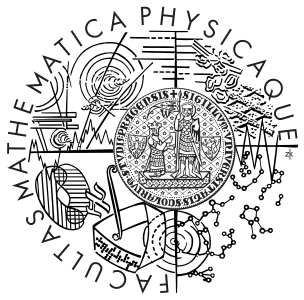


FACULTY
OF MATHEMATICS
AND PHYSICS
Charles University

MASTER THESIS

Jan Novák

Special Point-Free Spaces

Department of Applied Mathematics

Supervisor of the master thesis: prof. RNDr. Aleš Pultr, DrSc.

Study programme: Computer Science

Study branch: Discrete Models and Algorithms

Prague 2020

I declare that I carried out this master thesis independently, and only with the cited sources, literature and other professional sources. It has not been used toobtain another or the same degree.

I understand that my work relates to the rights and obligations under the Act No. 121/2000 Sb., the Copyright Act, as amended, in particular the fact that the Charles University has the right to conclude a license agreement on the use of this work as a school work pursuant to Section 60 subsection 1 of the Copyright Act.

In ..... date ....

Author's signature

I would like to thank Professor Aleš Pultr for his patience, invaluable lectures, and for the opportunity to work on this topic.

Title: Special point-free spaces

Author: Jan Novák

Department: Department of Applied Mathematics

Supervisor: prof. RNDr. Aleš Pultr, DrSc., Department of Applied Mathematics

Abstract: This thesis concerns separation axioms in point-free topology. We introduce a notion of weak inclusion, which is a relation on a frame that is weaker than the relation $\leq$. Weak inclusions provide a uniform way to work with standard separation axioms such as subfitness, fitness, and regularity. Proofs using weak inclusions often bring new insight into the nature of the axioms.

We focus on results related to the axiom of subfitness. We study a sublocale which is defined as the intersection of all the codense sublocales of a frame. We show that it need not be subfit. For spacial frames, it need not be spacial.

Keywords: point-free topology, frames and locales, separation, other specific conditions

Contents

1 Introduction 5

2 Preliminaries 7

2.1 Partial orders and lattices 7

2.2 Categories 8

2.3 Frames and locales 11

2.4 Sublocales 13

2.5 Axioms of separation 15

3 Weak Inclusions 17

3.1 Remarks on regularity 17

3.2 Weak inclusions 18

3.3 The subfit and the fit inclusion 20

3.4 Sublocales induced by weak inclusions 22

3.5 Codense sublocales 23

4 Weak Implications 25

4.1 Weak implications 25

4.2 The first weak Heyting formula 26

4.3 The second weak Heyting formula 28

4.4 The subfit implication 31

4.5 Sublocales related to weak implications 33

5 More on the Subfit Inclusion 35

5.1 Subfit inclusion and sublocales 35

5.2 Specialization order 38

6 Sublocales Induced by Weak Inclusions 39

6.1 Inclusions modulo finite set 39

6.2 More on inclusions modulo finite set 40

6.3 Inclusions modulo an ideal 42

6.4 Inflation of spaces 45

6.5 Weak inclusions on inflated spaces 46

6.6 Applications 48

7 Hausdorff-Type Axioms 52

7.1 Frame of antitone maps 52

7.2 Heyting maps 54

7.3 The Isbell-Hausdorff inclusion 56

A Bibliography 58

1 Introduction

The basic structure in point-free topology is called a *locale*. A locale can be viewed as an extension of the classical notion of a topological space. From a point-free point of view, properties of classic spaces are described via their lattices of open sets. Proofs in the theory of locales bring usually new insight into the topic, and they are often constructive. If we assume certain low separation axioms, we lose no information when we forget the points.

In classical topology, there is a well understood hierarchy of separation axioms

$$T_{3\frac{1}{2}} \quad \Rightarrow \quad T_3 \quad \Rightarrow \quad T_2 \quad \Rightarrow \quad T_1.$$

The axioms of regularity and complete regularity can be faithfully defined in locales. On the other hand, axioms which are below regularity are heavily dependent on points. There are numerous suggestions for a point-free Hausdorff axiom, and it is very difficult to imitate T_1.

In the pioneering paper by Isbell [1], he introduced the axiom of *subfitness*. A locale is subfit if each of its open sublocales is a join of closed ones, and it is strictly weaker than T_1. Subfitness was, however, rather neglected at first, because it behaves badly in constructions. Sublocales of subfit frames need not be subfit, and subfitness is not preserved when constructing localic products. Isbell also proposed a hereditary variant of subfitness, which is called *fitness*. It turned out that this axiom is much stronger that subfitness. It is already very close to regularity.

Recently, subfitness started to be intensively studied, and large part of this thesis is also devoted to subfitness. It does not claim to mention all known facts about subfitness. It describes many well known facts from a different perspective by introducing a calculus of so called *weak inclusions* and *weak implications*. In same cases, the results are generalized, and several new facts are derived. Other axioms, such as fitness, regularity, and the Isbell-Hausdorff axiom, are treated using this calculus as well.

Organization of the thesis

In the next chapter, we recall some basic facts from point-free topology and category theory.

In Chapter 3, we show that many of the axioms of our interest can be written using a relation of a special type, which we call a weak inclusion. In a certain sense, a weak inclusion measures, how much the axiom holds. In this way, we define the subfit inclusion, the fit inclusion, and the regular inclusion. We present their basic properties.

In Chapter 4, we introduce weak implications. We present several axioms how they relate to weak inclusions. Then, we show that some weak inclusions naturally define a subframe that can be used to construct a coreflection.

Chapter 5 concerns the subfit inclusion. We introduce a interesting weak inclusion on the coframe of sublocales, and we relate it to fitness and subfitness. Some

results in this chapter can be compared to that of a recent article [2], where they describe the structure of those sublocales which are join of closed ones.

In Chapter 6, we introduce a method for creating special point-free spaces with certain properties. Each weak inclusion induces a sublocale, and we discuss the nature of the sublocale which is induced by the subfit inclusion. We show that it need not be subfit. For spacial frames, it need not be spacial. Some of the results are related to recent articles [3][4].

In Chapter 7, we describe the Isbell-Hausdorff axiom in terms of weak inclusions.

2 Preliminaries

In this section, we recall some well-known facts from category theory and point-free topology. Most of the time, we will follow the standard monograph of Picado and Pultr [5].

We assume that the reader is acquainted with basic notions of set-theoretic topology. Otherwise, we refer the reader to some standard source [6]. Mainly, we are interested in separation axioms weaker than regularity. Throughout the whole thesis,

all topological spaces are assumed to be T_0.

Let $f : X \to Y$ be an arbitrary mapping, and let $A \subseteq X$ and $B \subseteq Y$. We denote the image and the preimage of a set by $f[A] = \{f(a);\ a \in A\}$ and $f^{-1}[B] = \{x \in X;\ f(x) \in B\}$, respectively.

2.1. Partial orders and lattices

A set P with a binary relation $\leq$ is called a *partially ordered set* if the following holds for all $x, y, z \in P$.

(1) $x \leq x$ (reflexivity)

(2) If $x \leq y$ and $y \leq z$, then $x \leq z$ (transitivity)

(3) If $x \leq y$ and $y \leq x$, then $x = y$ (antisymmetry)

If $a \in P$, we define $\uparrow a = \{b \in P;\ a \leq b\}$ and $\downarrow a = \{b \in P;\ b \leq a\}$. Similarly, for $A \subseteq P$, we set $\uparrow A = \bigcup\{\uparrow a;\ a \in A\}$ and $\downarrow A = \bigcup\{\downarrow a;\ a \in A\}$.

An element $m \in A$ is called *a minimal element of* A if $(\downarrow m) \cap A = \{m\}$. An element $m \in A$ is called *the least element of* A if $A \subseteq \uparrow m$. If the least element of A exists, it is unique. An element $b \in P$ is called *an upper bound of* A if $A \subseteq \downarrow b$. An element $s \in P$ is called *the supremum of* A if it is the minimal element of all the upper bounds of A.

The *dual order* $\leq^{\mathrm{op}}$ related to $\leq$ is given by

$$a \leq^{\mathrm{op}} b \quad \text{iff} \quad b \leq a.$$

All notions defined on P using $\leq$ can be also defined on the dual order by replacing $\leq$ by $\leq^{\mathrm{op}}$. In this way we define *maximal elements*, *greatest elements*, *lower bounds*, and *infima*, respectively.

2.1.1. Complete lattices. A partially ordered L set is called a *complete lattice* if each subset of L has a supremum and an infimum. If $A \subseteq L$, the supremum of A is denoted by $\bigvee A$, and the infimum by $\bigwedge A$.

When L has suprema of all of its subsets, then it is already a complete lattice, where infima are given by $\bigwedge A = \bigvee\{x;\ x \text{ lower bound of } A\}$. In a complete lattice, there is the *top element* 1, which is the infimum of an empty set. The *bottom element*, denoted by 0, is the supremum of an empty set.

2.1.2. Monotone maps. Let P, Q be partially ordered sets. A map $f : P \to Q$ is called *monotone* if

$$a \leq b \;\Rightarrow\; f(a) \leq f(b).$$

Assume, moreover, that P and Q are complete lattices. We say that f *preserves arbitrary joins* if $f(\bigvee A) = \bigvee f[A]$, for each $A \subseteq P$. In particular, $f(1) = 1$. We say that f *preserves finite joins* if $f(a \vee b) = f(a) \vee f(b)$ and $f(0) = 0$, for each $a, b \in P$. Dually, we define the *preservation of arbitrary meets* and the *preservation of finite meets*.

2.1.3. Adjunction. Let $f : P \to Q$ and $g : Q \to P$ be monotone maps of partially ordered sets P and Q. We say that f *is a left adjoint of* g, and that g *is a right adjoint of* f, if

$$f(x) \leq y \quad \text{iff} \quad x \leq g(y),$$

for each $x \in P$ and $y \in Q$. For a monotone mapping g, there is at most one left adjoint. If f is left adjoint of g, then

$$fgf = f \quad \text{and} \quad gfg = g.$$

We often use an alternative description that f is a left adjoint of g if and only if

$$x \leq gf(x) \quad \text{and} \quad fg(y) \leq y,$$

for each $x \in P$ and $y \in Q$. These formulas are called *the adjunction units*.

If P is a complete lattice, then $f : P \to Q$ has a right adjoint if and only if it preserves arbitrary joins. Then, its right adjoint g is given by

$$g(y) = \bigvee\{x;\; f(x) \leq y\}.$$

2.2. Categories

In this section, we recall some basic notions of category theory. For more detailed explanation, we refer to the Appendix II of [5], or some standard book on category theory [7].

2.2.1. Definition. A *category* $\mathcal{C}$ is a collection of *objects*, which are denoted by $X, Y, Z, ...$, together with a collection of *morphisms* $f, g, h, ...$, with the following structure:

(1) Each morphism f is assigned a pair of objects called the *domain of* f and the *codomain of* f. We write $f : X \to Y$ to denote that X and Y are the domain and the codomain of f, respectively.

(2) There is a binary operation on morphisms, called the *composition*, which assigns to a pair of morphisms $f : X \to Y$ and $g : Y \to Z$ a morphism $gf : X \to Z$ such that $f(gh) = (fg)h$, whenever the composition makes sense.

(3) For each object X, there is an *identity morphism* $1_X : X \to X$ such that for each $f : X \to Y$, $f1_X = 1_Y f$.

2.2.2. Basic notions. Let $\mathcal{C}$ be a category. The *category dual to* $\mathcal{C}$ is a category denoted by $\mathcal{C}^{\mathrm{op}}$, which has the same class of objects, but where the domains and codomains are swapped.

The class of all the morphisms from X to Y is denoted by

$$\mathcal{C}(X, Y).$$

A category $\mathcal{D}$ is a *full subcategory* of $\mathcal{C}$ if each object of $\mathcal{D}$ is also an object of $\mathcal{C}$, and $\mathcal{D}(X, Y) = \mathcal{C}(X, Y)$.

Let $f : X \to Y$ be a morphism. We say that f is an *isomorphism* if there is mapping $g : Y \to X$ such that $fg = 1_Y$ and $gf = 1_X$. We say that f is an *epimorphism* if for each pair $g, h : Y \to Z$ such that $gf = hf$, it holds that $h = g$. A morphism which satisfies the dual property is called a *monomorphism*.

2.2.3. Functors. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *functor* $F : \mathcal{C} \to \mathcal{D}$ assigns to each object X of $\mathcal{C}$ an object $F(X)$ of $\mathcal{D}$, and which assigns each morphism $f : X \to Y$ of $\mathcal{C}$ a morphism $F(f) : F(X) \to F(Y)$ of $\mathcal{D}$ such that

$$F(gf) = F(g)F(f) \quad \text{and} \quad F(1_X) = 1_{F(X)}.$$

We often omit the parentheses, and we write FX and Ff instead of $F(X)$ and $F(f)$, respectively. Functors can be composed in an obvious way. For each category $\mathcal{C}$, there is an *identity functor* $I_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$, where $I_{\mathcal{C}}(X) = X$ and $I_{\mathcal{C}}(f) = f$.

Let $\mathcal{C}$ be a category such that each $\mathcal{C}(X, Y)$ is a set. The assignment $\mathcal{C}(-, -)$ from 2.2.2 defines a functor $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$, where $\mathbf{Set}$ denotes the category of sets. It send each pair (X, Y) to the set of all the morphisms from X to Y. If $f : X_2 \to X_1$, $g : Y_1 \to Y_2$, and $\alpha \in \mathcal{C}(X_1, Y_1)$, we set $\mathcal{C}(f, g)(\alpha) = g\alpha f$.

2.2.4. Natural transformations. Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors. A *natural transformation* $\mu : F \to G$ consists of a collection $\{\mu_X : FX \to GX;\ X \text{ object of } \mathcal{C}\}$ of morphisms of $\mathcal{D}$ such that for each morphism $f : X \to Y$ of $\mathcal{C}$, we have, in $\mathcal{D}$, a commutative square

$$\begin{array}{ccc} F(X) & \xrightarrow{\mu_X} & G(X) \\ \big\downarrow{\scriptstyle F(f)} & & \big\downarrow{\scriptstyle G(f)} \\ F(Y) & \xrightarrow{\mu_Y} & G(Y). \end{array}$$

A *natural equivalence* is a natural transformation μ such that each μ_X is an isomorphism.

2.2.5. Adjunctions. Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors. The functor F is called *a left adjoint* of G, and G *a right adjoint* of F, if there there is a equivalence

$$\mathcal{D}(FX, Y) \simeq \mathcal{C}(X, GY),$$

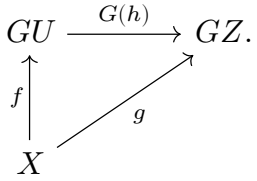
which is natural in both X and Y. Up to natural equivalence, for a functor G there is at most one left adjoint F. One often uses the following equivalent description

of an adjoint situation. A functor F is a left adjoint of G if there are natural transformations $\mu : I_{\mathcal{C}} \to FG$ and $\nu : GF \to I_{\mathcal{D}}$ such that

$$(F\nu_X)\mu_{FX} = 1_{FX} \quad \text{and} \quad \nu_{GY}(G\mu_Y) = 1_{GY}.$$

The transformations are called the *unit* and the *counit* of the adjunction, respectively. Given the units, one can reconstruct the isomorphism above, and vice-versa.

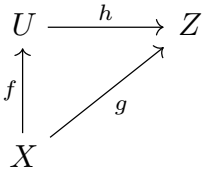
2.2.6. Universal morphisms. Let $G : \mathcal{D} \to \mathcal{C}$ be a functor, and let X be an object of $\mathcal{C}$. A morphism $f : X \to GU$ is *universal from X to G* when for each morphism $g : X \to GZ$, there is a unique $h : U \to Z$ such that



Adjoint situations can be described in terms of universal morphisms. Let $G : \mathcal{D} \to \mathcal{C}$ be a functor. For each object X of $\mathcal{C}$, let there be a mapping $\mu_X : X \to G(U_X)$ which is universal from X to G. Then the mapping $X \mapsto U_X$ can be extended to a functor $F : \mathcal{C} \to \mathcal{D}$. The mappings μ_X constitute a natural transformation. Furthermore, using the universality of μ_{GY} on the identity morphism 1_{GY}, we obtain a unique $\nu_Y : FGY \to Y$. This collection forms the counit. It can be shown that they satisfy the required identities, which completes the description of an adjunction.

2.2.7. Reflective subcategories. Let $\mathcal{D}$ be a full subcategory of $\mathcal{C}$. Denote by $G : \mathcal{D} \to \mathcal{C}$ the inclusion functor. We say that $\mathcal{D}$ is *a reflective subcategory of $\mathcal{C}$* if the functor G has a left adjoint.

By 2.2.6, this happens when for each X in $\mathcal{C}$, there is an object U of $\mathcal{D}$ and a morphism $f : X \to U$ of $\mathcal{C}$ such that each morphism $g : X \to Z$ in $\mathcal{C}$ can be extended to a morphism $h : U \to Z$ of $\mathcal{D}$ such that the diagram



commutes. Dually, we speak of a *coreflective subcategory* if G has a right adjoint.

2.3. Frames and locales

In this section, we summarize basic definitions from point-free topology. For more detailed explanation, see [8] or [5].

2.3.1. Frames. A complete lattice L is a *frame* if

$$(\bigvee a_i) \wedge b = (\bigvee a_i \wedge b),$$

for each family $\{a_i;\ i \in J\}$ and each $b \in L$. The *Heyting operation* on a frame L is a binary operation $(- \to -)$, which is given by the formula

$$x \wedge b \leq y \quad \text{iff} \quad x \leq b \to y.$$

Its existence follows from 2.1.3, because each mapping $(- \wedge b)$ preserves arbitrary joins. Given an element a of a frame L, the element $a^* = a \to 0$ is called the *pseudocomplement of* a. The Heyting operation is monotone in the second coordinate and antitone in the first. Furthermore, we will use the following standard formulas without mentioning:

(H1) $a \leq b$ iff $a \to b = 1$.

(H2) $a \to (\bigwedge b_i) = \bigwedge(a \to b_i)$.

(H3) $(\bigvee a_i) \to b = \bigvee(a_i \to b)$.

(H4) $(a_1 \wedge a_2) \to b = a_1 \to (a_2 \to b)$.

(H5) $x = (x \vee a) \wedge (a \to x)$.

(H6) If $a \vee x = 1$, then $a \to x \leq x$.

2.3.2. Homomorphisms. If L and M are frames, a mapping $h : L \to M$ is a *frame homomorphism* if it preserves arbitrary joins (including 0) and finite meets (including 1). Frame homomorphisms need not preserve the Heyting operation nor the pseudocomplement. We have only the obvious inequalities $h(x \to y) \leq h(x) \to h(y)$ and $h(x^*) \leq h(x)^*$. As in algebras, the subset $h[L]$ of M is a subframe of L. A frame homomorphism $f : L \to M$ is an isomorphism if and only if it is one-one and onto. If L and M are isomorphic, we write $L \simeq M$.

The category of all frames with frame homomorphism is denoted by **Frm**. The category dual to **Frm** is called the category of *locales*, we denote it by **Loc**. Since a frame homomorphism $h : L \to M$ preserves arbitrary joins, it has a unique right adjoint $h^+ : M \to L$, which we call a *localic morphism*.

2.3.3. Frames of open sets. When X is a topological space, the complete lattice $\mathcal{O}(X)$ of all open sets of X forms a frame. Given a continuous mapping $f : X \to Y$, the mapping $f^{-1}[-] : \mathcal{O}(Y) \to \mathcal{O}(X)$, going in the opposite direction, is a frame homomorphism. We write $\mathsf{Lc}(X) = \mathcal{O}(X)$, and by $\mathsf{Lc}(f) : \mathsf{Lc}(X) \to \mathsf{Lc}(Y)$ we denote the localic morphism associated to frame homomorphism $f^{-1}[-]$. This assignment defines a functor

$$\mathsf{Lc} : \mathbf{Top} \to \mathbf{Loc}.$$

2.3.4. Spectra. We say that an element p of L is *prime* if

$$a \not\leq p,\ b \not\leq p \quad \text{implies} \quad a \wedge b \not\leq p,$$

for all a, b in L. In the frame of open sets of X, a typical example of a prime element is an open set of the form of $X \setminus \overline{\{x\}}$, for some $x \in X$.

By $\mathsf{Sp}(L)$ we denote the set of all the prime elements of a frame L. It is easy to see that there is a topology on $\mathsf{Sp}(L)$ whose open sets are of the form of $\mathsf{sp}(a)$, for some $a \in L$. The resulting space is called *the spectrum of* L.

If $h : L \to M$ is a frame homomorphism, and p is a prime element of M, then $h^+(p)$ is a prime element of L. Therefore, to each localic map $h^+ : L \to M$ we can assign a continuous mapping $h^+(-) : \mathsf{Sp}(M) \to \mathsf{Sp}(L)$. Indeed, we have $(h^+)^{-1}[\mathsf{sp}(y)] = \mathsf{sp}(h(y))$, because $h^+(p) \in \mathsf{sp}(y)$ iff $y \not\leq h^+(p)$ iff $h(y) \not\leq p$ iff $p \in \mathsf{Sp}(h(y))$. It is easy to check that we have defined a functor

$$\mathsf{Sp} : \mathbf{Loc} \to \mathbf{Top}.$$

2.3.5. The functors Lc and Sp are related as follows. There is an adjoint situation

$$\mathbf{Top}(\mathsf{Sp}(L), X) \simeq \mathbf{Loc}(L, \mathsf{Lc}(X)).$$

The adjoint unit is a collection of continuous mappings

$$\rho_X : X \to \mathsf{SpLc}(X), \quad x \mapsto X \setminus \overline{\{x\}},$$

and the counit is a collection of localic morphisms

$$\lambda_L^+ : \mathsf{LcSp}(L) \to L, \quad \mathsf{sp}(a) \mapsto a.$$

2.3.6. Spaciality and sobriety. We say that a frame L is *spacial* when $L \simeq \mathcal{O}(X)$, for some X. It holds that L is spacial if and only if the frame homomorphism $\lambda_L : L \to \mathsf{LcSp}(L)$ is one-one, that is, if L is isomorphic to its own spectrum. The spacial frames form a reflective subcategory of **Frm**.

