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BACHELOR THESIS

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Deriving a pseudomanifold of dimension 3 from nonassociative triples

Department of Algebra

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I declare that I carried out this bachelor thesis independently, and only with the cited sources, literature and other professional sources.

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Název práce: Odvození pseudovariety dimenze 3 z neasociativních trojic

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Abstrakt: Neasociativita kvázigrúp je užitočná vlastnosť pre kryptografiu. A. Drápal and I. M. Wanless vo svojej nedávnej práci študovali existenciu maximálne neasociatívnych kvázigrúp, no táto otázka ostáva pre niektoré rády nezopovedaná. Táto práca je úvodom do novej metódy riešenia tejto otázky. Po rekapitulácii najnovších zistení a naznačení využitia v kryptografii vyloží práca konštrukciu abstraktného simpliciálneho komplexu dimenzie 3 z neasociatívnych trojíc konečnej kvázigrupy. Ukážeme, že tento komplex má formu zjednotenia uzavretých orientovateľných pseudovariet dimenzie 3. Pre rády do 6 nezávisle overíme zistenia Ježka and Kepku o spektre asociativity a klasifikujeme možné rozklady komplexu neasociativity na silne súvislé komponenty analýzou ich duálnych grafov. Hlavným výsledok práce je prvý krok k riešeniu singularít v komplexe neasociativity. Ukážeme, že linky vrcholov v komplexe majú riešiteľné singularity, čo nám umožní normalizovať ich algoritmicky. Nakoniec spočítame rody komponent v linkoch a ilustrujeme typy linkov na príkladoch malých kvázigrúp.

Klíčová slova: kvazigrupa, asociativní trojice, orientovaný komplex, pseudovarieta, kombinatorický povrch

Title: Deriving a pseudomanifold of dimension 3 from nonassociative triples

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Abstract: The non-associative properties of quasigroups are useful in cryptography. A. Drápal and I. M. Wanless have recently analyzed the existence of a maximally non-associative quasigroup of order n in their work, but there remain orders n for which the existence is not known. This thesis is an introduction to a new method of tackling the problem. After presenting the most recent results and hinting at a possible cryptographic application, the thesis proposes the construction of a 3-dimensional abstract simplicial complex from non-associative triples of a finite quasigroup. It shows that the complex forms of a union of closed orientable pseudomanifolds of dimension 3. For orders up to 6, we independently verify the findings of Ježek and Kepka regarding the associativity spectrum of n and classify possible decompositions of the non-associativity complexes into strongly connected components by analyzing their dual graphs. The main result of the thesis performs the first step towards resolving the singularities in the complex. We show that links of vertices in the complex have solvable singularities, enabling us to normalize the links of vertices algorithmically. Lastly, we illustrate the types of vertex neighborhoods on examples of small quasigroups by calculating the genera of their components.

Keywords: quasigroup, associative triple, oriented complex, pseudomanifold, combinatorial surface

Contents

Introduction

Ever since ancient times, mathematicians have been intrigued by Latin squares. A Latin square can be interpreted as a multiplication table of an algebraic structure called quasigroup, which resembles a group, but differs from it mainly in that it does not need to be associative. Quasigroups may be used in cryptography, where their non-associative properties are essential for hash functions to be attack-resistant. Finding quasigroups with a small number of associative triples, and even determining the number triples that a quasigroup of a given order must contain, is, therefore, an important problem.

A. Drápal and I. M. Wanless have analyzed the existence of a maximally non-associative quasigroup of order n in Maximally non-associative quasi-groups via quadratic orthomorphisms (Drápal and Wanless (2020)), but there remain orders n for which the question stays open. Other partial results regarding the count of associative triples have been achieved; however, due to the computational complexity of these problems, the topic is yet to be explored completely.

This thesis aims to start investigations of the topic through combinatorial topology.

Chapter 1

Quasigroups

1.1 Terminology

The terminology notes in this section are according to Valent (2016).

Definition 1. *A* quasigroup $(Q, \cdot)$ *is a set* Q *closed under* $\cdot$ *such that equations* $a \cdot x = b$ *and* $y \cdot a = b$ *have unique solutions for every* $a, b \in Q$.
The order *of* Q *(i.e., the number of elements of* Q*) is denoted by* $|Q|$*. Unless stated otherwise,* Q *will mean a finite quasigroup of order* n.

Note. Throughout this thesis, we shall write ax instead of more formal $a \cdot x$ when the binary operation is $\cdot$. Similarly, we may write $a(bc)$ or $a \cdot bc$ instead of $a \cdot (b \cdot c)$. Also, when the quasigroup operation is not stated, it is assumed to be $\cdot$.

Definition 2. *A* loop *is a quasigroup with a unit; that is, an element* $e \in Q$ *such that* $x \cdot e = x$ *and* $e \cdot x = x$ *for every* $x \in Q$.

Definition 3. *A* Latin square *is an* $n \times n$ *array filled with* n *different symbols, each occurring exactly once in each column and exactly once in each row.*

Note. Every multiplication table of a finite quasigroup is a Latin square. Conversely, every Latin square can be taken as the table of a quasigroup in many ways: both the border row (containing the column headers) and the border column (containing the row headers) can carry any permutation of elements of Q.

1.2 Associativity

Definition 4. *For a quasigroup* $(Q, \cdot)$*, a triple* (a, b, c)*,* $a, b, c \in Q$*, is called* associative *if* $(a \cdot b) \cdot c = a \cdot (b \cdot c)$*. The total number of associative triples in* Q *is denoted* $a(Q)$*. This number is also called the* associativity index *of* Q.
Set $a(n) = \min\{a(Q);\ Q \text{ is a quasigroup of order } n\}$.
A quasigroup Q *of order* n *is called* maximally non-associative *if* $a(Q) = n$.

Proposition 1 (Ježek and Kepka (1990), p. 15)**.** *Let* Q *be a quasigroup of order* n*. Then* $n \leq a(Q) \leq n^3$.

Proof. For every $x \in Q$ we can define two elements $f(x), e(x) \in Q$ by $f(x) \cdot x = x = x \cdot e(x)$. Since $(f(x) \cdot x) \cdot e(x) = x = f(x) \cdot (x \cdot e(x))$, the set $\{(f(x), x, e(x));\ x \in Q\}$ is contained in the set of all associative triples of Q.

□

Definition 5. *An element $q \in Q$ is called* idempotent *if $qq = q$. The quasigroup Q is called* idempotent *if every $q \in Q$ is an idempotent element.*

We list two of the most recent results regarding the associativity index.

Theorem 2 (Grošek and Horák (2012), Theorem 1.1.)**.** *Denote by $I(Q)$ the set of all idempotent elements of Q and set $i(Q) = |I(Q)|$. Let Q be a quasigroup of order n. Then $a(Q) \geq 2n - i(Q)$.*

Proof. See Grošek and Horák (2012).

□

Corollary. If $a(Q) = n$, then Q is idempotent.

Proof. By Theorem 2, we get $n = a(Q) \geq 2n - i(Q)$, equivalently $i(Q) \geq n$. However, by its definition, $i(Q) \leq n$, therefore $i(Q) = n$.

□

Note. The fact, that $a(Q) = |Q|$ implies idempotency of Q was already shown in Kepka (1980).

Note. The lower bound from Theorem 2 was significantly improved by Drápal and Valent (2020), Theorem 2.5.. The improvement meant a new possible approach to search for maximally non-associative quasigroups (which had been, previously, impossible for orders nine and higher due to the computational complexity of the problem). Using this approach, the authors discovered the first known maximally non-associative quasigroup (of order 9) by computer search.

Theorem 3 (Drápal and Wanless (2020), Theorem 1.1.)**.** *A maximally non-associative quasigroup of order n exists for all $n > 9$, with the possible exception of $n \in \{11, 12, 15, 40, 42, 44, 56, 66, 77, 88, 90, 110\}$ and orders of the form $n = 2p_1$ or $n = 2p_1p_2$ for odd primes p_1, p_2 with $p_1 \leq p_2 < 2p_1$.*

Proof. See Drápal and Wanless (2020).

□

1.3 Possible application in cryptography - hash functions

Quasigroups find many applications in cryptography. A broad range of such applications - spanning substitution boxes, block and stream ciphers, pseudo-random number generators, and hash functions - is listed and profoundly analyzed in

Markovski (2015). In this section, we describe an application of quasigroups in hash functions as presented by Valent (2016) and discuss the particular importance of quasigroups with a small associativity index.

A cryptographic *hash function* is a function that maps data of arbitrary size to data of fixed size. It is intended to be a *one-way function*, that being a function, which is infeasible to invert. A cryptographic hash function must have the following properties:

- **Pre-image resistance:** Given a hash value h, it should be difficult to find any message m such that $h = \text{hash}(m)$.
- **Second pre-image resistance:** Given a message m_1, it should be difficult to find a message $m_2 \neq m_1$ such that $\text{hash}(m_2) = \text{hash}(m_1)$.
- **Collision resistance:** It should be difficult to come up with a pair $m_1 \neq m_2$ such that $\text{hash}(m_1) = \text{hash}(m_2)$.

As an example, we propose the hash function as mentioned in Slaminková and Vojvoda (2010) and Vojvoda (2004).

Definition 6. *Let $(Q, \cdot)$ be a quasigroup of order n. Let $(m_1, m_2, ..., m_k)$, $m_i \in Q, 1 \leq i \leq k$ be the message to be hashed. Further let Q^* be the set of all finite strings over Q. The hash function $H_a : Q \times Q^* \to Q, a \in Q$ is defined by following relation:*

$$H_a(m_1, m_2, ..., m_k) = (((...(am_1)m_2)...)m_{k-1})m_k,$$

where a plays the role of an initialization vector.

The usage of general quasigroup is often impossible due to high memory requirements. For the hash length 256 bits, the multiplication table of its 2^{256} elements would require over 4×10^{143} TB of data. The use of the quasigroup of modular subtraction was proposed in Dvorský et al. (2001) to overcome this problem. The operation $\ominus$, defined on $\{0, ..., n-1\}$, is given as $a \ominus b = a - b \pmod{n}$. However, the usage of quasigroup of modular subtraction as the only operation for the hash function is insecure, as was shown in Vojvoda (2004). That is why it was recommended in Dvorský et al. (2001) to use the quasigroups isotopic with the quasigroup of modular subtraction.

For other examples of hash function constructions, we refer to Markovski (2015).

Second pre-image is one of the basic requirements for a cryptographic hash function. We show, similarly to Grošek and Horák (2012), that hash function as described above is not second pre-image resistant if Q has many associative triples. Let $m = (m_1, m_2, m_3)$ be a message to hash. Let H_a be constructed from Q as proposed in Definition 6. Find $x, y \in Q$ so that (a, x, y) is an associative triple. Set $m_1 = xy$. Then

$$\begin{aligned} H_a(m_1, m_2, m_3) &= ((am_1)m_2)m_3 \\ &= ((a(xy))m_2)m_3 \\ &= (((ax)y)m_2)m_3 \\ &= H_a(x, y, m_2, m_3). \end{aligned}$$

Thus, the magnitude of the second pre-image resistance of H_a is inversely proportional to $a(Q)$. However, in practice, such hash functions can only be used if $|Q|$ is very large; otherwise, it would not be resistant to brute force attacks. As discussed earlier in this chapter, such a hash function could require an impossible amount of storage, so a practical use case, where a small associativity index would prove vital, is not immediately apparent. Any practical usage would require finding sufficiently large quasigroups with small associativity index, or even sufficiently large maximally non-associative quasigroups (existence of which have been proved in Drápal and Wanless (2020)), but at the same time necessarily requiring a method to store the quasigroup efficiently. Finding examples of such efficient large quasigroups with a small associativity index may, therefore, be a problem worth addressing in the future. Finding methods utilizing this property, however, would likely prove vital for applied cryptography of the future.

Chapter 2

Combinatorial geometry

The purpose of this chapter is to introduce basic notions of combinatorial geometry and topology. The chapter is divided into three sections. The first section summarizes the basic terminology of graph theory (we refer to Matoušek and Nešetřil (2019)). The second section introduces key objects of computational topology - abstract simplicial complexes. The third section introduces fundamental concepts of topology and two-dimensional surfaces and then generalizes the theory into different dimensions and transforms it to form a connection with abstract simplicial complexes.

2.1 Graphs

Definition 7. *For a set S and $k \in \mathbb{N}_0$, the set $\binom{S}{k}$ is the collection of all $X \subseteq S$ such that $|X| = k$.*

Definition 8. *A (simple undirected)* graph *is a pair $G = (V, E)$, where V is a set, and $E \subseteq \binom{V}{2}$. Elements of V are called* vertices *or* nodes *and elements of E are called* edges*.*

Note. Unless stated otherwise, we always assume graphs to be finite. The following terminology and theorems may be adjusted for infinite graphs as well.

Definition 9. *A vertex is* incident *to an edge of a graph if the vertex is one of the two vertices the edge connects.*
The degree *or* valency *of a vertex of a graph (denoted $\deg(v)$) is the number of edges that the vertex is incident to.*
A graph is regular *if all of its vertices have the same degree. A regular graph with vertices of degree k is called a* k-regular graph *or* regular graph of degree k*.*

Definition 10. *Graphs $G = (V, E)$, $G' = (V', E')$ are* isomorphic *if there exists a bijection $f : V \to V'$ for which $\{x, y\} \in E \iff \{f(x), f(y)\} \in E'$.*
The mapping f is called an isomorphism*.*

Definition 11. *A graph $H = (V_H, E_H)$ is called a* subgraph *of a graph $G = (V_G, E_G)$, if $V_H \subseteq V_G$ and $E_H \subseteq E_G$. H is called an* induced subgraph *of G, if $V_H \subseteq V_G$ and $E_H = E_G \cap \binom{V_H}{2}$.*

Definition 12. *A (finite)* walk *is a sequence of vertices and edges* $(v_0, e_1, v_1, e_2, ..., e_n, v_n)$ *such that* $e_i = \{v_{i-1}, v_i\}$ *for* $i = 1, 2, ..., n$. *The sequence* $(v_0, v_1, ..., v_n)$ *is called the* vertex sequence *of the walk.*
A (finite) trail *is a walk in which all edges are distinct.*
A (finite) path *is a trail in which all vertices (and therefore also all edges) are distinct.*

Definition 13. *Two vertices* u *and* v *of a graph* G *are called* connected *if* G *contains a path from* u *to* v. *Otherwise, they are called* disconnected.
A graph is said to be connected *if every pair of vertices in the graph is connected.*
A connected component *is a maximal connected subgraph of a graph.*

Definition 14. *A* vertex cut *or* separating set *of a connected graph* G *is a set of vertices whose removal renders* G *disconnected.*

Definition 15. *A graph* G *is said to be* k-vertex-connected *or* k-connected *if it contains at least* $k+1$ *vertices, but does not contain a vertex cut of size* $k-1$ *or less. The* vertex connectivity $\kappa(G)$ *is defined as the largest* k *such that* G *is* k*-connected.*

Example. A *complete graph* (sometimes called a *clique*) with n vertices, denoted K_n, has no vertex cuts at all, but $\kappa(K_n) = n - 1$.

Definition 16. *An* edge cut *of a connected graph* G *is a set of edges whose removal renders the graph disconnected.*

Definition 17. *A graph* G *is said to be* k-edge-connected *if it does not contain an edge cut of size* $k-1$ *or less. The* edge-connectivity $\lambda(G)$ *is the size of a smallest edge cut.*

Definition 18. *A graph is said to be* maximally (vertex-)connected *if its (vertex) connectivity equals its minimum degree.*
A graph is said to be maximally edge-connected *if its edge-connectivity equals its minimum degree.*

Definition 19. *An* n-partite *graph is a graph* $G = (V, E)$ *whose set of vertices can be partitioned into* n *subsets* $V_1, ..., V_n$ *in a such way that for each* $i \in \{1, ..., n\}$ *no edge belongs to the induced subgraph of* G *given by* V_i *(equivalently, no edge of* G *has both endpoints from the same* V_i*).*

Definition 20. *For a graph* $G = (V, E)$ *with vertex set* $V = \{v_1, ..., v_n\}$, *the* adjacency matrix *is a square* $n \times n$ *matrix* $A = (a_{ij})$ *such that*

$$a_{ij} = \begin{cases} 1, & \{v_i, v_j\} \in E \\ 0, & \{v_i, v_j\} \notin E. \end{cases}$$

2.2 Abstract simplicial complexes

2.2.1 Basic terminology

Definition 21. *Let V be a countable set. A collection X of finite subsets of V is called an* abstract simplicial complex *with vertices in V, if*

1. $\forall v \in V: \ \{v\} \in X$
2. $\forall x \in X: \ y \subseteq x \implies y \in X.$

An element $x \in X$ is called a simplex *X or a* face *of X. When $x \subseteq y$ holds, x is a* face *of y and y is a* coface *of x. A face x is a* maximal face *of X if there is no face y such that $x \subsetneq y$.*

Note. The definition of a (general topological) simplicial complex expects vertices of each simplex to be affinely independent (the exact definition is omitted). A simplicial complex is, therefore, strictly speaking, a different structure than an abstract simplicial complex. However, because this text only deals with abstract simplicial complexes, the terms complex/simplicial complex will be used instead of the more formal abstract simplicial complex.

