FACULTY
OF MATHEMATICS
AND PHYSICS
Charles University

MASTER THESIS

Ester Sgallová

Periodicity of Jacobi–Perron algorithm

Department of algebra

Supervisor of the master thesis: Mgr. Vítězslav Kala, Ph.D.
Study programme: Mathematics
Study branch: Mathematical Structures

Prague 2021

I declare that I carried out this master thesis independently, and only with the cited sources, literature and other professional sources. It has not been used to obtain another or the same degree.

I understand that my work relates to the rights and obligations under the Act No. 121/2000 Sb., the Copyright Act, as amended, in particular the fact that the Charles University has the right to conclude a license agreement on the use of this work as a school work pursuant to Section 60 subsection 1 of the Copyright Act.

In ..... date ....

Author's signature

I would like to thank my supervisor, Mgr. Vítězslav Kala, Ph.D., and my coadvisor Ing. Magdaléna Tinková for the great patience with which they led this work and the amount of time they devoted to me.

Title: Periodicity of Jacobi–Perron algorithm

Author: Ester Sgallová

Department: Department of algebra

Supervisor: Mgr. Vítězslav Kala, Ph.D., Department of Algebra
Coadvisor: Ing. Magdaléna Tinková, Department of Algebra

Abstract: This thesis aims to study a connection between indecomposable elements in the cubic fields and the Jacobi-Perron algorithm (JPA). JPA is a multidimensional generalization of the usual continued fractions algorithm. We work in the family of Ennola's cubic fields and we examine how the indecomposable elements are related to elements originating from this algorithm and whether some of these elements generate all indecomposable elements in the fields. We formulate conjectures on how to determine which elements will generate the indecomposable elements. We also prove some necessary conditions that have to hold for elements originating from this algorithm to generate indecomposable elements.

Keywords: continued fractions, Jacobi-Perron algorithm, indecomposable elements, cubic fields

Contents

Introduction

The topic of this work is motivated by the study of quadratic forms. A quadratic form over totally real number fields is a sum $\sum_{i,j=1}^{n} a_{ij}x_ix_j$ where a_{ij} lie in the ring of algebraic integers. A quadratic form is universal if it represents all totally positive integers. For example, a famous universal quadratic form appears in Lagrange's four-squares theorem which states that every positive integer in $\mathbb{Z}$ is a sum of four squares.

My supervisor Vítězslav Kala has been working on extending the studies of universal forms in the totally real number fields. In [2], [10], he with Valentin Blomer proved that for every positive integer M there are infinitely many real quadratic fields $K = \mathbb{Q}(\sqrt{D})$ such that every classical universal quadratic form over K has at least M variables. In the proof of this theorem, they used the connection between so-called indecomposable elements in $\mathbb{Q}(\sqrt{D})$ (see Chapter 2) and the continued fractions, which was proved by Andreas Dress and Rudolf Scharlau in [3] and by Oskar Perron in [17].

A natural question is if this connection between indecomposable elements and continued fractions could be generalized into fields of higher degrees. In this thesis, we will examine this in cubic fields. However, in these fields, the continued fractions do not work well. Instead, one can study the Jacobi–Perron algorithm (JPA), which is a generalization of the continued fractions algorithm.

Jacobi–Perron algorithm was studied by many researchers, for instance [1], [13], [16], [21], [20]. The important question is under which conditions our JPA expansions are periodic. That is still open, but some partial results are known. Claude Levesque and Georges Rhin [13] and Paul Voutier [21] proved that period could have an arbitrary length. In [16], Roger Paysant-Le Roux and Eugene Dubois proved that for any real number field of order $n+1$ there exists a n-tuple with a purely periodic JPA expansion.

In this thesis, we will study a connection between indecomposable elements and elements originating from the algorithms on elements in the family of Ennola's cubic fields [7]. Concretely, we work with the family of fields $\mathbb{Q}(\rho)$, where ρ is one of the roots of a polynomial $x^3-(a-1)x^2-ax-1$ where $a \in \mathbb{N}, a \geq 5$. We try to figure out how the indecomposable elements are related to elements originating from this algorithm, whether some of these elements generate all indecomposable elements in the field, and how to determine which of these elements will generate indecomposable elements.

In Chapter 1, we give definitions of the JPA and of the homogeneous form of this algorithm (hJPA). Moreover, we show a relationship between these two algorithms. In Chapter 2, there are some facts about number fields and we introduce here the family of Ennola's cubic fields. In Chapter 3, we state some basic properties of JPA expansion (some of them without proof) and then using them to present the proof from [16] that for any real number field there exists a basis for which we have purely periodic JPA expansion.

In Chapters 4 and 5 we compute the JPA expansions of $(|\rho|, \rho^2)$ and the hJPA expansion of $(1, |\rho|, \rho^2)$. Here we will prove that these expansions are periodic for every root ρ.

In Chapter 6, we compare the indecomposable elements in the field $\mathbb{Q}(\rho)$ with

the elements originating from the algorithms from the previous chapter. We will determine which iterations in the algorithm generate the indecomposable elements and we will find some rules for this. Finally, we formulate several Conjectures 1, 2, 3 that give sufficient conditions for generating indecomposable elements by the iterations of hJPA expansion. At the same time, we prove some necessary conditions that have to hold for an iteration in the algorithm to generate these elements in Theorems 19, 20, 21.

The author's contribution in this thesis consists mainly of Chapters 4, 5, and 6, where the author computed the periodic JPA expansions and determined how to use them for obtaining indecomposable elements. These are original results and will be published in a paper with coauthors V. Kala and M. Tinková.

1. Definition of Jacobi–Perron Algorithms

This chapter focuses on the basic theory of the Jacobi–Perron Algorithm. In the first section, we introduce the definition of the Jacobi–Perron Algorithm (JPA). The second section describes the definition of the homogenous Jacobi–Perron Algorithm (hJPA) and the relationship between JPA and hJPA.

1.1 Jacobi–Perron Algorithm

The Jacobi–Perron algorithm [21] is a generalization of the continued fraction algorithm to higher dimensions. For fixed $n \in \mathbb{N}$ and $\alpha \in \mathbb{R}^{n-1}$ we define the sequence $\langle \alpha^{(\nu)} \rangle_{\nu \geq 0}$ of complete coefficients

$$\alpha^{(\nu)} = (\alpha_1^{(\nu)}, \alpha_2^{(\nu)}, \ldots, \alpha_{n-1}^{(\nu)})$$

where $\alpha_i^{(0)} = \alpha$ and sequences of incomplete coefficients

$$a^{(\nu)} = (a_1^{(\nu)}, a_2^{(\nu)}, \ldots, a_{n-1}^{(\nu)}) = (\lfloor \alpha_1^{(\nu)} \rfloor, \lfloor \alpha_2^{(\nu)} \rfloor, \ldots, \lfloor \alpha_{n-1}^{(\nu)} \rfloor),$$

where for all indices ν holds

$$\alpha_1^{(\nu)} = a_1^{(\nu)} + \frac{1}{\alpha_{n-1}^{(\nu+1)}}, \qquad \alpha_n^{(\nu+1)} > 1,$$

$$\alpha_i^{(\nu)} = a_i^{(\nu)} + \frac{\alpha_{i-1}^{(\nu+1)}}{\alpha_{n-1}^{(\nu+1)}}, \qquad (2 \leq i \leq n-1), \qquad 0 \leq \alpha_{i-1}^{(\nu+1)} \leq \alpha_{n-1}^{(\nu)}.$$

Then the JPA expansion of α is the sequence of the complete coefficients. We can verify that this is equivalent to the following definition, from which we can more easily see how the elements of the algorithm look like.

Definition 1. *Let $\alpha^{(0)} = (\alpha_1^{(0)}, \alpha_2^{(0)}, \ldots, \alpha_{n-1}^{(0)}) \in \mathbb{R}^{n-1}$ for $n \geq 2$. The Jacobi–Perron Algorithm (JPA) expansion of $\alpha^{(0)}$ is the sequence of elements of $\mathbb{R}^{n-1}$, $\langle \alpha^{(\nu)} \rangle_{\nu \geq 0}$, defined by*

$$\alpha^{(\nu+1)} = (\alpha_1^{(\nu+1)}, \ldots, \alpha_{n-2}^{(\nu+1)}, \alpha_{n-1}^{(\nu+1)}) = \left(\frac{\alpha_2^{(\nu)} - a_2^{(\nu)}}{\alpha_1^{(\nu)} - a_1^{(\nu)}}, \ldots, \frac{\alpha_{n-1}^{(\nu)} - a_{n-1}^{(\nu)}}{\alpha_1^{(\nu)} - a_1^{(\nu)}}, \frac{1}{\alpha_1^{(\nu)} - a_1^{(\nu)}} \right),$$

where $\alpha_1^{(\nu)} \neq a_1^{(\nu)}$ and $a_i^{(j)} = \lfloor \alpha_i^{(j)} \rfloor$. Here $\lfloor \cdot \rfloor$ denotes the floor function, i.e. for $x \in \mathbb{R}$ we have $\lfloor x \rfloor = \max\{m \in \mathbb{Z} \mid m \leq x\}$.

We can easily see that for $n = 2$ is this expansion exactly the continued fraction expansion for the real number $\alpha \in \mathbb{R}$.

The JPA expansion of $\alpha^{(0)}$ is *periodic* if there exist two integers l_0, l_1, with $l_0 \geq 0$ and $l_1 \geq 1$ such that

$$\alpha^{(\nu+l_1)} = \alpha^{(\nu)}$$

for every $\nu \geq l_0$. If l_0 and l_1 are the smallest integers satisfying these conditions, then $\alpha^{(0)}, \alpha^{(1)}, \ldots, \alpha^{(l_0-1)}$ and $\alpha^{(l_0)}, \alpha^{(l_0+1)}, \ldots, \alpha^{(l_0+l_1-1)}$ are called the *preperiod* and the *period* of the periodic JPA expansion, and l_0, l_1 are their *lengths*. If $l_0 = 0$, then the JPA expansion of $\alpha^{(0)}$ is *purely periodic*.

For the periodic JPA expansion of algebraic integers $\alpha_i^{(0)}$ the element

$$\epsilon = \prod_{i=l_0}^{l_0+l_1-1} \alpha_{n-1}^{(i)}$$

is a unit in the ring of algebraic integers of $\mathbb{Q}\left(\alpha_1^{(0)}, \alpha_2^{(0)}, \ldots, \alpha_{n-1}^{(0)}\right)$, named Hasse–Bernstein unit. This was proved by Hasse and Bernstein in [9].

1.2 The homogenous Jacobi–Perron Algorithm

For us it is more convenient to use the homogenous Jacobi–Perron Algorithm, which is defined in the following way:

Definition 2. *Let $\beta^{(0)} = (\beta_1^{(0)}, \beta_2^{(0)}, \ldots, \beta_n^{(0)}) \in \mathbb{R}^n$ for $n \geq 2$. The hJPA expansion of $\beta^{(0)}$ is the sequence of elements of $\mathbb{R}^n$, $\langle \beta^{(\nu)} \rangle_{\nu \geq 0}$, defined by*

$$\beta^{(i+1)} = \left(\beta_2^{(i)} - \left\lfloor \frac{\beta_2^{(i)}}{\beta_1^{(i)}} \right\rfloor \beta_1^{(i)}, \beta_3^{(i)} - \left\lfloor \frac{\beta_3^{(i)}}{\beta_1^{(i)}} \right\rfloor \beta_1^{(i)}, \ldots, \beta_n^{(i)} - \left\lfloor \frac{\beta_n^{(i)}}{\beta_1^{(i)}} \right\rfloor \beta_1^{(i)}, \beta_1^{(i)}\right).$$

For this expansion, unlike JPA expansion, if the starting elements are algebraic integers, then the elements in the expansion are always algebraic integers. We call the elements $\beta_i^{(j)}$ where $i, j \in \mathbb{N}$ the *convergents of the expansion* or simply *convergents*. These convergences sometimes generate so-called indecomposable elements in some field, this will be shown in Chapter 6. The following lemma gives us the relationship between JPA and hJPA expansion of a similar vector. In the latter chapter, we will calculate the hJPA expansion for some specific vectors. However in the JPA expansions, we have easily seen when is this expansion periodic, so first, we will calculate the JPA expansion, and then we use the following relationship.

Lemma 1. *Let $\alpha^{(i)}$ be the JPA expansion of $(\theta_1, \theta_2, \ldots, \theta_{n-1})$. Then the hJPA expansion of $\beta^{(0)} = (1, \theta_1, \theta_2, \ldots, \theta_{n-1})$ is equal to*

$$\beta^{(i)} = (\delta_i, \alpha_1^{(i)}\delta_i, \ldots, \alpha_{n-2}^{(i)}\delta_i, \alpha_{n-1}^{(i)}\delta_i),$$

where $\delta_k = \frac{1}{\alpha_{n-1}^{(1)} \cdot \ldots \cdot \alpha_{n-1}^{(k)}}$ for $k \geq 0$, $\delta_0 = 1$.

Proof. We will prove this by induction on i.

For $i = 0$ we have

$$\beta_1^{(0)} = 1 = \delta_0,$$

$$\beta_n^{(0)} = \theta_{n-1} = \alpha_{n-1}^{(0)}\delta_0$$

and for $1 < j < n$ it is

$$\beta_j^{(0)} = \theta_{j-1} = \alpha_{j-1}^{(0)} = \alpha_{j-1}^{(0)}\delta_0.$$

Therefore

$$\beta^{(0)} = (\delta_0, \alpha_1^{(0)}\delta_0, \ldots, \alpha_{n-1}^{(0)}\delta_0).$$

Let us assume that the formula holds for i, we will prove it for $i+1$:

$$\begin{aligned}
\beta_1^{(i+1)} &= \beta_2^{(i)} - \left\lfloor \frac{\beta_2^{(i)}}{\beta_1^{(i)}} \right\rfloor \beta_1^{(i)} = \alpha_1^{(i)}\delta_i - \left\lfloor \frac{\alpha_1^{(i)}\delta_i}{\delta_i} \right\rfloor \delta_i = (\alpha_1^{(i)} - \lfloor \alpha_1^{(i)} \rfloor)\delta_i \\
&= \frac{1}{\alpha_{n-1}^{(i+1)}}\delta_i = \delta_{i+1}, \\
\beta_n^{(i+1)} &= \beta_1^{(i)} = \delta_i = \alpha_{n-1}^{(i+1)}\delta_{i+1},
\end{aligned}$$

and for $1 < j < n$ we get

$$\begin{aligned}
\beta_j^{(i+1)} &= \beta_{j+1}^{(i)} - \left\lfloor \frac{\beta_{j+1}^{(i)}}{\beta_1^{(i)}} \right\rfloor \beta_1^{(i)} = \alpha_j^{(i)}\delta_i - \left\lfloor \frac{\alpha_j^{(i)}\delta_i}{\delta_i} \right\rfloor \delta_i = (\alpha_j^{(i)} - \lfloor \alpha_j^{(i)} \rfloor)\delta_i \\
&= \frac{(\alpha_j^{(i)} - \lfloor \alpha_j^{(i)} \rfloor)}{(\alpha_1^{(i)} - \lfloor \alpha_1^{(i)} \rfloor)}(\alpha_1^{(i)} - \lfloor \alpha_1^{(i)} \rfloor)\delta_i = \alpha_{j-1}^{(i+1)}\frac{1}{\alpha_{n-1}^{(i+1)}}\delta_i = \alpha_{j-1}^{(i+1)}\delta_{i+1}.
\end{aligned}$$

Hence,

$$\beta^{(i+1)} = (\delta_{i+1}, \alpha_1^{(i+1)}\delta_{i+1}, \ldots, \alpha_{n-1}^{(i+1)}\delta_{i+1}, \alpha_{n-1}^{(i+1)}\delta_{i+1}).$$

□

The hJPA expansion of $\beta^{(0)}$ is *periodic*, if there exist two integers l_0, l_1, with $l_0 \geq 0$ and $l_1 \geq 1$ and a unit ϵ such that

$$\beta^{(l_0+i+jl_1)} = \beta_k^{(i)}\epsilon^j$$

for $i = 0, 1, \ldots, l_1 - 1$, $j \geq 0$. If l_0 and l_1 are the smallest integers satisfying these conditions, then $\beta^{(0)}, \beta^{(1)}, \ldots, \beta^{(l_0-1)}$ and $\beta^{(l_0)}, \beta^{(l_0+1)}, \ldots, \beta^{(l_0+l_1-1)}$ are called the *preperiod* and the *period* of the periodic hJPA expansion, and l_0, l_1 are their *lengths*. If $l_0 = 0$, then the hJPA expansion of $\beta^{(0)}$ is *purely periodic*. The following lemma give us the relationship between periodic JPA expansion and periodic hJPA expansion of the similar vectors.

Lemma 2. *When JPA expansion of $(\theta_1, \theta_2, \ldots, \theta_{n-1})$ is periodic with preperiod and period lenght l_0 and l_1, then the hJPA expansion of $(1, \theta_1, \theta_2, \ldots, \theta_{n-1})$ is also periodic with the same preperiod and period length and the unit ϵ is the inverse of Hasse-Bernstein unit.*

Proof. It follows from the previous lemma. It is easy to see that

$$\delta_{l_0+i+jl_1} = \frac{1}{\alpha_{n-1}^{(1)} \cdots \alpha_{n-1}^{(l_0)} \cdots \alpha_{n-1}^{(l_0+i)} \cdots \alpha_{n-1}^{(l_0+i+jl_1)}} = \delta_{l_0+i}\frac{1}{\alpha_{n-1}^{(l_0+i+1)} \cdots \alpha_{n-1}^{(l_0+i+jl_1)}}.$$

So hJPA expansion is periodic and ϵ is the inverse of Hasse-Bernstein unit. □

2. Number fields

In this chapter, we will recall some basic facts about algebraic number fields. Then, we will introduce a family of Ennola's cubic fields and the indecomposable elements in these fields.

The field K is a *number field* if it is a finite-dimensional extension of rational numbers $\mathbb{Q}$. And the number field is *totally real* if all embeddings of K into $\mathbb{C}$ are real embeddings. For the element $\alpha \in K$ the *norm* is $N_{K/\mathbb{Q}}(\alpha) = \prod_{\lambda_i} \lambda_i(\alpha)$, where λ_i runs over all embeddings K into $\mathbb{C}$. For α we define a *signature*, $Sgn(\alpha)$ as a vector of sings, where the i-th sign is $+$ when $\lambda_i(\alpha) \geq 0$ and $-$ when $\lambda_i(\alpha) < 0$. We say that the element α is *totally positive* if $\lambda(\alpha) > 0$ for all embedings λ, i.e. $Sgn(\alpha) = (+, +, \cdots, +)$. The element α is *indecomposable* if it is an algebraic integer and it cannot be expressed as $\alpha = \alpha_1 + \alpha_2$ where α_1 a α_2 are algebraic integer and have the same signature as the α. When α is indecomposable and $Sgn(\alpha) = \sigma$, we can say that α is σ-indecomposable. However, in this thesis we will call them simple indecomposable elements, because it will not cause any confusion.

2.1 Ennola's number fields

In this thesis we will work with the family of Ennola's cubic fields [7], [8], i.e. the fields generated by a root ψ of the polynomial $g(x) = x^3 + (b-1)x^2 - bx - 1$ where $b \in \mathbb{N}$. However, we will instead work with a root of the similar polynomial

$$f(x) = f_a(x) = x^3 - (a-1)x^2 - ax - 1$$

where $a \in \mathbb{N}, a \geq 5$, this roots generate the same fields. Let us denote by ρ_i for $i \in \{1, 2, 3\}$ the roots of polynomial f (in is easy to see that they are all real), where $\rho_1 < \rho_2 < \rho_3$. The relationship between roots of the two polynomials is when ρ is root of polynomial f then $-(a-1) - (a-1)\rho + \rho^2$ is a root of polynomial g. In the later chapters we will look at the convergents of the hJPA expansion of the triple $(1, |\rho_i|, \rho_i^2)$ and compare them with the indecomposable elements of the field $\mathbb{Q}(\rho_i)$.

For roots of the polynomial f, we can easily see that

$$-1 < -\frac{a-1}{a} < \rho_1 < -\frac{a-2}{a-1} < 0,$$

$$-1 < -\frac{1}{a-2} < \rho_2 < -\frac{1}{a-1} < 0,$$

$$a < a + \frac{a-1}{a^3} < \rho_3 < a + \frac{a^2-1}{a^4} < a + \frac{1}{a^2}.$$

The field $\mathbb{Q}(\rho_i)$ is a totally real number field of degree three and with basis $\{1, \rho_i, \rho_i^2\}$. For every $i, j \in \{1, 2, 3\}$ there is a $\mathbb{Q}$-isomorphisms $\lambda_{i,j}$ between $\mathbb{Q}(\rho_i)$ and $\mathbb{Q}(\rho_j)$ such that ρ_i is mapped on ρ_j. Actually, every embedding $\mathbb{Q}(\rho_i)$ into $\mathbb{R}$ (or $\mathbb{C}$) is one of these isomorphism, i.e. for element $q_{\rho_i} = b + c\rho_i + d\rho_i^2 \in \mathbb{Q}(\rho_i)$, the imagines of these embedings from $\mathbb{Q}(\rho_i)$ into $\mathbb{R}$ are

1. $\lambda_{i,1}(q_{\rho_i}) = b + c\rho_1 + d\rho_1^2$,

2. $\lambda_{i,2}(q_{\rho_i}) = b + c\rho_2 + d\rho_2^2$,

3. $\lambda_{i,3}(q_{\rho_i}) = b + c\rho_3 + d\rho_3^2$.

From this we get that

$$N(q_{\rho_i}) = \lambda_{i,1}(q_{\rho_i})\lambda_{i,2}(q_{\rho_i})\lambda_{i,3}(q_{\rho_i}).$$

From the previous paragraphs it follows that if $q_{\rho_i} \in \mathbb{Q}(\rho_i)$ is an indecomposable element, then also $\lambda_{i,j}(q_{\rho_i}) \in \mathbb{Q}(\rho_j)$ is an indecomposable element in $\mathbb{Q}(\rho_j)$. At the same time, $N(q_{\rho_i}) = N(\lambda_{i,j}(q_{\rho_i}))$ and $Sgn(q_{\rho_i}) = Sgn(\lambda_{i,j}(q_{\rho_i}))$ for every $i, j \in \{1, 2, 3\}$. From this, we can see that we only need to examine the indecomposable elements in one of these fields, let us denote this field by $\mathbb{Q}(\rho)$.