We say that a space X is *sober* if each prime element U of $\mathcal{O}(X)$ is of the form of $X \setminus \overline{\{x\}}$, for some $x \in X$. A spectrum of a frame is always a sober space. From the adjunction it can be shown that a space X is sober if and only if it is homeomorphic to a spectrum of some frame. Therefore, the sober spaces are the spaces where we lose no information when we forget the point-structure. They form a reflective subcategory of **Top**.

2.4. Sublocales

In the category of locales, subobjects are, up to isomorphism, suitably represented by extremal monomorphisms. A monomorphism $f : A \to B$ is called an *extremal monomorphism* if it cannot be decomposed as $f = ge$, where e is an epimorphism and g is any morphism, unless e is an isomorphism. Dually, we speak of *extremal epimorphisms*. In **Frm**, these can be conveniently characterized, because the extremal epimorphisms in **Frm** are precisely onto frame homomorphisms.

If $X \to Y$ is an inclusion of a subspace, then there is an onto frame homomorphism $\mathcal{O}(Y) \to \mathcal{O}(X)$ given by $U \mapsto U \cap X$.

2.4.1. Congruences. A binary relation E on a frame L is a *congruence* if the following conditions hold:

(1) If $a_i E b_i$ for each $i \in I$, then $\bigvee a_i E \bigvee b_i$.

(2) If $a_1 E b_1$ and $a_2 E b_2$, then $(a_1 \wedge b_1) E (a_2 \wedge b_2)$.

Given a frame homomorphism $h : L \to M$, there is a congruence E_h given by

$$x E_h y \quad \text{iff} \quad h(x) = h(y).$$

When describing the quotient frame, we make use of the fact that a congruence class of an element x has the greatest element $\bigvee\{y;\ yEx\}$. That leads to a notion of a nucleus and a sublocale.

2.4.2. Nuclei. A mapping $\nu : L \to L$ is a *nucleus on* L if

(1) $a \leq \nu(a)$,

(2) $\nu(a) \wedge \nu(b) = \nu(a \wedge b)$,

(3) $\nu\nu(a) \leq \nu(a)$.

Note that a nucleus, as a mapping $L \to L$, is usually not a frame homomorphism. Given a congruence E on L, the mapping $\nu_E(x) = \bigvee\{y;\ xEy\}$ is a nucleus. In the opposite way, if ν is a nucleus on L, there is a congruence E_ν given by

$$x E_\nu y \quad \text{iff} \quad \nu(x) = \nu(y).$$

This defines a one-one correspondence between congruences and nuclei on L.

2.4.3. Sublocales. A subset $S \subseteq L$ is a *sublocale of* L if

(1) If $s \in S$ and $x \in L$, then $x \to s \in S$.

(2) If $A \subseteq S$, then $\bigwedge A \in S$.

In particular, $1 \in S$. If ν is a nucleus on L, then the subset $S_\nu = \nu[L] = \{a;\ \nu(a) \leq a\}$ of L is a sublocale of L. On the other hand, if S is a sublocale, we

define a nucleus ν_S by setting $\nu_S(a) = \bigwedge\{s \in L;\ s \in S,\ a \leq s\}$. It can be proved that this defines a one-one correspondence between nuclei and sublocales of L.

A sublocale S, ordered by the order $\leq$ of L, is a frame. By (2), meets in S coincides with that of L. The join $\bigvee s_i$, in S, is equal to $\nu_S(\bigvee s_i)$, taken in L. A nucleus ν, when regarded as a mapping $\nu : L \to S_\nu$, is a frame homomorphism.

2.4.4. The coframe of sublocales. Let L be a frame. The set $\mathsf{S}(L)$ of all the sublocales of L, ordered by inclusion, is a structure which satisfies the axioms which are dual to that of frames. Such structure is called a *coframe*. It is a complete lattice which satisfies the coframe distributivity

$$(\bigcap S_i) \vee Q = \bigcap (S_i \vee Q).$$

There is the co-Heyting operation denoted by $\setminus$, which satisfies

$$S \setminus Q \subseteq P \quad \text{iff} \quad S \subseteq Q \vee P.$$

The sublocale $S^\# = L \setminus S$ is called the *supplement* of a sublocale S in $\mathsf{S}(L)$. It is given by the formula $S^\# = \bigcap\{P;\ S \vee P = L\}$.

The one-one correspondence between sublocales and nuclei is actually an order-reversing isomorphism; nuclei are ordered point-wise. Therefore, the set of all the nuclei on L is a frame. We choose the representation which is more convenient in a given situation.

2.4.5. Booleanization of a frame. An element a of a frame L is called *complemented* when $a \vee a^* = 1$. A frame where each element is complemented is called a *complete Boolean algebra*.

If L is a frame, there is a nucleus $x \mapsto x^{**}$. The associated sublocale $\{x^*;\ x \in L\}$ is a complete Boolean algebra. In spaces, its elements are also known as *regular open sets*. This sublocale is dense: A sublocale S is called *dense* when A sublocale $S$is called *dense* when $0 \in S0 \in S$.

2.4.6. Open and closed sublocales. If a is an element of L, there is a sublocale

$$\mathfrak{o}(a) = \{x;\ a \to x \leq x\} = \{a \to x;\ x \in L\}.$$

These are referred to as *open sublocales*. The corresponding nucleus is given by $\nu(x) = a \to x$.

Closed sublocales are sublocales of the form of

$$\mathfrak{c}(a) = \{x;\ a \leq x\} = \{a \vee x;\ x \in L\},$$

for some $a \in L$. They can be also defined using a nucleus $\nu(x) = a \vee x$. In the coframe $\mathsf{S}(L)$, open sublocales and complemented by closed ones.

2.4.7. Prenuclei. A *prenucleus* on a frame L is a mapping $\pi : L \to L$ satisfying

(1) $a \leq \pi(a)$,

(2) $a \leq b \;\Rightarrow\; \pi(a) \leq \pi(b)$,

(3) $\pi(a) \wedge b \leq \pi(a \wedge b)$.

Note that some authors use a stronger definition replacing the axiom (3) by $\pi(a) \wedge \pi(b) \leq \pi(a \wedge b)$.

If π is a prenucleus on a frame L, the set $S_\pi = \{a;\ \pi(a) \leq a\}$ is a sublocale. The nucleus associated to S_π has an illustrative description. For each ordinal α, set

$$\pi^{\alpha+1}(b) = \pi(\pi^\alpha(b)), \quad \text{and} \quad \pi^\alpha(b) = \bigvee\{\pi^\beta(b);\ \beta < \alpha\} \text{ if } \alpha \text{ is limit.}$$

For each $\alpha > 0$, the mapping π^α is a prenucleus. The sequence increases. At some ordinal α, the prenucleus π^α becomes a nucleus.

2.5. Axioms of separation

In the sequel, we will be concerned with axioms which are weaker than regularity.

2.5.1. Regularity. We say that a frame L is *regular* if

$$a \not\leq b \;\Rightarrow\; \exists c,\ a \vee c^* = 1,\ c \not\leq b.$$

This formula will be treated in more detail later. It is easy to show that a space X is regular if and only if the frame $\mathcal{O}(X)$ is regular.

2.5.2. Subfitness and fitness. The axiom T_1 in spaces is heavily dependent on points. In point-free topology, one often uses the following weaker axiom. A frame L is *subfit* if

$$a \not\leq b \;\Rightarrow\; \exists c,\ a \vee c = 1,\ b \vee c \neq 1.$$

Each regular space is subfit. If a space X is T_1, then the frame $\mathcal{O}(X)$ is subfit. However, it is easy to find a space which is subfit but not T_1.

Subfitness behaves badly in constructions. A sublocale of a subfit space need not be subfit. A strengthening of subfitness which is inherited by sublocales is called fitness: A frame L is *fit* if

$$a \not\leq b \;\Rightarrow\; \exists c,\ a \vee c = 1,\ c \to b \not\leq b.$$

Each regular space is fit. Each fit space is T_1. Fitness is, however, substantially stronger then subfitness.

In applications, subfitness if often used as a supplementary axiom to other properties. For example, it holds that each normal subfit frame is regular. The reader can compare the statement with its classical formulation in set-theoretic topology, where T_1 is used instead. For another example, see 3.1.4. Other uses of subfitness were presented in [9] and [10].

2.5.3. Compactness. A frame is *compact* when for each $A \subseteq L$ such that $\bigvee A = 1$, there is a finite subset $\{a_1, ..., a_n\}$ of A such that $a_1 \vee \cdots \vee a_n = 1$. An element $c \neq 1$ of L is a *coatom* if $a \not\leq c$ implies $a \vee c = 1$.

Assuming the axiom of choice, each compact subfit frame is spatial: If $a \not\leq b$, there is a c such that $a \vee c = 1$ and $b \vee c \neq 1$. Using the axiom of choice and compactness, there is coatom d such that $b \vee c \leq d \neq 1$. Each coatom is prime, and we have $a \not\leq d$ and $b \leq d$.

2.5.4. Subfitness and fitness was introduced by Isbell in [1]. Some authors use the term conjunctivity, especially when working with general lattices [11].

3 Weak Inclusions

In the final part of the last chapter, we discussed several important separation axioms in point-free topology. Each of them was given as a first-order formula in the language of frames. In this chapter, we will compare different variations of these formulas. This will lead us to the notion of weak inclusion, which is central for the following chapters.

3.1. Remarks on regularity

Recall that a frame L is regular if and only if for each $a, b \in L$,

$$\forall a, b, \quad \text{if} \quad a \not\leq b, \quad \text{then} \quad \exists c,\ a \vee c^* = 1,\ c \not\leq b. \tag{Reg}$$

In an article concerning fitness [9], the referee pointed out that fitness is implied by the condition

$$\forall a, b, \quad \text{if} \quad a \not\leq b, \quad \text{then} \quad \exists c,\ a \vee b \vee c^* = 1,\ c \not\leq b. \tag{BB}$$

It resembles regularity, and a question arose whether we obtained a new axiom. It turns out that it is just a reformulation of regularity:

3.1.1. Proposition. *Let L be a frame. Then L is regular if and only if it satisfies the axiom* (BB).

Proof: Let L satisfy the axiom (BB). We prove that it is regular. Suppose that $a \not\leq b$. Then, we have $a \not\leq a \wedge b$. Therefore, by (BB), there is a c such that $a \vee (a \wedge b) \vee c^* = 1$ and $c \not\leq a \wedge b$. It follows that $a \vee c^* = 1$. Finally, since $c \leq a$, it cannot be $c \leq b$. □

3.1.2. The axiom (BB) can be formally weakened in the following way:

$$\forall a, b, \quad \text{if} \quad b < a, \quad \text{then} \quad \exists c,\ a \vee c^* = 1,\ c \not\leq b. \tag{BB1}$$

This axiom still implies (BB). Indeed, suppose that $a \not\leq b$. Then $b < a \vee b$, and so (BB1) yields a c such that $a \vee b \vee c^* = 1$ and $b \not\leq c$, which was required.

3.1.3. Next, we consider yet another weaker variation of the formula:

$$\forall a, b, \quad \text{if} \quad a \vee b = 1,\ b \neq 1, \quad \text{then} \quad \exists c,\ a \vee c^* = 1,\ c \not\leq b. \tag{BB2}$$

It is easy to see that in spaces, the axiom (BB2) holds if and only if for each non-empty closed set F,

$$F \cap \overline{\{x\}} = \emptyset \ \Rightarrow\ \exists\, U, V \text{ open disjoint},\ F \subseteq U,\ x \in V.$$

This time, the axiom is strictly weaker than the regularity condition. Consider the topological space of natural numbers where open sets are down-sets in the usual ordering. The axiom (BB2) is satisfied in a trivial way, because each two non-empty closed sets intersect. The space, however, is not even subfit. On the other hand, we have:

3.1.4. Proposition. (BB2) + *Subfitness = Regularity*

Proof: Suppose that $a \not\leq b$. By subfitness, there is a d such that $a \vee d = 1$ and $b \vee d \neq 1$, and so $a \vee (b \vee d) = 1$. Using (BB2), we obtain a c such that $a \vee c^* = 1$ and $c \not\leq b \vee d$. It follows that $c \not\leq b$. □

3.2. Weak inclusions

Although it is not immediately apparent, the reasoning in the previous section can be done without any property which is inherent to regularity. We will show in 3.2.4 that a similar result can be derived for any axiom which can be formulated using a binary relation satisfying certain properties.

3.2.1. Definition. Let L be a frame. A binary relation $\lesssim$ on L is called a *weak inclusion on* L if for each a, b in L,

$$a \leq b \quad \text{implies} \quad a \lesssim b.$$

By definition, a weak inclusion is always a reflexive relation, but it need not be transitive, in general. We say that

$\lesssim$ is *transitive* if $a \lesssim b$ and $b \lesssim c$ implies $a \lesssim c$.
$\lesssim$ is *upward-transitive* if $a \lesssim b$ and $b \leq c$ implies $a \lesssim c$.
$\lesssim$ is *downward-transitive* if $a \leq b$ and $b \lesssim c$ implies $a \lesssim c$.

$\lesssim$ is $\wedge$*-preserving* if $a \lesssim b_i$, for $i = 1, 2$, implies $a \lesssim b_1 \wedge b_2$.
$\lesssim$ is $\bigwedge$*-preserving* if $a \lesssim b_i$, for $i \in J$, implies $a \lesssim \bigwedge\{b_i;\ i \in J\}$.

$\lesssim$ is $\vee$*-preserving* if $a_i \lesssim b$, for $i = 1, 2$, implies $a_1 \vee a_2 \lesssim b$.
$\lesssim$ is $\bigvee$*-preserving* if $a_i \lesssim b$, for $i \in J$, implies $\bigvee\{a_i;\ i \in J\} \lesssim b$.

Clearly, each transitive weak inclusion is both upward-transitive and downward-transitive.

3.2.2. The regular inclusion. Let L be a frame. The *regular inclusion* $\lesssim_{\mathsf{Reg}}$ is weak inclusion on L given by

$$a \lesssim b \quad \text{iff} \quad \forall x,\ x^* \vee a = 1 \ \Rightarrow\ x \leq b.$$

3.2.3. Observation. *The regular inclusion on a frame is downward-transitive, upward-transitive and* $\bigwedge$*-preserving.* □

The regular inclusion will be denoted by $\lesssim_{\mathsf{Reg}}$. Clearly, a frame L is regular if and only if the regular inclusion on L coincides with the relation $\leq$. Using the regular inclusion, the axiom (BB) from 3.1.1 can be conveniently expressed by the formula

$$\forall a, b,\ a \vee b \lesssim_{\mathsf{Reg}} b \ \Rightarrow\ a \leq b.$$

Now, the equivalences 3.1.1 and 3.1.2 are special cases of the following proposition.

3.2.4. Proposition. *Let $\lesssim$ be $\wedge$-preserving, downward-transitive weak inclusion on a frame L. Let a be a fixed element of L. Then the following statements are equivalent.*

(1) $\forall b,\ a \lesssim b \ \Rightarrow\ a \leq b.$

(2) $\forall b,\ a \vee b \lesssim b \ \Rightarrow\ a \leq b.$

(3) $\forall b,\ a \lesssim b,\ b \leq a \ \Rightarrow\ a \leq b.$

(4) $\forall b,\ b < a \ \Rightarrow\ a \not\lesssim b.$

Proof: (1)⇒(2): If $a \vee b \lesssim b$, then by downward-transitivity we have $a \lesssim b$. Using (1), it follows that $a \leq b$. (2)⇒(3): If $a \lesssim b$ and $b \leq a$, then $a \vee b = a \lesssim b$, and hence $a \leq b$ by (2). The equivalence (3)⇔(4) is trivial. (3)⇒(1): Suppose that $a \lesssim b$. By $\wedge$-preservation, we have $a \lesssim a \wedge b$. Since $a \wedge b \leq a$, by (3) it follows that $a \leq a \wedge b$. □

In the remainder of this section, we present two properties of the regular inclusion to motivate our further development. Simply told, a well behaved weak inclusion should naturally define a subframe and a sublocale. One of the objectives of the next chapters is to describe sufficient conditions for a weak inclusion to be such well behaved.

3.2.5. Proposition. *Let $\lesssim$ be the regular inclusion on a frame L. Then the set $M = \{a \in L;\ \forall b,\ a \lesssim b \ \Rightarrow\ a \leq b\}$ forms a subframe of L.*

Proof: Let $a_i \in M$, for each i. Suppose that $\bigvee a_i \lesssim b$. By downward-transitivity, we have $a_i \lesssim b$, for each i. It follows that $a_i \leq b$, for each i, and therefore $\bigvee a_i \leq b$. We proved that $\bigvee a_i \in M$.

Next, let $a_1, a_2 \in M$, and suppose that $a_1 \wedge a_2 \lesssim b$. We need to prove that $a_1 \wedge a_2 \leq b$. It suffices to show that $a_1 \lesssim a_2 \to b$. Suppose that $a_1 \vee x^* = 1$. It remains to show that $x \leq a_2 \to b$. Again, it suffices to prove that $a_2 \lesssim x \to b$. Let $a_2 \vee y^* = 1$. Since $(a_1 \wedge a_2) \vee (x \wedge y)^*$, we use the assumption to conclude that $x \wedge y \leq b$. It follows that $y \leq x \to b$, which was required. □

Note that the elements of this subframe can be described by any of the conditions in 3.2.4. By exchanging the roles of a and b in 3.2.5, we obtain a sublocale:

3.2.6. Proposition. *Let $\lesssim$ be the regular inclusion on a frame L. Then the set $S = \{b \in L;\ \forall a,\ a \lesssim b \ \Rightarrow\ a \leq b\}$ forms a sublocale of L.*

Proof: Let $b_i \in S$, for each i. Suppose that $a \lesssim \bigwedge b_i$. By upward-transitivity, we see that $a \lesssim b_i$, for each i. It follows that $a \leq b_i$, and hence $a \leq \bigwedge b_i$. We proved that $\bigwedge b_i \in S$.

Next, suppose that $b \in S$ and $a \lesssim x \to b$. We need to prove that $a \leq x \to b$. It suffices to show that $a \wedge x \lesssim b$. Suppose that $y^* \vee (a \wedge x) = 1$. Then, as $y^* \vee a = 1$, from the assumption we infer that $y \leq x \to b$. Since $y^* \vee x = 1$, we have $y \leq x$, and hence $y \leq b$. We proved that $a \wedge x \lesssim b$, as required. □

3.3. The subfit and the fit inclusion

In this chapter, we introduce another two weak inclusions and prove their properties.

3.3.1. The subfit inclusion. Let L be a frame. The *subfit inclusion* $\lesssim_{\mathsf{Subfit}}$ is given by

$$a \lesssim_{\mathsf{Subfit}} b \quad \text{iff} \quad \forall c,\ c \vee a = 1 \ \Rightarrow\ c \vee b = 1.$$

A frame L is subfit if and only if the subfit inclusion on L coincides with $\leq$. A frame is called *weakly subfit* when $T_{\lesssim}(L)$ is dense in L.

3.3.2. Lemma. *If a, b are elements of a frame L, the following statements are equivalent:*

(1) $\forall c,\ a \vee c = 1 \ \Rightarrow\ b \vee c = 1.$

(2) $\forall c,\ a \vee c = 1,\ b \leq c \ \Rightarrow\ c = 1.$

(3) $\forall c,\ a \vee c = 1 \ \Rightarrow\ b \to c \leq c.$

Proof: The implication (1)⇒(2) is trivial. (2)⇒(3): Let $a \vee c = 1$ and $x \leq b \to c$. It follows that $a \vee (x \to c) = 1$ and $b \leq x \to c$. Using (2), we obtain $x \leq c$. (3)⇒(1): Let $a \vee c = 1$. Using (3), as $a \vee (b \vee c) = 1$ and $b \leq b \vee c$, we conclude that $b \to (b \vee c) \leq b \vee c$. Since $b \to (b \vee c) = 1$, the result follows. □

3.3.3. Proposition. *The subfit inclusion on a frame L is a transitive, $\wedge$-preserving, and $\vee$-preserving weak inclusion. Moreover, $1 \lesssim b$ implies $b = 1$.*

Proof: Transitivity is trivial. For the second property, suppose that $a \lesssim b_i$ for $i = 1, 2$, and let $a \vee c = 1$, $b_1 \wedge b_2 \leq c$. Then $a \vee (b_2 \to c) = 1$, $b_1 \leq b_2 \to c$. Since $a \lesssim b_1$, it follows that $b_2 \leq c$. Then, as $a \lesssim b_2$, we get $c = 1$.

Finally suppose that $a_i \lesssim b$ for $i = 1, 2$, and let $(a_1 \vee a_2) \vee c = 1$ and $b \leq c$. Then $a_1 \vee (a_2 \vee c) = 1$ and $b \leq a_2 \vee c$. Since $a_1 \lesssim b$, it follows that $a_2 \vee c = 1$. As $a_2 \lesssim b$, we get $c = 1$. □

3.3.4. The fit inclusion. The *fit inclusion* on a frame L is a weak inclusion $\lesssim_{\mathsf{Fit}}$ given by

$$a \lesssim_{\mathsf{Fit}} b \quad \text{iff} \quad \forall x,\ a \vee c = 1 \ \Rightarrow\ c \to b \leq b.$$

A frame L is fit if and only if the fit inclusion on L coincides with $\leq$.

