Supervisor's report on the Master Thesis

Special Point-Free Spaces

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Topological spaces can be studied to advantage from the point of view of the lattices $\Omega(X)$ of their open sets. The specific property of the lattice $L = \Omega(X)$ is that it satisfies the distributivity law

$$\text{(frm)} \qquad (\bigvee_{i\in J} a_i) \wedge b = \bigvee_{i\in J}(a_i \wedge b)$$

for all subsets $\{a_i \mid i \in J\} \subseteq L$ (including the empty one) and $b \in B$. General lattices L satisfying (frm) are called *frames* or *locales*. They are more general than the $\Omega(X)$'s; considering them as spaces leads to a (very useful) extension of the concept. Continuous maps are modeled by *frame homomorphisms* preserving all joins (suprema) $\bigvee A$ and binary meets (the classical continuous maps $f : X \to Y$ appear as the frame homomorphisms $\Omega(f) = (U \mapsto f^{-1}[U]) : \Omega(Y) \to \Omega(Y)$, which represents the continuity very well, and in the sober case precisely).

It may come as a surprise that the neglect of points does not lead to a substantial loss of information (for a broad class of spaces, the original can be reconstructed from the lattice data, if we wish so). On the contrary, it leads to a very important advantage of the theory (not the only one): many classical results depending on the Axiom of Choice become in the new context constructive (roughly speaking, this can be seen as the following phenomenon: if we think of a solution of a problem as the system of the diminishing approximation neighborhood it turns out that the Axiom of Choice is in fact needed only for assuming a "center" of the system). However, it is of a principal interest how to model properties of special (generalized) spaces – in the case of this Thesis the so called separation axioms – where the points in the classical context play a basic theoretical role. It can come as a surprise that it is basically always possible; moreover the hardship with some cases leads to new specific very useful conditions.

The condition studied in this Thesis are the subfitness, fitness, the Isbell-Haudorff axiom, and regularity (the last one modelling precisely the homonymous classical property, the first two are specific: subfitness is slightly weaker that T_1 and has a lot of very important consequences, the also important fitness would need some detailed explanation, not suitable for this report). To discuss and analyze them the author uses the techniques of *weak inclusions* and *weak implications* and he does it very well, obtaining very useful insights, and interesting result. His formulation in the introduction is in fact too modest ("Some of the results are related to recent articles ..." – in fact some of the results are new and after article style remake may be published).

It is a very good Thesis. The author proved he can work well with literature, creatively absorb new facts, and that he has considerable abilities for research. Also, the text is well written and can be in future a help for other students interested in the field. I recommend it to be accepted as a Master Thesis.

September 2020 Aleš Pultr