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Basic sequences in Banach spaces

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Abstract: An ordering on bases in Banach spaces is defined as a natural generalization of the notion of equivalence. Its theory is developed with emphasis on its behavior with respect to shrinking and boundedly-complete bases. We prove that a bounded operator mapping a shrinking basis to a boundedly-complete one is weakly compact. A well-known result concerning the factorization of a weakly compact operator through a reflexive space is then reinterpreted in terms of the ordering.

Next, we introduce a class of Banach spaces whose norm is constructed from a given two-dimensional norm N. We prove that any such space X_N is isomorphic to an Orlicz sequence space. A key step in obtaining this correspondence is to describe the unit circle in the norm N with a convex function φ. The canonical unit vectors form a basis of a subspace Y_N of X_N. We characterize the equivalence of these bases and the situation when the basis is boundedly-complete. The criteria are formulated in terms of the norm N and the function φ.

Keywords: boundedly-complete basis, shrinking basis, Orlicz sequence spaces, equivalence of bases, ordering on bases

Contents

Introduction

We investigate two concepts related to the theory of basic sequences in Banach spaces. The first one is a definition of an ordering on bases. The relation is defined in such a way that it naturally generalizes the equivalence of bases. Then we study its properties with special emphasis on its behavior with respect to shrinking and boundedly-complete bases. The second concept we introduce is a definition of a special class of Banach spaces which we came to call spaces with iterative norms. The most important results are summarized in the following two sections of the introduction.

In the first chapter of the thesis we set up notations and review the basic definitions, the second chapter is devoted to the ordering on bases and the third one to spaces with iterative norms.

0.1 Ordering on bases

The capital letters X, Y and Z always denote Banach spaces. For other pieces of notation the reader is referred to Section 1.1.

Definition 0.1. Let (x_n) be a basis of X and (y_n) be a basis of Y. We say that (x_n) *dominates* (y_n), or that (y_n) *is dominated by* (x_n), if

$$\sum_{n=1}^{\infty} a_n x_n \text{ converges} \Rightarrow \sum_{n=1}^{\infty} a_n y_n \text{ converges}$$

for all sequences of scalars (a_n). If this is the case we write $(y_n) \preceq (x_n)$.

It will be seen that if $(y_n) \preceq (x_n)$ and $(x_n) \preceq (y_n)$, the two bases are equivalent. We will also show later (Proposition 2.1) that $(y_n) \preceq (x_n)$ if and only if the linear operator $T : X \to Y$ mapping x_n to y_n is bounded.

It turns out that the case when a shrinking basis dominates a boundedly-complete basis is of special interest. Then the operator T has a nontrivial property captured by the next theorem.

Theorem 0.2. *If (x_n) is a shrinking basis of X and (z_n) is a boundedly-complete basis of Z such that $(z_n) \preceq (x_n)$, then the operator $T : X \to Z$ mapping x_n to z_n is weakly compact.*

This allows us to connect the theory with a famous result of Davis et al. [3] that every weakly compact operator between Banach spaces factors through a reflexive Banach space. We prove that in our setting the weakly compact operator factors through a reflexive space with a basis. According to a theorem of James (Theorem 1.26), if a reflexive Banach space has a basis, then the basis is both shrinking and boundedly-complete. The conclusion can be stated purely in terms of bases.

Theorem 0.3. *Let (x_n) be a shrinking basis of X and (z_n) a boundedly-complete basis of Z such that $(z_n) \preceq (x_n)$. Then there exists a space Y with a basis (y_n) such that $(z_n) \preceq (y_n) \preceq (x_n)$, where (y_n) is both shrinking and boundedly-complete.*

0.2 Spaces with iterative norms

We consider a class of Banach spaces X_N whose norm is defined by a certain iterative process. Their exact definition will be provided in Chapter 3. It can be informally described as follows. We work over a field $\mathbb{F}$ of scalars ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$). Starting with a norm N on $\mathbb{F}^2$ we inductively extend it to $\mathbb{F}^n$ for all $n > 2$. Thus the norms N_n are obtained. A scalar sequence (a_i) belongs to the space X_N if and only if the supremum of $N_n(a_1, \dots, a_n)$ is finite. We also define Y_N to be $[e_i]$, the closed linear span of the canonical unit vectors in X_N. The sequence (e_i) is a basis of Y_N which is in fact unconditional.

If we begin the construction with the p-norm on $\mathbb{F}^2$, given by

$$N(a,b) = \begin{cases} \left(|a|^p + |b|^p\right)^{1/p}, & 1 \le p < \infty, \\ \max(|a|, |b|), & p = \infty, \end{cases}$$

then the resulting space X_N will be ℓ_p. In this sense the spaces X_N generalize the sequence spaces ℓ_p.

The norm N will always be assumed to satisfy

$$N(a,b) = N(|a|, |b|), \qquad (a,b) \in \mathbb{F}^2. \tag{0.1}$$

Such norms are called *absolute* in [7]. It is further assumed that the norm is *normalized*, i.e. $N(a,0) = N(0,a) = |a|$ for $a \in \mathbb{F}$.

The norm N is uniquely determined by its unit ball (or rather the unit circle if the norm is defined on $\mathbb{R}^2$). In describing the unit sphere we can restrict ourselves to the first quadrant $\{(a,b) \in \mathbb{F}^2;\ a \ge 0, b \ge 0\}$ since the norm is absolute. We show how the unit sphere in this set can be viewed as the graph of a certain convex function φ associated with N. One of the key insights of this thesis is to connect various properties of the space X_N to the properties of the function φ.

If X_{N^1} and X_{N^2} are two spaces constructed from the norms N^1 and N^2, then we can ask whether they consist of the same sequences. This is equivalent to the identity mapping between them being an isomorphism. Let us denote by φ_1 and φ_2 the functions associated with N^1 and N^2. It turns out that one possible characterization is as follows: there exists a positive constant K and $b_0 > 0$ such that if $b \in [0, b_0]$, then

$$\varphi_2\left(\frac{b}{K}\right) \le \varphi_1(y) \le \varphi_2(Kb). \tag{0.2}$$

Theorem 3.35 lists other equivalent criteria. In particular, this means that the properties of X_N preserved by isomorphism depend only on the behavior of φ on some neighborhood of the origin.

The function φ enables us to find a correspondence between the spaces X_N and Orlicz sequence spaces (the basic properties of which are reviewed in Section 3.1). We show that every space X_N is isomorphic to an Orlicz sequence space ℓ_Φ and vice versa, the isomorphism being the identity mapping.

Lastly, we ask the question when the basis (e_i) of Y_N is boundedly-complete. The link with Orlicz sequence spaces provides an answer immediately: Precisely when φ satisfies the so-called Δ_2-condition at zero, i.e.

$$\limsup_{y \to 0+} \frac{\varphi(2y)}{\varphi(y)} < \infty.$$

1. Basic sequences and biorthogonal systems

In this chapter we present some of the fundamental definitions and theorems concerning Schauder bases in Banach spaces, basic sequences and biorthogonal systems. The material is quite standard, which is why we do not find it necessary to give proofs of the propositions. A more thorough exposition of bases and basic sequences can be found for example in Chapters 1 and 3 of [1]. Section 1.f of [4] is devoted to biorthogonal systems. We want to take this opportunity to collect the results we build on in later chapters.

1.1 Notations and standard definitions

The letters X, Y, Z always denote a Banach space. All Banach spaces are assumed to be over a field $\mathbb{F}$ which is either $\mathbb{R}$ or $\mathbb{C}$. A sequence of elements x_n in X is denoted (x_n). Here it is implicitly understood that $n = 1, 2, \dots$ takes values in the natural numbers. If we want to emphasize this fact or use a different numbering of the indices we write $(x_n)_{n=1}^{\infty}$, $(x_n)_{n=0}^{\infty}$ and so on. The closed linear span of (x_n) is denoted by $[x_n]$, hence $[x_n] = \overline{\text{span}}(x_n)$. Similarly, we use the symbols (a_n) and (b_n) for scalar sequences. A finite scalar sequence is denoted $(a_i)_{i=1}^{n}$. The closed unit ball of X is denoted by B_X while X^* denotes the dual space of X. By an operator $T : X \to Y$ we always mean a bounded linear operator between the Banach spaces X and Y. Of course, the fact that T is bounded needs to be proved in some cases. If Z is a direct sum of X and Y, we write $Z = X \oplus Y$. Then Z is equipped with the norm $\|z\|_Z = \|(x, y)\|_Z = \|x\|_X + \|y\|_Y$, where $\|\cdot\|_X$ and $\|\cdot\|_Y$ are the norms on X and Y, respectively.

The weak topology on X is the topology whose basis consists of the sets $V_\varepsilon(x_0; x_1^*, x_2^*, \dots, x_n^*) = \{x \in X;\ |x_i^*(x_0) - x_i^*(x)| < \varepsilon, i = 1, \dots, n\}$, where $\varepsilon > 0$ and $x_1^*, \dots, x_n^* \in X^*$. The sets $V_\varepsilon^*(x_0^*, x_1, x_2, \dots, x_n) = \{x^* \in X^*;\ |x_0^*(x_i) - x^*(x_i)| < \varepsilon\}$, where $\varepsilon > 0$ and $x_1, \dots, x_n \in X$ form a basis of the weak* topology on X^*. An operator $T : X \to Y$ is called weakly compact if $\overline{T(B_X)}$ is a weakly compact set.

The space c_0 is the Banach space of all scalar sequences (a_n) such that $\lim_{n\to\infty} a_n = 0$. A norm on c_0 is given by $\|(a_n)\|_\infty = \sup |a_n|$. For $1 \le p < \infty$, the space ℓ_p is the Banach space of all scalar sequences (a_n) such that $\sum_{n=1}^{\infty} |a_n|^p < \infty$ equipped with the norm $\|(a_n)\|_p = \left(\sum_{n=1}^{\infty} |a_n|^p\right)^{1/p}$.

1.2 Equivalence of bases

One of the aims of this thesis is to define an ordering on bases and to examine its properties. We must first introduce the notion of equivalence of bases.

Definition 1.1. A sequence (x_n) in a Banach space X is called a *basis* of X if every element $x \in X$ can be uniquely expressed as $x = \sum_{n=1}^{\infty} a_n x_n$, where (a_n) is a sequence of scalars. The sequence (x_n) is called a *basic sequence* if (x_n) is a basis of $[x_n]$.

The term *Schauder basis* is often used in place of *basis* to distinguish it from the basis of a vector space (called *Hammel basis*). If (x_n) is a basis of X, then there exists a sequence of linear functionals (x_n^*) such that $x_m^*(x_n) = \delta_{m,n}$ for all n, m, where $\delta_{n,m}$ is the Kronecker delta symbol. An element $x \in X$ can be expressed as

$$x = \sum_{n=1}^{\infty} x_n^*(x) x_n.$$

To a basis (x_n) of X we associate the projections $P_n : X \to X$ defined by $P_n(x) = \sum_{i=1}^{n} x_i^*(x) x_i$ for $n \in \mathbb{N} \cup \{0\}$. These operators are uniformly bounded, meaning that $K = \sup_n \|P_n\|$ is finite. The number K is called the *basis constant.*

We will often work with basic sequences which are (semi)normalized.

Definition 1.2. A basic sequence (x_n) in X is said to be normalized if $\|x_n\| = 1$ for all n. It is said to be seminormalized if there exist positive constants c_1 and c_2 such that $c_1 \leq \|x_n\| \leq c_2$ for all n.

In Section 2.2, it will be easier for us to work with the projections associated with a basis rather than with the basis itself. It is convenient to introduce the more general notion of a Schauder decomposition [4, Def 1.g.1].

Definition 1.3. A *Schauder decomposition* of X is a sequence (X_n) of closed subspaces of X such that every $x \in X$ can be uniquely written as $x = \sum_{n=1}^{\infty} x_n$ with $x \in X_n$ for every n.

We associate the projections P_n defined as $P_n(\sum_{i=1}^{\infty} x_i) = \sum_{i=1}^{n} x_i$ with a given Schauder decomposition (X_n) of X. These projections clearly satisfy $P_n P_m = P_{\min(m,n)}$ and $P_n x \to x$ for any $x \in X$. When a sequence of projections (P_n) satisfies the last two conditions, a Schauder decomposition (X_n) of X can be defined by putting $X_1 = P_1(X)$ and $X_n = (P_n - P_{n-1})(X)$ for $n > 1$. The special case when $\dim X_n = 1$ for each n is described by the following Proposition [1, Prop. 1.1.7].

Proposition 1.4. *If a sequence (P_n) of bounded linear projections on X satisfies*

(i) $\dim P_n(X) = n$ *for all* n*,*

(ii) $P_n P_m = P_{\min(m,n)}$ *for any* m *and* n*,*

(iii) $P_n(x) \to x$ *for all* $x \in X$*,*

then any sequence (x_n) of nonzero vectors such that $x_1 \in P_1(X)$ and $x_n \in (P_n - P_{n-1})(X)$ for $n \geq 2$ is a basis of X whose associated projections are P_n.

To determine whether a given sequence (x_n) is basic we find the next proposition, called *Grundblum's criterion* [1, Prop. 1.1.9], very useful.

Proposition 1.5. *Let (x_n) be a sequence of nonzero elements in X. Then (x_n) is basic if and only if there exists a constant $C > 0$ such that*

$$\left\| \sum_{i=1}^{m} a_i x_i \right\| \leq C \left\| \sum_{i=1}^{n} a_i x_i \right\| \tag{1.1}$$

for any scalar sequence (a_n) and any $m \leq n$.

We note that if (x_n) is a basis of X, then the least C for which (1.1) holds is K, the basis constant of (x_n). Now we are ready to define what it means for two bases to be equivalent.

Definition 1.6. Let (x_n) be a basis of X and (y_n) be a basis of Y. Then (x_n) and (y_n) are said to be *equivalent* if

$$\sum_{n=1}^{\infty} a_n x_n \text{ converges} \Leftrightarrow \sum_{n=1}^{\infty} a_n y_n \text{ converges}$$

for all scalar sequences (a_n). If this is the case we write $(x_n) \sim (y_n)$.

The proof of the next proposition will follow from Proposition 2.1.

Proposition 1.7. *Let (x_n) and (y_n) be bases of X and Y, respectively. Then $(x_n) \sim (y_n)$ if and only if there exists an isomorphism $T : X \to Y$ such that $T(x_n) = y_n$ for all n.*

Corollary 1.8. *Let (x_n) and (y_n) be bases of X and Y, respectively. Then $(x_n) \sim (y_n)$ if and only if there exist positive constants C_1 and C_2 such that*

$$C_1 \left\| \sum_{i=1}^{n} a_i y_i \right\| \leq \left\| \sum_{i=1}^{n} a_i x_i \right\| \leq C_2 \left\| \sum_{i=1}^{n} a_i y_i \right\| .$$

for any finite sequence of scalars $(a_i)_{i=1}^n$.

So far, we defined the equivalence of bases. Proposition 1.7 provides a way of generalizing the notion to arbitrary sequences.

Definition 1.9. We say that two sequences (x_n) in X and (y_n) in Y are *equivalent* if there is an isomorphism $T : [x_n] \to [y_n]$ such that $T(x_n) = y_n$ for all n.

The last definition does not appear in either one of the books [1, 4] but it is the natural one. Indeed, if (x_n) and (y_n) are basic sequences, they are equivalent in the sense of Definition 1.9 if and only if they are equivalent as bases of $[x_n]$ and $[y_n]$ respectively. We continue with a discussion of biorthogonal systems.

Definition 1.10. If (x_n) is a sequence in X and there exists a sequence (x_n^*) in X^* such that $x_m^*(x_n) = \delta_{m,n}$ for all m, n, then (x_n) is called a *minimal system* and the pair $((x_n), (x_n^*))$ is called a *biorthogonal system.*

Minimal systems are called equivalent in [4, p. 46] precisely when they satisfy the condition from Definition 1.9.

Definition 1.11. A minimal system (x_n) in X satisfying $[x_n] = X$ is said to be *fundamental.* If (x_n^*) is a minimal system in X^* such that $x_n^*(x) = 0$ for all n implies $x = 0$, then (x_n^*) is said to be *total.*

Remark 1.12. For the sake of completeness, we should mention a slightly different notion of equivalence of sequences. We say that two sequences $(x_n) \subset X$ and $(y_n) \subset Y$ are *congruent with respect to* (X, Y) if there is an invertible operator $T : X \to Y$ such that $T(x_n) = y_n$ for all n [1, Def. 1.3.8]. Clearly, (x_n) and (y_n) are equivalent if and only if they are congruent with respect to $([x_n], [y_n])$.

1.3 Unconditional, subsymmetric and symmetric bases

The special types of bases discussed in this section will be useful for the investigation of iteratively defined norms. A series $\sum_{n=1}^{\infty} x_n$ in X is said to be *unconditionally convergent* if $\sum_{n=1}^{\infty} x_{\pi(n)}$ converges for every permutation π of $\mathbb{N}$. By [4, Proposition 1.c.1], the series $\sum_{n=1}^{\infty} x_n$ converges unconditionally if and only if the series $\sum_{n=1}^{\infty} \theta_n x_n$ converges for every choice of signs $\theta_n = \pm 1$.

Definition 1.13. A basis (x_n) of X with biorthogonal functionals (x_n^*) is called *unconditional* if the series $\sum_{n=1}^{\infty} x_n^*(x) x_n$ converges unconditionally for all $x \in X$.

As in [4, p. 18], we define the natural projections $P_\sigma : X \to X$ associated to the unconditional basis (x_n) by $P_\sigma(\sum_{n=1}^{\infty} a_n x_n) = \sum_{n\in\sigma} a_n x_n$ for each $\sigma \subset \mathbb{N}$. We also define the operators $M_\theta : X \to X$ by $M_\theta(\sum_{n=1}^{\infty} a_n x_n) = \sum_{n=1}^{\infty} \theta_n x_n^*(x) x_n$ for all sequences of signs $\theta = (\theta_n)$. (If X is defined over $\mathbb{R}$, then $\theta_n \in \{-1, 1\}$ and if X is complex, then θ_n are complex numbers such that $|\theta_n| = 1$.) It is not difficult to show $\sup_\sigma \|P_\sigma\| < \infty$ and $\sup_\theta \|M_\theta\| < \infty$ by the uniform boundedness principle. The constant $K_u = \sup_\theta \|M_\theta\|$ is called the *unconditional constant* of (x_n). The following simple proposition characterizes unconditional bases in terms of finite linear combinations of their elements [1, Prop. 3.1.3].

Proposition 1.14. *A basis (x_n) of X is unconditional if and only if there exists a constant $K \geq 1$ such that*

$$\left\| \sum_{i=1}^{n} a_i x_i \right\| \leq K \left\| \sum_{i=1}^{n} b_i x_i \right\| \tag{1.2}$$

for any two finite sequences of scalars $(a_i)_{i=1}^n$ and $(b_i)_{i=1}^n$ satisfying $|a_i| \leq |b_i|$, $i = 1, \ldots, n$.

Remark 1.15. The least constant K for which (1.2) holds is K_u. In fact, this is how the unconditional constant is defined in [1, Def. 3.1.4].

We proceed to define symmetric and subsymmetric bases.

Definition 1.16. A basis (x_n) of X is called *symmetric* if $(x_{\pi(n)})$ is equivalent to (x_n) for any permutation π of the integers.

It is immediately seen that every symmetric basis is unconditional. Let (x_n) be a symmetric basis. Following [4, p. 113], we define the operators $V_\pi : X \to X$ by $V_\pi(\sum_{n=1}^{\infty} a_n x_n) = \sum_{n=1}^{\infty} a_n x_{\pi(n)}$ for every permutation π of the integers. Again by the uniform boundedness principle, $\sup_\pi \|V_\pi\| < \infty$. The constant $K = \sup_{\theta,\pi} \|M_\theta V_\pi\|$, which is consequently also finite, is called the *symmetric constant* of (x_n). It turns out that the bases which will be studied in Chapter 3 satisfy the following weaker property.

Definition 1.17. A basis (x_n) of X is called *subsymmetric* if it is unconditional and (x_{n_i}) is equivalent to (x_n) for any increasing sequence (n_i) of integers.

Assume that (x_n) is a subsymmetric basis. In the same spirit as before we define the operators $S_{(n_i)} : X \to X$ by $S_{(n_i)}(\sum_{n=1}^{\infty} a_n x_n) = \sum_{i=1}^{\infty} a_i x_{n_i}$ for every increasing sequence (n_i) of integers [4, p. 114]. Since the family of these operators is uniformly bounded, we can set $K_1 = \sup_{\theta,(n_i)} \|M_\theta S_{(n_i)}\| < \infty$ as the *subsymmetric constant* of (x_n). The term subsymmetric is justified by the following proposition.