Definition 22. *$Y \subseteq X$ is called a* subcomplex*, if Y is a complex.*

Definition 23. *The* dimension *of a simplex $x \in X$ is defined as* $\dim(x) := |x| - 1$*. The* dimension *of a complex X is defined as* $\dim(X) := \max_{x \in X} \dim(x)$*.*

Note. When $\dim(x) = k \in \mathbb{N}$, x is called a *k-simplex*. Similarly, if $\dim(X) = k$, X is a *k-complex*. (see Figure 2.1)
The empty set is a (-1)-simplex. The vertices are 0-simplices.

Definition 24. *The 1-dimensional faces of X are called* edges*. A simplicial complex of dimension at most 1 is a (simple and loopless)* graph*.*

Definition 25. *A subcomplex of all faces of X of dimension at most d is called a d-*skeleton*.*

Note. Formally, the vertex set and the 0-skeleton of X are not the same - while the former is the set of all vertices, the latter is the set containing a 1-element set for each vertex of X. However, since an apparent bijection between the two exists, we may use the terms interchangeably. Similarly, the terms 1-skeleton of X and the *underlying graph* of X will often be used to describe the graph $G = (V, E)$, where V is the vertex set of X, and E is the set of 1-simplices of X (the obvious bijection maps $\{a\} \in X$ to $a \in V$).

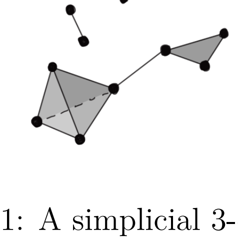


Figure 2.1: A simplicial 3-complex.

Definition 26. *A k-complex X is* pure *or* homogeneous*, if every maximal face of X is of dimension k.*

Note. The rest of this paper assumes the set of vertices V of any given complex X to be finite. All of the definitions and theorems listed for finite cases can (with occasional minor adjustments) be formulated for the countable cases.

Proposition 4. *Let X be a complex, $x \in X$ be a k-simplex.*
The number of faces in x of dimension l is equal to $\binom{k+1}{l+1}$.
The number of faces in x is equal to $\sum_{l=-1}^{k} \binom{k+1}{l+1} = 2^{k+1}$.

Proof. By definition, the number of faces of dimension l is the number of subsets of x of size $l+1$, which is $\binom{k+1}{l+1}$. The second part follows from the binomial theorem.

□

2.2.2 Notable subcomplexes

Throughout this subsection, let X be a simplicial complex and let S be a collection of simplices in X. We refer to Wikipedia contributors (2021).

Definition 27. *The* closure *of S (denoted* Cl S*) is the smallest subcomplex of X that contains each simplex in S.*

Note. Cl S is obtained by repeatedly adding to S each face of every simplex in S (see Figure 2.2).

Definition 28. *For a simplex x in S, the* star *of x is the set of all cofaces of x* (see Figure 2.3).
The star *of S (denoted* st S*) is the union of the stars of each simplex in S.*

Note. The star of S is generally not a simplicial complex itself, so some authors define the *closed star* of S (denoted St S) as Cl st S, i.e. the closure of the star of S.

Definition 29. *The* link *of S (denoted* Lk S*) is defined as* Cl st $S \setminus$ st Cl S.

Note. The link of S is the closed star of S minus the stars of all faces of S (see Figure 2.4).

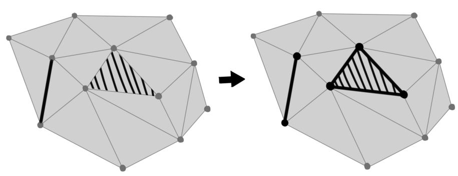


Figure 2.2: Two simplices and their closure.

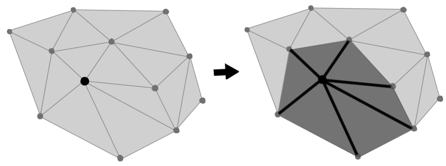


Figure 2.3: A vertex and its star.

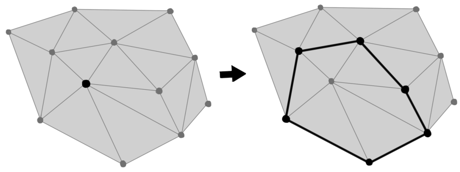


Figure 2.4: A vertex and its link.

2.2.3 Orientation

Definition 30. *Let x be a k-simplex. We denote O_x the set of all orderings of the vertex set of x. On O_x, set relation $\sim_\rightarrow$ by $\forall o_1, o_2 \in O_x : \; o_1 \sim_\rightarrow o_2$ iff they differ by an even permutation.*

Lemma 5. *$\sim_\rightarrow$ is an equivalence relation.*

Proof. The identity permutation is even, which implies reflexivity of the relation. Symmetry follows from the fact that the inverse of an even permutation is even. The composition of a pair of even permutations (in either order) is even, implying transitivity of the relation.

□

Definition 31. *An* orientation *of a k-simplex x is an equivalence class of O_x by $\sim_\rightarrow$. An* oriented k-simplex *is a k-simplex x together with an orientation of x.*

Note. Let $x = \{v_0, ..., v_k\}$ be a k-simplex. We will denote an orientation of x (the oriented simplex x with a given orientation) by $(v_0, ..., v_k)$. By previous definition, two orientations $(v_{i_0}, ..., v_{i_k}), (v_{j_0}, ..., v_{j_k})$ of x are equivalent if and only if they differ by an even permutation.

Note. Let x be a k-simplex. Then

$$|O_x/\sim_\rightarrow| = \begin{cases} 1, & k = -1, 0; \\ 2, & k \geq 1. \end{cases}$$

Vertices have only a single possible orientation. A simplex consisting of at least two vertices has two possible orientations.

Definition 32. *An oriented k-simplex $(v_0, ..., v_k)$ determines (*induces*) an orientation of each of its $(k-1)$-faces, called the* induced orientation*, by the following rule: the induced orientation on the face $\{v_0, ..., v_{i-1}, v_{i+1}, ..., v_k\}$ is defined to be $(-1)^i(v_0, ..., v_{i-1}, v_{i+1}, ..., v_k)$.*

Note. Let $x = (v_0, ..., v_k)$ be an oriented k-simplex.

1. if $k \geq 1$, then the orientations of $(-1)^i x$ and x are
   (a) equivalent, if i is even;
   (b) different, if i is odd;
2. if $k < 1$, then the orientations of $(-1)^i x$ and x are equivalent for all i.

2.3 Combinatorial (hyper)surfaces

A *surface* is, informally, an object similar to a plane but that need not be flat. We can not only define the term rigorously through topology, but also extrapolate the meaning of surfaces in higher dimension - *hypersurfaces*. For purposes of this thesis, most of the terminology will be formulated in an abstract combinatorial sense. An obvious disadvantage of this approach is the lack of visual intuition of combinatorial surfaces. However, this can be (at least to some extent) solved by envisioning a combinatorial surface as a *triangulation* (i.e., a ”finite approximation”) of a topological surface (the rigorous definition of triangulation is omitted). Understanding the connection between combinatorial and topological surfaces is important; however, for our purposes, it mainly provides us with examples and visual insight of the key structures of the thesis - *pseudomanifolds* (the exact definition will be presented in Definition 34).

2.3.1 Manifolds

We begin by presenting the basic notion of (hyper)surface in topology - a *manifold* and several related results.

Definition 33. *An n-dimensional* manifold*, or n-*manifold*, is a topological space with the property that each point has a neighborhood that is homeomorphic to the Euclidean space of dimension n.*

Higher-dimensional manifolds are very difficult to visualize, but in dimension 2, we can imagine them as surfaces of 3-dimensional objects (though with possible self-intersections). The classification of 2-manifolds, listed here for illustrative purposes in a simplified version and without proof, is a classical topology result. The original proof for open 2-manifolds by Kerékjártó (1923) has been generalized for all 2-manifolds and improved over the years, notably by Brown and Messer (1979). However, the classification of 3-manifolds is currently a significant open problem, and since recognizing whether two triangulated manifolds of dimension 4 are homeomorphic is undecidable according to Markov (1958), the generalization of the theorem to higher dimensions is impossible.

Theorem 6 (Classification Theorem for Compact 2-manifolds, Edelsbrunner and Harer (2010), p. 35)**.** *The two infinite families $\mathbb{S}^2, \mathbb{T}^2, \mathbb{T}^2 \# \mathbb{T}^2, ...$ and $\mathbb{P}^2, \mathbb{P}^2 \# \mathbb{P}^2, ...$ exhaust the family of compact 2-manifolds without boundary.*

The first family of *orientable*, compact 2-manifolds consists of the *sphere*, the *torus*, the *double torus*, and so on. The second family of *non-orientable*, compact 2-manifolds consists of the *projective plane*, the *Klein bottle*, the *triple projective plane*, and so on. Every 2-manifold can be obtained by repeatedly "gluing" *handles* and *cross-caps* to a sphere (exact definitions are omitted, but illustrative examples are provided in Figure 2.5).

In the above examples, we have mentioned *orientability*, a fundamental property of topological manifolds, without a proper introduction. In layman's terms, orientability is closely related to the ability to embed the manifold in space. The definition of orientability used in a general topological context typically deals with continuous deformations of closed curves to establish a sense of "clockwise" and "counter-clockwise" direction (the exact definition is omitted). This approach seemingly introduces ambiguity to the term, as the term of orientation has been rigorously defined for k-simplices in Definition 32, and the term of orientability will be defined for particular types of k-complexes in Definition 37 using the terminology laid out in Definition 32. However, the topological and combinatorial versions of these definitions can be unified thanks to a fundamental result of topology, which grants the existence of isomorphic refinements of triangulations of two homeomorphic manifolds (exact details are omitted). The idea behind the unification of the topological and combinatorial versions is then to find a triangulation of a given topological manifold, view the triangulation as a complex, and determine its orientability in the means of Definition 37.

Orientability in dimension 2 has a straightforward idea. Informally, a 2-manifold is orientable if we can consistently assign what is "inside" and what is "outside". Orientability turns out to be equivalent to whether the surface contains no subset that is homeomorphic to the Möbius strip. Thus, for 2-manifolds, the Möbius strip may be considered the source of all non-orientability.

Furthermore, the number of handles (cross-caps) attached to a sphere is an invariant property of the orientable (non-orientable) 2-manifold. This number is called the *genus*. Informally speaking, the genus of a 2-manifold is the number of "holes" in an orientable case and the number of "self-intersections" in a non-orientable case. As we will see in a moment, the genus of a surface is closely related to another essential invariant - the Euler characteristic of a surface. Euler characteristic extends the definition of genus for manifolds in higher dimensions.

Figure 2.5: Left: a sphere with three handles attached = a triple torus = a pretzel. Right: a sphere with a cross-cap = a projective plane.

To get a classification of the connected, compact 2-manifolds with boundary, we can take one without boundary and make h holes by removing the same number of open disks. Each starting compact 2-manifold and each $h \geq 1$ give a different surface, and they exhaust all possibilities.

More information on the subject can be found in Chapter 2 of Edelsbrunner and Harer (2010).

2.3.2 Pseudomanifolds

Having presented the general notion of topological surface and its orientation, we may formally build the theory of (hyper)surfaces in a combinatorial sense. This approach has several advantages:

- it can be extrapolated to higher dimensions,
- it generalizes the notions by enabling singularities, and
- it is suitable for working with abstract simplicial complexes, as it naturally enables combinatorial computations.

Definition 34. *A finite simplicial k-complex is called a k-dimensional* pseudomanifold*, if it satisfies the following properties:*

1. *it is* non-branching*: Each $(k-1)$-dimensional simplex is a face of precisely one or two k-dimensional simplices;*
2. *it is* strongly connected*: Any two k-dimensional simplices can be joined by a "chain" of k-dimensional simplices in which each pair of neighboring simplices have a common $(k-1)$-dimensional face;*
3. *it has* dimensional homogeneity*: Each simplex is a face of some k-dimensional simplex.*

A pseudomanifold is called normal *or a* combinatorial manifold *if the link of each simplex with codimension ≥ 2 is a pseudomanifold.*

Note. The non-branching property can be made more restrictive by requiring each $(k-1)$-dimensional simplex to be a face of precisely two k-dimensional simplices. A k-dimensional pseudomanifold satisfying this more restrictive non-branching property is *closed*.

Note. A pseudomanifold is a combinatorial generalization of the general idea of a manifold with singularities. It can be regarded as a particular type of topological space (because given a topological pseudomanifold, one can obtain its combinatorial counterpart via triangulation).

Definition 35. *Let X be a k-pseudomanifold. A pair of oriented k-simplices $x, y \in X$ is* consistently oriented*, if $|x \cap y| \neq k$, or x and y induce different orientations on $x \cap y$.*
A k-dimensional pseudomanifold X is orientable *if there exists a mapping f that assigns an orientation to every k-simplex in X in such way that each pair of k-simplices in X is oriented consistently.*

The pseudomanifold encompasses essential structural properties of the corresponding topological structure. A notable example of such property is orientability, defined rigorously in this subsection for pseudomanifolds and hinted in the previous subsection in the context of (topological) manifolds. As mentioned earlier, it is possible to equivalently define the orientation and orientability in the topological sense by assigning orientations to refined triangulations according to Definition 35. For purposes of this thesis, it is only necessary to understand that this justifies informally mixing the terms of combinatorial and topological pseudomanifolds, which enables us to list visual examples for otherwise very abstract combinatorial structures.

Example. A manifold is a normal pseudomanifold. Therefore, a sphere and a torus, shown in Figure 2.6 (a projective plane and a Klein bottle) are examples of normal, orientable (non-orientable), compact 2-dimensional pseudomanifold.

Example. A pinched torus, shown in Figure 2.6 is an example of an orientable, closed 2-pseudomanifold. It is not a normal pseudomanifold because the link of the "pinched" vertex is not connected (hence not a pseudomanifold).

In a 2-pseudomanifold, every singularity is obtained by "gluing" together several points. However, nothing more complicated (e.g., "gluing" two edges together) is permitted (see Figure 2.7). We formulate and prove a theorem and a proposition that expand on this statement, and describe a method to normalize a particular type of union of 2-pseudomanifolds, where no singularity is more complicated than a vertex.

Figure 2.6: Left: a triangulated torus. Right: a pinched torus.

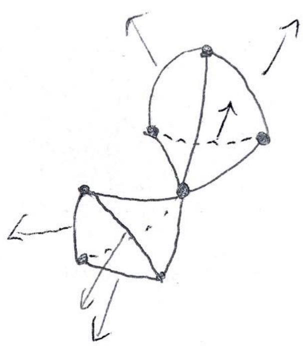


Figure 2.7: The only acceptable case of singularity in a 2-pseudomanifold. Higher dimensional pseudomanifolds allow for far more complicated cases of singularities.

Theorem 7. *Let A be a vertex in a closed 2-pseudomanifold S. Then* $\operatorname{Lk} A$ *is a union of disjoint cycles.*

Proof. The closed star of A is a 2-complex containing every face of every 2-simplex that contains A as its vertex. To obtain $\operatorname{Lk} A$, we remove any face that contains A as its vertex. We may observe that $\operatorname{Lk} A$ has the following properties:

- It is a 1-complex (a graph), because it contains vertices from $\operatorname{Cl}\ \operatorname{st} A$ other than A and exactly one edge for every 2-face in $\operatorname{Cl}\ \operatorname{st} A$ - the edge opposite the vertex A. $\operatorname{Lk} A$ contains no 2-simplex, as every 2-face in $\operatorname{Cl}\ \operatorname{st} A$ contains A as its vertex.

- It is closed and non-branching: a vertex C in $\operatorname{Lk} A$ is a neighbor of vertex B if and only if $\{A, B, C\} \in \operatorname{Cl}\ \operatorname{st} A$. Because S is closed and non-branching, there exist precisely two 2-faces of S that contain the edge $\{A, B\}$, namely $\{A, B, C\}$ and $\{A, B, D\}$, and since both of them are included in $\operatorname{Cl}\ \operatorname{st} A$, the vertex B has precisely two neighbors in $\operatorname{Lk} A$.

It follows that $\operatorname{Lk} A$ is a union of disjoint cycles.