It was proved by Ennola [7] and by Thomas [18] that the fundamental units in the order $\mathbb{Z}(\rho)$ are λ and $\lambda - 1$. From the relationship between roots of polynomial f and g we get that the fundamental units in the order $\mathbb{Z}(\rho)$ are $-(a-1) - (a-1)\rho + \rho^2$ and $-a - (a-1)\rho + \rho^2$. It is easy to show that these units have the signature $(-, -, +)$:

$$\begin{aligned} -a \;-\; (a-1)\rho_1 + \rho_1^2 \;&<\; -(a-1) - (a-1)\rho_1 + \rho_1^2 \\ &<\; -(a-1) - (a-1)\frac{a-1}{a} + \frac{(a-1)^2}{a^2} \;=\; \frac{-a+1}{a^2} \;<\; 0, \end{aligned}$$

$$\begin{aligned} -a \;-\; (a-1)\rho_2 + \rho_2^2 \;&<\; -(a-1) - (a-1)\rho_2 + \rho_2^2 \\ &<\; -(a-1) - (a-1)\frac{1}{a-2} + \frac{1}{(a-2)^2} \;=\; \frac{-a^3 + 6a^2 - 11a + 8}{(a-2)^2} \;<\; 0 \end{aligned}$$

and

$$\begin{aligned} &-\; (a-1) - (a-1)\rho_3 + \rho_3^2 \;>\; -a - (a-1)\rho_3 + \rho_3^2 \\ &>\; -a - (a-1)(a + \frac{a^2-1}{a^4}) + a(a + \frac{a-1}{a^3})^2 \;=\; \frac{1 - 2a + a^3 - a^4 + a^5}{a^6} \;>\; 0. \end{aligned}$$

Hence, $Sgn(-(a-1) - (a-1)\rho + \rho^2) = Sgn(-a - (a-1)\rho + \rho^2) = (-, -, +)$. These units generate all units in this order, so all units have signatures $(+, +, +)$, $(-, -, -)$, $(-, -, +)$ or $(+, +, -)$.

The following theorem gives us all indecomposable elements in the order $\mathbb{Z}(\rho)$, which was proved by Kala and Tinková in [12] and in [11].

Theorem 3. *([12], Proposition 8.1) Let ρ be root of the polynomial $x^3 - (a-1)x^2 - ax - 1$, $a \in \mathbb{N}, a \geq 5$. Then all indecoposable elements up to multiplication by units in $\mathbb{Z}[\rho]$ are*

1. $\gamma_w = 1 - w + aw + (1 - w + aw)\rho - w\rho^2$ *where* $1 \leq w \leq a - 3$,

2. $\delta_{v,u} = 1 + v - u + au + (a - u + au)\rho - (u+1)\rho^2$ *where* $1 \leq v \leq a-3, 0 \leq u \leq v$,

3. $\zeta_z = z + 2 + (z+4)\rho + \rho^2$ *where* $0 \leq z \leq a - 4$.

Let us call these elements basic indecomposable elements.

One of the most important tools for comparing convergents and indecomposable elements is their signatures and norms. Indecomposable elements are described in the previous theorem up to multiplication by a unit, hence if we take an element from the field, it is not easy to see whether it is an indecomposable element. However, each indecomposable element has in the absolute value the same norm as some element from the previous theorem. Also, each indecomposable element has the same signature as some basic indecomposable element, up to multiplication by a unit. Therefore, to distinguish indecomposable elements in some hJPA expansion, it is necessary to know the norms and the signatures of the basic indecomposable elements.

For the computation of norms, we use the Vieta's formulas for polynomial f, i.e. $\rho_1\rho_2\rho_3 = 1$ and $\rho_1\rho_2 + \rho_1\rho_3 + \rho_2\rho_3 = -a$. Then with help by computer program we get that

1. $N(\gamma_w) = 1 - w^2 - w^3 + a(w + w^2)$,
2. $N(\delta_{v,u}) = 1 + 3v + 2v^2 + v^3 - 5u - 3vu - 3v^2u - 2u^2 + 2vu^2 - u^3 + (-v - v^2 + 4u + 5vu + v^2u - 2u^2 - vu^2)a + (-u - vu + u^2)a^2$,
3. $N(\zeta_z) = 25 + 20z + 7z^2 + z^3 - az + (-2 - z)a^2$.

For the computation of signatures we use the fact that $-\frac{a-1}{a} < \rho_1 < -\frac{a-2}{a-1}$, $-\frac{1}{a-2} < \rho_2 < -\frac{1}{a-1}$ and $a < a + \frac{a-1}{a^3} < \rho_3 < a + \frac{a^2-1}{a^4} < a + \frac{1}{a^2}$. We will show that $Sgn(\gamma_w) = (+, +, +)$. We have that:

$$\begin{aligned} 1 &- w + aw + (1 - w + aw)\rho_1 - w\rho_1^2 \\ &> 1 - w + aw + (1 - w + aw)\frac{-a+1}{a} - w\frac{(a-1)^2}{a^2} \\ &= \frac{a + 3wa - w}{a^2} > \frac{6}{a^2} > 0; \end{aligned}$$

in the inequality we used that $1 \leq w \leq a - 3$. For the second root we get

$$\begin{aligned} 1 &- w + aw + (1 - w + aw)\rho_2 - w\rho_2^2 \\ &> 1 - w + aw + (1 - w + aw)\frac{-1}{a-2} - w\frac{1}{(a-2)^2} \\ &= \frac{w(a^3 - 6a^2 + 11a - 7) + a^2 - 5a + 6}{(a-2)^2} > 0, \end{aligned}$$

in the inequality we used that $w > 0$, $a^3 - 6a^2 + 11a - 7 > 0$ and $a^2 - 5a + 6 > 0$ for $a \geq 5$. And for the last root we have

$$\begin{aligned} 1 &- w + aw + (1 - w + aw)\rho_3 - w\rho_3^2 \\ &> 1 - w + aw + (1 - w + aw)a - w(a + \frac{1}{a^2})^2 \\ &= 1 + a - w - \frac{2wa^3 + w}{a^4} > 1 + a - (a-3) - \frac{2(a-3)a^3 + (a-3)}{a^4} > 0. \end{aligned}$$

Therefore the signature of the element γ_w is $(+,+,+)$. For the $\delta_{v,u}$ we get that

$$\begin{aligned}
1 \;+\; & v-u+au+(a-u+au)\rho_1-(u+1)\rho_1^2 \\
< \; & 1+v-u+au+(a-u+au)\frac{-a+2}{a-1}-(u+1)\frac{(a-2)^2}{(a-1)^2} \\
= \; & \frac{-3-a^3+3a^2+(2a-3)u+(a^2-2a+1)v}{(a-1)^2} \\
< \; & \frac{-3-a^3+3a^2+(2a-3)v+(a^2-2a+1)v}{(a-1)^2} \\
< \; & \frac{-3-a^3+3a^2+(a^2-2)v}{(a-1)^2} \;<\; \frac{-3-a^3+3a^2+(a^2-2)(a-3)}{(a-1)^2} \\
= \; & \frac{-2a+3}{(a-1)^2} \;<\; 0.
\end{aligned}$$

Here, we used that $1 \leq v \leq a-3$ and $0 \leq u \leq v$. Then we have that

$$\begin{aligned}
1 \;+\; & v-u+au+(a-u+au)\rho_2-(u+1)\rho_2^2 \\
> \; & 1+v-u+au+(a-u+au)\frac{-1}{a-2}-(u+1)\frac{1}{(a-2)^2} \\
= \; & 3-2a+(a^3-6a^2+11a-7)u+(a^2-4a+4)v \\
> \; & 3-2a+(a^3-6a^2+11a-7)0+(a^2-4a+4)1 \\
= \; & a^2-6a+7 \;>\; 0.
\end{aligned}$$

For the last root, we get

$$\begin{aligned}
1 \;+\; & v-u+au+(a-u+au)\rho_3-(u+1)\rho_3^2 \\
> \; & 1+v-u+au+(a-u+au)(a+\frac{a-1}{a^3}\rho_3)-(u+1)(a+\frac{a^2-1}{a^4})^2 \\
= \; & -\frac{1-2a^2+a^4-2a^5+a^6+a^7-a^8}{a^8}+v-u\frac{1-2a^2+a^4-3a^5+2a^6+a^7+a^8}{a^8} \\
\geq \; & -\frac{1-2a^2+a^4-2a^5+a^6+a^7-a^8}{a^8}-v\frac{1-2a^2+a^4-3a^5+2a^6+a^7}{a^8} \\
\geq \; & -\frac{1-2a^2+a^4-2a^5+a^6+a^7-a^8}{a^8}-(a-3)\frac{1-2a^2+a^4-3a^5+2a^6+a^7}{a^8} \\
= \; & \frac{2-a-4a^2+2a^3+2a^4-8a^5+8a^6}{a^8} \;>\; 0
\end{aligned}$$

We used that $u \leq v \leq a-3$ Hence, we have that $Sgn(\delta_{v,u}) = (-,+,+)$. The elements ζ_z have the signature $(-,+,+)$, because

$$\begin{aligned} z &+ 2+(z+4)\rho_1+\rho_1^2 \\ &< z+2+(z+4)\frac{-a+2}{a-1}+\frac{(a-1)^2}{a^2} \\ &= \frac{-a^3+3a^2+3a-1+za^2}{a^3-a^2} < \frac{-a^3+3a^2+3a-1+(a-4)a^2}{a^3-a^2} \\ &= \frac{-a^2+3a-1}{a^3-a^2} < 0, \end{aligned}$$

$$\begin{aligned} z &+ 2+(z+4)\rho_2+\rho_2^2 \\ &> z+2+(z+4)\frac{-1}{a-2}+\frac{1}{(a-1)^2} \\ &= \frac{2a^3-12a^2+19a-10+(a^3+5a^2+7a-3)z}{(a-2)(a-1)^2} \\ &> \frac{2a^3-12a^2+19a-10+(a^3+5a^2+7a-3)0}{(a-2)(a-1)^2} \\ &= \frac{2a^3-12a^2+19a-10}{(a-2)(a-1)^2} > 0, \end{aligned}$$

and

$$\begin{aligned} z &+ 2+(z+4)\rho_3+\rho_3^2 \\ &> z+2+(z+4)a+a^2 \\ &= a^2+4a+2+(a+1)z > a^2+4a+2+(a+1)0 > 0. \end{aligned}$$

Therefore, we have $Sgn(\zeta_z)=(-,+,+)$.

We can see that the elements γ_w lie on a line. Moreover, it will be helpful for us to know the direction of this line, i.e. what is the difference between γ_{w+l} and γ_w. Because in Chapter 6 we will look at the convergents and discuss whether they generate the incomposable elements, we will try to find such elements which also generate this line. So, define $\tilde{\gamma}_l$ as

$$\tilde{\gamma}_l=\gamma_{w+l}-\gamma_w=l(a-1)-l\rho-l\rho^2,$$

and it is easy to prove that

$$N(\tilde{\gamma}_l)=-n^3, Sgn(\tilde{\gamma}_l)=(+,+,-) \qquad \text{if} \qquad l>0.$$

Similarly the elements ζ_z lie on a line and define $\tilde{\zeta_k}$ as

$$\tilde{\zeta_k}=\zeta_{z+k}-\zeta_k=k+k\rho$$

and their norm and signatures are

$$N(\tilde{\zeta_k})=k^3, Sgn(\tilde{\zeta_k})=(+,+,+) \qquad \text{if} \qquad k>0.$$

Elements $\delta_{v,u}$ form a triangle and the elements

$$\tilde{\delta}_{n,m}=\delta_{v+n,u+m}-\delta_{v,u}=-m+am+n+(-m+am)\rho-m\rho^2,$$

have the norm $-m^3+2m^2n-am^2n+(a-3)mn^2+n^3$. Signature of these elements depends on the values of n,m and below in Figure 2.1, we can see some of these elements $\tilde{\delta}_{n,m}$ and their norms and signatures.

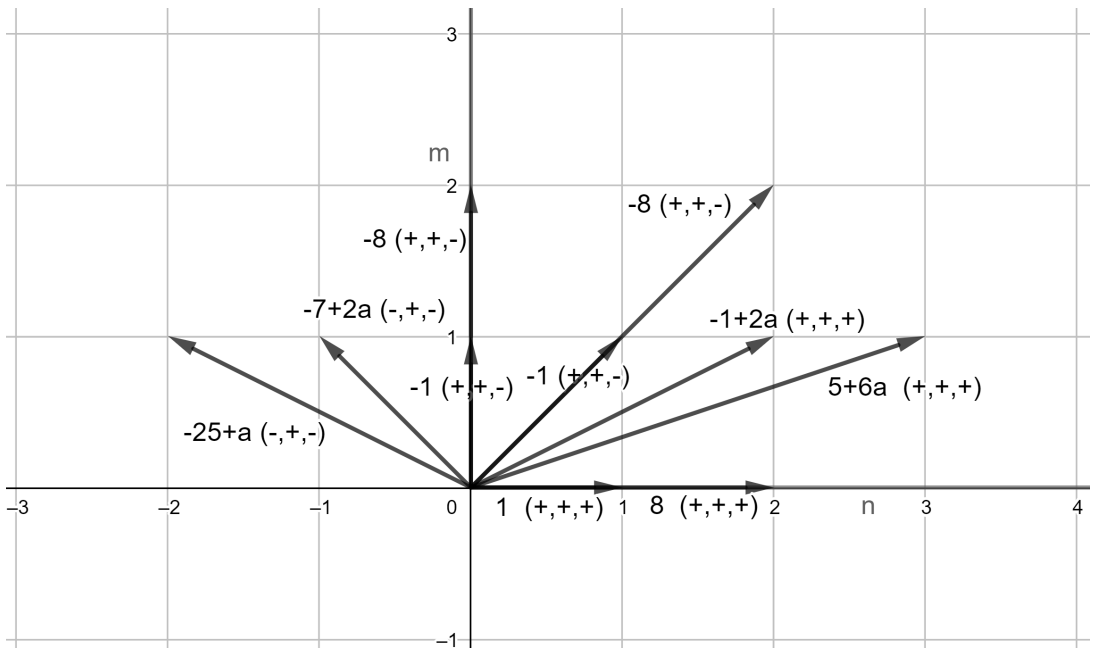

Figure 2.1: Some of elements $\tilde{\delta}_{n,m}$

3. Purely periodic JPA

In this chapter, we will show the proof from the [16] that for any real number field there exists a basis for which the JPA expansion of this basis is purely periodic. In the first section, we will show some properties of the purely periodic JPA expansion from [5], [14], [16], and [20]. In the second section, we introduce the Pisot numbers and some of their properties and in the third section, we prove the main theorem. In the last section, we show an example of purely periodic JPA expansion in one concrete field.

3.1 Properties of periodic JPA expansion

First, we need to define a set of matrices for a given JPA expansion. So, let us consider the JPA expansion of elements $\alpha^{(\nu)} = (\alpha_1^{(\nu)}, \dots, \alpha_n^{(\nu)})$ with incomplete coefficients $a^{(\nu)} = (a_1^{(\nu)}, \dots, a_n^{(\nu)})$ for some fixed n, then define $(n+1)$-component vectors

$$A^{(\nu)^T} = (A_0^{(\nu)}, A_1^{(\nu)}, \dots, A_n^{(\nu)}) \qquad \nu \in \mathbb{N},$$

where $A_i^j = 0$ if $i \neq j$, $A_i^{(j)} = 1$ if $i = j$ for $0 \leq i, j \leq n$ and

$$A_i^{(\nu+n+1)} = A_i^{(\nu)} + a_1^{(\nu)} A_i^{(\nu+1)} + \dots + a_n^{(\nu)} A_i^{(\nu+n)}.$$

Thus, for instance $(A^{n+1})^T = (1, a_1^{(0)}, \dots, a_n^{(0)})$. Then $M^{(\nu)}, B^{(\nu)}$ are $(n+1) \times (n+1)$ matrix such that

$$B^{(\nu)} = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 & a_1^{(\nu)} \\ 0 & 1 & \dots & 0 & a_2^{(\nu)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & a_n^{(\nu)} \end{pmatrix}$$

and

$$M^{(\nu)} = (A^{(\nu)} | A^{(\nu+1)} | \dots | A^{(\nu+n)}).$$

For matrices $M^{(\nu)}, B^{(\nu)}$ clearly hold that

$$M^{(\nu+1)} = M^{(\nu)} B^{(\nu)}.$$

We will prove that for purely periodic JPA expansion of the elements $\alpha_1, \dots, \alpha_n$ the vector $(1, \alpha_1, \dots, \alpha_n)$ is an eigenvector of the matrix $M^{(l)}$ where l is the period length of the expansion. This result follows from the following lemma, which was shown in [20].

Lemma 4. *([20], page 262) Let us have the matrices $B^{(i)}$, $M^{(i)}$ defined as above and vectors $v^{(i)} = (1, \alpha_1^{(i)}, \dots, \alpha_n^{(i)})^T$, where $i \in \mathbb{N}$. Then the following equalities hold*

$$B^{(k)}v^{(k+1)} = \alpha_n^{(k+1)}v^{(k)},$$
$$v^{(0)} = \prod_{j=1}^{k} \frac{1}{\alpha_n^{(j)}} M^{(k)}v^{(k)}.$$

Proof. We have that

$$\begin{aligned} B^{(k)} \cdot v^{(k+1)} &= \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 & a_1^{(k)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & a_n^{(k)} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ \alpha_1^{(k+1)} \\ \vdots \\ \alpha_n^{(k+1)} \end{pmatrix} = \begin{pmatrix} \alpha_n^{(k+1)} \\ 1 + a_1^{(k)}\alpha_n^{(k+1)} \\ \vdots \\ \alpha_{n-1}^{(k+1)} + a_n^{(k)}\alpha_n^{(k+1)} \end{pmatrix} \\ &= \alpha_n^{(k+1)} \cdot \begin{pmatrix} 1 \\ \alpha_1^{(k)} \\ \vdots \\ \alpha_{n-1}^{(k)} \end{pmatrix} = \alpha_n^{(k+1)}v^{(k)}. \end{aligned}$$

The third equality follows from the definition of the JPA expansion:

$$\begin{aligned} \alpha_1^{(k)} &= a_1^{(k)} + \frac{1}{\alpha_n^{(k+1)}}, \\ \alpha_i^{(k)} &= a_i^{(k)} + \frac{\alpha_{i-1}^{(k+1)}}{\alpha_n^{(k+1)}} \qquad (2 \le i \le n-1). \end{aligned}$$

This proved the first equation. For the second equation, we have that

$$\begin{aligned} \prod_{j=1}^{k} \frac{1}{\alpha_n^{(j)}} M^{(k)}v^{(k)} &= \prod_{j=1}^{k} \frac{1}{\alpha_n^{(j)}} M^{(k-1)}B^{(k-1)}v^{(k)} = \prod_{j=1}^{k} \frac{1}{\alpha_n^{(j)}} B^{(1)}B^{(2)}\dots B^{(k-1)}v^{(k)} \\ &= \prod_{j=1}^{k} \frac{1}{\alpha_n^{(j)}} B^{(1)}B^{(2)}\dots B^{(k-2)}\alpha_n^{(k)}v^{(k-1)} = \dots = v^{(0)}. \end{aligned}$$

□

When we consider this lemma for purely periodic JPA expansion, we get the following corollary.

Corollary 5. *([16], page 146) If $\alpha^{(\nu)}$ is purely periodic JPA with incomplete coefficients $a^{(\nu)}$ and period length l, then vector $v^{(0)}$ is an eigenvector of the matrix $M^{(l)}$*

Proof. This follows from the previous lemma and the fact that $v^{(l)} = v^{(0)}$. □

For this JPA expansion the polynomial $f(x) = (-1)^{n+1}det(M^{(l)} - xI)$, where I is identity matrix, is the *characteristic polynomial* of this expansion. Let us denote the matrix $M^{(l)} - xI$ by M and by ϱ_0 the largest root of this polynomial. From [5] (page 244) and [14] (page 101) we know that ϱ_0 is simple root and $\varrho_0 = \prod_{j=1}^{k} \alpha_n^{(j)}$.

From [14] we know that for the characteristic polynomial of the JPA expansion, the following theorem holds. We write here just the statement of this theorem.

Theorem 6. *([14], page 104) The characteristic polynomial of JPA of* $(\alpha_1, \cdots, \alpha_d)$ *is irreducible if and only if* $(1, \alpha_1, \cdots, \alpha_d)$ *are linearly independent over* $\mathbb{Q}$.

Now we show some properties of incomplete coefficients $a^{(\nu)} = (a_1^{(\nu)}, \ldots, a_n^{(\nu)})$ of some JPA expansion $\alpha^{(\nu)}$. We know that $a_i^{(\nu)} = \lfloor \alpha_i^{(\nu)} \rfloor$ for every i, ν and from this we can get the following inequalities.

Lemma 7. *([14], page 72) For a purely periodic JPA expansion* $\alpha^{(\nu)}$ *with incomplete coefficients* $a^{(\nu)}$ *and periodic length* l*, it holds that:*

$$a_n^{(\nu)} \geq 1, a_n^{(\nu)} \geq a_i^{(\nu)} \geq 0, \qquad (0 \leq i \leq n) \qquad (3.1)$$

$$(a_n^{(\nu)}, a_{n-1}^{(\nu+1)}, \ldots, a_{n-i}^{(\nu+i)}) \geq (a_i^{(\nu)}, a_{i-1}^{(\nu+1)}, \ldots, a_1^{(\nu+i-1)}, 1) \qquad (0 \leq i \leq n) \qquad (3.2)$$

where $\geq$ *denotes the lexicographic order, i.e. for some two vectors* $v = (v_1, \ldots, v_n)$ *and* $u = (u_1, \ldots, u_n)$ *it holds that* $v < u$ *if there exists some* $j \in \mathbb{N}$, $1 \leq j \leq n$ *such that* $v_j < u_j$ *and for all* $k \in \mathbb{N}$, $k < j$ *is* $v_k = u_k$.

Proof. This follows from the definition of JPA expansions. We know that for every ν it holds that

$$\alpha_1^{(\nu-1)} = a_1^{(\nu-1)} + \frac{1}{\alpha_n^{(\nu)}} \text{ and } a_1^{(\nu-1)} = \lfloor \alpha_1^{(\nu-1)} \rfloor.$$

Hence,

$$\frac{1}{\alpha_n^{(\nu)}} < 1$$

and

$$\alpha_n^{(\nu)} > 1.$$

From the fact that $a_n^{(\nu)} = \lfloor \alpha_n^{(\nu)} \rfloor$, we get that $a_n^{(\nu)} \geq 1$. Similarly, we know that

$$\alpha_i^{(\nu-1)} = a_i^{(\nu-1)} + \frac{\alpha_{i-1}^{(\nu)}}{\alpha_n^{(\nu)}} \text{ and } a_i^{(\nu-1)} = \lfloor \alpha_i^{(\nu-1)} \rfloor.$$

From this we get that

$$\frac{\alpha_{i-1}^{(\nu)}}{\alpha_n^{(\nu)}} < 1 \text{ and } \alpha_n^{(\nu)} > \alpha_{i-1}^{(\nu)}.$$

Thus

$$a_n^{(\nu)} \geq a_{i-1}^{(\nu)}$$

for all $2 \leq i \leq n$. This gives us inequality (3.1). For the inequality (3.2) we know that $a_n^{(\nu)} \geq a_i^{(\nu)}$. If $a_n^{(\nu)} > a_i^{(\nu)}$, we know that the inequality (3.2) holds, and if $a_n^{(\nu)} = a_i^{(\nu)}$ we can see that

$$a_i^{(\nu)} + \frac{\alpha_{i-1}^{(\nu+1)}}{\alpha_n^{(\nu+1)}} = \alpha_i^{(\nu)} < \alpha_n^{(\nu)} = a_n^{(\nu)} + \frac{\alpha_{n-1}^{(\nu+1)}}{\alpha_n^{(\nu+1)}}.$$

Hence, $\alpha_{i-1}^{(\nu+1)} < \alpha_{n-1}^{(\nu+1)}$ and $a_{i-1}^{(\nu+1)} \leq a_{n-1}^{(\nu+1)}$. We can proceed the same way and prove the inequality (3.2). $\square$

On the contrary, if we have incomplete coefficients $a^{(\nu)}$ satisfying the inequalities (3.1), (3.2), we can find $(1, \alpha_1, \dots, \alpha_n)$ such that the JPA expansion of these elements has incomplete coefficients $a^{(\nu)}$. This was showed in [4], we write here just a statement of this lemma.