Proposition 1.18 ([4, Prop. 3.a.3])**.** *Every symmetric basis is subsymmetric.*

1.4 Shrinking and boundedly-complete bases

Next we summarize some of the results regarding bases and duality. Since we have already defined biorthogonal systems we can use them to formulate the next well-known fact (see for instance [1, Prop. 3.2.1]).

Proposition 1.19. *Let $(x_n, x_n^*) \subset X \times X^*$ be a biorthogonal system. If (x_n) is a basis of X with basis constant K, then (x_n^*) is a basic sequence in X^* with basis constant less than or equal to K.*

If (x_n) is a minimal system in X, we will denote by H the closed linear space spanned by the biorthogonal functionals x_n^* in X^*, i.e. $H = [x_n^*] \subset X^*$. The following definition is adapted from [1, Def. 3.2.2].

Definition 1.20. Let X be a Banach space and Y a subspace of X^*. If the functional given by

$$\|x\|_Y = \sup\{|y^*(x)| : y^* \in Y, \|y^*\| = 1\}$$

satisfies

$$c\|x\| \leq \|x\|_Y \leq \|x\|$$

for some positive constant $c \leq 1$ and all $x \in X$, then Y is said to be a *c-norming subspace for X in X^*.*

The situation when H is c-norming will be of some importance in the next chapter. It turns out that H meets the condition from Definition 1.20 in particular if (x_n) is a basis [1, Prop. 3.2.3].

Proposition 1.21. *Let $(x_n, x_n^*) \subset X \times X^*$ be a biorthogonal system. If (x_n) is a basis of X with basis constant K, then H is a K^{-1}-norming subspace of X^*.*

Now we introduce two special types of bases.

Definition 1.22. A basis (x_n) of X with biorthogonal functionals (x_n^*) is called *shrinking* if $[x_n^*] = X^*$. It is called *boundedly-complete* if $\sup_n \|\sum_{i=1}^{n} a_i x_i\| < \infty$ implies the convergence of $\sum_{i=1}^{\infty} a_i x_i$ for all scalar sequences (a_n).

A simple characterization of shrinking bases can be stated in terms of block basic sequences, which we define next.

Definition 1.23. Let $(x_n) \subset X$ be a basic sequence in X. If (k_n) is a sequence of scalars such that $0 = k_0 < k_1 < \ldots$, (a_i) is an arbitrary sequence of scalars and the vectors

$$u_n = \sum_{i=k_{n-1}+1}^{k_{n+1}} a_i x_i$$

are nonzero, then (u_n) is called a *block basic sequence* of (x_n).

The proof of the next proposition can be found in [1, Prop. 3.2.7].

Proposition 1.24. *A basis (x_n) of X is shrinking if and only if every bounded block basic sequence (u_n) of (x_n) satisfies $u_n \xrightarrow{w} 0$.*

The notions of shrinking and boundedly-complete basis are dual in some sense. Our statement of this fact is a combination of Theorems 3.2.10 and 3.2.12 from [1].

Proposition 1.25. *Let (x_n) be a basis of X with biorthogonal functionals (x_n^*). Then*

(i) (x_n) is shrinking if and only if (x_n^*) is a boundedly-complete basis of X^*,

(ii) (x_n) is boundedly-complete if and only if (x_n^*) is a shrinking basis of H.

The two types of bases provide an elegant characterization of reflexivity in the next famous theorem of James [1, Thm. 3.2.13].

Theorem 1.26. *A Banach space X with a basis (x_n) is reflexive if and only if (x_n) is both shrinking and boundedly-complete.*

If we know that a given basis is unconditional, we can decide whether it is shrinking or boundedly-complete by looking at its block basic sequences. We list two criteria from [1, Thm. 3.3.1 and 3.3.2].

Theorem 1.27. *Let (x_n) be an unconditional basis of a Banach space X. Then*

(i) (x_n) fails to be shrinking if and only X contains a complemented subspace isomorphic to ℓ_1,

(ii) (x_n) fails to be boundedly-complete if and only if X contains a complemented subspace isomorphic to c_0.

We note that the word "complemented" can be dropped, cf. [1, Thm. 3.3.1 and 3.3.2].

Remark 1.28. Equivalent criteria can also be formulated in terms of block basic sequences. Under the assumptions of the theorem, (x_n) fails to be shrinking or boundedly-complete if and only if there is a block basic sequence (u_n) of (x_n) equivalent to the canonical basis of ℓ_1 or c_0, respectively.

2. Ordering on bases

2.1 Basic properties

Definition 0.1 states what it means for one basis to dominate another. Its immediate consequences are examined in this section.

Proposition 2.1. *Two bases (x_n) of X and (y_n) of Y satisfy $(y_n) \preceq (x_n)$ if and only if there exists $T : X \to Y$ such that $T(x_n) = y_n$ for all n.*

Proof. Assume first that the operator T exists. If (a_n) is a sequence of scalars, then

$$\sum_{i=1}^{n} a_i y_i = \sum_{i=1}^{n} a_i T(x_i) = T\left(\sum_{i=1}^{n} a_i x_i\right)$$

for all n. Hence the convergence of the series $\sum_{n=1}^{\infty} a_n x_n$ implies the convergence of $\sum_{n=1}^{\infty} a_n y_n$. Conversely, if $(y_n) \preceq (x_n)$, then we define the operator T by

$$T\left(\sum_{n=1}^{\infty} a_n x_n\right) = \sum_{n=1}^{\infty} a_n y_n.$$

The mapping is well-defined and linear. Next, we use the Closed Graph Theorem to show that T is continuous. Let (u_j) be a sequence in X such that $u_j \to u$ in X and $T(u_j) \to v$ in Y. The expansions of u_j and u in the basis (x_n) are $u_j = \sum_{n=1}^{\infty} x_n^*(u_j) x_n$ and $u = \sum_{n=1}^{\infty} x_n^*(u) x_n$. The continuity of the biorthogonal functionals associated to the bases (x_n) and (y_n) gives $x_n^*(u_j) \to x_n^*(u)$ and also $x_n^*(u_j) = y_n^*(Tu_j) \to y_n^*(v)$ for all n. By the uniqueness of limit $x_n^*(u) = y_n^*(v)$ for all n. This means $T(u) = v$ and T is therefore continuous. □

Corollary 2.2. *Let (x_n) and (y_n) be bases of X and Y, respectively. Then $(y_n) \preceq (x_n)$ if and only if there exists a positive constant C such that*

$$\left\|\sum_{i=1}^{n} a_i y_i\right\| \leq C \left\|\sum_{i=1}^{n} a_i x_i\right\| \tag{2.1}$$

for any finite sequence of scalars $(a_i)_{i=1}^{n}$.

Definition 0.1 can be generalized to arbitrary sequences similarly to the definition of equivalence.

Definition 2.3. We say that a sequence (x_n) in X *dominates* a sequence (y_n) in Y if there is an operator $T : [x_n] \to [y_n]$ such that $T(x_n) = y_n$ for all n. We again write $(y_n) \preceq (x_n)$.

Lemma 2.4. *Let (x_n), (y_n) and (z_n) be sequences in X, Y, Z respectively.*

1) $(x_n) \sim (x_n)$

2) If $(z_n) \preceq (y_n)$ and $(y_n) \preceq (x_n)$, then $(z_n) \preceq (x_n)$.

3) $(y_n) \preceq (x_n)$ and $(x_n) \preceq (y_n)$ if and only if $(x_n) \sim (y_n)$.

Proof. The lemma is an immediate consequence of Definitions 1.9 and 2.3. □

Assume that (x_n) is a sequence in X whose elements are linearly independent and (y_n) is an arbitrary sequence in Y. We note that $(x_n) \preceq (y_n)$ if and only if there is a constant $C > 0$ such that (2.1) holds. Indeed, the existence of $T : [x_n] \to [y_n]$, $T(x_n) = y_n$ for all n implies

$$\left\| \sum_{i=1}^{n} a_i y_i \right\| = \left\| T\left(\sum_{i=1}^{n} a_i x_i \right) \right\| \le \|T\| \cdot \left\| \sum_{i=1}^{n} a_i x_i \right\|.$$

On the other hand, the linear independence assumption enables us to define $T : \mathrm{span}(x_n) \to \mathrm{span}(y_n)$ by

$$T\left(\sum_{i=1}^{n} a_i x_i \right) = \sum_{i=1}^{n} a_i y_i.$$

If (2.1) holds, then T is continuous with $\|T\| \le C$. Because $\mathrm{span}(x_n)$ is dense in $[x_n]$ the operator T has a unique extension to $[x_n]$. This also proves Corollary 2.2.

We next examine how $\preceq$ behaves with respect to duality.

Lemma 2.5. *Let $(x_n, x_n^*) \subset X \times X^*$ and $(y_n, y_n^*) \subset Y \times Y^*$ be two biorthogonal systems. Denote by $j_X : X \to X^{**}$ the canonical embedding of X into its second dual.*

- *(i) If $(y_n) \preceq (x_n)$ and there exists a constant $c > 0$ such that $[j_X(x_n)]$ is c-norming for $[x_n^*]$, then $(x_n^*) \preceq (y_n^*)$.*
- *(ii) If $(x_n^*) \preceq (y_n^*)$ and there exists a constant $c > 0$ such that $[y_n^*]$ is c-norming for $[y_n]$, then $(y_n) \preceq (x_n)$.*

Proof. (i). Let us define the operator $r_X : X^* \to [x_n]^*$ as the restriction of each functional in X^* to $[x_n]$. The operator $r_Y : Y^* \to [y_n]^*$ is defined analogously. Since $(y_n) \preceq (x_n)$, there exists $T : [x_n] \to [y_n]$ such that $T(x_n) = y_n$ for all n. Consider the dual operator $T^* : [y_n]^* \to [x_n]^*$. We get

$$(T^*(r_Y y_m^*))(x_n) = r_Y y_m^*(T(x_n)) = y_m^*(y_n) = \delta_{m,n}$$

for any m, n. In other words, T^* maps $r_Y y_m^*$ to $r_X x_m^*$. Now we take an arbitrary $x^* = \sum_{i=1}^{n} a_i x_i^*$ and define $y^* = \sum_{i=1}^{n} a_i y_i^*$. By what we just showed,

$$T^*(r_Y y^*) = T^*\left(\sum_{i=1}^{n} a_i r_Y y_i^* \right) = \sum_{i=1}^{n} a_i r_X x_i^* = r_X x^*.$$

Now the assumption that $[j_X(x_n)]$ is c-norming for $[x_n^*]$ implies

$$\begin{aligned} c\|x^*\| &\le \sup\{|z(x^*)|;\ z \in [j_X(x_n)], \|z\| = 1\} \\ &= \sup\{|x^*(x)|;\ x \in [x_n], \|x\| = 1\} = \|r_X x^*\|, \end{aligned}$$

where we used that the canonical embedding j_X is an isometry. We obtain

$$\|x^*\| \le \frac{1}{c}\|r_X x^*\| = \frac{1}{c}\|T^*(r_Y y^*)\| \le \frac{\|T^*\|}{c}\|r_Y y^*\| \le \frac{\|T^*\|}{c}\|y^*\|.$$

Because $x^* = \sum_{i=1}^n a_i x_i$ was arbitrary we proved (2.1) with $C = \frac{\|T\|}{c}$. By the discussion preceding this proof, $(x_n^*) \preceq (y_n^*)$ as claimed.

(ii). We show that the second statement follows from the first. For any Banach space Z, let $j_Z : Z \to Z^{**}$ denote the canonical embedding. We claim that the condition on $[y_n^*]$ implies $[j_{Y^*}(y_n^*)]$ is c-norming for $[j_Y(y_n)]$. Indeed,

$$\begin{aligned} c\|j_Y(y)\| = c\|y\| &\leq \sup\{y^*(y);\ y^* \in [y_n^*], \|y^*\| = 1\} \\ &= \sup\{(j_{Y^*}(y^*))(j_Y(y));\ y^* \in [y_n^*], \|y^*\| = 1\} \\ &= \sup\{z(j_Y(y));\ z \in [j_{Y^*}(y_n^*)], \|z\| = 1\} \end{aligned}$$

for any $y \in Y$. From the first part of the lemma applied on $(\tilde{y_n}) = (x_n^*)$ and $(\tilde{x_n}) = (y_n^*)$ instead of (y_n) and (x_n) it follows that $(j_Y(y_n)) \preceq (j_X(x_n))$. This is the same as $(y_n) \preceq (x_n)$. □

Remark 2.6. The condition of Lemma 2.5 (i) is satisfied in particular when (x_n) is fundamental. In that case $X = [x_n]$, and it follows that $[j_X(x_n)]$ is 1-norming for $[x_n^*]$.

Proposition 2.7. *Let (x_n) be a basis of X and (y_n) be a basis of Y. Then $(y_n) \preceq (x_n)$ if and only if $(x_n^*) \preceq (y_n^*)$.*

Proof. If (y_n) has basis constant K, then $[y_n^*]$ is $1/K$-norming for Y by Proposition 1.21. The implication "⇐" now follows from Lemma 2.5 (ii). On the other hand, $j_X(X)$ is 1-norming for $[x_n^*]$ because (x_n) is a basis of X. Therefore, Lemma 2.5 (i) implies "⇒". □

Proposition 2.8. *Let $(x_n, x_n^*) \subset X \times X^*$ be a biorthogonal system.*

(i) If (e_n) denotes the canonical basis of ℓ_1, then $(x_n) \preceq (e_n)$ is equivalent to $\sup\limits_n \|x_n\| < \infty$.

(ii) If (e_n) denotes the canonical basis of c_0 and (x_n) is fundamental, then $(e_n) \preceq (x_n)$ is equivalent to $\sup\limits_n \|x_n^*\| < \infty$.

Proof. We first prove (i). By the remarks made after Definition 2.3, $(x_n) \preceq (e_n)$ is equivalent to the existence of $C > 0$ such that

$$\left\| \sum_{i=1}^n a_i x_i \right\| \leq C \sum_{i=1}^n |a_i| \tag{2.2}$$

for all finite sequences of scalars $(a_i)_{i=1}^n$. But (2.2) is clearly equivalent to $\|x_n\| \leq C$ for all n. Next we prove (ii). By Remark 2.6, we can apply Lemma 2.5 (i) to show that $(e_n) \preceq (x_n)$ implies $(x_n^*) \preceq (e_n^*)$. On the other hand, (e_n^*) is the canonical basis of ℓ_1, and $[e_n^*]$ is therefore 1-norming for $c_0 = [e_n]$. Hence we can use Lemma 2.5 (ii) to deduce $(x_n^*) \preceq (e_n^*)$ implies $(e_n) \preceq (x_n)$. By part (i), $(x_n^*) \preceq (e_n^*)$ is equivalent to $\sup_n \|x_n^*\| < \infty$. □

The assumption made on (x_n) in the second statement of Proposition 2.8 is not restrictive. If (x_n) is not fundamental, we simply take the restrictions of x_n^* to $[x_n]$ instead of the functionals themselves.

Remark 2.9. If (e_n) denotes the canonical basis of c_0, then by Proposition 2.8

$$(e_n) \preceq (x_n) \preceq (e_n^*)$$

for any seminormalized basis (x_n). Assume that (x_n) has basis constant K and its elements satisfy $c_1 \leq \|x_n\| \leq c_2$, where c_1 and c_2 are some positive constants. Then

$$c_1|x_n^*(x)| \leq \|x_n^*(x)x_n\| = \|P_n(x) - P_{n-1}(x)\| \leq 2K\|x\|$$

for $x \in X$, and hence $\|x_n^*\| \leq \frac{2K}{c_1}$ for all n. Thus we have both $\sup_n \|x_n\| < \infty$ and $\sup_n \|x_n^*\| < \infty$.

In the Section 2.2 a situation arises in which $(z_n) \preceq (y_n) \preceq (x_n)$, where (x_n) and (z_n) are known to be bases. We may ask whether (y_n) must be a basis of $[y_n]$. The next example shows that the answer is negative.

Example 2.10. We consider the space $C_{2\pi}$ of all real-valued continuous functions on $\mathbb{R}$ having period 2π equipped with the norm $\|x\| = \max_{t\in[0,2\pi]} |x(t)|$. Let us define

$$x_0(t) = \frac{1}{2}, \quad x_{2n-1}(t) = \sin nt, \quad x_{2n}(t) = \cos nt$$

for $t \in \mathbb{R}$ and $n \in \mathbb{N}$. The biorthogonal functionals x_n^* associated to the elements x_n are given by

$$x_{2n}^*(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} x(t)\cos(nt)dt, \quad x_{2n+1}^*(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} x(t)\sin((n+1)t)dt$$

for $x \in C_{2\pi}$ and $n \in \mathbb{N} \cup \{0\}$. The sequence $(x_n)_{n=0}^{\infty}$ is fundamental but not basic [8, Example 4.1.]. We crudely estimate

$$\|x_{2n}^*\| \leq \frac{1}{\pi}\int_{-\pi}^{\pi} |\cos(nt)|dt \leq 2, \quad \|x_{2n+1}^*\| \leq \frac{1}{\pi}\int_{-\pi}^{\pi} |\sin((n+1)t)|dt \leq 2.$$

Since $\|x_n\| \leq 1$ and $\|x_n^*\| \leq 2$ for all $n \in \mathbb{N} \cup \{0\}$, Proposition 2.8 shows $(e_n) \preceq (x_n) \preceq (e_n^*)$, where $(e_n)_{n=0}^{\infty}$ is the canonical basis of c_0.

If (x_n) is a basis of X, then it is possible to consider the dual functionals associated to its dual functionals. Let us explain what we mean exactly. The sequence (x_n^*) is basic in X^*, hence a basis of $[x_n^*]$. Then the dual functionals associated to x_n^* are elements $x_m^{**} \in [x_n^*]^*$.

Lemma 2.11. *If (x_n) is a basis of X, then $(x_n^{**}) \sim (x_n)$.*

Proof. We show $T : X \to [x_n^*]^*$ defined by $Tx_n = x_n^{**}$ is an isomorphism. If $j : X \to X^{**}$ denotes the canonical embedding, then x_n^{**} can be identified with the restriction of $j(x_n)$ to $[x_n^*]$. If $r : X^{**} \to [x_n^*]^*$ denotes the restriction operator, then $T = r \circ j$, which proves that T is bounded. By Proposition 1.21, $[x_n^*]$ is $1/K$-norming for X (K being the basis constant of (x_n)). Hence

$$\frac{1}{K}\|x\| \leq \{|x^*(x)|;\ x^* \in [x_n^*], \|x^*\| = 1\} = \|r(jx)\|$$

for all $x \in X$, which proves T is bounded from below. □

We will be concerned with the system of all bases in the remainder of this section. We must first explain why this system can be viewed as a set. By the Banach-Mazur Theorem, every separable Banach space X embeds isometrically into $C([0,1])$ [1, Thm. 1.4.3]. Hence any basis can be identified with a basic sequence in $C([0,1])$. These comments justify the following definition.

Definition 2.12. Let B denote the set of all basic sequences in $C([0,1])$. We designate by $\mathfrak{B}$ the quotient set $\mathfrak{B} = B/\sim$ and call it the *set of all bases*. The equivalence class of $(x_n) \in B$ is denoted $[(x_n)]_\sim \in \mathfrak{B}$. We define the relation $\preceq$ on $\mathfrak{B}$ by

$$[(y_n)]_\sim \preceq [(x_n)]_\sim \Leftrightarrow (y_n) \preceq (x_n).$$

Lemma 2.13. *The relation $\preceq$ on $\mathfrak{B}$ is well defined and $(\mathfrak{B}, \preceq)$ is a partially ordered set.*

Proof. Let $(x_n^1) \sim (x_n^2)$, $(y_n^1) \sim (y_n^2)$ and suppose that $(y_n^1) \preceq (x_n^1)$. It follow from Lemma 2.4 that $(y_n^2) \preceq (x_n^2)$, which shows that the relation $\preceq$ on $\mathfrak{B}$ is well defined. The fact that $(\mathfrak{B}, \preceq)$ is a partially ordered set is also a consequence of Lemma 2.4. □

As was explained above Definition 2.12, if (x_n) is a basis of some Banach space X, then there exists a unique equivalence class $[(y_n)]_\sim$ in $\mathfrak{B}$ such that $(x_n) \sim (y_n)$. By an abuse of notation, we identify (x_n) with this class and write simply $(x_n) \in \mathfrak{B}$.

We show that it is possible to define the operations of supremum and infimum on $\mathfrak{B}$ so that $(\mathfrak{B}, \preceq)$ together with these operations forms a lattice.