□

Proposition 8. *Let S be a union of closed 2-pseudomanifolds such that for every vertex $V \in S$ is* $\operatorname{Lk} V$ *a union of disjoint cycles. Let X be a vertex in S. Choose a cycle $\mathcal{C}$ in* $\operatorname{Lk} X$ *and suppose* $\operatorname{Lk} X \setminus \mathcal{C} \neq \emptyset$. *Modify S using the following two steps:*

1. *create a new 1-skeleton by connecting a new vertex $\tilde{X}$ with every vertex in $\mathcal{C}$ and disconnecting X from every vertex in $\mathcal{C}$*

2. *perform an operation of closure on the new 1-skeleton*

This process creates a modified version of S by "cutting at singularity X" once. Repeating this process a finite number of times transforms S into a union of disjoint closed 2-manifold.

Proof. Because S is finite, there are only finitely many singularities. Every step disconnects one cyclic part of the link of a singularity, and therefore after finitely many steps, the link is a cycle for every vertex. It remains to be shown that after any number of iterations, the link of any vertex in the current modified version of S is a union of disjoint cycles. We prove this using induction and only have to prove the induction step, as the first step is the same.

The process does not connect already disconnected cycles in links of vertices by cutting "somewhere else". To elaborate, if K, L are vertices in the link of some vertex M and they had belonged to two different cycles before performing a cut at singularity Z and are now connected via a new copy $\tilde{Z}$, then K and L had to have belonged to the same cycle in the link of Z. Therefore, they had to have been connected even before performing a cut at Z, a contradiction.

Finally, the process does not disconnect the cycles: if a cycle in the link of vertex M gets disconnected by performing a cut at Z, by definition of "cutting" at singularity Z we get that M had to have belonged to two different cycles in the link of Z as it is now connected to both Z and $\tilde{Z}$, a contradiction.

□

Theorem 7 and Proposition 8 grant resolvability of singularities in closed 2-pseudomanifolds by showing that it can be done by simply "cutting" at vertices. Because of this, it was possible to derive one of the most important results of combinatorial topology, the classification theorem for 2-pseudomanifolds, first given by Dehn and Heegaard (1910). We formulate and prove the classification theorem by replicating the steps proposed in Stillwell (1993).

We introduce terminology that enables us to formulate and prove the classification theorem intelligibly.

As defined in Dehn and Heegaard (1910), a *closed surface* (or just *surface*) is a closed 2-pseudomanifold (Definition 34 for $k = 2$). It will be convenient to build surfaces from polygons other than triangles, so we now go to an alternative definition.

A (finite) closed surface is a (finite) set of polygons with oriented edges identified in pairs (meaning that there exist precisely two identical copies of an edge in the entire system; the pair of identified edges may or may not be on a single polygon). Such a system is called a *schema*. Both definitions of surface are equivalent. Suppose the polygons are given sufficiently fine simplicial decompositions, which are compatible on identified edges (i.e., an edge from an identified pair is decomposed exactly the same way as its identified counterpart). In that case, each edge in the decomposition will be either an interior edge of a polygon, hence incident with two triangles, or else a pair of subedges identified by the schema, hence also incident with two triangles. This reasoning justifies transforming polygons into complexes and vice versa by adding and removing interior points and edges. We denote the edges of polygons in a schema using letters, assigning the same letter to identified edges. A polygon in a schema is represented by a word made of edges, for example $aba^{-1}b$ or $a^{-1}bc$ (see Figure 2.8). A portion of a boundary of a polygon with the form in Figure 2.9a will be called a *handle* and denoted symbolically by $aba^{-1}b^{-1}$ (reading labels and orientations clockwise). Similarly, a portion like that in Figure 2.9b, will be called a *cross-cap* and denoted by aa or a^2.

We formulate the classification using the notions of words, handles and cross-caps as follows:

Theorem 9 (Classification of closed 2–pseudomanifolds)**.** *Every closed 2-pseudomanifold is represented by a polygon of one of the following three types:*

- *a* sphere (aa^{-1})
- *a sphere with n handles* $(a_1a_2a_1^{-1}a_2^{-1}...a_{2n-1}a_{2n}a_{2n-1}^{-1}a_{2n}^{-1})$
- *a sphere with n cross-caps* $(a_1a_1a_2a_2...a_na_n)$

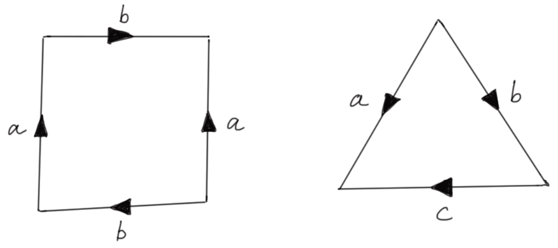
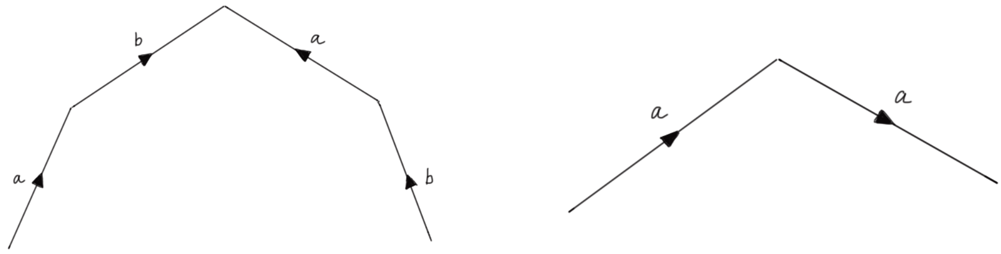


Figure 2.8: Left: polygon $aba^{-1}b$. Right: polygon $a^{-1}bc$.



(a) A handle. (b) A cross-cap.

Figure 2.9: Notable portions of a boundary of a polygon

Proof. (by Stillwell (1993), p. 69-74) Assume the polygons in the schema define a connected surface. Then, it is possible to amalgamate all of them into a single polygon by repeatedly gluing them together along identified edges of separate polygons. The resulting polygon has an even number of edges divided into pairs of identified edges, which follows from the fact that by performing a triangulation on the polygonal schema, we increase the total number of edges by an even number. A closed 2-pseudomanifold obtained from the polygonal schema via triangulation is a schema whose polygons have an even number of edges in total because every edge in a closed 2-pseudomanifold is a face of precisely two triangles.

The schema of a sphere (Figure 2.10) has 2 distinct vertices A, B and is exceptional in having only a "canceling pair" of edges, aa^{-1}. We show that any other polygon with more than one pair of identified edges and more than one vertex can be transformed to one of the above types by performing the following steps:

1. In the first step, we reduce the number of unique vertices of the polygon until we are either left with a schema of a sphere or with a single unique vertex. The step consists of repeatedly performing two moves:

   (a) Anytime the polygon consists of at least three vertices and a "canceling pair" of edges, we collapse one such pair. This move reduces the number of unique vertices by one.

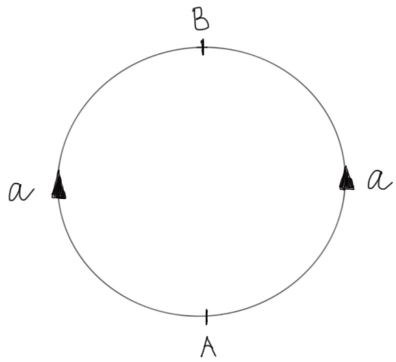


Figure 2.10: A sphere aa^{-1}.

(b) Divide the apparent vertices of the polygon into equivalence classes of vertices identified with each other, then, assuming there are at least two equivalence classes, consider an edge a whose endpoints A, B belong to different classes. We choose one of them, for example, B. If there are at least two vertices in the equivalence class of B, by repeating the step (*) described in Figure 2.11, we reduce the number of vertices in the equivalence class containing the vertex B until we are left with a single vertex of class B. Then, it means that edges leading to (or from) it make a "canceling pair", and so we are either left with a sphere (and then we are done), or we collapse them, removing the last vertex of equivalence class B.

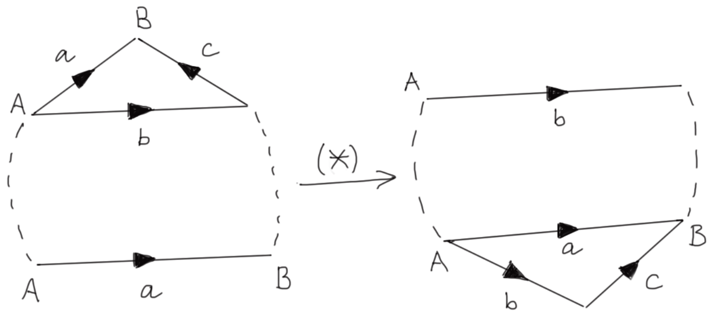


Figure 2.11: Reducing one vertex of class B. The step denoted (*) cuts along b and pastes the cut triangle along edge a.

When we can no longer perform these steps, we are either left with a sphere (and then we are done) or with a polygon, whose all vertices are in the same equivalence class (that is, a polygon with only a single vertex). Any time we cut and paste the polygon from now on, its vertices remain in the same equivalence class.

2. The second step consists of repeatedly performing the move described in Figure 2.12 in order to group together all the pairs of identified edges of the polygon with the same orientation. This will form a cross-cap from every pair of like-oriented identified edges.

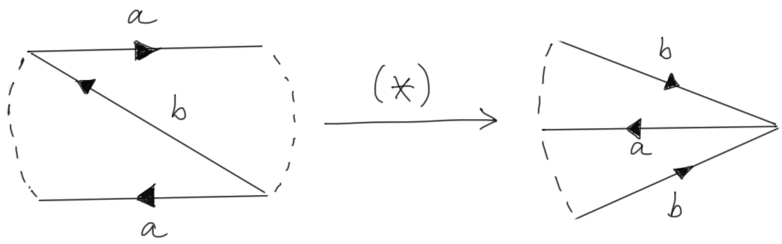


Figure 2.12: Forming a cross-cap from identified edges with the same orientation. The step denoted (*) cuts the polygon along b and pastes the cut polygon along edge a.

Note that the move does not affect the ordering of other edges; notably, it does not disrupt the pairs already ordered. This process will be repeated until we have turned every pair of identified edges with the same orientation into cross-caps.

3. If any pairs of oppositely oriented identified edges remain after the previous step, they must occur as "crossed pairs" $...a...b...a^{-1}...b^{-1}...$ in the boundary, because if, for example, the pair $...a...a^{-1}...$ is not separated by any other pair of oppositely oriented identified edges, then the situation is the one depicted in Figure 2.13.

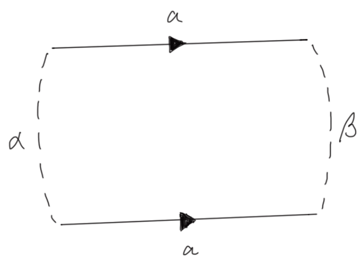


Figure 2.13: A non-separated pair.

In this situation, each edge in α is identified with another edge in α, and analogously for edges in β because all like-oriented edges were made adjacent in the previous step. Then, however, the corresponding pairs of ends of a cannot be identified, which contradicts the first step.

Therefore, in the third step, we perform the process described in Figure 2.14 a finite number of times, each time replacing a crossed pair by a handle. The process will exhaust all such crossed pairs after a finite number of repetitions.

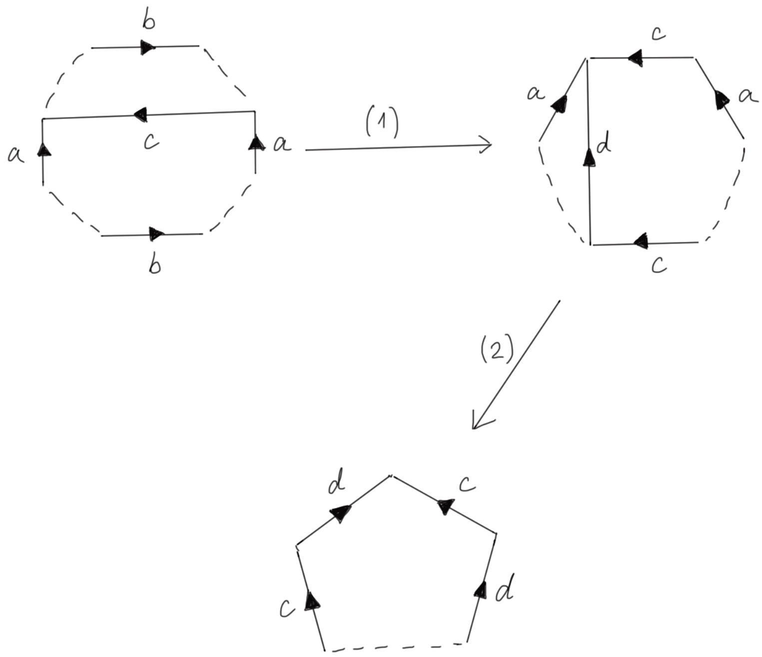


Figure 2.14: Creating a handle from two crossed pairs. The move (1) cuts along c and pastes along b. The move (2) cuts along d and pastes along a.

4. Having completed the previous steps, we are left with a polygon consisting either only of handles, or of both cross-caps and handles. If it consists only of handles, we are finished. Else, the boundary must contain a sequence $..aabcb^{-1}c^{-1}...$, which we convert to three cross-caps by performing a step shown in Figure 2.15, then replacing the three like-oriented pairs with adjacent pairs as in the second step.

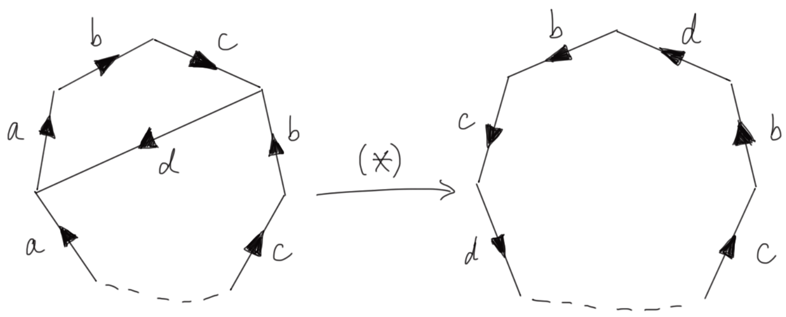


Figure 2.15: Creating three cross-caps from a cross-cap and a handle. The step denoted (*) cuts along d and pastes along a.

This step does not disturb the dashed part of the boundary, and the handle will not reappear if we normalize the cross-caps in the right order (one such order is b, c, d). Repeating this step a finite number of times yields a word from one of the categories from the Theorem.

□

In recent years, the classification of compact two-dimensional topological pseudomanifolds has been finalized (see Banagl and Friedman (2004), p. 527-528.).

In dimension 3, a valid singularity is obtained, for example, by "gluing" two tetrahedra (3-simplices) together by an edge and then "gluing" the two vertices of the edge together. Generally speaking, a singularity in a k-pseudomanifold is obtained by "gluing" together some combination of up to $(k-2)$-simplices in some order. With an increasing dimension, the number of possibilities (even their cardinality) becomes incredibly high, resulting in far too complicated and far too many possible pseudomanifold structures.

Classification of 3-pseudomanifold is an open problem that requires the classification of 3-manifolds to be solved first. Classification of pseudomanifolds of dimension 4 or higher cannot exist (again, according to Markov (1958)).

There exist relatively easily accessible properties, enabling us to divide pseudomanifolds into larger categories. These properties provide basic insight into otherwise very complicated structures. Euler Characteristic is an important example of such invariant:

Definition 36. *Let X be a complex of dimension k. For $i \in \{0, ..., k\}$ set $s_i = |\{x \in X;\ \dim(x) = i\}|$. The number $\chi(X) := \sum_{i=0}^{k}(-1)^i s_i$ is called* Euler characteristic *of X.*

Note. As mentioned earlier, Euler characteristic extends the definition of the genus to higher dimensions. The following proposition describes their relation in dimension 2.

Proposition 10 (Euler Characteristic and genus of Compact 2-manifolds)**.** *A sphere with g tubes (an orientable compact 2-manifold of genus g) has $\chi = 2-2g$. A sphere with g cross-caps (a non-orientable compact 2-manifold of genus g) has $\chi = 2 - g$.*

Proof. See Edelsbrunner and Harer (2010), p. 30.

□

Note. Euler characteristic can be generalized for pseudomanifolds by taking into account the multiplicities of singularities.