Lemma 8. *([4], Section 2.) Let us fix constant $n \in \mathbb{N}$, and assume that some positive integers $a_i^{(\nu)}$ where $1 \le i \le n$ satisfy the inequalities (3.1), (3.2). Assume that the matrix M is defined as above. Define the elements*

$$\alpha_i = \frac{g_{j,i}(\varrho_0)}{g_{j,0}(\varrho_0)}, \qquad 1 \le i \le n$$

where $g_{j,i}(x)$ is the minor of matrix M without $(j+1)^{st}$ column and $(i+1)^{st}$ row and ϱ_0 is the largest root of the characteristic polynomial of matrix M. Then vector $(\alpha_1, \dots, \alpha_n)$ have purely periodic JPA expansion with incomplete coefficients $a^{(\nu)}$.

Hence, now we can find a matrix M for some purely periodic JPA expansion, and for some specific matrix M we can find elements $(\alpha_1, \dots, \alpha_n)$ such that this matrix is related to the JPA expansion of these elements. We will use this in the main theorem of this chapter.

In Theorem 9, we will derive the form of the characteristic polynomial of purely periodic JPA expansion with specific incomplete coefficients. This is Proposition II from the article [16]. However, the statement of the theorem is a little bit different, because the definition of the coefficients c_i in the article is not correct (see remark under the proof).

Theorem 9. *([16], Proposition II) Let $a_1, a_2, \dots, a_n$ are positive integers such that*

$$a_n > Max(a_1, a_2, \dots, a_{n-1}). \tag{3.3}$$

Then the purely periodic JPA with period length $n+1$ and incomplete coefficients

$$a^{(\nu)} = (0, \dots, 0, 1), \qquad 0 \le \nu \le n-1, \qquad a^{(n)} = (a_1, a_2, \dots, a_n) \tag{3.4}$$

has the characteristic polynomial

$$x^{n+1} - c_n x^n + c_{n-1} x^{n-1} - \dots + (-1)^n c_1 x + (-1)^{n+1} = 0, \tag{3.5}$$

where for $1 \le j \le n$ we have

$$c_n = \binom{n+1}{1} + \sum_{k=1}^{n} a_k, \qquad c_i = \binom{n+1}{i} + \sum_{k=1}^{i} \binom{n-k}{n-i} a_k. \tag{3.6}$$

Proof. From the assumptions we have that $A^0 = (1, 0, \dots, 0), \dots, A^n = (0, \dots, 0, 1)$. We compute that $A^{n+1} = (1, 0, \dots, 0, 1)$, $A^{n+2} = (1, 1, \dots, 0, 1)$ and $A^{2n+1} = (s_n, s_{n-1}, \dots, 1, 1 + s_n)$, where

$$s_j = \sum_{k=n-j+1}^{n} a_k, \qquad 1 \le j \le n.$$

Then the characteristic polynomial of this expansion is

$$\begin{aligned} f(x) &= (-1)^{n+1}\det\begin{pmatrix} 1-x & 1 & \dots & 1 & s_n \\ 0 & 1-x & \dots & 1 & s_{n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1-x & s_1 \\ 1 & 1 & \dots & 1 & 1+s_n-x \end{pmatrix} \\ &= (-1)^{n+1}\det\begin{pmatrix} 1-x & 0 & \dots & 0 & 1 \\ 1 & 1-x & \dots & 0 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \dots & 1-x & 1 \\ s_n & s_{n-1} & \dots & s_1 & 1+s_n-x \end{pmatrix}. \end{aligned}$$

Now, we subtract the first column of the matrix from the last column, and by the expansion along the last column, we get that

$$\begin{aligned} &\det\begin{pmatrix} 1-x & 0 & \dots & 0 & 1 \\ 1 & 1-x & \dots & 0 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \dots & 1-x & 1 \\ s_n & s_{n-1} & \dots & s_1 & 1+s_n-x \end{pmatrix} \\ = &\det\begin{pmatrix} 1-x & 0 & \dots & 0 & x \\ 1 & 1-x & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \dots & 1-x & 0 \\ s_n & s_{n-1} & \dots & s_1 & 1-x \end{pmatrix} \\ = &(1-x)^{n+1} + (-1)^{n+2}x\det\begin{pmatrix} 1 & 1-x & \dots & 0 & 0 \\ 1 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \dots & 1 & 1-x \\ s_n & s_{n-1} & \dots & s_2 & s_1 \end{pmatrix}. \end{aligned}$$

Let us denote

$$D = \begin{pmatrix} 1 & 1-x & \dots & 0 & 0 \\ 1 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \dots & 1 & 1-x \\ s_n & s_{n-1} & \dots & s_2 & s_1 \end{pmatrix}.$$

Then we have that

$$f(x) = (-1)^{n+1}((1-x)^{n+1} + (-1)^{n+2}x\det D) = (x-1)^{n+1} - x\det D.$$

We will compute the determinant of D. Firstly, we subtract the first column of the matrix from the every other column and then we multiplied every column,

except the first one, by -1. We get

$$\begin{aligned}
\det D &= \det \begin{pmatrix} 1 & 1-x & \dots & 0 & 0 \\ 1 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \dots & 1 & 1-x \\ s_n & s_{n-1} & \dots & s_2 & s_1 \end{pmatrix} \\
&= (-1)^{n-1} \det \begin{pmatrix} 1 & x & \dots & 1 & 1 \\ 1 & 0 & \dots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & \dots & 0 & x \\ s_n & s_n - s_{n-1} & \dots & s_n - s_2 & s_n - s_1 \end{pmatrix}.
\end{aligned}$$

Let us denote

$$\sigma_j = s_n - s_{n-j} = \sum_{k=1}^{n} a_k - \sum_{k=j+1}^{n} a_k = \sum_{k=1}^{j} a_k, \qquad 1 \le j \le n-1, \qquad \sigma_n = s_n.$$

Now, we compute the determinant of D by expansion along the last row. Here A_j denotes the matrix

$$\begin{pmatrix} 1 & x & \dots & 1 & 1 \\ 1 & 0 & \dots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & \dots & 0 & x \\ s_n & s_n - s_{n-1} & \dots & s_n - s_2 & s_n - s_1 \end{pmatrix} = \begin{pmatrix} 1 & x & \dots & 1 & 1 \\ 1 & 0 & \dots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & \dots & 0 & x \\ \sigma_n & \sigma_1 & \dots & \sigma_{n-2} & \sigma_{n-1} \end{pmatrix}$$

without last row and the jth column for $2 \le j \le n$. Then by the expansion along the last row we get:

$$\det D = \sigma_n x^{n-1} + \sum_{j=2}^{n} (-1)^{j-1} \sigma_{j-1} \det A_j.$$

We compute determinant A_j as follows

$$\begin{aligned}
\det A_j &= \det \begin{pmatrix} 1 & x & \dots & 1 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \dots & x & 1 & \dots & 1 \\ 1 & 0 & \dots & 0 & 1 & \dots & 1 \\ 1 & 0 & \dots & 0 & x & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \dots & 0 & 0 & \dots & x \end{pmatrix} \\
&= \det \begin{pmatrix} 1 & x & \dots & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \dots & x & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 & x-1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \dots & 0 & -1 & \dots & x-1 \end{pmatrix} = (-1)^{j-1} x^{j-2} (x-1)^{n-j}.
\end{aligned}$$

Hence,

$$\begin{aligned}
f(x) &= (-1)^{n+1}((1-x)^{n+1} + (-1)^{n+2}x\,\det D) \;=\; (x-1)^{n+1} - x\,\det D \\
&= (x-1)^{n+1} - x\left(\sigma_n x^{n-1} + \sum_{j=2}^{n}(-1)^{j-1}\sigma_{j-1}\det A_j\right) \\
&= (x-1)^{n+1} - x\left(\sigma_n x^{n-1} + \sum_{j=2}^{n}\sigma_{j-1}x^{j-1}(x-2)^{n-j}\right) \\
&= \sum_{i=0}^{n+1}\binom{n+1}{i}(-1)^{n+1-i}x^i - \sigma_n x^n + \sum_{j=1}^{n-1}\sigma_j x^j \sum_{i=0}^{n-1-j}\binom{n-1-j}{i}(-1)^{n-1-j-i}x^i \\
&= \sum_{i=0}^{n+1}\binom{n+1}{i}(-1)^{n+1-i}x^i - \sigma_n x^n + \sum_{j=1}^{n-1}\sigma_j \sum_{i=j}^{n-1}\binom{n-1-j}{i-j}(-1)^{n-1-i}x^i \\
&= x^{n+1} - c_n x^n + c_{n-1}x^{n-1} - \cdots + (-1)^n c_1 x + (-1)^{n+1}.
\end{aligned}$$

where

$$c_n \;=\; \binom{n+1}{n} + \sigma_n \;=\; \binom{n+1}{n} + \sum_{k=1}^{n} a_k,$$

$$\begin{aligned}
c_i &= \binom{n+1}{i} + \sum_{j=1}^{i}\sigma_j\binom{n-1-j}{i-j} \;=\; \binom{n+1}{i} + \sum_{j=1}^{i}\sum_{k=1}^{j}a_k\binom{n-1-j}{i-j} \\
&= \binom{n+1}{i} + \sum_{k=1}^{i}a_k\sum_{j=k}^{i}\binom{n-1-j}{i-j} \;=\; \binom{n+1}{i} + \sum_{k=1}^{i}a_k\binom{n-k}{n-i}.
\end{aligned}$$

Hence, the theorem was proved. □

Remark. In the article [16] we have the coefficients $c_i = \binom{n+1}{i} + \sum_{k=1}^{i}\binom{n-1-k}{n-1-i}a_k$ but this cannot work. For instance, when we take $n = 3$ then we have

$$c_2 = 6 + a_1 + a_2,$$

but the characteristic polynomial of the expansion is

$$x^4 - (4 + a_1 + a_2 + a_3)x^3 + (6 + 2a_1 + a_2)x^2 - (4 + a_1)x + 1.$$

Hence, c_2 is not correct. In this thesis we proved that $c_i = \binom{n+1}{i} + \sum_{k=1}^{i}\binom{n-k}{n-i}a_k$.

3.2 Pisot numbers

For the rest of the proof, we will need some properties of the *Pisot numbers*, which are a positive algebraic integer greater than 1 whose all conjugates have absolute value less than 1. All lemmas and theorem in this section are stated without proof.

Theorem 10. *([16], Théorème III.) Let q_0 be a Pisot number, $q_1, \ldots, q_n$ be its conjugates. If there exist $k \in \mathbb{Z}$, $0 \leq k \leq n$ and integers l_j, i_j where $j \in \{0, \ldots, k\}$ such that*

$$0 \leq i_0 < i_1 \ldots < i_k \leq n,$$
$$q_{i_0}^{l_0} \cdot \ldots \cdot q_{i_k}^{l_k} = 1,$$

then either $l_0 = \ldots = l_k = 0$ or $k = n$ and $l_0 = \ldots = l_n$.

Corollary 11. *([16], Corollaire III.1) Let q_0 be a Pisot number, $q_1, \ldots, q_n$ be its conjugates. If $|q_i| = |q_j|$ and $i \neq j$, then q_i and q_j are complex conjugates.*

Corollary 12. *([16], Corollaire III.2) Let q_0 be a Pisot number, $\theta_1, \ldots, \theta_s$ be the arguments of non-real conjugates $q_{i_1}, \ldots, q_{i_s}$ of number q_0. Then $2\pi, \theta_1, \ldots, \theta_s$ are $\mathbb{Q}$-linearly independent.*

Theorem 13. *([16], Proposition IV.) Let K be a real number field of degree $n+1$ ($n \geq 0$). Then for arbitrary positive constant λ there exists infinite many Pisot numbers which generate K and are the roots of the following polynomial:*

$$x^{n+1} - c_n x^n + c_{n-1} x^{n-1} - \ldots + (-1)^n c_1 x + (-1)^{n+1} = 0$$

where $c_{i+1} > \lambda c_i$, $i = 0, \ldots, n-1$, ($c_0 = 1$).

3.3 Existence of the periodic expansion in any real number field

In this section, we will prove that for every real number field there exist a basis for which the JPA expansion is purely periodic. Firstly, we will show how for concrete $c_1, \ldots, c_n$ and $a_1, \ldots, a_n$ from Theorem 9 we can find a vector $(1, \alpha_1, \ldots, \alpha_n)$ which is a basis of the field $\mathbb{Q}(\alpha_1, \ldots, \alpha_n)$ and have a purely periodic JPA expansion. This we will need the theorem from [15], and the consequence of Rouché's theorem from [22] and the Perron's irreducibility criterion [23], we put here just the statement of this theorems.

Theorem 14. *([6], Théorème 2) Let be $\alpha_1, \alpha_2, \ldots, \alpha_n$ an n-tuple with periodic JPA expansion, where $1, \alpha_1, \alpha_2, \ldots, \alpha_n$ are linearly independent over $\mathbb{Q}$ and let the characteristic polynomial of this expansion be irreducible with a Pisot number as root ϱ_0. Then $\mathbb{Q}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \mathbb{Q}(\varrho_0)$ and the degree of $\mathbb{Q}(\varrho_0)$ is $n+1$.*

Corollary 15. *(Corollary of Rouché's theorem, [22]) If for the polynomial $a_n x^n + \ldots + a_1 x + a_0$ exist a positive real number R and an integer $0 \leq k \leq n$ such that*

$$|a_k| R^k > |a_0| + \cdots + |a_{k-1}| R^{k-1} + |a_{k+1}| R^{k+1} + \cdots + |a_n| R^n,$$

then there are exactly k roots, counted with multiplicity, of absolute value less than R.

Lemma 16. *(Perron's irreducibility criterion], [23]) Suppose that we have the following polynomial with integer coefficients*

$$f(x) = x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

where $a_0 \neq 0$. If either of the following two conditions applies:

$$|a_{n-1}| > 1 + |a_{n-2}| + \cdots + |a_0|, \text{ or}$$

$$|a_{n-1}| \geq 1 + |a_{n-2}| + \cdots + |a_0|, \quad f(\pm 1) \neq 0,$$

then f is irreducible over the integers (and by Gauss's lemma also over the rational numbers).

From these theorems and Theorem 8 we can prove the following theorem where we will find the purely periodic JPA expansion for concrete coefficients c_i from Theorem 9.

Theorem 17. *([16], Théorème II.) Let $c_1, \ldots, c_n$ be positive integers. Assume, that integers $a_1, \ldots, a_n$ defined as in (3.6) are non-negative and satisfies inequality (3.3). Let ϱ_0 be the largest positive root of polynomial (3.5). Then we can find $\alpha_1, \ldots, \alpha_n \in \mathbb{Q}(\varrho_0)$ such that their JPA expansion is purely periodic with period length $n+1$. Then the coefficients of the expansion are $a^{(0)}, a^{(1)}, \ldots, a^{(n)}$, where $a^{(i)}$ are defined in (3.4).*

If in addition, $c_n > c_{n-1} + \ldots + c_1 + 1$, then the polynomial (3.5) is irreducible, ϱ_0 is a Pisot number and $1, \alpha_1, \ldots, \alpha_n$ is a basis of the field $\mathbb{Q}(\varrho_0)$.

Proof. It is easy to see that α_i defined in Theorem 8 satisfy the condition of the theorem. Moreover, we get that $\varrho_0 = \prod_{j=1}^{k} \alpha_n^{(j)} > 1$, because from the inequality (3.1) we have $a_n^{(j)} \geq 1$ for every $j \in \mathbb{N}$.

The second part of the statement is clear from the previous corollary and theorem. When we assume that $c_n > c_{n-1} + \ldots + c_1 + 1$ and take $R = 1$ then we can see from Corollary 14 that the characteristic polynomial has exactly n roots, counted with multiplicity, of absolute value less than 1. Hence, we get that ϱ_0 is a Pisot number. And from Perron's irreducibility criterion we get that this polynomial is irreducible.

Then from the Theorem 6 and Theorem 14 we get that $\mathbb{Q}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \mathbb{Q}(\varrho_0)$ and the degree of $\mathbb{Q}(\varrho_0)$ is $n+1$. Hence, $1, \alpha_1, \cdots, \alpha_n$ is a basis of the field $\mathbb{Q}(\varrho_0)$. □

Finally, we will prove the main theorem of this chapter, where we show that in every real number field exists a purely periodic JPA expansion.

Theorem 18. *([16], Théoréme IV.) In every real number field K with degree $n+1$, there exist infinitely many n-tuples $(\alpha_1, \ldots, \alpha_n)$ such that their JPA expansion is purely periodic and $(1, \alpha_1, \ldots, \alpha_n)$ is a basis of the field K.*

Proof. The idea of the proof is to find infinitely many constants λ from Theorem 13 such that for every $\{c_1, \ldots, c_n\}$ where

$$c_{i+1} > \lambda c_i, i = 0, \cdots, n-1 \text{ and } c_0 = 1$$

it has to hold that

$$a_i \geq 0, \text{ for } i = 0, \cdots, n-1 \text{ and } a_n > Max(a_1, a_2, \ldots, a_{n-1})$$

where a_i are from the equations (3.6). Then we will use Theorem 13 and Theorem 17.

From the fact that all coefficients c_i are positive we can easily see that there exists constant λ_1 such that when $c_{i+1} > \lambda_1 c_i$, $i = 0, \cdots, n-1$ and $c_0 = 1$. We get that all coefficients a_i are non negative. For instance, for a_1 we know that

$$c_1 = \binom{n+1}{1} + \tau_0 a_1$$

where τ_0 is some positive constant which depends only on n. Hence, if we take $\lambda_1 > n+2$, then we get that $c_1 > (n+2)c_0 = n+2$ and so $\tau_0 a_1 > 0$ and a_1 is positive. Then for a_2 we have that

$$c_2 = \binom{n+1}{2} + \tau_1 a_1 + \tau_2 a_2$$

where τ_1, τ_2 are some positive constant which depend only on n. Hence, if we take that $\lambda_1 > \frac{n}{2} + \frac{\tau_1}{\tau_0}$ we get that

$$\begin{aligned} \binom{n+1}{2} + \tau_1 a_1 + \tau_2 a_2 &= c_2 > \left(\frac{n}{2} + \frac{\tau_1}{\tau_0}\right) c_1 > \left(\frac{n}{2} + \frac{\tau_1}{\tau_0}\right)(n+1+\tau_0 a_1) \\ &= \binom{n+1}{2} + \tau_1 a_1 + \frac{n}{2}\tau_0 a_1 + \frac{\tau_1}{\tau_0}(n+1). \end{aligned}$$

Thus, $\tau_2 a_2$ is positive and also a_2 is positive. Therefore, we can proceed in the same way and for every a_i we can find some condition. Then we can fix some λ_1.

For the condition $a_n > Max(a_1, a_2, \ldots, a_{n-1})$ we find λ_2 such that for all $\lambda > \lambda_2$ this condition holds. We know that a_n depends only on c_n. We have that

$$c_n = n + 1 + \sum_{k=1}^{n} a_k$$

and

$$c_{n-1} = \binom{n+1}{n-1} + \sum_{k=1}^{n-1} \mu_k a_k$$

where $\mu_k \geq 1$. Hence, when we take $\lambda_2 > 4$, then we get

$$n + 1 + \sum_{k=1}^{n} a_k \;>\; 4\left(\binom{n+1}{n-1} + \sum_{k=1}^{n-1} \mu_k a_k\right) \;=\; 2(n+1)(n) + \sum_{k=1}^{n-1} 4\mu_k a_k.$$

Therefore,

$$a_n > \sum_{k=1}^{n-1} a_k,$$

so also $a_n > Max(a_1, a_2, \ldots, a_{n-1})$. Then we take $\lambda = Max(\lambda_1, \lambda_2)$.

For this λ we can find a Pisot number ϱ_0 and c_i, a_i from Theorem 13. Then by Theorem 17 we can find $\alpha_1, \ldots, \alpha_n \in \mathbb{Q}(\varrho_0)$ which have a purely periodic JPA expansion with period length $n+1$. Moreover, we can easily see that the condition $c_n > c_{n-1} + \ldots + c_1 + 1$ holds (because $\lambda_2 > 4$), hence ϱ_0 is a Pisot number and the elements $1, \alpha_1, \ldots, \alpha_n$ form a basis of the field $\mathbb{Q}(\varrho_0)$. $\square$

3.4 Example

In this section, we will show the concrete example of purely periodic expansion in one real number field. In the previous chapter, we say that in this thesis, we work with the fields $\mathbb{Q}(\rho)$ where ρ is one of the roots of polynomial $x^3-(a-1)x^2-ax-1$ where $a \in \mathbb{N}$, $a \geq 5$, so we will find a purely periodic JPA expansion in one of these fields.

Let us take $a = 5$ and $f(x) = x^3 - 4x^2 - 5x - 1$ and let ρ be the largest root of the polynomial f. We know that this polynomial has other roots less than 1 in absolute value, hence ρ is a Pisot number. In the previous chapter, we showed how the norm of the element looks like in this field and it is easy to show that $N(\rho) = -1$ and $N(\rho^{2k}) = 1$ for every $k \in \mathbb{N}$. Hence, ρ^{2k} is a Pisot number of the norm 1. From [16] (page 150) we know that $\mathbb{Q}(\rho) = \mathbb{Q}(\rho^l)$ for every $l \in \mathbb{N}$, hence we can use any Pisot number of the form ρ^{2k}. Also, we know that degree of this field is 3, thus $n = 2$.

We need to find polynomial from Theorem 17 of the form

$$x^3 - c_2x^2 + c_1x - 1, \text{ where } c_2 > c_1 + 1$$

and a_1, a_2 such that $a_2 > a_1 > 0$ where

$$c_2 = 3 + a_1 + a_2 \text{ and } c_1 = 3 + a_1$$

or equivalently

$$a_1 = c_1 - 3 \text{ and } a_n = c_2 - c_1.$$

First, we look at the minimal polynomial of ρ^2 which is

$$x^3 - 26x^2 + 17x - 1.$$

This polynomial is in the right form, but the condition $a_2 > a_1 > 0$ is not satisfied because here we get $a_1 = 14$ and $a_2 = 9$. Thus this polynomial does not work.

So, we try the minimal polynomial of ρ^4:

$$x^3 - 642x^2 + 237x - 1.$$

For this polynomial we get that $a_1 = 234$ and $a_2 = 405$ which satisfies the condition $a_2 > a_1 > 0$. Hence, from Theorem 9 we know that the purely periodic JPA expansion with period length 3 and incomplete quotient

$$a^{(0)} = (0, 1) \qquad a^{(1)} = (0, 1), \qquad a^{(2)} = (a_1, a_2).$$

has the characteristic polynomial

$$x^3 - 642x^2 + 237x - 1.$$

Now we just have to find the α_1, α_2 from Theorem 17. We have that

$$M = \begin{pmatrix} 1-x & 1 & a_1 + a_2 \\ 0 & 1-x & a_2 \\ 1 & 1 & 1 + a_1 + a_2 - x \end{pmatrix} = \begin{pmatrix} 1-x & 1 & 639 \\ 0 & 1-x & 405 \\ 1 & 1 & 640 - x \end{pmatrix}.$$

From this we can compute that for $j = 2$ we get

$$\alpha_1 = \frac{g_{2,1}(\rho^4)}{g_{2,0}(\rho^4)} = \frac{405(\rho^4 - 1)}{639\rho^4 - 1044}$$

and

$$\alpha_2 = \frac{g_{2,2}(\rho)}{g_{2,0}(\rho)} = \frac{(\rho^4 - 1)^2}{639\rho^4 - 1044}.$$

Therefore, by Theorem 18 the pair (α_1, α_2) has purely periodic JPA expansion with period length 3 and with incomplete coefficients

$$a^{(0)} = (0, 1) \qquad a^{(1)} = (0, 1), \qquad a^{(2)} = (a_1, a_2).$$

Moreover, $(1, \alpha_1, \alpha_2)$ is a basis of the field $\mathbb{Q}(\rho)$.