Definition 2.14. Let (x_n), (y_n) be bases of X, Y respectively. The supremum of (x_n) and (y_n) is defined as

$$(x_n) \vee (y_n) = ((x_n, y_n)), \tag{2.3}$$

where $((x_n, y_n))$ is a sequence in $X \oplus Y$ consisting of the ordered tuples (x_n, y_n). The infimum of (x_n) and (y_n) is defined as

$$(x_n) \wedge (y_n) = ((x_n^*, y_n^*)^*), \tag{2.4}$$

where $((x_n^*, y_n^*))$ is the sequence of dual functionals associated with the sequence $((x_n^*, y_n^*))$ in $X^* \oplus Y^*$.

Lemma 2.15. *The operations $\vee$ and $\wedge$ are well defined on $\mathfrak{B}$.*

Proof. The Grundblum's criterion (Proposition 1.5) will be used to prove that (x_n, y_n) is basic if (x_n) and (y_n) are basic. Denote by K_1, K_2 the basis constants of (x_n), (y_n) respectively and take any m, n such that $m \leq n$. Then any sequence (a_i) of scalars satisfies

$$\left\|\sum_{i=1}^{m} a_i x_i\right\| \leq K_1 \left\|\sum_{i=1}^{n} a_i x_i\right\|, \qquad \left\|\sum_{i=1}^{m} a_i y_i\right\| \leq K_2 \left\|\sum_{i=1}^{n} a_i y_i\right\|$$

and the definition of norm in $X \oplus Y$ gives

$$\left\|\sum_{i=1}^{m} a_i (x_i, y_i)\right\| \leq \max\{K_1, K_2\} \left\|\sum_{i=1}^{n} a_i (x_i, y_i)\right\|.$$

Application of Proposition 1.5 proves the claim.

We know that (x_n^*) and (y_n^*) are basic sequences from Proposition 1.19. The same argument as before shows that (x_n^*, y_n^*) is a basic sequence in $X^* \oplus Y^*$. The sequence of dual functionals $((x_n^*, y_n^*))^*)$ is again basic in $(X^* \oplus Y^*)^*$.

Let $(x_{i,n})$, $(y_{i,n})$ be bases of X_i, Y_i respectively for $i = 1, 2$. Let us assume $(x_{i,n}) \sim (x_{2,n})$ and $(y_{1,n}) \sim (y_{2,n})$.

We first show that $\vee$ is well defined as an operation on $\mathfrak{B}$. By Proposition 1.7, there exist isomorphisms $T_1 : X_1 \to X_2$ and $T_2 : Y_1 \to Y_2$ such that $T_1(x_{1,n}) = x_{2,n}$ and $T_2(y_{1,n}) = y_{2,n}$ for all n. We let $T : X_1 \oplus Y_1 \to X_2 \oplus Y_2$ be given by $T((x, y)) = (T_1(x), T_2(y))$. Then T is an isomorphism mapping $(x_{1,n}, y_{1,n})$ to $(x_{2,n}, y_{2,n})$, which proves $((x_{1,n}, y_{1,n})) \sim ((x_{2,n}, y_{2,n}))$.

Next, we show that $\wedge$ is well defined on $\mathfrak{B}$. By Proposition 2.7, $(x_{1,n}^*) \sim (y_{1,n}^*)$ and $(x_{2,n}^*) \sim (y_{2,n}^*)$. Hence $((x_{1,n}^*, y_{1,n}^*)) \sim ((x_{2,n}^*, y_{2,n}^*))$ and using Proposition 2.7 once more we get $((x_{1,n}^*, y_{1,n}^*)^*) \sim ((x_{2,n}^*, y_{2,n}^*)^*)$. □

Theorem 2.16. *The partially ordered set $(\mathfrak{B}, \preceq)$ together with the operations $\vee$ and $\wedge$ forms a lattice.*

Proof. Let $(x_n), (y_n) \in \mathfrak{B}$ be fixed. Firstly, $(x_n) \preceq ((x_n, y_n))$ holds because

$$\left\| \sum_{i=1}^{n} a_i x_i \right\| \leq \left\| \sum_{i=1}^{n} a_i (x_i, y_i) \right\|$$

for any sequence of scalars (a_i) (we are using Corollary 2.2). Analogously, $(y_n) \preceq ((x_n, y_n))$. Secondly, if (z_n) is a basis of Z such that $(x_n) \preceq (z_n)$ and $(y_n) \preceq (z_n)$, then $((x_n, y_n)) \preceq (z_n)$. Indeed, according to Corollary 2.2, there are positive constants C_1 and C_2 such that

$$\left\| \sum_{i=1}^{n} a_i x_i \right\| \leq C_1 \left\| \sum_{i=1}^{n} a_i z_i \right\|, \qquad \left\| \sum_{i=1}^{n} a_i y_i \right\| \leq C_2 \left\| \sum_{i=1}^{n} a_i z_i \right\|$$

for any sequence of scalars (a_i). The sequence $((x_n, y_n))$ then satisfies

$$\left\| \sum_{i=1}^{n} a_i (x_i, y_i) \right\| = \left\| \sum_{i=1}^{n} a_i x_i \right\| + \left\| \sum_{i=1}^{n} a_i y_i \right\| \leq (C_1 + C_2) \left\| \sum_{i=1}^{n} a_i z_i \right\|,$$

which proves that $((x_n, y_n))$ is the supremum of (x_n) and (y_n).

The result on infimum is proved by exploiting duality. Let $Z = [(x_n^*, y_n^*)]$. By the first part of the proof, $(x_n^*) \preceq ((x_n^*, y_n^*))$. Because (x_n^*) is a basis of $[x_n^*]$ and $((x_n^*, y_n^*))$ is a basis of Z, we obtain by Proposition 2.7 and Lemma 2.11

$$((x_n^*, y_n^*)^*) \preceq (x_n^{**}) \sim (x_n).$$

Analogously, $((x_n^*, y_n^*)^*) \preceq (y_n)$.

What remains is to show $(w_n) \preceq (x_n)$ and $(w_n) \preceq (y_n)$ implies $(w_n) \preceq ((x_n^*, y_n^*))^*$ for any $(w_n) \in \mathfrak{B}$. Again by Proposition 2.7, $(x_n^*) \preceq (w_n^*)$ and $(y_n^*) \preceq (w_n^*)$. Then also $((x_n^*, y_n^*)) \preceq (w_n^*)$ since $((x_n^*, y_n^*))$ is the supremum of (x_n^*) and (y_n^*). Finally, Lemma 2.11 and a third application of Proposition 2.7 yield

$$(w_n) \sim (w_n^{**}) \preceq ((x_n^*, y_n^*)^*),$$

which concludes the proof. □

It turns out the operations $\vee$ and $\wedge$ preserve the properties "being shrinking" and "being boundedly-complete". We prove this for the supremum as a lemma.

Lemma 2.17. *Assume (x_n) is a basis of X and (y_n) is a basis of Y. Denote $Z = [(x_n, y_n)] \subset X \oplus Y$ so that $((x_n, y_n))$ is a basis of Z. If (x_n) and (y_n) are shrinking, then $((x_n, y_n))$ is shrinking. If (x_n) and (y_n) are boundedly-complete, then $((x_n, y_n))$ is boundedly-complete.*

Proof. The characterization of shrinking bases (Proposition 1.24) is used to prove the first claim. We let $z_n = (x_n, y_n)$ and take a block basic sequence (u_n) of (z_n) given by

$$u_n = \sum_{i=k_n}^{k_{n+1}} a_i z_i.$$

Putting $u_n^1 = \sum_{i=k_n}^{k_{n+1}} a_i x_i$ and $u_n^2 = \sum_{i=k_n}^{k_{n+1}} a_i y_i$ we can write $u_n = (u_n^1, u_n^2)$. Because Z is a subspace of $X \oplus Y$, any functional $z^* \in Z^*$ can be extended to $X \oplus Y$ by the Hahn-Banach Theorem. Every bounded functional on $X \oplus Y$ is represented as (x^*, y^*), where $x^* \in X^*$, $y^* \in Y^*$ and

$$(x^*, y^*)(x, y) = x^*(x) + y^*(y), \qquad (x, y) \in X \oplus Y.$$

Now we fix z^* and find $x^* \in X^*$, $y^* \in Y^*$ such that $(x^*, y^*)|_Z = z^*$. Then

$$z^*(u_n) = x^*(u_n^1) + y^*(u_n^2). \tag{2.5}$$

Since $(x_n) \preceq ((x_n, y_n))$, the block basic sequence (u_n^1) is bounded. The same holds for (u_n^2). By Proposition 1.24, (u_n^1) and (u_n^2) are weakly null because (x_n) and (y_n) are shrinking. It follows from (2.5) that (u_n) is weakly null and (z_n) is therefore shrinking.

To prove the second claim, we assume that (a_n) is a sequence for scalars for which there exists a constant $C > 0$ such that

$$\left\| \sum_{i=1}^{n} a_i (x_i, y_i) \right\| \leq C$$

for all n. Then also $\| \sum_{i=1}^{n} a_i x_i \| \leq C$ and $\| \sum_{i=1}^{n} a_i y_i \| \leq C$ for all n. If (x_n) and (y_n) are boundedly-complete, the series $\sum_{i=1}^{\infty} a_i x_i$ and $\sum_{i=1}^{\infty} a_i y_i$ converge. Consequently, $\sum_{i=1}^{\infty} a_i (x_i, y_i)$ converges in $X \oplus Y$ and $((x_n, y_n))$ is therefore boundedly-complete. □

Definition 2.18. We let $\mathfrak{S} \subset \mathfrak{B}$ denote the subset of all shrinking bases and $\mathfrak{Bc} \subset \mathfrak{B}$ denote the subset of all boundedly-complete bases.

Theorem 2.19. *The partially ordered sets $(\mathfrak{S}, \preceq)$ and $(\mathfrak{Bc}, \preceq)$ form sublattices of $(\mathfrak{B}, \preceq)$ with the operations $\vee$ and $\wedge$.*

Proof. We prove the theorem for shrinking bases since the proof for boundedly-complete bases is the same. If $(x_n), (y_n) \in \mathfrak{S}$, then $(x_n) \vee (y_n) \in \mathfrak{S}$ by Lemma 2.17. It follows from Proposition 1.25 that the dual bases (x_n^*) and (y_n^*) are boundedly-complete. Again by Lemma 2.17, $((x_n^*, y_n^*))$ is also boundedly-complete. The second part of Proposition 1.25 shows that $(x_n) \wedge (y_n) = ((x_n^*, y_n^*)^*)$ is shrinking. □

2.2 Factoring the inclusion operator

2.2.1 Factoring through a reflexive space

The main goal of this section is to examine how the ordering on bases behaves with respect to the properties "being shrinking" and "being boundedly-complete". We focus on the situation when a boundedly-complete basis (z_n) of Z and a shrinking basis (x_n) of X satisfy $(z_n) \preceq (x_n)$. Let us make the additional assumption that the bases are normalized. We first present an example which shows that (z_n) is not necessarily shrinking and (x_n) is not necessarily boundedly-complete.

Definition 2.20 ([4, Def. 4.e.1])**.** Let $1 \leq p < \infty$. For a sequence $w = (w_n)$ tending to 0 such that $1 = w_1 \geq w_2 \ldots \geq w_n \geq \ldots > 0$ and $\sum_{n=1}^{\infty} w_n = \infty$, let

$$\|(a_n)\|_{w,p} = \sup_{\pi} \left(\sum_{n=1}^{\infty} |a_{\pi(n)}|^p w_n \right)^{\frac{1}{p}},$$

where the supremum is taken over all permutations π of the integers. The Banach space of all scalar sequences (a_n) for which $\|(a_n)\|_{w,p} < \infty$ is called a *Lorentz sequence space* and is denoted by $d(w,p)$.

Two facts about Lorentz spaces are used in the subsequent example:

(F1) The sequence (e_n) of canonical unit vectors is a normalized basis of the space $d(w,p)$ [1, Problems 9.4].

(F2) The space $d(w,p)$ is reflexive if and only if $p > 1$ [4, p. 178].

Example 2.21. Let us assume that the sequence (w_n) from the previous definition additionally satisfies $\sum_{n=1}^{\infty} w_n^2 < \infty$ (we set for example $w_n = \frac{1}{n}$). We check that the canonical basis (e_n) of $d(w,1)$ is boundedly-complete. Let (a_i) be a sequence of scalars such that

$$\left\| \sum_{i=1}^{n} a_i e_i \right\|_{d(w,p)} \leq C$$

for all n. Then we have $\sum_{i=1}^{n} |a_{\pi(i)}| w_i \leq C$ for any permutation π of the integers and hence $\sum_{i=1}^{\infty} |a_{\pi(i)}| w_i \leq C$. This proves $(a_n) \in d(w,1)$ and (e_n) is therefore boundedly-complete. On the other hand, $d(w,1)$ is not reflexive by (F2). It follows from James' theorem 1.26 that (e_n) is not shrinking.

Now we take (x_n) to be the canonical basis of ℓ_2. If π is a fixed permutation of the integers, then by Hölder's inequality

$$\sum_{i=1}^{n} |a_{\pi(i)}| w_i \leq \left(\sum_{i=1}^{n} w_i^2 \right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} |a_{\pi(i)}|^2 \right)^{\frac{1}{2}} \leq \left(\sum_{i=1}^{\infty} w_i^2 \right)^{\frac{1}{2}} \left\| \sum_{i=1}^{\infty} a_i x_i \right\|_{\ell_2}.$$

Passing to the supremum over all π yields

$$\left\| \sum_{i=1}^{\infty} a_i e_i \right\|_{d(w,p)} \leq \left(\sum_{i=1}^{\infty} w_i^2 \right)^{\frac{1}{2}} \left\| \sum_{i=1}^{\infty} a_i x_i \right\|_{\ell_2}$$

and since we assumed $\sum_{i=1}^{\infty} w_i^2 < \infty$, we proved $(e_n) \preceq (x_n)$. The basis (x_n) is shrinking while (e_n) is not.

We conclude the example by taking the sequences of dual functionals. Proposition 2.7 tells us that $(x_n^*) \preceq (e_n^*)$. The basis (x_n^*) is equivalent to the canonical basis of ℓ_2 and hence boundedly-complete. The basis (e_n^*) is shrinking but not boundedly-complete (Proposition 1.25).

According to Proposition 2.1, $(z_n) \preceq (x_n)$ is equivalent to the existence of a bounded operator $T : X \to Z$ with $T(x_n) = z_n$ for all n. Now we prove that this operator is weakly compact in the setting under discussion, which was stated already in the Introduction as Theorem 0.2.

Proof of Theorem 0.2. Because (x_n) is shrinking, the dual functionals (x_n^*) form a basis of X^* and $x^* = \sum_{i=1}^{\infty} x^*(x_n)x_n^*$ for $x^* \in X^*$. If we identify each x_n with its image under the canonical embedding of X into X^{**}, we can regard (x_n) as a sequence in X^{**}. Let $x^{**} \in X^{**}$. Then

$$\lim_{n\to\infty} \left\langle \sum_{i=1}^{n} x^{**}(x_n^*)x_n, x^* \right\rangle = \lim_{n\to\infty} x^{**} \left(\sum_{i=1}^{n} x^*(x_n)x_n^* \right) = x^{**}(x^*)$$

for any $x^* \in X^*$. Hence the sequence $(\sum_{i=1}^{n} x^{**}(x_n^*)x_n)$ w^*-converges to x^{**} (in the topology $\sigma(X^{**}, X^*)$). In particular, it is bounded. Because T^{**} is a dual operator, it is continuous from the topology $\sigma(X^{**}, X^*)$ to $\sigma(Z^{**}, Z^*)$, and so

$$T^{**} \left(\sum_{i=1}^{n} x^{**}(x_n^*)x_n \right) \xrightarrow{w^*} T^{**}(x^{**}). \tag{2.6}$$

Since $(z_n) \preceq (x_n)$, the sequence $(\sum_{i=1}^{n} x^{**}(x_n^*)z_n)$ is also bounded. The assumption that (z_n) is boundedly-complete implies that $\sum_{i=1}^{\infty} x^{**}(x_n^*)z_n$ converges to some $z \in Z$, hence

$$T^{**} \left(\sum_{i=1}^{n} x^{**}(x_n^*)x_n \right) = \sum_{i=1}^{n} x^{**}(x_n^*)z_n \to z. \tag{2.7}$$

From (2.6) and (2.7) we obtain $T^{**}(x^{**}) = z$. This proves $T^{**}(X^{**}) \subset Z$. By Gantmacher's theorem, the operator T is weakly compact. □

Weakly compact operators can be characterized as the operators which factor through a reflexive space. This result was first proved by Davis et al [3]. The basis of the proof is an interpolation method. We want to show that in our setting, the operator T factors through a space Y with a basis (y_n) such that $(z_n) \preceq (y_n) \preceq (x_n)$. Instead of following the proof in [3] directly, we work with a fairly general interpolation scheme described in [5, Section 2.g].

Definition 2.22. Let Z and E be a Banach spaces such that E has a normalized unconditional basis (e_i) with unconditional constant one. Assume that $\|\cdot\|_i$ is an equivalent norm on Z for all i and denote $Z_i = (Z, \|\cdot\|_i)$. If $z \in Z$, we set

$$\|z\|_Y = \sup_m \left\| \sum_{i=1}^{m} \|z\|_i e_i \right\|_E .$$

Then we define the space

$$\tilde{Y}((Z_i), E) = \{z \in Z;\ \|z\|_Y < \infty\}$$

equipped with the norm $\|\cdot\|_Y$ and its subspace $Y((Z_i), E)$ of all $y \in \tilde{Y}((Z_i), E)$ such that $\sum_{i=1}^{\infty} \|y\|_i e_i$ converges.

The assumptions on E guarantee that (1.2) in Proposition 1.14 holds with $K = 1$. This is used in the subsequent proof several times.

Proposition 2.23. *The functional $\|\cdot\|_Y$ is a norm on $\tilde{Y}((Z_i), E)$ and $\tilde{Y}((Z_i), E)$ with this norm is a Banach space. $Y((Z_i), E)$ is a closed subspace of $\tilde{Y}((Z_i), E)$.*

Proof. We begin by proving that $\|\cdot\|_Y$ satisfies the properties of a norm. If $y \in \tilde{Y}((Z_i), E)$, then $\|y\|_Y = 0$ if and only if $\|y\|_i = 0$ for all i and thus if and only if $y = 0$. Next, for any scalar λ,

$$\|\lambda y\|_Y = \sup_m \left\| \sum_{i=1}^m \|\lambda y\|_i e_i \right\|_E = \sup_m \left\| \sum_{i=1}^m |\lambda| \cdot \|y\|_i e_i \right\|_E$$

and $|\lambda|$ can be factored out of the norm, which proves homogeneity. Lastly, if $y_1, y_2 \in \tilde{Y}((Z_i), E)$, then by the fact that (e_i) is unconditional with unconditional constant one,

$$\sup_m \left\| \sum_{i=1}^m \|y_1 + y_2\|_i e_i \right\|_E \leq \sup_m \left\| \sum_{i=1}^m (\|y_1\|_i + \|y_2\|_i) e_i \right\|_E .$$

Using the subadditivity of the norm of E gives the subadditivity of $\|\cdot\|_Y$.

We prove next that $\tilde{Y}((Z_i), E)$ is a Banach space. Let (y_n) be a Cauchy sequence in $\tilde{Y}((Z_i), E)$. The norm $\|\cdot\|_Y$ majorizes $\|\cdot\|_i$, which is an equivalent norm on Z. Hence (y_n) is a Cauchy sequence in Z and therefore convergent to some $y \in Z$.

Each of the norms $\|\cdot\|_i$ is continuous with respect to the norm on Z. In particular,

$$\|y_n\|_i \to \|y\|_i \tag{2.8}$$

for all i. Also, for any fixed m, $\| \sum_{i=1}^m \|\cdot\|_i e_i \|_E$ is continuous with respect to the norm on Z. The norm $\|\cdot\|_Y$, being a supremum of continuous functions, is lower semicontinuous. This yields

$$\|y\|_Y \leq \liminf_{n\to\infty} \|y_n\|_Y < \infty \Rightarrow y \in \tilde{Y}((Z_i), E),$$

where we used that (y_n) is Cauchy and therefore bounded.

It remains to show y_n converge to y in $\tilde{Y}((Z_i), E)$. If $\varepsilon > 0$ is fixed, then

$$\exists n_0 \; \forall n_1, n_2 \geq n_0 : \|y_{n_1} - y_{n_2}\|_Y < \frac{\varepsilon}{2}.$$

Taking such an n_0, we get from (2.8)

$$\forall m \; \exists n_m \geq n_0 : \left\| \sum_{i=1}^m \|y - y_{n_m}\|_i e_i \right\|_E < \frac{\varepsilon}{2}.$$

The last two inequalities imply

$$\left\| \sum_{i=1}^m \|y - y_n\|_i e_i \right\|_E \leq \left\| \sum_{i=1}^m \|y - y_{n_m}\|_i e_i \right\|_E + \left\| \sum_{i=1}^m \|y_{n_m} - y_n\|_i e_i \right\|_E < \varepsilon$$

for any m and any $n \geq n_0$. Finally, $\|y - y_n\|_Y < \varepsilon$ for $n \geq n_0$ follows by taking the supremum over m on the left-hand side.