2.3.3 Colorability

Definition 37. *Let X be a simplicial k-complex, V be its vertex set, and F be the set of all k-simplices in X.*
*X is n-*vertex-colorable*, if there exists a mapping $g : V \to \{1, 2, ..., n\}$ such that $\{v_1, v_2\} \in X \implies g(v_1) \neq g(v_2)$ (vertices connected by an edge must be assigned different colors), equivalently if the underlying graph of X is n-partite.*
*X is n-*face-colorable*, if there exists a mapping $g : F \to \{1, 2, ..., n\}$ such that $f_1, f_2 \in F,\ |f_1 \cap f_2| = k \implies g(f_1) \neq g(f_2)$ (if two k-simplices share a $(k-1)$-face, they must be assigned different colors).*

Proposition 11. *The minimal n, for which a given k-complex is n-vertex-colorable, is at least $k + 1$.*

Proof. The vertices and edges in any k-simplex form K_{k+1}. It follows that any k-simplex must be colored using $k+1$ colors, thus requiring at least $k+1$ colors for the entire k-complex.

□

The following theorem describes a connection between colorability and orientability for pseudomanifolds.

Theorem 12. *A closed k-dimensional pseudomanifold X is 2-face-colorable if and only if it is orientable.*

Proof. $\implies$: Let X be a closed 2-face-colorable k-pseudomanifold, whose 3-simplices are colored using colors 0 and 1. For a k-simplex $\{v_0, v_1, ..., v_k\}$ in X, we define its orientation as follows:

$$\begin{cases} (v_0, v_1, ..., v_k), & \text{if } \{v_0, v_1, ..., v_k\} \text{ is colored } 0; \\ (v_0, v_1, ...v_{k-2}, v_k, v_{k-1}), & \text{if } \{v_0, v_1, ..., v_k\} \text{ is colored } 1. \end{cases}$$

By Definition 34 and the following note, X is non-branching, and therefore, every $(k-1)$-face of X is a face of exactly two k-simplices. These two k-simplices have different colors (because X is 2-face-colorable), and therefore, by construction, induce different orientations on the common $(k-1)$-face. It follows that the orientation mapping defined above assigns orientations in a way that each pair of k-simplices is oriented consistently. X is orientable by Definition 35.
$\impliedby$: X is orientable and non-branching, so every $(k-1)$-face of X is a face of precisely two consistently oriented k-simplices. We may, therefore, choose a k-simplex and assign color 0 to it and every other k-simplex in X with the same orientation, and assign color 1 to the remaining k-simplices. This mapping satisfies the requirements set in Definition 37 because neighboring k-simplices have different orientations, and therefore X is 2-face-colorable.

□

2.3.4 Dual graphs

Definition 38. *The* dual graph *of a k-pseudomanifold X is a graph $G_X = (V_X, E_X)$, where V_X is the set of all k-simplices of X, and for all $x, y \in V_X$: $\{x, y\} \in E_X \iff |x \cap y| = k$.*

Proposition 13. *The dual graph of a closed k-pseudomanifold is $(k+1)$-regular.*

Proof. By Proposition 4, every k-simplex in a k-pseudomanifold has exactly $\binom{k+1}{(k-1)+1} = k+1$ $(k-1)$-faces. By definition of closed pseudomanifold (34, more precisely its non-branching property, and the following note), every $(k-1)$-face is a face of exactly 2 k-simplices, implying that every k-simplex has exactly $k+1$ neighbors. Therefore, by definition, each vertex of the dual graph has degree $k+1$.

□

Chapter 3

Complex of non-associativity

This chapter presents two constructions of abstract simplicial complexes that describe the property of non-associativity of a given quasigroup. We then show that one of the constructions is a simplification of the other and that the simpler simplicial complex possesses some nice combinatorial and topological properties, enabling us to transform the problem of studying (non)-associative properties of a quasigroup to studying the topological and combinatorial properties of the generated complex.

We will always assume the given quasigroup to be finite. Given a quasigroup $(Q, \cdot)$ of order $n \in \mathbb{N}$, we will without loss of generality presume Q to be the set $\{0, ..., n-1\}$.

3.1 Requirements

Strictly speaking, there are many different complexes to be obtained from a given structure because we can choose which faces to include or not. When building an abstract simplicial complex from an algebraic structure to describe the desired property of the structure, one needs to consider several factors that determine the properties of the resulting complex. We shall discuss our reasoning behind the choices we made:

- **Existence and uniqueness:** The ultimate goal of this paper is to determine whether the associative property of a given quasigroup can be interpreted in a language of computational topology. To get information about a quasigroup and to be able to compare that information with information about other quasigroups of the same size, we must first and foremost be able to construct a complex from any given quasigroup. We must always build these complexes in the same way (i.e., using the same algorithm), meaning the algorithm must be deterministic and must output only a single complex for every input.

- **Maximality:** We want the complexes to encompass the maximum amount of information about the quasigroups. Therefore, every complex we build should ”span” the whole structure of the given quasigroup.

- **Specifics of the problem:** The associative property of a given quasigroup is uniquely determined by values of $a(bc)$ and $(ab)c$ for all triples $a, b, c \in Q$. We shall, therefore, create a complex in such way, that it encapsulates information on whether $a(bc) = (ab)c$ or not for each triple.

3.2 Construction

The associative property of Q can be described by constructing a 3-complex by taking 3-simplices, whose first three vertices are elements a, b, c of Q in a fixed order and the fourth vertex is either $(ab)c$ or $a(bc)$. Given a quasigroup Q, we will construct its associativity 3-complex as follows:

Start with four identical copies of Q; we will denote these copies Q_1 through Q_4. The union of these sets will form the vertex set of the complex.

Next, we define the structure of the 3-complex by building all possible 3-simplices. Take all triplets $a \in Q_1,\ b \in Q_2,\ c \in Q_3$, and for each of them construct either a set $\{[a,1],[b,2],[c,3],[(ab)c,4]\}$ if $(ab)c = a(bc)$, where $(ab)c,\ a(bc) \in Q_4$, or a pair of sets $\{[a,1],[b,2],[c,3],[(ab)c,4]\}$ and $\{[a,1],[b,2],[c,3],[a(bc),4]\}$ if $(ab)c \neq a(bc)$ (again, $(ab)c,\ a(bc) \in Q_4$). The union of the power sets of these sets forms the simplicial 3-complex. We will call it the *associativity complex* of Q and denote it $\texttt{AC}(Q)$.

By modifying this construction, we obtain a 3-complex that ignores the associative triples but still encompasses the non-associative properties.

Again, the union of Q_1 through Q_4 forms the vertex set, but now, we ignore the sets $\{[a,1],[b,2],[c,3],[(ab)c,4]\}$ if $(ab)c = a(bc)$, and only take sets $\{[a,1],[b,2],[c,3],[(ab)c,4]\}$ and $\{[a,1],[b,2],[c,3],[a(bc),4]\}$ if $(ab)c \neq a(bc)$. The union of the power sets of these sets again forms a simplicial 3-complex. We will call this the *non-associativity complex* of Q and denote it $\texttt{NAC}(Q)$.

Note. In our data structure, $\texttt{AC}(Q)$ will be represented by an array of ordered lists. In each step, the algorithm of construction of $\texttt{AC}(Q)$ will append to the array either a pair of ordered lists $[a,b,c,(ab)c],\ [a,b,c,a(bc)]$ if $(ab)c \neq a(bc)$, or an ordered list $[a,b,c,(ab)c]$ if $(ab)c = a(bc)$. These ordered lists store the information about the 3-simplices of our complex, because the index in ordered list corresponds to the index of the copy of Q, from which the given vertex comes. Because of this, a simplified notation $[a,b,c,d]$ for a 3-simplex $\{[a,1],[b,2],[c,3],[d,4]\}$ is justified and will be used through this thesis.

Similarly, $\texttt{NAC}(Q)$ will also be represented by an array of ordered lists, with the only difference being in the construction, where the algorithm will only append a pair of ordered lists $[a,b,c,(ab)c],\ [a,b,c,a(bc)]$ if $(ab)c \neq a(bc)$, skipping the ordered lists generated from associative triples.

Note. For a pair of 3-simplices $A = [a_1,a_2,a_3,a_4],\ B = [b_1,b_2,b_3,b_4]$, we will say "$B$ is a k-neighbor of A", or "B is a neighbor of A opposite the kth vertex", if $a_j = b_j\ \forall j \in \{1,...,4\} \setminus \{k\}$ and $a_k \neq b_k$. If we imagine A and B as tetrahedrons, this means that A and B are connected by a face opposite the kth vertex.

3.3 Trivial properties

This section lists several observations about the created complexes. Properties listed here mostly follow immediately from the constructions, but they are, nonetheless, important to realize in order for us to understand the more complicated structural properties of the complexes.

3.3.1 Satisfaction of requirements

The presented constructions accomplish every requirement formulated in Section 3.1:

- **Existence:** It is clear that for every quasigroup Q, both of the constructions output a 3-complex.
- **Uniqueness and maximality:** The construction of $\mathtt{AC}(Q)$ generates all ordered quadruplets that represent an instance of multiplication of a triple (in one of two possible orders) in Q, therefore spanning all possible 3-simplices and creating, in a sense, a maximally connected complex (meaning that every relevant face is included). It follows that this construction yields the same complex every time we input the same quasigroup.

  The same is true for the construction of $\mathtt{NAC}(Q)$, with the only difference being that this time $\mathtt{NAC}(Q)$ spans all 3-simplices corresponding to non-associative triples. In this construction, the relevant faces all correspond to these 3-simplices.
- **Specifics of the problem:** The resulting complexes represent the associative properties in the quasigroup well: there exists a 4-neighbor to a 3-simplex $[a, b, c, d]$ if and only if the triple a, b, c is not associative. The following section shows that both complexes encompass even more information about the (non-)associative properties.

3.3.2 Associative property in the complexes

We now proceed by showing that both $\mathtt{AC}(Q)$ and $\mathtt{NAC}(Q)$ provide us with information about the count of associative and non-associative triples in Q.

It follows from the construction of $\mathtt{AC}(Q)$ that for a quasigroup Q of order n

$$|AC| = |A| + 2|N|, \tag{3.1}$$

where $|AC|$ is the number of 3-simplices in $\mathtt{AC}(Q)$, $|A|$ is the number of associative triples and $|N|$ is the number of non-associative triples. Furthermore, because

$$|A| + |N| = n^3 \tag{3.2}$$

is the number of all triples in Q, we have proved the following proposition:

Proposition 14. *Let Q be a quasigroup of order n. Then*

$$|A| = 2n^3 - |AC|, \tag{3.3}$$

where $|A|$ is the number of associative triples in Q and $|AC|$ is the number of 3-simplices in $\mathtt{AC}(Q)$.

Furthermore, it follows from the construction of $\mathtt{NAC}(Q)$ that

$$|AC| = |A| + |NAC|, \tag{3.4}$$

where $|NAC|$ is the number of 3-simplices in $\mathtt{NAC}(Q)$. It immediately follows that

$$|NAC| = 2|N|, \tag{3.5}$$

and by combining the formulas 3.2 and 3.5, we obtain a link between the number of associative triples of Q and the number of 3-simplices in $\mathtt{NAC}(Q)$:

Proposition 15. *Let Q be a quasigroup of order n. Then*

$$|NAC| = 2(n^3 - |A|) \tag{3.6}$$

where $|A|$ is the number of associative triples in Q and $|NAC|$ is the number of 3-simplices in $\mathtt{NAC}(Q)$.

We have shown that both $\mathtt{AC}(Q)$ and $\mathtt{NAC}(Q)$ have an elegant link to the number of associative triples in Q, because $|A|$ is uniquely defined by the number of 3-simplices in them.

3.3.3 Maximal faces and dimensional homogenity

It follows from the constructions that every maximal face of both $\mathtt{AC}(Q)$ and $\mathtt{NAC}(Q)$ is a 3-simplex. By definition, this means that both complexes are pure.

Furthermore, it follows from the constructions that every face of a 3-simplex (either in $\mathtt{AC}(Q)$, or in $\mathtt{NAC}(Q)$) is included in the respective complex. Therefore, both complexes have dimensional homogenity (see Definition 34).

3.3.4 Number of neighbors

The logical next step in our study of structural properties of the complexes is to understand how the 3-simplices in them connect. Our goal in this subsection is, given a quasigroup, to express the number of neighboring 3-simplexes (meaning sharing a common 2-face, see 3.2) of each 3-simplex in the (non-)associativity complex of the quasigroup.

Let Q be a quasigroup of order n. We start by counting the number of neighbors of 3-simplices in $\mathtt{AC}(Q)$ and $\mathtt{NAC}(Q)$.

In $\mathtt{NAC}(Q)$, no 3-simplex corresponds to an associative triple of Q.
Let $[a, b, c, (ab)c]$ be a 3-simplex in $\mathtt{AC}(Q)$ corresponding to an associative triple of Q. It follows from the construction that no of $[a, b, c, (ab)c]$ 4-neighbor exists. Next, we analyze cases for neighbors opposite the remaining vertices:

1. (a) If $(xb)c = (ab)c$, then it follows from the definition of quasigroup, that $xb = ab$, and finally $x = a$.

   (b) If $x(bc) = (ab)c$, then from $(ab)c = a(bc)$ and from the definition of quasigroup it follows $x = a$.

2. (a) If $(ay)c = (ab)c$, then analogously to the case 1.(a) we get $y = b$.
   (b) If $a(yc) = (ab)c$, then analogously to the case 1.(b) we get $y = b$.

3. (a) If $(ab)z = (ab)c$, we get $z = c$ from the definition.
   (b) If $a(bz) = (ab)c$, the case is once again similar to e.g. 1.(b) and we get $z = c$.

We have shown that a 3-simplex in $\mathtt{AC}(Q)$ corresponding to an associative triple has no neighbors.

Now, let $[a, b, c, d]$ be a 3-simplex (either in $\mathtt{AC}(Q)$, or in $\mathtt{NAC}(Q)$) corresponding to a non-associative triple and let us assume $d = (ab)c$ without loss of generality (we could analogously derive everything for $d = a(bc)$). It follows from the constructions that there exists exactly one 4-neighbor of $[a, b, c, (ab)c]$, namely $[a, b, c, a(bc)]$. We now analyze cases for neighbors opposite the remaining vertices:

1. (a) If $(xb)c = (ab)c$, we get the same case as 1.(a) of the associative case, which implies $x = a$.
   (b) If $x(bc) = (ab)c \neq a(bc)$, by definition of quasigroup there exists a unique solution for x and $x \neq a$.

2. (a) If $(ay)c = (ab)c$, we get the same case as 2.(a) of the associative case, which implies $y = b$.
   (b) If $a(yc) = (ab)c \neq a(bc)$, by definition of quasigroup there exists a unique solution for y and $y \neq b$.

3. (a) If $(ab)z = (ab)c$, we get the same case as 3.(a) of the associative case, which implies $z = c$.
   (b) If $a(bz) = (ab)c \neq a(bc)$, by definition of quasigroup there exists a unique solution for z and $z \neq c$.

We have shown that every 3-simplex (either in $\mathtt{AC}(Q)$, or in $\mathtt{NAC}(Q)$) corresponding to a non-associative triple has precisely four neighbors, one opposite every vertex. Furthermore, every two neighbors of $[a, b, c, d]$ are necessarily different because they share precisely three vertices with $[a, b, c, d]$ and these triples of vertices are different. It follows that the two neighbors have exactly two vertices in common (they share an edge, but nothing more).

We have derived and proved the following proposition.

Proposition 16. *Let Q be a quasigroup. The number of neighbors of a 3-simplex $[a, b, c, d]$ in $\mathtt{AC}(Q)$ is*

- *0, if $(ab)c = a(bc)$, or*
- *4, if $(ab)c \neq a(bc)$.*

The number of neighbors of a 3-simplex $[a, b, c, d]$ in $\mathtt{NAC}(Q)$ is 4.
If $(ab)c \neq a(bc)$, $[a, b, c, d]$ has exactly one neighbor opposite each vertex, and all its neighbors are different.

Proposition 16 explains, why $\mathtt{NAC}(Q)$ is a simplification of $\mathtt{AC}(Q)$ with better topological properties, while still encompassing all the important properties about the associative properties of Q that $\mathtt{AC}(Q)$ describes. To elaborate:

- by its construction, $\texttt{NAC}(Q)$ is a subcomplex of $\texttt{AC}(Q)$,
- by Propositions 14 and 15, both $\texttt{AC}(Q)$ and $\texttt{NAC}(Q)$ possess the information about the number of associative and non-associative triples in Q,
- and by Proposition 16, every 3-simplex in $\texttt{AC}(Q)$ corresponding to an associative triple in Q is disconnected from the rest of the complex. On the other hand, every 3-simplex (either in $\texttt{AC}(Q)$, or in $\texttt{NAC}(Q)$) corresponding to a non-associative triple in Q is neighbored by at least 4 other 3-simplices. Therefore, $\texttt{NAC}(Q)$ is a simplification of $\texttt{AC}(Q)$ that ignores 3-simplices without neighbor. These 3-simplices are disconnected from the rest of the complex and, as shown in Subsection 3.3.2, possess no additional information about Q.