4. The JPA expansion

It this chapter we will find the JPA expansion of the couple $(|\rho|, \rho^2)$, where ρ is one of the roots of polynomial $x^3 - (a-1)x^2 - ax - 1$ where $a \in \mathbb{N}$, $a \geq 5$. Let us denote this polynomial by $f(x)$ (or $f_a(x)$) and the roots of the polynomial by ρ_1, ρ_2, ρ_3, where $\rho_1 < \rho_2 < \rho_3$.

From Chapter 2 we know that

$$-1 < -\frac{a-1}{a} < \rho_1 < -\frac{a-2}{a-1} < 0,$$

$$-1 < -\frac{1}{a-2} < \rho_2 < -\frac{1}{a-1} < 0,$$

$$a < a + \frac{a-1}{a^3} < \rho_3 < a + \frac{a^2-1}{a^4} < a + \frac{1}{a^2}.$$

4.1 The first root

We will prove that the JPA expansion of the couple $(|\rho_1|, \rho_1^2)$ is periodic with preperiod length 4, period length 7, and the first eleven iterations are

$$\begin{pmatrix} \alpha^{(0)} \\ \alpha^{(1)} \\ \alpha^{(2)} \\ \alpha^{(3)} \\ \alpha^{(4)} \\ \alpha^{(5)} \\ \alpha^{(6)} \\ \alpha^{(7)} \\ \alpha^{(8)} \\ \alpha^{(9)} \\ \alpha^{(10)} \\ \alpha^{(11)} \end{pmatrix} = \begin{pmatrix} -\rho_1 & \rho_1^2 \\ -\rho_1 & a + (a-1)\rho_1 - \rho_1^2 \\ a^2 - 2a + 1 + (a^2 - 2a + 2)\rho_1 + (-a+1)\rho_1^2 & a + (a-1)\rho_1 - \rho_1^2 \\ -\rho_1 & -1 + (-a+1)\rho_1 + \rho_1^2 \\ -a^2 + 3a - 1 + (-a^2 + 3a - 3)\rho_1 + (a-2)\rho_1^2 & a + (a-1)\rho_1 - \rho_1^2 \\ \frac{1}{2a-7}(-2 - 3\rho_1 + \rho_1^2) & \frac{1}{2a-7}(-2 + (-2a+4)\rho_1 + \rho_1^2) \\ a - 1 + (a-1)\rho_1 - \rho_1^2 & -a + (-2a+1)\rho_1 + 2\rho_1^2 \\ -3 + (-a+1)\rho_1 + \rho_1^2 & -1 + (-a)\rho_1 + \rho_1^2 \\ \frac{1}{2a-7}(2a - 5 + 3\rho_1 - \rho_1^2) & \frac{1}{2a-7}(-4 + (-2a+1)\rho_1 + 2\rho_1^2) \\ \frac{1}{a^2-5a+5}(-(a-2)^2 - (a^2 - 3a + 3)\rho_1 + (a-2)\rho_1^2) & \frac{1}{a^2-5a+5}(a^2 - 6a + 7 - (2a-4)\rho_1 + \rho_1^2) \\ -\rho_1 & -2 + (-a+1)\rho_1 + \rho_1^2 \\ -a^2 + 3a - 1 + (-a^2 + 3a - 3)\rho_1 + (a-2)\rho_1^2 & a + (a-1)\rho_1 - \rho_1^2 \end{pmatrix}.$$

We will calculate the expansion directly from the definition of JPA, so the ith iteration is

$$\alpha^{(i)} = \left(\frac{\alpha_2^{(i-1)} - a_2^{(i-1)}}{\alpha_1^{(i-1)} - a_1^{(i-1)}}, \frac{\alpha_3^{(i-1)} - a_3^{(i-1)}}{\alpha_1^{(i-1)} - a_1^{(i-1)}}, \frac{1}{\alpha_1^{(i-1)} - a_1^{(i-1)}} \right)$$

where $a_j^{(k)} = \lfloor \alpha_k^{(k)} \rfloor$. It follows straight away from the definition that $a_j^{(k)} \geq 0$ for all j, k, since the elements $\alpha_j^{(k)}$ is always non-negative which follows from the fact that the elements $\alpha_j^{(0)}$ are non-negative.

First iteration

From the fact that $-1 < \rho_1 < 0$ we directly get the value of $a^{(0)}$, where

$$a_1^{(0)} = \lfloor \alpha_1^{(0)} \rfloor = \lfloor -\rho_1 \rfloor = 0,$$
$$a_2^{(0)} = \lfloor \alpha_2^{(0)} \rfloor = \lfloor \rho_1^2 \rfloor = 0.$$

Now, it is easy to see that

$$\alpha_1^{(1)} \;=\; \frac{\alpha_2^{(0)} - a_2^{(0)}}{\alpha_1^{(0)} - a_1^{(0)}} \;=\; \frac{\rho_1^2}{-\rho_1} \;=\; -\rho_1$$

and

$$\alpha_2^{(1)} \;=\; \frac{1}{\alpha_1^{(0)} - a_1^{(0)}} \;=\; \frac{1}{-\rho_1} \;=\; a + (a-1)\rho_1 - \rho_1^2.$$

In the last equality we used that ρ_1 is a root of the polynomial $x^3 - (a-1)x^2 - ax - 1$. So, it also implies

$$(a + (a-1)\rho_1 - \rho_1^2)(-\rho_1) = \rho_1^3 - (a-1)\rho_1^2 - a\rho_1 = 1.$$

Hence we get $\alpha^{(1)} = (-\rho_1, a + (a-1)\rho_1 - \rho_1^2)$.

Second iteration

Let us count first $a^{(1)}$. From the first iteration we know that $a_1^{(1)} = \lfloor -\rho_1 \rfloor = 0$. So, we only need to compute $a_2^{(1)} = \lfloor \alpha_2^{(1)} \rfloor$:

$$\alpha_2^{(1)} \;=\; a + (a-1)\rho_1 - \rho_1^2 \;>\; a + (a-1)\frac{-a+1}{a} - \frac{a^2 - 2a + 1}{a^2} \;=\; 1 + \frac{a-1}{a^2} \;>\; 1,$$

$$\begin{aligned} \alpha_2^{(1)} \;&=\; a + (a-1)\rho_1 - \rho_1^2 \;<\; a + (a-1)\frac{-a+2}{a-1} - \frac{a^2 - 4a + 4}{a^2 - 2a + 1} \\ &=\; 2 - 1 + \frac{2a-3}{a^2 - 2a + 1} \;<\; 2. \end{aligned}$$

Hence, we have that $a^{(1)} \;=\; (0, 1)$.

It simply follows the identities

$$\begin{aligned} \alpha_1^{(2)} \;&=\; \frac{\alpha_2^{(1)} - a_2^{(1)}}{\alpha_1^{(1)} - a_1^{(1)}} \;=\; \frac{a + (a-1)\rho_1 - \rho_1^2 - 1}{-\rho_1} \\ &=\; a^2 - 2a + 1 + (a^2 - 2a + 2)\rho_1 + (-a+1)\rho_1^2 \end{aligned}$$

and

$$\alpha_2^{(2)} \;=\; \frac{1}{\alpha_1^{(1)} - a_1^{(1)}} \;=\; \frac{1}{-\rho_1} \;=\; a + (a-1)\rho_1 - \rho_1^2.$$

In the both equations we again used that $(-\rho_1)^{-1} = a + (a-1)\rho_1 - \rho_1^2$.

This means that $\alpha^{(2)} = (a^2 - 2a + 1 + (a^2 - 2a + 2)\rho_1 + (-a+1)\rho_1^2, a + (a-1)\rho_1 - \rho_1^2)$.

Third iteration

We will repeat the same step as in the previous iteration. So, we first show that $a^{(2)} = (0, 1)$:

$$
\begin{aligned}
\alpha_1^{(2)} &= a^2-2a+1+(a^2-2a+2)\rho_1+(-a+1)\rho_1^2 \\
&< a^2-2a+1+(a^2-2a+2)\frac{-a+2}{a-1}+(-a+1)\frac{a^2-4a+4}{a^2-2a+1} \\
&= a^2-2a+1-(a^2-2a+2)+\frac{a^2-2a+2}{a-1}-\frac{a^2-4a+4}{a-1} \\
&= -1+\frac{2a-2}{a-1} \;=\; -1+2 \;=\; 1.
\end{aligned}
$$

Since the element $\alpha_1^{(2)}$ is non-negative, we have $a_1^{(2)}=0$. And from the second iteration we have that $a_2^{(2)}=a_2^{(1)}=1$.

To calculate $\alpha_1^{(3)}$ we use that $(a^2-2a+1+(a^2-2a+2)\rho_1+(-a+1)\rho_1^2)(-1+(-a+1)\rho_1+\rho_1^2)=1$. This follows from the fact that $\rho_1^3=(a-1)\rho_1^2+a\rho_1+1$. Hence,

$$
\alpha_1^{(3)} \;=\; \frac{\alpha_2^{(2)}-a_2^{(2)}}{\alpha_1^{(2)}-a_1^{(2)}} \;=\; \frac{a+(a-1)\rho_1-\rho_1^2-1}{a^2-2a+1+(a^2-2a+2)\rho_1+(-a+1)\rho_1^2} \;=\; -\rho_1
$$

and

$$
\begin{aligned}
\alpha_2^{(3)} &= \frac{1}{\alpha_1^{(2)}-a_1^{(2)}} \;=\; \frac{1}{a^2-2a+1+(a^2-2a+2)\rho_1+(-a+1)\rho_1^2} \\
&= -1+(-a+1)\rho_1+\rho_1^2.
\end{aligned}
$$

We verified that $\alpha^{(3)}=(-\rho_1,-1+(-a+1)\rho_1+\rho_1^2)$.

Fourth iteration

As usual we first calculate $a^{(3)}$. From the previous iteration it is clear that $a_1^{(3)}=0$ and so we need to compute only $a_2^{(3)}$:

$$
\begin{aligned}
\alpha_2^{(3)} &= -1+(-a+1)\rho_1+\rho_1^2 \;>\; -1+(-a+1)\frac{-a+2}{a-1}+\frac{a^2-4a+4}{a^2-2a+1} \\
&= -1+a-2++\frac{a^2-4a+4}{a^2-2a+1} \;>\; a-3
\end{aligned}
$$

and

$$
\begin{aligned}
\alpha_2^{(3)} &= -1+(-a+1)\rho_1+\rho_1^2 \;<\; -1+(-a+1)\frac{-a+1}{a}+\frac{a^2-2a+1}{a^2} \\
&= -1+a-2+\frac{1}{a}+1+\frac{-2a+1}{a^2} \;=\; a-2-\frac{a+1}{a^2} \;<\; a-2.
\end{aligned}
$$

Thus, we get that $a^{(3)}=(0,a-3)$. A trivial verification shows that

$$
\begin{aligned}
\alpha_1^{(4)} &= \frac{\alpha_2^{(3)}-a_2^{(3)}}{\alpha_1^{(3)}-a_1^{(3)}} \;=\; \frac{-1+(-a+1)\rho_1+\rho_1^2-a+3}{-\rho_1} \\
&= -a^2+3a-1+(-a^2+3a-3)\rho_1+(a-2)\rho_1^2
\end{aligned}
$$

and

$$\alpha_2^{(4)} = \frac{1}{\alpha_1^{(3)} - a_1^{(3)}} = \frac{1}{-\rho_1} = a + (a-1)\rho_1 - \rho_1^2.$$

In the both equations we again used $(-\rho_1)^{-1} = a + (a-1)\rho_1 - \rho_1^2$. Therefore we have that $\alpha^{(4)} = (-a^2+3a-1+(-a^2+3a-3)\rho_1+(a-2)\rho_1^2, a+(a-1)\rho_1-\rho_1^2)$.

Fifth iteration

Firstly, we will show that $a^{(4)} = (0,1)$. The element $\alpha_1^{(4)}$ is greater or equal to zero, so we just need to show that $\alpha_1^{(4)} < 1$. Here we need to use the fact that $\frac{-a^4+a^3+2a+2}{a^4} < \rho_1$.

$$\begin{aligned}
\alpha_1^{(4)} &= -a^2+3a-1+(-a^2+3a-3)\rho_1+(a-2)\rho_1^2 \\
&< -a^2+3a-1+(-a^2+3a-3)\frac{-a^4+a^3+2a+2}{a^4} + (a-2)\left(\frac{-a^4+a^3+2a+2}{a^4}\right)^2 \\
&= \frac{-8-12a-4a^3-2a^4+8a^5-2a^6+a^8}{a^8} < 1.
\end{aligned}$$

Hence, $a_1^{(4)} = 0$ and from the second iteration we know that $a_2^{(4)} = 1$. When we realize that $\frac{1}{2a-7}(-a^2+3a-1+(-a^2+3a-3)\rho_1+(a-2)\rho_1^2)(-2+(-2a+4)\rho_1+\rho_1^2) = 1$, it easy to see that

$$\begin{aligned}
\alpha_1^{(5)} &= \frac{\alpha_2^{(4)} - a_2^{(4)}}{\alpha_1^{(4)} - a_1^{(4)}} = \frac{a+(a-1)\rho_1-\rho_1^2-1}{-a^2+3a-1+(-a^2+3a-3)\rho_1+(a-2)\rho_1^2} \\
&= \frac{1}{2a-7}(-2-3\rho_1+\rho_1^2)
\end{aligned}$$

and

$$\begin{aligned}
\alpha_2^{(5)} &= \frac{1}{\alpha_1^{(4)} - a_1^{(4)}} = \frac{1}{-a^2+3a-1+(-a^2+3a-3)\rho_1+(a-2)\rho_1^2} \\
&= \frac{1}{2a-7}(-2+(-2a+4)\rho_1+\rho_1^2).
\end{aligned}$$

Therefore, we get that $\alpha^{(5)} = (\frac{1}{2a-7}(-2-3\rho_1+\rho_1^2), \frac{1}{2a-7}(-2+(-2a+4)\rho_1+\rho_1^2))$.

Sixth iteration

Now, we will compute $a^{(5)}$. For $\alpha_1^{(5)}$, we can see that

$$\begin{aligned}
\alpha_1^{(5)} &= \frac{1}{2a-7}(-2-3\rho_1+\rho_1^2) < \frac{1}{2a-7}\left(-2-3\frac{-a+1}{a}+\frac{a^2-2a+1}{a^2}\right) \\
&= \frac{1}{2a-7}\left(-2+3-\frac{3}{a}+1+\frac{-2a+1}{a^2}\right) = \frac{1}{2a-7}\frac{2a^2-5a+1}{a^2} < 1
\end{aligned}$$

and because $\alpha_1^{(5)} \geq 0$, we get that $a_1^{(5)} = 0$. For $\alpha_2^{(5)}$, it holds the following

$$\begin{aligned}
\alpha_2^{(5)} &= \frac{1}{2a-7}\left(-2+(-2a+4)\rho_1+\rho_1^2\right) \\
&> \frac{1}{2a-7}\left(-2+(-2a+4)\frac{-a+2}{a-1}+\frac{a^2-4a+4}{a^2-2a+1}\right) \\
&= \frac{1}{2a-7}\left(-2+2a-6+\frac{2}{a-1}+1+\frac{-2a+3}{a^2-2a+1}\right) \\
&= 1+\frac{1}{2a-7}\frac{1}{a^2-2a+1} \;>\; 1
\end{aligned}$$

and

$$\begin{aligned}
\alpha_2^{(5)} &= \frac{1}{2a-7}\left(-2+(-2a+4)\rho_1+\rho_1^2\right) \\
&< \frac{1}{2a-7}\left(-2+(-2a+4)\frac{-a+1}{a}+\frac{a^2-2a+1}{a^2}\right) \\
&= \frac{1}{2a-7}\left(-2+2a-6+\frac{4}{a}+1+\frac{-2a+1}{a^2}\right) \\
&= 1+\frac{1}{2a-7}\frac{2a+1}{a^2} \;<\; 2.
\end{aligned}$$

Hence, $a^{(5)} = (0,1)$. Now it is simply to verify that

$$\begin{aligned}
\alpha_1^{(6)} &= \frac{\alpha_2^{(5)}-a_2^{(5)}}{\alpha_1^{(5)}-a_1^{(5)}} = \frac{\frac{1}{2a-7}\left(-2+(-2a+4)\rho_1+\rho_1^2\right)-1}{\frac{1}{2a-7}\left(-2-3\rho_1+\rho_1^2\right)} \\
&= a-1+(a-1)\rho_1-\rho_1^2
\end{aligned}$$

and

$$\begin{aligned}
\alpha_2^{(6)} &= \frac{1}{\alpha_1^{(5)}-a_1^{(5)}} = \frac{1}{\frac{1}{2a-7}\left(-2-3\rho_1+\rho_1^2\right)} \\
&= -a+(-2a+1)\rho_1+2\rho_1^2.
\end{aligned}$$

We used that $\frac{1}{2a-7}\left(-2-3\rho_1+\rho_1^2\right)\left(-a+(-2a+1)\rho_1+2\rho_1^2\right) = 1$. Therefore, we get the following identity $\alpha^{(6)} = (a-1+(a-1)\rho_1-\rho_1^2, -a+(-2a+1)\rho_1+2\rho_1^2)$.

Seventh iteration

From the second iteration we know that $a_2^{(1)} = 1 = \lfloor a+(a-1)\rho_1-\rho_1^2 \rfloor$. Hence it is clear that $a_1^{(6)} = \lfloor a-1+(a-1)\rho_1-\rho_1^2 \rfloor = 0$. So we only need to compute $a_2^{(6)}$. For $a > 5$ it holds

$$\begin{aligned}
\alpha_2^{(6)} &= -a+(-2a+1)\rho_1+2\rho_1^2 \;>\; -a+(-2a+1)\frac{-a+2}{a-1}+2\frac{a^2-4a+4}{a^2-2a+1} \\
&= -a+2a-3+\frac{-1}{a-1}+2+\frac{-4a+6}{a^2-2a+1} \;=\; a-1+\frac{-5a+7}{a^2-2a+1} \\
&> a-2
\end{aligned}$$

and for $a = 5$ we have $f_5(x) = x^3 - 4x^2 - 5x - 1$ and $\rho_1 < \frac{-31}{40}$, then we get that

$$\begin{aligned}\alpha_2^{(6)} &= -5 - 9\rho_1 + 2\rho_1^2 \;>\; -5 + 9\frac{31}{40} + 2\frac{31^2}{40^2} \\ &= \frac{2541}{800} \;>\; 3 \;=\; a - 2.\end{aligned}$$

Together we get that $\alpha_2^{(6)} > a - 2$ for all $a \geq 5$. For $\alpha_2^{(6)}$, it holds that

$$\begin{aligned}\alpha_2^{(6)} &= -a + (-2a+1)\rho_1 + 2\rho_1^2 \;<\; -a + (-2a+1)\frac{-a+1}{a} + 2\frac{a^2-2a+1}{a^2} \\ &= -a + 2a - 3 + \frac{-1}{a-1} + 2 + \frac{-4a+2}{a^2} \;=\; a - 1 + \frac{-3a+2}{a^2} \;<\; a - 1.\end{aligned}$$

Hence, we have $a^{(6)} = (0, a-2)$. We will use the following identity $(a - 1 + (a - 1)\rho_1 - \rho_1^2)(-1 + (-a)\rho_1 + \rho_1^2) = 1$ to verify that

$$\begin{aligned}\alpha_1^{(7)} &= \frac{\alpha_2^{(6)} - a_2^{(6)}}{\alpha_1^{(6)} - a_1^{(6)}} \;=\; \frac{-a + (-2a+1)\rho_1 + 2\rho_1^2 - a + 2}{a - 1 + (a-1)\rho_1 - \rho_1^2} \\ &= -3 + (-a+1)\rho_1 + \rho_1^2\end{aligned}$$

and

$$\begin{aligned}\alpha_2^{(7)} &= \frac{1}{\alpha_1^{(6)} - a_1^{(6)}} \;=\; \frac{1}{a - 1 + (a-1)\rho_1 - \rho_1^2} \\ &= -1 + (-a)\rho_1 + \rho_1^2.\end{aligned}$$

Therefore, $\alpha^{(7)} = (-3 + (-a+1)\rho_1 + \rho_1^2, -1 + (-a)\rho_1 + \rho_1^2)$.

Eighth iteration

We will show that $a^{(7)} = (a - 5, a - 2)$. For $\alpha_1^{(7)}$ we have that

$$\begin{aligned}\alpha_1^{(7)} &= -3 + (-a+1)\rho_1 + \rho_1^2 \;>\; -3 + (-a+1)\frac{-a+2}{a-1} + \frac{a^2-4a+4}{a^2-2a+1} \\ &= -3 + a - 2 + 1 + \frac{-2a+3}{a^2-2a+1} \;=\; a - 4 + \frac{-2a+3}{a^2-2a+1} \;>\; a - 5\end{aligned}$$

and

$$\begin{aligned}\alpha_1^{(7)} &= -3 + (-a+1)\rho_1 + \rho_1^2 \;<\; -3 + (-a+1)\frac{-a+1}{a} + \frac{a^2-2a+1}{a^2} \\ &= -3 + a - 2 + \frac{1}{a} + 1 + \frac{-2a+1}{a^2} \;=\; a - 4 + \frac{-a+1}{a^2} \;<\; a - 4.\end{aligned}$$

For $\alpha_2^{(7)}$ we can see the following

$$\begin{aligned}\alpha_2^{(7)} &= -1 + (-a)\rho_1 + \rho_1^2 \;>\; -1 + (-a)\frac{-a+2}{a-1} + \frac{a^2-4a+4}{a^2-2a+1} \\ &= -1 + a - 1 - \frac{1}{a-1} + 1 + \frac{-2a+3}{a^2-2a+1} \;=\; a - 1 + \frac{-3a+4}{a^2-2a+1} \\ &> a - 2\end{aligned}$$

and

$$\begin{aligned}\alpha_2^{(7)} &= -1+(-a)\rho_1+\rho_1^2 \;<\; -1+(-a)\frac{-a+1}{a}+\frac{a^2-2a+1}{a^2}\\ &= -1+a-1+1+\frac{-2a+1}{a^2} \;<\; a-1.\end{aligned}$$

Now, when we used the identity $\frac{1}{2a-7}(-a+2+(-a+1)\rho_1+\rho_1^2)(-4+(-2a+1)\rho_1+2\rho_1^2)=1$, it is easy to verify the following equalities:

$$\begin{aligned}\alpha_1^{(8)} &= \frac{\alpha_2^{(7)}-a_2^{(7)}}{\alpha_1^{(7)}-a_1^{(7)}} = \frac{-1+(-a)\rho_1+\rho_1^2-a+2}{-3+(-a+1)\rho_1+\rho_1^2-a+5}\\ &= \frac{1}{2a-7}(2a-5+3\rho_1-\rho_1^2),\end{aligned}$$

$$\begin{aligned}\alpha_2^{(8)} &= \frac{1}{\alpha_1^{(7)}-a_1^{(7)}} = \frac{1}{-3+(-a+1)\rho_1+\rho_1^2-a+5}\\ &= \frac{1}{2a-7}(-4+(-2a+1)\rho_1+2\rho_1^2).\end{aligned}$$

Therefore, $\alpha^{(8)}=\left(\frac{1}{2a-7}(2a-5+3\rho_1-\rho_1^2),\frac{1}{2a-7}(-4+(-2a+1)\rho_1+2\rho_1^2)\right)$.