To prove that $Y((Z_i), E)$ is a closed subspace of $\tilde{Y}((Z_i), E)$, take a sequence (y_n) in $Y((Z_i), E)$ converging to some y in $\tilde{Y}((Z_i), E)$. If $\varepsilon > 0$ is fixed, there exists y_n such that $\|y - y_n\|_Y < \frac{\varepsilon}{2}$. We can find m_0 such that for all $m_1 \geq m_0$

$$\left\| \sum_{i=m_1+1}^{\infty} \|y_n\|_i e_i \right\|_E < \frac{\varepsilon}{2}.$$

Then for all $m_2 > m_1 \geq m_0$

$$\left\| \sum_{i=m_1+1}^{m_2} \|y\|_i e_i \right\|_E \leq \left\| \sum_{i=m_1+1}^{m_2} \|y - y_n\|_i e_i \right\|_E + \left\| \sum_{i=m_1+1}^{m_2} \|y_n\|_i e_i \right\|_E$$

where the first norm on the right-hand side can be estimated as

$$\left\| \sum_{i=m_1+1}^{m_2} \|y - y_n\|_i e_i \right\|_E \leq \left\| \sum_{i=1}^{m_2} \|y - y_n\|_i e_i \right\|_E < \frac{\varepsilon}{2}$$

and the second norm is less than $\frac{\varepsilon}{2}$. By letting $m_2 \to \infty$, we obtain for any $m_1 \geq m_0$

$$\left\| \sum_{i=m_1+1}^{\infty} \|y\|_i e_i \right\|_E < \varepsilon.$$

Hence the sum $\sum_{i=1}^{\infty} \|y\|_i e_i$ converges and $y \in Y((Z_i), E)$. □

Lemma 2.24. *Under the assumptions of Definition 2.22, let $\tilde{Y} = \tilde{Y}((Z_i), E)$. The operator $J : \tilde{Y} \hookrightarrow Z$ given by inclusion is bounded.*

Proof. The space $\tilde{Y}$ is defined as a subset of Z. The norms $\|\cdot\|_i$ are by definition equivalent to the norm on Z, hence there exists a constant $c > 0$ such that for any $y \in \tilde{Y}$,

$$\|Jy\|_Z \leq c\|Jy\|_1 = c\| \|y\|_1 e_1\|_E \leq c\|y\|_Y,$$

where we used that (e_i) is a normalized unconditional basis with unconditional constant one. □

Proposition 2.25. *In the setting of Definition 2.22, let $Y = Y((Z_i), E)$ and let $j : Y \hookrightarrow Z$ denote the operator given by inclusion. Assume that Z has a basis (z_n) and let S_n denote the projections associated to (z_n). Suppose that the following two conditions are satisfied.*

(i) $S_n(Z) \subset j(Y)$ for all n.

(ii) The projection $S_n : Z_i \to Z_i$ has norm $\|S_n\|_i \leq 1$ for all i and all n.

Then the vectors $y_n = j^{-1} z_n$ form a basic sequence in Y.

Proof. We set $Q_n = j^{-1} S_n j$ for all n. By condition (i), the operators Q_n are well defined. We have

$$Q_m Q_n = j^{-1} S_m S_n j = j^{-1} S_m j = Q_m$$

for $m \leq n$. In particular, $Q_n^2 = Q_n$, and hence Q_n are projections on Y. For any of the elements y_i,

$$Q_n y_i = (j^{-1} S_n j)(j^{-1} z_i) = j^{-1}(S_n z_i).$$

Now $S_n z_i$ equals z_i if $i \leq n$ and zero otherwise. Thus $Q_n y_i = y_i$ for $i \leq n$ and $Q_n y_i = 0$ for $i > n$.

We show next that the projections Q_n all have norm one. If $y \in Y$, then

$$\|Q_n y\|_Y = \sup_m \left\| \sum_{i=1}^m \|S_n j y\|_i e_i \right\|_E \leq \sup_m \left\| \sum_{i=1}^m \|jy\|_i e_i \right\|_E = \|y\|_Y,$$

where we used $jQ_n = S_n j$ and assumption (ii). We conclude the proof with the help of Grundblum's criterion 1.5. Let $y = \sum_{i=1}^n a_i y_i$ be any finite linear combination of elements in (y_i). If $m < n$, then

$$\left\| \sum_{i=1}^m a_i y_i \right\|_Y = \|Q_m y\|_Y \leq \|y\|_Y = \left\| \sum_{i=1}^n a_i y_i \right\|_Y .$$

Proposition 1.5 proves the claim. □

Definition 2.26. Let X and Z be Banach spaces such that there exists a bounded operator $T : X \to Z$. Assume that a and b are positive scalars. We define an equivalent norm on Z by setting

$$k(w, a, b) = \inf\{a\|x\|_X + b\|z\|_Z;\ w = Tx + z\}$$

for all $w \in Z$.

This definition corresponds to the first part of Definition 2.g.3 from [5].

Lemma 2.27. *If $a, b > 0$, then $k(\cdot, a, b)$ is an equivalent norm on Z.*

Proof. If $w \in Z$ satisfies $w = Tx + z$ for $x \in X$ and $z \in Z$, then

$$\min(a, b)\|w\|_Z \leq \min(a, b)(\|Tx\|_Z + \|z\|_Z) \leq a\|T\| \cdot \|x\|_X + b\|z\|_Z$$

and the last expression can be estimated as $\max(\|T\|, 1) \cdot (a\|x\|_X + b\|z\|_Z)$. By passing to the infimum, we obtain

$$\min(a, b)\|w\|_Z \leq \max(\|T\|, 1)k(w, a, b).$$

On the other hand, $k(w, a, b) \leq b\|w\|_Z$ because every $w \in Z$ can be written as $w = T0 + w$. It is now clear that $k(w, a, b) = 0$ if and only if $w = 0$. The other two defining properties of norm, namely homogeneity and the triangle inequality, follow directly from the definition. □

What follows is the second part of Definition 2.g.3 from [5]. Spaces analogous to $\tilde{Y}((Z_i), E)$ and $Y((Z_i), E)$ are only given there for the following special choice of norms.

Definition 2.28. Assume that $T : X \to Z$ is a bounded operator between Banach spaces. Let E be a Banach space with a normalized unconditional basis (e_i) whose unconditional constant is one and let (a_i) and (b_i) be two sequences of scalars such that $\sum_{i=1}^\infty \min(a_i, b_i) < \infty$. We set $\| \cdot \|_i = k(\cdot, a_i, b_i)$ and $Z_i = (Z, \| \cdot \|_i)$.

Lemma 2.27 ensures that if the norms $\|\cdot\|_i$ are defined as in 2.26, then the assumptions of Definition 2.22 are satisfied and it makes sense to consider the space $\widetilde{Y}((Z_i), E)$. The next lemma captures a fact stated in [5, p. 219].

Lemma 2.29. *Suppose that the conditions in Definition 2.28 are satisfied. Let* $Y = Y((Z_i), E)$ *and let* $j : Y \hookrightarrow Z$ *denote the operator given by inclusion. Then the operator* $T_1 : X \to Y$ *given by* $T_1(x) = j^{-1}(Tx)$ *for* $x \in X$ *is bounded.*

In a diagram form, $T : X \overset{T_1}{\to} Y \overset{j}{\hookrightarrow} Z$.

Proof. The operator j is bounded because it is a restriction of the operator J from Lemma 2.24. If $x \in X$, then its image under T satisfies

$$k(Tx, a_i, b_i) \leq \min(a_i\|x\|_X, b_i\|Tx\|_Z) \leq \min(a_i, b_i) \cdot \max(\|x\|_X, \|Tx\|_Z)$$

for all i. The sum $\sum_{i=1}^{\infty} \|Tx\|_i e_i = \sum_{i=1}^{\infty} k(Tx, a_i, b_i)e_i$ converges because it converges absolutely:

$$\sum_{i=1}^{\infty} \|k(Tx, a_i, b_i)e_i\|_E \leq \sum_{i=1}^{\infty} \min(a_i, b_i) \cdot \max(\|x\|_X, \|Tx\|_Z)$$

and $\sum_{i=1}^{\infty} \min(a_i, b_i)$ converges by assumption. Thus Tx is in the range of j and the norm of $j^{-1}(Tx)$ satisfies

$$\|j^{-1}(Tx)\|_Y \leq \sum_{i=1}^{\infty} \min(a_i, b_i) \cdot \max(\|T\|, 1)\|x\|_X.$$

□

Lemma 2.30. *In the setting of Definition 2.28, let us assume further that* X *has a basis* (x_n), Z *has a basis* (z_n) *and the operator* $T : X \to Z$ *is such that* $Tx_n = z_n$. *Let* $Y = Y((Z_i), E)$ *and let* $j : Y \to Z$ *denote the operator given by inclusion. If we set* $y_n = j^{-1}z_n$, *then* (y_n) *is a basic sequence in* Y *and* $(z_n) \preceq (y_n) \preceq (x_n)$.

Proof. We denote by P_n be the projections associated to the basis (x_n) and by S_n the projections associated to (z_n). Without loss of generality, it can be assumed that the basis constants of (x_n) and (z_n) are both equal to one. Clearly, $TP_n = S_nT$ for all n.

We prove the lemma by verifying the conditions from Proposition 2.25. The range of S_n is contained in the range of T because for $z = \sum_{i=1}^{\infty} a_i z_i \in Z$ we have

$$S_n(z) = \sum_{i=1}^{n} a_i z_i = T\left(\sum_{i=1}^{n} a_i x_i\right).$$

By Lemma 2.29, $S_n(Z) \subset T(X) \subset j(Y)$.

If $w \in Z$, then

$$k(w, a, b) = \inf\{a\|x\|_X + b\|z\|_Z;\ w = Tx + z\}$$

for any $a, b > 0$. Assuming $x \in X$ and $z \in Z$ satisfy $w = Tx + z$ we have

$$S_n w = (S_nT)x + S_n z = T(P_n x) + S_n z.$$

Therefore,

$$\begin{aligned}k(S_n w, a, b) &= \inf\{a\|x_1\|_X + b\|z_1\|_Z;\ S_n w = Tx_1 + z_1\}\\ &\le a\|P_n x\|_X + b\|S_n z\| \le a\|x\|_X + b\|z\|_Z,\end{aligned}$$

where we used the assumption that the basis constants of both bases are one. Because x and z were chosen arbitrarily in the decomposition of w, by passing to the infimum on the right-hand side we obtain

$$k(S_n w, a, b) \le k(w, a, b)$$

for any $a, b > 0$. Since $\|\cdot\|_i = k(\cdot, a_i, b_i)$, we proved $\|S_n w\|_i \le \|w\|_i$ for all i.

The operator j maps y_n to z_n, hence $(z_n) \preceq (y_n)$. The operator $T_1 : X \to Y$ defined as $T_1(x) = j^{-1}(Tx)$ is bounded by Lemma 2.29 and maps x_n to y_n, which shows $(y_n) \preceq (x_n)$. □

Now we are ready to prove the main theorem (Theorem 0.3). We follow in part the proof that a weakly compact operator factors through a reflexive space [5, Theorem 2.g.11]. This proof uses the space $Y = Y((Z_i), \ell_2) = \tilde{Y}((Z_i), \ell_2)$, which means we take ℓ_2 for E in Definition 2.26. There is a simpler description of Y available in this case. Let L be defined as the ℓ_2-sum of the spaces $Z_i = (Z, \|\cdot\|_i)$, so that

$$L = \Big(\bigoplus Z_i\Big)_{\ell_2}, \qquad \|(z_i)\|_L = \left(\sum_{i=1}^{\infty} \|z_i\|_i^2\right)^{\frac{1}{2}}.$$

Then Y can be identified with the "diagonal" of L, i.e. the set of sequences $(z, z, z, \dots) \in L$ with all elements the same. By Lemma 2.24, there is an injective operator $j : Y \to Z$. An element $y \in Y$ corresponds to $(jy, jy, \dots) \in L$. The dual norm of $\|\cdot\|_i$ is given by

$$\|z^*\|_i^* = \sup\{|z^*(z)|;\ z \in Z, \|z\|_i \le 1\}, \qquad z^* \in Z^*.$$

The dual space L^* can be identified with the ℓ_2-sum of the spaces $(Z^*, \|\cdot\|_i^*)$. Every bounded linear functional y^* on Y can be extended to a linear functional on L with the same norm by the Hahn-Banach theorem and hence can be regarded as a sequence (z_i^*) of elements in Z^* such that $\sum_{i=1}^{\infty} (\|z_i^*\|_i^*)^2 < \infty$. Then for $y \in Y$ which corresponds to $(z, z, \dots) \in L$ we have

$$y^*(y) = \sum_{i=1}^{\infty} z_i^*(z).$$

The last fact is used in the following proof.

Proof of Theorem 0.3. By Theorem 0.2, the operator $T : X \to Z$ such that $T(x_n) = z_n$ is weakly compact. It follows from definition that the set $\overline{TB_X}$ is weakly compact (we are using the fact that TB_X is convex, hence the closures in the norm topology and the weak topology coincide).

Let us choose sequences (a_i) and (b_i) of positive scalars such that $\sum_{i=1}^{\infty} a_i < \infty$ and (b_i) is an increasing sequence growing to infinity. If we define $Y = Y((Z_i), \ell_2)$ with the help of the norms $\|\cdot\|_i = k(\cdot, a_i, b_i)$, then according to Lemma 2.30 there is a basic sequence (y_n) in Y which satisfies $(x_n) \preceq (y_n) \preceq (z_n)$.

Let (w_n) be a sequence in Y such that $\|w_n\|_Y < 1$ for all n. Let $j : Y \to Z$ denote the operator given by inclusion. To prove that Y is reflexive it suffices to show that (w_n) has a weakly convergent subsequence. Since $\|w_n\|_Y < 1$ we have $k(w_n, a_i, b_i) < 1$ for all i and any n, which means there exist elements $x_{n,i}$ in X and $z_{n,i}$ in Z such that

$$jw_n = Tx_{n,i} + z_{n,i}, \qquad a_i\|x_{n,i}\|_X + b_i\|z_{n,i}\|_Z < 1. \tag{2.9}$$

For each fixed i, the sequence $(x_{n,i})_{n=1}^{\infty}$ is bounded in X. By the weak compactness of $\overline{TB_X}$, we can assume (by passing to a subsequence) that $Tx_{n,i}$ weakly converges to some $z_i \in Z$. Additionally, (2.9) implies that $\|z_{n,i}\|_Z < b_i^{-1}$. If n is fixed, then $Tx_{n,i} - Tx_{n,j} = z_{n,j} - z_{n,i}$ for any i and j, hence

$$\|z_i - z_j\|_Z \le \sup_n \|Tx_{n,i} - Tx_{n,j}\|_Z \le \sup_n \|z_{n,j} - z_{n,i}\|_Z < b_i^{-1} + b_j^{-1}.$$

The sequence (z_i) therefore converges in Z to an element $z \in Z$.

A diagonal argument shows that $Tx_{n,n}$ converges weakly to z. Moreover, $\|z_{n,n}\|_Z < b_n^{-1}$ and $b_n \to 0$. Altogether, we see that jw_n converges weakly to z in Z. Setting $y = j^{-1}z$, we want to show w_n converges weakly to y in Y. According to the discussion preceding this proof, this is the same as showing

$$\lim_{n\to\infty} \sum_{i=1}^{\infty} z_i^*(jw_n) = \sum_{i=1}^{\infty} z_i^*(z) \tag{2.10}$$

if $z_i^* \in Z^*$ are such that $\sum_{i=1}^{\infty}(\|z_i^*\|_i^*)^2 < \infty$. Let $\varepsilon > 0$ and i_0 be such that $\sum_{i=i_0}^{\infty}(\|z_i^*\|_i^*)^2 < \varepsilon$. Then

$$\left|\sum_{i=i_0}^{\infty} z_i^*(jw_n) - \sum_{i=i_0}^{\infty} z_i^*(z)\right| = \left|\sum_{i=i_0}^{\infty} z_i^*(jw_n - z)\right| \le \varepsilon\|w_n - y\|_Y \le C\varepsilon \tag{2.11}$$

where $C = \sup_n \|w_n - y\| < \infty$. Since jw_n converges weakly to z in Z we have

$$\lim_{n\to\infty} \sum_{i=1}^{i_0} z_i^*(jw_n) = \sum_{i=1}^{i_0} z_i^*(z). \tag{2.12}$$

The equality (2.10) now follows from (2.11) and (2.12). □

2.2.2 Factoring through a non-reflexive space

Let (e_n^1) and (e_n^2) be the respective canonical bases of ℓ_p and ℓ_q, where $1 \le p < q < \infty$. We show there exists a basis (y_n) which is neither shrinking nor boundedly-complete and satisfies $(e_n^2) \preceq (y_n) \preceq (e_n^1)$.

The result is a corollary of Proposition 3.b.4 in [4]. The proof of this proposition uses ideas from the article [2] by Davis, which in turn is based on the technique developed in [3] to factor a weakly compact operator. We closely follow the approach in [4] but provide detailed proofs of the facts which we state as Lemmas 2.32 and 2.34.

Let X and Z be Banach spaces such that their respective bases (x_n) and (z_n) are symmetric with symmetric constant 1. Suppose that $(z_n) \preceq (x_n)$, i.e. $T : X \to Z$ mapping x_n to z_n is bounded. Let $\lambda_X(n) = \|\sum_{i=1}^{n} x_i\|_X$ and $\lambda_Z(n) = \|\sum_{i=1}^{n} z_i\|_Z$. We make the following two assumptions.

1) $\|T\| \le 1$

2) $\lim_{n\to\infty} \frac{\lambda_Z(n)}{\lambda_X(n)} = 0$

The first assumption is not crucial but simplifies some of the estimates. We note that both conditions are satisfied in particular when $X = \ell_p$ and $Z = \ell_q$, where $1 \le p < q < \infty$.

Definition 2.31 ([4, p. 125])**.** Let $m \ge 1$. Then we define a norm $\|\cdot\|_m$ on Z as

$$\|w\|_m = \inf\left\{\left(\|x\|_X^2 + \|z\|_Z^2\right)^{1/2};\ w = m^{-1}z + mTx \text{ with } x \in X, z \in Z\right\}$$

for $w \in Z$.

Lemma 2.32. *If $m \ge 1$, then $\|\cdot\|_m$ is an equivalent norm on Z. The basis (z_n) is symmetric with respect to $\|\cdot\|_m$ and has symmetric constant one.*

Proof. It is straightforward to verify that $\|\cdot\|_m$ is a norm. If $w = m^{-1}z + mTx$ for some $x \in X$ and $z \in Z$, then

$$\begin{aligned}\|w\|_Z \le m^{-1}\|z\|_Z + m\|Tx\|_Z &\le (m^{-2} + m^2)^{1/2}(\|z\|_Z^2 + \|Tx\|_Z^2)^{1/2} \\ &\le (m^{-2} + m^2)^{1/2}(\|z\|_Z^2 + \|x\|_X^2)^{1/2}.\end{aligned}$$

We obtain $\|w\|_Z \le (m^{-2} + m^2)^{1/2}\|w\|_m$ by passing to the infimum. Choosing $z = mw$ and $x = 0$ yields $\|w\|_m \le m\|w\|_Z$.

Now let $w = \sum_{k=1}^\infty a_k z_k \in Z$. We show $\|\sum_{k=1}^\infty a_{\pi(k)} z_k\|_m \le \|w\|_m$ for any permutation π of the integers. Let $w = m^{-1}z + mTx$ with $x = \sum_{k=1}^\infty b_k x_k$ and $z = \sum_{k=1}^\infty c_k z_k$. Then

$$\sum_{k=1}^\infty a_{\pi(k)} z_k = m^{-1}\sum_{k=1}^\infty c_{\pi(k)} z_k + m\sum_{k=1}^\infty b_{\pi(k)} x_k,$$

hence

$$\left\|\sum_{k=1}^\infty a_{\pi(k)} z_k\right\|_m^2 \le \left\|\sum_{k=1}^\infty b_{\pi(k)} x_k\right\|_X^2 + \left\|\sum_{k=1}^\infty c_{\pi(k)} z_k\right\|_Z^2 = \left\|\sum_{k=1}^\infty b_k x_k\right\|_X^2 + \left\|\sum_{k=1}^\infty c_k z_k\right\|_Z^2,$$

where we used that (x_k) and (z_k) are symmetric with symmetric constant one. The desired inequality follows by passing to the infimum on the right-hand side. □

Definition 2.33. Let E be a Banach space such that E has a normalized unconditional basis (e_i) with unconditional constant one. Let (m_i) be an increasing sequence of numbers which are greater than or equal to 1 such that $\sum_{i=1}^\infty m_i^{-1}$ converges. In the setting of Definition 2.22, let $Z_i = (Z, \|\cdot\|_{m_i})$.

