0}^{\infty} (-1)^n q^{(n+1/2)^2} \sin[(2n + 1)z], \\
\theta_2[z, q] &= 2 \sum_{n=0}^{\infty} q^{(n+1/2)^2} \cos[(2n + 1)z], \\
\theta_3[z, q] &= 1 + 2 \sum_{n=1}^{\infty} q^{n^2} \cos[2nz], \\
\theta_4[z, q] &= 1 + 2 \sum_{n=1}^{\infty} (-1)^n q^{n^2} \cos[2nz].
\end{align*}
\] (C.1)

For the graphs of the theta functions, see the supplement F.1.

Another approach is to use the Weierstrass’s Zeta function and the other three Weierstrass’s functions derived from the Zeta function. The theta functions may be obtained from the Weierstrass’s Zeta function too.

To establish convergence of the series (C.1), i.e. the condition $|q| < 1$, it is useful to replace goniometric functions by exponentials. It leads to alternative definitions by doubly infinite series

\[
\begin{align*}
\theta_1[z, q] &= -i \sum_{n=-\infty}^{\infty} (-1)^n q^{(n+1/2)^2} e^{i(2n+1)z}, \\
\theta_2[z, q] &= \sum_{n=-\infty}^{\infty} q^{(n+1/2)^2} e^{i(2n+1)z}, \\
\theta_3[z, q] &= \sum_{n=-\infty}^{\infty} q^{n^2} e^{2inz}, \\
\theta_4[z, q] &= \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} e^{2inz}.
\end{align*}
\] (C.2)

Let us denote $u_n$ the $n$-th term of these series. Then D’Alembert criterion shows that for example for the case of $\theta_1$

\[
\lim_{n \to \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \to \infty} |q^{2n+2} e^{2iz}| = \lim_{n \to \infty} |q|^{2n+2} e^{-2y},
\]

where $z = x + iy$. This limit is zero if $|q| < 1$. Therefore, the series converges at $+\infty$ for such $q$. Analogous argument can be made for $n \to -\infty$, hence the series is absolutely
convergent for $|q| < 1$. The convergence is even locally uniform. For other theta functions the calculation is similar. The theta functions are regular for all finite values of $z$.

Clearly, for real values of $z$ and $q$, the theta functions are real-valued. From the definitions (C.1) it is obvious that $\theta_1$ and $\theta_2$ are periodic with a period of $2\pi$, whereas the period of $\theta_3$ and $\theta_4$ is $\pi$. It also directly follows that $\theta_1$ is an odd function of $z$, whereas $\theta_2, \theta_3$ and $\theta_4$ are even. Further, incrementing the argument $z$ of $\theta_1[z, q]$ by a quarter period $\pi/2$ and using the addition formula for the sine function, gives

$$\theta_1[z + \frac{\pi}{2}, q] = \theta_2[z, q].$$

(C.3)

Similarly one can derive

$$\theta_4[z + \frac{\pi}{2}, q] = \theta_3[z, q].$$

(C.4)

A consequence of these relations is, that we were sufficient with the functions $\theta_1, \theta_4$ in our calculations. For the following we shall introduce the parameter $\tau$ by

$$q = e^{i\tau},$$

(C.5)

where the imaginary part of $\tau$ must be positive to give $|q| < 1$. If we further require $\theta_i$ to be real-valued, then $\tau$ must be purely imaginary. The theta functions have also the property that if their argument $z$ is incremented by $\pi \tau$, they remain unaffected except for a multiplication by a simple factor. For example

$$\theta_1[z + \pi \tau, q] = -i \sum_{n=-\infty}^{\infty} (-1)^n \exp[(n + \frac{1}{2})^2 \pi \tau + i(2n + 1)(z + \pi \tau)]$$

$$= i \exp[-i\pi \tau - 2iz] \sum_{n=-\infty}^{\infty} (-1)^{n+1} \exp[(n + \frac{3}{2})^2 \pi \tau + i(2n + 3)z]$$

(C.6)

$$= -(qe^{2iz})^{-1} \theta_1[z, q].$$

Analogous calculation can be made for the other theta functions as follows

$$\theta_1[z + \pi \tau, q] = -\lambda \theta_1[z + \pi \tau, q] = \lambda \theta_1[z + \pi + \pi \tau, q],$$

(C.7)

$$\theta_2[z + \pi \tau, q] = -\lambda \theta_2[z + \pi + \pi \tau, q] = -\lambda \theta_2[z + \pi + \pi \tau, q],$$

(C.8)

$$\theta_3[z + \pi \tau, q] = \lambda \theta_3[z + \pi + \pi \tau, q] = \lambda \theta_3[z + \pi + \pi \tau, q],$$