Ninth iteration

Firstly, we will count $a^{(8)}$:

$$\begin{aligned}\alpha_1^{(8)} &= \frac{1}{2a-7}(2a-5+3\rho_1-\rho_1^2) \;<\; \frac{1}{2a-7}\left(2a-5+3\frac{-a+2}{a-1}-\frac{a^2-4a+4}{a^2-2a+1}\right)\\ &= \frac{1}{2a-7}\left(2a-5-3+\frac{3}{a-1}-1-\frac{-2a+3}{a^2-2a+1}\right)\\ &= 1+\frac{1}{2a-7}\frac{-2a^2+9a-8}{a^2-2a+1} \;<\; 1\end{aligned}$$

and because $\alpha_1^{(8)}\geq 0$, we have that $a_1^{(8)}=0$. We will show $a_2^{(8)}=1$:

$$\begin{aligned}\alpha_2^{(8)} &= \frac{1}{2a-7}(-4+(-2a+1)\rho_1+2\rho_1^2)\\ &> \frac{1}{2a-7}\left(-4+(-2a+1)\frac{-a+2}{a-1}+2\frac{a^2-4a+4}{a^2-2a+1}\right)\\ &= \frac{1}{2a-7}\left(-4+2a-3-\frac{1}{a-1}+2+\frac{-4a+6}{a^2-2a+1}\right)\\ &= 1+\frac{1}{2a-7}\left(2+\frac{-5a+7}{a^2-2a+1}\right) \;>\; 1\end{aligned}$$

and

$$
\begin{aligned}
\alpha_2^{(8)} &= \frac{1}{2a-7}(-4+(-2a+1)\rho_1+2\rho_1^2) \\
&< \frac{1}{2a-7}\left(-4+(-2a+1)\frac{-a+1}{a}+2\frac{a^2-2a+1}{a^2}\right) \\
&= \frac{1}{2a-7}\left(-4+2a-3+\frac{1}{a}+2+\frac{-4a+2}{a^2}\right) \\
&= 1+\frac{1}{2a-7}\left(2+\frac{-3a+2}{a^2}\right) \ < \ 2.
\end{aligned}
$$

Hence, $a^{(8)} = (0,1)$. Now it is easy to find the value of $\alpha^{(9)}$:

$$
\begin{aligned}
\alpha_1^{(9)} &= \frac{\alpha_2^{(8)}-a_2^{(8)}}{\alpha_1^{(8)}-a_1^{(8)}} = \frac{\frac{1}{2a-7}(-4+(-2a+1)\rho_1+2\rho_1^2)-1}{\frac{1}{2a-7}(2a-5+3\rho_1-\rho_1^2)} \\
&= \frac{1}{a^2-5a+5}(-(a-2)^2-(a^2-3a+3)\rho_1+(a-2)\rho_1^2)
\end{aligned}
$$

and

$$
\begin{aligned}
\alpha_2^{(9)} &= \frac{1}{\alpha_1^{(8)}-a_1^{(8)}} = \frac{1}{\frac{1}{2a-7}(2a-5+3\rho_1-\rho_1^2)} \\
&= \frac{1}{a^2-5a+5}(a^2-6a+7-(2a-4)\rho_1+\rho_1^2).
\end{aligned}
$$

We used that the inverse of the element $\frac{1}{2a-7}(2a-5+3\rho_1-\rho_1^2)$ is $\frac{1}{a^2-5a+5}(a^2-6a+7-(2a-4)\rho_1+\rho_1^2)$. Therefore, we get: $\alpha^{(9)} = (\frac{1}{a^2-5a+5}(-(a-2)^2-(a^2-3a+3)\rho_1+(a-2)\rho_1^2), \frac{1}{a^2-5a+5}(a^2-6a+7-(2a-4)\rho_1+\rho_1^2))$.

Tenth iteration

We will show that $a^{(9)} = (0,1)$:

$$
\begin{aligned}
\alpha_1^{(9)} &= \frac{1}{a^2-5a+5}(-(a-2)^2-(a^2-3a+3)\rho_1+(a-2)\rho_1^2) \\
&< \frac{1}{a^2-5a+5}\left(-(a-2)^2-(a^2-3a+3)\frac{-a+1}{a}+(a-2)\frac{a^2-2a+1}{a^2}\right) \\
&= \frac{1}{a^2-5a+5}\left(-a^2+4a-4+a^2-4a+6-\frac{3}{a}+a-4+\frac{5a-2}{a^2}\right) \\
&= \frac{1}{a^2-5a+5}\left(a-2+\frac{2a-2}{a^2}\right) \ < \ 1.
\end{aligned}
$$

So, $a_1^{(9)} = 0$, because $\alpha_1^{(9)}$ is non-negative. Then we get the following inequalities:

$$
\begin{aligned}
\alpha_2^{(9)} &= \frac{1}{a^2-5a+5}(a^2-6a+7-(2a-4)\rho_1+\rho_1^2) \\
&> \frac{1}{a^2-5a+5}\left(a^2-6a+7-(2a-4)\frac{-a+2}{a-1}+\frac{a^2-4a+4}{a^2-2a+1}\right) \\
&= \frac{1}{a^2-5a+5}\left(a^2-6a+7+2a-4-2+\frac{2}{a-1}+1+\frac{-2a+3}{a^2-2a+1}\right) \\
&= 1+\frac{1}{a^2-5a+5}\left(a-3+\frac{1}{a^2-2a+1}\right) \ > \ 1
\end{aligned}
$$

and

$$
\begin{aligned}
\alpha_2^{(9)} &= \frac{1}{a^2-5a+5}(a^2-6a+7-(2a-4)\rho_1+\rho_1^2) \\
&< \frac{1}{a^2-5a+5}\left(a^2-6a+7-(2a-4)\frac{-a+1}{a}+\frac{a^2-2a+1}{a^2}\right) \\
&= \frac{1}{a^2-5a+5}\left(a^2-6a+7+2a-6+\frac{4}{a}+1+\frac{-2a+1}{a^2}\right) \\
&= 1+\frac{1}{a^2-5a+5}\left(a-3+\frac{2a+1}{a^2}\right) \ < \ 2.
\end{aligned}
$$

Thus, $a^{(9)} = (0, 1)$.

From the fact that $\frac{1}{a^2-5a+5}(-(a-2)^2-(a^2-3a+3)\rho_1+(a-2)\rho_1^2)(-2+(-a+1)\rho_1+\rho_1^2) = 1$ we can see that

$$
\begin{aligned}
\alpha_1^{(10)} &= \frac{\alpha_2^{(9)}-a_2^{(9)}}{\alpha_1^{(9)}-a_1^{(9)}} = \frac{\frac{1}{a^2-5a+5}(a^2-6a+7-(2a-4)\rho_1+\rho_1^2)-1}{\frac{1}{a^2-5a+5}(-(a-2)^2-(a^2-3a+3)\rho_1+(a-2)\rho_1^2)} \\
&= -\rho_1,
\end{aligned}
$$

$$
\begin{aligned}
\alpha_2^{(10)} &= \frac{1}{\alpha_1^{(9)}-a_1^{(9)}} = \frac{1}{\frac{1}{a^2-5a+5}(-(a-2)^2-(a^2-3a+3)\rho_1+(a-2)\rho_1^2)} \\
&= -2+(-a+1)\rho_1+\rho_1^2.
\end{aligned}
$$

Hence, the tenth iteration is $\alpha^{(10)} = (-\rho_1, -2+(-a+1)\rho_1+\rho_1^2)$.

Eleventh iteration

From the first iteration we know that $a_1^{(10)} = \lfloor -\rho_1 \rfloor = 0$. From the fourth iteration we know that $\lfloor -1+(-a+1)\rho_1+\rho_1^2 \rfloor = a-3$, hence $a_2^{(10)} = \lfloor -2+(-a+1)\rho_1+\rho_1^2 \rfloor = a-4$. A trivial verification shows the following identities

$$
\begin{aligned}
\alpha_1^{(11)} &= \frac{\alpha_2^{(10)}-a_2^{(10)}}{\alpha_1^{(10)}-a_1^{(10)}} = \frac{-2+(-a+1)\rho_1+\rho_1^2-a+4}{-\rho_1} \\
&= -a^2+3a-1+(-a^2+3a-3)\rho_1+(a-2)\rho_1^2,
\end{aligned}
$$

$$\begin{aligned}\alpha_2^{(11)} &= \frac{1}{\alpha_1^{(10)} - a_1^{(10)}} = \frac{1}{-\rho_1} \\ &= a + (a-1)\rho_1 - \rho_1^2.\end{aligned}$$

Therefore, $\alpha^{(11)} = (-a^2+3a-1+(-a^2+3a-3)\rho_1+(a-2)\rho_1^2, a+(a-1)\rho_1-\rho_1^2)$ and $\alpha^{(4)} = \alpha^{(11)}$. We showed that JPA expansion of the couple $(|\rho_1|, \rho_1^2)$ is periodic with preperiod length 4 and period length 7 and derived the first eleven iterations.

4.2 The second root

We will prove that the JPA expansion of the couple $(|\rho_2|, \rho_2^2)$ is periodic with preperiod length 3, period length 3, and the first six iterations are

$$\begin{pmatrix}\alpha^{(0)}\\ \alpha^{(1)}\\ \alpha^{(2)}\\ \alpha^{(3)}\\ \alpha^{(4)}\\ \alpha^{(5)}\\ \alpha^{(6)}\end{pmatrix} = \begin{pmatrix} -\rho_2 & \rho_2^2 \\ -\rho_2 & a+(a-1)\rho_2-\rho_2^2 \\ a+1+(2a-1)\rho_2+(-2)\rho_2^2 & a+(a-1)\rho_2-\rho_2^2 \\ \frac{1}{g}(2a^2-9a+10+(a^2-2a-1)\rho_2+(2-a)\rho_2^2) & \frac{1}{g}(2a^2-8a+8+(2a^2-7a+6)\rho_2+(5-2a)\rho_2^2) \\ 1-\rho_2 & a-2+a\rho_2-\rho_2^2 \\ a+(2a-1)\rho_2+(-2)\rho_2^2 & a+(a-1)\rho_2-\rho_2^2 \\ \frac{1}{g}(2a^2-9a+10+(a^2-2a-1)\rho_2+(2-a)\rho_2^2) & \frac{1}{g}(2a^2-8a+8+(2a^2-7a+6)\rho_2+(5-2a)\rho_2^2)\end{pmatrix}$$

where $g = 2a^2 - 12a + 17$.

We use the same technique as for the first root. For the second root we know that $\frac{-1}{a-2} < \rho_2 < \frac{-1}{a-1}$ and from the definition of the JPA expansion we know that $\alpha_k^{(j)} > 0$ for all $k, j \in \mathbb{N}$.

First iteration

The first iteration is the same as the first iteration of the first root. We can use exactly the same proof, because for the first root we only used the fact that $-1 < \rho_1 < 0$, which also holds for the second root. Hence, $\alpha^{(1)} = (-\rho_2, a+(a-1)\rho_2-\rho_2^2)$.

Second iteration

We will prove that $a^{(1)} = (0, a-2)$. From the previous iteration we know that $a_1^{(1)} = \lfloor \alpha_1^{(1)} \rfloor = 0$. So it is enough to verify that $a_2^{(1)} = \lfloor \alpha_2^{(1)} \rfloor = a-2$:

$$\begin{aligned}\alpha_2^{(1)} &= a + (a-1)\rho_2 - \rho_2^2 \;>\; a + (a-1)\frac{-1}{a-2} - \frac{1}{a^2-4a+4} \\ &= a - 1 - \frac{1}{a-2} - \frac{1}{a^2-4a+4} \;=\; a - 1 - \frac{a-1}{a^2-4a+4} \;>\; a-2\end{aligned}$$

and

$$\begin{aligned}\alpha_2^{(1)} &= a + (a-1)\rho_2 - \rho_2^2 \;<\; a + (a-1)\frac{-1}{a-1} - \frac{1}{a^2-2a+1} \\ &= a - 1 - \frac{1}{a^2-2a+1} \;< a-1.\end{aligned}$$

For the calculation of $\alpha^{(2)}$ we will use identity $(-\rho_2)^{-1} = a + (a-1)\rho_2 - \rho_2^2$, so

$$\begin{aligned}\alpha_1^{(2)} &= \frac{\alpha_2^{(1)} - a_2^{(1)}}{\alpha_1^{(1)} - a_1^{(1)}} = \frac{a + (a-1)\rho_2 - \rho_2^2 - a + 2}{-\rho_2} \\ &= a + 1 + (2a-1)\rho_2 - 2\rho_2^2\end{aligned}$$

and

$$\alpha_2^{(2)} = \frac{1}{\alpha_1^{(1)} - a_1^{(1)}} = \frac{1}{-\rho_2} = a + (a-1)\rho_2 - \rho_2^2.$$

Hence, we get that $\alpha^{(2)} = (a + 1 + (2a-1)\rho_2 + (-2)\rho_2^2, a + (a-1)\rho_2 - \rho_2^2)$.

Third iteration

First we will compute that $a^{(2)} = (a-2, a-2)$. From the previous iteration we know that $a_2^{(2)} = a - 2$, so it is enough to calculate $a_1^{(2)}$:

$$\begin{aligned}\alpha_1^{(2)} &= a + 1 + (2a-1)\rho_2 + (-2)\rho_2^2 > a + 1 + (2a-1)\frac{-1}{a-2} + (-2)\frac{1}{a^2 - 4a + 4} \\ &= a + 1 + \frac{-2a+1}{a-2} + \frac{-2}{a^2 - 4a + 4} = a - 1 + \frac{-3a+4}{a^2 - 4a + 4} > a - 2\end{aligned}$$

and

$$\begin{aligned}\alpha_1^{(2)} &= a + 1 + (2a-1)\rho_2 + (-2)\rho_2^2 < a + 1 + (2a-1)\frac{-1}{a-1} + (-2)\frac{1}{a^2 - 2a + 1} \\ &= a + 1 + \frac{-2a+1}{a-1} + \frac{-2}{a^2 - 2a + 1} = a - 1 + \frac{-a-1}{a^2 - 2a + 1} < a - 1.\end{aligned}$$

Now, it is easy to verify that

$$\begin{aligned}\alpha_1^{(3)} &= \frac{\alpha_2^{(2)} - a_2^{(2)}}{\alpha_1^{(2)} - a_1^{(2)}} = \frac{a + (a-1)\rho_2 - \rho_2^2 - a + 2}{a + 1 + (2a-1)\rho_2 + (-2)\rho_2^2 - a + 2} \\ &= \frac{1}{2a^2 - 12a + 17}(2a^2 - 9a + 10 + (a^2 - 2a - 1)\rho_2 + (2-a)\rho_2^2)\end{aligned}$$

and

$$\begin{aligned}\alpha_2^{(3)} &= \frac{1}{\alpha_1^{(2)} - a_1^{(2)}} = \frac{1}{a + 1 + (2a-1)\rho_2 + (-2)\rho_2^2 - a + 2} \\ &= \frac{1}{2a^2 - 12a + 17}(2a^2 - 8a + 8 + (2a^2 - 7a + 6)\rho_2 + (5-2a)\rho_2^2)).\end{aligned}$$

In the both equations we used that $\frac{1}{2a^2-12a+17}(3 + (2a-1)\rho_2 + (-2)\rho_2^2)(2a^2 - 8a + 8 + (2a^2 - 7a + 6)\rho_2 + (5-2a)\rho_2^2)) = 1$. Therefore, $\alpha^{(3)} = (\frac{1}{2a^2-12a+17}(2a^2 - 9a + 10 + (a^2 - 2a - 1)\rho_2 + (2-a)\rho_2^2), \frac{1}{2a^2-12a+17}(2a^2 - 8a + 8 + (2a^2 - 7a + 6)\rho_2 + (5-2a)\rho_2^2))$.

Fourth iteration

We will show that $a^{(3)} = (1,1)$. For $\alpha_1^{(3)}$ we have the following

$$\begin{aligned}
\alpha_1^{(3)} &= \frac{1}{2a^2-12a+17}(2a^2-9a+10+(a^2-2a-1)\rho_2+(2-a)\rho_2^2 \\
&> \frac{1}{2a^2-12a+17}\left(2a^2-9a+10+(a^2-2a-1)\frac{-1}{a-2}+(2-a)\frac{1}{a^2-4a+4}\right) \\
&= \frac{1}{2a^2-12a+17}\left(2a^2-9a+10-a+\frac{1}{a-2}+\frac{2-a}{a^2-4a+4}\right) \\
&= 1+\frac{1}{2a^2-12a+17}\left(4a-7+\frac{-2a+4}{a^2-4a+4}\right) \;>\; 1
\end{aligned}$$

and

$$\begin{aligned}
\alpha_1^{(3)} &= \frac{1}{2a^2-12a+17}(2a^2-9a+10+(a^2-2a-1)\rho_2+(2-a)\rho_2^2) \\
&< \frac{1}{2a^2-12a+17}\left(2a^2-9a+10+(a^2-2a-1)\frac{-1}{a-1}+(2-a)\frac{1}{a^2-2a+1}\right) \\
&= \frac{1}{2a^2-12a+17}\left(2a^2-9a+10-a+1+\frac{-2}{a-1}+\frac{2-a}{a^2-2a+1}\right) \\
&= 1+\frac{1}{2a^2-12a+17}\left(4a-8+\frac{a}{a^2-2a+1}\right) \;<\; 2.
\end{aligned}$$

For $\alpha_2^{(3)}$ we get

$$\begin{aligned}
\alpha_2^{(3)} &= \frac{1}{2a^2-12a+17}(2a^2-8a+8+(2a^2-7a+6)\rho_2+(5-2a)\rho_2^2) \\
&> \frac{1}{2a^2-12a+17}\left(2a^2-8a+8+(2a^2-7a+6)\frac{-1}{a-2}+(5-2a)\frac{1}{a^2-4a+4}\right) \\
&= \frac{1}{2a^2-12a+17}\left({}^2-8a+8-2a+3+(5-2a)\frac{1}{a^2-4a+4}\right) \\
&= 1+\frac{1}{2a^2-12a+17}\left(-6+(5-2a)\frac{1}{a^2-4a+4}\right) \;>\; 1
\end{aligned}$$

and

$$\begin{aligned}
\alpha_2^{(3)} &= \frac{1}{2a^2-12a+17}(2a^2-8a+8+(2a^2-7a+6)\rho_2+(5-2a)\rho_2^2) \\
&< \frac{1}{2a^2-12a+17}\left(2a^2-8a+8+(2a^2-7a+6)\frac{-1}{a-1}+(5-2a)\frac{1}{a^2-2a+1}\right) \\
&= \frac{1}{2a^2-12a+17}\left(2a^2-8a+8-2a+5+\frac{-1}{a-1}+(5-2a)\frac{1}{a^2-2a+1}\right) \\
&= 1+\frac{1}{2a^2-12a+17}(2a-4+(6-3a)\frac{1}{a^2-2a+1}) \;<\; 2.
\end{aligned}$$

Hence, $a^{(3)} = (1,1)$. Now, when we used that $\frac{1}{2a^2-12a+17}(3a-7+(a^2-2a-1)\rho_2+(2-a)\rho_2^2)(a-2+a\rho_2-\rho_2^2) = 1$, it is easy to check that

$$\begin{aligned}\alpha_1^{(4)} &= \frac{\alpha_2^{(3)} - a_2^{(3)}}{\alpha_1^{(3)} - a_1^{(3)}} = \frac{\frac{1}{2a^2-12a+17}(2a^2 - 8a + 8 + (2a^2 - 7a + 6)\rho_2 + (5 - 2a)\rho_2^2) - 1}{\frac{1}{2a^2-12a+17}(2a^2 - 9a + 10 + (a^2 - 2a - 1)\rho_2 + (2 - a)\rho_2^2) - 1} \\ &= 1 - \rho_2\end{aligned}$$

and

$$\begin{aligned}\alpha_2^{(4)} &= \frac{1}{\alpha_1^{(3)} - a_1^{(3)}} = \frac{1}{\frac{1}{2a^2-12a+17}(2a^2 - 9a + 10 + (a^2 - 2a - 1)\rho_2 + (2 - a)\rho_2^2) - 1} \\ &= a - 2 + a\rho_2 - \rho_2^2.\end{aligned}$$

So, the fourth iteration is $(1 - \rho_2, a - 2 + a\rho_2 - \rho_2^2)$.

Fifth iteration

We will show the following identity $a^{(4)} = (1, a - 4)$. From the first iteration we know that $\lfloor -\rho_2 \rfloor = 0$, then it is clear that $a_1^{(4)} = \lfloor 1 - \rho_2 \rfloor = 1$. So it is enough to show $a_2^{(4)} = a - 4$:

$$\begin{aligned}\alpha_2^{(4)} &= a - 2 + a\rho_2 - \rho_2^2 > a - 2 + a\frac{-1}{a-2} - \frac{1}{a^2 - 4a + 4} \\ &= a - 2 - 1 + \frac{-2}{a-2} - \frac{1}{a^2 - 4a + 4} = a - 3 - \frac{2a - 3}{a^2 - 4a + 4} > a - 4\end{aligned}$$

and

$$\begin{aligned}\alpha_2^{(4)} &= a - 2 + a\rho_2 - \rho_2^2 < a - 2 + a\frac{-1}{a-1} - \frac{1}{a^2 - 2a + 1} \\ &= a - 2 - 1 + \frac{-1}{a-1} - \frac{1}{a^2 - 2a + 1} = a - 3 - \frac{a}{a^2 - 2a + 1} < a - 3.\end{aligned}$$

It is easy to verify that

$$\begin{aligned}\alpha_1^{(5)} &= \frac{\alpha_2^{(4)} - a_2^{(4)}}{\alpha_1^{(4)} - a_1^{(4)}} = \frac{a - 2 + a\rho_2 - \rho_2^2 - a + 4}{1 - \rho_2 - 1} \\ &= a + (2a - 1)\rho_2 + (-2)\rho_2^2\end{aligned}$$

and

$$\begin{aligned}\alpha_2^{(5)} &= \frac{1}{\alpha_1^{(4)} - a_1^{(4)}} = \frac{1}{1 - \rho_2 - 1} \\ &= a + (a - 1)\rho_2 - \rho_2^2.\end{aligned}$$

We used the fact that $(1 - \rho_2 - 1)(a + (a - 1)\rho_2 - \rho_2^2) = 1$. Therefore, $\alpha^{(5)} = (a + (2a - 1)\rho_2 + (-2)\rho_2^2, a + (a - 1)\rho_2 - \rho_2^2)$.