The space $Y((Z_i), E)$ for this particular choice of norms is defined in [4, p. 126].

Lemma 2.34. *Let Z_i be given as in Definition 2.33. Let $Y = Y((Z_i), E)$ and let $j : Y \hookrightarrow Z$ denote the operator given by inclusion. The operator $T_1 : X \to Y$ defined as $T_1(x) = j^{-1}(Tx)$ for $x \in X$ is bounded. The vectors $y_n = j^{-1}z_n$ form a basic sequence in Y which is in fact symmetric with symmetric constant one.*

Proof. If $m \geq 1$ and $x \in X$ are fixed, then Tx can be expressed as $Tx = m^{-1}0 + mT(\frac{x}{m})$, which shows $\|Tx\|_m \leq m^{-1}\|x\|_X$. The sum $\sum_{i=1}^{\infty} \|Tx\|_{m_i} e_i$ converges absolutely because

$$\sum_{i=1}^{\infty} \|Tx\|_{m_i} \leq \left(\sum_{i=1}^{\infty} m_i^{-1}\right) \|x\|_X$$

and $C = \sum_{i=1}^{\infty} m_i^{-1} < \infty$ by assumption.

It follows from the definition of norm in $Y((Z_i), E)$ that

$$\|j^{-1}(Tx)\|_Y \leq \left\| \sum_{i=1}^{\infty} \|Tx\|_{m_i} e_i \right\|_E \leq C\|x\|_X,$$

which shows that $T_1 = j^{-1}T$ is bounded.

Checking the conditions from Proposition 2.25 to show that (y_n) is basic is completely analogous to the proof of Lemma 2.30.

If $y = \sum_{n=1}^{\infty} a_n y_n \in Y$, then $jy = \sum_{n=1}^{\infty} a_n z_n$ and by Lemma 2.32,

$$\left\| \sum_{n=1}^{\infty} a_{\pi(n)} y_n \right\|_Y = \left\| \sum_{i=1}^{\infty} \left\| \sum_{n=1}^{\infty} a_{\pi(n)} z_n \right\|_{m_i} e_i \right\|_E = \left\| \sum_{i=1}^{\infty} \left\| \sum_{n=1}^{\infty} a_n z_n \right\|_{m_i} e_i \right\|_E = \|y\|_Y.$$

This proves (y_n) is symmetric with symmetric constant one. □

Now we state the aforementioned Proposition 3.b.4. from [4]. In our notation it may be formulated as follows.

Proposition 2.35. *There exists an increasing sequence (m_i) of numbers greater or equal to 1 such that $\sum_{i=1}^{\infty} m_i^{-1} < \infty$ and $Y = Y((Z_i), E))$ contains a complemented subspace isomorphic to E. Moreover, every infinite dimensional subspace of Y contains an infinite dimensional subspace which is isomorphic to a subspace of X or to a subspace of E.*

Remark 2.36. A block basic sequence (u_n) of (y_n) is constructed in the proof which is equivalent to the basis of E. Hence the subspace $[y_n]$ of Y contains a complemented subspace isomorphic to E.

Proposition 2.37. *Let $1 \leq p < q < \infty$. Let (e_n^1) and (e_n^2) denote the respective canonical bases of ℓ_p and ℓ_q. There exists a basis (y_n) which is neither shrinking nor boundedly-complete and satisfies $(e_n^2) \preceq (y_n) \preceq (e_n^1)$.*

Proof. We choose $E = \ell_1 \oplus c_0$ as the space in Definition 2.33. Since the canonical vectors form a normalized unconditional basis in both ℓ_1 and c_0, the sequence $(e_1, 0), (0, e_1), (e_2, 0), (0, e_2), \ldots$ is a normalized unconditional basis of E. The remark following Proposition 2.35 asserts that the subspace $[y_n]$ of Y contains a complemented subspace isomorphic to E. Consequently, $[y_n]$ contains a subspace isomorphic to ℓ_1 and also a subspace isomorphic to c_0. The basic sequence (y_n) is symmetric by Lemma 2.32, and hence unconditional. Theorem 1.27 now shows that (y_n) is neither shrinking nor boundedly-complete. □

Remark 2.38. The proof above gives more, namely that the dual of $[y_n]$ is not separable. Indeed, we showed that $[y_n]$ contains a complemented subspace isomorphic to ℓ_1, hence $[y_n]^*$ contains a complemented subspace isomorphic to ℓ_∞.

2.3 Summary

For the convenience of the reader, we repeat the main results about ordering on bases from this chapter. Here (x_n), (y_n), (z_n) always denote the respective bases of some Banach spaces X, Y, Z.

- Two bases satisfy $(y_n) \preceq (x_n)$ if and only if there exists an operator $T : X \to Y$ such that $T(x_n) = y_n$ for all n (Proposition 2.1).
- We have $(y_n) \preceq (x_n)$ if and only if $(x_n^*) \preceq (y_n^*)$ (Proposition 2.7).
- The system of all bases may be viewed as a set $\mathfrak{B}$. The relation $\preceq$ is a partial ordering on $\mathfrak{B}$ (Lemma 2.4).
- Any (semi)normalized basis always satisfies $(e_n) \preceq (x_n) \preceq (e_n^*)$, where (e_n) is the canonical basis of c_0 (Remark 2.9).
- The supremum of two bases (x_n), (y_n) is defined as $(x_n) \vee (y_n) = ((x_n, y_n))$ and their infimum as $(x_n) \wedge (y_n) = ((x_n^*, y_n^*)^*)$.
- $(\mathfrak{B}, \preceq)$ with the operations $\vee$ and $\wedge$ forms a lattice (Theorem 2.16).
- If $\mathfrak{S} \subset \mathfrak{B}$ denotes the subsets of all shrinking bases and $\mathfrak{Bc} \subset \mathfrak{B}$ the subset of all boundedly-complete bases, then $\mathfrak{S}$ and $\mathfrak{Bc}$ form sublattices of $\mathfrak{B}$ (Theorem 2.19).
- Let two normalized bases satisfy $(z_n) \preceq (x_n)$, where (x_n) is shrinking and (z_n) is boundedly-complete. Example 2.21 shows that (x_n) is not necessarily boundedly-complete and (z_n) is not necessarily shrinking.
- Let (x_n) be shrinking and (z_n) boundedly-complete. If $(z_n) \preceq (x_n)$, then the operator $T : X \to Z$ mapping x_n to z_n is weakly compact (Theorem 0.2).
- If a shrinking basis (x_n) and a boundedly-complete basis (z_n) satisfy $(z_n) \preceq (x_n)$, then there exists a basis (y_n) such that $(z_n) \preceq (y_n) \preceq (x_n)$, where (y_n) is both shrinking and boundedly-complete (Theorem 0.3).
- Let (e_n^1) and (e_n^2) denote the respective canonical bases of ℓ_p and ℓ_q, where $1 \leq p < q < \infty$. There exists a basis (y_n) which is neither shrinking nor boundedly-complete but satisfies $(e_n^1) \preceq (y_n) \preceq (e_n^2)$ (Proposition 2.37).

3. Spaces with iterative norms

3.1 Orlicz sequence spaces

The relevant material on Orlicz sequence spaces is reviewed in this section. Our exposition is partly based on Chapter 4 of [4]. We supply some additional propositions and proofs and also take this opportunity to apply the notion of ordering on bases introduced earlier.

3.1.1 Definition of Orlicz sequence spaces

Definition 3.1. A convex non-decreasing function Φ defined on $[0, \infty)$ such that $\Phi(0) = 0$ and $\lim_{t\to\infty} \Phi(t) = \infty$ is called an *Orlicz function*.

Remark 3.2. An Orlicz function Φ is continuous. Convexity implies its continuity on the interval $(0, \infty)$. Because it is non-decreasing and non-negative, it has a limit $\lim_{t\to 0+} \Phi(t)$. If this limit was greater than zero, the function would not be convex. This proves that Φ is right-continuous at zero.

Definition 3.3. Let Φ be an Orlicz function. The linear space ℓ_Φ of all scalar sequences (a_i) such that $\sum_{i=1}^{\infty} \Phi(|a_i|/\rho) < \infty$ for some $\rho > 0$ equipped with the norm

$$\|(a_i)\|_\Phi = \inf \left\{ \rho > 0;\ \sum_{i=1}^{\infty} \Phi\left(\frac{|a_i|}{\rho}\right) \leq 1 \right\} \tag{3.1}$$

is called an *Orlicz sequence space*. We let h_Φ denote its subspace consisting of all sequences (a_i) which satisfy

$$\sum_{i=1}^{\infty} \Phi\left(\frac{|a_i|}{\rho}\right) < \infty \tag{3.2}$$

for any $\rho > 0$.

It is a standard exercise to show that ℓ_Φ is a Banach space [1, Exercise 3.4]. The content of the next proposition matches that of Proposition 4.a.2 in [4].

Proposition 3.4. *If Φ is an Orlicz function, then $h_\Phi = [e_i]$, the closed linear span of the canonical unit vectors in ℓ_Φ. The sequence (e_i) is a symmetric basis of h_Φ with symmetric constant one.*

Proof. If (a_i) is a sequence of scalars and $m \leq n$, then

$$\sum_{i=1}^{n} \Phi\left(\frac{|a_i|}{\rho}\right) \leq 1 \Rightarrow \sum_{i=1}^{m} \Phi\left(\frac{|a_i|}{\rho}\right) \leq 1.$$

Hence

$$\left\| \sum_{i=1}^{m} a_i e_i \right\|_\Phi \leq \left\| \sum_{i=1}^{n} a_i e_i \right\|_\Phi$$

and (e_i) is a basic sequence according to the Grundblum's criterion. It is clear from (3.1) that $\|(a_i)\|_\Phi = \|(a_{\pi(i)})\|_\Phi$ for any permutation π of the integers. By definition, (e_i) is symmetric with symmetric constant one. Finally, $(a_i) \in [e_n]$ if and only if for every $\rho > 0$ there exists i_0 such that

$$\left\| \sum_{i=i_0}^{\infty} a_i e_i \right\|_\Phi < \rho.$$

This holds if and only if for every $\rho > 0$ there exists i_0 such that

$$\sum_{i=i_0}^{\infty} \Phi\left(\frac{|a_i|}{\rho}\right) \leq 1,$$

and the last condition is equivalent to (3.2). $\square$

The interest of the next simple proposition is in the analogy between (3.3) and the formula for the norm of X_N given in Definition 3.27.

Proposition 3.5. *Let Φ be an Orlicz function. If $(a_i) \in \ell_\Phi$, then*

$$\|(a_i)\|_\Phi = \sup \left\| \sum_{i=1}^{n} a_i e_i \right\|_\Phi . \tag{3.3}$$

Proof. We can assume that (a_i) is nonzero. Let c denote the supremum on the right-hand side of (3.3) and $\rho = \|(a_i)\|_\Phi$. Then $\sum_{i=1}^n \Phi(|a_i|/\rho) \leq 1$ for all n which implies

$$\left\| \sum_{i=1}^{n} a_i e_i \right\|_\Phi \leq \rho$$

for all n. Conversely, $\sum_{i=1}^n \Phi(|a_i|/c) \leq 1$ for all n and therefore $\|(a_i)\|_\Phi \leq c$. $\square$

An Orlicz function Φ is called *degenerate* if $\Phi(t_0) = 0$ for some $t_0 > 0$.

Lemma 3.6. *If Φ is an Orlicz function, the following conditions are equivalent.*

(i) The identity mapping $Id\colon \ell_\infty \to \ell_\Phi$ is bounded.

(ii) The canonical bases (e_n^1) of c_0 and (e_n^2) of h_Φ satisfy $(e_n^2) \preceq (e_n^1)$.

(iii) Φ is degenerate.

Proof. (i) $\Rightarrow$ (ii). The mapping Id restricted to h_Φ is also bounded and maps each basis vector e_n^1 to e_n^2. This is equivalent to $(e_n^2) \preceq (e_n^1)$ by Proposition 2.1.

(ii) $\Rightarrow$ (iii). If $(e_n^2) \preceq (e_n^1)$, then there exists a constant $\rho > 0$ such that

$$\left\| \sum_{i=1}^{n} e_i^2 \right\|_\Phi \leq \left\| \sum_{i=1}^{n} e_i^1 \right\|_\infty \leq \rho$$

for all n. It follows from the definition of norm in ℓ_Φ that

$$n\Phi\left(\frac{1}{\rho}\right) \leq 1$$

for all n, hence $\Phi(1/\rho) = 0$ and Φ is degenerate.

(iii) $\Rightarrow$ (i). Now assume Φ is degenerate and let $(a_i) \in \ell_\infty$ be nonzero. The function Φ is zero on some interval $[0, t_0]$ where $t_0 > 0$. If we set $\rho = \frac{1}{t_0}\|(a_i)\|_\infty$, then $\sum_{i=1}^\infty \Phi(|a_i|/\rho) = 0$ because $\frac{|a_i|}{\rho} \leq t_0$ for all i. It follows that $\|(a_i)\|_\Phi \leq \rho = \frac{1}{t_0}\|(a_i)\|_\infty$. $\square$

3.1.2 Ordering on bases

Definition 3.7. We call two Orlicz functions Φ_1 and Φ_2 equivalent at zero if there exists a positive constants K and $t_0 > 0$ such that for all $t \in [0, t_0]$

$$\Phi_2\left(\frac{t}{K}\right) \leq \Phi_1(t) \leq \Phi_2(Kt). \tag{3.4}$$

This notion can be weakened similarly to that of equivalence of bases.

Definition 3.8. Let Φ_1 and Φ_2 be two Orlicz functions. We write $\Phi_2 \preceq \Phi_1$ if there exists a positive constant K and $t_0 > 0$ such that for all $t \in [0, t_0]$

$$\Phi_2(t) \leq \Phi_1(Kt). \tag{3.5}$$

An analogous notion appears in [6, p. 15] with the difference that (3.5) is assumed to hold for $t \geq t_0$. The behavior away from zero is relevant in the context of Orlicz spaces of functions.

Remark 3.9. The previous definition can be reformulated as follows. We have $\Phi_1 \preceq \Phi_2$ if and only if there exist positive constants k_1, k_2 and $t_0 > 0$ such that $\Phi_2(t) \leq k_1\Phi_1(k_2 t)$ for $t \in [0, t_0]$. To get (3.5) we simply set $k_1 = 1$ and $k_2 = K$. Conversely, if the latter condition holds, then we can assume $k_1 \geq 1$ (otherwise we take larger k_1). By the convexity of Φ_1, $k_1\Phi_1(k_2 t) \leq \Phi_1(k_1 k_2 t)$ and we set $K = k_1 k_2$.

For example, if Φ_1 is a positive multiple of Φ_2, they are equivalent.

Lemma 3.10. *If two Orlicz functions Φ_1 and Φ_2 satisfy $\Phi_1 \preceq \Phi_2$ and $\Phi_2 \preceq \Phi_1$ they are equivalent at zero.*

Proof. Because $\Phi_2 \preceq \Phi_1$, there exist positive constant K_1 and $s_1 > 0$ such that $\Phi_2(s) \leq \Phi_1(K_1 s)$ for $s \in [0, s_1]$. If we set $t_1 = K_1 s_1$, then

$$\Phi_2\left(\frac{t}{K_1}\right) \leq \Phi_1(t)$$

for $t \in [0, t_1]$. The relation $\Phi_1 \preceq \Phi_2$ implies the existence of positive constant K_2 and $t_2 > 0$ such that $\Phi_1(t) \leq \Phi_2(K_2 t)$ for $t \in [0, t_2]$. We let $t_0 = \min(t_1, t_2)$ and observe that (3.4) is satisfied for $K = \max(K_1, K_2)$. □

Proposition 3.11. *If Φ_1 and Φ_2 are Orlicz functions, the following conditions are equivalent.*

(i) The identity mapping $Id \colon \ell_{\Phi_1} \to \ell_{\Phi_2}$ is bounded.

(ii) The canonical bases (e_n^1) of h_{Φ_1} and (e_n^2) of h_{Φ_2} satisfy $(e_n^2) \preceq (e_n^1)$.

(iii) $\Phi_2 \preceq \Phi_1$.

Proof. (i) ⇒ (ii). The restriction of Id to h_{Φ_1} is also bounded, hence $(e_n^2) \preceq (e_n^1)$ by Proposition 2.1.

(ii) ⇒ (iii). Let us assume first that Φ_1 is degenerate and let (e_n) denote the canonical basis of c_0. By Lemma 3.6, $(e_n^1) \preceq (e_n)$. If $(e_n^2) \preceq (e_n^1)$, then $(e_n^2) \preceq (e_n)$ and a second application of Lemma 3.6 shows that Φ_2 is degenerate.

We next consider the case where Φ_1 is non-degenerate. Assume for contradiction that $\Phi_2 \preceq \Phi_1$ is not satisfied. Then for all positive K and all $t_0 > 0$ there exists $t \in [0, t_0]$ such that $\Phi_2(t) > \Phi_1(Kt)$. In particular, there exists a sequence (t_n) of positive numbers $t_n < \frac{1}{n^2}$ such that $\Phi_2(t_n) > \Phi_1(nt_n)$. Setting $s_n = nt_n$ we obtain $\Phi_2(\frac{s_n}{n}) > \Phi_1(s_n)$ for each n while $s_n < \frac{1}{n}$. All the basis vectors e_n^1 have the same norm which we denote by c.

Recall that (e_n) denotes the canonical basis of c_0. Since Φ_1 is non-degenerate, Lemma 3.6 shows that $(e_n^1) \preceq (e_n)$ does not hold. It follows that the norm of $\sum_{i=1}^k e_i^1$ grows to infinity with k.

Thus there exists k such that $\| \sum_{i=1}^k s_n e_i^1 \|_{\Phi_1} > 1$ for each n. Let k_n be the smallest integer with this property. Then

$$\left\| \sum_{i=1}^{k_n} s_n e_i^1 \right\|_{\Phi_1} \leq \left\| \sum_{i=1}^{k_{n-1}} s_n e_i^1 \right\|_{\Phi_1} + \|s_n e_i^1\|_{\Phi_1} \leq 1 + \frac{c}{n}. \tag{3.6}$$

On the other hand,

$$\sum_{i=1}^{k_n} \Phi_2 \left(\frac{s_n}{2^n} \right) \geq \sum_{i=1}^{k_n} \Phi_1(s_n) > 1,$$

where the last inequality follows from the fact that the norm of $\sum_{i=1}^{k_n} s_n e_i^1$ in ℓ_{Φ_1} is greater than one. Hence $\| \sum_{i=1}^{k_n} s_n e_i^2 \|_{\Phi_2} > 2^n$. This together with (3.6) constitutes a contradiction with (ii).

(iii) $\Rightarrow$ (i). Suppose (3.5) for some positive constant K and $t \in [0, t_0]$. Let us denote again by c the norm of the basis vectors e_n^1. If (a_i) is a sequence in ℓ_{Φ_1} of norm $\|(a_i)\|_{\Phi_1} = 1$, then $\|a_i e_i\|_{\Phi_1} \leq 1$ implies $|a_i| \leq \frac{1}{c}$ for all i. Choosing $\rho \geq \frac{1}{cKt_0}$ we have

$$\sum_{i=1}^{\infty} \Phi_2 \left(\frac{|a_i|}{K\rho} \right) \leq \sum_{i=1}^{\infty} \Phi_1 \left(\frac{|a_i|}{\rho} \right) \leq 1,$$

where we used $\frac{|a_i|}{K\rho} \leq \frac{1}{cK\rho} \leq t_0$. This proves $\|(a_i)\|_{\Phi_2} \leq \frac{1}{ct_0}$ and since (a_i) was an arbitrary sequence in ℓ_{Φ_1} of norm one, the identity mapping is bounded. □

Following [4, p. 139] we summarize the properties of an Orlicz function Φ. The convexity of Φ implies that it has a right derivative for all $t \in [0, \infty)$ which is often denoted by $p(t)$. The function p is non-negative, non-decreasing and right-continuous. Furthermore, $\Phi(t) = \int_0^t p(s)ds$ for $t \geq 0$. From the last formula it follows that $\Phi(t) \leq tp(t)$ for $t \geq 0$. Since Φ is convex and $\Phi(0) = 0$, we deduce $\frac{t_2}{t_1}\Phi(t_1) \leq \Phi(t_2)$ for $0 < t_1 < t_2$. This shows that $\frac{\Phi(t)}{t}$ is non-decreasing.