(C.9)

$$\theta_4[z + \pi \tau, q] = -\lambda \theta_4[z + \pi + \pi \tau, q] = -\lambda \theta_4[z + \pi + \pi \tau, q],$$

(C.10)

where $\lambda = qe^{2iz}$. Usefulness of these relations will be clear in the appendix D. The term $\pi \tau$ is called a quasi-period of the theta functions. Similar identities may be also verified for the half of the quasi-period $\frac{\pi}{2} \tau$ and for the term $\frac{\pi}{2} + \frac{\pi}{2} \tau$. Since these results were used in the section (4.3), we reproduce the list from [14]

$$\theta_1[z, q] = -i \mu \theta_4[z + \frac{1}{2} \pi \tau, q] = -i \mu \theta_3[z + \frac{1}{2} \pi + \frac{1}{2} \pi \tau, q],$$

(C.11)

$$\theta_2[z, q] = \mu \theta_3[z + \frac{1}{2} \pi \tau, q] = \mu \theta_4[z + \frac{1}{2} \pi + \frac{1}{2} \pi \tau, q],$$

(C.12)

$$\theta_3[z, q] = \mu \theta_2[z + \frac{1}{2} \pi \tau, q] = \mu \theta_1[z + \frac{1}{2} \pi + \frac{1}{2} \pi \tau, q],$$

(C.13)

$$\theta_4[z, q] = -i \mu \theta_1[z + \frac{1}{2} \pi \tau, q] = i \mu \theta_2[z + \frac{1}{2} \pi + \frac{1}{2} \pi \tau, q],$$

(C.14)
where \( \mu = q^{1/4} e^{i \tau} \).

From the following paragraph on we shall not write the second argument \( q \), which we consider to be the same for all the theta functions. For products of the theta functions, one can obtain a number of important identities by multiplication of two of their series, followed by a rearrangement of the terms in the product series (which is permissible, since the series are absolutely convergent). We list these particular identities, which we will use in the Appendices D, E:

\[
\begin{align*}
\theta_1(x + y)\theta_1(x - y)\theta_4^2(0) & = \theta_3^2(x)\theta_2^2(y) - \theta_2^2(x)\theta_3^2(y) = \theta_1^2(x)\theta_2^2(y) - \theta_2^2(x)\theta_1^2(y), \\
\theta_2(x + y)\theta_2(x - y)\theta_4^2(0) & = \theta_3^2(x)\theta_2^2(y) - \theta_2^2(x)\theta_3^2(y) = \theta_2^2(x)\theta_2^2(y) - \theta_2^2(x)\theta_2^2(y), \\
\theta_3(x + y)\theta_3(x - y)\theta_4^2(0) & = \theta_3^2(x)\theta_3^2(y) - \theta_3^2(x)\theta_3^2(y) = \theta_3^2(x)\theta_3^2(y) - \theta_3^2(x)\theta_3^2(y), \\
\theta_4(x + y)\theta_4(x - y)\theta_4^2(0) & = \theta_3^2(x)\theta_3^2(y) - \theta_3^2(x)\theta_3^2(y) = \theta_3^2(x)\theta_3^2(y) - \theta_3^2(x)\theta_3^2(y).
\end{align*}
\]  

(C.15) 

(C.16) 

(C.17) 

(C.18)

The upper index number denotes square. For a derivation of these identities see [14], chapter 1.4. If we substitute \( y = 0 \) to (C.15)–(C.18), we get only four distinct relationships

\[
\begin{align*}
\theta_1^2(x)\theta_4^2(0) & = \theta_2^2(x)\theta_2^2(0) - \theta_2^2(x)\theta_3^2(0), \\
\theta_2^2(x)\theta_4^2(0) & = \theta_3^2(x)\theta_2^2(0) - \theta_3^2(x)\theta_3^2(0), \\
\theta_3^2(x)\theta_4^2(0) & = \theta_3^2(x)\theta_2^2(0) - \theta_3^2(x)\theta_3^2(0), \\
\theta_1^2(x)\theta_4^2(0) & = \theta_3^2(x)\theta_3^2(0) - \theta_3^2(x)\theta_3^2(0).
\end{align*}
\]  

(C.19) 

(C.20) 

(C.21) 

(C.22)

From a detailed investigation of these four identities it follows that no theta function is a trivial multiple of any other.