Sixth iteration

From the third iteration we know the following: $\lfloor a+1+(2a-1)\rho_2+(-2)\rho_2^2\rfloor = a-2$, so $a_1^{(5)} = \lfloor a+(2a-1)\rho_2+(-2)\rho_2^2\rfloor = -1+\lfloor a+1+(2a-1)\rho_2+(-2)\rho_2^2\rfloor = a-3$. And also from the third iteration we know that $a_2^{(5)} = \lfloor a+(a-1)\rho_2-\rho_2^2\rfloor = a-2$. So, $a^{(5)} = (a-3, a-2)$.

Now it easy to check that

$$\begin{aligned}\alpha_1^{(6)} &= \frac{\alpha_2^{(5)} - a_2^{(5)}}{\alpha_1^{(5)} - a_1^{(5)}} = \frac{a+(a-1)\rho_2-\rho_2^2-a+3}{a+(2a-1)\rho_2+(-2)\rho_2^2-a+2} \\ &= \frac{1}{2a^2-12a+17}(2a^2-9a+10+(a^2-2a-1)\rho_2+(2-a)\rho_2^2)\end{aligned}$$

and

$$\begin{aligned}\alpha_2^{(6)} &= \frac{1}{\alpha_1^{(5)} - a_1^{(5)}} = \frac{1}{a+(2a-1)\rho_2+(-2)\rho_2^2-a+2} \\ &= \frac{1}{2a^2-12a+17}(2a^2-8a+8+(2a^2-7a+6)\rho_2+(5-2a)\rho_2^2).\end{aligned}$$

Therefore, we prove that $\alpha^{(6)} = (\frac{1}{2a^2-12a+17}(2a^2-9a+10+(a^2-2a-1)\rho_2+(2-a)\rho_2^2), \frac{1}{2a^2-12a+17}(2a^2-8a+8+(2a^2-7a+6)\rho_2+(5-2a)\rho_2^2))$. We can see that $\alpha^{(6)} = \alpha^{(3)}$, which means that we derived the form of the JPA expansion of the couple $(|\rho_2|, \rho_2^2)$. It is periodic with period length 3 and preperiod length 3.

4.3 The third root

The JPA expansion of the couple (ρ_3, ρ_3^2) is periodic with preperiod length 2, period length 1, and the first three iterations are

$$\begin{pmatrix}\alpha^{(0)}\\ \alpha^{(1)}\\ \alpha^{(2)}\\ \alpha^{(3)}\end{pmatrix} = \begin{pmatrix}\rho_3 & \rho_3^2\\ a+\rho_3 & \rho_3+\rho_3^2\\ a+1+\rho_3 & \rho_3+\rho_3^2\\ a+1+\rho_3 & \rho_3+\rho_3^2\end{pmatrix}$$

We will use the same technique as for the other roots. Here we know that $a < \rho_3 < \frac{a^3+1}{a^2}$ (and all $a_i^{(j)} \geq 0$ from the definition of the JPA).

First iteration

We prove that $a^{(0)} = (a, a^2)$.

We know that $a < \rho_3 < a+1$, so $a_1^{(0)} = \lfloor \rho_3 \rfloor = a$. And $a^2 < \rho_3^2 < \frac{a^6+2a^3+1}{a^4} = a^2 + \frac{2a^3+1}{a^4} < a^2+1$. So, $a_2^{(0)} = \lfloor \rho_3^2 \rfloor = a^2$. From this we get that

$$\alpha_1^{(1)} = \frac{\alpha_2^{(0)} - a_2^{(0)}}{\alpha_1^{(0)} - a_1^{(0)}} = \frac{\rho_3^2 - a^2}{\rho_3 - a} = a + \rho_3$$

and

$$\alpha_2^{(1)} \;=\; \frac{1}{\alpha_1^{(0)} - a_1^{(0)}} \;=\; \frac{1}{\rho_3 - a} \;=\; \rho_3 + \rho_3^2.$$

Hence, the first iteration is $(a + \rho_3, \rho_3 + \rho_3^2)$.

Second iteration

We prove the following identity: $a^{(1)} = (2a, a + a^2)$.

From the first iteration we know that $\lfloor \rho_3 \rfloor = a$, so $a_1^{(1)} = \lfloor a + \rho_3 \rfloor = a + \lfloor \rho_3 \rfloor = 2a$. For the $a_2^{(1)}$ we compute

$$\alpha_2^{(1)} \;=\; \rho_3 + \rho_3^2 \;>\; a^2 + a$$

and

$$\alpha_2^{(1)} \;=\; \rho_3 + \rho_3^2 \;<\; \frac{a^3+1}{a^2} + \frac{a^6+2a^3+1}{a^4} \;=\; \frac{a^6+a^5+2a^3+a^2+1}{a^4} \;<\; a^2+a+1.$$

So, $a_2^{(1)} = a^2 + a$. Now it is easy to show that

$$\alpha_1^{(2)} \;=\; \frac{\alpha_2^{(1)} - a_2^{(1)}}{\alpha_1^{(1)} - a_1^{(1)}} \;=\; \frac{\rho_3 + \rho_3^2 - a^2 - a}{a + \rho_3 - 2a} \;=\; a + 1 + \rho_3$$

and

$$\alpha_2^{(2)} \;=\; \frac{1}{\alpha_1^{(1)} - a_1^{(1)}} \;=\; \frac{1}{a + \rho_3 - 2a} \;=\; \rho_3 + \rho_3^2.$$

Therefore, $\alpha^{(2)} = (a + 1 + \rho_3, \rho_3 + \rho_3^2)$.

Third iteration

From the second iteration we know that $a_1^{(2)} = \lfloor a + 1 + \rho_3 \rfloor = 1 + \lfloor a + \rho_3 \rfloor = 2a + 1$ and $a_2^{(2)} = \lfloor \rho_3 + \rho_3^2 \rfloor = a^2 + a$. Hence, $a^{(2)} = (2a + 1, a + a^2)$. And we can easily check that

$$\alpha_1^{(3)} \;=\; \frac{\alpha_2^{(2)} - a_2^{(2)}}{\alpha_1^{(2)} - a_1^{(2)}} \;=\; \frac{\rho_3 + \rho_3^2 - a^2 - a}{a + 1 + \rho_3 - 2a - 1} \;=\; a + 1 + \rho_3$$

and

$$\alpha_2^{(3)} \;=\; \frac{1}{\alpha_1^{(2)} - a_1^{(2)}} \;=\; \frac{1}{a + 1 + \rho_3 - 2a - 1} \;=\; \rho_3 + \rho_3^2.$$

So, $\alpha^{(2)} = (a + 1 + \rho_3, \rho_3 + \rho_3^2)$. We can see that the second iteration is the same as the third iteration, so we proved that the JPA expansion of the couple (ρ_3, ρ_3^2) is periodic with preperiod length 2 and period length 1.

5. The hJPA expansion

In this chapter we will find the hJPA expansion of the triple $(1, |\rho|, \rho^2)$, where ρ is one of the roots of the polynomial $x^3 - (a-1)x^2 - ax - 1$ where $a \in \mathbb{N}$, $a \geq 5$. We are use the expansions from Chapter 4 and Lemma 1, which gives us the relationship between JPA expansion and hJPA expansion. It holds that if $\alpha^{(i)}$ is the JPA expansion of the pair $(|\rho|, \rho^2)$, then i^{th} iteration of the hJPA expansion of the triple $(1, |\rho|, \rho^2)$ is given by the following relation:

$$(\delta_i, \alpha_1^{(i)}\delta_i, \alpha_2^{(i)}\delta_i)$$

where $\delta_k = \frac{1}{\alpha_2^{(1)} \cdot \ldots \cdot \alpha_2^{(k)}}$ for $k \geq 0$, $\delta_0 = 1$.

5.1 The first root

We will show that the first eleven iterations of hJPA expansion of the sequence $(1, |\rho_1|, \rho_1^2)$ are

$$\begin{pmatrix}
1 & -\rho_1 & \rho_1^2 \\
-\rho_1 & \rho_1^2 & 1 \\
\rho_1^2 & 1+\rho_1 & -\rho_1 \\
1+\rho_1 & -\rho_1-\rho_1^2 & \rho_1^2 \\
-\rho_1-\rho_1^2 & -a+3+(-a+3)\rho_1+\rho_1^2 & 1+\rho_1 \\
-a+3+(-a+3)\rho_1+\rho_1^2 & 1+2\rho_1+\rho_1^2 & -\rho_1-\rho_1^2 \\
1+2\rho_1+\rho_1^2 & a-3+(a-4)\rho_1-2\rho_1^2 & -a+3+(-a+3)\rho_1+\rho_1^2 \\
a-3+(a-4)\rho_1-2\rho_1^2 & -2a+5+(-3a+7)\rho_1+(-a+3)\rho_1^2 & 1+2\rho_1+\rho_1^2 \\
-a^2+6a-10+(-a^2+6a-13)\rho_1+(a-7)\rho_1^2 & -a^2+5a-5+(-a^2+6a-6)\rho_1+(2a-3)\rho_1^2 & a-3+(a-4)\rho_1-2\rho_1^2 \\
-a^2+5a-5+(-a^2+6a-6)\rho_1+(2a-3)\rho_1^2 & a^2-5a+7+(a^2-5a+9)\rho_1+(-a+5)\rho_1^2 & \beta_3^{(9)} \\
a^2-5a+7+(a^2-5a+9)\rho+(-a+5)\rho^2 & a-5-7\rho+(-a-4)\rho^2 & \beta_3^{(10)} \\
a-5-7\rho+(-a-4)\rho^2 & -a^3+8a^2-22a+23+(-a^3+8a^2-23a+30)\rho+(a^2-7a+17)\rho^2 & \beta_3^{(11)}
\end{pmatrix}$$

where

$$\begin{aligned}
\beta_3^{(9)} &= -a^2+6a-10+(-a^2+6a-13)\rho_1+(a-7)\rho_1^2, \\
\beta_3^{(10)} &= -a^2+5a-5+(-a^2+6a-6)\rho+(2a-3)\rho^2, \\
\beta_3^{(11)} &= a^2-5a+7+(a^2-5a+9)\rho+(-a+5)\rho^2.
\end{aligned}$$

First iteration

Firstly, we find δ_1 from Lemma 1:

$$\delta_1 = \frac{1}{\alpha_2^{(1)}} = \frac{1}{a+(a-1)\rho_1-\rho_1^2} = -\rho_1.$$

Then we count the first iteration:

$$\begin{aligned}
\beta^{(1)} &= (\delta_1, \alpha_1^{(1)}\delta_1, \alpha_1^{(i)}\delta_1) = (-\rho_1, -\rho_1(-\rho_1), -\rho_1(a+(a-1)\rho_1-\rho_1^2)) \\
&= (-\rho, \rho^2, 1).
\end{aligned}$$

In the last equality we used that $\rho_1^3 - (a-1)\rho_1^2 - a\rho_1 = 1$.

Second iteration

For δ_2 we have

$$\delta_2 = \frac{1}{\alpha_2^{(1)}\alpha_2^{(2)}} = \delta_1\frac{1}{\alpha_2^{(2)}} = -\rho_1\frac{1}{a+(a-1)\rho_1-\rho_1^2} = -\rho_1(-\rho_1) = \rho_1^2$$

and then we get

$$\beta^{(2)} = (\delta_2, \alpha_1^{(2)}\delta_2, \alpha_2^{(2)}\delta_2) = (\rho_1^2, \rho_1^2(a^2 - 2a + 1 + (a^2 - 2a + 2)\rho_1 + (-a + 1)\rho_1^2),$$
$$\rho_1^2(a + (a - 1)\rho_1 - \rho_1^2)) = (\rho_1^2, 1 + \rho_1, -\rho_1).$$

Third iteration

In the same way we find $\beta^{(3)}$,

$$\begin{aligned}\delta_3 &= \delta_2 \frac{1}{\alpha_2^{(3)}} = \rho_1^2 \frac{1}{-1 + (-a + 1)\rho_1 + \rho_1^2} \\ &= \rho_1^2(1 - 2a + a^2 + (2 - 2a + a^2)\rho_1 + (1 - a)\rho_1^2) = 1 + \rho_1\end{aligned}$$

and

$$\begin{aligned}\beta^{(3)} &= (\delta_3, \alpha_1^{(3)}\delta_3, \alpha_2^{(3)}\delta_3) \\ &= (1 + \rho_1, (1 + \rho_1)(-\rho_1), (1 + \rho_1)(-1 + (-a + 1)\rho_1 + \rho_1^2)) \\ &= (1 + \rho_1, -\rho_1 - \rho_1^2, \rho_1^2).\end{aligned}$$

Fourth iteration

For δ_4 and $\beta^{(4)}$ we have

$$\begin{aligned}\delta_4 &= \delta_3 \frac{1}{\alpha_2^{(4)}} = (1 + \rho_1) \frac{1}{a + (a - 1)\rho_1 - \rho_1^2} \\ &= (1 + \rho_1)(-\rho_1) = -\rho_1 - \rho_1^2\end{aligned}$$

and

$$\begin{aligned}\beta^{(4)} &= (\delta_4, \alpha_1^{(4)}\delta_4, \alpha_2^{(4)}\delta_4) \\ &= (-\rho_1 - \rho_1^2, (-\rho_1 - \rho_1^2)(-a^2 + 3a - 1 + (-a^2 + 3a - 3)\rho_1 + (a - 2)\rho_1^2), \\ &\quad (-\rho_1 - \rho_1^2)(a + (a - 1)\rho_1 - \rho_1^2)) \\ &= (-\rho_1 - \rho_1^2, -a + 3 + (-a + 3)\rho_1 + \rho_1^2, 1 + \rho_1).\end{aligned}$$

Fifth iteration

We will compute δ_5:

$$\begin{aligned}\delta_5 &= \delta_4 \frac{1}{\alpha_2^{(5)}} = (-\rho_1 - \rho_1^2) \frac{1}{\frac{1}{2a-7}(-2 + (-2a + 4)\rho_1 + \rho_1^2)} \\ &= (1 + \rho_1)(-1 + 3a - a^2 + (-3 + 3a - a^2)\rho_1 + (-2 + a)\rho_1^2) \\ &= -a + 3 + (-a + 3)\rho_1 + \rho_1^2.\end{aligned}$$

Then we get

$$\begin{aligned}\beta^{(5)} &= (\delta_5, \alpha_1^{(5)}\delta_5, \alpha_2^{(5)}\delta_5) \\ &= (-a + 3 + (-a + 3)\rho_1 + \rho_1^2, \\ &\quad (-a + 3 + (-a + 3)\rho_1 + \rho_1^2) \frac{1}{2a - 7}(-2 - 3\rho_1 + \rho_1^2)), \\ &\quad (-a + 3 + (-a + 3)\rho_1 + \rho_1^2) \frac{1}{2a - 7}(-2 + (-2a + 4)\rho_1 + \rho_1^2)) \\ &= (-a + 3 + (-a + 3)\rho_1 + \rho_1^2, 1 + 2\rho_1 + \rho_1^2, -\rho_1 - \rho_1^2).\end{aligned}$$

Sixth iteration

In the sixth iteration we have that

$$\begin{aligned}
\delta_6 &= \delta_5 \frac{1}{\alpha_2^{(6)}} = (-a+3+(-a+3)\rho_1+\rho_1^2)\frac{1}{-a+(-2a+1)\rho_1+2\rho_1^2} \\
&= (-a+3+(-a+3)\rho_1+\rho_1^2)\frac{-2-3\rho_1+\rho_1^2}{2a-7} \\
&= 1+2\rho_1+\rho_1^2
\end{aligned}$$

and

$$\begin{aligned}
\beta^{(6)} &= (\delta_6, \alpha_1^{(6)}\delta_6, \alpha_2^{(6)}\delta_6) \\
&= (1+2\rho_1+\rho_1^2, (1+2\rho_1+\rho_1^2)(a-1+(a-1)\rho_1-\rho_1^2)), \\
&\quad (1+2\rho_1+\rho_1^2)(-a+(-2a+1)\rho_1+2\rho_1^2)) \\
&= (1+2\rho_1+\rho_1^2, a-3+(a-4)\rho_1-2\rho_1^2, -a+3+(-a+3)\rho_1+\rho_1^2).
\end{aligned}$$

Seventh iteration

We will show that $\delta_7 = a-3+(a-4)\rho_1-2\rho_1^2$:

$$\begin{aligned}
\delta_7 &= \delta_6 \frac{1}{\alpha_2^{(7)}} = (1+2\rho_1+\rho_1^2)\frac{1}{-1+(-a)\rho_1+\rho_1^2} \\
&= (1+2\rho_1+\rho_1^2)(-1+a+(-1+a)\rho_1-\rho_1^2) \\
&= a-3+(a-4)\rho_1-2\rho_1^2
\end{aligned}$$

and then

$$\begin{aligned}
\beta^{(7)} &= (\delta_7, \alpha_1^{(7)}\delta_7, \alpha_2^{(7)}\delta_7) \\
&= (a-3+(a-4)\rho_1-2\rho_1^2, \\
&\quad (a-3+(a-4)\rho_1-2\rho_1^2)(-3+(-a+1)\rho_1+\rho_1^2)), \\
&\quad (a-3+(a-4)\rho_1-2\rho_1^2)(-1+(-a)\rho_1+\rho_1^2) \\
&= (a-3+(a-4)\rho_1-2\rho_1^2, -2a+5+(-3a+7)\rho_1+(-a+3)\rho_1^2, \\
&\quad 1+2\rho_1+\rho_1^2).
\end{aligned}$$

Eighth iteration

It is easy to see that

$$\begin{aligned}
\delta_8 &= \delta_7 \frac{1}{\alpha_2^{(8)}} = (a-3+(a-4)\rho_1-2\rho_1^2)\frac{1}{\frac{1}{2a-7}(-4+(-2a+1)\rho_1+2\rho_1^2)} \\
&= (a-3+(a-4)\rho_1-2\rho_1^2)(2-a+(1-a)\rho_1+\rho_1^2) \\
&= -a^2+6a-10+(-a^2+6a-13)\rho_1+(a-7)\rho_1^2
\end{aligned}$$

and

$$\begin{aligned}
\beta^{(8)} &= (\delta_8, \alpha_1^{(8)}\delta_8, \alpha_2^{(8)}\delta_8) \\
&= (-a^2+6a-10+(-a^2+6a-13)\rho_1+(a-7)\rho_1^2, \\
&\quad (-a^2+6a-10+(-a^2+6a-13)\rho_1+(a-7)\rho_1^2)\frac{(2a-5+3\rho_1-\rho_1^2)}{2a-7}, \\
&\quad (-a^2+6a-10+(-a^2+6a-13)\rho_1+(a-7)\rho_1^2)\frac{(-4+(-2a+1)\rho_1+2\rho_1^2)}{2a-7}) \\
&= (-a^2+6a-10+(-a^2+6a-13)\rho_1+(a-7)\rho_1^2, \\
&\quad -a^2+5a-5+(-a^2+6a-6)\rho_1+(2a-3)\rho_1^2, \\
&\quad a-3+(a-4)\rho_1-2\rho_1^2).
\end{aligned}$$

Ninth iteration
For δ_9 we have

$$\begin{aligned}
\delta_9 &= \delta_8 \frac{1}{\alpha_2^{(9)}} \\
&= (-a^2+6a-10+(-a^2+6a-13)\rho_1+(a-7)\rho_1^2)\frac{1}{\frac{(a^2-6a+7-(2a-4)\rho_1+\rho_1^2)}{a^2-5a+5}} \\
&= (-a^2+6a-10+(-a^2+6a-13)\rho_1+(a-7)\rho_1^2)\frac{-5+2a+3\rho_1-\rho_1^2}{2a-7} \\
&= -a^2+5a-5+(-a^2+6a-6)\rho_1+(2a-3)\rho_1^2
\end{aligned}$$

and we get

$$\begin{aligned}
\beta^{(9)} &= (\delta_9, \alpha_1^{(9)}\delta_9, \alpha_2^{(9)}\delta_9) \\
&= (-a^2+5a-5+(-a^2+6a-6)\rho_1+(2a-3)\rho_1^2, \\
&\quad (-a^2+5a-5+(-a^2+6a-6)\rho_1+(2a-3)\rho_1^2)\frac{-(a-2)^2-(a^2-3a+3)\rho_1+(a-2)\rho_1^2}{a^2-5a+5}, \\
&\quad (-a^2+5a-5+(-a^2+6a-6)\rho_1+(2a-3)\rho_1^2)\frac{a^2-6a+7-(2a-4)\rho_1+\rho_1^2}{a^2-5a+5}) \\
&= (-a^2+5a-5+(-a^2+6a-6)\rho_1+(2a-3)\rho_1^2, \\
&\quad a^2-5a+7+(a^2-5a+9)\rho_1+(-a+5)\rho_1^2, \\
&\quad -a^2+6a-10+(-a^2+6a-13)\rho_1+(a-7)\rho_1^2).
\end{aligned}$$

Tenth iteration
We can verify that

$$\begin{aligned}
\delta_{10} &= \delta_9 \frac{1}{\alpha_2^{(10)}} \\
&= \delta_9 \frac{1}{-2+(-a+1)\rho_1+\rho_1^2} \\
&= \delta_9 \frac{8-10a+6a^2-a^3+(8-10a+5a^2-a^3)\rho_1+(4-4a+a^2)\rho_1^2}{-8+8a-6a^2+a^3} \\
&= a^2-5a+7+(a^2-5a+9)\rho_1+(-a+5)\rho_1^2
\end{aligned}$$

and

$$\begin{aligned}
\beta^{(10)} &= (\delta_{10}, \alpha_1^{(10)}\delta_{10}, \alpha_2^{(10)}\delta_{10}) \\
&= (a^2-5a+7+(a^2-5a+9)\rho_1+(-a+5)\rho_1^2, \\
&\quad (a^2-5a+7+(a^2-5a+9)\rho_1+(-a+5)\rho_1^2)(-\rho_1), \\
&\quad (a^2-5a+7+(a^2-5a+9)\rho_1+(-a+5)\rho_1^2)(-2+(-a+1)\rho_1+\rho_1^2)) \\
&= (a^2-5a+7+(a^2-5a+9)\rho_1+(-a+5)\rho_1^2, a-5-7\rho_1+(-a-4)\rho_1^2, \\
&\quad -a^2+5a-5+(-a^2+6a-6)\rho_1+(2a-3)\rho_1^2).
\end{aligned}$$

Eleventh iteration
The last iteration is

$$\begin{aligned}
\delta_{11} &= \delta_{10}\frac{1}{\alpha_2^{(11)}} \\
&= (a^2-5a+7+(a^2-5a+9)\rho+(-a+5)\rho^2)\frac{1}{a+(a-1)\rho_1-\rho_1^2} \\
&= (a^2-5a+7+(a^2-5a+9)\rho+(-a+5)\rho^2)(-\rho_1) \\
&= a-5-7\rho+(-a-4)\rho^2
\end{aligned}$$

and

$$\begin{aligned}
\beta^{(11)} &= (\delta_{11}, \alpha_1^{(11)}\delta_{11}, \alpha_2^{(11)}\delta_{11}) \\
&= (a-5-7\rho+(-a-4)\rho^2, \\
&\quad (a-5-7\rho+(-a-4)\rho^2)(-a^2+3a-1+(-a^2+3a-3)\rho_1+(a-2)\rho_1^2), \\
&\quad (a-5-7\rho+(-a-4)\rho^2)(a+(a-1)\rho_1-\rho_1^2)) \\
&= (a-5-7\rho+(-a-4)\rho^2, \\
&\quad -a^3+8a^2-22a+23+(-a^3+8a^2-23a+30)\rho+(a^2-7a+17)\rho^2, \\
&\quad a^2-5a+7+(a^2-5a+9)\rho+(-a+5)\rho^2).
\end{aligned}$$

We found how the first eleven iterations of the hJPA expansion of the sequence $(1, |\rho_1|, \rho_1^2)$ look like. From Lemma 2 it follows that this expansion is periodic with preperiod length 4 and period length 7. It is easy to verify that for $\epsilon = \prod_{i=4}^{10} \alpha_2^{(i)} = 2-a^2+(-1+3a+a^2-a^3)\rho_1+(-2-a+a^2)\rho_1^2$ is $\beta^{(4)}\epsilon^{-1} = \beta^{(11)}$.