Corollary 3.12. *The identity mapping $Id : \ell_1 \to \ell_\Phi$ is bounded for any Orlicz function Φ. It is an isomorphism if and only if the right derivative of Φ satisfies $p(0) > 0$.*

Proof. Setting $\Psi(t) = t$ for $t \geq 0$ we have $\ell_1 = \ell_\Psi$. Let us fix arbitrary $t_0 > 0$ and define $c_0 = \frac{\Phi(t_0)}{t_0}$. Then $\frac{\Phi(t)}{t} \leq c$ for all $t \in [0, t_0]$ by the fact that $\frac{\Phi(t)}{t}$ is non-decreasing. Hence $\Phi(t) \leq c\Psi(t)$ for $t \in [0, t_0]$. This means $\Phi \preceq \Psi$ and the mapping $\mathrm{Id} : \ell_1 \to \ell_\Phi$ is bounded by Proposition 3.11.

The mapping is an isomorphism if and only if $\Psi \preceq \Phi$. The last relation is equivalent to the existence of $K > 0$ and $t_1 > 0$ such that $Kt \leq \Phi(t)$ for $t \in [0, t_1]$. It follows from the comments above that $K \leq \frac{\Phi(t)}{t} \leq p(t)$ for $t \in (0, t_1]$ and $K \leq p(0)$. Conversely, $0 < K = p(0)$ implies $K \leq p(s)$ for all s and hence $\Phi(t) = \int_0^t p(s)ds \geq Kt$. □

3.1.3 Boundedly-complete bases

Definition 3.13. An Orlicz function Φ satisfies the Δ_2*-condition at zero* if

$$\limsup_{t\to 0+} \frac{\Phi(2t)}{\Phi(t)} < \infty. \tag{3.7}$$

Remark 3.14. Let $Q > 1$. An Orlicz function Φ satisfies the Δ_2-condition at zero if and only if

$$\limsup_{t\to 0+} \frac{\Phi(Qt)}{\Phi(t)} < \infty. \tag{3.8}$$

The proof of this fact is entirely analogous to the proof of Lemma 3.45 which will be given later.

It is also not difficult to show that the Δ_2-condition at zero is equivalent to

$$\limsup_{t\to 0+} \frac{tp(t)}{\Phi(t)} < \infty.$$

We refer the reader to [4, p. 140].

Proposition 3.15. *Let Φ be a nondegenerate Orlicz function. The basis (e_n) of h_Φ is boundedly-complete if and only if Φ satisfies the Δ_2-condition at zero.*

A proof together with some equivalent criteria can be found in [4, Prop. 4.a.4]. Here we only note that (e_n) is boundedly-complete if and only if $\ell_\Phi = h_\Phi$. This is an immediate consequence of Proposition 3.5.

3.1.4 Duality

The dual space of an Orlicz space h_Φ is described in terms of the complementary function to Φ which is denoted by Φ^*. If we do not want Φ^* to attain the value infinity, it is sometimes necessary to suitably redefine Φ away from zero. The next lemma states that this is possible. The given formulation is slightly more general than required (it will be helpful later).

Lemma 3.16. *Let φ be a convex non-decreasing function defined on an interval $[0, t_0)$ for some $t_0 > 0$ such that $\varphi(0) = 0$. If $t_1 \in [0, t_0)$, then there exists an Orlicz function Φ such that*

(i) $\varphi(t) = \Phi(t)$ *for all* $t \in [0, t_1]$,

(ii) $\Phi(t) > 0$ *for* $t > t_1$,

(iii) $\lim_{t\to\infty} \frac{\Phi(t)}{t} = \infty$.

Proof. The convexity of φ implies that it a has right derivative p on the interval $[0, t_0)$. Let $\tilde{p}$ be given by

$$\tilde{p}(t) = \begin{cases} p(t), & t \in [0, t_1], \\ p(t_1) + t - t_1, & t \in (t_1, \infty). \end{cases}$$

The function $\tilde{p}$ is non-decreasing and right-continuous. If we set $\Phi(t) = \int_0^t \tilde{p}(s)ds$ for $t \in [0, \infty)$, then Φ is an Orlicz function whose right derivative is $\tilde{p}$. Now (i) holds because $\tilde{p} = p$ on $[0, t_1]$, (ii) holds because $\tilde{p}(t) > 0$ for $t > t_1$ and (iii) follows from the fact that $\frac{\Phi(t)}{t} \geq \tilde{p}(t)$ for $t > 0$ and $\lim_{t\to\infty} \tilde{p}(t) = \infty$. □

Thus a given Orlicz function Φ_1 can always be redefined to satisfy condition (iii) from Lemma 3.16. If we let $t_1 \in (0, +\infty)$ such that $\Phi_1(t_1) > 1$, the lemma provides an Orlicz function Φ_2 satisfying (iii) and $\Phi_1(t) = \Phi_2(t)$ for $t \in [0, t_1]$. In particular, $\Phi_1(t) \leq 1$ if and only if $\Phi_2(t) \leq 1$. By the definition of norm in an Orlicz space (3.1), $\ell_{\Phi_1} = \ell_{\Phi_2}$.

In the reminder of this section, Φ is assumed to satisfy $\lim_{t\to\infty} \frac{\Phi(t)}{t}$.

Definition 3.17. If Φ is an Orlicz function, then the function Φ^* defined for $u \geq 0$ as

$$\Phi^*(u) = \sup\{tu - \Phi(t);\ 0 \leq t < \infty\}$$

is called the *complementary function* to Φ.

The complementary function Φ^* is convex, non-decreasing, $\Phi^*(0) = 0$ and $\lim_{t\to\infty} \Phi^*(t) = \infty$. We claim that for each $u \geq 0$, there exists a constant C_u and $t_0 > 0$ such that $tu - \Phi(t) \leq C_u$ for $t \geq t_0$ and $\Phi^*(u)$ is therefore finite. This is a consequence of the assumption $\lim_{t\to\infty} \frac{\Phi(t)}{t} = \infty$.

Proposition 3.18. *Let Φ be an Orlicz function satisfying $\lim_{t\to\infty} \frac{\Phi(t)}{t} = \infty$. Then h_Φ^* is isomorphic to ℓ_{Φ^*}.*

For a proof we refer the reader to [4, Prop. 4.b.1].

3.2 Spaces with iterative norms

3.2.1 Basic properties of finite-dimensional norms

A two-dimensional norm N will be used to define the spaces with iterative norms that we are interested in. We introduce absolute and normalized norms on $\mathbb{F}^n$ before moving to the two-dimensional case. The terminology is used in the article [7] where a certain characterization of these norms is provided. In what follows $x = (a_1, a_2, \ldots, a_n)$ denotes a vector in $\mathbb{F}^n$.

Definition 3.19. Let N_n be a norm on $\mathbb{F}^n$. We call N_n *absolute* if

$$N_n(a_1, a_2, \ldots, a_n) = N_n(|a_1|, |a_2|, \ldots, |a_n|), \qquad (a_1, a_2, \ldots, a_n) \in \mathbb{F}^n.$$

Lemma 3.20. *Let N_n is an absolute norm on $\mathbb{F}^n$. Then*

$$N_n(a_1, a_2, \ldots, a_n) \leq N_n(b_1, b_2, \ldots, b_n) \tag{3.9}$$

if $|a_i| \leq |b_i|$ for $i = 1, \ldots, n$.

Proof. It can be assumed without loss of generality that the scalars a_i and b_i are real and non-negative. There exists $\lambda \in [0, 1]$ such that $a_1 = \lambda_1 b_1 + (1 - \lambda_1)(-b_1)$. Then by the triangle inequality and homogeneity,

$$\begin{aligned} N_n(a_1, a_2, \ldots, a_n) &= N_n(\lambda_1 b_1 + (1 - \lambda_1)(-b_1), a_2, \ldots, a_n) \\ &\leq \lambda N_n(b_1, a_2, \ldots, a_n) + (1 - \lambda) N_n(-b_1, a_2, \ldots, a_n) \\ &= N_n(b_1, a_2, \ldots, a_n). \end{aligned}$$

Repeating the argument for each $i = 2, \ldots, n$ we obtain (3.9). □

The last lemma can be thought of as a finite-dimensional analogue of Proposition 1.14.

Definition 3.21. A norm N_n on $\mathbb{F}^n$ is said to be *normalized* if $N_n(e_i) = 1$ for each canonical unit vector $e_i \in \mathbb{F}^n$.

Lemma 3.22. *Let N_n be an absolute norm on $\mathbb{F}^n$. Then the following statements are equivalent.*

(i) N_n is normalized.

(ii) $\|x\|_\infty \leq N_n(x) \leq \|x\|_1$ for all $x \in \mathbb{F}^n$.

Proof. Assume N_n is normalized and let $(a_1, \ldots, a_n) \in \mathbb{F}^n$. If $1 \leq i \leq n$, then $|a_i| = N_n(a_i e_i)$ because N_n is normalized and $N_n(a_i e_i) \leq N_n(a_1, \ldots, a_n)$ by Lemma 3.20. Putting the two inequalities together we have

$$\sup_{1 \leq i \leq n} |a_i| \leq N_n(a_1, \ldots, a_n).$$

By the triangle inequality

$$N_n(a_1, \ldots, a_n) \leq N_n(a_1 e_1) + \ldots + N_n(a_n e_n) = |a_1| + \ldots + |a_n|$$

and (i) $\Rightarrow$ (ii) is proved. The converse follows by substituting e_i for x. □

From now on we always work with a norm N on $\mathbb{F}^2$ which is assumed to be absolute and normalized. We proceed with the description of the unit circle in the norm N.

Definition 3.23. Suppose that N is an absolute and normalized norm on $\mathbb{F}^2$. Let φ be the function given by

$$\varphi(b) = 1 - a$$

for $b \in [0, 1)$, where $a \in (0, 1]$ is such that $N(a, b) = 1$. We also set

$$\varphi(1) = \lim_{b \to 1-} \varphi(b).$$

We call φ the *function associated with N*.

It remains to verify that φ is well defined. Then it is clear that $\varphi(b) \in [0, 1]$ for $b \in [0, 1]$.

Proposition 3.24. *Let N be an absolute normalized norm on $\mathbb{F}^2$. Then the function φ associated with N has the following properties.*

(i) φ is well defined, convex, non-decreasing, $\varphi(0) = 0$ and

$$\varphi(b) = 1 - a \Leftrightarrow N(a, b) = 1 \tag{3.10}$$

for $(a, b) \in (0, 1] \times [0, 1)$.

(ii) $N(a, 1) = 1 \Rightarrow a = 0$ if and only if $\varphi(1) = 1$. In this case (3.10) holds for $(a, b) \in [0, 1] \times [0, 1]$.

(iii) $N(1,b)=1 \Rightarrow b=0$ *if and only if* $\varphi(b)>0$ *for all* $b\in(0,1]$.

Proof. (i). Suppose $b\in[0,1)$ is fixed and let f be given by $f(a)=N(a,b)$ for $a\geq 0$. Then $f(0)=N(0,b)=b<1$ and $f(1)=N(1,b)\geq 1$. Since f is continuous, there exists $a\in(0,1]$ such that $f(a)=1$ by the Intermediate Value Theorem. Additionally, f is non-decreasing (Lemma 3.20) and convex because

$$N(ta_1+(1-t)a_2,b)\leq tN(a_1,b)+(1-t)N(a_2,b)$$

for $t\in[0,1]$ and $a_1,a_2\geq 0$. Then f must be increasing except possibly on some interval $[0,a_0]$ where it is constant. Hence there exists only one value $a\in(0,1]$ for which $f(a)=1$ and $\varphi(b)=1-a$ for this a. This proves that φ is well defined on the interval $[0,1)$ and also the equivalence (3.10).

To prove that φ is convex on $[0,1)$, let $b_1,b_2\in[0,1)$ and $a_1,a_2\in(0,1]$ satisfy $\varphi(b_1)=1-a_1$ and $\varphi(b_2)=1-a_2$. If $(a,b)=t(a_1,b_1)+(1-t)(a_2,b_2)$ for some $t\in[0,1]$, then $N(a,b)\leq 1$. Consequently, the unique $a_0\in(0,1]$ satisfying $N(a_0,b)=1$ is greater than or equal to a, which implies

$$\varphi(b)=1-a_0\leq 1-a=t(1-a_1)+(1-t)(1-a_2)=t\varphi(b_1)+(1-t)\varphi(b_2)$$

and convexity is proved. Clearly $\varphi(0)=0$ because $N(1,0)=1$ due to N being normalized. Since φ is non-negative and convex, it must be non-decreasing.

(ii). We assume first that $N(a,1)=1\Rightarrow a=0$. If we let $a_1=1-\lim_{b\to 1-}\varphi(b)$, then the continuity of norm gives

$$N(a_1,1)=\lim_{b\to 1-}N(1-\varphi(b),b)=1.$$

By our assumption, $a_1=0$ and therefore $\varphi(1)=\lim_{b\to 1-}\varphi(b)=1$.

To prove the second implication, assume there exists $a\in(0,1]$ such that $N(a,1)=1$. Then $N(a,b)\leq 1$ for $b\in[0,1)$ and thus $\varphi(b)\leq 1-a$. From this we conclude $\varphi(1)=\lim_{b\to 1-}\varphi(b)\leq 1-a<1$.

(iii). By definition, $\varphi(b)=0$ is equivalent to $N(1,b)=1$ for $b\in[0,1)$. If $N(1,b)=1\Rightarrow b=0$, then $\varphi(b)=0$ only for $b=0$. □

Remark 3.25. Let us point out that $\varphi(b)\leq 1-a \Leftrightarrow N(a,b)\leq 1$ for $(a,b)\in(0,1]\times[0,1)$. Indeed, $\varphi(b)=1-a_0\leq 1-a$ if and only if $N(a,b)\leq N(a_0,b)=1$.

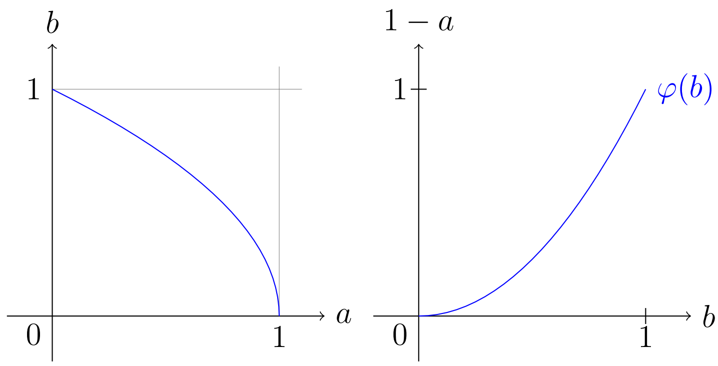

Figure 3.1: Function φ associated with N.

3.2.2 Definition of spaces with iterative norms

A sequence of norms N_n on $\mathbb{F}^n$ is defined as follows. For $n = 1$, let $N_1(a_1) = |a_1|$ be the absolute value on $\mathbb{F}$. For $n = 2$, let $N_2(a_1, a_2) = N(a_1, a_2)$, where N is a given absolute and normalized norm on $\mathbb{F}^2$. For each $n > 2$, we define inductively

$$N_n(a_1, a_2, \ldots, a_n) = N(N_{n-1}(a_1, a_2, \ldots, a_{n-1}), a_n), \tag{3.11}$$

where $(a_1, a_2, \ldots, a_n) \in \mathbb{F}^n$.

Lemma 3.26. *The formula (3.11) defines a norm on $\mathbb{F}^n$ for all $n > 2$. The norms N_n are absolute and normalized.*

Proof. The proof is by induction on n. First, $N_2 = N$ has the required properties by definition. Next, let $n > 2$ and assume that N_{n-1} is a norm. Throughout the proof, a vector $x \in \mathbb{F}^n$ will be written as $x = (\tilde{x}, a_n)$, where $\tilde{x} \in \mathbb{F}^{n-1}$.

Now $N_n(x) = N(N_{n-1}(\tilde{x}), a_n) = 0$ if and only if $N_{n-1}(\tilde{x}) = 0$ and $a_n = 0$. By the induction hypothesis, this is equivalent to $\tilde{x} = 0$ and $a_n = 0$.

To show homogeneity we let $\lambda \in \mathbb{F}$ and use that N is absolute to deduce

$$N_n(\lambda x) = N(N_{n-1}(\lambda\tilde{x}), \lambda a_n) = N(|\lambda| N_{n-1}(\tilde{x}), \lambda a_n) = |\lambda| N_n(N_{n-1}(\tilde{x}), a_n),$$

which proves $N_n(\lambda x) = |\lambda| N_n(x)$.

Let $y = (\tilde{y}, b_n) \in \mathbb{F}^n$ be a second vector. It follows from $N_{n-1}(\tilde{x} + \tilde{y}) \leq N_{n-1}(\tilde{x}) + N_{n-1}(\tilde{y})$ and Lemma 3.20 that

$$\begin{aligned} N(N_{n-1}(\tilde{x} + \tilde{y}), a_n + b_n) &\leq N(N_{n-1}(\tilde{x}) + N_{n-1}(\tilde{y}), a_n + b_n) \\ &\leq N(N_{n-1}(\tilde{x}), a_n) + N(N_{n-1}(\tilde{y}), b_n). \end{aligned}$$

This proves the triangle inequality for N_n. Finally, the norm N_n is absolute and normalized because N_{n-1} and N are. □

We are ready to introduce the main definition.

Definition 3.27. Suppose that N is an absolute normalized norm on $\mathbb{F}^2$ and N_n are the norms given by (3.11). If (a_i) is a sequence of scalars, let

$$\|(a_i)\|_N = \sup_n N_n(a_1, a_2, \ldots, a_n).$$

We define X_N to be the Banach space of all sequences of scalars (a_i) such that $\|(a_i)\|_N < \infty$ equipped with the norm $\|\cdot\|_N$. Its subspace $Y_N = [e_n]$ is defined as the closed linear span of the canonical unit vectors. We call X_N and Y_N *spaces with iterative norms.*

We prove that X_N really is a Banach space as part (ii) of Proposition 3.29.

Lemma 3.28. *Let N be a norm on $\mathbb{F}^2$ which is absolute and normalized. If (a_i) is a sequence of scalars and $m \leq n$, then*

(i) $N_m(a_1, \ldots, a_m) \leq N_n(a_1, \ldots, a_n)$,

(ii) $N_n(a_1, \ldots, a_m, 0, \ldots, 0) = N_m(a_1, \ldots, a_m)$,

(iii) $a_i = 0$ *for* $i > n$ *implies* $\|(a_i)\|_N = N_n(a_1, \ldots, a_n)$.

Proof. By Lemma 3.22 applied to N,

$$N_{n-1}(a_1, \ldots, a_{n-1}) \leq N(N_{n-1}(a_1, \ldots, a_{n-1}), a_n) = N_n(a_1, \ldots, a_n).$$

The inequality in (i) now follows by induction.

If $m = n - 1$, then Lemma 3.22 shows that

$$N(N_{n-1}(a_1, \ldots, a_{n-1}), 0) = N_{n-1}(a_1, \ldots, a_{n-1})$$

and (ii) follows again by induction.

If $k \leq n$, then $N_k(a_1, \ldots, a_k) \leq N_n(a_1, \ldots, a_n)$ by (i) and if $k > n$, then $N_k(a_1, \ldots, a_k) = N_n(a_1, \ldots, a_n)$ by (ii). Hence

$$\|(a_i)\|_N = \sup N_k(a_1, \ldots, a_k) = N_n(a_1, \ldots, a_n),$$

which concludes the proof of (iii). □

It is convenient for the purposes of our next proof to extend the norms N_n to the linear space $\mathbb{F}^{\mathbb{N}}$ of all scalar sequences. This extension is again denoted N_n by an abuse of notation. Thus we write $N_n(x) = N_n(a_1, \ldots, a_n)$ for a sequence of scalars $x = (a_i)$. The formula for norm becomes $\|x\|_N = \sup_n N_n(x)$ for $x \in X_N$. It is clear that N_n is a pseudonorm on $\mathbb{F}^{\mathbb{N}}$.

Proposition 3.29. *Let N be a norm on $\mathbb{F}^2$ which is absolute and normalized. Then*

(i) $\| \cdot \|_N$ is a norm on X_N and $\|x\|_\infty \leq \|x\|_N \leq \|x\|_1$ for any sequence of scalars $x = (a_i)$ (with the convention that each of the three functionals can also attain the value infinity),

(ii) X_N is a Banach space and Y_N is a closed subspace of X_N,

(iii) (e_n) is an unconditional basis of Y_N with unconditional constant one.