Finally substitution \( x = 0 \) into (C.22) yields the fundamental result

\[
\theta_3^2(0) = \theta_4^2(0) + \theta_4^2(0),
\]

which we shall use in the following section.
Appendix D

Jacobi’s Elliptic Functions

Before introducing the elliptic functions, we shall denote the following ratios of the theta functions by $k, k'$:

$$k = k(q) = \frac{\theta_3^2[0, q]}{\theta_2^2[0, q]} = 4 \frac{\sum_{n=0}^{\infty} q^{(n+1/2)^2}}{1 + 2 \sum_{n=1}^{\infty} q^{n^2}},$$  \hspace{1cm} (D.1)

$$k' = k'(q) = \frac{\theta_4^2[0, q]}{\theta_6^2[0, q]} = \frac{1 + 2 \sum_{n=1}^{\infty} (-1)^n q^{n^2}}{1 + 2 \sum_{n=1}^{\infty} q^{n^2}}.$$  \hspace{1cm} (D.2)

The reason for this notation will be clear from the following paragraph. If $0 \leq q < 1$, then $0 \leq k < 1, 0 < k' \leq 1$. When working with the elliptic functions, usually the parameters $k, k'$ are regarded as given and the quantity $q$ is to be derived from the last pair of equations. More precisely, a simple formula for $q$ as a function of $k$ will be presented below. Given the values of $k, k'$ in the interval $(0,1)$, there is a unique value of $q$ satisfying both of the equations (D.1), (D.2), see the graph Fig. F.3, supplement F.1. The equation (C.23) is then equivalent to

$$k^2 + k'^2 = 1.$$  \hspace{1cm} (D.3)

The parameter $k$ is called the elliptic modulus and $k'$ is the complementary modulus.

The Jacobi’s elliptic functions are defined in terms of the theta functions as:

$$\begin{align*}
\text{sn}[u, k] &= \frac{\theta_3[0, q(k)]}{\theta_2[0, q(k)]} \cdot \frac{\theta_1[z, q(k)]}{\theta_1[z, q(k)]}, \\
\text{cn}[u, k] &= \frac{\theta_2[0, q(k)]}{\theta_4[0, q(k)]} \cdot \frac{\theta_3[z, q(k)]}{\theta_3[z, q(k)]}, \\
\text{dn}[u, k] &= \frac{\theta_3[0, q(k)]}{\theta_4[0, q(k)]} \cdot \frac{\theta_4[z, q(k)]}{\theta_4[z, q(k)]},
\end{align*}$$  \hspace{1cm} (D.4)

where $z = u/\theta_3^2(0)$. The elliptic functions are regular except for poles. They are real-valued for $0 \leq k \leq 1$. For the graphs of these functions, see the supplement F.2.

Now the identity

$$\text{sn}^2[u, k] + \text{cn}^2[u, k] = 1.$$  \hspace{1cm} (D.5)
follows from the identity (C.20). Similarly the identity
\[ \text{dn}^2[u, k] + k'^2 \text{sn}^2[u, k] = 1 \]  
(D.6)
is equivalent to the relation (C.21) and the equation
\[ \text{dn}^2[u, k] - k^2 \text{cn}^2[u, k] = k'^2 \]  
(D.7)
can be obtained from (C.19).

Using the identities (C.7)–(C.10), we see that the elliptic functions have two distinct periods
\[
\begin{align*}
\text{sn}[u, k] &= \text{sn}[u + 2\pi \theta_3^2(0), k] = \text{sn}[u + \pi \tau \theta_3^2(0), k], \\
\text{cn}[u, k] &= \text{cn}[u + 2\pi \theta_3^2(0), k] = \text{cn}[u + \pi \theta_3^2(0) + \pi \tau \theta_3^2(0), k], \\
\text{dn}[u, k] &= \text{dn}[u + \pi \theta_3^2(0), k] = \text{dn}[u + 2\pi \tau \theta_3^2(0), k],
\end{align*}
\]  
(D.8)
where \( \tau \) is considered to be positive and purely imaginary, see (C.5). Then the first period is real and the second purely imaginary. If we denote
\[
\mathcal{K} = \frac{\pi}{2} \theta_3^2[0, k], \quad i\mathcal{K}' = \frac{\pi}{2} \tau \theta_3^2[0, k] = \tau \mathcal{K},
\]  
(D.9)
then both \( \mathcal{K} \) and \( \mathcal{K}' \) are real and positive. Clearly \( \mathcal{K}, \mathcal{K}' \) are functions of \( k \). The periods of \( \text{sn}[u, k] \) are then \( 4\mathcal{K} \) and \( 2i\mathcal{K}' \). \( \text{cn}[u, k] \) has periods \( 4\mathcal{K} \) and \( 2\mathcal{K} + 2i\mathcal{K}' \) and \( \text{dn}[u, k] \) has periods \( 4\mathcal{K} \) and \( 2\mathcal{K} + 2i\mathcal{K}' \). From the equation (D.9) we also see that
\[
q = e^{i\pi \tau} = e^{-\pi \mathcal{K}'/\mathcal{K}},
\]  
(D.10)
by which we fulfill the promise given above to prove. Notice that \( \mathcal{K}(k) \) is also the value of the elliptic integral of the first kind. It follows from the equality
\[
\text{sn}[\mathcal{K}(k), k] = 1
\]  
(D.11)
and the definition of the elliptic integral of the first kind (4.2.6), i.e. \( \mathcal{K}(k) = \text{sn}^{-1}[1, k] \). We conclude our introduction to the elliptic functions by the remark that for \( k = 0, \ k = 1 \)
\[
\begin{align*}
\text{sn}[u, 0] &= \sin(u) \quad \text{cn}[u, 0] = \cos(u) \quad \text{dn}[u, 0] = 1, \\
\text{sn}[u, 1] &= \sinh(u) \quad \text{cn}[u, 1] = \cosh(u) \quad \text{dn}[u, 1] = 0.
\end{align*}
\]  
(D.12)
Appendix E