3.3.5. Proposition. *The fit inclusion is a $\bigwedge$-preserving, downward-transitive weak inclusion, which satisfies the following conditions.*

(1) *If $a \lesssim b$, then $a \vee b \lesssim b$,*

(2) *If $a \lesssim b$, then $a \lesssim x \to b$, for each $x \in L$,*

(3) *If $1 \lesssim b$, then $b = 1$.*

Proof: First, it is a weak inclusion, since if $a \leq b$ and $a \vee c = 1$, then $b \vee c = 1$, and hence $c \to b = b$. Second, let us show that it is downward-transitive. If $a_1 \leq a_2 \lesssim b$ and $a_1 \vee c = 1$, then $a_2 \vee c = 1$, and so $c \to b \lesssim b$. It follows that $a_1 \lesssim b$. Now the $\bigwedge$-preservation. Suppose that $a \lesssim b_i$ for each $i \in J$. If $a \vee c = 1$, then $c \to b_i \leq b_i$ for each $i \in J$. Consequently we have $c \to \bigwedge b_i = \bigwedge(c \to b_i) \leq \bigwedge b_i$, and hence $a \lesssim \bigwedge b_i$.

(1): Suppose that $a \lesssim b$ and $a \vee b \vee c = 1$. Then $b \to c \leq (b \vee c) \to b \leq b$. (2): Suppose that $a \lesssim b$. If $a \vee c = 1$, then $b \to (x \to c) = x \to (b \to c) \leq x \to c$. (3) is trivial. □

The proposition 3.3.5 asserts that there is a sublocale

$$\mathfrak{c}_{\mathsf{Fit}}(a) = \{b;\ a \lesssim_{\mathsf{Fit}} b\} = \bigcap\{\mathfrak{o}(b);\ \mathfrak{c}(a) \subseteq \mathfrak{o}(b)\} \supseteq \mathfrak{c}(a).$$

Therefore, a frame is fit if and only if $\forall a,\ \mathfrak{c}_{\mathsf{Fit}}(a) \subseteq \mathfrak{c}(a)$.

3.3.6. Proposition. *Let a, b be elements of a frame L. The following propositions are equivalent.*

(1) $a \lesssim_{\mathsf{Subfit}} b$

(2) $\forall c,\ a \lesssim_{\mathsf{Fit}} b \vee c$

(3) $\mathfrak{c}(b) \subseteq \mathfrak{c}_{\mathsf{Fit}}(a)$

Proof: (1)⇒(2): If $a \lesssim_{\mathsf{Subfit}} b$, then $a \lesssim_{\mathsf{Subfit}} b \vee c$, and so $a \lesssim_{\mathsf{Fit}} b \vee c$. (2)⇒(1): Suppose that $a \vee c = 1$. Then from (2) we infer that $1 = a \vee c \lesssim_{\mathsf{Fit}} b \vee c$. It follows that $a \lesssim_{\mathsf{Subfit}} b$. The equivalence (2)⇔(3) is trivial. □

3.3.7. Proposition. *The fit inclusion is the largest weak inclusion which satisfies conditions (1), (2), and (3) from* 3.3.5.

Proof: By 3.3.5, the fit inclusion satisfies the listed properties. Suppose that they are satisfied by a weak inclusion $\lesssim$. Suppose that $a \lesssim b$, $a \vee c = 1$, and $x \leq b \to c$. Then $b \leq x \to c$, and so $a \lesssim x \to c$. Therefore, we have $1 = a \vee c \lesssim x \to c$. It follows that $1 \leq x \to c$, and hence $x \leq c$. We proved that $a \lesssim_{\mathsf{Fit}} b$. □

3.3.8. Proposition. *The subfit inclusion is the largest upward-transitive weak-inclusion which satisfies conditions (1), (2), and (3) from* 3.3.5.

Proof: By 3.3.3, the conditions are satisfied by the subfit inclusion. On the other hand, let a weak inclusion $\lesssim$ satisfy the listed properties. Suppose that $a \lesssim b$, $a \vee c = 1$, and $b \leq c$. Since $\lesssim$ is upward-transitive, we have $a \lesssim c$. By the last property, it follows that $1 = a \vee c \lesssim c$. Finally, we conclude that $c = 1$, because $\lesssim$ is codense. We proved that $a \lesssim_{\mathsf{Subfit}} b$. □

3.4. Sublocales induced by weak inclusions

Given a binary relation on a frame, we are usually interested in the congruence it generates. Such congruence is obtained as the least congruence which extends the binary relation in question. In the context of weak inclusions this makes no sense, since $0 \lesssim 1$. Therefore we rather think of weak inclusion as generalized nuclei.

3.4.1. Definitions. Let $\lesssim$ be a weak inclusion on a frame L. The *nucleus* $\nu_\lesssim$ *induced by* $\lesssim$ is the smallest nucleus on L such that for each a, b in L,

$$a \lesssim b \quad \text{implies} \quad a \leq \nu_\lesssim(b),$$

The congruence associated to $\nu_\lesssim$ is called the *congruence induced by* $\lesssim$. The sublocale associated to $\nu_\lesssim$ is called the *sublocale induced by* $\lesssim$, and we denote it by $T_\lesssim(L)$.

The *prenucleus* $\pi_\lesssim$ *induced by* $\lesssim$ is the smallest prenucleus on L such that for each a, b in L,

$$a \lesssim b \quad \text{implies} \quad a \leq \pi_\lesssim(b),$$

3.4.2. Proposition. *Let $\lesssim$ be a $\wedge$-preserving weak inclusion on L. Then the congruence induced by $\lesssim$ is the smallest congruence E such that for each a, b in L,*

$$a \lesssim b \;\&\; b \lesssim a \;\Rightarrow\; aEb.$$

Proof: Let $\nu_\lesssim$ be the nucleus induced by $\lesssim$, and denote by ν_E the nucleus associated with E. If $a \lesssim b$ and $b \lesssim a$, then $\nu_\lesssim(a) = \nu_\lesssim(b)$, and therefore $\nu_E \leq \nu_\lesssim$.
It remains to show the opposite inequality. Suppose that $a \lesssim b$. Since $\lesssim$ is $\wedge$-preserving, we have $a \lesssim a \wedge b$, and also $a \wedge b \lesssim a$. Therefore we conclude that $aE(a \wedge b)$, that is, $\nu_E(a) = \nu_E(a \wedge b) = \nu_E(a) \wedge \nu_E(b)$, and so $a \leq \nu_E(b)$. By the definition of $\nu_\lesssim$, it follows that $\nu_\lesssim \leq \nu_E$. □

3.4.3. Proposition. *Let L be a frame. Let $\lesssim$ be a weak inclusion on L which is $\wedge$-preserving, downward-transitive, and upward-transitive. Then*

$$\pi_\lesssim(b) = \bigvee\{a;\ a \lesssim b\} \quad \textit{and} \quad T_\lesssim(L) = \{b;\ \forall a,\ a \lesssim b \ \Rightarrow\ a \leq b\}.$$

Proof: Since $\bigvee\{a;\ a \lesssim b\} \leq \pi_{\lesssim}(b)$, it is enough to show that the mapping $b \mapsto \bigvee\{a;\ a \lesssim b\}$ is a prenucleus. Since $\lesssim$ is a weak inclusion, we have $b \leq \bigvee\{a;\ a \lesssim b\}$. Using upward transitivity, we see that if $b_1 \leq b_2$, then $\bigvee\{a;\ a \lesssim b_1\} \leq \bigvee\{a;\ a \lesssim b_2\}$. Finally, using frame distributivity and the fact that $\lesssim$ is $\wedge$-preserving and downward-transitive, we have $\bigvee\{a;\ a \lesssim b\} \wedge x = \bigvee\{a \wedge x;\ a \lesssim b\} \leq \bigvee\{a \wedge x;\ a \wedge x \lesssim b \wedge x\} \leq \bigvee\{y;\ y \lesssim b \wedge x\}$. □

3.4.4. Proposition. *Under the mapping $\lesssim \mapsto \pi_{\lesssim}$, which assigns to each weak inclusion the prenucleus induced by $\lesssim$, the following statements hold:*

(1) *Prenuclei are in one-one correspondence with those weak inclusions which are $\bigvee$-preserving, $\wedge$-preserving, downward-transitive and upward-transitive.*

(2) *Nuclei on L are in one-one correspondence with the weak inclusions which are $\bigvee$-preserving, $\wedge$-preserving, and transitive.*

Proof: By 3.4.3, $\pi_{\lesssim}$ is a prenucleus. Consider a mapping $\pi \mapsto \lesssim_{\pi}$, which assigns to each prenucleus a weak inclusion such that $a \lesssim_{\pi} b$ iff $a \leq \pi(b)$. We show that the mappings $\lesssim \mapsto \pi_{\lesssim}$ and $\pi \mapsto \lesssim_{\pi}$ are mutually inverse when restricted to domains from (1) and (2), respectively:

(1): If π is prenucleus, then $\lesssim_{\pi}$ is clearly $\bigvee$-preserving and downward-transitive. Since π is monotone, $\lesssim_{\pi}$ is upward-transitive. Since π is $\wedge$-preserving, so is $\lesssim_{\pi}$. Next, if π is a prenucleus, we have $\pi(a) = \bigvee\{x;\ x \leq \pi(a)\}$. On the other hand, if $\lesssim$ is upward-transitive and join-preserving, then we have $a \lesssim b$ if and only if $a \leq \bigvee\{x;\ x \lesssim b\}$.

(2): Since $\lesssim$ is $\bigvee$-preserving, we have $\pi_{\lesssim}(b) \lesssim b$, and $\pi_{\lesssim}(\pi_{\lesssim}(b)) \lesssim \pi_{\lesssim}(b)$. Using transitivity, we have $\pi_{\lesssim}(\pi_{\lesssim}(b)) \lesssim b$, and so $\pi(\pi(b)) \leq \pi(b)$. On the other hand, let π be a nucleus. We show that the weak inclusion $\lesssim_{\pi}$ is transitive. If $a \leq \pi(b)$ and $b \leq \pi(c)$, then $a \leq \pi(b) \leq \pi(\pi(c)) = \pi(c)$. □

3.5. Codense sublocales

A mapping $f : P \to Q$ between partially ordered sets is called *codense* when $f(a) = 1$ implies $a = 1$, for each $a \in P$. A sublocale S of a frame L is *codense* when $\uparrow a \cap S = \{1\}$ implies $a = 1$. This happens precisely when the nucleus $\nu_S : L \to L$ is codense.

3.5.1. Proposition. *Let $\lesssim$ be the subfit inclusion on a frame L. Then the following statements are equivalent.*

(1) *$\lesssim$ is $\bigvee$-preserving.*

(2) *The prenucleus $\pi_{\lesssim}$ is codense.*

(3) *The sublocale $T_{\lesssim}(L)$ is codense.*

Proof: If $\lesssim$ is $\bigvee$-preserving, then $\pi_{\lesssim}$ is clearly codense. Using 3.4.4, we infer that it is already a nucleus. Therefore, the implications (1)⇒(2) and (1)⇒(3) are done.

The implication (3)⇒(2) is trivial. It remains to prove the implication (2)⇒(1). Let $a_i \lesssim b$, for each i. We prove that $\bigvee a_i \lesssim b$. Suppose that $\bigvee a_i \vee c = 1$ and $b \leq c$. By upward-transitivity, we have $a_i \lesssim c$. Since $\lesssim$ is $\vee$-preserving, we have $a_i \vee c \lesssim c$. It follows that $1 = \bigvee a_i \vee c \leq \pi_{\lesssim}(c)$. By (2), we conclude that $c = 1$, which was required. □

Clearly, the subfit inclusion on a space X is $\bigvee$-preserving if and only if for each open cover $\{U_i;\ i \in I\}$ of a space X, and each $x \in X$, there is an $i \in I$ and $y \in X$ such that $\overline{\{y\}} \subseteq \overline{\{x\}} \cap U_i$.

The following facts have already been observed in [3].

3.5.2. Proposition. *If the subfit inclusion $\lesssim$ on a frame L is $\bigvee$-preserving, then the induced sublocale $T_{\lesssim}(L)$ is subfit and the induced prenucleus $\pi_{\lesssim}$ is a nucleus.*

Proof: Suppose that for $a, b \in T_{\lesssim}(L)$ it holds in $T_{\lesssim}(L)$ that $a \lesssim b$. By 3.5.1, $T_{\lesssim}(L)$ is codense, and so we have, in L, $a \lesssim b$. It follows that $a \leq b$, and hence L is subfit.

The fact that π is a nucleus follows from 3.4.4. □

3.5.3. Proposition. *If a frame L is compact, then the subfit inclusion on L is $\bigvee$-preserving.*

Proof: By 3.5.1, it suffices to prove that the induced prenucleus π is codense. If $\pi_{\lesssim}(b) = 1$, then there are $a_1, ..., a_n$ such that $a_1 \vee \cdots \vee a_n = 1$ and $a_i \lesssim b$, for each $1 \leq i \leq n$. Since the subfit inclusion is $\vee$-preserving, the result follows. □

4 Weak Implications

In the previous chapter, we have seen that if $\lesssim$ is the regular inclusion on a frame L, then the set

$$\{a \in L;\ \forall b,\ a \lesssim b\ \Rightarrow\ a \leq b\}$$

is a subframe of L. We ask whether a similar result can be derived for the fit inclusion.

Inspecting the proof of 3.2.5, we find that it is essential to be able to rewrite a formula containing a binary meet using the Heyting operation. This consideration leads us to a notion of weak implication.

We will be able to write proofs using standard frame-theoretic reasoning in a first-order language, which is enriched by weak inclusions a weak implications. One has to have in mind, though, that weak inclusions are not transitive relations. Each formula which holds for the standard Heyting operation must be carefully verified.

4.1. Weak implications

Let L be a frame. A binary operation $(- \rightsquigarrow -)$ on L is called a *weak implication* on L if for all a, b in L,

$$a \to b \leq a \rightsquigarrow b.$$

If $\rightsquigarrow$ is a weak implication on a frame, we say that a weak inclusion $\lesssim$ is *induced by implication* $\rightsquigarrow$ when

$$a \lesssim b \quad \text{iff} \quad a \rightsquigarrow b = 1.$$

If there is no danger of confusion,

the weak inclusion induced by $\rightsquigarrow$ will be always denoted by $\lesssim$.

4.1.1. Weak modus ponens. Let $\rightsquigarrow$ be a weak implication on a frame L. We say that $\rightsquigarrow$ satisfies the *weak modus ponens* when

$$a \wedge (a \rightsquigarrow b) \lesssim b.$$

We show that when $\rightsquigarrow$ satisfies the weak modus ponens, the sublocale induced by $\lesssim$ can be described directly using $\rightsquigarrow$.

4.1.2. Proposition. *Let $\rightsquigarrow$ be a weak implication on a frame L. Then the nucleus $\nu_{\lesssim}$ induced by $\lesssim$ is the least nucleus such that*

$$a \rightsquigarrow b \leq a \to \nu_{\lesssim}(b).$$

Proof: Denote by $\nu_{\rightsquigarrow}$ the least nucleus which satisfies $a \rightsquigarrow b \leq a \to \nu_{\rightsquigarrow}(b)$. If $a \lesssim b$, then $a \leq a \rightsquigarrow b \leq a \to \nu_{\rightsquigarrow}(b)$, that is, $a \leq \nu_{\rightsquigarrow}(b)$. It follows that $\nu_{\lesssim} \leq \nu_{\rightsquigarrow}$. On the other hand, since $a \wedge (a \rightsquigarrow b) \lesssim b$, we have $a \wedge (a \rightsquigarrow b) \leq \nu_{\lesssim}(b)$. We conclude that $a \rightsquigarrow b\ \Rightarrow\ a \to \nu_{\lesssim}(b)$, and so $\nu_{\rightsquigarrow} \leq \nu_{\lesssim}$. □

4.1.3. Proposition. *Let $\rightsquigarrow$ be a weak implication which satisfies the weak modus ponens. If b is an element of L, then the following conditions are equivalent.*

(1) $\forall a,\ a \lesssim b \ \Rightarrow\ a \leq b.$

(2) $\forall a,\ a \rightsquigarrow b \leq a \to b.$

Proof: The implication from (2) to (1) is trivial. It remains to prove (1)⇒(2). Let a be arbitrary. Since $a \wedge (a \rightsquigarrow b) \lesssim b$, we have $a \wedge (a \rightsquigarrow b) \leq b$. Consequently $a \rightsquigarrow b \leq a \to b$. □

4.2. The first weak Heyting formula

Let $\rightsquigarrow$ be a weak implication on a frame L. We say that $\rightsquigarrow$ satisfies the *first weak Heyting formula* if for all y, a, b in L,

$$y \wedge a \lesssim b \quad \text{iff} \quad y \lesssim a \rightsquigarrow b.$$

The formula resembles the standard adjunction formula of the Heyting operation. In general, however, we cannot speak of an adjunction, because the relation $\lesssim$ need not be transitive. Even if it were, the mapping $(a \rightsquigarrow -) : (L, \lesssim) \to (L, \lesssim)$ is not always monotone. Some usual facts can be derived nevertheless.

4.2.1. Proposition. *If $\rightsquigarrow$ is a weak implication satisfying the first weak Heyting formula on a frame L, then:*

(1) *$\lesssim$ is downward-transitive,*

(2) *$y \lesssim a \rightsquigarrow b$ iff $a \lesssim y \rightsquigarrow b$,*

(3) *$y \leq a \rightsquigarrow (y \wedge a)$,*

(4) *$a \wedge (a \rightsquigarrow b) \lesssim b$ (weak modus ponens).*

Proof: (1): Suppose that $x \leq a$ and $a \lesssim b$. Since $x \lesssim 1 = a \rightsquigarrow b$, we have $x = x \wedge a \lesssim b$. Parts (2), (3), and (4) follows immediately from the formula. □

Formulas (3) and (4) in 4.2.1 resembles the adjunction units formulas. In general, however, they are weaker then the first weak Heyting formula (see, e.g., 4.4.2).

4.2.2. Proposition. *Let ν be a nucleus on a frame L. Then the weak implication $\rightsquigarrow$ given by $a \rightsquigarrow b = a \to \nu(b)$ satisfies the first weak Heyting formula.*

Proof: It satisfies the first weak Heyting formula, because we have $y \wedge a \leq \nu(b)$ iff $y \leq a \to \nu(b) = \nu(a \to \nu(b))$. □

4.2.3. The regular implication. Let L be frame. The *regular implication* $\rightsquigarrow_{\mathsf{Reg}}$ *on* L is a weak implication given by

$$a \rightsquigarrow_{\mathsf{Reg}} b = \bigvee\{c;\ a \vee c^* = 1\} \to b = \bigwedge\{c \to b;\ a \vee c^* = 1\}.$$

Clearly, it is a weak implication, and it induces the regular inclusion. The fact that it satisfies the first weak Heyting formula is a consequence of a more general fact:

4.2.4. Lemma. *Let $\rightsquigarrow$ be a weak implication on a frame L. Suppose that there is mapping $\lambda : L \to L$ such that $a \rightsquigarrow b = \lambda(a) \to b$. Then $\rightsquigarrow$ satisfies the first weak Heyting formula if and only if $\lambda(a) \wedge \lambda(b) = \lambda(a \wedge b)$, for each $a, b \in L$.*

Proof: The fact that $\rightsquigarrow$ satisfies the weak Heyting formula can be rewritten as an equivalence $\lambda(y \wedge a) \leq b$ iff $\lambda(y) \leq \lambda(a) \to b$. The result easily follows. □

4.2.5. Proposition. *The regular implication satisfies the first weak Heyting formula.*

Proof: We use 4.2.4. Set $\lambda(a) = \bigvee\{c;\ a \vee c^* = 1\}$. First, λ is monotone. Second, using the frame distributivity, we have $\lambda(a) \wedge \lambda(b) = \bigvee\{c \wedge d;\ a \vee c^* = 1,\ b \vee d^* = 1\} \leq \bigvee\{c \wedge d;\ a \vee (c \wedge d)^* = 1,\ b \vee (c \wedge d)^* = 1\} \leq \bigvee\{c \wedge d;\ (a \wedge b) \vee (c \wedge d)^* = 1\} \leq \lambda(a \wedge b)$. □

4.2.6. The fit implication. Let L be frame. The *fit implication* $\rightsquigarrow_{\mathsf{Fit}}$ *on* L is a weak implication given by

$$a \rightsquigarrow_{\mathsf{Fit}} b = \bigvee\{c \to b;\ a \vee c = 1\} \to b = \bigwedge\{(c \to b) \to b;\ a \vee c = 1\}.$$

Clearly, it induces the fit inclusion.

4.2.7. Proposition. *The fit implication satisfies the first weak Heyting formula.*

Proof: Suppose $y \lesssim a \rightsquigarrow b$; we shall prove that $y \wedge a \lesssim b$. Let $(y \wedge a) \vee c = 1$; we want to show that that $c \to b \leq b$. Since $y \vee c = 1$, we have, using the assumption, $c \to (a \rightsquigarrow b) \leq a \rightsquigarrow b$. From this formula, because of $a \vee c = 1$, we obtain $c \to b \leq (c \to (a \rightsquigarrow b)) \to b$. Thus, $c \to b \leq b$, since $c \leq a \rightsquigarrow b$.