Proof. To prove (i), let $x = (a_i)$ denote a sequence of scalars. If x is zero, then $N_n(x) = 0$ for all n and therefore $\|x\|_N = 0$. If $a_n \neq 0$ for some n, then $N_n(x) = N_n(a_1, \ldots, a_n) > 0$ and hence $\|x\|_N > 0$. It is also not difficult to prove the other two properties, namely homogeneity and the triangle inequality, from the corresponding properties of N_n.

Lemma 3.26 asserts that the norms N_n are normalized. By Lemma 3.22

$$\|(a_1, \ldots, a_n)\|_\infty \leq N_n(a_1, \ldots, a_n) \leq \|(a_1, \ldots, a_n)\|_1, \qquad (a_1, \ldots, a_n) \in \mathbb{F}^n$$

for all n and the inequalities in 1) follow by passing to the supremum.

We begin the proof of (ii) by showing that N_n is continuous with respect to the supremum norm $\| \cdot \|_\infty$ for each n. If $x_k \to x$ in ℓ_∞, then the sequences x_k converge to x in each component and we deduce $N_n(x - x_k) \to 0$ for $k \to \infty$.

Now we let (x_k) be a Cauchy sequence in X_N. By (i), (x_k) is Cauchy in ℓ_∞ and hence converges in ℓ_∞ to some $x \in \ell_\infty$. Because N_n are continuous with respect to $\| \cdot \|_\infty$, the norm $\| \cdot \|_N$ is lower-semicontinous on ℓ_∞ as their supremum. Therefore

$$\|x\|_N \leq \liminf_{k \to \infty} \|x_k\|_N < \infty,$$

which proves $x \in X_N$. Let $\varepsilon > 0$ and choose k_0 such that if $k, j \geq k_0$, then $\|x_k - x_j\|_N \leq \frac{\varepsilon}{2}$. For a fixed n, there exists $j \geq k_0$ such that $N_n(x - x_j) \leq \frac{\varepsilon}{2}$. Then for $k \geq k_0$

$$N_n(x - x_k) \leq N_n(x - x_j) + N_n(x_j - x_k) \leq N_n(x - x_j) + \|x_j - x_k\|_N \leq \varepsilon.$$

Taking the supremum over n on the left-hand side we get $\|x - x_k\|_N \leq \varepsilon$ for $k \geq k_0$ and so $x_k \to x$ in X_N, which proves X_N is Banach. The subspace $Y_N = [e_n]$ is closed by definition.

Next, we show (iii). Let (a_i) be a sequence of scalars and $m \leq n$. Then

$$\left\| \sum_{i=1}^{m} a_i e_i \right\| = N_m(a_1, \ldots, a_m) \leq N_n(a_1, \ldots, a_n) = \left\| \sum_{i=1}^{n} a_i e_i \right\|.$$

The sequence (e_i) is basic by the Grundblum's criterion and hence a basis of Y_N. If (b_i) is another scalar sequence with $|a_i| \leq |b_i|$ for all i, then $N_n(a_1, \ldots, a_n) \leq N_n(b_1, \ldots, b_n)$ by Lemma 3.20. Therefore

$$\left\| \sum_{i=1}^{n} a_i e_i \right\|_N \leq \left\| \sum_{i=1}^{n} b_i e_i \right\|_N$$

and this is equivalent to (e_i) being unconditional with unconditional constant one by Proposition 1.14. □

We recall that a basis (e_i) is said to be subsymmetric if it is equivalent to (e_{k_i}) for any increasing sequence of integers (k_i). It is not very difficult to show with the help of Lemma 3.28 that the canonical basis (e_i) of Y_N is subsymmetric (with subsymmetric constant one). This is because the zero elements in a sequence (a_i) can be omitted without changing its norm $\|(a_i)\|_N$. At this point however, it is entirely unclear whether (e_i) is symmetric. We will show this only indirectly after proving Theorem 3.40.

3.2.3 Ordering on bases

We let N^1 and N^2 denote two norms on $\mathbb{F}^2$ which are absolute and normalized. In this section, we study the situation when the respective canonical bases (e_n^1) and (e_n^2) of Y_{N^1} and Y_{N^2} satisfy $(e_n^2) \preceq (e_n^1)$. Our goal is to find equivalent criteria formulated in terms of the norms N^1, N^2 and in terms of the functions φ_1, φ_2 associated with them.

Definition 3.30. We write $N^2 \preceq N^1$ if there exists a constant $K > 0$ such that

$$N^2(Ka, b) \leq K N^1(a, b) \tag{3.12}$$

for all $a, b \in \mathbb{F}$. We also write $N^1 \sim N^2$ if there exists $K > 0$ such that

$$N^2\left(a, \frac{b}{K}\right) \leq N^1(a, b) \leq N^2(a, Kb) \tag{3.13}$$

for all $a, b \in \mathbb{F}$.

Remark 3.31. The inequality (3.12) may be rewritten as $N^2(a, \frac{b}{K}) \le N^1(a,b)$. It remains valid if K is replaced by a larger constant because N^2 is absolute. This observation allows one to prove

$$N^2 \preceq N^1 \text{ and } N^2 \preceq N^1 \Leftrightarrow N^1 \sim N^2. \tag{3.14}$$

The next lemma will simplify the subsequent arguments.

Lemma 3.32. *If there is a constant $K > 0$ such that $N^2(K,c) \le KN^1(1,c)$ for all $c \ge 0$, then $N^2 \preceq N^1$.*

Proof. Since the norms N^1 and N^2 are absolute, it is enough to show (3.12) for $a \ge 0$ and $b \ge 0$. If $a > 0$, then we substitute $\frac{b}{a}$ for c and multiply by a to get (3.12). By Remark 3.31, we can assume $K \ge 1$, in which case (3.12) holds also for $a = 0$. □

Theorem 3.33. *Let N^1, N^2 be two norms on $\mathbb{F}^2$ which are absolute and normalized. The following statements are equivalent.*

(i) The identity mapping $Id : X_{N^1} \to X_{N^2}$ is bounded.

(ii) If (e_n^1), (e_n^2) are the respective canonical bases of Y_{N^1}, Y_{N^2}, then $(e_n^2) \preceq (e_n^1)$.

(iii) $N^2 \preceq N^1$.

Proof. Proposition 2.1 asserts that $(e_n^2) \preceq (e_n^1)$ is equivalent to the mapping $Id : Y_{N^1} \to Y_{N^2}$ being bounded. The implication (i) ⇒ (ii) follows.

(ii) ⇒ (iii). Suppose $Id : Y_{N^1} \to Y_{N^2}$ is bounded of norm K and let $\varepsilon > 0$. Then there exists a sequence $y = (a_i)$ in Y_{N^1} such that $\|y\|_{N^1} = 1$ and $\|y\|_{N^2} > K - \varepsilon$. By the definition of the norm $\|\cdot\|_{N^2}$, $N_n^2(a_1, \dots, a_n) \ge K - \varepsilon$ for some n. Then for any $c \in \mathbb{F}$

$$\begin{aligned} N^2(K-\varepsilon, c) &\le N^2(N_n^2(a_1,\dots,a_n), c) = N_{n+1}^2(a_1,\dots,a_n,c) \\ &\le KN_{n+1}^1(a_1,\dots,a_n,c) = KN^1(N_n^1(a_1,\dots,a_n),c) = KN^1(1,c), \end{aligned}$$

where we used that the norm of Id : $Y_{N^1} \to Y_{N^2}$ is K and the properties of N_n. Letting $\varepsilon \to 0$ we have $N^2(K,c) \le KN^1(1,c)$ and the conclusion follows from Lemma 3.32.

(iii) ⇒ (i). Again, there is no loss of generality in assuming $K \ge 1$. Now let $(a_i) \in X_{N^1}$. We prove by induction

$$N_n^2(a_1,\dots,a_n) \le KN_n^1(a_1,\dots,a_n), \qquad n \ge 2. \tag{3.15}$$

For $n = 2$ we have

$$N^2(a_1,a_2) \le N^2(Ka_1,a_2) \le KN^1(a_1,a_2).$$

If the inequality holds for n, then

$$\begin{aligned} N_{n+1}^2(a_1,\dots,a_{n+1}) &= N^2(N_n^2(a_1,\dots,a_n),a_{n+1}) \le N^2(KN_n^1(a_1,\dots,a_n),a_{n+1}) \\ &\le KN^1(N_n^1(a_1,\dots,a_n),a_{n+1}) = KN_{n+1}^1(a_1,\dots,a_{n+1}). \end{aligned}$$

We used (iii) in the third inequality. Taking the supremum on both sides of (3.15) we have $\|(a_i)\|_{N^2} \le K\|(a_i)\|_{N^1}$. □

The next step is to express the condition (iii) in Theorem 3.33 in terms of the functions φ_1, φ_2 associated with N^1, N^2.

Proposition 3.34. *Let N^1, N^2 be two norms on $\mathbb{F}^2$ which are absolute and normalized and let φ_1, φ_2 be the functions associated with N^1, N^2. Then the following statements are equivalent.*

(i) $N^2 \preceq N^1$.

(ii) There exist $K > 0$ and $b_0 \in (0,1)$ such that $\varphi_2(b) \leq \varphi_1(Kb)$ for all $b \in [0, b_0]$.

(iii) There exist $K, C > 0$ such that

$$N^2(K, b) \leq KN^1(1, b) \tag{3.16}$$

for $0 \leq b \leq C$.

Proof. (i) $\Rightarrow$ (ii). We can assume that there exists $K > 1$ such that for $a, b \in \mathbb{F}$

$$N^2\left(a, \frac{b}{K}\right) \leq N^1(a, b). \tag{3.17}$$

Let us fix $b \in [0, \frac{1}{K})$ and define $b_1 = Kb$. Then $b_1 \in [0, 1)$ and by definition of φ_1, there is some $a_1 \in (0, 1]$ such that $\varphi_1(b_1) = 1 - a_1$. By the equivalence (3.10) in Proposition 3.24

$$\varphi_1(b_1) = 1 - a_1 \Rightarrow N^1(a_1, b_1) = 1.$$

Then $N^2(a_1, \frac{b_1}{K}) \leq N^1(a_1, b_1) = 1$ from (3.17). Since $\frac{b_1}{K} \leq b_1 < 1$, Remark 3.25 applies and

$$N^2\left(a_1, \frac{b_1}{K}\right) \leq 1 \Rightarrow \varphi_2\left(\frac{b_1}{K}\right) \leq 1 - a_1 = \varphi(b_1).$$

We obtained $\varphi_2(\frac{b_1}{K}) \leq \varphi(b_1)$ and this is $\varphi_2(b) \leq \varphi_1(Kb)$ for $b \in [0, \frac{1}{K})$.

(ii) $\Rightarrow$ (iii). We can assume $K \geq 1$. Let $C = \frac{b_0}{K} < 1$. The inequality in (ii) becomes

$$\varphi_2\left(\frac{b}{K}\right) \leq \varphi_1(b), \qquad b \in [0, C]. \tag{3.18}$$

We take a fixed $b \in [0, C]$ and define $\lambda = N^1(1, b) \geq 1$. Using (3.10),

$$N^1\left(\frac{1}{\lambda}, \frac{b}{\lambda}\right) = 1 \Rightarrow \varphi_1\left(\frac{b}{\lambda}\right) = 1 - \frac{1}{\lambda}.$$

By (3.18), $\varphi_2(\frac{b}{K\lambda}) \leq \varphi_2(\frac{b}{\lambda})$ and therefore

$$\varphi_2\left(\frac{b}{K\lambda}\right) \leq 1 - \frac{1}{\lambda} \Rightarrow N^2\left(\frac{1}{\lambda}, \frac{b}{K\lambda}\right) \leq 1 \Rightarrow N^2\left(1, \frac{b}{K}\right) \leq \lambda.$$

We proved $N^2\left(1, \frac{b}{K}\right) \leq N^1(1, b)$ for $b \in [0, C]$ which is equivalent to (iii).

(iii) $\Rightarrow$ (i). We begin by rewriting (iii) as

$$N^2(K_1, b) \leq K_1 N^1(1, b), \qquad b \in [0, C] \tag{3.19}$$

with K_1 in place of K. We prove that there exists some $K \geq 1$ such that $N^2(K, b) \leq KN^1(1, b)$ for all $b \geq 0$. Let us fix $K_3 > 1$ and set $C_2 = \frac{K_3}{K_3 - 1}$. Then $b \geq C_2 \Leftrightarrow K_3 b \geq K_3 + b$. Using Lemma 3.22 we estimate

$$N^2(K_3, b) \leq K_3 + b \leq K_3 b \leq K_3 N^1(1, b), \qquad b \in [C_2, \infty). \tag{3.20}$$

We claim there exists $K_2 > 0$ such that

$$N^2\left(1, \frac{C_2}{K_2}\right) \leq N^1(1, C_1). \tag{3.21}$$

There are two cases to consider. If $N^1(1, C_1) > 1$, then the inequality holds for K_2 large enough by the fact that $N^2(1, 0) = 1$. If $N^1(1, C_1) = 1$, then $N^1(1, b) = 1$ for all $b \in [0, C_1]$ and (3.19) implies $N^2(K_1, b) = K_1$ for $b \in [0, C_1]$. Dividing by K_1, we get $N^2(1, b) = 1$ for $b \in [0, \frac{C_1}{K_1}]$. Hence we can find $K_2 > 0$ such that $N^2(1, \frac{C_2}{K_2}) = 1 = N^1(1, C_1)$. This proves the claim. If $b \in [C_1, C_2]$, the inequality (3.21) gives

$$N^2(K_2, b) \leq K_2 N^2\left(1, \frac{C_2}{K_2}\right) \leq K_2 N^1(1, C_1) \leq K_2 N^1(1, b). \tag{3.22}$$

Finally, we put (3.19), (3.20) and (3.22) together. Each inequality remains valid if we replace K_i by $K = \max(K_1, K_2, K_3)$, therefore

$$N^2(K, b) \leq KN^1(1, b)$$

for all $b \in [0, \infty)$. Application of Lemma 3.32 concludes the proof. □

Theorem 3.35. *Let N^1, N^2 be two norms on $\mathbb{F}^2$ which are absolute and normalized and let φ_1, φ_2 denote the functions associated with N^1, N^2. Then the following statements are equivalent.*

(i) The identity mapping $Id\colon X_{N^1} \to X_{N^2}$ is an isomorphism.

(ii) The canonical bases (e_n^1) of Y_{N^1} and (e_n^2) of Y_{N^2} are equivalent.

(iii) $N^1 \sim N^2$.

(iv) There exists $K > 0$ and $b_0 \in (0, 1)$ such that $\varphi_2\left(\frac{b}{K}\right) \leq \varphi_1(b) \leq \varphi_2(Kb)$ for all $b \in [0, b_0]$.

Proof. The proof follows from Theorem 3.33 and Proposition 3.34. □

3.2.4 Correspondence with Orlicz sequence spaces

Lemma 3.36. *Let φ be a convex, non-decreasing function defined on $[0, 1]$ such that $\varphi(0) = 0$ and $\lim_{t \to 1-} \varphi(t) = \varphi(1) = 1$. Then φ is associated with a norm N on $\mathbb{F}^2$ which is absolute and normalized.*

Proof. Define $C = \{(a, b) \in \mathbb{F}^2;\ |b| \leq 1,\ \varphi(|b|) \leq 1 - |a|\}$.

C is convex: Let $(a_1, b_1), (a_2, b_2) \in C$ and $t \in [0,1]$. We have $\varphi(|b_1|) \leq 1 - |a_1|$ and $\varphi(|b_2|) \leq 1 - |a_2|$. Since φ is convex and non-decreasing,

$$\begin{aligned}\varphi(|tb_1 + (1-t)b_2|) &\leq \varphi(t|b_1| + (1-t)|b_2|) \leq t\varphi(|b_1|) + (1-t)\varphi(|b_2|) \\ &\leq t(1-|a_1|) + (1-t)(1-|a_2|) = 1 - (t|a_1| + (1-t)|a_2|) \\ &\leq 1 - |ta_1 + (1-t)a_2|,\end{aligned}$$

which proves the point $(a, b) = t(a_1, b_1) + (1-t)(a_2, b_2)$ is contained in C.

C is balanced: Let $(a, b) \in C$ and $\alpha \in \mathbb{F}$, $|\alpha| \leq 1$. The points $(|a|, |b|)$ and $(0, 0)$ belong to C and so does their convex combination

$$(|\alpha a|, |\alpha b|) = |\alpha|(|a|, |b|) + (1 - |\alpha|)(0, 0).$$

It follows that $\alpha(a, b) \in C$.

C is absorbing: Let $(a, b) \in \mathbb{F}^2$. If $\lambda > 0$ is large enough, the inequalities $\frac{|b|}{\lambda} \leq 1$ and $\varphi\left(\frac{|b|}{\lambda}\right) \leq 1 - \frac{|a|}{\lambda}$ are satisfied, which implies $\frac{1}{\lambda}(a, b) \in C$.

We define N to be the Minkowski functional of C, i.e.,

$$N(a, b) = \inf\{\lambda > 0;\ (a, b) \in \lambda C\}.$$

Since C is convex, balanced and absorbing, N is a pseudonorm. Let us show that it is a norm. If $N(a, b) = 0$, then $\frac{1}{\lambda}(a, b) \in C$ for all $\lambda > 0$. In particular, $\frac{|b|}{\lambda} \leq 1$ for all $\lambda > 0$, which implies $b = 0$. Moreover, $0 \leq 1 - \frac{|a|}{\lambda}$ for all $\lambda > 0$, hence $a = 0$.

Now the norm N is absolute because $(a, b) \in \lambda C$ if and only if $(|a|, |b|) \in \lambda C$. Since

$$\frac{1}{\lambda}(1, 0) \in C \Leftrightarrow \varphi(0) \leq 1 - \frac{1}{\lambda} \Leftrightarrow \lambda \geq 1,$$

we have $N(1, 0) = 1$. The equality $N(0, 1) = 1$ can be proved similarly and we deduce that N is normalized.

Let $b \in [0, 1)$ and $a \in (0, 1]$. Then $N(a, b) = 1$ if and only if $(a, b) \in \lambda C$ for $\lambda > 1$ and $(a, b) \notin \lambda C$ for $\lambda < 1$. Equivalently, $\varphi(\frac{b}{\lambda}) \leq 1 - \frac{a}{\lambda}$ for $\lambda > 1$ and $\varphi(\frac{b}{\lambda}) > 1 - \frac{a}{\lambda}$ for $\lambda < 1$ such that $\frac{b}{\lambda} \leq 1$. These two inequalities hold if and only if $\varphi(b) = 1 - a$. This proves $N(a, b) = 1 \Leftrightarrow \varphi(b) = 1 - a$ for $b \in [0, 1)$ and $a \in (0, 1]$. We also have $\varphi(1) = \lim_{t \to 1-} \varphi(t)$ by assumption. By Definition 3.23, φ is the function associated with N. □

If the function φ associated with N satisfies $\varphi(1) = 1$, then by Proposition 3.24, $\varphi(b) = 1 - a \Leftrightarrow N(a, b) = 1$ for all $(a, b) \in [0, 1] \times [0, 1]$. More generally

$$N(a, b) = \lambda \Leftrightarrow N\left(\frac{a}{\lambda}, \frac{b}{\lambda}\right) = 1 \Leftrightarrow \varphi\left(\frac{b}{\lambda}\right) = 1 - \frac{a}{\lambda} \Leftrightarrow \lambda\varphi\left(\frac{b}{\lambda}\right) = \lambda - a \quad (3.23)$$

for $a, b \geq 0$. This is used in the next proof.

Proposition 3.37. *Let N be a norm on $\mathbb{F}^2$ which is absolute and normalized. Assume that $\varphi(1) = 1$, where φ is the function associated with N. If $x = (a_i)$ is a sequence of scalars such that $a_1 \neq 0$ and we denote $\lambda_n = N_n(a_1, \ldots, a_n)$ for all n, then*

$$\|x\|_N = \sum_{i=1}^{\infty} \lambda_i \varphi\left(\frac{|a_i|}{\lambda_i}\right). \quad (3.24)$$

In particular, $x \in X_N$ if and only if the sum converges.

Proof. We prove by induction

$$\lambda_n = \sum_{i=1}^{n} \lambda_i \varphi\left(\frac{|a_i|}{\lambda_i}\right) \tag{3.25}$$

for all $n \geq 1$. The equality (3.24) then follows from $\|x\|_N = \sup_n \lambda_n$. For $n = 1$, $\lambda_1 = N_1(a_1) = |a_1|$. Since $\varphi(1) = 1$, $\lambda_1 = \lambda_1 \varphi(\frac{|a_1|}{\lambda_1})$ is satisfied.