Complete elliptic integral of the second kind $E$ in terms of the theta functions

We are going to derive the equality (4.2.7), which was fundamental in the integration of $\Pi(u, a, k)$. The proof is made according to [14], chapter 3.5.

When we differentiate the identity (C.18) twice with respect to $y$ and then set $y = 0$, we get the equality

$$\frac{d}{dx} \left( \frac{\theta_4'(x)}{\theta_4(x)} \right) = \frac{\theta_4''(0)}{\theta_4(0)} - \theta_2'(0) \sin^2(\theta_2'(0)x).$$

Using the definition of the elliptic modulus $k$ in the dependence on theta functions $k = \theta_2'(0)/\theta_3'(0)$, see (D.1) and the identity $d\theta^2 = 1 - k^2 \sin^2 \theta$, we derive an auxiliary expression

$$d\theta^2(\theta_2'(0)x) = \frac{1}{\theta_3'(0)} \frac{d}{dx} \left( \frac{\theta_4'(x)}{\theta_4(x)} \right) + 1 - \frac{\theta_4''(0)}{\theta_3'(0) \theta_4(0)}.$$

After an integration, it follows for $E(u)$

$$E(u) = \frac{1}{\theta_3'(0) \theta_4'(0)} \theta_4'(\theta_2'(0)x) + \left(1 - \frac{\theta_4'(0)}{\theta_3'(0) \theta_4(0)} \right) u,$$  

(E.1)

where $u = \theta_2'(0)x$. Substitution of $u = K = \frac{1}{2} \pi \theta_3'(0)$ into the relation (E.1), leads to the relation between the complete elliptic integral of the second kind $E$ and $K$.

$$E = \left(1 - \frac{\theta_4'(0)}{\theta_3'(0) \theta_4(0)} \right) K,$$

since $\theta_4'(\pi/2) = 0$. Equation (E.1) can now be rewritten

$$E(u) = \frac{1}{\theta_3'(0) \theta_4'(0)} \theta_4'(x) + \frac{2E_x}{\pi} = \frac{d}{du} \log \theta_4 \left( \frac{\pi u}{2K} \right) + \frac{E}{K} u,$$  

(E.2)

which we wanted to show.
Appendix F
Supplements
F.1 Graphs of the theta functions

For all of the graphs, the blue color corresponds to the lowest value of $k$, whereas the green color corresponds to the highest value of $k$.

Figure F.1: The graphs of the function $\theta_1[u, k]$ for choices of the elliptic modulus $k$ from the Table (3.2).

Figure F.2: The graphs of the function $\theta_4[u, k]$ for choices of the elliptic modulus $k$ from the Table (3.2).

Figure F.3: The graph of the dependence of the elliptic modulus $k$ as a function $k(q)$. 
F.2 Graphs of the Jacobi’s elliptic functions

For all of the graphs, the blue color corresponds to the lowest value of $k$, whereas the green color corresponds to the highest value of $k$.

![Graph of the elliptic sine](image)

Figure F.4: The graphs of the elliptic sine for choices of the elliptic modulus $k$ from the Table (3.2).

![Graph of the elliptic cosine](image)

Figure F.5: The graphs of the elliptic cosine for choices of the elliptic modulus $k$ from the Table (3.2).

![Graph of the elliptic cn](image)

Figure F.6: The graphs of the elliptic $cn$ for choices of the elliptic modulus $k$ from the Table (3.2)
Bibliography