For the other implication, assume that $y \wedge a \lesssim b$, and we claim that $y \lesssim a \rightsquigarrow b$. Let $c \vee y = 1$; it suffices to show that $c \to (a \rightsquigarrow b) \leq a \rightsquigarrow b$. For that, let $d \vee a = 1$, and we need to prove that $d \to b \leq (c \to (a \rightsquigarrow b)) \to b$, that is, $d \to b \wedge (c \to (a \rightsquigarrow b)) \leq b$. So let $x \leq d \to b$ and $x \leq c \to (a \rightsquigarrow b)$; it is enough to show $x \leq b$. We have $x \wedge c \leq a \rightsquigarrow b$. Since $d \vee a = 1$, we have $d \to b \leq (x \wedge c) \to b$. Thus, $x \leq (x \wedge c) \to b$, and $x \leq c \to b$. Summed up, we have obtained $x \leq (c \to b) \wedge (d \to b)$.

Finally, using the original assumption $y \wedge a \lesssim b$ and the fact that $y \vee (c \vee d) = 1$ and $a \vee (c \vee d) = 1$, we obtain $(c \vee d) \to b \leq b$, that is, $(c \to b) \wedge (d \to b) \leq b$. Now, it follows that $x \leq b$. □

4.3. The second weak Heyting formula

Let $\rightsquigarrow$ be a weak implication on a frame L. We say that $\rightsquigarrow$ satisfies *the second weak Heyting formula* when

$$a \lesssim y \to b \quad \text{iff} \quad y \leq a \rightsquigarrow b.$$

4.3.1. Proposition. *If $\rightsquigarrow$ satisfies the second weak Heyting formula, then*

(1) $a \rightsquigarrow b = \bigvee\{y;\ a \lesssim y \to b\}$.

(2) $y \leq (y \to b) \rightsquigarrow b$,

(3) $a \lesssim (a \rightsquigarrow b) \to b$.

Proof: (1): Clearly, we have $\bigvee\{y;\ a \lesssim y \to b\} \leq a \rightsquigarrow b$. To prove the other inequality, suppose that $z \leq a \rightsquigarrow b$. It follows that $a \lesssim z \to b$, and hence $z \leq \bigvee\{y;\ a \lesssim y \to b\}$. Statements (2) and (3) are trivial. □

4.3.2. Proposition. *Let $\lesssim$ be a $\bigwedge$-preserving weak inclusion which satisfies the formula $a \lesssim b \ \Rightarrow\ a \lesssim x \to b$. Then it is induced by a weak implication which satisfies the second weak Heyting formula.*

Proof: Set $a \rightsquigarrow b = \bigvee\{y;\ a \lesssim y \to b\}$. The implication $a \lesssim y \to b \ \Rightarrow\ y \leq a \rightsquigarrow b$ it trivial. Let us prove the opposite one. First, we observe that since $\lesssim$ is $\bigwedge$-preserving, we have $a \lesssim \bigwedge\{y \to b;\ a \lesssim y \to b\} = \bigvee\{y;\ a \lesssim y \to b\} \to b = (a \rightsquigarrow b) \to b$. Then, if we assume that $y \leq a \rightsquigarrow b$, we can infer that $a \lesssim y \to ((a \rightsquigarrow b) \to b) = y \to b$. □

4.3.3. Proposition. *Let ν be a nucleus on L, and let $\rightsquigarrow$ be a weak implication on L such that $a \rightsquigarrow b = a \to \nu(b)$. Then $\rightsquigarrow$ satisfies the second weak Heyting formula if and only if for all $a, b \in L$, $\nu(a) \to \nu(b) \leq \nu(a \to b)$.*

Proof: Indeed, we have $a \leq \nu(y \to b)$ iff $y \leq a \to \nu(b)$ iff $a \leq y \to \nu(b) = \nu(y) \to \nu(b)$. The inequality $\nu(a \to b) \leq \nu(a) \to \nu(b)$ is trivial. □

4.3.4. Proposition. *The regular implication satisfies the second weak Heyting formula.*

Proof: The regular implication is given by $a \rightsquigarrow_{\mathsf{Reg}} b = \lambda(a) \to b$, where $\lambda(a) = \bigvee\{c;\ a \vee c^* = 1\}$. Next, we have $\lambda(a) \leq y \to b$ iff $a \leq \lambda(a) \to b$. □

4.3.5. Proposition. *The fit implication satisfies the second weak Heyting formula.*

Proof: First, for any $c, y, b \in L$, we have $c \to (y \to b) \leq y \to b$ iff $y \to (c \to b) \leq y \to b$ iff $c \to b \leq y \to b$ iff $y \leq (c \to b) \to b$.

Then, we have $a \lesssim_{\mathsf{Fit}} y \to b$ iff $\forall c,\ a \vee c = 1 \ \Rightarrow\ c \to (y \to b) \leq y \to b$ iff $y \leq \bigwedge\{(c \to b) \to b;\ a \vee c = 1\} = a \rightsquigarrow_{\mathsf{Fit}} b$. □

4.3.6. Proposition. *Let $\rightsquigarrow$ satisfies the second weak Heyting formula. Then:*

(1) $\lesssim$ *is downward-transitive iff* $a_1 \leq a_2$ *implies* $a_2 \rightsquigarrow b \leq a_1 \rightsquigarrow b$.

(2) $\lesssim$ *is upward-transitive iff* $b_1 \leq b_2$ *implies* $a \rightsquigarrow b_1 \leq a \rightsquigarrow b_2$.

(3) $\lesssim$ *is* $\to$*-transitive iff* $a \rightsquigarrow b \leq a \rightsquigarrow (x \to b)$.

(4) $\lesssim$ *is* $\bigvee$*-preserving iff* $\bigwedge(a_i \rightsquigarrow b) \leq (\bigvee a_i) \rightsquigarrow b$.

(5) $\lesssim$ *is* $\bigwedge$*-preserving iff* $\bigwedge(a \rightsquigarrow b_i) \leq a \rightsquigarrow \bigwedge b_i$.

(6) $(a \rightsquigarrow b) \wedge y \leq a \rightsquigarrow (b \wedge y)$.

(7) $a \rightsquigarrow b \leq a \rightsquigarrow (a \to b)$.

Proof: (1): If $a_2 \leq a_1$ and $y \leq a_2 \rightsquigarrow b$, then $a_2 \lesssim y \to b$. Now, by downward-transitivity, we have $a_1 \lesssim y \to b$, hence $y \leq a_1 \rightsquigarrow b$. For the other implication, let $x \leq a \lesssim b$. Then $1 = a \rightsquigarrow b \leq x \to b$, and hence $x \lesssim b$.
(2): If $b_1 \leq b_2$ and $y \leq a \rightsquigarrow b_1$, then $a \lesssim y \to b_1$. Now, by upward-transitivity, we have $a \lesssim y \to b_2$, hence $y \leq a \rightsquigarrow b_2$. For the other implication, let $a \lesssim b \leq x$. Then $1 = a \rightsquigarrow b \leq a \rightsquigarrow b$, hence $a \lesssim x$.
(3): We have $a \rightsquigarrow b \leq a \rightsquigarrow (x \to b)$ iff $a \lesssim (a \rightsquigarrow b) \to (x \to b)$ iff $a \lesssim (x \wedge (a \rightsquigarrow b)) \to b$ iff $x \wedge (a \rightsquigarrow b) \leq a \rightsquigarrow b$, which is true. For the other implication, if $a \lesssim b$ and x is arbitrary, then $1 \leq a \rightsquigarrow b \leq a \rightsquigarrow (x \to b)$, hence $a \lesssim x \to b$.
(4): Let $\lesssim$ be $\bigvee$-preserving. If $y \leq \bigwedge_i (a_i \rightsquigarrow b)$, then $\forall i,\ y \leq a_i \rightsquigarrow b$, and $\forall i,\ a_i \lesssim a \to b$, and $\bigvee a_i \lesssim y \to b$, so $y \leq (\bigvee a_i) \rightsquigarrow b$. For the other implication, if $a_i \lesssim b$ for each i, then $1 = \bigwedge(a_i \rightsquigarrow b) = (\bigvee a_i) \rightsquigarrow b$, hence $\bigvee a_i \lesssim b$.
(5): Let $\lesssim$ be $\bigwedge$-preserving. If $y \leq \bigwedge(a \rightsquigarrow b_i)$, then $\forall i,\ y \leq a \rightsquigarrow b_i$, and $\forall i,\ a \lesssim y \to b_i$, and $a \lesssim \bigwedge y \to b_i = y \to \bigwedge b_i$, so $y \leq a \rightsquigarrow \bigwedge b_i$. On the other hand, if $a \lesssim b_i$ for each i, then $1 = \bigwedge(a \rightsquigarrow b_i) = a \rightsquigarrow \bigwedge b_i$, hence $a \lesssim \bigwedge b_i$.
(6): Let $x \leq a \rightsquigarrow b$ and $x \leq y$. Then $a \rightsquigarrow x \to b = (x \to b) \wedge (x \to y) = x \to (b \wedge y)$. Hence, $x \leq a \rightsquigarrow (b \wedge y)$.
(7): We have $a \rightsquigarrow b \leq a \rightsquigarrow (a \to b)$ iff $a \lesssim (a \rightsquigarrow b) \to (a \to b) = (a \wedge (a \rightsquigarrow b)) \to b$ iff $a \wedge (a \rightsquigarrow b) \leq a \rightsquigarrow b$, which is true. □

There is a "dual version" of 4.1.3.

4.3.7. Proposition. *Let a weak implication $\rightsquigarrow$ on a frame L satisfy the second weak Heyting formula. If a is an element of L, then the following statements are equivalent:*

(1) $\forall b,\ a \lesssim b \ \Rightarrow\ a \leq b$.

(2) $\forall b,\ a \rightsquigarrow b \leq a \to b$.

(3) $\forall b,\ a \to b \leq b \ \Rightarrow\ a \rightsquigarrow b \leq b$.

Proof: The implications (2)⇒(1) and (2)⇒(3) are trivial. (1)⇒(2): Let $y \leq a \rightsquigarrow b$. Then $a \lesssim y \to b$. Therefore, using (1), we obtain $a \leq y \to b$, hence $y \leq a \to b$. (3)⇒(2): Since $a \to (a \to b) \leq a \to b$, we have $a \rightsquigarrow (a \to b) \leq a \to b$. Then, by 4.3.6, we get $a \rightsquigarrow b \leq a \to b$. □

4.3.8. Proposition. *Let $\rightsquigarrow$ be a weak implication on a frame L satisfying the the first and the second weak Heyting formula. Then the set*

$$F(L) = \{a \in L;\ \forall b,\ a \lesssim b\ \Rightarrow\ a \leq b\}$$

is a subframe of L.

Proof: Since $\lesssim$ is downward transitive, it is closed under joins. Trivially, it contains the element 1. We show that it is closed under binary meets. Suppose that $a_1 \wedge a_2 \lesssim b$, with $a_1, a_2 \in F(L)$. Then $a_1 \lesssim a_2 \rightsquigarrow b$, and so $a_1 \leq a_2 \rightsquigarrow b$. It follows that $a_2 \lesssim a_1 \to b$, and hence $a_2 \leq a_1 \to b$. Consequently, $a_1 \wedge a_2 \leq b$. □

4.3.9. Proposition. *On each frame L, let there be a weak inclusion $\lesssim_L$ such that set $F(L) = \{a;\ \forall b;\ a \lesssim_L b\ \Rightarrow\ a \leq b\}$ is a subframe of L. Suppose that for each frame homomorphism $h : M \to L$,*

$$a \lesssim_L b \quad \textit{implies} \quad h^+(a) \lesssim_M h^+(b).$$

Then the full subcategory of all the frames L where $\lesssim_L$ coincides with $\leq$ is a coreflective subcategory of **Frm**.

Proof: First, we observe that for any frame homomorphism $h : M \to L$ such that $\lesssim_M\ = \leq$, it holds that $h[M] \subseteq F(M)$. Indeed, if $h(a) \lesssim_L b$, then $h^+h(a) \lesssim_M h^+(b)$. Since $\lesssim_M\ = \leq$, we have $a \leq h^+h(a) \leq h^+(b)$, and hence $h(a) \leq b$. It follows that $h(a) \in F(L)$.

Next, let L be a frame. Set $F^0(L) = L$. For each ordinal α, set $F^{\alpha+1}(L) = F(F^\alpha(L))$, and when α is limit, $F^\alpha(L) = \bigcap\{F^\beta(L);\ \beta < \alpha\}$. We have defined a decreasing sequence of subframes of L. Denote by $G(L)$ the first $F^\alpha(L)$ such that $F^{\alpha+1}(L) = F^\alpha(L)$. The weak inclusion $\lesssim_{G(L)}$ on $G(L)$ equals $\leq$.

We prove that the inclusions $G(L) \to L$ constitute the coreflection. Suppose that $h : M \to L$ be a frame homomorphism such that $\lesssim_M$ is equal to $\leq$. By the first part of the proof we infer that $h[M] \subseteq F^\alpha(L)$, for each α, and therefore $h[M] \subseteq G(L)$. □

4.3.10. Proposition. *Each localic map $h^+ : L \to M$ preserves the fit and the regular inclusion in the sense of* 4.3.9.

Proof: First, let us prove the statement for the fit inclusion. Let $h(a) \lesssim b$ and $a \vee c = 1$. Then $h(a) \leq h(c) = 1$, and so $h(c) \to b \leq b$. It follows that $h(c \to h^+(b)) \leq h(c) \to hh^+(b) \leq h(c) \to b \leq b$. We have proved that $a \lesssim h^+(b)$.

For the regular inclusion, the proof is similar. Let $h(a) \lesssim b$ and $a \vee c = 1$. Then $h(a) \leq h(c) = 1$, and so $h(c)^* \leq b$. It follows that $h(c^*) \leq h(c)^* \leq b$. □

Consequently, we obtained the well known fact that the full subcategory of fit frames, and the full subcategory of regular frames are coreflective subcategories of **Frm**.

4.4. The subfit implication

Let L be a frame. The *subfit implication* on L is a weak implication given by

$$a \rightsquigarrow b = \bigwedge\{c;\ a \vee c = 1,\ b \leq c\} = \bigwedge\{b \vee c;\ a \vee c = 1\}.$$

It is a weak implication, because for each c such that $a \vee c = 1$ and $b \leq c$, we have $a \to b \leq a \to c = c$, and so $a \to b \leq a \rightsquigarrow b$. Clearly, it induces the subfit inclusion.

4.4.1. Proposition. *The subfit implication satisfies:*

(1) *If $a \vee c = 1$, then $a \rightsquigarrow c \leq c$*

(2) $(a \rightsquigarrow b_1) \wedge (a \rightsquigarrow b_2) = a \rightsquigarrow (b_1 \wedge b_2)$.

(3) $y \lesssim a \rightsquigarrow b \ \Rightarrow\ y \wedge a \lesssim b$.

(4) $a \lesssim y \to b \ \Rightarrow\ y \leq a \rightsquigarrow b$.

(5) $a \wedge (a \rightsquigarrow b) \lesssim b$ *(weak modus ponens).*

Proof: (1) is trivial. (2): First, let $b_1 \leq b_2$. We show that $a \rightsquigarrow b_1 \leq a \rightsquigarrow b_2$. If $y \leq a \rightsquigarrow b_1$, $a \vee c = 1$ and $b_2 \leq c$, then $b_1 \leq c$. Consequently, $y \leq c$. Next, let $y \leq a \rightsquigarrow b_1$ and $y \leq a \rightsquigarrow b_2$. Suppose that $a \vee c = 1$ and $b_1 \wedge b_2 \leq c$. Since $a \vee (b_2 \to c) = 1$ and $b_1 \leq b_2 \to c$, it follows that $y \leq b_2 \to c$. Therefore $a \vee (y \to c) = 1$ and $b_2 \leq y \to c$, and so $y \leq y \to c$. As $y \leq c$, we proved that $y \leq a \rightsquigarrow (b_1 \wedge b_2)$.

(3): Let $a \lesssim y \to b$, $a \vee c = 1$ and $b \leq c$. We shall prove that $y \leq c$. Since $a \vee c = 1$, we have $c \vee (y \to b) = 1$. Since $b \leq c$, we have also $c \vee (y \to c) = 1$ and hence $1 = (c \to (y \to c) \leq y \to c$. This yields $y \leq c$.

(4): Suppose that $a \lesssim y \to b$, $a \vee c = 1$, and $b \leq c$. We have $a \lesssim y \to c$, and therefore $y \to c = 1$. It follows that $y \leq c$. (5) Follows from (3). □

4.4.2. Example. *There is a weakly subfit compact space X such that the subfit implication on X satisfies neither the first nor the second weak Heyting formula.*

Set $X = [-1, 1]$, the interval on the real line. A subset U of X is open if it is open in the standard euclidean topology and for each $x \in U \cap (0, 1)$, there is an $\varepsilon \geq 0$ such that $(-\varepsilon, x) \setminus \{0\} \subseteq U$. It is clearly a T_0 topology, and the space X is compact. For the further development, it is useful to realize that if $x \in (0, 1]$, then $\overline{\{x\}} = [x, 1]$, and each point $x \in [-1, 0]$ is closed. It is easy to see that X is weakly subfit.

We prove that the subfit implication does not satisfy the weak Heyting operation. Set $a = [-1, 1] \setminus \{0\}$, $y = [-1, 1)$, and $b = [-1, 0)$. Then, we have $a \wedge y = [-1, 1) \setminus \{0\} \lesssim b$. For each natural $n \geq 1$, it holds that $([-1, 1] \setminus \{0\}) \rightsquigarrow [-1, 0) \leq [-1, 1/n)$. It follows that $a \rightsquigarrow b = [-1, 0)$. At the same time, we have $[-1, 1) \not\leq [-1, 0)$, and the result follows.

Next, we show that the subfit implication on X does not satisfy the second weak Heyting formula. Set $a = [-1, 1)$, $y = (-1, 1]$, and $b = (-1, 0)$. First, we have $a \rightsquigarrow b = y$. Second, we have $y \to b = [-1, 0)$. Therefore, we see that $a \not\leq y \to b$. It follows that $\rightsquigarrow$ does not satisfy the second weak Heyting formula. □

4.4.3. The fact that the regular inclusion satisfies the second weak Heyting formula stems from the property that the it is a $\bigwedge$-preserving weak inclusion. On the other hand, we will show that the subfit implication does not need to satisfy the second weak Heyting formula even when it is $\bigwedge$-preserving. Thus, by 4.3.2 the subfit inclusion may be induced by a weak implication which satisfies the second weak Heyting formula, but this weak implication may be different from the subfit implication.

4.4.4. Proposition. *If the subfit inclusion on a frame L is $\bigwedge$-preserving, then it satisfies the first weak Heyting formula.*

Proof: By 4.4.1(3), we need to prove that $a \wedge y \lesssim b$ implies $y \lesssim a \rightsquigarrow b$. Suppose that $a \wedge y \lesssim b$, $a \vee c = 1$, and $b \leq c$. We prove that $y \lesssim c$ and then the result will follow using $\bigwedge$-preservation. Since the subfit inclusion is upward-transitive and $\vee$-preserving, we have $(y \wedge a) \vee c \lesssim c$. Therefore, $y = y \wedge (a \vee c) \leq (y \wedge a) \vee c \lesssim c$. □

4.4.5. Proposition. *Let L be both a frame and a coframe, and let $\rightsquigarrow$ be the subfit implication on L. Then $\rightsquigarrow$ satisfies the second weak Heyting formula if and only if $\forall a,\ a^{\diamond\diamond} \wedge a^{\diamond} = 0$, where $a^{\diamond} = a \rightsquigarrow 0$.*

Proof: Since L is a coframe, we observe that $a \rightsquigarrow b = \bigwedge\{b \vee c;\ a \vee c = 1\} = b \vee \bigwedge\{c;\ a \vee c = 1\} = a^{\diamond} \vee b$. Also, we have $a \lesssim b$ iff $1 \leq a^{\diamond} \vee b$ iff $a^{\diamond\diamond} \leq b$.

We infer that the subfit implication satisfies the second weak Heyting formula if and only if $\forall a, b, y,\ y \leq a \rightsquigarrow b \ \Rightarrow\ a \lesssim y \to b$ if and only if $\forall a, b, y,\ y \leq a^{\diamond} \vee b \ \Rightarrow\ y \leq a^{\diamond\diamond} \to b$ if and only if $\forall a, b, y,\ a^{\diamond} \vee b \leq a^{\diamond\diamond} \to b$ if and only if $\forall a, b,\ a^{\diamond} \wedge a^{\diamond\diamond} \leq b$ if and only if $\forall a,\ a^{\diamond} \wedge a^{\diamond\diamond} \leq 0$. □

4.4.6. Example. *There is a frame L where the subfit inclusion is $\bigwedge$-preserving but it does not satisfy the second weak Heyting formula.*

Consider a three-element set $P = \{p, q, t\}$ ordered such that $p < t$ and $q < t$. Set L to be the frame of up-sets on P. It is also a coframe. Moreover, $\{p, t\}^{\diamond} = \{q, t\}$ and $\{q, t\}^{\diamond} = \{p, t\}$. Therefore, we have $\{p, t\}^{\diamond\diamond} \wedge \{p, t\}^{\diamond} = \{t\} \neq \emptyset$. The result follows from 4.4.5. □

4.4.7. Proposition. *Let $a, b \in L$. Then $a \lesssim_{\mathsf{Subfit}} b$ if and only if*

$$\exists y,\ \mathfrak{o}(a) \cap \mathfrak{c}(b) \subseteq \mathfrak{o}(y) \cap \mathfrak{c}_{\mathsf{Fit}}(y).$$

Proof: Suppose that $a \lesssim_{\mathsf{Subfit}} b$. By 3.3.6, we have $\mathfrak{c}(b) \subseteq \mathfrak{c}_{\mathsf{Fit}}(a)$, and therefore $\mathfrak{o}(a) \cap \mathfrak{c}(b) \subseteq \mathfrak{o}(a) \cap \mathfrak{c}_{\mathsf{Fit}}(a)$. To prove the opposite implication, we will use 3.3.6. Suppose that $b \leq x$, we prove that $a \lesssim_{\mathsf{Fit}} x$. Since $\mathfrak{o}(a) \cap \mathfrak{c}(b) \subseteq \mathfrak{o}(y)$, we have $a \leq y \vee b \leq y \vee x$. Next, since $b \leq a \to x$, we have $y \lesssim_{\mathsf{Fit}} a \to x$, and therefore $a \leq y \rightsquigarrow_{\mathsf{Fit}} x$. Thus, we conclude that $a \leq (y \vee x) \wedge (y \rightsquigarrow_{\mathsf{Fit}} x) = (y \wedge (y \rightsquigarrow_{\mathsf{Fit}} x)) \vee x$. Using the weak modus ponens it follows that $a \lesssim_{\mathsf{Fit}} x$. □

In particular, from the last proposition it follows that L is weakly subfit if and only if $\forall y,\ \mathrm{int}(\mathfrak{c}_{\mathsf{Fit}}(y) \cap \mathfrak{o}(y)) = \emptyset$. This equivalence was already observed in [10].