5.2 The second root

We will show that the first six iterations of hJPA expansion of the sequence $(1, |\rho_2|, \rho_2^2)$ are

$$\begin{pmatrix}
1 & -\rho_2 & \rho_2^2 \\
-\rho_2 & \rho_2^2 & 1 \\
\rho_2^2 & 1+(a-2)\rho_2 & -\rho_2 \\
1+(-2+a)\rho_2+(-a+2)\rho_2^2 & -\rho_2+(-a+2)\rho_2^2 & \rho_2^2 \\
-1+(-a+1)\rho_2 & -1+(-a+2)\rho_2+(a-1)\rho_2^2 & 1+(-2+a)\rho_2+(-a+2)\rho_2^2 \\
\rho_2+(a-1)\rho_2^2 & a-3+(a^2-4a+2)\rho_2+(-a+2)\rho_2^2 & -1+(-a+1)\rho_2 \\
a-3+(a^2-5a+5)\rho_2+(-a^2+3a-1)\rho_2^2 & -1+(-2a+3)\rho_2+(-a^2+3a-2)\rho_2^2 & \rho_2+(a-1)\rho_2^2
\end{pmatrix}.$$

Again, we will use Lemma 1 that gives us the relationship between JPA and hJPA expansion of the sequences $(|\rho_2|, \rho_2^2)$ and $(1, |\rho_2|, \rho_2^2)$.

First iteration
Firstly, we find δ_1

$$\delta_1 = \frac{1}{\alpha_2^1} = \frac{1}{a+(a-1)\rho_2-\rho_2^2} = -\rho_2.$$

Then we get

$$\begin{aligned}
\beta^{(1)} &= (\delta_1, \alpha_1^{(1)}\delta_1, \alpha_2^{(1)}\delta_1) \;=\; (-\rho_2, -\rho_2(-\rho_2), -\rho_2(a+(a-1)\rho_2-\rho_2^2)) \\
&= (-\rho_2, \rho_2^2, 1).
\end{aligned}$$

Second iteration
The second iteration is

$$\delta_2 \;=\; \delta_1 \frac{1}{\alpha_2^{(2)}} \;=\; -\rho_2 \frac{1}{a+(a-1)\rho_2-\rho_2^2} = -\rho_2(-\rho_2) \;=\; \rho_2^2$$

and

$$\begin{aligned}
\beta^{(2)} &= (\delta_2, \alpha_1^{(2)}\delta_2, \alpha_2^{(2)}\delta_2) \\
&= (\rho_2^2, \rho_2^2(a+1+(2a-1)\rho_2+(-2)\rho_2^2), \rho_2^2(a+(a-1)\rho_2-\rho_2^2)) \\
&= (\rho_2^2, 1+(a-2)\rho_2, -\rho_2).
\end{aligned}$$

Third iteration
We will compute δ_3 and $\beta^{(3)}$:

$$\begin{aligned}
\delta_3 &= \delta_2 \frac{1}{\alpha_2^{(3)}} \;=\; \rho_2^2 \frac{1}{\frac{1}{2a^2-12a+17}(2a^2-8a+8+(2a^2-7a+6)\rho_2+(5-2a)\rho_2^2)} \\
&= \rho_2^2(3+(2a-1)\rho_2-2\rho_2^2) \;=\; 1+(-2+a)\rho_2+(-a+2)\rho_2^2,
\end{aligned}$$

$$\begin{aligned}
\beta^{(3)} &= (\delta_3, \alpha_1^{(3)}\delta_3, \alpha_2^{(3)}\delta_3) \\
&= (1+(-2+a)\rho_2+(-a+2)\rho_2^2, \\
&\quad ((1+(-2+a)\rho_2+(-a+2)\rho_2^2)\frac{2a^2-9a+10+(a^2-2a-1)\rho_2+(2-a)\rho_2^2}{2a^2-12a+17}, \\
&\quad (1+(-2+a)\rho_2+(-a+2)\rho_2^2)\frac{2a^2-8a+8+(2a^2-7a+6)\rho_2+(5-2a)\rho_2^2}{2a^2-12a+17}) \\
&= (1+(-2+a)\rho_2+(-a+2)\rho_2^2, -\rho_2+(-a+2)\rho_2^2, \rho_2^2).
\end{aligned}$$

Fourth iteration
As usual we find δ_4:

$$\begin{aligned}
\delta_4 &= \delta_3 \frac{1}{\alpha_2^{(4)}} \;=\; (1+(-2+a)\rho_2+(-a+2)\rho_2^2)\frac{1}{a-2+a\rho_2-\rho_2^2} \\
&= (1+(-2+a)\rho_2+(-a+2)\rho_2^2)\frac{-7+3a+(-1-2a+a^2)\rho_2+(2-a)\rho_2^2}{17-12a+2a^2} \\
&= -1+(-a+1)\rho_2
\end{aligned}$$

and then we get

$$\begin{aligned}
\beta^{(4)} &= (\delta_4, \alpha_1^{(4)}\delta_4, \alpha_2^{(4)}\delta_4) \\
&= (-1+(-a+1)\rho_2, (-1+(-a+1)\rho_2)(1-\rho_2), \\
&\quad (-1+(-a+1)\rho_2)(a-2+a\rho_2-\rho_2^2)) \\
&= (-1+(-a+1)\rho_2, -1+(-a+2)\rho_2+(a-1)\rho_2^2, \\
&\quad 1+(-2+a)\rho_2+(-a+2)\rho_2^2).
\end{aligned}$$

Fifth iteration
We can see that

$$\begin{aligned}\delta_5 &= \delta_4\frac{1}{\alpha_2^{(5)}} = (-1+(-a+1)\rho_2)\frac{1}{a+(a-1)\rho_2-\rho_2^2}\\ &= (-1+(-a+1)\rho_2)(-\rho_2)\\ &= \rho_2+(a-1)\rho_2^2\end{aligned}$$

and

$$\begin{aligned}\beta^{(5)} &= (\delta_5, \alpha_1^{(5)}\delta_5, \alpha_2^{(5)}\delta_5)\\ &= (\rho_2+(a-1)\rho_2^2, (\rho_2+(a-1)\rho_2^2)(a+(2a-1)\rho_2+(-2)\rho_2^2),\\ &\quad (\rho_2+(a-1)\rho_2^2)(a+(a-1)\rho_2-\rho_2^2))\\ &= (\rho_2+(a-1)\rho_2^2, a-3+(a^2-4a+2)\rho_2+(-a+2)\rho_2^2, -1+(-a+1)\rho_2).\end{aligned}$$

Sixth iteration
The last iteration is

$$\begin{aligned}\delta_6 &= \delta_5\frac{1}{\alpha_2^{(6)}}\\ &= (\rho_2+(a-1)\rho_2^2)\frac{1}{\frac{1}{2a^2-12a+17}(2a^2-8a+8+(2a^2-7a+6)\rho_2+(5-2a)\rho_2^2)}\\ &= (\rho_2+(a-1)\rho_2^2)(3+(-1+2a)\rho_2-2\rho_2^2)\\ &= a-3+(a^2-5a+5)\rho_2+(-a^2+3a-1)\rho_2^2\end{aligned}$$

and

$$\begin{aligned}\beta^{(6)} &= (\delta_6, \alpha_1^{(6)}\delta_6, \alpha_2^{(6)}\delta_6)\\ &= (a-3+(a^2-5a+5)\rho_2+(-a^2+3a-1)\rho_2^2,\\ &\quad \delta_6\frac{2a^2-9a+10+(a^2-2a-1)\rho_2+(2-a)\rho_2^2}{2a^2-12a+17},\\ &\quad \delta_6\frac{2a^2-8a+8+(2a^2-7a+6)\rho_2+(5-2a)\rho_2^2}{2a^2-12a+17})\\ &= (a-3+(a^2-5a+5)\rho_2+(-a^2+3a-1)\rho_2^2,\\ &\quad -1+(-2a+3)\rho_2+(-a^2+3a-2)\rho_2^2,\\ &\quad \rho_2+(a-1)\rho_2^2).\end{aligned}$$

We determined the first sixth iterations of the hJPA expansion of the sequence $(1, |\rho_2|, \rho_2^2)$. From Lemma 2 it follows that this expansion is periodic with preperiod length 3 and period length 3. We can verify that $(1+(-2+a)\rho_2+(-a+2)\rho_2^2, -\rho_2+(-a+2)\rho_2^2, \rho_2^2)\epsilon^{-1} = (a-3+(a^2-5a+5)\rho_2+(-a^2+3a-1)\rho_2^2, -1+(-2a+3)\rho_2+(-a^2+3a-2)\rho_2^2, \rho_2+(a-1)\rho_2^2)$, where $\epsilon = \prod_{i=3}^{6}\alpha_2^{(i)} = -1+(-1+a)\rho_2+(-2+2a-a^2)\rho_2^2$. Hence, $\beta^{(3)}\epsilon^{-1} = \beta^{(6)}$ and the Hasse-Bernstein unit is $-1+(-1+a)\rho_2+(-2+2a-a^2)\rho_2^2$.

5.3 The third root

We will show that the first three iterations of hJPA expansion of the sequence $(1, |\rho_3|, \rho_2^3)$ are

$$\begin{pmatrix} 1 & -\rho_3 & \rho_3^2 \\ -a+\rho_3 & -a^2+\rho_3^2 & 1 \\ a^2-2a\rho_3+\rho_3^2 & 1+a^2+a^3+(-a-a^2)\rho_3 & -a+\rho_3 \\ 1-a^3+(a+3a^2)\rho_3+(-1-2a)\rho_3^2 & -a-a^3-a^4+(1+2a^2+2a^3)\rho_3+(-a-a^2)\rho_3^2 & a^2-2a\rho_3+\rho_3^2 \end{pmatrix}.$$

First iteration

In the first iteration we have that

$$\delta_1 = \frac{1}{\alpha_2^{(1)}} = \frac{1}{\rho_3+\rho_3^2} = -a+\rho_3$$

and

$$\begin{aligned} \beta^{(1)} &= (\delta_1, \alpha_1^{(1)}\delta_1, \alpha_1^{(1)}\delta_1) \\ &= (-a+\rho_3, (-a+\rho_3)(a+\rho_3), (-a+\rho_3)(\rho_3+\rho_3^2)) \\ &= (-a+\rho_3, -a^2+\rho_3^2, 1). \end{aligned}$$

Second iteration

Then, we get

$$\delta_2 = \delta_1\frac{1}{\alpha_2^{(2)}} = (-a+\rho_3)\frac{1}{\rho_3+\rho_3^2} = (-a+\rho_3)(-a+\rho_3) = a^2-2a\rho_3+\rho_3^2$$

and

$$\begin{aligned} \beta^{(2)} &= (\delta_2, \alpha_1^{(2)}\delta_2, \alpha_2^{(2)}\delta_2) \\ &= (a^2-2a\rho_3+\rho_3^2, (a^2-2a\rho_3+\rho_3^2)(a+1+\rho_3), (a^2-2a\rho_3+\rho_3^2)(\rho_3+\rho_3^2)) \\ &= (a^2-2a\rho_3+\rho_3^2, 1+a^2+a^3+(-a-a^2)\rho_3, -a+\rho_3). \end{aligned}$$

Third iteration

We can see that the last equation is

$$\begin{aligned} \delta_3 &= \delta_2\frac{1}{\alpha_2^{(3)}} = (a^2-2a\rho_3+\rho_3^2)\frac{1}{\rho_3+\rho_3^2} \\ &= (a^2-2a\rho_3+\rho_3^2)(-a+\rho_3) = 1-a^3+(a+3a^2)\rho_3+(-1-2a)\rho_3^2, \end{aligned}$$

$$\begin{aligned} \beta^{(3)} &= (\delta_3, \alpha_1^{(3)}\delta_3, \alpha_2^{(3)}\delta_3) \\ &= (1-a^3+(a+3a^2)\rho_3+(-1-2a)\rho_3^2, \\ &\quad (1-a^3+(a+3a^2)\rho_3+(-1-2a)\rho_3^2)(a+1+\rho_3), \\ &\quad (1-a^3+(a+3a^2)\rho_3+(-1-2a)\rho_3^2)(\rho_3+\rho_3^2)) \\ &= (1-a^3+(a+3a^2)\rho_3+(-1-2a)\rho_3^2, \\ &\quad -a-a^3-a^4+(1+2a^2+2a^3)\rho_3+(-a-a^2)\rho_3^2, \\ &\quad a^2-2a\rho_3+\rho_3^2). \end{aligned}$$

We derived the first three iterations of the hJPA expansion of the sequence $(1, |\rho_3|, \rho_3^2)$. From Lemma 2 it follows that this expansion is periodic with preperiod length 2 and period length 1. We can verify that $(a^2-2a\rho_3+\rho_3^2, 1+a^2+a^3+(-a-a^2)\rho_3, -a+\rho_3)\epsilon^{-1} = (1-a^3+(a+3a^2)\rho_3+(-1-2a)\rho_3^2, -a-a^3-a^4+(1+2a^2+2a^3)\rho_3+(-a-a^2)\rho_3^2, a^2-2a\rho_3+\rho_3^2)$, where $\epsilon = \alpha_2^{(2)} = \rho_3+\rho_3^2$. Hence, $\beta^{(2)}\epsilon^{-1} = \beta^{(3)}$ and the Hasse-Bernstein unit is $\rho_3+\rho_3^2$.

6. Indecomposable elements in the expansions

In this chapter, we will compare the convergents from the previous Chapter and the indecomposable elements from Chapter 2. As we say in Chapter 2 the important tools for comparing these elements are their norms and signatures. So, we will start with computing norms and signatures of convergents from Chapter 5.

6.1 Norms and signatures of convergents

We know that the hJPA expansion of the triple $(1, |\rho_1|, \rho_1^2)$ is periodic with preperiod length 4, period length 7, and the first eleven iterations are

6.1.1 First root

$$\begin{pmatrix}
1 & -\rho_1 & \rho_1^2 \\
-\rho_1 & \rho_1^2 & 1 \\
\rho_1^2 & 1+\rho_1 & -\rho_1 \\
1+\rho_1 & -\rho_1-\rho_1^2 & \rho_1^2 \\
-\rho_1-\rho_1^2 & -a+3+(-a+3)\rho_1+\rho_1^2 & 1+\rho_1 \\
-a+3+(-a+3)\rho_1+\rho_1^2 & 1+2\rho_1+\rho_1^2 & -\rho_1-\rho_1^2 \\
1+2\rho_1+\rho_1^2 & a-3+(a-4)\rho_1-2\rho_1^2 & -a+3+(-a+3)\rho_1+\rho_1^2 \\
a-3+(a-4)\rho_1-2\rho_1^2 & -2a+5+(-3a+7)\rho_1+(-a+3)\rho_1^2 & 1+2\rho_1+\rho_1^2 \\
-a^2+6a-10+(-a^2+6a-13)\rho_1+(a-7)\rho_1^2 & -a^2+5a-5+(-a^2+6a-6)\rho_1+(2a-3)\rho_1^2 & a-3+(a-4)\rho_1-2\rho_1^2 \\
-a^2+5a-5+(-a^2+6a-6)\rho_1+(2a-3)\rho_1^2 & a^2-5a+7+(a^2-5a+9)\rho_1+(-a+5)\rho_1^2 & \beta_3^{(9)} \\
a^2-5a+7+(a^2-5a+9)\rho+(-a+5)\rho^2 & a-5-7\rho+(-a-4)\rho^2 & \beta_3^{(10)} \\
a-5-7\rho+(-a-4)\rho^2 & -a^3+8a^2-22a+23+(-a^3+8a^2-23a+30)\rho+(a^2-7a+17)\rho^2 & \beta_3^{(11)}
\end{pmatrix}$$

where

$$\begin{aligned}
\beta_3^{(9)} &= -a^2+6a-10+(-a^2+6a-13)\rho_1+(a-7)\rho_1^2, \\
\beta_3^{(10)} &= -a^2+5a-5+(-a^2+6a-6)\rho+(2a-3)\rho^2, \\
\beta_3^{(11)} &= a^2-5a+7+(a^2-5a+9)\rho+(-a+5)\rho^2.
\end{aligned}$$

Norms and signatures of these elements are in the following table. We will calculate them in the same way as norms and signatures of the indecomposable elements in Chapter 2. We write here only the determination of signatures for some more complicated elements, i.e. the elements $\beta_2^{(4)}, \beta_2^{(6)}, \beta_2^{(7)}, \beta_1^{(8)}, \beta_2^{(8)}$ and $\beta_2^{(9)}$.

	$N(\beta_1)$	$Sgn(\beta_1)$	$N(\beta_2)$	$Sgn(\beta_2)$	$N(\beta_3)$	$Sgn(\beta_3)$
$\beta^{(0)}$	1	$(+,+,+)$	-1	$(+,+,-)$	1	$(+,+,+)$
$\beta^{(1)}$	-1	$(+,+,-)$	1	$(+,+,+)$	1	$(+,+,+)$
$\beta^{(2)}$	1	$(+,+,+)$	1	$(+,+,+)$	-1	$(+,+,-)$
$\beta^{(3)}$	1	$(+,+,+)$	-1	$(+,+,-)$	1	$(+,+,+)$
$\beta^{(4)}$	-1	$(+,+,-)$	$-2a+7$	$(+,-,+)$	1	$(+,+,+)$
$\beta^{(5)}$	$-2a+7$	$(+,-,+)$	1	$(+,+,+)$	-1	$(+,+,-)$
$\beta^{(6)}$	1	$(+,+,+)$	-1	$(+,+,-)$	$-2a+7$	$(+,-,+)$
$\beta^{(7)}$	-1	$(+,+,-)$	$2a^2-14a+23$	$(+,-,-)$	1	$(+,+,+)$
$\beta^{(8)}$	$2a-7$	$(+,-,-)$	$-a^2+5a-5$	$(+,-,+)$	-1	$(+,+,-)$
$\beta^{(9)}$	$-a^2+5a-5$	$(+,-,+)$	1	$(+,+,+)$	$2a-7$	$(+,-,-)$
$\beta^{(10)}$	1	$(+,+,+)$	-1	$(+,+,-)$	$-a^2+5a-5$	$(+,-,+)$
$\beta^{(11)}$	-1	$(+,+,-)$	$-2a+7$	$(+,-,+)$	1	$(+,+,+)$

For the computation of signatures we use the fact that every convergent is positive and that $-1 < -\frac{1}{a-2} < \rho_2 < -\frac{1}{a-1} < 0$ and $a < \rho_3 < a + \frac{1}{a^2}$.

The element $\beta_2^{(4)}$ has the signature $(+,-,+)$, which we can check in the following way

$$\lambda_{1,2}(\beta_2^{(4)}) \;=\; -a+3+(-a+3)\rho_2+\rho_2^2 \;<\; -a+3+(-a+3)(-1)+0^2 \;=\; 0$$

and

$$\begin{aligned}\lambda_{1,3}(\beta_2^{(4)}) &= -a+3+(-a+3)\rho_3+\rho_3^2 \;>\; -a+3+(-a+3)(a+\frac{1}{a^2})+a^2\\ &= \frac{2a^3+3a^2-a+3}{a^2} \;>\; 0.\end{aligned}$$

We can see that $Sgn(\beta_2^{(6)}) = (+,+,-)$:

$$\begin{aligned}\lambda_{1,2}(\beta_2^{(6)}) &= a-3+(a-4)\rho_2-2\rho_2^2 > a-3+(a-4)\frac{-1}{a-2}-2\frac{1}{(a-2)^2}\\ &= a-4+\frac{2a-6}{(a-2)^2} \;>\; 0,\end{aligned}$$

$$\begin{aligned}\lambda_{1,3}(\beta_2^{(6)}) &= a-3+(a-4)\rho_3-2\rho_3^2 < a-3+(a-4)(a+\frac{1}{a^2})-2a^2\\ &= -a^2-3a-3+\frac{a-4}{a^2} \;<\; 0.\end{aligned}$$

The element $\beta_2^{(7)}$ has the signature $(+,-,-)$, because

$$\begin{aligned}\lambda_{1,2}(\beta_2^{(7)}) &= -2a+5+(-3a+7)\rho_2+(-a+3)\rho_2^2\\ &< -2a+5+(-3a+7)\frac{-1}{a-2}+(-a+3)\frac{1}{(a-1)^2}\\ &= -2a+8+\frac{-2a+4}{(a-1)^2} \;<\; 0,\end{aligned}$$

$$\begin{aligned}\lambda_{1,3}(\beta_2^{(7)}) &= -2a+5+(-3a+7)\rho_3+(-a+3)\rho_3^2\\ &< -2a+5+(-3a+7)a+(-a+3)a^2\\ &= -a^3+5a+5 \ < \ 0.\end{aligned}$$

We can verify that the element $\beta_1^{(8)}$ has the signature $(+,-,-)$:

$$\begin{aligned}\lambda_{1,2}(\beta_1^{(8)}) &= -a^2+6a-10+(-a^2+6a-13)\rho_2+(a-7)\rho_2^2\\ &< -a^2+6a-10+(-a^2+6a-13)\frac{-1}{a-2}+(a-7)\frac{1}{(a-2)^2}\\ &= \frac{-a^4+11a^3-46a^2+90a-73)}{(-2+a)^2} \ < \ 0,\end{aligned}$$

$$\begin{aligned}\lambda_{1,3}(\beta_1^{(8)}) &= -a^2+6a-10+(-a^2+6a-13)\rho_3+(a-7)\rho_3^2\\ &< -a^2+6a-10+(-a^2+6a-13)a+(a-7)(a+\frac{1}{a^2})^2\\ &= -a^3+6a^2-14a-10+\frac{a-7}{a^4} \ < \ 0.\end{aligned}$$

We can calculate that $Sgn(\beta_2^{(8)})=(+,-,+)$:

$$\begin{aligned}\lambda_{1,2}(\beta_2^{(8)}) &= -a^2+5a-5+(-a^2+6a-6)\rho_2+(2a-3)\rho_2^2\\ &< -a^2+5a-5+(-a^2+6a-6)\frac{-1}{a-2}+(2a-3)\frac{1}{(a-2)^2}\\ &= \frac{-a^4+10a^3-37a^2+60a-35}{(-2+a)^2} \ < \ 0\end{aligned}$$

and

$$\begin{aligned}\lambda_{1,3}(\beta_2^{(8)}) &= -a^2+5a-5+(-a^2+6a-6)\rho_3+(2a-3)\rho_3^2\\ &> -a^2+5a-5+(-a^2+6a-6)(a+\frac{1}{a^2})+(2a-3)a^2\\ &= a^3+2a^2-a-6+\frac{6a-6}{a^2} \ > \ 0.\end{aligned}$$

And finally the signature of the element $\beta_2^{(9)}$ is $(+,+,+)$:

$$\begin{aligned}\lambda_{1,2}(\beta_2^{(9)}) &= a^2-5a+7+(a^2-5a+9)\rho_2+(-a+5)\rho_2^2\\ &> a^2-5a+7+(a^2-5a+9)\frac{-1}{a-2}+(-a+5)\frac{1}{(a-2)^2}\\ &= \frac{a^4-10a^3+38a^2-68a+51}{(-2+a)^2} \ > \ 0\end{aligned}$$

and

$$\begin{aligned}\lambda_{1,3}(\beta_2^{(9)}) &= a^2-5a+7+(a^2-5a+9)\rho_3+(-a+5)\rho_3^2\\ &> a^2-5a+7+(a^2-5a+9)a+(-a+5)(a+\frac{1}{a^2})^2\\ &= a^2+4a+5+\frac{10a^3-a+5}{a^4} \ > \ 0.\end{aligned}$$

From the fifth chapter we know that the eleventh iteration is the same as the fourth iteration multiplied by the unit $a^2-4a+4+(a^2-4a+5)\rho_1+(-a+3)\rho_1^2$. It is easy to verify that this unit has the signature $(+,+,+)$, so the elements in the eleventh iteration have the same norms and signatures as the element in the fourth iteration.