From this point on, we will only study the properties of $\texttt{NAC}(Q)$. We have already justified its use, as it simplifies $\texttt{AC}(Q)$ while preserving its important properties. Furthermore, because Proposition 16 states that every 3-simplex in $\texttt{NAC}(Q)$ has exactly one neighbor opposite each of its vertices, we obtain the following proposition:

Proposition 17. $\texttt{NAC}(Q)$ *is a non-branching complex for every quasigroup* Q.

3.3.5 Dual graph and strongly connected components

For a k-complex to be a pseudomanifold, it has to be strongly connected. To reiterate, this means that any two k-simplices can be joined by a sequence of k-simplices in such a way that each pair of neighboring simplices share a $(k-1)$-dimensional face.

It is important to realize a significant difference between connectivity and strong connectivity of a k-complex - a connected k-complex need not necessarily be strongly connected, even if it is closed and pure. For a simple example, take two spheres joined at a single vertex. This is an example of a closed, pure 2-complex that is connected, but not strongly connected, as for any given pair of 2-faces of this complex such that the first 2-face lies in the first sphere and the second 2-face of the pair lies in the second sphere, there exists no sequence of 2-faces with the property that each pair of consecutive 2-faces in the sequence share an edge. Therefore, determining whether a complex is strongly connected is not equivalent to analyzing the connectivity of the complex.

To address the question of strong connectivity for a pure k-complex X, we can generalize the Definition 38 and instead of analyzing whether a k-complex is strongly connected, analyze the connectivity of its dual graph.

We have shown in Subsection 3.3.3 that $\texttt{NAC}(Q)$ is a pure 3-complex. We build its dual graph according to Definition 38: The 3-simplices of $\texttt{NAC}(Q)$ will form the vertex set of the dual graph, and the edges set of the dual graph will consist of every pair of 3-simplices $\{A, B\}$, that for which $|A \cap B| = 3$, that is, we connect the 3-simplices by their common 2-faces.

We can picture a 3-simplex as a tetrahedron with vertices a, b, c, d (from the ordered list $[a, b, c, d]$). Connecting two tetrahedra together then means choosing a face on each of them and then identifying the two faces (i.e. "gluing" the two faces together). Formally, the way we connect two 3-simplices together will be similar to this: we will attach together all pairs of different 3-simplices $A = [a_1, a_2, a_3, a_4]$, $B = [b_1, b_2, b_3, b_4]$, for which there exists $i \in \{1, .., 4\}$ so that $a_j = b_j \ \forall j \in \{1, ..., 4\} \setminus \{i\}$ and $a_i \neq b_i$. In our implementation, the structure of the dual graph will be represented by pointers to the neighboring 3-simplices. The above condition will be used to determine which vertices of the dual graph will be connected.

The above construction uniquely determines the dual graph of $\mathtt{NAC}(Q)$. We will denote the dual graph of $\mathtt{NAC}(Q)$ by $\mathtt{G}(Q)$.

The following proposition follows from the reasoning in this section.

Proposition 18. *Let K be a connected component in $\mathtt{G}(Q)$. The 3-simplices corresponding to the vertices in K then form a strongly connected pure subcomplex in $\mathtt{NAC}(Q)$. This subcomplex is maximal in $\mathtt{NAC}(Q)$ in a sense that by appending an additional 3-simplex to it, the subcomplex will no longer be strongly connected.*

Note. Later in this work, by analyzing the connectedness of $\mathtt{G}(Q)$ for small quasigroups Q, we will show that there exist quasigroups, for which $\mathtt{G}(Q)$ is disconnected. It remains to be determined whether or not the maximal strongly connected subcomplexes in $\mathtt{NAC}(Q)$ can be disjoint.

3.3.6 Vertex colorability and cliques

The 3-complex $\mathtt{NAC}(Q)$ is constructed in such a way that every edge is a proper face of some 3-simplex, whose vertices all lie in different copies of Q: the first vertex in Q_1, the second in Q_2 etcetera. Therefore, the underlying graph of $\mathtt{NAC}(Q)$, its 1-skeleton, is 4-partite. Equivalently, $\mathtt{NAC}(Q)$ is 4-vertex-colorable. It follows that K_5 is not a subgraph of the 1-skeleton of $\mathtt{NAC}(Q)$ for any Q.

At the same time, since $\mathtt{NAC}(Q)$ is a pure 3-complex, the existence of K_4 in every component of the 1-skeleton of $\mathtt{NAC}(Q)$ is guaranteed. It follows that $\mathtt{NAC}(Q)$ is not 3-vertex-colorable.

3.3.7 Face colorability

We show that $\mathtt{NAC}(Q)$ is 2-face colorable (by face, we mean a maximal face, i.e. a 3-simplex in $\mathtt{NAC}(Q)$).

For a 3-simplex $[a, b, c, d]$ in $\mathtt{NAC}(Q)$, we assign its color to be:

- 0, if $d = (ab)c$,
- 1, if $d = a(bc)$.

In Subsection 3.3.4, by analyzing the number of neighbors of 3-simplices corresponding to a non-associative triple a, b, c, we have simultaneously shown, that the neighbors of $[a, b, c, (ab)c]$ are exactly $[x, b, c, x(bc)]$, $[a, y, c, a(yc)]$, $[a, b, z, a(bz)]$, and the neighbors of $[a, b, c, a(bc)]$ are $[x, b, c, (xb)c]$, $[a, y, c, (ay)c]$, $[a, b, z, (ab)z]$. Therefore, if two different 3-simplices in $\mathtt{NAC}(Q)$ share a 2-face, they are assigned different colors by the face coloring proposed above (i.e. all neighbors of a 3-simplex of color 0 are assigned color 1, and vice versa). By Definition 37, $\mathtt{NAC}(Q)$ is 2-face-colorable.

3.4 Topological structure of $\mathtt{NAC}(Q)$

The following theorem is the resulting theorem of this chapter. It summarizes the observations made in Section 3.3 and describes the topological structure of $\mathtt{NAC}(Q)$.

Theorem 19. *Let Q be a quasigroup. $\mathtt{NAC}(Q)$ is a union of closed orientable 3-pseudomanifolds. Each of these 3-pseudomanifolds consists of at least 5 different 3-simplices that together consist of at least 8 vertices.*

Proof. By Subsection 3.3.3, the 3-complex $\mathtt{NAC}(Q)$ is pure and has dimensional homogenity, by Proposition 17 it is non-branching, and by Proposition 18 it is a union of strongly connected subcomplexes. As a result, $\mathtt{NAC}(Q)$ is a union of 3-pseudomanifolds. Moreover, it follows from Proposition 16 that these 3-pseudomanifolds are closed.

Subsection 3.3.6 states that $\mathtt{NAC}(Q)$ is 4-vertex-colorable and subsection 3.3.7 states that $\mathtt{NAC}(Q)$ is 2-face-colorable. Therefore, it follows from Theorem 12 that each of the 3-pseudomanifolds is orientable.

The part about 5 different 3-simplices follows from the fact that all neighbors of a 3-simplex in $\mathtt{NAC}(Q)$ are different (see Proposition 16), and the reasoning in the same section proves the part about 8 vertices.

□

3.5 Implementation

We present a possible implementation of the construction of the non-associativity complex $\mathtt{NAC}(Q)$ of a quasigroup Q and its dual graph $\mathtt{G}(Q)$ as mentioned earlier.

Note. The input Latin square consists of numbers 0 through $n-1$. All arrays in this work are indexed from 0.

Note. In this implementation, the structure of the complex is stored in an array of objects of data type Simplex. An instance of Simplex represents a 3-simplex of the non-associativity complex, and at the same time, the corresponding vertex of its dual graph and has the following attributes:

- *name*: integer, describing the position = index of object in the array Complex
- *values*: array of 4 integers, to save vertices of the 3-simplex
- *neighbors*: array of 4 arrays of integers (in the variant with self-connections, neighbors would also consist of 4 arrays of integer, but each of them would be one element longer than in the version without self-connections). *neighbors*$[i]$ consists of integers describing the positions of $(i+1)$-neighbors of the object in the array Complex.

Algorithm 1: Build $\mathtt{NAC}(Q)$ and $\mathtt{G}(Q)$

```
Input  : L = Latin square in a form of 2-dimensional array
Output: Complex = array of Simplex
begin
    n := order of L;
    Complex := [ ];
    for i := 0 to n - 1 do
        for j := 0 to n - 1 do
            for k := 0 to n - 1 do
                if L[i][L[j][k]] ≠ L[L[i][j]][k] then
                    initialize new simplex S;
                    S.name := length(Complex);
                    S.values := [i, j, k, L[L[i][j]][k] ];
                    S.neighbors := [[ ], [ ], [ ], [length(Complex) + 1]];
                    append S to Complex;
                    initialize new simplex S;
                    S.name := length(Complex);
                    S.values := [i, j, k, L[i][L[j][k]] ];
                    S.neighbors := [[ ], [ ], [ ], [length(Complex) - 1]];
                    append S to Complex;
                end
            end
        end
    end
    for i := 0 to length(Complex) - 1 do
        for j := 0 to length(Complex) - 1 do
            if i ≠ j then
                if Complex[i].values[k] = Complex[j].values[k] ∀k = 1, 2, 3
                 then
                    append j to Complex[i].neighbors[0];
                end
                if Complex[i].values[k] = Complex[j].values[k] ∀k = 0, 2, 3
                 then
                    append j to Complex[i].neighbors[1];
                end
                if Complex[i].values[k] = Complex[j].values[k] ∀k = 0, 1, 3
                 then
                    append j to Complex[i].neighbors[2];
                end
            end
        end
    end
end
```

Note. In the construction of the dual graph, we could allow the connection of a 3-simplex to itself. In that case, every 3-simplex A would be its neighbor opposite every vertex. Both constructions are equivalent, as we can easily traverse between them by allowing/disallowing the self-neighboring property, and it is evident that this difference does not change any relevant properties of the dual graph. The difference between them would require, for example, different implementations of search algorithms, possibly making one construction favorable over the other for implementation in the desired programming language.

We end this section with a quick analysis of the algorithm. The algorithm consists of two parts.

1. The first part initializes the non-associativity complex in the form of all the 3-simplices and connects pairs of 3-simplices corresponding to the same non-associative triple to create the first edges of the dual graph. It cycles through all possible triples of elements of the quasigroup, therefore having time complexity $T_1 = O(n^3)$, where n is the order of the quasigroup.

2. The second part connects the remaining 3-simplices to its neighbors, finalizing the construction of the dual graph. It requires all the 3-simplices to be already initialized. Therefore, it cannot be run in parallel with the first part. The time complexity of this part is $T_2 = O((\text{length}(Complex))^2)$ because it requires checking each pair of 3-simplices for whether they have a common face. To specify this even more, it follows from the first part of the algorithm that $n^3 \leq \text{length}(Complex) \leq 2n^3$, thus giving us an approximate time complexity of the second part as a function of n: $O(n^6) \leq T_2 \leq O(4n^6)$

In total, the algorithm has a worst-case time complexity $O(4n^6 + n^3)$, which is acceptable for our purposes, where n is relatively small.

Chapter 4

Global and local properties of $\texttt{NAC}(Q)$

This chapter analyzes structural properties on $\texttt{NAC}(Q)$. The first section describes the relation between $\texttt{G}(Q)$ and the associativity index of Q and answers the question, whether $\texttt{NAC}(Q)$ is a 3-pseudomanifold, or merely a union of several 3-pseudomanifolds. The question is answered by analyzing the dual graph $\texttt{G}(Q)$ of the complex for small order quasigroups. The second section describes local properties of $\texttt{NAC}(Q)$ by studying the properties of links of vertices in $\texttt{NAC}(Q)$. The findings of this section are the first steps to determining whether singularities in $\texttt{NAC}(Q)$ can be resolved. The third section calculates Euler characteristic of links of vertices in $\texttt{NAC}(Q)$ in order to classify neighborhoods of vertices by their genus.

4.1 Properties of $\texttt{G}(Q)$

4.1.1 Size of $\texttt{G}(Q)$ and its relation to associativity index

Having constructed $\texttt{NAC}(Q)$ and subsequently its dual graph $\texttt{G}(Q)$ by Algorithm 1, we extract the number of nodes in $\texttt{G}(Q)$ using the following elementary algorithm.

Algorithm 2: Count number of nodes in $\texttt{G}(Q)$

```
Input  : Complex = array of Simplex
Output: m = number of 3-simplices
begin
  |  m = length(Complex);
end
```

Because the nodes of $\texttt{G}(Q)$ (represented by 3-simplices of $\texttt{NAC}(Q)$) are stored in an array, their count is returned in constant time $O(1)$. However, despite its simplicity, the algorithm provides us with valuable information about the quasigroup.

To get first information about the structure of non-associativity complexes of small quasigroups, we start by building the complexes for every quasigroup of order up to 6 and counting nodes in their dual graphs.

The most challenging part - generating representatives for all possible quasigroups - was accomplished by generating Latin squares isomorphic to Latin squares of loops using brute force. We started with Latin square representatives of loops parsed from data by McKay, and for each representative of a loop and each possible permutation of rows and columns that fixates the first row and the first column, a Latin square representing a quasigroup was formed. This method provides us with a list of representatives of each quasigroup (though with numerous duplicities). By permuting rows and columns, we can transform the Caley table of a quasigroup to have element 0 in the top left corner. Our algorithm, therefore, returns a representative for every quasigroup of the given order.

The algorithm is much faster than backtracking since it does not require checking whether the obtained Latin squares are valid because every table generated from a multiplication table of a loop by permuting its rows and columns is necessarily a Latin square.

Table 4.1 lists all possible counts of nodes of which $\mathtt{G}(Q)$ can consist for Q of order n.

n	1	2	3	4	5					
$\lvert\mathtt{G}(Q)\rvert$	0	0	0	0	0	140	160	180	200	220
			36	64	72	142	162	182	202	
				80	90	144	164	184	204	
				96	92	146	166	186	206	
					102	148	168	188	208	
					124	150	170	190	210	
					126	152	172	192	212	
					132	154	174	194	214	
					136	156	176	196	216	
					138	158	178	198	218	

n	6									
$\lvert\mathtt{G}(Q)\rvert$	0	160	190	220	246	272	298	324	350	376
	54	162	192	222	248	274	300	326	352	378
	64	164	196	224	250	276	302	328	354	380
	88	166	198	226	252	278	304	330	356	382
	96	168	200	228	254	280	306	332	358	384
	108	170	204	230	256	282	308	334	360	386
	112	172	206	232	258	284	310	336	362	388
	128	176	208	234	260	286	312	338	364	390
	136	178	210	236	262	288	314	340	366	392
	144	180	212	238	264	290	316	342	368	394
	148	182	214	240	266	292	318	344	370	400
	150	184	216	242	268	294	320	346	372	
	158	188	218	244	270	296	322	348	374	

Table 4.1: Possible counts of nodes in $\mathtt{G}(Q)$ for quasigroups Q of size n.

The numbers in 4.1 may appear seemingly complicated and random, but Proposition 15 explains, that $|\mathtt{G}(Q)|$ corresponds to $a(Q)$ by the following relation:

$$a(Q) = n^3 - \frac{|\mathtt{G}(Q)|}{2}. \tag{4.1}$$

The values in Table 4.1, therefore, uniquely describe all possible values of $a(Q)$ for Q of order n. The set of all such values for n is called the *associativity spectrum* of n and denoted $\mathrm{assspec}(n)$. We have:

$$\begin{aligned}
\mathrm{assspec}(1) &= \{1\}\\
\mathrm{assspec}(2) &= \{8\}\\
\mathrm{assspec}(3) &= \{9, 27\}\\
\mathrm{assspec}(4) &= \{16, 24, 32, 64\}\\
\mathrm{assspec}(5) &= \{15, ..., 57, 59, 62, 63, 74, 79, 80, 89, 125\}\\
\mathrm{assspec}(6) &= \{16, 19, ..., 114, 116, ..., 118, 120, ..., 122, 124, ..., 128, 130, ...\\
&\quad ..., 137, 141, 142, 144, 148, 152, 160, 162, 168, 172, 184, 189, 216\}
\end{aligned} \tag{4.2}$$

Our calculations independently verify the findings of Ježek and Kepka (1990).

4.1.2 Connected components of $\mathtt{G}(Q)$ and strong connectivity of $\mathtt{NAC}(Q)$

This subsection aims to answer the question of the strong connectivity of $\mathtt{NAC}(Q)$. It accomplishes this by analyzing the connectivity of $\mathtt{G}(Q)$ for small order quasigroups. First, we formulate a corollary of Theorem 19 in context of $\mathtt{G}(Q)$.

Corollary. (a) $\mathtt{G}(Q)$ is a 4-regular graph.

(b) Every component K of $\mathtt{G}(Q)$ consists of at least 5 vertices.