4.5. Sublocales related to weak implications

The Heyting operation on a frame gives rise to open sublocales. In this section we study an analogous statement about weak Heyting operations. Let us begin with a simple lemma.

4.5.1. Lemma. *If $\rightsquigarrow$ is a weak implication on a frame L, then the mapping $(a \rightsquigarrow -)$ is a nucleus on L if and only if both*

(1) $b_1 \leq b_2 \;\Rightarrow\; a \rightsquigarrow b_1 \leq a \rightsquigarrow b_2$, *and*

(2) $x \wedge (a \rightsquigarrow b) \leq a \rightsquigarrow (x \wedge b)$.

Proof: The left-to-right implication is trivial. To prove that $(a \rightsquigarrow -)$ is a nucleus, it remains to check the following two conditions. First, the axiom $b \leq a \rightsquigarrow b$ is satisfied by each weak inclusion. Second, since $(a \rightsquigarrow (a \rightsquigarrow b)) \wedge (a \rightsquigarrow b) = a \rightsquigarrow ((a \rightsquigarrow b) \wedge b) = a \rightsquigarrow b$, it follows that $a \rightsquigarrow (a \rightsquigarrow b) \leq a \rightsquigarrow b$. □

In particular, if $(a \rightsquigarrow -)$ is a nucleus, the induced weak inclusion has to be upward-transitive and $\wedge$-preserving.

4.5.2. If $(a \rightsquigarrow -)$ is a nucleus, the associated sublocale is denoted by $\mathfrak{o}_{a\rightsquigarrow}(a)$. Therefore, we have

$$\mathfrak{o}_{a\rightsquigarrow}(a) = \{b;\ a \rightsquigarrow b \leq b\} = \{a \rightsquigarrow b;\ b \in L\} \subseteq \mathfrak{o}(a).$$

4.5.3. Proposition. *Let $\rightsquigarrow$ be a weak implication satisfying the weak modus ponens such that $(a \rightsquigarrow -)$ is a nucleus. Then, $T_{\lesssim}(L) \cap \mathfrak{o}(a) \subseteq \mathfrak{o}_{a\rightsquigarrow}(a) \subseteq \mathfrak{o}(a)$.*

Proof: If $b \in T_{\lesssim}(L) \cap \mathfrak{o}(a)$, then $a \rightsquigarrow b = a \to b = b$, and therefore $b \in \mathfrak{o}_{a\rightsquigarrow}(a)$. □

Let ν be a nucleus on a frame L. The weak implication $a \rightsquigarrow b = a \to \nu(b)$ is $\wedge$-preserving: We have $a \to \nu(b_1 \wedge b_2) = a \to (\nu(b_1) \wedge \nu(b_2)) = (a \to \nu(b_1)) \wedge (a \to \nu(b_2))$. Clearly, $\mathfrak{o}_{a\rightsquigarrow}(a) \subseteq T_{\lesssim}(L)$. Hence, by the previous proposition, we have $\mathfrak{o}_{a\rightsquigarrow}(a) = T_{\lesssim}(L) \cap \mathfrak{o}(a)$.

4.5.4. Let $\rightsquigarrow$ be the subfit implication. By 4.4.1, the mapping $(a \rightsquigarrow -)$ is a nucleus. As $a \rightsquigarrow b = \bigwedge\{b \vee c;\ \mathfrak{c}(c) \subseteq \mathfrak{o}(a)\}$, it induces the familiar sublocales

$$\mathfrak{o}_{a\rightsquigarrow}(a) = \bigvee\{\mathfrak{c}(c);\ \mathfrak{c}(c) \subseteq \mathfrak{o}(a)\}.$$

Also observe that $\mathfrak{o}_{a\rightsquigarrow}(1) = L$. Hence the subfit implication is a weak inclusion satisfying the weak modus ponens such that the opposite inclusion of 4.5.3 does not hold.

4.5.5. If $\rightsquigarrow$ is the regular implication, then $(a \rightsquigarrow -)$ is a nucleus and

$$\mathfrak{o}_{\rightsquigarrow}(a) = \mathfrak{o}(\lambda(a)) = \bigvee\{\mathfrak{o}(x);\ \overline{\mathfrak{o}(x)} \subseteq \mathfrak{o}(a)\},$$

where $\lambda(a) = \bigvee\{c;\ c^* \vee a = 1\}$.

Since the fit inclusion is not generally upward-transitive, it does not seem to admit weak open sublocales.

4.5.6. Proposition. *Suppose that a weak implication $\rightsquigarrow$ on a frame L satisfies the weak modus ponens. If $(a \rightsquigarrow -)$ is a nucleus, then*

$$T_{\lesssim}(L) = \bigcap\{\mathfrak{c}(a) \vee \mathfrak{o}_{\rightsquigarrow}(a);\ a \in L\}.$$

Proof: $\subseteq$: Let b an element of $T_{\rightsquigarrow}(L)$ and $a \in L$ be arbitrary. Since $a \rightsquigarrow b = a \to b$ and $b = (a \vee b) \wedge (a \to b)$, it follows that $b = (a \vee b) \wedge (a \rightsquigarrow b) \in \mathfrak{c}(a) \vee \mathfrak{o}_{\rightsquigarrow}(a)$.
$\supseteq$: Suppose that $b \in \bigcap\{\mathfrak{c}(a) \vee \mathfrak{o}_{\rightsquigarrow}(a);\ a \in L\}$, that is, $b = x \wedge (a \rightsquigarrow y)$ with $a \leq x$. Then, we have $a \rightsquigarrow b = a \rightsquigarrow (x \wedge (a \rightsquigarrow y)) = (a \rightsquigarrow x) \wedge (a \rightsquigarrow (a \rightsquigarrow y)) = 1 \wedge a \rightsquigarrow y = a \rightsquigarrow y \leq x \to b \leq a \to b$. It follows that $b \in T_{\rightsquigarrow}(L)$. □

In the coframe of sublocales of a frame L, open sublocales are complemented by closed ones.

4.5.7. Proposition. *Let L be a frame with the subfit implication. Then the sublocale $\mathfrak{c}(a) \cap \mathfrak{o}_{\rightsquigarrow}(a)$ is the void sublocale, and $\mathfrak{c}(a) \vee \mathfrak{o}_{\rightsquigarrow}(a)$ is codense in L.*

Proof: The first part of the proposition trivial, since $\mathfrak{o}_{\rightsquigarrow}(a) \subseteq \mathfrak{o}(a)$. To prove the second part, let z be an element of L. Set $b = (a \vee z) \wedge (a \rightsquigarrow z) \in \mathfrak{c}(a) \vee \mathfrak{o}_{\rightsquigarrow}(a)$. Assume that $b = 1$, that is, $a \vee z = 1$ and $a \lesssim z$. Then it follows that $z = 1$, and hence the sublocale is codense. □

We already know that the sublocale induced by the subfit inclusion is contained in every codense sublocale. Now, we proved the opposite.

4.5.8. Corollary. *The sublocale induced by the subfit inclusion on a frame L is the intersection of all the codense sublocales of L.* □

5 More on the Subfit Inclusion

In the first part of this chapter, we introduce a weak inclusion on the congruence frame of a frame L. We show that it is tightly connected with subfitness and fitness. Furthermore, when given a sublocale S, we characterize the sublocale $T_{\mathsf{Subfit}}(S)$ as an intersection of certain sublocales of L, which generalizes 4.5.8.

In the second part, we present a property of topological spaces, which can be compared to subfitness.

5.1. Subfit inclusion and sublocales

5.1.1. Let S be a sublocale of a frame L. We define a weak inclusion $\lesssim^S$ on L by setting

$$a \lesssim^S b \quad \text{iff} \quad \forall c \in S,\ \nu_S(a \vee c) = 1,\ b \leq c \ \Rightarrow\ c = 1.$$

We see that the restriction of $\lesssim^S$ to S is the subfit inclusion on the sublocale S. The sublocale induced by $\lesssim^S$ will be denoted by $T^S(L)$. It is clear that $a \lesssim^S b$ if and only if $\nu_S(a) \lesssim^S \nu_S(b)$. Using this equivalence, we easily see that $\lesssim^S$ inherits the fundamental properties of the subfit inclusion on S: The weak inclusion $\lesssim^S$ on L is transitive, $\vee$-preserving, and $\wedge$-preserving.

5.1.2. Proposition. *If S is a sublocale of a frame L, then $T_{\mathsf{Subfit}}(S) = T^S(L)$.*

Proof: First, let us show that $T^S(L) \subseteq S$. Suppose that $b \in T^S(L)$ and $a \leq \nu_S(b)$. It suffices to show that $a \lesssim^S b$. Suppose that $\nu_S(a \vee c) = 1$ and $b \leq c$. Then we have $a \leq \nu_S(b) \leq \nu_S(c)$. It follows that $\nu_S(c) = 1$, which was required.

Now suppose that $b \in S$. The required equality easily follows from the fact that, in S, $\nu(a) \lesssim_{\mathsf{Subfit}} b$ holds if and only if, in L, $a \lesssim^S b$. □

5.1.3. Let S be a sublocale of a frame L. We set

$$a \mathrel{\rightsquigarrow}^S b = \bigwedge\{c \in S;\ \nu_S(a \vee c) = 1,\ b \leq c\}.$$

Again, the restriction of $\mathrel{\rightsquigarrow}^S$ to S is the subfit implication on the sublocale S, and we have $a \mathrel{\rightsquigarrow}^S b = \nu_S(a) \mathrel{\rightsquigarrow}^S \nu_S(b)$. Using the properties of the subfit inclusion in S, we easily infer that $a \wedge (a \mathrel{\rightsquigarrow}^S b) \lesssim^S b$ and $(a \mathrel{\rightsquigarrow}^S b_1) \wedge (a \mathrel{\rightsquigarrow}^S b_2) = a \mathrel{\rightsquigarrow}^S (b_1 \wedge b_2)$.

The sublocale associated with the nucleus $(a \mathrel{\rightsquigarrow}^S -)$ will be denoted by $\mathfrak{o}^S_{\rightsquigarrow}(a)$.

5.1.4. Weak inclusion of sublocales. If S, Q are sublocales of a frame L, we set

$$S \lesssim Q \quad \text{iff} \quad \forall x \in S,\ \nu_S\nu_Q(x) = 1 \ \Rightarrow\ x = 1.$$

We are dealing with a weak inclusion on the coframe $\mathsf{S}(L)$ of sublocales of L. The inclusion has the following properties.

5.1.5. Proposition. *Let S, Q, P be sublocales of a frame L.*

(1) *If $S \subseteq Q$, then $S \lesssim Q$.*

(2) *If $S \lesssim P$ and $P \subseteq Q$, then $S \lesssim Q$.*

(3) *If $S \lesssim Q$ and $P \lesssim Q$, then $S \vee P \lesssim Q$.*

(4) *If $L \lesssim S$ and $S \lesssim Q$, then $L \lesssim Q$.*

Proof: (1): If $\nu_S\nu_Q(x) = 1$ with $x \in S \subseteq Q$, then $1 = \nu_S(x) = x$. (2): Suppose that $\nu_S\nu_Q(x) = 1$, $x \in S$. Using the assumption, we have $\nu_S\nu_P(x) = 1$, and therefore $x = 1$. (3): Suppose that $x \in S \vee P$ and $\nu_{S\vee P}\nu_Q(x) = 1$. Then $\nu_S\nu_Q(x) = 1$ and $\nu_P\nu_Q(x) = 1$. We can write $x = s \wedge p$, where $s \in S$ and $p \in P$. By the assumption we conclude that $s = 1$ and $p = 1$. (4): Suppose that $\nu_Q(x) = 1$. We need to prove that $x = 1$. Since $\nu_S(\nu_Q(\nu_S(x)) = 1$, $\nu_S(x) \in S$, and $S \lesssim Q$, we have $\nu_S(x) = 1$. Finally, as $L \lesssim S$, it follows that $x = 1$. □

5.1.6. Proposition. *Let L be a frame, let S, Q be sublocales of L.*

(1) *$L \lesssim S$ if and only if S is codense in L.*

(2) *$S \lesssim \mathfrak{o}^S_{\lesssim}(a) \vee \mathfrak{c}(a)$.*

(3) *$T_{\mathsf{Subfit}}(S) = \bigcap\{Q;\ S \lesssim Q\}$.*

(4) *S is subfit if and only if $\forall Q,\ S \lesssim Q \ \Rightarrow\ S \subseteq Q$.*

(5) *L is fit if and only if $\lesssim$ equals $\subseteq$.*

Proof: (1) is trivial. (2): Denote the sublocale $\mathfrak{o}^S_{\lesssim}(a) \vee \mathfrak{c}(a)$ by Q. Suppose that $x \in S$ and $\nu_S\nu_Q(x) = 1$. Therefore $\nu_S(a \rightsquigarrow^S x) = 1$ and $\nu_S(a \vee x) = 1$. Since $a \rightsquigarrow^S x \in S$, we have $a \rightsquigarrow^S x$. Then $a \vee x \lesssim^S x$, and hence $1 = \nu_S(a \vee x) \lesssim^S x$. We conclude that $\nu_S(x) = 1$, which was required.

(3): It is a consequence of (2) and 4.5.8. (4): The statement follows immediately from (3). (5): We use (4) and the fact that a frame is fit if and only if each of its sublocales is subfit. □

5.1.7. Proposition. *Let S be a sublocale of a frame L. If $T(L)$ is codense in L, then*

$$T(L) \setminus (T(L) \setminus S) \subseteq T(S).$$

Consequently, if L is subfit, then $S^{\#\#} \subseteq T(S)$.

Proof: We prove that for a sublocale P of an arbitrary frame L it holds that

$$L \lesssim P \vee S \ \Rightarrow\ T(L) \subseteq T(P) \vee S.$$

Under the assumptions of the proposition, the result will easily follow. Suppose that $L \lesssim P \vee S$. Then $T(P) \vee S = \bigcap\{U;\ P \lesssim U\} \vee S = \bigcap\{U \vee S;\ P \lesssim U\} \supseteq \bigcap\{U \vee S;\ P \vee S \lesssim U \vee S\} \supseteq \bigcap\{U \vee S;\ L \lesssim U \vee S\} \supseteq T(L)$. We used the coframe distributivity and Proposition 5.1.5. □

5.1.8. Weak difference of sublocales. If S, Q are sublocales of a frame L, we set

$$S \wr Q = \mathsf{M}\{x \in S;\ \nu_S \nu_Q(x) = 1\},$$

where $\mathsf{M}(X) = \{\bigwedge A;\ A \subseteq X\}$. The following proposition asserts that we have defined a "weak difference" on the coframe $\mathsf{S}(L)$.

5.1.9. Proposition. *Let S, Q be sublocales of a frame L. Then:*

(1) *$S \wr Q$ is a sublocale of L.*

(2) *$S \wr Q \subseteq S \setminus Q$.*

(3) *$L \wr Q = \bigvee\{\mathfrak{c}(x);\ \mathfrak{c}(x) \cap Q = \{1\}\}$*

Proof: (1): First, it is closed under meets. Second, suppose that $\nu_S \nu_Q(x_i) = 1$, for each $i \in I$. Since we have $1 \leq \nu_S \nu_Q(x_i) \leq \nu_S \nu_Q(y \to x_i)$, it follows that $y \to (\bigwedge x_i) = \bigwedge (y \to x_i) \in S \wr Q$

(2): It suffice to show that $\{x \in S;\ \nu_S \nu_Q(x) = 1\} \subseteq \bigwedge\{P;\ S \subseteq Q \vee P\}$. Let $x \in S$ and $\nu_S \nu_Q(x) = 1$. If $S \subseteq Q \vee P$, then $\nu_Q \wedge \nu_P \leq \nu_S$. Therefore, $\nu_P(x) \leq \nu_Q(x) \to \nu_S(x) \leq \nu_S \nu_U(x) \to \nu_S(x) = 1 \to \nu_S(x) = x$.

(3): $L \wr Q = \{x;\ \nu(x) = 1 \ \Rightarrow\ x = 1\}$. □

5.1.10. Proposition. *Let S be a sublocale of a frame L. Then the following statements are equivalent.*

(1) *S is subfit.*

(2) *$\forall Q,\ S \wr Q = S \setminus Q$.*

Consequently, L is fit if and only if $\forall S, Q,\ S \wr Q = S \setminus Q$.

Proof: If $x \in S$, then $\nu_Q(x) \rightsquigarrow^S x = \bigwedge\{c \in S;\ \nu_S(\nu_Q(x) \vee c)1,\ x \leq c\}$ belongs to the sublocale $S \wr Q$, because for all the c in the meet, we have $\nu_Q(c) = 1$.

Now, suppose that the sublocale S is subfit. For each x in S, it holds that $x = (x \vee \nu_Q(x)) \wedge (\nu_Q(x) \to x) = \nu_Q(x) \wedge (\nu_Q(x) \rightsquigarrow^S x) \in Q \vee (S \wr Q)$. It follows that $S \subseteq Q \vee (S \wr Q)$, and therefore $S \setminus Q \subseteq S \wr Q$. □

5.1.11. Proposition. *Let L be a frame, let Q be a sublocale of L. The following statements are equivalent.*

(1) *L is subfit.*

(2) *$\forall Q,\ L \wr Q = Q^{\#}$.*

(3) *$\forall a,\ L \wr (L \wr \mathfrak{c}(a)) = \mathfrak{c}(a)$.*

Proof: Equivalence (1)⇔(2) follows immediately from 5.1.10. (1)⇔(3): Observe that we have $L \wr (L \wr \mathfrak{c}(a)) \subseteq \mathfrak{c}(a)$ iff $\forall b,\ \mathfrak{c}(b) \cap L \wr \mathfrak{c}(a) = \{1\} \ \Rightarrow\ \mathfrak{c}(b) \subseteq \mathfrak{c}(a)$ iff $\forall b,\ L \wr \mathfrak{c}(a) \subseteq \mathfrak{o}(b) \ \Rightarrow\ a \leq b$ iff $\forall b,\ a \lesssim_{\mathsf{Subfit}} b \ \Rightarrow\ a \leq b$. □

5.2. Specialization order

In this section, we discuss a property of topological spaces which has a similar looking formula to that of subfitness.

5.2.1. Let P be a partial order. Elements a and b of P are called *compatible*, we denote this by $a \parallel b$, if there is some $c \in P$ such that $c \leq a$ and $c \leq b$. A partial order P is called *separative* if

$$x \not\leq y \quad \text{implies} \quad \exists z,\ z \leq x,\ z \nparallel y.$$

The condition on the right-hand side can be rephrazed in the following way.

5.2.2. Lemma. *If x, y are elements of a partial order P, then the following statements are equivalent:*

(1) $\forall z,\ z \parallel x \ \Rightarrow\ z \parallel y.$

(2) $\forall z,\ z \leq x \ \Rightarrow\ z \parallel y.$

Proof: (1)⇒(2) is trivial. Let us prove the opposite implication. If $z \parallel x$, then there is a v such that $v \leq z$ and $v \leq x$. By (2), we have $v \parallel y$, so there is a w such that $w \leq v$ and $w \leq y$. We obtained $w \leq z$ and $w \leq y$, and the result follows. □

5.2.3. Specialization order. Let X be a topological space. *The specialization order* on X is a partial order $\sqsubseteq$ on the points of X given by the formula

$$x \sqsubseteq y \quad \text{iff} \quad x \in \overline{\{y\}} \quad \text{iff} \quad \overline{\{x\}} \subseteq \overline{\{y\}}.$$

The relation is reflexive and transitive. By T_0, it is antisymmetric. If X is T_1, then the specialization order on X is discrete, and hence separative.