Now we assume (3.25) for n and prove that it holds for $n+1$. Since $\lambda_{n+1} = N(\lambda_n, a_{n+1})$, (3.23) shows

$$\lambda_{n+1} - \lambda_n = \lambda_{n+1} \varphi\left(\frac{a_{n+1}}{\lambda_{n+1}}\right).$$

Using the induction hypothesis we express λ_{n+1} as

$$\lambda_{n+1} = \lambda_n + \lambda_{n+1}\varphi\left(\frac{a_{n+1}}{\lambda_{n+1}}\right) = \sum_{i=1}^{n} \lambda_i \varphi\left(\frac{|a_i|}{\lambda_i}\right) + \lambda_{n+1}\varphi\left(\frac{a_{n+1}}{\lambda_{n+1}}\right),$$

which establishes the formula. □

Proposition 3.38. *Let N be a norm on $\mathbb{F}^2$ which is absolute and normalized. Suppose that the function φ associated to N satisfies $\varphi(1) = 1$ and there exists an Orlicz function Φ such that $\varphi = \Phi|_{[0,1]}$. Then X_N is isomorphic to ℓ_Φ and the isomorphism is given by the identity mapping $Id\colon X_N \to \ell_\Phi$.*

Proof. We first show that $\mathrm{Id} : X_N \to \ell_\Phi$ is bounded. Let $x = (a_i) \in X_N$ be nonzero. It can be assumed that $a_1 \neq 0$ because if a_n is the first nonzero element, then $\|x\|_N = \|(a_i)_{i=n}^\infty\|_N$ and $\|x\|_\Phi = \|(a_i)_{i=n}^\infty\|_N$ as well. If we denote $\lambda = \|x\|_N$ and $\lambda_n = N_n(a_1, \ldots, a_n)$, then $0 < \lambda_n \leq \lambda$ for all n. Using the inequality $\lambda\varphi(\frac{|a_n|}{\lambda}) \leq \lambda_n \varphi(\frac{|a_n|}{\lambda_n})$ which follows from the convexity of φ we obtain

$$\sum_{n=1}^{\infty} \lambda \Phi\left(\frac{|a_n|}{\lambda}\right) \leq \sum_{n=1}^{\infty} \lambda \varphi\left(\frac{|a_n|}{\lambda}\right) \leq \sum_{n=1}^{\infty} \lambda_n \varphi\left(\frac{|a_n|}{\lambda_n}\right) = \lambda,$$

where the last equality is (3.24) from Proposition 3.37. Hence

$$\sum_{n=1}^{\infty} \Phi\left(\frac{|a_n|}{\lambda}\right) \leq 1.$$

By the definition of norm in an Orlicz sequence space, $\|(a_n)\|_\Phi \leq \lambda$.

We next prove that the inverse mapping $\mathrm{Id} : \ell_\Phi \to X_N$ is bounded. By the Closed Graph Theorem, it suffices to show that if $(a_i) \in \ell_\Phi$, then $(a_i) \in X_N$. Let $x = (a_i) \in \ell_\Phi$ be nonzero and denote $\rho = \|x\|_\Phi$. As before, we can assume $a_1 \neq 0$. Suppose, to derive a contradiction, that $\|(a_i)\|_N = \infty$. Then the sequence of $\lambda_n = N_n(a_1, \ldots, a_n)$ grows to infinity. There exists n_0 such that $\lambda_n \geq \rho$ for all $n \geq n_0$. Then by the convexity of φ,

$$\sum_{n=n_0}^{\infty} \lambda_n \varphi\left(\frac{|a_n|}{\lambda_n}\right) \leq \sum_{n=n_0}^{\infty} \rho\varphi\left(\frac{|a_n|}{\rho}\right) \leq \rho.$$

In the last inequality, we used $\sum_{n=1}^{\infty} \varphi(\frac{|a_n|}{\rho}) \leq 1$. Now for all $m \geq n_0$,

$$\lambda_m = \sum_{n=1}^{m} \lambda_n \varphi\left(\frac{|a_n|}{\lambda_n}\right) \leq \sum_{n=1}^{n_0-1} \lambda_n \varphi\left(\frac{|a_n|}{\lambda_n}\right) + \rho.$$

Therefore, the sequence of λ_m is bounded, which is a contradiction. □

Lemma 3.39. *Let N^1 be a norm on $\mathbb{F}^2$ which is absolute and normalized. Then there is a norm $N^2 \sim N^1$ and an Orlicz function Φ_2 such that the function φ_2 associated to N^2 satisfies $\varphi_2(1) = 1$ and $\varphi_2 = \Phi_2|_{[0,1]}$.*

Proof. Let $t_1 \in (0, 1)$ be arbitrary. We know from Lemma 3.16 that there exists an Orlicz function Φ_1 such that $\varphi_1(t) = \Phi_1(t)$ for $t \in [0, t_1]$ and $\Phi_1(t) > 0$ for $t > t_1$. We set $\Phi_2(t) = \frac{\Phi_1(t)}{\Phi_1(1)}$. Then Φ_2 is an Orlicz function equivalent to Φ_1 and $\Phi_2(1) = 1$. We define further $\varphi_2 = \Phi|_{[0,1]}$. By Lemma 3.36, there is a norm N^2 whose associated function is φ_2. Since the functions Φ_1 and Φ_2 are equivalent at zero, their restrictions φ_1 and φ_2 satisfy condition (iv) from Theorem 3.35 and therefore $N^2 \sim N^1$. □

Theorem 3.40. *Every Orlicz space ℓ_Φ is isomorphic to a space with an iterative norm X_N and vice versa. The isomorphism is given by the identity mapping $Id : \ell_\Phi \to X_N$.*

Proof. Let Φ be an Orlicz function. If Φ is degenerate, then ℓ_Φ is isomorphic to ℓ_∞ and we can take $N = \|\cdot\|_\infty$. If Φ is not degenerate then it can be assumed to satisfy $\Phi(1) = 1$ (otherwise we take the function $\Phi_1(t) = \frac{\Phi(t)}{\Phi(1)}$ equivalent to Φ at zero). The function φ given by $\varphi = \Phi|_{[0,1]}$ satisfies the assumptions of Lemma 3.36. Hence there is a norm N such that φ is the function associated with N. By Proposition 3.38, $\mathrm{Id} : X_N \to \ell_\Phi$ is an isomorphism.

Conversely, assume that N is a norm on $\mathbb{F}^2$ which is absolute and normalized. By Lemma 3.39, there exists a norm $N^2 \sim N$ whose associated function φ_2 satisfies $\varphi_2(1) = 1$ and $\varphi_2 = \Phi_2|_{[0,1]}$ for some Orlicz function Φ_2. Theorem 3.35 shows that $Id : X_N \to X_{N^2}$ is an isomorphism and Proposition 3.38 yields that $\mathrm{Id} : X_{N^2} \to \ell_{\Phi_2}$ is an isomorphism, which concludes the proof. □

Corollary 3.41. *The canonical basis (e_n) of Y_N is symmetric.*

3.2.5 Duality

The dual norm to a given absolute and normalized norm N on $\mathbb{F}^2$ is given by

$$N^*(a_1, a_2) = \sup\{|a_1 b_1 + a_2 b_2|;\ N(b_1, b_2) \leq 1\}.$$

We characterize the dual space of Y_N as X_{N^*}.

Lemma 3.42. *Let N be a norm on $\mathbb{F}^2$ which is normalized and absolute. If N^* denotes the dual norm of N, then the dual norm of N_n is $(N_n)^* = (N^*)_n$ for all $n \geq 2$.*

Proof. In the case $n = 2$, $(N^*)_2 = N^*$ by definition. We suppose that the statement holds for n and prove it for $n + 1$. The assumption allows us to omit the brackets and write simply N_n^* for the dual norm of N_n.

Fix $(a_1, \ldots, a_{n+1}) \in \mathbb{F}^{n+1}$ and let $(b_1, \ldots, b_{n+1}) \in \mathbb{F}^{n+1}$ of norm

$$N_{n+1}(b_1, \ldots, b_{n+1}) \leq 1.$$

If we let $a = N_n^*(a_1, \ldots, a_n)$ and $b = N_n(b_1, \ldots, b_n)$, then

$$N(b, b_{n+1}) = N_{n+1}(b_1, \ldots, b_{n+1}) \leq 1$$

and

$$\left|\sum_{i=1}^{n} a_i b_i\right| \leq ab.$$

Thus we get

$$\begin{aligned}\left|\sum_{i=1}^{n+1} a_i b_i\right| &\leq \left|\sum_{i=1}^{n} a_i b_i\right| + |a_{n+1}b_{n+1}| \leq ab + |a_{n+1}b_{n+1}| \\ &\leq N^*(a, a_{n+1})N(b, b_{n+1}) \leq (N^*)_{n+1}(a_1, \ldots, a_{n+1}),\end{aligned}$$

which proves $(N_{n+1})^* \leq (N^*)_{n+1}$. To prove the other inequality, let $(b_1, \ldots, b_n) \in \mathbb{F}^n$ of norm 1 satisfy

$$a = N_n^*(a_1, \ldots, a_n) = \sum_{i=1}^{n} a_i b_i.$$

Further, let $(b, b_{n+1}) \in \mathbb{F}^2$ of norm 1 be such that

$$N^*(a, a_{n+1}) = ab + a_{n+1}b_{n+1}.$$

The norm of $(bb_1, \ldots, bb_n, b_{n+1}) \in \mathbb{F}^{n+1}$ is

$$N(N_n(bb_1, \ldots, bb_n), b_{n+1}) = N(b, b_n) = 1$$

and

$$(N^*)_{n+1}(a_1, \ldots, a_{n+1}) = N^*(a, a_{n+1}) = ab + a_{n+1}b_{n+1} = \sum_{i=1}^{n} a_i bb_i + a_{n+1}b_{n+1},$$

which shows $(N^*)_{n+1} \leq (N_{n+1})^*$. □

Proposition 3.43. *The dual of Y_N is isometrically isomorphic to X_{N^*}. The duality is given by*

$$\langle x^*, x\rangle = \sum_{i=1}^{\infty} a_i b_i$$

for $x^* = (a_i) \in X_{N^*}$ and $x = (b_i) \in Y_N$.

Proof. First let us take $x^* \in (Y_N)^*$ and let $a_i = x^*(e_i)$. If $x = (b_i) \in Y_N$, then

$$x^*(x) = x^*\left(\sum_{i=1}^{\infty} b_i e_i\right) = \lim_{n\to\infty} x^*\left(\sum_{i=1}^{n} b_i e_i\right) = \sum_{i=1}^{\infty} a_i b_i.$$

For every n we have

$$N_n^*(a_1, \ldots, a_n) = \sup\left\{\left|\sum_{i=1}^{n} a_i b_i\right|;\ N_n(b_1, \ldots, b_n) \leq 1\right\} \leq \|x^*\|,$$

hence $\|(a_i)\|_{N^*} \leq \|x^*\|$. Conversely, let $(a_i) \in X_{N^*}$ and define

$$x^*(x) = \sum_{i=1}^{\infty} a_i b_i$$

for $x = (b_i) \in Y_N$. For any n we have

$$\left|\sum_{i=1}^{n} a_i b_i\right| \leq \sum_{i=1}^{n} |a_i||b_i| \leq N_n^*(a_1, \ldots, a_n)N_n(b_1, \ldots, b_n),$$

where we used that the norms N_n and N_n^* are absolute. It follows that $\|x^*\| \leq \|(a_i)\|_{N^*}$, which concludes the proof. □

3.2.6 Boundedly-complete bases

Our next aim is to characterize the situation in which the canonical basis (e_n) of the space Y_N is boundedly-complete. Boundedly-complete bases of Orlicz sequence spaces have already been described in Section 3.1.3 which allows us to exploit the correspondence between these spaces and spaces with iterative norms.

If an Orlicz function Φ is non-degenerate, then the canonical basis (e_n) of an Orlicz sequence space h_Φ is boundedly-complete if and only if Φ satisfies the Δ_2-condition at zero given in Definition 3.13. Since the condition depends only on the behavior of Φ on some neighborhood of zero, we propose to extend the definition to the function φ associated with N.

There are two cases to be considered. The first possibility is that $\varphi(b) = 0$ for some $b \in (0, 1]$, in which case we will say that φ is *degenerate*. By Theorem 3.35, (e_n) is equivalent to the canonical basis of c_0 and is therefore not boundedly-complete.

The other possibility is that $\varphi(b) > 0$ on the interval $(0, 1]$. The third part of Proposition 3.24 asserts that this is equivalent to $N(1, b) = 1 \Rightarrow b = 0$. We will say that φ satisfies the Δ_2-condition at zero if

$$\limsup_{b\to 0+} \frac{\varphi(2b)}{\varphi(b)} < \infty.$$

Similarly, φ satisfies the Δ_Q-condition at zero if

$$\limsup_{b\to 0+} \frac{\varphi(Qb)}{\varphi(b)} < \infty.$$

Next we find an equivalent formulation of the Δ_2-condition at zero for φ in terms of the norm N.

Definition 3.44. Let N be a norm on $\mathbb{F}^2$ which is absolute and normalized and let $Q > 1$. We say that N satisfies the *C_Q-condition* if there exists a positive constant K such that

$$N(a, Qb) - a \leq K(N(a, b) - a) \tag{3.26}$$

for all $a, b \geq 0$.

Lemma 3.45. *Let $Q_1, Q_2 > 1$. The norm N satisfies the C_{Q_1}-condition if and only if it satisfies the C_{Q_2}-condition.*

Proof. Since $Q_1 > 1$, we have $Q_2 \leq Q_1^r$ for some $r \in \mathbb{N}$ large enough. Applying the C_{Q_1}-condition r times yields

$$N(a, Q_2 b) - a \leq N(a, Q_1^r b) - a \leq K^r(N(a, b) - a),$$

which proves the C_{Q_2}-condition with the constant K^r in place of K. □

Proposition 3.46. *Let N be a norm on $\mathbb{F}^2$ which is absolute and normalized and φ be the function associated with N. Let us assume that φ is non-degenerate. Then the following conditions are equivalent.*

(i) The norm N satisfies the C_2-condition.

(ii) The function φ satisfies the Δ_2-condition at zero.

(iii) There exist constants $K \geq 1$ and $C > 0$ such that

$$N(1, 2b) - 1 \leq K(N(1, b) - 1)$$

for all $b \in [0, C]$.

Proof. (i) $\Rightarrow$ (ii). Fix $Q > 2$ and $b \in [0, \frac{1}{Q})$. If N satisfies the C_2-condition, then it also satisfies the C_Q-condition by Lemma 3.45. Let $\lambda_1 = N(1, b)$ and $\lambda_2 = N(1, Qb)$. Since $Qb < 1$, we have $1 \leq \lambda_1 \leq \lambda_2 \leq 2$. By Proposition 3.24,

$$N\left(\frac{1}{\lambda_2}, \frac{Qb}{\lambda_2}\right) = 1 \Rightarrow \varphi\left(\frac{Qb}{\lambda_2}\right) = 1 - \frac{1}{\lambda_2}, \tag{3.27}$$

where we used $\frac{Qb}{\lambda_2} \leq Qb < 1$. Analogously,

$$N\left(\frac{1}{\lambda_1}, \frac{b}{\lambda_1}\right) = 1 \Rightarrow \varphi\left(\frac{b}{\lambda_1}\right) = 1 - \frac{1}{\lambda_1}. \tag{3.28}$$

Inequality (3.26) for $a = 1$ becomes

$$\lambda_2 - 1 \leq K(\lambda_1 - 1).$$

Expressing both sides from (3.27) and (3.28) we get

$$\lambda_2 \varphi\left(\frac{Qb}{\lambda_2}\right) \leq K\lambda_1 \varphi\left(\frac{b}{\lambda_1}\right).$$

From the last inequality and the bounds for λ_i,

$$\varphi\left(\frac{Q}{2}b\right) \leq \lambda_2 \varphi\left(\frac{Qb}{\lambda_2}\right) \leq K\lambda_1 \varphi\left(\frac{b}{\lambda_1}\right) \leq 2K\varphi(b),$$

which proves φ satisfies $\Delta_{Q/2}$-condition with constant $2K$.

(ii) $\Rightarrow$ (iii). If φ satisfies the Δ_2-condition at zero, then there exists a constant $K \geq 1$ such that

$$\varphi(2b) \leq K\varphi(b) \tag{3.29}$$

for b in some interval $[0, b_0]$. We can assume that $b_0 < \frac{1}{2}$ so that $2b < 1$ for $b \in [0, b_0]$. If we let $\lambda_1 = N(1, b)$ and $\lambda_2 = N(1, 2b)$, then $1 \leq \lambda_1 \leq \lambda_2 \leq 2$. We have

$$N\left(\frac{1}{\lambda_1}, \frac{b}{\lambda_1}\right) = 1 \quad \text{and} \quad N\left(\frac{1}{\lambda_2}, \frac{2b}{\lambda_2}\right) = 1,$$

hence

$$\varphi\left(\frac{b}{\lambda_1}\right) = 1 - \frac{1}{\lambda_1} \quad \text{and} \quad \varphi\left(\frac{2b}{\lambda_2}\right) = 1 - \frac{1}{\lambda_2} \tag{3.30}$$

analogously as before. Using first the convexity of φ and then (3.29) we get

$$\lambda_2 \varphi\left(\frac{2b}{\lambda_2}\right) \leq \lambda_1 \varphi\left(\frac{2b}{\lambda_1}\right) \leq K\lambda_1 \varphi\left(\frac{b}{\lambda_1}\right).$$

Now we express the two sides of this inequality from (3.30) to obtain

$$N(1,2b)-1=\lambda_2-1=\lambda_2\varphi\left(\frac{2b}{\lambda_2}\right)\leq K\lambda_1\varphi\left(\frac{b}{\lambda_1}\right)=K(\lambda_1-1)=K(N(1,b)-1),$$

which proves (iii).

(iii) $\Rightarrow$ (i): It suffices to show there exists a constant $K \geq 1$ such that

$$N(1,2b)-1 \leq K(N(1,b)-1) \tag{3.31}$$

for all $b \geq 0$. We know from (iii) that there exist constants $K_1 \geq 1$ and $C > 0$ such that

$$N(1,2b)-1 \leq K_1(N(1,b)-1) \tag{3.32}$$

for $b \in [0, C]$. Let $K_3 > 2$ be fixed and set $C_2 = \frac{K_3}{K_3-2}$. Then $b \geq C_2$ if and only if $2b \leq K_3(b-1)$. Consequently,

$$N(1,2b)-1 \leq 1+2b-1 = 2b \leq K_3(b-1) \leq K_3(N(1,b)-1) \tag{3.33}$$

for $b \in [C_2, \infty)$. Now we use the assumption that φ is non-degenerate to obtain $N(1, C_1) > 1$ from Proposition 3.24 (iii). Hence there exists a constant $K_2 \geq 1$ such that

$$N(1,2b)-1 \leq N(1,2C_2)-1 \leq K_2(N(1,C_1)-1) \leq K_2(N(1,b)-1) \tag{3.34}$$

for $b \in [C_1, C_2]$. Putting (3.32), (3.33) and (3.34) together we get (3.31) with $K = \max(K_1, K_2, K_3)$. □

Theorem 3.47. *Let N be a norm on $\mathbb{F}^2$ which is absolute and normalized. Assume that the function φ associated with N is non-degenerate. The canonical basis (e_n) of Y_N is boundedly-complete if and only if φ satisfies the Δ_2-condition at zero.*

Proof. By Lemma 3.39, there exists a norm $N^2 \sim N$ and an Orlicz function Φ_2 such that the function φ_2 associated to N^2 satisfies $\varphi_2(1) = 1$ and $\varphi_2 = \Phi_2|_{[0,1]}$. Proposition 3.38 states that $\mathrm{Id} : X_{N^2} \to \ell_{\Phi_2}$ is an isomorphism.

By Theorem 3.35, the bases of Y_N and Y_{N^2} are equivalent. Therefore the bases of Y_N and h_{Φ_2} are equivalent. The basis of h_{Φ_2} is boundedly-complete if and only if Φ_2, and hence φ_2, satisfies the Δ_2-condition at zero.

Theorem 3.35 (iv) essentially states that φ and φ_2 are equivalent at zero (except that equivalence at zero was defined only for Orlicz functions). It follows that the Δ_2-condition at zero holds for φ if and only if it holds for φ_2, which concludes the proof. □

The last theorem also indirectly characterizes the situation in which the basis (e_n) of Y_N is shrinking. It follows from Proposition 3.43 that the sequence of dual functionals associated to (e_n) is the canonical basis (e_n^*) of Y_{N^*}. By Proposition 1.25, (e_n) is shrinking if and only if (e_n^*) is boundedly-complete.

Corollary 3.48. *Let ψ denote the function associated with N^*. Then the canonical basis (e_n) of Y_N is shrinking if and only if ψ is non-degenerate and satisfies the Δ_2-condition at zero.*

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