Proof. Follows immediately from Theorem 19 by translating the relevant parts according to the correspondence between 3-simplices of $\mathtt{NAC}(Q)$ and nodes of $\mathtt{G}(Q)$ (see Subsection 3.3.5).

□

The following proposition describes the relation between the order of Q, $a(Q)$, the number of edges of $\mathtt{G}(Q)$ and components of $\mathtt{G}(Q)$.

Proposition 20. *Let* $\mathtt{G}(Q) = (V, E)$ *be the dual graph of* $\mathtt{NAC}(Q)$ *for a quasigroup* Q *of order* n*. Then*

$$|E| = 2|V| = \sum_{X \text{ component of } \mathtt{G}(Q)} 2|X| = 4(n^3 - a(Q)). \tag{4.3}$$

Proof. Follows from the previous Corollary and Proposition 15.

□

We will now describe an algorithm, that for a given complex $\mathtt{NAC}(Q)$ returns an array of integers containing the sizes of all components of the dual graph $\mathtt{G}(Q)$. To do this, we first need to expand the data type Simplex from Section 3.5 to also include a boolean attribute *visited* (with initial value $= False$). With this adjustment, we present a Depth-First Search (DFS) version of the algorithm:

Algorithm 3: DFS visit every vertex in a component

```
Input  : Complex = array of Simplex, simplex, temp = array of Simplex
Output: temp
begin
    simplex.visited := True;
    append simplex to temp;
    for i := 0 to 3 do
        for j := 0 to length(simplex.neighbors[i]) do
            if Complex[simplex.neighbors[i][j]].visited = False then
                temp := DFS(Complex, Complex[simplex.neighbors[i][j]],
                  temp);
            end
        end
    end
end
```

Algorithm 4: Return component sizes

```
Input  : Complex = array of Simplex
Output: ComponentSizes = array of integers
begin
    ComponentSizes := [ ];
    for i := 0 to length(Complex) - 1 do
        if Complex[i].visited = False then
            temp := [ ];
            append length(DFS(Complex, Complex[i], temp)) to
              ComponentSizes;
        end
    end
end
```

If needed, Algorithm 4 can be easily modified to save entire components of $\mathtt{G}(Q)$ instead of their sizes.

The time complexity of DFS is well known, $O(|V|+|E|)$. More interestingly, thanks to Proposition 20, we may also describe the time complexity in terms of the number of vertices of $\mathtt{G}(Q)=(V,E)$ (or equivalently in terms of number of 3-simplices of $\mathtt{NAC}(Q)$) as $O(3|V|)$. From Proposition 19 we also get time complexity in terms of $|Q|=n$ and $a(Q)$: $O(3|V|)=O(6(n^3-a(Q)))$. Finally, because $a(Q)\geq n$, we obtain an upper bound for the time complexity $O(6n^3-6n)=O(6n(n^2-1))$.

Algorithm 4 was run on the same data set as Algorithm 2. Appendix A extends the data from Table 4.1 by enlisting every possible configuration of component sizes in $\mathtt{G}(Q)$ for Q of order up to 6 (orders up to 5 are discussed separately because of their simplicity, and possibilities for order 6 are listed in Table 8). For every unique configuration with more than one component, a single representing Latin square was saved in order to be studied later.

In our implementation, we first generated Latin square representatives of all quasigroups and then ran Algorithm 4 in a single thread Python 3.9 application. This approach is suboptimal for several reasons - parallelization is possible when generating Latin square representatives of quasigroups, and the data set of the Latin squares needed for further calculations will be impossibly large from n around 9. Therefore, a much better implementation would be one that sequentially generates a Latin square and then calculates component sizes for the dual graph of its complex.

The calculations proved the following theorem about the structure of $\mathtt{NAC}(Q)$:

Theorem 21. *$\mathtt{G}(Q)$ is not necessarily connected. Generally, $\mathtt{NAC}(Q)$ is not strongly connected. It is, therefore, merely a union of 3-pseudomanifolds.*

Proof. Computationally using Algorithm 4. The smallest order of Q, for which $\mathtt{G}(Q)$ can be disconnected, is $n = 4$. Several examples of Q, for which $\mathtt{NAC}(Q)$ is not strongly connected, are listed in Appendix B.

$\square$

4.2 Local properties of $\mathtt{NAC}(Q)$

Understanding the global structural properties of $\mathtt{NAC}(Q)$ is complicated. $\mathtt{NAC}(Q)$ cannot be embedded in 3D, making it very difficult to visualize it as the union of 3-pseudomanifolds. No classification of 3-pseudomanifolds is known (see Subsection 2.3.2), therefore making it difficult to categorize the non-associativity complexes of different Q by common structural properties. Finally, as discussed in Subsection 2.3.2, singularities in 3-pseudomanifolds can be very complicated and are not always resolvable.

We believe that being able to address these problems for $\mathtt{NAC}(Q)$ and solving them could lead to finding invariant properties of the complex and form criteria that would grant or deny the existence of maximally non-associative quasigroups for some orders n.

The main result of this thesis is presented in the following two sections and addresses the aforementioned problems locally. In this section, we derive theoretical observations about links of vertices in $\mathtt{NAC}(Q)$. These observations are used in the following section to develop an algorithm to resolve singularities in vertex neighborhoods (links) and calculate the Euler characteristic and genus of the links.

We begin by describing the structure of links of vertices in $\mathtt{NAC}(Q)$.

Proposition 22. *Let A be a vertex in $\mathtt{NAC}(Q)$. $\operatorname{Lk} A$ is a union of closed orientable 3-vertex-colorable 2-pseudomanifolds.*

Proof. Follows from the fact that $\operatorname{Lk} A$ is induced from $\mathtt{NAC}(Q)$, and the fact that $\mathtt{NAC}(Q)$ is a union of closed orientable 3-pseudomanifolds according to Theorem 19:

- It is a union of closed and non-branching 2-complexes with dimensional homogenity. This follows from the definition of link (Definition 29), because two 2-faces in $\operatorname{Lk} A$ share an edge if and only if the 3-simplexes in $\mathtt{NAC}(Q)$ from which they are induced share a 2-face. This also implies orientability because if two 3-simplices with a common vertex A are consistently oriented, then the induced 2-faces in $\operatorname{Lk} A$ are consistently oriented.

- 3-vertex-colorability follows from the fact, that $\operatorname{Lk} A$ is (in case of $\mathtt{NAC}(Q)$) a subcomplex of $\operatorname{st} A$, because $\operatorname{st} A$ is 4-vertex-colorable and A is the common vertex of all 3-simplices of $\operatorname{st} A$.

$\square$

The previous proposition naturally raises the question of whether $\operatorname{Lk} A$ is strongly connected. As it turns out, not necessarily.

Proposition 23. *Let A be a vertex in $\mathtt{NAC}(Q)$. $\operatorname{Lk} A$ is not necessarily strongly connected, therefore, generally not a 2-pseudomanifold, but merely a union of 2-pseudomanifolds.*

Proof. Computationally, using an algorithm similar to Algorithm 4 for the dual graphs of links of vertices in $\mathtt{NAC}(Q)$. The links and their dual graphs can be easily obtained from data exported from 1 (from all 3-simplices containing the fixed vertex, we generate an array of 2-faces to form the link and then connect them to form the dual graph of the link by including relevant connections from $\mathtt{G}(Q)$). Table 11 provides an example of quasigroup with links of vertices not strongly connected. It was found using this method by searching the representatives of quasigroups with disconnected $\mathtt{G}(Q)$ created in the previous section.

$\square$

To end this section, we formulate a critical theorem and a proposition about the structures of neighborhoods of vertices in $\operatorname{Lk} A$. They are, for the most part, direct applications of Theorem 7 and Proposition 8 for the case of $\mathtt{NAC}(Q)$ and they enable us to formulate an algorithm to normalize $\operatorname{Lk} A$ by cutting at singularities. This normalization algorithm will turn a union of 2-pseudomanifolds into several disjoint 2-manifolds, which we can classify by calculating their genus and identifying intersection points of these components in $\operatorname{Lk} A$. This process will yield us with the description of links and is the first step toward resolving singularities in the entirety of $\mathtt{NAC}(Q)$.

Theorem 24. *Let A be a vertex in $\mathtt{NAC}(Q)$ and B be a vertex in $\operatorname{Lk} A$. Then link of B in $\operatorname{Lk} A$ (denoted $\operatorname{Lk}_A B$) is a union of disjoint cycles of even length.*

Proof. By Proposition 22, $\operatorname{Lk} A$ is a union of closed orientable 3-vertex-colorable 2-pseudomanifolds.

If B belongs to only one of these 2-pseudomanifold parts of $\operatorname{Lk} A$, then by Theorem 7 $\operatorname{Lk}_A B$ is a union of disjoint cycles. However, Theorem 7 does not hold for a union of closed 2-pseudomanifolds (the link of a vertex in a union of closed 2-pseudomanifolds forms cycles that need not be disjoint), as can be easily observed. Proving that $\operatorname{Lk}_A B$ is a union of disjoint cycles must, therefore, be done using a different method.

Without loss of generality assume $\mathrm{Lk}_A B$ is induced from all 3-simplices of form $[A, B, y, z]$, because $\mathrm{Lk}_A B$ is a neighborhood of a fixed vertex in $\mathrm{Lk}\, A$. Choose a pair of cycles $\mathcal{C}_1, \mathcal{C}_2$ in $\mathrm{Lk}_A B$. Assume $\mathcal{C}_1, \mathcal{C}_2$ have a common vertex C (without loss of generality assume the 3-simplices in NAC(Q) containing the point A, B, C are of form $[A, B, C, z]$). Let P_1, Q_1 be the neighbors of C in $\mathcal{C}_1$ and P_2, Q_2 be the neighbors of C in $\mathcal{C}_2$, as shown in Figure 4.1.

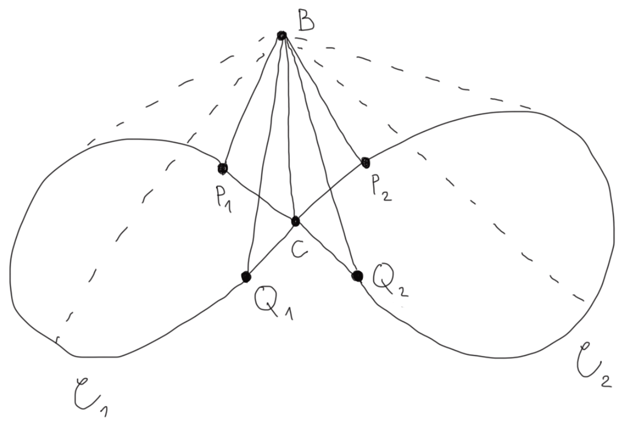


Figure 4.1: Cycles $\mathcal{C}_1, \mathcal{C}_2$ in $\mathrm{Lk}_A B$ with a common vertex C.

In the illustration, an edge between B and a vertex in one of the cycles represents a 2-face of the 3-simplices $[A, B, C, z]$ in `NAC`(Q) given by vertices A, B, C and a triangle represents a 3-simplex in `NAC`(Q). For example, BC actually represents the 2-face ABC, and similarly, for example, the triangle BCP_1 represents the 3-simplex $[A, B, C, P_1]$.

It follows from Proposition 16 that a vertex in $\mathrm{Lk}_A B$ has exactly two different neighbors, because every 2-face in `NAC`(Q) is shared by exactly two 3-simplices. This holds in our situation, where we fix two of the vertices of the 2-face and then search for neighbors of the third vertex in $\mathrm{Lk}_A B$. To formalize this idea, F and G are neighbors of C in $\mathrm{Lk}_A B$ if and only if $[A, B, C, F]$ and $[A, B, C, G]$ share a 2-face in `NAC`(Q), and by Proposition 16 and reasoning earlier in this paragraph we get that F and G are the only neighbors of C in $\mathrm{Lk}_A B$.

Because of this, components in $\mathrm{Lk}_A B$ are non-branching. Therefore, if $\mathcal{C}_1$ and $\mathcal{C}_2$ share vertex C, they belong to the same component of $\mathrm{Lk}_A B$ and are therefore identical (if they were not, a branching would have to occur in a common point of $\mathcal{C}_1$ and $\mathcal{C}_2$, but that cannot happen).

Finally, $\mathrm{Lk}_A B$ is orientable because its orientability is induced from the orientability of $\mathrm{Lk}\, A$ (analogous reasoning was performed in the first part of the proof of Proposition 22). The even length of the cycles follows from their orientability.

□

Proposition 25. *Let A be a vertex in* $\texttt{NAC}(Q)$ *and B be a vertex in* $\operatorname{Lk} A$*. Choose a cycle $\mathcal{C}$ in* $\operatorname{Lk}_{\mathrm{A}} B$ *and suppose* $\operatorname{Lk}_{\mathrm{A}} B \setminus \mathcal{C} \neq \emptyset$*. Modify* $\operatorname{Lk} A$ *using the following two steps:*

1. *create a new* 1*-skeleton by connecting a new vertex $\tilde{B}$ with every vertex in $\mathcal{C}$ and disconnecting B from every vertex in $\mathcal{C}$,*

2. *perform an operation of closure on the new* 1*-skeleton.*

This process creates a modified version of $\operatorname{Lk} A$ *by "cutting at singularity B" once. Repeating this process a finite number of times transforms* $\operatorname{Lk} A$ *into a union of disjoint closed* 2*-manifolds.*

Proof. The proposition is a special case of Proposition 8 for $S = \operatorname{Lk} A$, because $\operatorname{Lk} A$ satisfies the requirements of Proposition 8.

□

4.3 Euler characteristic of links

4.3.1 Algorithm

Propositions 22 and 23, Theorem 24 and Proposition 25 allow us to describe and classify links of vertices in $\texttt{NAC}(Q)$ in three steps:

1. determine singularities in the given link, i.e. vertices with links consisting of several cycles,

2. normalize the link, i.e. transform it into a union of disjoint 2-manifolds using a sequence of cuts, as described in Proposition 25,

3. calculate the genus of each of them.

Every 2-manifold in this union is orientable, because the process described in Proposition 25 does not change orientability. According to Proposition 10 we can, therefore, calculate the genus of the 2-manifold from its Euler characteristic as

$$g = 1 - \frac{\chi}{2}, \tag{4.4}$$

where χ is the Euler characteristic of the 2-manifold.

To calculate Euler characteristic and genus of each component in the link of vertex A in $\texttt{NAC}(Q)$ and their identified vertices, only the 1-skeleton of $\operatorname{Lk} A$ is needed. Theorem 24 and Proposition 25 explain, why this important observation holds.

To transform the link of a vertex in $\texttt{NAC}(Q)$, a union of 2-pseudomanifolds, into a union of 2-manifolds, we need to cut at singularities until there are no singularities left. Cutting at a singularity, a step described in Proposition 25 increases the number of vertices by one (by creating a copy of vertex in the singularity) but does not alter the edge count, nor the face count of the link complex, and neither its orientability. Because of this invariant-preserving property of the step, the fact that the link of a vertex in 2-manifold is a cycle, and Theorem 24, the edge count in each component 2-manifold can be expressed as

$$e = \frac{1}{2} \sum_{v} \deg v, \tag{4.5}$$

where v are vertices in the component because $\deg v$ is equal to the length of the link cycle of v and every edge consists of exactly 2 vertices, The face count can be always be expressed as a function of edge count, because of the properties of 2-pseudomanifolds - every edge belongs to precisely two faces, and each face has precisely three edges, leading to the relation between the counts of edges and faces

$$f = \frac{2}{3} \cdot e. \tag{4.6}$$

The last issue remaining to address is the after-cut singularity identification. To fully understand the properties of the link, it is necessary to be able to glue the 2-pseudomanifold back together from the normalized parts, i.e., identify the vertices in the union of 2-manifolds in a way that forms the original union of 2-pseudomanifolds. To see how this can be encapsulated in the 1-skeleton and preserved throughout the cut steps, it is only important to realize that the copy created by the cut will have the same position (i.e., the same y in the pair $[x, y]$ describing the vertex) as the original, and we have to preserve the information about the value (i.e., the x in $[x, y]$). Because the values of x are from the range $\{0, ..., |Q| - 1\}$, by naming the copy $[\tilde{x}, y]$ of the vertex $[x, y]$ in such way that $\tilde{x} \equiv x \pmod{|Q|}$, the new vertex will keep the information about its origin. Conversely, when reconstructing the link from the 2-manifold components, we will identify vertices $[x, y]$ with the same value of y and values of x congruent modulo the order of the quasigroup.