5.2.4. Proposition. *If a space X is subfit, then the specialization order on X is separative.*

Proof: In the language of closed sets, X is subfit if and only if

$$\overline{\{x\}} \not\subseteq F \quad \text{implies} \quad \exists z \in \overline{\{x\}},\ \overline{\{z\}} \cap F = \emptyset,$$

for each closed set F. The specialization order of X is separative precisely when

$$\overline{\{x\}} \not\subseteq \overline{\{y\}} \quad \text{implies} \quad \exists z \in \overline{\{x\}},\ \overline{\{z\}} \cap \overline{\{y\}} = \emptyset. \qquad \square$$

5.2.5. Example. *There is a non-subfit space whose specialization order is separative.*

Denote by ω the set of natural numbers with the usual order. Let a, b be two distinct sets. Define a topological space on the set $X = \{a, b\} \times [0, \omega)$ as follows. Basic open sets are $\{a\} \times [0, n]$ and $(\{a\} \times [0, n]) \cup \{(b, n)\}$, for $n \in \omega$. We clearly see that this is a T_0 space, each set $\{(b, n)\}$ is closed, and $\overline{\{(a, n)\}} = \{a, b\} \times [n, \omega)$, for each n. The space X is not subfit, because $\{a, b\} \times [0, n] \precsim_{\mathsf{Subfit}} \{a\} \times [0, n]$. It is easy to verify that the specialization order of X is separative. □

6 Sublocales Induced by Weak Inclusions

The sublocale induced by the regular inclusion is always regular (recall 4.3.10). We ask whether a similar statement holds for the subfit inclusion:

Is the sublocale induced by the subfit inclusion always subfit?

This is true for compact frames. More generally, it holds for those frames where the sublocale induced by the subfit inclusion is codense (recall 3.5.2). Another question concerns the subfit inclusion on spacial frames:

If L is spacial, is the sublocale induced by the subfit inclusion also spacial?

In this chapter, we show that the answers are both negative. Regarding the former question, that means there is a non-subfit frame L such that $L \simeq T_{\mathsf{Subfit}}(M)$, for some other frame M. Of course, a single counter-example L would suffice, but there arises an interesting embedding problem: Given a non-subfit frame L, we want to decide whether it can embedded (as a sublocale) in some larger frame M such that $L \simeq T_{\mathsf{Subfit}}(M)$. In other words, we want to describe the class

$$\{L;\ \exists M \text{ frame},\ L \simeq T_{\mathsf{Subfit}}(M)\}.$$

It will turn out that the class is much larger than that of subfit frames. It contains, for example, all the spacial frames. Note, on the contrary, that the class $\{L;\ \exists M \text{ frame},\ L \simeq T_{\mathsf{Regular}}(M)\}$ is precisely the class of all regular frames.

We begin this chapter by introducing yet another weak inclusion. We will be able to answer some of the questions above for this particular weak inclusion. Later on, we show how the results can be transferred to the subfit inclusion.

6.1. Inclusions modulo finite set

Let X be a topological space. There is weak inclusion $\lesssim_{\mathsf{Fin}}$ on the frame $\mathcal{O}(X)$ given by

$$U \lesssim_{\mathsf{Fin}} V \quad \text{if and only if} \quad U \setminus V \text{ is finite.}$$

The sublocale of $\mathcal{O}(X)$ induced by $\lesssim_{\mathsf{Fin}}$ (X) will be denoted by $T_{\mathsf{Fin}}(X)$.

6.1.1. Observation. *If X is a space, then $\lesssim_{\mathsf{Fin}}$ is a transitive, $\wedge$-preserving, and $\vee$-preserving weak inclusion on $\mathcal{O}(X)$.* □

6.1.2. Proposition. *If a space X is T_{D} and sober, then the spectrum of $T_{\mathsf{Fin}}(X)$ is empty.*

Proof: First, let us prove the following simple fact: If S is a sublocale of L, and if $p \in S$ is a prime element of S, then p is prime in L. Indeed, suppose that $a \wedge b \leq p$. We have $\nu_S(a) \wedge \nu_S(b) = \nu_S(a \wedge b) \leq p$. It follows that either $a \leq \nu_S(a) \leq p$ or $b \leq \nu_S(b) \leq p$, and the statement follows.

Since X is sober, prime elements of $\mathcal{O}(X)$ are of the form of $X \setminus \overline{\{x\}}$, for some $x \in X$. The set $X \setminus \overline{\{x\}} \cup \{x\}$ is open, because X is T_{D}. Therefore, the set $X \setminus \overline{\{x\}}$ does not belong to $T_{\mathsf{Fin}}(X)$. □

As an example, consider the space R of real numbers. Each regular open set is certainly in $T_{\mathsf{Fin}}(R)$. But $T_{\mathsf{Fin}}(R)$ has more elements than that. For instance, set $V = R \setminus C$, where C is the Cantor set. Clearly, if $U \cap C \neq 0$, where U is open, then $U \cap C$ is infinite. Therefore, V belongs to $T_{\mathsf{Fin}}(R)$.

6.1.3. Proposition. *There is a topological space X such that $T_{\mathsf{Subfit}}(X) = \mathcal{O}(X)$ but $T_{\mathsf{Fin}}(X) = \{1\}$.*

Proof: Pick any countable discrete space. □

6.1.4. Proposition. *There is a topological space X such that $T_{\mathsf{Fin}}(X) = \mathcal{O}(X)$ but $T_{\mathsf{Subfit}}(X) = \{1\}$.*

Proof: Let Q be the set of rational numbers, and open sets are of the form of $\{x \in Q;\ x < y\}$, for some $y \in Q$, where $<$ is the usual order on Q. □

Now, for formulate the main proposition, which will be used in sequel.

6.1.5. Proposition. *If a frame L is spacial, then there is a space X such that $L \simeq T_{\mathsf{Fin}}(X)$. Moreover, the space X can be chosen such that the prenucleus induced by $\lesssim_{\mathsf{Fin}}$ is nucleus.*

Proof: Let Y be a topological space. For each $y \in Y$, let Z_y be a copy of the set of integers. Set X to be a space on the disjoint union of all the X_y, for $y \in Y$, with the following topology. A set $U \subseteq X$ is open in X if and only if both each $Z_y \cap U$ is an lower-set of Z_y, and $\{y \in Y;\ Z_y \cap U \neq 0\}$ is open in Y. The space X is clearly T_0. Take the ideal $\mathcal{J}$ of finite subsets of X. Then, elements of $T_{\mathcal{J}}(X)$ are of the form of $\bigcup\{Z_y;\ y \in U\}$ for U open in Y. This sublocale is isomorphic to $\mathcal{O}(Y)$. □

6.2. More on inclusions modulo finite set

In this section, we show that the inclusion $\lesssim_{\mathsf{Fin}}$ can be extended to all frames. The content of this section is not needed for further development

6.2.1. Let L be a frame. If a and b are elements of L, we set

$$[a, b] = \{y \in L;\ a \leq x \leq b\}.$$

Observe that $a \leq b$ if and only if $[a \wedge b, a] = \{a\}$ if and only if the set $[a \wedge b, a]$ has at most one element. We relax the condition in the following way. We define a weak inclusion $\lesssim_{\mathsf{Fini}}$ on the frame L by setting

$$a \lesssim_{\mathsf{Fini}} b \quad \text{iff} \quad \text{the set } [a \wedge b, a] \text{ is finite.}$$

6.2.2. Proposition. *The relation $\lesssim_{\mathsf{Fini}}$ on a frame L is transitive, $\vee$-preserving, and $\wedge$-preserving weak inclusion.*

Proof: Transitivity: Suppose that $a \lesssim b$ and $b \lesssim c$. We need to prove that the set $A = [a \wedge c, c]$ is finite. We define mappings $f_1 : A \to [a \wedge b, b]$, $f_1(x) = a \wedge (x \vee b)$, and $f_2 : A \to [b \wedge c, b]$, $f_2(x) = b \wedge (x \vee c)$.

Then, we show that there are the following formulas. First, $f_1(x) \wedge f_2(x) = a \wedge b \wedge (x \vee c) \wedge (x \vee c) = a \wedge b \wedge (x \vee c) = b \wedge ((a \wedge x) \vee (a \wedge c)) = b \wedge x$. Second, we have $b \vee f_1(x) \vee f_2(x) = b \vee (a \wedge x) \vee (a \wedge b) \vee (b \wedge x) \vee (b \wedge c) = b \vee x$.

Consequently, we can show that the assignment $x \mapsto (f_1(x), f_2(x)) : A \to [a \wedge b, a] \times [b \wedge c, b]$ is one-one. Indeed, if $f_i(x) = f_i(y)$, for $i = 1, 2$ and $x, y \in A$, then, using the previous formulas, we have $b \wedge x = b \wedge y$ and $b \vee x = b \vee y$. It follows that $x = y$.

$\wedge$-preservation: Suppose that $a \lesssim b_i$, for $i = 1, 2$. We prove that the set $A = [a \wedge b_1 \wedge b_2, a]$ is finite. We define mappings $f_i : A \to [a \wedge b_i, a]$, for $i = 1, 2$, by setting $f_i(y) = y \vee (a \wedge b_i)$. If $y \in A$, then we have $f_1(y) \wedge f_2(y) = (y \vee (a \wedge b_1) \wedge (y \vee (a \wedge b_2)) = y \vee (a \wedge b_1 \wedge b_2) = y$. We prove that the assignment $y \mapsto (f_1(y), f_2(y)) : A \to [a \wedge b_1] \times [a \wedge b_2]$ is one-one. Indeed, if $f_i(y) = f_i(z)$, for $i = 1, 2$ and $y, z \in A$, then $y = f_1(y) \wedge f_2(y) = f_1(z) \wedge f_2(z) = z$.

$\vee$-preservation: Suppose that $a_i \lesssim b$, for $i = 1, 2$. We prove that $A = [(a_1 \vee a_2) \wedge b, a_1 \vee a_2]$ is finite. We define mappings $f_i : A \to [a_i \wedge b, a_i]$ by setting $f_i(y) = y \wedge a_i$. When $y \in A$, we have $f_1(y) \vee f_2(y) = (y \wedge a_1) \vee (y \wedge a_2) = y \wedge (a_1 \vee a_2) = y$. It easily follows that the mapping $y \mapsto (f_1(y), f_2(y)) : A \to [a_1 \wedge b, a_1] \times [a_2 \wedge b, a_2]$ is one-one. Therefore, the set A is finite. □

6.2.3. Proposition. *If U and V are open in a topological space X, then*

$$U \lesssim_{\mathsf{Fini}} V \quad \textit{if and only if} \quad V \setminus U \textit{ is finite.}$$

Proof: First, we observe that if $W_0 < W_1$ and $W_1 \setminus W_0$ is infinte, there is an open set W_2 such that $W_0 < W_2 < W_1$, and also either $W_1 \setminus W_2$ is infinite or $W_2 \setminus W_1$ is infinite. Indeed, since $W_1 \setminus W_0$ is infinite, there are distinct points x, y in $W_1 \setminus W_0$. All spaces are assumed to be T_0, so there is an open set W_2 such that, say, $x \in W_2$ and $y \notin W_2$. Without loss of generality, we may assume that $W_0 \subseteq W_2 \subseteq W_1$. Therefore, we have $x \in W_2 \setminus X_0$ and $y \in W_1 \setminus W_2$, and the remaining part of the observation is clear.

Now, suppose that $V \setminus U$ be infinite. Set $W = U \cup V$. Since $V < W$ and $V \setminus W$ is also infinite, using the observation we construct by induction an infinite family of distinct open sets in $[V, W]$.

On the other hand, suppose that $V \setminus U$ is finite. Since $V \setminus (U \cap V)$ is finite, the set $[U \cap V, U]$ must also be finite. □

6.3. Inclusions modulo an ideal

In this section, we generalize the inclusion modulo finite set.

6.3.1. Let X be a topological space. An *ideal $\mathcal{J}$ on X* is a collection of arbitrary subsets of X such that

(1) $\emptyset \in \mathcal{J}$.

(2) $A \in \mathcal{J}$, $B \in \mathcal{J}$ implies $A \cap B \in \mathcal{J}$,

(3) $A \in \mathcal{J}$, $B \subseteq A$ implies $B \in \mathcal{J}$,

We will be mostly interested in the ideals satisfying the following property: An ideal $\mathcal{J}$ on X is called *free* if it contains all the finite subsets of X, that is, if $\bigcup \mathcal{J} = X$.

6.3.2. If $\mathcal{J}$ is an ideal on a space X, we define a weak inclusion $\lesssim_{\mathcal{J}}$ on the frame $\mathcal{O}(X)$ by setting

$$U \lesssim_{\mathcal{J}} V \quad \text{iff} \quad U \setminus V \in \mathcal{J}.$$

The sublocale induced by $\lesssim_{\mathcal{J}}$ will be denoted by $T_{\mathcal{J}}(X)$.

6.3.3. Proposition. *If $\mathcal{J}$ is an ideal on a space X, then $\lesssim_{\mathcal{J}}$ is a transitive, $\wedge$-preserving, and $\vee$-preserving weak inclusion.*

Proof: Suppose that the sets $U \setminus V$ and $V \setminus W$ are in $\mathcal{J}$. As $U = (U \setminus V) \cup (U \cap V)$, we have $U \setminus W = ((U \setminus V) \setminus W) \cup ((U \cap V) \setminus W) \subseteq (U \setminus V) \cup (V \setminus W)$. If follows that the set $U \setminus W$ is also in $\mathcal{J}$.

Next, if $U \setminus V_i$ is in $\mathcal{J}$, for $i = 1, 2$, then the set $U \setminus (V_1 \cap V_2) = (U \setminus V_1) \cup (U \setminus V_2)$ is also in $\mathcal{J}$.

Similarly, when $U_i \setminus V \in$ is in $\mathcal{J}$, for $i = 1, 2$, then so is $(U_1 \cup U_2) \setminus V = (U_1 \setminus V) \cup (U_2 \setminus V)$. □

Let us recall several notions from the point-set topology. Let X be a topological space. A subset A of X is called *nowhere dense* if $\mathrm{int}(\overline{A}) = 0$.

Let us show the well known fact that the collection of all the nowhere dense sets forms an ideal on X. First, each subset of a nowhere dense set is clearly nowhere dense. Second, suppose that A and B are nowhere dense, and let $U \subseteq \overline{A \cup B} = \overline{A} \cup \overline{B}$ be open. Then $U \setminus \overline{A} \subseteq \overline{B}$ is also open, and so $U \setminus \overline{A} = 0$. Therefore $U \subseteq \overline{A}$, and hence $U = 0$. We can conclude that $A \cup B$ is also nowhere dense.

6.3.4. Observation. *The sublocale induced by the ideal of nowhere dense sets is always dense.* □

The ideal of nowhere dense sets need not be free: Consider, for example, any discrete space X. The ideal of nowhere dense sets contains only the empty set.

6.3.5. Construction. Let Y be a T_1 topological space. For each point $y \in Y$, let A_y be a copy of the set of integers. Let X be the set $\bigcup\{A_y;\ y \in Y\}$ endowed with the following topology: Open sets are of the form of $\bigcup\{D_y;\ y \in V\}$, where $V \subseteq Y$ is open in Y, and each D_y is a non-empty down-set of A_y in the usual order on integers.

There is a quotient mapping $q : X \to Y$, $q[A_y] = \{y\}$. When M is any subset of Y, we set $\mathsf{e}(M) = q^{-1}[M] = \bigcup\{A_y;\ y \in M\}$. The mapping $\mathsf{e} : \mathcal{O}(Y) \to \mathcal{O}(X)$ is a one-one frame homomorphism. It is immediate from the definition that q is an open map. Therefore, for arbitrary subsets M, N of Y we have formulas

(1) $\mathsf{e}(M) \setminus \mathsf{e}(N) = \mathsf{e}(M \setminus N)$,

(2) $\overline{\mathsf{e}(M)} = \mathsf{e}(\overline{M})$,

(3) $\mathrm{int}(\mathsf{e}(M)) = \mathsf{e}(\mathrm{int}(M))$,

6.3.6. Proposition. *We continue using the notation from* 6.3.5. *Let V_1, V_2 be open sets of Y. Denote by $\lesssim$ the inclusion modulo a nowhere dense set.*

(1) *$V_1 \lesssim V_2$ if and only if $\mathsf{e}(V_1) \lesssim \mathsf{e}(V_2)$.*

(2) *The set $\overline{\{x\}}$ is nowhere dense, for all $x \in X$.*

(3) *The restriction $\mathsf{e} : T_{\lesssim}(Y) \to T_{\lesssim}(X)$ is an isomorphism.*

Proof: (1): We have $\mathrm{int}(\overline{\mathsf{e}(V_1) \setminus \mathsf{e}(V_2)}) = \mathrm{int}(\overline{\mathsf{e}(V_1 \setminus V_2)}) = \mathrm{int}(\mathsf{e}(\overline{V_1 \setminus V_2})) = \mathsf{e}(\mathrm{int}(\overline{V_1 \setminus V_2}))$. Since $\mathrm{int}(\overline{V_1 \setminus V_2}) = \emptyset$ if and only if $\mathsf{e}(\mathrm{int}(\overline{V_1 \setminus V_2})) = \emptyset$, the statements follows.

(2): Suppose that $x \in A_y$, $y \in Y$. The set $\mathsf{e}(Y \setminus \{y\}) \cup \{z \in A_y;\ z < x\}$ is open in X, because Y is T_1. Consequently, we have $\overline{\{x\}} = \{z \in A_y;\ x \leq z\}$, whose interior is empty.

(3): $T_{\lesssim}(X) \subseteq \mathsf{e}[T_{\lesssim}(Y)]$: Suppose that $U \in T_{\lesssim}(X)$, and $U = \bigcup\{D_y;\ y \in V_2\}$, where $V_2 \subseteq Y$ is open in Y, and each D_y is a non-empty down-set of A_y. By (2), the set $A_y \setminus D_y$ is nowhere dense. As $U \cup A_y$ is open, it follows that $A_y \subseteq U$. Therefore, $U = \mathsf{e}(V_2)$. It remains to show that V_2 is in $T_{\lesssim}(Y)$. Using (1), if $V_1 \lesssim V_2$, then $\mathsf{e}(V_1) \lesssim \mathsf{e}(V_2) = U$, and so $\mathsf{e}(V_1) \subseteq \mathsf{e}(V_2)$. We proved that $V_1 \subseteq V_2$, because e is one-one.

$\mathsf{e}[T_{\lesssim}(Y)] \subseteq T_{\lesssim}(X)$: Suppose that $V_2 \in T_{\lesssim}(Y)$. We prove that $\mathsf{e}(V_2) \in T_{\lesssim}(X)$. Suppose that $U \lesssim \mathsf{e}(V_2)$ such that $U = \bigcup\{D_y;\ y \in V_1\}$, V_1 is open in X, and each D_y is a non-empty down-set of A_y. Clearly, if $x \in A_y \setminus D_y$, $y \in V_1$, then $x \in \overline{U}$. It follows that $\overline{U \setminus \mathsf{e}(V_2)} = \overline{\mathsf{e}(V_1) \setminus \mathsf{e}(V_2)}$. Therefore, $\mathsf{e}(V_1) \lesssim \mathsf{e}(V_2)$. Since $V_2 \in T_{\lesssim}(Y)$, using (1) we obtain $U \subseteq \mathsf{e}(V_1) \subseteq \mathsf{e}(V_2)$.

Finally, we have to show that the restriction of e to $T_{\lesssim}(Y)$ is a frame homomorphism. We use the restriction of the localic map e^+. Suppose that $U \in T_{\lesssim}(X)$. It suffices to show that $\mathsf{e}^+(U) \subseteq T_{\lesssim}(Y)$. Let $V \lesssim \mathsf{e}^+(U)$. Then $\mathsf{e}(V) \lesssim \mathsf{e}\mathsf{e}^+(U) \subseteq U$. Therefore $\mathsf{e}(V) \lesssim U$, and so $\mathsf{e}(V) \subseteq U$. It follows that $V \subseteq \mathsf{e}^+(U)$. □

6.3.7. Corollary. *Let $\lesssim$ denote the inclusion modulo a nowhere dense set. Let L be a frame such that $L \simeq T_{\lesssim}(Y)$, for some T_1 space Y. Then, there is a space X such that $L \simeq T_{\lesssim}(X)$, where the ideal of nowhere dense sets on X is free.* □

Open set U is called *regular open* if $\mathrm{int}(\overline{U}) \subseteq U$. A set is *clopen* if it is both closed and open. A topological space is *extremally disconnected* if the closure of each open set is also open. Therefore, in such spaces, regular open sets are precisely clopen sets.

6.3.8. Proposition. *Let $\mathcal{J}$ be the ideal of nowhere dense sets on an extremally disconnected space X. If V is open in X, then the following statements are equivalent.*

(1) *V is regular open.*

(2) *V belongs to $T_{\mathcal{J}}(X)$.*

(3) *V is clopen.*

Proof: The implication (1)⇒(2) follows from 6.3.4. Since X is extremally disconnected, the implication (3)⇒(1) is immediate. It remains to prove (2)⇒(3): Assume that $V \in T_{\mathcal{J}}(X)$. Since $\overline{V} \setminus V$ is closed and has empty interior, it means that $\overline{V} \setminus V$ is nowhere dense. The set $\overline{V}$ is open. It follows that $\overline{V} \lesssim_{\mathcal{J}} V$, and hence $\overline{V} \subseteq V$. □

Let us recall several well known facts about Boolean algebras. A Boolean algebra is a distributive lattice whose each element is complemented. If B is a Boolean algebra, the *Stone space* $\mathsf{St}(B)$ *of* B is a space on the set of all the ultrafilters on B, whose basic open sets are of the form of

$$\mathsf{st}(a) = \{F;\ F \text{ ultrafilter},\ a \in F\}.$$

The space $\mathsf{St}(B)$ is compact Hausdorff, and B is isomorphic to the algebra of clopen sets $\mathsf{St}(B)$. Moreover, if B is complete lattice, then $\mathsf{St}(B)$ is extremally disconnected. For more detailed explanation, see [12].

