A Review of the Bachelor Thesis “Combinatorial Algorithms for Flow Problems” by Richard Hladík

Summary

The topic of the thesis are two generalizations of the famous Maximum Flow problem: the Multicommodity Flow Problem (MCF), and the Length-Bounded Flow Problem (LBF). Both problems are known to be solvable in polynomial time by a relatively straightforward application of linear programming. However, no combinatorial algorithm is known for either problem. (Curiously, a combinatorial algorithm for LBF has been claimed in the ’70s by Koubek and Říha, but the correctness of this algorithm has been recently disproved by Altmanová in her bachelor thesis.)

The author proposes two algorithms. The first, dubbed “Helpful-MCF”, is an adaptation of Weintraub’s algorithm for Min-Cost Max-Flow to MCF. The second is an interpretation of the Away-Steps Frank-Wolfe algorithm for a natural penalty formulation of MCF. While the author does not manage to prove that either algorithm converges “linearly” (i.e., that it achieves error ϵ in $\text{poly}\log(1/\epsilon)$ iterations; this is necessary to converge close enough to an optimum in polynomial time), he identifies two sufficient conditions for linear convergence. The first condition is a bound on the “inflation rate” of a certain polyhedron, related to so-called Hoffman bounds. The second condition are Hölder Error Bounds (HEB) which serve, e.g., as a generalization of strong convexity in the study of Frank-Wolfe-type algorithms. Moreover, the author shows that, with respect to the linear convergence of the studied algorithms, these two conditions are equivalent. Moreover, he shows that a good bound on the inflation rate would be implied by a good bound on the so-called circuits of the constraint matrix of the corresponding LP. Lastly, he shows an exponential lower bound on the circuits, ruling out this angle of attack to this problem.

While the author does not prove linear convergence of the studied algorithms, he does show that the studied algorithms achieve error $O(\epsilon)$ in at most $O(\text{poly}(1/\epsilon))$ iterations if the capacities are polynomial.

As for LBF, most results in the thesis carry over, with the exception of the Helpful-MCF algorithm. Namely, the author also shows an exponential lower bound on the LBF circuits, and also gives a specialization of Away-Steps FW for LBF.

Evaluation

The thesis is structured appropriately, it is well-written, accompanied by helpful figures, and it is easy to navigate.

Strenghts

MCF has sometimes been called "the easiest problem without a combinatorial algorithm". LBF has not received that much attention, perhaps because a combinatorial algorithm seemed to have exist until very recently. Even so, Altmanová was not able to fix the gap in the Koubek-Říha algorithm, and Voborník was not able to construct one from scratch in his master thesis, despite considerable effort.

As such, even though the author has not found a combinatorial algorithm for MCF, I believe the contributions of the thesis are significant. In particular, to the best of my knowledge, HEB and Hoffman bounds have not been previously connected, and their connection to circuits is also new. Moreover, I believe some claims can be strenghtened from implications to equivalences. For example, I believe it might be possible to show that a large circuit implies large inflation rate, which implies large HEB. This would show that any combinatorial algorithm for MCF (or LBF) somehow has to get around these issues; even without knowing that this holds the thesis suggests that these questions and notions should be of much interest.

What I found most interesting about the results is that this is perhaps the first "natural" problem with a $-1/0/1$ constraint matrix for which an exponential lower bound on circuits has been shown. It opens up the question whether there is something inherent about the difficulty of designing a combinatorial algorithm for a problem whose natural LP has exponential circuits.

Summing up, the thesis makes interesting and new connections, and suggests several good research questions, in particular the exact relationship between circuit bounds, HEB, and inflation rate bounds.

Weaknesses

I don't think there is any weakness *per se*. One thing I have noticed is that some ideas could have perhaps been taken a step further, but time did not permit. This is also because the thesis goes in several directions and doesn't explore each of them quite exhaustively. However, the thesis already goes well beyond the scope of a typical bachelor thesis, and I believe it will serve as a basis for at least two new papers. Moreover, the wide scope of the thesis is simply the result of exploring an open-ended research question; if anything, the fact that the author was able to explore several relatively unrelated directions is a testament to his ability to adapt and quickly learn new topics.

Conclusion

Overall, I think this is an excellent bachelor thesis and it deserves the best possible grade "1".

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