The following algorithm utilizes the reasoning from this section:

Algorithm 5: Normalize Lk A

```
Input  : S = 1-skeleton of Lk A, |Q|
Output: S' = 1-skeleton of normalized Lk A
begin
    k := 0;
    S' := S;
    n := |Q|;
    for node v = [v1, v2] in S do
        if neighbors of v in S' form ≥ 2 cycles then
            for each cycle do
                k += 1;
                append a copy vk = [v1 + n · k, v2] of v = [v1, v2] to S';
                for vertex w in cycle do
                    disconnect v and w;
                    connect vk and w;
                end
            end
        end
    end
end
```

The output from Algorithm 5 can be easily analyzed to reveal the structural type of Lk A:

- The output consists of several disjoint graphs. Therefore the count of vertices and edges in the output can be determined using graph search algorithms such as DFS. As mentioned earlier in this section, the counts of vertices and edges uniquely determine the genus of the component.
- The information about resolved singularities is stored in the names of the vertices, making it convenient to glue the components back together.

4.3.2 Examplary calculations

For the quasigroup given by Table 4.2 (also listed in Appendix B as Table 9) (the order of this quasigroup is 4), we describe the structure of Lk $[1,1]$ and Lk $[1,2]$.

	0	1	2	3
0	0	3	2	1
1	1	2	3	0
2	2	1	0	3
3	3	0	1	2

Table 4.2: A quasigroup of order 4 with $\mathtt{G}(Q)$ consisting of four 16-vertex components.

Note. $[1,1]$ means value 1 at position 1. $[a,b,c,d]$ is a shortened notation for the 3-simplex $\{[a,1],[b,2],[c,3],[d,4]\}$.

Structure of Lk $[1,1]$

The non-associativity complex contains the following 3-simplices with fixed vertex $[1,1]$: $[0,1,1,0],[0,1,1,2],[0,1,3,0],[0,1,3,2],[1,1,1,1],[1,1,1,3]$, $[1,1,3,1],[1,1,3,3],[2,1,1,0],[2,1,1,2],[2,1,3,0],[2,1,3,2],[3,1,1,1],[3,1,1,3]$, $[3,1,3,1],[3,1,3,3]$. Lk $[1,1]$ is induced from these 3-simplices.

The vertex set of Lk$[1,1]$ can be obtained by taking every vertex at positions $0,2,3$ in these 3-simplices. Lk $[1,1]$ contains 10 different vertices: $[0,0],...,[3,0]$, $[1,2],[3,2],[0,3],...,[3,3]$.

To normalize the link, we look at links of each of the vertices in Lk $[1,1]$ and more specifically at the cycles of 3-simplices in the non-associativity complex from which they are induced. For every vertex in Lk $[1,1]$, we obtain several cycles of 3-simplices where every two neighboring 3-simplices share a 2-face and all of these 3-simplices have a fixed vertex $[1,1]$:

$\mathrm{Lk}_{[1,1]}\ [0,0]$: $[0,1,1,2] \leftrightarrow [0,1,1,0] \leftrightarrow [0,1,3,0] \leftrightarrow [0,1,3,2] \leftrightarrow [0,1,1,2]$

$\mathrm{Lk}_{[1,1]}\ [1,0]$: $[1,1,1,1] \leftrightarrow [1,1,1,3] \leftrightarrow [1,1,3,3] \leftrightarrow [1,1,3,1] \leftrightarrow [1,1,1,1]$

$\mathrm{Lk}_{[1,1]}\ [2,0]$: $[2,1,1,0] \leftrightarrow [2,1,1,2] \leftrightarrow [2,1,3,2] \leftrightarrow [2,1,3,0] \leftrightarrow [2,1,1,0]$

$\mathrm{Lk}_{[1,1]}\ [3,0]$: $[3,1,1,1] \leftrightarrow [3,1,1,3] \leftrightarrow [3,1,3,3] \leftrightarrow [3,1,3,1] \leftrightarrow [3,1,1,1]$

$\mathrm{Lk}_{[1,1]}$ $[1,2]$:

1. $[0,1,1,0] \leftrightarrow [0,1,1,2] \leftrightarrow [2,1,1,2] \leftrightarrow [2,1,1,0] \leftrightarrow [0,1,1,0]$
2. $[1,1,1,1] \leftrightarrow [1,1,1,3] \leftrightarrow [3,1,1,3] \leftrightarrow [3,1,1,1] \leftrightarrow [1,1,1,1]$

$\mathrm{Lk}_{[1,1]}$ $[3,2]$:

1. $[0,1,3,0] \leftrightarrow [0,1,3,2] \leftrightarrow [2,1,3,2] \leftrightarrow [2,1,3,0] \leftrightarrow [0,1,3,0]$
2. $[1,1,3,1] \leftrightarrow [1,1,3,3] \leftrightarrow [3,1,3,3] \leftrightarrow [3,1,3,1] \leftrightarrow [1,1,3,1]$

$\mathrm{Lk}_{[1,1]}$ $[0,3]$: $[0,1,1,0] \leftrightarrow [0,1,3,0] \leftrightarrow [2,1,3,0] \leftrightarrow [2,1,1,0] \leftrightarrow [0,1,1,0]$
$\mathrm{Lk}_{[1,1]}$ $[1,3]$: $[1,1,1,1] \leftrightarrow [1,1,3,1] \leftrightarrow [3,1,3,1] \leftrightarrow [3,1,1,1] \leftrightarrow [1,1,1,1]$
$\mathrm{Lk}_{[1,1]}$ $[2,3]$: $[0,1,1,2] \leftrightarrow [0,1,3,2] \leftrightarrow [2,1,3,2] \leftrightarrow [2,1,1,2] \leftrightarrow [0,1,1,2]$
$\mathrm{Lk}_{[1,1]}$ $[3,3]$: $[1,1,1,3] \leftrightarrow [1,1,3,3] \leftrightarrow [3,1,3,3] \leftrightarrow [3,1,1,3] \leftrightarrow [1,1,1,3]$

There are two singularities, namely vertices $[1,2]$ and $[3,2]$, which have to be normalized. Algorithm 5 performed on Lk $[1,1]$ cuts at $[1,2]$ and $[3,2]$ and changes the 3-simplices $[1,1,1,1],[1,1,1,3],[3,1,1,3],[3,1,1,1]$ for $[1,1,5,1],[1,1,5,3],[3,1,5,3],[3,1,5,1]$, and the 3-simplices $[1,1,3,1],[1,1,3,3],$ $[3,1,3,3],[3,1,3,1]$ for $[1,1,11,1],[1,1,11,3],[3,1,11,3],[3,1,11,1]$.

The modified link induced from the updated 3-simplices consists of two disjoint components of 6 vertices:

1. $[0,0],[2,0],[1,2],[3,2],[0,3],[2,3]$
2. $[1,0],[3,0],[5,2],[11,2],[1,3],[3,3]$.

The two components can be glued back together by identifying pairs of vertices with identical positions (y in $[x,y]$) and values at the position (x in $[x,y]$) congruent modulo order of the quasigroup. In our case, this means identifying the pairs $[1,2]-[5,2]$ and $[3,2]-[11,2]$.

Finally, we calculate the Euler characteristic of each of the components and their genera. Each of the components is orientable because the algorithm does not change orientability. We get $v=6$ vertices in each of the components and each of these vertices has 4 neighbors (because the number of neighbors of a vertex in the modified link is equal to the length of the link cycle of the vertex in the modified link). Every edge has two vertices, giving us $e=\frac{4\times 6}{2}=12$ edges in each component. Lastly, every face consists of 3 edges and every edge belongs to exactly two faces. This yields us $f=\frac{2\times 12}{3}=8$ faces in each component. Therefore, each component has Euler characteristic $\chi=v-e+f=6-12+8=2$. By Proposition 10, because the components are orientable, we get $\chi=2-2g$, from which we obtain $g=2$.

To summarize, Lk $[1,1]$ is a union of two spheres identified at points $[1,2]$ and $[3,2]$ (illustrated in Figure 4.2).

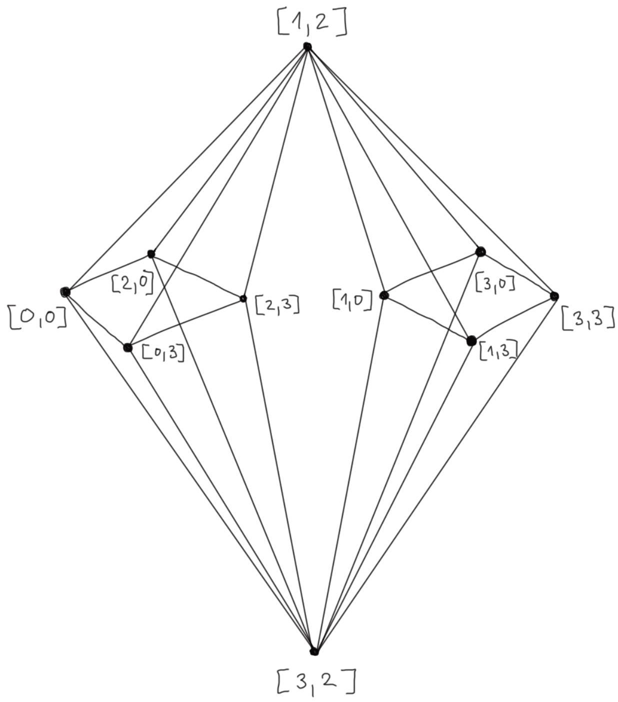


Figure 4.2: Visualization of Lk $[1,1]$ in the non-associativity complex of the quasigroup given by Table 9.

Structure of Lk $[1,2]$

Lk $[1,2]$ contains 12 vertices: $[0,0]...[3,0], [0,1]...[3,1], [0,3]...[3,3]$. By analyzing links of these vertices in Lk $[1,2]$, we see that the link of each vertex consists of two cycles of length 4, therefore every vertex is a singularity obtained by gluing two vertices. Using Algorithm 5, we cut at each of the singularities to obtain a modified link of $[1,2]$. The modified link consists of 24 vertices divided into four disjoint components of 6 vertices:

1. $[0,0], [2,0], [0,1], [2,1], [1,3], [3,3]$
2. $[4,0], [14,0], [1,1], [3,1], [0,3], [2,3]$
3. $[1,0], [3,0], [20,1], [30,1], [36,3], [46,3]$
4. $[9,0], [19,0], [25,1], [35,1], [41,3], [51,3]$

The disjoint components can be glued together by identifying the following pairs:

- 1. and 2.: $[0,0]-[4,0]$ and $[2,0]-[14,0]$
- 1. and 3.: $[0,1]-[20,1]$ and $[2,1]-[30,1]$
- 1. and 4.: $[1,3]-[41,3]$ and $[3,3]-[51,3]$
- 2. and 3.: $[0,3]-[36,3]$ and $[2,3]-[46,3]$
- 2. and 4.: $[1,1]-[25,1]$ and $[3,1]-[35,1]$
- 3. and 4.: $[1,0]-[9,0]$ and $[3,0]-[19,0]$

Euler characteristic and genera are the same for all of the components: Every components consists of $v = 6$ vertices connected by $e = 12$ edges and forming $f = 8$ triangles, therefore $\chi = 2$ and $g = 0$ for each component.

To summarize, Lk $[1, 2]$ is a union of four spheres such that every vertex of Lk $[1, 2]$ connects exactly 2 spheres and each pair of spheres is connected via exactly 2 vertices. A visualization could be obtained in a way similar to the case of Lk $[1, 1]$, though the increased number of components and singularities would render the visualization very cluttered.

4.3.3 Computational results

Using the methodology of Subsection 4.3.1, a Python 3.9 script was developed to automate the calculations of structural properties of the links of vertices in $\mathtt{NAC}(Q)$. Given a Latin square representing a quasigroup Q, the algorithm returns the following information about every component of the normalized link of each vertex in $\mathtt{NAC}(Q)$:

- names of vertices in the component,
- the number of vertices, edges, and faces in the component,
- Euler characteristic of the component

and

- genus of the component.

As the last result of this thesis, we present data about interesting examples of small order quasigroups listed in Appendix B obtained using the Python script.

Example of order 4

	0	1	2	3
0	0	3	2	1
1	1	2	3	0
2	2	1	0	3
3	3	0	1	2

Table 4.3: A quasigroup of order 4 with $\mathtt{G}(Q)$ consisting of four 16-vertex components (also listed as Table 9).

The links of vertices in the non-associativity complex of this quasigroup are one of the following three types:

1. The empty complex (in case of vertices $[0, 2]$ and $[2, 2]$). This means that all triples of forms $a, b, 0$ and $a, b, 2$ are associative.
2. A union of four spheres (in case of vertices $[1, 2]$ and $[3, 2]$). These spheres have 6 vertices, 12 edges, and 8 faces and are joined in a way that every pair of spheres is glued in precisely 2 vertices of the same color (i.e., with the same y in $[x, y]$) and every singularity is an intersection of exactly two spheres. As it turns out, every vertex in the complex is a singularity.

3. A union of two spheres. These spheres have vertex-edge-face configuration 6-12-8 and are joined in two vertices of the same color.

Examples of the types of links mentioned above have been shown in the previous subsection.

Example of order 5

	0	1	2	3	4
0	0	1	2	3	4
1	4	0	1	2	3
2	3	4	0	1	2
3	2	3	4	0	1
4	1	2	3	4	0

Table 4.4: A quasigroup of order 5 with $\mathtt{G}(Q)$ consisting of two 100-vertex components (also listed as Table 10).

The links of vertices in the non-associativity complex of this quasigroup are one of the following three types:

1. The empty complex (in case of vertex $[0, 0]$),
2. A 6-torus with vertex-edge-face configuration 15-75-50 (in case of vertices $[0, 1]...[0, 4]$).
3. A union of two spheres with vertex-edge-face configurations 12-30-20 (in case of vertices $[0, 1], ...[4, 1], [0, 2], ..., [4, 2], [0, 3], ...[4, 3]$). The two spheres are joined in 10 vertices (all vertices of color other than 0).

Example of order 6

	0	1	2	3	4	5
0	0	1	3	2	5	4
1	1	0	2	3	4	5
2	3	5	4	1	2	0
3	2	4	5	0	3	1
4	4	3	0	5	1	2
5	5	2	1	4	0	3

Table 4.5: A quasigroup of order 6 with $\mathtt{G}(Q)$ consisting of seven components of sizes $16, 16, 16, 16, 16, 16, 200$ (also listed as Table 11).

The following table lists the vertex-edge-face configurations and genera of components of the normalized links of vertices in the non-associativity complex of this quasigroup.

Type	#V	#E	#F	Genus	Surface	#Comps	Vertices with this Lk
1	6	12	8	0	sphere	x4	$[0, 0], [1, 0], [0, 2]$
2	6	12	8	0	sphere	x3	$[3, 1]$
	10	24	16	0	sphere	x1	
3	6	12	8	0	sphere	x4	$[4, 2]$
	10	24	16	0	sphere	x2	
4	6	12	8	0	sphere	x3	$[5, 1], [3, 2]$
	12	30	20	0	sphere	x1	
5	6	12	8	0	sphere	x3	$[2, 1]$
	14	36	24	0	sphere	x1	
6	6	12	8	0	sphere	x4	$[1, 1]$
	14	36	24	0	sphere	x1	
7	6	12	8	0	sphere	x2	$[4, 3], [5, 3]$
	15	39	26	0	sphere	x1	
8	6	12	8	0	sphere	x4	$[0, 3], [1, 3]$
	15	39	26	0	sphere	x1	
9	6	12	8	0	sphere	x3	$[4, 0], [5, 0]$
	18	48	32	0	sphere	x1	
10	6	12	8	0	sphere	x2	$[5, 2]$
	20	54	36	0	sphere	x1	
11	6	12	8	0	sphere	x1	$[2, 3], [3, 3]$
	20	60	40	1	torus	x1	
12	6	12	8	0	sphere	x2	$[0, 1]$
	20	60	40	1	torus	x1	
13	6	12	8	0	sphere	x1	$[4, 1]$
	20	66	44	2	2-torus	x1	
14	6	12	8	0	sphere	x2	$[2, 2]$
	24	72	48	1	torus	x1	
15	6	12	8	0	sphere	x1	$[2, 0], [3, 0]$
	24	78	52	2	2-torus	x1	
16	12	30	20	0	sphere	x2	$[1, 2]$

Table 4.6: Structural types of the links of vertices in the non-associativity complex.

Maximally non-associative quasigroup of order 9

	0	1	2	3	4	5	6	7	8
0	0	3	6	2	7	4	1	8	5
1	7	1	4	5	0	8	3	2	6
2	5	8	2	6	3	1	7	4	0
3	4	2	8	3	6	0	5	1	7
4	6	5	0	1	4	7	8	3	2
5	1	7	3	8	2	5	0	6	4
6	8	4	1	7	5	2	6	0	3
7	2	6	5	0	8	3	4	7	1
8	3	0	7	4	1	6	2	5	8

Table 4.7: A maximally non-associative quasigroup of order 9 discovered by Drápal and Valent (2020) (also listed as Table 12).