6.3.9. Proposition. *Each Boolean frame L is a isomorphic to $T_{\mathcal{J}}(X)$, for some space X and some free ideal $\mathcal{J}$ on X.*

Proof: Consider the Stone space of L. By 6.3.7, it can be modified such that the ideal of nowhere dense sets is free. □

6.4. Inflation of spaces

Let Y be a topological space. On the set Y, one can consider a topology whose basic open sets are of the form of $V \setminus K$, where $V \subseteq Y$ is open in Y, and F is arbitrary finite subset of Y. The resulting space is T_1, and the topology is finer than the original one. We think of $V \setminus K$ as an open set with finitely many holes. The idea of open set with holes is generalized in the follwing construction, where each point of Y is replaced by another topological space.

6.4.1. Definition. Let Y be a topological space. Let $\mathcal{J}$ be a free ideal on Y in the sense of 6.3.1. Let $\{A_y;\ y \in Y\}$ be a family topological spaces. We assume that they are pair-wise disjoint. Notice that we have one space for each point of Y. The *inflation of the space Y by the family $\{A_y;\ y \in Y\}$ using the ideal $\mathcal{J}$* is a topological space X which is defined as follows. The set of points of X is the union of all the A_y, for $y \in Y$. If $V \subseteq Y$ is open in Y, set

$$\mathsf{in}(V) = \bigcup\{A_y;\ y \in V\}.$$

Basic open sets of X are of the form of $\mathsf{in}(V) \setminus \bigcup\{F_y;\ y \in K\}$, where $V \subseteq Y$ is open in Y, $K \in \mathcal{J}$, and each $F_y \subseteq A_y$ is closed in A_y. Since $\mathcal{J}$ is an ideal, this is indeed a base.

Since the ideal $\mathcal{J}$ is free, the space is T_0, and each A_y is a closed subspace of X. Moreover, the mapping $q : X \to Y$ given by $q[A_y] = \{y\}$ is continuous, and therefore in is a one-one frame homomorphism

$$\mathsf{in} : \mathcal{O}(Y) \to \mathcal{O}(X).$$

6.4.2. Support of a set. We shall describe the topology more explicitely. Let Z be any subset of X. A set $V \subseteq Y$ is said to be a *basic support of Z* if V is open in Y, and if there is a set $K \in \mathcal{J}$ such that

$$\forall y \in V \setminus K,\ A_y \subseteq Z.$$

If V is a basic support of Z, and $W \subseteq V$, then W is also a basic support of Z. The *support of Z*, denoted by $\mathsf{supp}(Z)$, is the union of all the basic supports of Z.

6.4.3. Lemma. *Let X be the inflation of a space Y by a family $\{A_y;\ y \in J\}$ using an ideal $\mathcal{J}$ on Y. If U_1 and U_2 are open in X, then*

(1) $\mathsf{supp}(U_1 \cap U_2) = \mathsf{supp}(U_1) \cap \mathsf{supp}(U_2)$,

(2) $\mathsf{supp}(\mathsf{in}(V)) = \pi_{\mathcal{J}}(V)$, *for each open V in Y.*

Proof: (1): The mapping supp is clearly monotone. Now, let $y \in \mathsf{supp}(U_1) \cap \mathsf{supp}(U_2)$. Then, there is a basic support V_1 of U_1 and a basic support V_2 of U_2 such that $y \in V_1 \cap V_2$. The set $V_1 \cap V_2$ is a basic support of both U_1 and U_2. Thus, $\{y \in V_1 \cap V_2;\ X_y \subseteq U_1 \cap U_2\} \subseteq \{y \in V_1 \cap V_2;\ \subseteq U_1\} \cup \{y \in V_1 \cap V_2;\ \subseteq U_2\}$, which is in $\mathcal{J}$.

(2): For each $y \in Y$, we have $X_y \not\subseteq \mathsf{in}(V)$ if and only if $y \notin V$. Consequently, an open set W in Y is a basic support of $\mathsf{in}(V)$ if and only if $W \setminus V \in \mathcal{J}$ if and only if $W \lesssim_{\mathcal{J}} V$. Therefore, the statement follows. □

6.4.4. Proposition. *Let X be the inflation of a space Y by a family $\{A_y;\ y \in J\}$ using an ideal $\mathcal{J}$ on Y. A set $U \subseteq X$ is open if and only if both*

(1) $U \subseteq \mathsf{in}(\mathsf{supp}(U))$ *and*

(2) *the set $X_y \cap U$ is open in X_y for each $y \in Y$.*

Proof: Let $\mathcal{O}_2$ denote the collection of sets satisfying (1) and (2). The condition (2) is clearly preserved under joins and finite meets. Since supp and in are monotone, the condition (1) is preserved under joins. The set $\mathcal{O}_2$ is closed under finite meets, because supp and in preserve them. Therefore it is a topology.

Let $\mathcal{O}_1$ be the topology from the definition of $\mathsf{Infl}_{\mathcal{J}}(Y, X)$. It is given by a base $\mathcal{B}$ consisting of all the U for which there exists a basic support $V \subseteq Y$ of U such that $U \subseteq \mathsf{in}(V)$ and $U \cap X_y$ is open for all $y \in Y$. Therefore, $\mathcal{O}_1 \subseteq \mathcal{O}_2$. On the other hand, assume that $x \in U \in \mathcal{O}_2$, where $x \in X_y$ for some y. Since $y \in \mathsf{supp}(U)$, there is a basic support V of U such that $y \in V$. Then $y \in \mathsf{in}(V) \cap U$, which belongs to $\mathcal{B}$. We proved that $\mathcal{O}_2 \subseteq \mathcal{O}_1$. □

6.5. Weak inclusions on inflated spaces

In this section, we describe the subfit inclusion and the inclusion modulo finite set on spaces which are created using the method of inflation. An inflated space consists of a family of pair-wise disjoint closed subspaces. The construction is tailored so that the weak inclusions can be, in some sense, computed on each of these closed subspaces separately.

To be able to treat the subfit inclusion and the inclusion modulo finite set at once, we formulate our results more generally. Consequently, we will be able to apply them to any weak inclusion which satisfies certain properties.

6.5.1. The properties. Let $\lesssim$ be a weak inclusion on a space X. We say that $\lesssim$ satisfies conditions (P1)–(P3) when the following three conditions hold.

(P1) $\lesssim$ is downward and upward transitive.

(P2) $\lesssim$ is $\wedge$-preserving and $\vee$-preserving.

(P3) If $U_1 \lesssim U_2$ for open sets U_1 and U_2, and if $\{A_i;\ i \in I\}$ is a family of pair-wise disjoint closed subsets of X, then there is a finite set $K \subseteq J$ such that

$$\forall i \in J \setminus K,\ (A_i \subseteq U_1 \ \Rightarrow\ A_i \subseteq U_2).$$

We show that the conditions are satisfied by the subfit inclusion and by the inclusion modulo finite set. The conditions (P1) and (P2) are trivial.

If $\lesssim$ is the subfit inclusion, the implication in (P3) actually holds for all $i \in J$. Indeed, if $A_i \setminus U_1$ and $A_i \not\subseteq U_2$, then the set $A_i \setminus U_2$ is a non-empty closed subset of $U_1 \setminus U_2$, which is not possible.

Next, if $\lesssim$ is the inclusion modulo finite set on X, then there is an infinite set $R \subseteq I$ of indices such that $A_i \setminus U_1$ and $A_i \not\subseteq U_2$ for each $i \in R$. Since the closed sets are pair-wise disjoint, the set $\bigcup\{A_i \setminus U_2;\ i \in R\}$ is an infinite subset of $U_1 \setminus U_2$, which is a contradiction.

6.5.2. Weak inclusions on closed subspaces. Let $\lesssim$ be a weak inclusion on a space X. Let $A \subseteq X$ be a closed subspace of X. Let $G_1 \subseteq A$ and $G_2 \subseteq A$ be open sets of A. The *restriction of $\lesssim$ to the closed subspace A* is a weak inclusion $\lesssim_A$ on the space A given by

$$G_1 \lesssim_A G_2 \quad \text{iff} \quad G_1 \cup (X \setminus A) \lesssim G_2 \cup (X \setminus A).$$

If $\lesssim$ is the subfit inclusion in X, then the weak inclusion $\lesssim_A$ is the weak inclusion on A. Similarly, is $\lesssim$ is the inclusion modulo finite set on X, then $\lesssim_A$ is the inclusion modulo finite set on A.

6.5.3. Lemma. *Let $\mathcal{J}$ be a free ideal on a space Y. Let X be the inflation of a space Y by a family $\{A_y;\ y \in Y\}$ of spaces using the ideal $\mathcal{J}$. Let $\lesssim$ be a weak inclusion on X which satisfies (P1)–(P3). Denote by π the prenucleus induced by $\lesssim$. Denote by $\lesssim_y$ the restriction of $\lesssim$ to the closed subspace A_y, and denote by π_y its induced prenucleus. Suppose that U_1, U_2, and U are open sets of X.*

(1) *If $U_1 \lesssim U$, then $\mathsf{supp}(U_1) \subseteq \mathsf{supp}(U)$.*

(2) $\pi(U) = \bigcup\{\pi_y(U \cap X_y);\ y \in \mathsf{supp}(U)\}$.

Proof: (1): Let V be a basic support of U_1. There is a set $K_1 \in \mathcal{J}$ such that $\forall y \in V \setminus K_1,\ A_y \subseteq U_1$. By (P3), there is a finite set K_2 such that $\forall y \in V \setminus K_2,\ (A_y \subseteq U_1 \ \Rightarrow\ A_y \subseteq U_2)$. We have $K_1 \cup K_2 \in \mathcal{J}$ and $\forall y \in V \setminus (K_1 \cup K_2),\ A_y \subseteq U_2$. Therefore, V is a basic support of U_2.

(2): Assume that $U' \lesssim U$, where U' is open in X. For the left-to-right inclusion it suffices to show that $U' \cap A_y \subseteq \pi_y(U \cap A_y)$, for each $y \in Y$. By 6.4.4, it is enough to assume that y is from $\mathsf{supp}(U')$. By (1), such y belongs to $\mathsf{supp}(U)$. Since $U' \cap A_y \lesssim_y U \cap A_y$, we have $U' \cap A_y \subseteq \pi_y(U \cap A_y)$.

For the other inclusion, let $y \in \mathsf{supp}(U)$. We show that $\pi_y(U \cap A_y) \subseteq \pi(U)$. Assume that $G \lesssim_y U \cap X_y$ for some $G \subseteq X_y$ open in X_y. Then $G \cup (X \setminus A_y) \lesssim_y U \cup (X \setminus A_y)$. Since $\lesssim$ is $\vee$-preserving, we have $(U \cup G) \cup (X \setminus A_y) \lesssim_y U \cup (X \setminus A_y)$. Now, using $\wedge$-stability, it follows that $U \cup G \lesssim_y (U \cup (X \setminus A_y)) \cap (U \cup G) \subseteq (U \cup (X \setminus A_y)) \cap (U \cup A_y) = U$. By upward transitivity, we conclude that $U \cup G \lesssim_y U$, and hence $G \subseteq U \cup G \subseteq \pi(U)$. $\square$

6.5.4. Proposition. *Let $\mathcal{J}$ be an ideal on a space Y. Let X be the inflation of a space Y by a family $\{A_y;\ y \in Y\}$ of spaces using the ideal $\mathcal{J}$. Let $\lesssim$ be a weak inclusion on X which satisfies (P1)–(P3), and denote by π the induced prenucleus. Assume, moreover, that $\pi_y(\emptyset) = A_y$, for each $y \in Y$.*

(1) *If U is open in X, then $\pi(U) = \mathsf{in}(\mathsf{supp}(U))$.*

(2) *If V is open in Y, then $\pi(\mathsf{in}(V)) = \mathsf{in}(\pi_{\mathcal{J}}(V))$.*

(3) *The frames $T_{\lesssim}(X)$ and $T_{\mathcal{J}}(Y)$ are isomorphic.*

Proof: The statement (1) follows immediately from 6.5.3.
(2): From 6.4.3(2) and (1) we infer that $\pi(\mathsf{in}(V)) = \mathsf{in}(\mathsf{supp}(\mathsf{in}(V))) = \mathsf{in}(\pi_{\mathcal{J}}(V))$.
(3): Using the one-one homomorphism $\mathsf{in} : \mathcal{O}(Y) \to \mathcal{O}(X)$, the sublocale $T_{\mathcal{J}}(Y)$ can be identifed with the frame $M = \{\mathsf{in}(V);\ V \in T_{\mathcal{J}}(Y)\}$. It suffices to show that $T_{\lesssim}(\mathcal{O}(X)) = M$. Using (2) and the fact that in is one-one, we see that

$$\pi_{\mathcal{J}}(V) \subseteq V \quad \text{iff} \quad \pi_{\lesssim}(\mathsf{in}(V)) = \mathsf{in}(\pi_{\mathcal{J}}(V)) \subseteq \mathsf{in}(V).$$

It follows that $\mathsf{in}(V) \in M$ if and only if $\mathsf{in}(V) \in T_{\lesssim}(X)$. By (1), each open set in $T_{\lesssim}(X)$ is of the form of $\mathsf{in}(V)$, for some open V in Y, and the statement follows. □

6.6. Applications

In [3], the author asks whether the prenucleus induced by the subfit inclusion is always a nucleus. Recently, this was answered in [4]. The author exhibits a space where this prenucleus is not a nucleus. As a prenucleus can be iterated until we obtain a nucleus (recall 2.4.7), the author also shows that the required number of iterations cannot by bounded. In this case, the constructed examples were not spacial.

Using the method of inflation, such examples can be construced easily. The following proposition can be applied either to the subfit inclusion or the inclusion modulo finite set.

6.6.1. Proposition. *Let each topological space X be equipped with a weak inclusion $\lesssim_X$ which satisfies (P1)–(P3). Then, for each ordinal $\alpha \geq 1$, there is a space X where the prenucleus induced by $\lesssim_X$ is not a nucleus.*

Proof: Let $\alpha \geq 1$ be an ordinal. Denote by Y the space on $\alpha \times \omega$ with the topology of down-sets with respect to its lexicographical order:

$$(0,0) \leq (0,1) \leq \cdots \quad (1,0) \leq (1,1) \leq \cdots \quad (\beta,0) \leq (\beta,1) \leq \cdots.$$

Denote by $\mathcal{J}$ the ideal of finite sets on Y. It is easy to see that $\pi_{\mathcal{J}}(\beta \times \omega) = (\beta+1) \times \omega$.

Let X be the inflation of Y by a space A of natural numbers with the topology of downsets. Denote by π the prenucleus induced by the weak inclusion $\lesssim_X$.

We prove by transfinite induction that $\pi^\beta(0) = \mathsf{in}(\beta \times \omega)$, for each $\beta < \alpha$. If β is isolated, using 6.5.4(2) it follows that $\pi^{\beta+1}(0) = \pi(\pi^\beta(0)) = \pi(\mathsf{in}(\beta \times \omega)) = \mathsf{in}((\beta+1) \times \omega)$. If β is limit, we have $\pi^\beta(0) = \bigcup\{\pi^\gamma(0);\ \gamma < \beta\} = \bigcup\{\mathsf{in}(\gamma \times \omega);\ \gamma < \beta\} = \mathsf{in}(\beta \times \omega)$.

We proved that α is the first ordinal such that $\pi^\alpha(0) = X$. □

We have seen that each complete Boolean algebra is isomorphic (as a frame) to some $T_{\mathcal{J}}(X)$, where X is a space and $\mathcal{J}$ is an ideal on X. We show that we may choose the space X such that $\mathcal{J}$ is the ideal of all the finite subsets of X.

6.6.2. Proposition. *If L is a frame, the following statements are equivalent.*

(1) *There is a space X such that $L \simeq T_{\mathsf{Fin}}(X)$.*

(2) *There is a space X and a free ideal $\mathcal{J}$ on X such that $L \simeq T_{\mathcal{J}}(X)$.*

Proof: The implication (1)⇒(2) is trivial. (1)⇒(2): Suppose that $L \simeq T_{\mathcal{J}}(Y)$ for some space Y and an ideal $\mathcal{J}$ on Y. Denote by Z the space of integers with the topology of downsets. Let X be the inflation of Y by the space Z using the ideal $\mathcal{J}$. Consider the inclusion modulo finite set on X. By the choice of the space Z, we use 6.5.4 to obtain $T_{\mathcal{J}}(Y) \simeq T_{\mathsf{Fin}}(X)$. □

6.6.3. Proposition. *Let L be a frame. If $L \simeq T_{\mathsf{Fin}}(Y)$, for some space Y, then there is a space X such that $L \simeq T_{\mathsf{Subfit}}(X)$.*

Proof: Let A be the space of all integers with the topology of downsets. Let X be the inflation of Y by A using the ideal of all the finite sets on Y. By 6.5.4(3), the statement follows. □

Now, we are ready to solve the problems from the beginning of this chapter.

6.6.4. Example. *There is a frame L where $T_{\mathsf{Subfit}}(L)$ is not subfit.*

Pick some non-subfit space X. By 6.6.3, it follows that $\mathcal{O}(X)$ is isomorphic to some $T_{\mathsf{Subfit}}(L)$, for some frame L. □

6.6.5. Example. *There is a spacial frame L such that $T_{\mathsf{Subfit}}(L)$ is not spacial.*

Consider some non-spatial complete Boolean algebra M, By 6.6.3, it follows that $\mathcal{O}(X)$ is isomorphic (as a frame) to some $T_{\mathsf{Subfit}}(L)$, for some frame L.

Thus, the class of the frames whose are isomorphic to some $T_{\mathsf{Subfit}}(M)$, for some frame M, contains all the spacial frames and all complete Boolean algebras.

6.6.6. In 3.5.2, we proved the that

$$\pi_{\mathsf{Subfit}} \text{ is a nucleus} \iff T_{\mathsf{Subfit}}(L) \text{ is codense} \implies T_{\mathsf{Subfit}}(L) \text{ is subfit.}$$

We show that none of the indicated implications can be reversed.

6.6.7. Example. *There is a space X where $T_{\mathsf{Subfit}}(X)$ is subfit, but the prenucleus π_{Subfit} is not a nucleus.*

Use the space X from 6.6.1. There, $T_{\mathsf{Subfit}}(X) = \{1\}$, which is subfit. □

6.6.8. Example. *There is a space X where the prenucleus π_{Subfit} is nucleus, but $T_{\mathsf{Subfit}}(X)$ is not subfit.*

Let Y be a non-subfit space Y where $T_{\mathsf{Fin}}(X) = \mathcal{O}(X)$. The existence of such Y follows from 6.1.4. Consider the inflation X of the space Y using the ideal of all the finite subsets of Y. Using 6.5.4(3) for the subfit inclusion on X, and using the choice of Y, $T_{\mathsf{Subfit}}(X)$ is isomorphic to $\mathcal{O}(Y)$, which is not subfit.

Using 6.5.4(1), each $\pi_{\mathsf{Fin}}(U)$ is of the form of $\mathsf{in}(V)$, for some open V in Y. By 6.5.4(2) and the choice of Y, each $\mathsf{in}(V)$ is $T_{\mathsf{Fin}}(X)$. It follows that the prenucleus π_{Fin} is a nucleus. □

Since T_{Subfit} need not be subfit, we ask how many iterations of the operation $T_{\mathsf{Subfit}}(-)$ is needed to obtain a subfit sublocale. Let us begin with a simple lemma.

6.6.9. Lemma. *Let $\{L_i;\ i \in J\}$ be a collection of frames, and let $L = \prod\{L_i;\ i \in J\}$ be their product in the category of frames. Denote by $p_i : L \to L_i$ the projections, and denote by $\lesssim$ the subfit inclusion. If $a, b \in L$, then*

(1) $a \lesssim b$ *iff* $\forall i \in J,\ p_i(a) \lesssim p_i(b)$,

(2) $b \in T_{\lesssim}(L)$ *iff* $\forall i \in J,\ p_i(b) \in T_{\lesssim}(L_i)$,

(3) $T_{\mathsf{Subfit}}(L) \simeq \prod\{T_{\mathsf{Subfit}}(L_i);\ i \in J\}$.

Proof: (1): Suppose that, in L, $a \lesssim b$, and that, in L_i, $p_i(a) \vee x = 1$. Consider $c \in L$ such that $p_i(c) = x$ and $p_j(c) = 1$ for $j \neq i$. Then $a \vee c = 1$, and so $b \vee c = 1$. It follows that $p_i(b) \vee x = p_i(b) \vee p_i(c) = 1$. On the other hand, suppose that $p_i(a) \lesssim p_i(b)$, for all $i \in J$, and $a \vee c = 1$, for some $c \in L$. Then $p_i(a) \vee p_i(c) = 1$, $p_i(b) \vee p_i(c) = 1$, for each i, and therefore $b \vee c = 1$.

(2): Suppose that $b \in T_{\lesssim}(L)$ and $x \lesssim p_i(b)$, where $x \in L_i$. Consider $a \in L$ such that $p_i(a) = x$ and $p_j(a) = 0$ for $j \neq i$. By (1), $a \lesssim b$, and so $a \leq b$. It follows that $x = p_i(a) \leq p_i(b)$. For the other implication, if $a \lesssim b$, then $p_i(a) \lesssim p_i(b)$, and $p_i(a) \leq p_i(b)$, and so $a \leq b$.