There exist only two types of links of vertices in the non-associativity complex of this quasigroup:

1. Type $[x, 0]$: the normalized links of vertices of form $[x, 0]$ and $[x, 2]$ have vertex-edge-face configuration 48-240-160. This means their genus is 17 - the normalized links are 17-tori.

   For every vertex of this form, the link includes all 27 vertices of the remaining colors (for example, the link of vertex $[4, 2]$ includes all vertices $[x, y]$ with $y \neq 2$) and can be reconstructed from the normalized link by gluing the cut singularities of the links back together.

   For $y = 0, 2$, the singularities in Lk $[x, y]$ can be described as follows: vertices $[x, 1]$ and $[x, 3]$ are one-fold singularities, vertices $[z, w], z \neq x, w \in \{0, 2\} \setminus \{y\}$ are two-fold singularities and vertex $[x, w], w \in \{0, 2\} \setminus \{y\}$ is a three-fold singularity.

   We illustrate the singularities in links of Type $[x, 0]$ on an example of Lk $[4, 2]$:

   - vertices of form $[z, 1], [z, 3]$ for $z \neq 4$ are not singularities,
   - vertices $[4, 1]$ and $[4, 3]$ are one-fold singularities,
   - vertices of form $[z, 0]$ for $z \neq 4$ are two-fold singularities,
   - vertex $[4, 0]$ is a three-fold singularity.

2. Type $[x, 1]$: the normalized links of vertices of form $[x, 1]$ and $[x, 3]$ have vertex-edge-face configuration 46-240-160. Therefore, their genus is 18 - the normalized links are 18-tori.

   As in the previous case, for every vertex of this form, the link includes all vertices of the remaining colors and can be reconstructed from the normalized link by gluing the cut singularities.

   For $y = 1, 3$, the singularities in Lk $[x, y]$ can be described as follows: vertices $[x, 0], [x, 2]$ and $[x, w], w \in \{1, 3\} \setminus \{y\}$ are one-fold singularities and vertices $[z, w], z \neq x, w \in \{1, 3\} \setminus \{y\}$ are two-fold singularities.

We illustrate the singularities in links of Type $[x, 1]$ on an example of Lk $[6, 1]$:

- vertices of form $[z, 0], [z, 2]$ for $z \neq 6$ are not singularities,
- vertices $[6, 0], [6, 2]$ and $[6, 3]$ are one-fold singularities,
- vertices of form $[z, 3]$ for $z \neq 6$ are two-fold singularities.

Unlike in the previous examples, the two types of links are very similar - both types contain 27 vertices, the largest possible amount, both types consist of a single strongly connected component, and they are structurally very similar, differing only in a link of a single significant vertex. We believe these properties are directly related to the (maximal) non-associativity of the quasigroup; however, the precise nature of this relation is yet to be unveiled.

Conclusion

The purpose of this thesis was to start investigations of associative properties of quasigroups using methods of combinatorial topology.

We began the text by presenting the most recent results regarding the existence of maximally non-associative quasigroups and then hinted at a possible cryptographic application of quasigroups with small associativity indexes.

Using the theory of abstract simplicial complexes and combinatorial surfaces presented in Chapter 2, we proposed construction of a 3-dimensional abstract simplicial complex $\mathtt{NAC}(Q)$ from non-associative triples of a finite quasigroup Q in Chapter 3. We have shown in Proposition 15 that the size of $\mathtt{NAC}(Q)$ has a direct link to the number of associative triples of Q and later, throughout the section, we have shown that $\mathtt{NAC}(Q)$ is a union of closed orientable 3-pseudomanifolds. This property of $\mathtt{NAC}(Q)$ has been summarized in the resulting theorem of the chapter, Theorem 19.

The third chapter also presented an algorithm to build $\mathtt{NAC}(Q)$ and its dual graph $\mathtt{G}(Q)$. By computationally analyzing the structure of $\mathtt{G}(Q)$ using a Python 3.9 program we have independently verified the findings of Ježek and Kepka (1990) regarding the associativity spectrum of n for $n \leq 6$ in Subsection 4.1.1 and discovered a complete classification of size configurations of strongly connected components in $\mathtt{NAC}(Q)$ for $|Q| \leq 6$ in Subsection 4.1.2. This result, listed as Theorem 21, is backed by computational data in Appendix A.

Understanding the global structural properties of $\mathtt{NAC}(Q)$ is complicated because of its almost impossible visualization, no known classification of 3-pseudomanifolds (see Subsection 2.3.2) and complex singularities. We believe that the resolvability of singularities in $\mathtt{NAC}(Q)$ could be closely related to the existence of maximally non-associative quasigroups for some orders n. The main result of the thesis, presented in Sections 4.2 and 4.3, focuses on neighborhoods of vertices in $\mathtt{NAC}(Q)$ to describe $\mathtt{NAC}(Q)$ locally and is the first step towards determining whether the singularities in $\mathtt{NAC}(Q)$ can be resolved.

Theorem 24 discovers that links of vertices in $\mathtt{NAC}(Q)$ have only solvable singularities, which enables us to formulate Proposition 25, that describes a way in which links of vertices in $\mathtt{NAC}(Q)$ can be normalized, i.e., transformed from a union of closed orientable 2-pseudomanifolds into a union of disjoint closed orientable 2-manifolds. The theoretical results of Section 4.2 are used in Section 4.3 to describe Algorithm 5 to normalize the link of a vertex in $\mathtt{NAC}(Q)$, from which we can calculate Euler characteristic and genus for each of the disjoint components of the normalized link.

The rest of the chapter performs exemplary calculations using this method on examples of quasigroups of small order listed in Appendix B. The analyses show great diversity in types of possible links. However, the exemplary case of maximally non-associative quasigroup of order 9 shows a large amount of symmetry in a relatively simple structure, which motivates further effort to address the topic through methods similar to our thesis.

In future work, we aim to continue our efforts regarding the study of resolvability of singularities in $\texttt{NAC}(Q)$, which may require further investigation of the structure of vertex links. An approach using means of simplicial homology may also be used in the future to tackle the problem of the existence of maximally non-associative quasigroups of the remaining orders. These approaches will require a substantial amount of work; however, we believe they may lead to decisively answering the question of the existence of maximally non-associative quasigroups (at least for some orders) in the future.

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Appendices

A Components of $\mathtt{G}(Q)$

For $|Q| = n \leq 5$, a non-empty $\mathtt{G}(Q)$ is always connected, except for two cases:

1. For $n = 4$, no connected $\mathtt{G}(Q)$ with 64 vertices exists, but there exists a case of $\mathtt{G}(Q)$ consisting of 4 components of size 16.

2. For $n = 5$, there exists not only a connected $\mathtt{G}(Q)$ with 200 vertices, but also one consisting of 2 components of size 100.

All possible configurations for order 6 are listed in Table 8.

A multiplication table of Q has been stored for every possible configuration of component sizes of $\mathtt{G}(Q)$ for all orders up to 6.

Table 8: Possible configurations of sizes of components of $\mathtt{G}(Q)$ for Q of order 6.

$\|\mathtt{G}(Q)\|$	Possible configurations of component sizes									
0	0									
54	54									
64	16	16	16	16						
88	16	16	56							
96	16	16	16	16	16	16				
	96									
108	36	36	36							
112	16	16	80							
	56	56								
128	16	16	16	16	16	16	16	16		
	128									
136	16	120								
144	144									
148	148									
150	150									
158	158									
160	16	16	16	16	16	16	16	16	16	16
	160									
162	36	36	36	54						
	54	54	54							
164	164									
166	166									
168	168									
170	170									
172	172									

$\|\mathtt{G}(Q)\|$	Possible configurations of component sizes													
176	16	16	16	16	16	16	80							
	16	16	16	16	16	96								
	16	16	16	16	56	56								
	16	160												
	176													
178	178													
180	36	36	36	36	36									
	180													
182	182													
184	184													
188	188													
190	190													
192	16	16	16	16	16	16	16	16	16	16	16	16		
	16	16	16	144										
	16	16	160											
	192													
196	196													
198	198													
200	16	16	16	16	16	120								
	16	16	16	16	56	80								
	16	16	16	16	136									
	200													
204	204													
206	206													
208	16	16	16	16	16	16	56	56						
	16	16	176											
	16	192												
	208													
210	210													
212	16	16	180											
	16	196												
	212													
214	214													
216	16	16	16	16	16	16	16	16	16	16	56			
	16	16	16	16	152									
	16	16	184											
	36	36	36	36	36	36								
	36	36	36	54	54									
	54	54	54	54										
	216													
218	218													
220	16	204												
	220													
222	16	16	190											
	222													
224	16	16	16	16	16	16	16	16	16	16	16	16	16	16
	16	16	16	16	16	16	16	16	96					
	16	16	16	16	16	16	128							
	16	16	16	16	16	144								
	16	16	16	16	80	80								
	16	16	16	16	160									
	224													
226	226													

$\|\mathtt{G}(Q)\|$	Possible configurations of component sizes										
228	16	212									
	228										
230	16	16	198								
	230										
232	16	16	16	16	16	16	16	120			
	16	16	16	16	16	16	56	80			
	16	16	16	16	16	16	136				
	16	16	16	16	56	56	56				
	16	16	16	16	168						
	16	80	136								
	16	216									
	232										
234	16	218									
	36	36	36	36	36	54					
	234										
236	236										
238	16	16	206								
	238										
240	16	16	16	16	16	16	16	16	16	16	80
	16	16	16	16	16	16	16	16	56	56	
	16	16	16	16	176						
	16	16	16	192							
	16	16	208								
	16	224									
	80	160									
	240										
242	16	226									
	242										
244	16	228									
	244										
246	16	16	214								
	16	230									
	246										
248	16	16	16	16	16	16	152				
	16	16	16	16	16	168					
	16	16	16	16	184						
	16	232									
	56	192									
	248										
250	16	234									
	250										
252	16	16	16	204							
	16	16	220								
	16	236									
	36	36	36	36	36	36	36				
	252										
254	16	16	222								
	254										

$\lvert\mathtt{G}(Q)\rvert$	Possible configurations of component sizes														
256	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16
	16	16	16	16	16	16	16	144								
	16	16	16	16	16	16	160									
	16	16	16	16	16	56	120									
	16	16	16	16	192											
	16	16	16	208												
	16	16	80	144												
	16	16	224													
	16	240														
	256															
258	16	242														
	258															
260	16	16	16	212												
	16	16	228													
	16	244														
	260															
262	16	16	230													
	16	246														
	262															
264	16	16	16	16	16	184										
	16	16	16	16	200											
	16	16	16	216												
	16	16	232													
	16	248														
	264															
266	16	16	234													
	16	250														
	266															
268	16	16	236													
	16	252														
	268															
270	16	16	16	16	206											
	16	16	238													
	16	254														
	36	36	36	36	36	36	54									
	36	36	36	54	54	54										
	54	54	54	54	54											
	270															
272	16	16	16	16	16	16	16	16	144							
	16	16	16	16	16	16	176									
	16	16	16	16	16	80	112									
	16	16	16	16	16	192										
	16	16	16	16	208											
	16	16	240													
	16	256														
	272															
274	16	16	242													
	16	258														
	274															
276	16	16	16	228												
	16	16	244													
	16	260														
	276															

$\lvert \mathtt{G}(Q) \rvert$	Possible configurations of component sizes									
278	16	16	16	16	214					
	16	16	246							
	278									
280	16	16	16	16	16	16	184			
	16	16	16	16	16	200				
	16	16	16	16	216					
	16	16	248							
	16	264								
	280									
282	16	16	250							
	16	266								
	282									
284	16	16	16	16	220					
	16	16	16	236						
	16	16	252							
	16	268								
	284									
286	16	16	254							
	16	270								
	286									
288	16	16	16	16	16	16	16	16	80	80
	16	16	16	16	16	208				
	16	16	16	16	56	168				
	16	16	16	16	224					
	16	16	16	240						
	16	16	256							
	16	272								
	36	36	36	36	36	36	36	36		
	36	36	36	36	36	54	54			
	144	144								
	288									
290	16	16	258							
	16	274								
	290									
292	16	16	16	16	228					
	16	16	16	244						
	16	16	260							
	16	276								
	292									
294	16	16	16	16	230					
	16	16	262							
	16	278								
	294									
296	16	16	16	16	16	16	200			
	16	16	16	16	16	216				
	16	16	16	16	232					
	16	16	16	248						
	16	16	264							
	16	280								
	296									
298	16	16	266							
	16	282								
	298									

$\lvert \mathtt{G}(Q) \rvert$	Possible configurations of component sizes							
300	16	16	16	16	236			
	16	16	16	252				
	16	16	268					
	16	284						
	300							
302	16	16	16	16	238			
	16	16	270					
	16	286						
	302							
304	16	16	16	16	16	16	208	
	16	16	16	16	16	224		
	16	16	16	16	240			
	16	16	16	256				
	16	16	272					
	16	288						
	304							
306	16	16	16	258				
	16	16	274					
	16	290						
	36	36	36	36	36	36	36	54
	306							
308	16	16	16	16	244			
	16	16	16	260				
	16	16	276					
	16	292						
	308							
310	16	16	278					
	16	294						
	310							
312	16	16	16	16	248			
	16	16	16	264				
	16	16	280					
	16	296						
	312							
314	16	16	282					
	16	298						
	314							
316	16	16	16	268				
	16	16	284					
	16	300						
	316							
318	16	16	286					
	16	302						
	318							
320	16	16	16	16	128	128		
	16	16	16	16	256			
	16	16	16	272				
	16	16	288					
	16	304						
	56	264						
	320							
322	16	16	290					
	16	306						
	322							

$\vert\mathtt{G}(Q)\vert$	Possible configurations of component sizes					
324	16	16	292			
	16	308				
	54	54	54	54	54	54
	324					
326	16	16	294			
	16	310				
	326					
328	16	16	16	16	264	
	16	16	16	280		
	16	16	296			
	16	312				
	328					
330	16	16	298			
	16	314				
	330					
332	16	16	300			
	16	316				
	332					
334	16	16	302			
	16	318				
	334					
336	16	16	16	16	272	
	16	16	304			
	16	320				
	336					
338	16	16	306			
	16	322				
	338					
340	16	16	308			
	16	324				
	340					
342	16	16	310			
	16	326				
	342					
344	16	16	16	16	280	
	16	16	312			
	16	328				
	344					
346	16	16	314			
	16	330				
	346					
348	16	16	316			
	16	332				
	348					
350	16	16	318			
	16	334				
	350					
352	16	16	16	16	288	
	16	16	320			
	16	336				
	352					
354	16	16	322			
	16	338				
	354					

$\|\mathtt{G}(Q)\|$	Possible configurations of component sizes				
356	16	16	324		
	16	340			
	356				
358	16	342			
	358				
360	16	16	328		
	16	344			
	360				
362	16	16	330		
	16	346			
	362				
364	16	16	332		
	16	348			
	364				
366	16	350			
	366				
368	16	16	16	16	304
	16	16	336		
	16	352			
	368				
370	370				
372	372				
374	374				
376	16	16	344		
	16	360			
	376				
378	378				
380	380				
382	382				
384	16	16	352		
	384				
386	386				
388	388				
390	390				
392	392				
394	394				
400	400				

B Examples of small order quasigroups with interesting properties

	0	1	2	3
0	0	3	2	1
1	1	2	3	0
2	2	1	0	3
3	3	0	1	2

Table 9: A quasigroup of order 4 with $\mathtt{G}(Q)$ consisting of four 16-vertex components.

	0	1	2	3	4
0	0	1	2	3	4
1	4	0	1	2	3
2	3	4	0	1	2
3	2	3	4	0	1
4	1	2	3	4	0

Table 10: A quasigroup of order 5 with $\mathtt{G}(Q)$ consisting of two 100-vertex components.

	0	1	2	3	4	5
0	0	1	3	2	5	4
1	1	0	2	3	4	5
2	3	5	4	1	2	0
3	2	4	5	0	3	1
4	4	3	0	5	1	2
5	5	2	1	4	0	3

Table 11: A quasigroup of order 6 with $\mathtt{G}(Q)$ consisting of seven components of sizes $16, 16, 16, 16, 16, 16, 200$.

	0	1	2	3	4	5	6	7	8
0	0	3	6	2	7	4	1	8	5
1	7	1	4	5	0	8	3	2	6
2	5	8	2	6	3	1	7	4	0
3	4	2	8	3	6	0	5	1	7
4	6	5	0	1	4	7	8	3	2
5	1	7	3	8	2	5	0	6	4
6	8	4	1	7	5	2	6	0	3
7	2	6	5	0	8	3	4	7	1
8	3	0	7	4	1	6	2	5	8

Table 12: A maximally non-associative quasigroup of order 9 discovered by Drápal and Valent (2020).