(3): The frame $\prod\{T_{\mathsf{Subfit}}(L_i);\ i \in J\}$ can be regarded as a sublocale of L. Then, the statement follows from (2). □

Let L be a frame. We define inductively

$$T^0_{\mathsf{Subfit}}(L) = L \quad \text{and} \quad T^{n+1}_{\mathsf{Subfit}}(L) = T_{\mathsf{Subfit}}(T^n_{\mathsf{Subfit}}(L)).$$

6.6.10. Example. *There is a space X such that $T^n_{\mathsf{Subfit}}(X)$ is not subfit for each natural n.*

Proof: Let Y_0 be any subfit space. We define spaces $Y_1, Y_2, ...$ by induction as follows. Given the space Y_n, we use 6.5.4, to obtain Y_{n+1} such that $T_{\mathsf{Subfit}}(Y_{n+1}) \simeq \mathcal{O}(Y_n)$.
For $n \geq 1$, we clearly see that $T^{n-1}(\mathcal{O}(Y_n)) = \mathcal{O}(Y_1)$, which is not subfit. Set $Y = \coprod\{Y_n;\ n \geq 0\}$, where $\coprod$ denotes the disjoint sum of topological spaces. We have $\mathcal{O}(Y) \simeq \prod\{\mathcal{O}(Y_n);\ n \geq 0\}$, where $\prod$ denotes the product in the category of frames. Using 6.6.9, the statement follows. □

It remains unsettled whether the sublocale $\bigcap T^n_{\mathsf{Subfit}}(L)$ of L can be non-subfit.

7 Hausdorff-Type Axioms

It is a basic result in classical topology, that a space X is Hausdorff iff the diagonal mapping $X \to X \times X$ is closed. The analogous formulation in the category of locales was studied by Isbell. In this chapter, we shall describe this axiom in terms of a weak inclusion.

7.1. Frame of antitone maps

In the category of frames, a coproduct of L and M is usually defined as a suitable sublocale of the free frame on $L \times M$. For our purposes, it will be convenient to work with another well-known representation of finite coproducts, which is mentioned in [8].

7.1.1. Contravariant adjoints. Let L and M be frames. A mapping $\alpha : L \to M$ is said to have a *contravariant adjoint* if it is antitone and there is an antitone mapping $\alpha^* : M \to L$ such that

$$y \leq \alpha(x) \quad \text{iff} \quad x \leq \alpha^*(y).$$

By standard arguments, $\alpha : L \to M$ has a contravariant adjoint if and only if

$$\alpha(\bigvee x_i) = \bigwedge \alpha(x_i).$$

In particular, $\alpha(0) = 1$.

7.1.2. The set of all the maps from L to M which have contravariant adjoints will be denoted by $L \oplus M$. We order the antitone maps point-wise. We are going to show that the set $L \oplus M$ is a frame, and it is the coproduct of L and M in the category of all frames.

7.1.3. Frame of antitone maps. Let $\mathsf{Ant}(L, M)$ denote the frame of all antitone maps from L to M. Both suprema and infima are computed point-wise. There is a mapping $(-)^* : \mathsf{Ant}(L, M) \to \mathsf{Ant}(M, L)$ assigning to each $\alpha : L \to M$ a mapping $\alpha^* : M \to L$ given by

$$\alpha^*(y) = \bigvee\{x;\ y \leq \alpha(x)\}.$$

7.1.4. Lemma.

(1) *If* $y \leq f(x)$, *then* $x \leq f^*(y)$.

(2) $(f \wedge g)^* = f^* \wedge g^*$.

Proof: (1) is trivial. (2): Let us show first that it is monotone. Suppose that $f \leq g$. Since $\forall x, y,\ y \leq f(x) \ \Rightarrow\ y \leq g(x)$, we have $\forall x, y,\ y \leq f(x) \ \Rightarrow\ x \leq g^*(y)$, and so $f^* \leq g^*$ by the definition of f^*. Now, by frame distributivity and using monotonicity of f and g, we have $f_1^*(y) \wedge f_2^*(y) = \bigvee\{x_1 \wedge x_2;\ y \leq f_i^*(x_i),\ i = 1, 2\} \leq \bigvee\{x_1 \wedge x_2;\ y \leq f_i^*(x_1 \wedge x_2),\ i = 1, 2\} \leq (f_1 \wedge f_2)(y)$. □

7.1.5. Applying the star operator twice to get back to $\mathsf{Ant}(L,M)$ gives a prenucleus $\nu : \mathsf{Ant}(L,M) \to \mathsf{Ant}(L,M)$ defined by $\nu(\alpha) = \alpha^{**}$. The conditions on prenucleus are easy to verify. Applying 7.1.4(1) twice, we get $f \le f^{**}$. From (2), ν is monotone and preserves finite meets. Finally, if $f^{**} = f$, then f^* is contravariant adjoint to f. Thus, the sublocale generated by ν is the set $L \oplus M$. Furthermore, the mapping * restricted to $L \oplus M$ is an isomorphism between $L \oplus M$ and $M \oplus L$.

7.1.6. The injections. Let $\alpha \in L \oplus M$. From the localic point of view, a projection of α to M is the element $\alpha(1)$, and the projection to L equals $\alpha^*(1)$. The tedious part is to describe the corresponding frame homomorphism.

Let φ be a frame-theoretic formula in one variable. A *characteristic function* of the formula φ is a mapping $\chi(\varphi) : L \to L$ defined by setting

$$\chi(\varphi)(x) = \begin{cases} 1 & \varphi(x) \text{ is true,} \\ 0 & \text{otherwise} \end{cases}$$

The *injection homomorphism* $i_M : M \to L \oplus M$ is defined by setting

$$i_M(a)(x) = a \vee \chi(x = 0), \quad \text{where} \quad (i_M(a))^*(y) = \chi(y \le a).$$

Both are clearly antitone and the adjunction unit formulas hold:

$$x \le \chi(\chi(x = 0) \le a) = \chi(a \vee \chi(x = 0) \le a) = i_M(a)((i_M(a))^*(x)),$$
$$y \le a \vee \chi(y \not\le a) = a \vee \chi(\chi(y \le a) = 0) = (i_M(a))^*(i_M(a)(y)).$$

Next, we need to show that i_M preserves finite meets. We have $i_M(a \wedge b)(x) = (a \wedge b) \vee \chi(x = 0) = (a \vee \chi(x = 0)) \wedge (b \vee \chi(x = 0)) = (i_M(a) \wedge i_M(b))(x)$.

The corresponding localic morphism is given by $i_M^+(\alpha) = \alpha(1)$. Moreover, the mapping i_M is is one-one and i_M^+ is surjective. Indeed, $i_M^+ i_M(a) = i^+(x \mapsto a \vee \chi(x = 0)) = a \vee \chi(1 = 0) = a$. For the other unit, $i_M i^+(\alpha) = x \mapsto \alpha(1) \vee \chi(x = 0) \le x \mapsto \alpha(x) = \alpha$.

7.1.7. Let $a \in L$ and $b \in M$. Set $a \oplus b = i_L(a) \wedge i_M(b)$. Clearly, for $z \ne 0$, we have

$$(a \oplus b)(z) = \begin{cases} b & \text{if } z \le a \\ 0 & \text{otherwise} \end{cases}$$

7.1.8. Proposition. *Let $\alpha \in L \oplus M$.*

(1) $y \le \alpha(x)$ *iff* $x \oplus y \le \alpha$.

(2) $\alpha = \bigvee\{x \oplus \alpha(x);\ x \in L\}$.

Proof: (1): Suppose that $y \le \alpha(x)$. If $0 \ne z \le x$, then $(x \oplus y)(z) = y \le \alpha(x) \le \alpha(z)$. The other cases are trivial. On the other hand, if $x \oplus y \le \alpha$, then $y = (x \oplus y)(x) \le \alpha(x)$.

(2): Since $\alpha(x) \le \alpha(x)$, by (1) we have $x \oplus \alpha(x) \le \alpha$, and hence α is an upper bound of the prescribed set. Next, let β be another upper bound. Since $x \oplus \alpha(x) \le \beta$, by (1) it follows that $\alpha(x) \le \beta(x)$. □

7.1.9. Proposition. *Let L, M be frames. The frame $L \oplus M$ is a coproduct of L and M in* **Frm**.

Proof: Let $f_L : N \to L$ and $f_M : N \to M$ be frame homomorphisms. We shall show that there exists a unique frame homomorphism $f : N \to L \oplus M$ such that $fi_L = f_L$ and $fi_M = f_M$. Note that if it exists, it must satisfy $f(a \oplus b) = f(i_L(a) \wedge i_M(b)) = fi_L(a) \wedge fi_M(b) = f_L(a) \wedge f_L(b)$. It follows that f is defined uniquely on a $\bigvee$-base of $L \oplus M$, and therefore it is unique.

Set $f(\alpha) = \bigvee\{f_L(x) \wedge f_M(\alpha(x));\ x \in L\}$. We have $f(\alpha) \leq n$ iff $\forall x,\ f_L(x) \wedge f_M(\alpha(x)) \leq n$ iff $\forall x, f_M(\alpha(x)) \leq f_L(x) \to n$ iff $\forall x, \alpha(x) \leq f_M^+(f_L(x) \to n)$. This means that f has a right adjoint given by $f^+(n) = f_M^+(f_L(-) \to n)$. Since f_L and f_M are $\wedge$-preserving, so is f. □

7.2. Heyting maps

This section concerns the square $L \oplus L$ of a frame L, where the Heyting structure on L provides important examples of maps which have contravariant adjoints.

The *diagonal morphism* is a localic morphism $\Delta : L \to L \oplus L$ defined by $\Delta(a) = (- \to a) : L \to L$. The maps of the form of $\Delta(a)$ are called *Heyting maps*. We see that the contravariant adjoint of a Heyting map is the mapping itself. The frame homomorphism $\nabla : L \oplus L \to L$ related to Δ is given by $\nabla(\alpha) = \bigvee\{x \wedge \alpha(x);\ x \in L\}$. First, there are adjunction units

(1) $\alpha \leq \Delta\nabla(\alpha)$, that is, $\forall a,\ \alpha(a) \leq a \to \nabla(\alpha)$.

(2) $\nabla\Delta(y) \leq y$, that is, $\forall a,\ a \wedge (a \to y) \leq y$.

Second, using frame distributivity, $\Delta(\alpha) \wedge \Delta(\beta) = \bigvee\{(x \wedge \alpha(x)) \wedge (y \wedge \alpha(y));\ x, y \in L\} = \bigvee\{(x \wedge y) \wedge (\alpha \wedge \beta)(x \wedge y);\ x, y \in L\} = \Delta(\alpha \wedge \beta)$.

The main object of investigation is the closed sublocale $\uparrow\Delta(0)$. However, it is more convenient to introduce a more general notion, which captures the essential properties of $\uparrow\Delta(0)$.

7.2.1. Heyting-closed families. Let $\alpha \in L \oplus L$, and let $c, d \in L$. Then the mapping $c \to \alpha(d \wedge -)$ has a contravariant adjoint which is given by $d \to \alpha^*(c \wedge -)$. Indeed, we have $y \leq c \to \alpha(d \wedge x)$ iff $y \wedge c \leq \alpha(d \wedge x)$ iff $d \wedge x \leq \alpha^*(c \wedge y)$ iff $x \leq d \to \alpha^*(c \wedge y)$.

We say that a subset $\Lambda \subseteq L \oplus L$ is *Heyting-closed* if

(1) Λ contains all the Heyting maps

(2) If $\alpha \in \Lambda$, then $\alpha^* \in \Lambda$.

(3) If $\alpha \in \Lambda$, then $c \to \alpha(d \wedge -) \in \Lambda$, for each $c, d \in L$.

The family $\Delta[L]$ of all Heyting maps is Heyting-closed. Indeed, (1) and (2) are trivial, and also $c \to ((- \wedge d) \to a) = \Delta((c \wedge d) \to a)$.

7.2.2. Lemma. *Let $\Lambda \subseteq L \oplus L$ be Heyting closed, and let a be an element of L. Then the following are equivalent.*

(1) $\forall \alpha \in \Lambda,\ \alpha(a) = a \to \alpha(1)$.

(2) $\forall \alpha \in \Lambda,\ a \oplus a \leq \alpha \ \Rightarrow\ 1 \oplus a \leq \alpha$.

Proof: (1)⇒(2): Let $a \oplus a \leq \alpha$. Then $a \leq \alpha(a) = a \to \alpha(1)$, and so $1 \oplus a \leq \alpha$. (2)⇒(1): Consider the mapping $(a \to \alpha(-)) \in \Lambda$. Since $\alpha(a) \leq 1 = a \to \alpha\alpha(a)$, we have $\alpha(a) \leq a \to \alpha(1)$. For the inequality $a \to \alpha(1) \leq \alpha(a)$, suppose that $x \leq a \to \alpha(1)$. It follows that $x \wedge a \leq \alpha(1)$, and so $a \leq 1 \leq \alpha^*(x \wedge a)$. Using the mapping $\alpha^*(x \wedge -) \in \Lambda$ we obtain $a \leq \alpha^*(x)$, and hence $x \leq \alpha(a)$. □

7.2.3. The previous lemma leads to the following definition.

$$\lambda_\Lambda(a) = \bigvee\{x;\ \forall \alpha \in \Lambda,\ a \oplus a \leq \alpha \ \Rightarrow\ 1 \oplus x \leq \alpha\}.$$

Clearly, $\lambda(a)$ is the largest element which satisfies $a \oplus a \leq \alpha \ \Rightarrow\ 1 \oplus \lambda(a) \leq \alpha$.

7.2.4. Lemma. *If Λ is a Heyting-closed subset of $L \oplus L$, then:*

(1) $\lambda_\Lambda(a) \leq a$.

(2) $\lambda_\Lambda(a_1) \wedge \lambda_\Lambda(a_2) = \lambda_\Lambda(a_1 \wedge a_2)$.

(3) $\lambda_\Lambda(a) = \bigwedge\{\alpha(1);\ a \oplus a \leq \alpha \in \Lambda\}$.

Proof: (1) Since $a \leq a \to a = \Delta(a) \in \Lambda$, we obtain $x \leq \Delta(1) = 1 \to a = a$.
(2): The fact that λ_Λ is monotone is obvious. Next, let $x \leq \lambda_\Lambda(a_1) \wedge \lambda_\Lambda(a_2)$. We want to prove that $x \leq \lambda(a_1 \wedge a_2)$, so suppose that $a_1 \wedge a_2 \leq \alpha(a_1 \wedge a_2)$. Then $a_1 \leq a_2 \to \alpha(a_2 \wedge a_1)$. Considering the mapping $a_2 \to \alpha(a_2 \wedge -) \in \Lambda$, we obtain $x \leq a_2 \to \alpha(a_2 \wedge 1)$, and then $a_2 \leq x \to \alpha(a_2)$. Finally, since $(x \to \alpha(-)) \in \Lambda$, we have $x \leq x \to \alpha(a_1)$, and the statement follows.
(3) follows immediately from the definition. □

7.2.5. Let Λ be a Heyting-closed subset of $L \oplus L$. By 7.2.4(1), the formula

$$a \lesssim_\Lambda b \quad \text{iff} \quad \lambda_\Lambda(a) \leq b$$

defines a $\bigwedge$-preserving, upward-transitive weak inclusion, which is induced by the then weak implication $a \rightsquigarrow \lambda_\Lambda(-)$. By 7.2.4(2) and 4.2.4, it satisfies the first and the second weak Heyting formula.

7.2.6. Lemma. *Let $h : L \to M$ be a frame homomorphism. If α is in $M \oplus M$, then $h^+\alpha h \in L \oplus L$, and its contravariant adjoint equals $h^+\alpha^* h$.*

Proof: Notice that $y \leq h^+\alpha(x)$ iff $h(y) \leq \alpha(x)$ iff $x \leq \alpha^* h(y)$. Then we have $y \leq h^+\alpha h(x)$ iff $h(x) \leq \alpha^* h(y)$ iff $x \leq h^+\alpha^* h(y)$. □

7.2.7. A frame homomorphism $h : L \to M$ is said to be Λ-*preserving* if for each $\alpha \in \Lambda_M$, $h^+\alpha h \in \Lambda_L$.

7.2.8. Proposition. *If a frame homomorphism $h : L \to M$ is Λ-preserving, then it is $\lesssim_\Lambda$-preserving.*

Proof: It is enough to show that $h\lambda(a) \leq \lambda h(a) = \bigwedge\{\alpha(1);\ h(a) \leq \alpha(h(a)),\ \alpha \in \Lambda_M\}$. If $h(a) \leq \alpha(h(a))$, then $a \leq h^+\alpha h(a)$. Consequently, $\lambda(a) \leq h^+\alpha h(1)$, and so $h\lambda(a) \leq \alpha h(1) = \alpha(1)$. □

By 4.3.9, we can conclude that if each frame L is equipped with a Heyting-closed set Λ_L, and if each frame homomorphism is Λ-preserving, then those frames where λ_L is the identity form coreflective subcategory of **Frm**.

7.3. The Isbell-Hausdorff inclusion

Recall that a frame L is Isbell-Hausdorff if the diagonal morphism Δ is closed. This amounts to saying that each mapping in the set $\Lambda = {\uparrow}\Delta(0)$ is Heyting. This set Λ is always Heyting-closed. The induced weak inclusion is called the *Isbell-Hausdorff inclusion*, and it is denoted by $\lesssim_{\mathsf{IH}}$. Explicitely,

$$a \lesssim_{\mathsf{IH}} b \quad \text{iff} \quad \forall x,\ 1 \oplus x \leq \Delta(0) \vee (a \oplus a) \ \Rightarrow\ x \leq b.$$

By lemma 7.2.2, L is Isbell-Hausdorff if and only if the Isbell-Hausdorff inclusion on L is trivial. A frame L is Isbell-Hausdorff if and only if

$$\forall a,\ 1 \oplus a \leq \Delta(0) \vee (a \oplus a).$$

The fact that each regular frame is Isbell-Hausdorff can be refined as follows.

7.3.1. Proposition. *For each frame, $\lesssim_{\mathsf{IH}} \subseteq \lesssim_{\mathsf{Reg}}$.*

Proof: (1): Let $x * \vee a = 1$, and let $a \leq \alpha(a)$ with $\Delta(0) \leq \alpha$. First, we have $x \leq a$, and so $\alpha^*(a) \leq \alpha^*(x)$. As $x \to 0 \leq \alpha^*(x)$ and $a \leq \alpha^*(a)$, it follows that $\alpha^*(x) \vee \alpha^*(a) = 1$. Thus, we conclude that $x \leq \alpha(1)$. □

7.3.2. Proposition. *If $\Delta(0) \vee (a \oplus a) \vee (c \oplus 1) = 1$, then $a \vee c = 1$.*

Proof: Consider the mapping $\Delta(a \vee c)$. Since $\Delta(0) \leq \Delta(a \vee c)$, $a \leq a \to (a \vee c)$, and $c \leq 1 \to (a \vee c)$, it follows that $1 \leq 1 \to (a \vee c)$. □

The opposite implication of the previous proposition has a characterization in terms of the fit inclusion.

7.3.3. Proposition. *Let $a \in L$. The following propositions are equivalent:*

(1) $\forall c,\ a \vee c = 1 \ \Rightarrow\ \Delta(0) \vee (a \oplus a) \vee (c \oplus 1) = 1.$

(2) $\forall \alpha,\ (a \oplus a) \vee \Delta(0) \leq \alpha \ \Rightarrow\ a \lesssim_{\mathsf{Fit}} \alpha(1).$

Proof: (1)⇒(2): Suppose that $\Delta(a) \vee (a \oplus a) \leq \alpha$ and $a \vee c = 1$. We need to prove that $c \to \alpha(1) \leq \alpha(1)$. Consider an adjointed mapping $\beta(x) = (c \to \alpha(1)) \to \alpha(x)$. We have $c \leq \beta(1) = (c \to \alpha(1)) \to \alpha(1)$, and so $c \oplus 1 \leq \beta$. Since $\alpha \leq \beta$, we obtained $\Delta(0) \vee (a \oplus a) \vee (c \oplus 1) \leq \beta$. By the assumption, $\beta = 1$. Since $\beta(1) = 1$, the statement follows.

(2)⇒(1): Suppose that $a \vee c = 1$ and $\Delta(0) \vee (a \oplus a) \vee (c \oplus 1) \leq \alpha$. From (2) we infer that $a \lesssim_{\mathsf{Fit}} \alpha(1)$. It follows that $c \to \alpha(1) \leq \alpha(1)$. Since $c \leq \alpha(1)$, we conclude that $\alpha(1) = 1$, and so $\alpha = 1$. □

Consequently, if a frame is fit, then 7.3.3(2) holds precisely in Isbell-Hausdorff frames.

A Bibliography

[1] J.R. Isbell, "Atomless Parts of Spaces", *MATHEMATICA SCANDINAVICA* 31, 5–32, 1972.

[2] J. Picado, A. Pultr and A. Tozzi, "Joins of closed sublocales", *Houst. J. Math* 45(1), 21–38, 2019.

[3] H. Simmons, "A curious nucleus", *Journal of Pure and Applied Algebra* 214(11), 2063–2073, 2010.

[4] L. Haykazyan, "More on a Curious Nucleus", *arXiv:1807.08020*, 2018.

[5] J. Picado and A. Pultr, *Frames and Locales: Topology Without Points*, Birkhäuser Basel, 2012.

[6] R. Engelking, *General topology*, Heldermann Verlag, 1989.

[7] S. Mac Lane, *Categories for the Working Mathematician*, Springer-Verlag New York, 1978.

[8] P.T. Johnstone, *Stone Spaces* (no. 3), Camb. Univ. Press, 1983.

[9] J. Picado and A. Pultr, "More on subfitness and fitness", *Appl. Categ. Structures* 23, 323–335, 2015.

[10] J. Picado and A. Pultr, "New aspects of subfitness in frames and spaces", *A. Appl Categor Struct* 24, 703–7014, 2016.

[11] H. Simmons, "The lattice theoretic part of topological separation properties", *Proceedings of the Edinburgh Mathematical Society* 21(1), 41–-48, 1978.

[12] J.D. Monk, R. Bonnet and S. Koppelberg, *Handbook of Boolean Algebras, Volume 1*, North-Holland, 1989.