6.1.2 Second root

For the second root, we know that the hJPA expansion of the triple $(1,|\rho_2|,\rho_2^2)$ is periodic with preperiod length 3, period length 3, and the first six iterations are

$$\begin{pmatrix} 1 & -\rho_2 & \rho_2^2 \\ -\rho_2 & \rho_2^2 & 1 \\ \rho_2^2 & 1+(a-2)\rho_2 & -\rho_2 \\ 1+(-2+a)\rho_2+(-a+2)\rho_2^2 & -\rho_2+(-a+2)\rho_2^2 & \rho_2^2 \\ -1+(-a+1)\rho_2 & -1+(-a+2)\rho_2+(a-1)\rho_2^2 & 1+(-2+a)\rho_2+(-a+2)\rho_2^2 \\ \rho_2+(a-1)\rho_2^2 & a-3+(a^2-4a+2)\rho_2+(-a+2)\rho_2^2 & -1+(-a+1)\rho_2 \\ a-3+(a^2-5a+5)\rho_2+(-a^2+3a-1)\rho_2^2 & -1+(-2a+3)\rho_2+(-a^2+3a-2)\rho_2^2 & \rho_2+(a-1)\rho_2^2 \end{pmatrix}.$$

The norms and signatures of these elements are in the following table. We again show just computation of signatures for some complicated elements, i.e. elements $\beta_1^{(3)}, \beta_2^{(4)}$ and $\beta_2^{(5)}$.

	$N(\beta_1)$	$Sgn(\beta_1)$	$N(\beta_2)$	$Sgn(\beta_2)$	$N(\beta_3)$	$Sgn(\beta_3)$
$\beta^{(0)}$	1	$(+,+,+)$	-1	$(+,+,-)$	1	$(+,+,+)$
$\beta^{(1)}$	-1	$(+,+,-)$	1	$(+,+,+)$	1	$(+,+,+)$
$\beta^{(2)}$	1	$(+,+,+)$	$-a^2+5a-5$	$(-,+,+)$	-1	$(+,+,-)$
$\beta^{(3)}$	$2a^2-12a+17$	$(-,+,-)$	a^2-5a+5	$(-,+,-)$	1	$(+,+,+)$
$\beta^{(4)}$	-1	$(+,+,-)$	$2a-1$	$(+,+,+)$	$2a^2-12a+17$	$(-,+,-)$
$\beta^{(5)}$	1	$(+,+,+)$	$2a-7$	$(-,+,-)$	-1	$(+,+,-)$
$\beta^{(6)}$	$2a^2-12a+17$	$(-,+,-)$	a^2-5a+5	$(-,+,-)$	1	$(+,+,+)$

For the computation of norms we again use that every convergent in the expansion is positive and that $-\frac{a-1}{a}<\rho_1<-\frac{a-2}{a-1}$, $a<\rho_3<a+\frac{1}{a^2}$.

We get that $Sgn(\beta_1^{(3)})=(-,+,-)$:

$$\begin{aligned} \lambda_{2,1}(\beta_1^{(3)}) &= 1+(-2+a)\rho_1+(-a+2)\rho_1^2 \\ &< 1+(-2+a)\frac{-a+2}{a-1}+(-a+2)\frac{(a-2)^2}{(a-1)^2} \\ &= \frac{-2a^3+12a^2-22a+13}{(-1+a)^2} \;<\; 0, \end{aligned}$$

$$\begin{aligned} \lambda_{2,3}(\beta_1^{(3)}) &= 1+(-2+a)\rho_3+(-a+2)\rho_3^2 \\ &< 1+(-2+a)\left(a+\frac{1}{a^2}\right)+(-a+2)a^2 \\ &= -a^3+3a^2-2a+1\frac{a-2}{a^2} \;<\; 0. \end{aligned}$$

We can verify that the $Sgn(\beta_2^{(4)})=(+,+,+)$:

$$\begin{aligned}
\lambda_{2,1}(\beta_2^{(4)}) &= -1+(-a+2)\rho_1+(a-1)\rho_1^2 \\
&> -1+(-a+2)\frac{-a+2}{a-1}+(a-1)\frac{(a-2)^2}{(a-1)^2} \\
&= \frac{2a^2-9a+9}{a-1} > 0,
\end{aligned}$$

$$\begin{aligned}
\lambda_{2,3}(\beta_2^{(4)}) &= -1+(-a+2)\rho_3+(a-1)\rho_3^2 \\
&> -1+(-a+2)\left(a+\frac{1}{a^2}\right)+(a-1)a^2 \\
&= a^3-2a^2+2a-1\frac{-1+2}{a^2} > 0.
\end{aligned}$$

And the last element is $\beta_2^{(5)}$, which has signature $(-,+,-)$:

$$\begin{aligned}
\lambda_{2,1}(\beta_2^{(5)}) &= a-3+(a^2-4a+2)\rho_1+(-a+2)\rho_1^2 \\
&< a-3+(a^2-4a+2)\frac{-a+2}{a-1}+(-a+2)\frac{(a-2)^2}{(a-1)^2} \\
&= \frac{-a^4+7a^3-15a^2+9a+1}{(-1+a)^2} < 0
\end{aligned}$$

and

$$\begin{aligned}
\lambda_{2,3}(\beta_2^{(5)}) &= a-3+(a^2-4a+2)\rho_3+(-a+2)\rho_3^2 \\
&< a-3+(a^2-4a+2)\left(a+\frac{1}{a^2}\right)+(-a+2)a^2 \\
&= -2a^2+3a-2+\frac{-4a+2}{a^2} < 0.
\end{aligned}$$

From the previous chapter we know, that the sixth iteration is equal to the third iteration multiplied by the unit $-1+(a-1)\rho_2+\rho_2^2$, which has signature $(+,+,+)$. Hence, the elements in the sixth iteration have the same signatures and norms as the element in the third iteration.

6.1.3 Third root

From the previous chapter, we know that expansion of the sequence $(1,|\rho_3|,\rho_2^3)$ are periodic with preperiod length 2, period length 1. The first three iterations are:

$$\begin{pmatrix}
1 & -\rho_3 & \rho_3^2 \\
-a+\rho_3 & -a^2+\rho_3^2 & 1 \\
a^2-2a\rho_3+\rho_3^2 & 1+a^2+a^3+(-a-a^2)\rho_3 & -a+\rho_3 \\
1-a^3+(a+3a^2)\rho_3+(-1-2a)\rho_3^2 & -a-a^3-a^4+(1+2a^2+2a^3)\rho_3+(-a-a^2)\rho_3^2 & a^2-2a\rho_3+\rho_3^2
\end{pmatrix}.$$

The norms and signatures of these elements are in the following table. The signatures here are relatively obvious, so we have here this table without any computation.

	$N(\beta_1)$	$Sgn(\beta_1)$	$N(\beta_2)$	$Sgn(\beta_2)$	$N(\beta_3)$	$Sgn(\beta_3)$
$\beta^{(0)}$	1	$(+,+,+)$	1	$(-,-,+)$	1	$(+,+,+)$
$\beta^{(1)}$	1	$(-,-,+)$	$1-2a+2a^3$	$(-,-,+)$	1	$(+,+,+)$
$\beta^{(2)}$	1	$(+,+,+)$	$1+a+3a^2+2a^3$	$(+,+,+)$	1	$(-,-,+)$
$\beta^{(3)}$	1	$(-,-,+)$	$1+a+3a^2+2a^3$	$(-,-,+)$	1	$(+,+,+)$

We know that the third iteration is the second iteration multiplied by the unit $-a+\rho_3$, which has signature $(-,-,+)$. (Hence, the signature of the third and second iteration are not the same.)

6.2 Comparing convergents and indecomposable element

From the third chapter we have that the indecoposable elements up to multiplication by units in the field $\mathbb{Q}(\rho)$ (where ρ is one of the roots of polynomial $x^3-(a-1)x^2-ax-1$ where $a\in\mathbb{N}$, $a\geq 5$) are

1. $\gamma_w = 1-w+aw+(1-w+aw)\rho-w\rho^2$ where $1\leq w\leq a-3$,
2. $\delta_{v,u} = 1+v-u+au+(a-u+au)\rho-(u+1)\rho^2$ where $1\leq v\leq a-3, 0\leq u\leq v$,
3. $\zeta_z = z+2+(z+4)\rho+\rho^2$ where $0\leq z\leq a-4$.

And the norms and signatures of these elements are

1. $N(\gamma_w) = 1-w^2-w^3+a(w+w^2)$, $Sgn(\gamma_w)=(+,+,+)$,
2. $N(\delta_{v,u}) = 1+3v+2v^2+v^3-5u-3vu-3v^2u-2u^2+2vu^2-u^3+(-v-v^2+4u+5vu+v^2u-2u^2-vu^2)a+(-u-vu+u^2)a^2$, $Sgn(\delta_{v,u})=(-,+,+)$,
3. $N(\zeta_z) = 25+20z+7z^2+z^3-az+(-2-z)a^2$, $Sgn(\zeta_z)=(-,+,+)$.

The elements γ_w lie on a line, where

$$\tilde{\gamma_l} = \gamma_{w+l}-\gamma_w = l(a-1)-l\rho-l\rho^2$$

and

$$N(\tilde{\gamma_l}) = -n^3,\ Sgn(\tilde{\gamma_l}) = (+,+,-) \qquad \text{for} \qquad l>0.$$

The elements $\delta_{v,u}$ form a triangle with

$$\tilde{\delta}_{n,m} = \delta_{v+n,u+m}-\delta_{v,u} = -m+am+n+(-m+am)\rho-m\rho^2,$$

and

$$N(\tilde{\delta}_{n,m}) = -m^3+2m^2n-am^2n+(a-3)mn^2+n^3$$

and the signatures of these elements are different depending on the m,n (see Figure 2.1). Similarly, the elements ζ_z lie on a line and

$$\tilde{\zeta_k} = \zeta_{z+k}-\zeta_k = k+k\rho$$

and the norm and signatures are

$$N(\tilde{\zeta_k}) = k^3,\ Sgn(\tilde{\zeta_k}) = (+,+,+) \qquad \text{if} \qquad k > 0.$$

In the expansion we will discuss whether the elements in one iteration give the indecomposable elements, concretely we will examine the elements of the form $\beta_2^{(i)} - h_1\beta_1^{(i)} - j_1\beta_3^{(i)}$ and $\beta_3^{(i)} - h_2\beta_1^{(i)} - j_2\beta_2^{(i)}$ where $h_1, i_2, h_1, j_2 \in \mathbb{Z}$. We will compare these elements with the indecomposable elements in the order $\mathbb{Z}[\rho_i]$. We will show that, for specific h_1, h_2, j_1, j_2, some of these elements generate indecomposable elements. For simplicity, we will omit the indices of the h, j in some of the following calculations, where it will be clear from the context which coefficient is meant.

6.2.1 First root

Firstly, we look at the convergent in the hJPA expansion of $(1, |\rho_1|, \rho_1^2)$. Here it is enough to examine just the first ten iterations because the hJPA expansion is periodic with preperiod length 4 and period length 7.

(i) Elements γ_w

Here we examine if some iteration generates elements γ_w and $\tilde{\gamma}_l$. First, by comparing of norms we try to figure out if some elements $\beta_t^{(i)}$ have the same norm as some element γ_w or $\tilde{\gamma}_l$. Then we will try to find a unit ϵ for which for some $w, l \in \mathbb{N}$ one of the following equations hold:

$$\beta_t^{(i)} = \gamma_w\epsilon \text{ or } \beta_t^{(i)} = \tilde{\gamma}_l\epsilon.$$

If there exists such a unit, then the element $\beta_t^{(i)}$ generates γ_w or $\tilde{\gamma}_l$. Then we will look at the elements $\beta_2^{(i)} - h_1\beta_1^{(i)} - j_1\beta_3^{(i)}$ and $\beta_3^{(i)} - h_2\beta_1^{(i)} - j_2\beta_2^{(i)}$ and again by comparing norms we will try to find some h_1, h_2, j_1, j_2, for which the norms are the same as norms of elements γ_w. Next we will try to find some unit ϵ for which it holds that

$$\beta_2^{(i)} - h_1\beta_1^{(i)} - j_1\beta_3^{(i)} = \gamma_w\epsilon,$$

or

$$\beta_3^{(i)} - h_2\beta_1^{(i)} - j_2\beta_2^{(i)} = \gamma_w\epsilon.$$

If we find a unit such that one of these equations is satisfied for all w (depending on the h_1, h_2, j_1, j_2), then elements from this iteration generate all indecomposable elements γ_w. Finally, we try to state some general rules when some iteration generates these elements.

For instance, in the first iteration we can see that $N(\beta_1^{(1)}) = N(\tilde{\gamma}_1)$, $N(\beta_2^{(1)}) = N(\gamma_0)$ and $N(\beta_3^{(1)}) = N(\gamma_0)$. Then the units $\epsilon_1, \epsilon_2, \epsilon_3$ satisfying $\beta_1^{(1)} = \tilde{\gamma}_1\epsilon_1$, $\beta_2^{(1)} = \gamma_0\epsilon_2$ and $\beta_3^{(1)} = \gamma_0\epsilon_3$ are $\epsilon_1 = \epsilon_2 = -1 + (1-a)\rho_1 + \rho_1^2$ and $\epsilon_3 = -a\rho_1 + \rho_1^2$. This means that the elements in the first iteration generate elements $\gamma_0, \tilde{\gamma}_1$. Note that for the elements $\beta_1^{(1)}$ and $\beta_2^{(1)}$ we found the same unit. Hence, it is clear that element $\beta_1^{(1)} + \beta_2^{(1)}$ give the element $\gamma_0 + \tilde{\gamma}_1 = \gamma_1$. Indeed the following two equalities hold.

$$N(\beta_2^{(1)} - h\beta_1^{(1)}) = N(\gamma_{-h}),$$

and

$$\beta_2^{(1)} - h\beta_1^{(1)} = \gamma_{-h}(-1 + (1-a)\rho_1 + \rho_1^2).$$

It means that the elements $\beta_2^{(i)} - h\beta_1^{(i)} - j\beta_3^{(i)}$, when we take $j = 0$ and $h \in \{-a+3, \dots, 1\}$, generate all indecomposable elements γ_w. For the elements $\beta_3^{(1)} - h\beta_1^{(1)} - j\beta_2^{(1)}$, we can see that

$$N(\beta_3^{(1)} + \beta_1^{(1)}) = N(\gamma_1), \qquad N(\beta_3^{(1)} + \beta_2^{(1)}) = N(\gamma_0)$$

and then

$$\beta_3^{(1)} + \beta_1^{(1)} = \gamma_1(-1 + (1-a)\rho_1 + \rho_1^2), \qquad \beta_3^{(1)} + \beta_2^{(1)} = \gamma_0(-a\rho_1 + \rho_1^2).$$

We probably do no get all the elements γ_w which can be expressed as $\beta_2^{(1)} - h\beta_1^{(1)} - j\beta_3^{(1)}$. We try to find such elements for small value of indices h, j and we did not find any intervals for h, j such that $\beta_2^{(1)} - h\beta_1^{(1)} - j\beta_3^{(1)}$ generates all elements γ_w for all $a \geq 5$.

For rest of iterations, we proceed in the same way and in the following tables we show the most interesting results which we got. We proved that the first and seventh iterations generate all elements γ_w. There are probably some more combinations of i, j, h such that elements $\beta_2^{(i)} - h_1\beta_1^{(i)} - j_1\beta_3^{(i)}$ and $\beta_3^{(i)} - h_2\beta_1^{(i)} - j_2\beta_2^{(i)}$ generate some elements of γ_w, but we did not find any other iteration which generate all the elements γ_w for all a.

The following tables give a list of some elements that each iteration generates. For each iteration, there are the elements of this iteration, their norms, and signatures. Moreover, we can find there some elements γ_w and $\tilde{\gamma}_l$ multiplied by units such that they are equal to some convergent in this iteration. Then, for iterations for which it is interesting, there are elements $\beta_2^{(i)} - h_1\beta_1^{(i)} - j_1\beta_3^{(i)}$, $\beta_3^{(i)} - h_2\beta_1^{(i)} - j_2\beta_2^{(i)}$ and the elements γ_w which correspond to them. In these tables, we show only the first ten iterations, because other iterations are the same as one of the first ten iterations up to multiplication by a unit, so they generate the same elements γ_w.

First iteration		
$\beta_1^{(1)} = -\rho_1$	$\beta_2^{(1)} = \rho_1^2$	$\beta_3^{(1)} = 1$
$N(\beta_1^{(1)}) = -1,\ Sgn(\beta_1^{(1)}) = (+,+,-)$	$N(\beta_2^{(1)}) = 1, Sgn(\beta_2^{(1)}) = (+,+,+)$	$N(\beta_3^{(1)}) = 1,\ Sgn(\beta_3^{(1)}) = (+,+,+)$
$\beta_1^{(1)} = \tilde{\gamma}_1(-1 + (1-a)\rho_1 + \rho_1^2)$	$\beta_2^{(1)} = \gamma_0(-1 + (1-a)\rho_1 + \rho_1^2)$	$\beta_3^{(1)} = \gamma_0(-a\rho_1 + \rho_1^2)$

In the first iteration, there are interesting elements $\beta_2^{(1)} - h\beta_1^{(1)} - j\beta_3^{(1)}$. When we take $j = 0$ and $-a+3 \leq h \leq -1$, the following equation holds:

$$\beta_2^{(1)} - h\beta_1^{(1)} = \gamma_{-h}(-1 + (1-a)\rho_1 + \rho_1^2).$$

Hence, these elements generate all the elements γ_w.

Second iteration		
$\beta_1^{(2)} = \rho_1^2$	$\beta_2^{(2)} = 1 + \rho_1$	$\beta_3^{(2)} = -\rho_1$
$N(\beta_1^{(2)}) = 1,\ Sgn(\beta_1^{(2)}) = (+,+,+)$	$N(\beta_2^{(2)}) = 1,\ Sgn(\beta_2^{(2)}) = (+,+,+)$	$N(\beta_3^{(2)}) = -1,\ Sgn(\beta_3^{(2)}) = (+,+,-)$
$\beta_1^{(2)} = \gamma_0(-1 + (1-a)\rho_1 + \rho_1^2)$	$\beta_2^{(2)} = \gamma_{-1}(1 + a\rho_1 - \rho_1^2)$	$\beta_3^{(2)} = \tilde{\gamma}_1(-1 + (1-a)\rho_1 + \rho_1^2)$
$\beta_1^{(2)} = \gamma_{-1}(-1 + a + (-2a + a^2)\rho_1 + (2-a)\rho_1^2$	$\beta_2^{(2)} = \gamma_0$	
Third iteration		
$\beta_1^{(3)} = 1 + \rho_1$	$\beta_2^{(3)} = -\rho_1 - \rho_1^2$	$\beta_3^{(3)} = \rho_1^2$
$N(\beta_1^{(3)}) = 1,\ Sgn(\beta_1^{(3)}) = (+,+,+)$	$N(\beta_2^{(3)}) = -1,\ Sgn(\beta_2^{(3)}) = (+,+,-)$	$N(\beta_3^{(3)}) = 1,\ Sgn(\beta_3^{(3)}) = (+,+,+)$
$\beta_1^{(3)} = \gamma_{-1}(1 + a\rho_1 - \rho_1^2)$	$\beta_2^{(3)} = \tilde{\gamma}_1\rho_1^2$	$\beta_3^{(3)} = \gamma_0(-1 + (1-a)\rho_1 + \rho_1^2)$
$\beta_1^{(3)} = \gamma_0$		$\beta_3^{(3)} = \gamma_{-1}(-1 + a + (-2a + a^2)\rho_1 + (2-a)\rho_1^2)$

Fourth iteration		
$\beta_1^{(4)} = -\rho_1 - \rho_1^2$	$\beta_2^{(4)} = -a+3+(-a+3)\rho_1+\rho_1^2$	$\beta_3^{(4)} = 1+\rho_1$
$N(\beta_1^{(4)}) = -1,\ Sgn(\beta_1^{(4)}) = (+,+,-)$	$N(\beta_2^{(4)}) = -2a+7,\ Sgn(\beta_2^{(4)}) = (+,-,+)$	$N(\beta_3^{(4)}) = 1,\ Sgn(\beta_3^{(4)}) = (+,+,+)$
$\beta_1^{(4)} = \tilde{\gamma}_1\rho^2$		$\beta_3^{(4)} = \gamma_{-1}(1+a\rho-\rho^2)$
		$\beta_3^{(4)} = \gamma_0$
Fifth iteration		
$\beta_1^{(5)} = -a+3+(-a+3)\rho_1+\rho_1^2$	$\beta_2^{(5)} = 1+2\rho_1+\rho_1^2$	$\beta_3^{(5)} = -\rho_1-\rho_1^2$
$N(\beta_1^{(5)}) = -2a+7,\ Sgn(\beta_1^{(5)}) = (+,-,+)$	$N(\beta_2^{(5)}) = 1,\ Sgn(\beta_2^{(5)}) = (+,+,+)$	$N(\beta_3^{(5)}) = -1,\ Sgn(\beta_3^{(5)}) = (+,+,-)$
	$\beta_2^{(5)} = \gamma_{-1}\rho_1$	$\beta_3^{(5)} = \tilde{\gamma}_1\rho_1^2$
	$\beta_2^{(5)} = \gamma_0(1+\rho_1)$	
Sixth iteration		
$\beta_1^{(6)} = 1+2\rho_1+\rho_1^2$	$\beta_2^{(6)} = a-3+(a-4)\rho_1-2\rho_1^2$	$\beta_3^{(6)} = -a+3+(-a+3)\rho_1+\rho_1^2$
$N(\beta_1^{(6)}) = 1,\ Sgn(\beta_1^{(6)}) = (+,+,+)$	$N(\beta_2^{(6)}) = -1,\ Sgn(\beta_2^{(6)}) = (+,+,-)$	$N(\beta_3^{(6)}) = -2a+7,\ Sgn(\beta_3^{(6)}) = (+,-,+)$
$\beta_1^{(6)} = \gamma_{-1}\rho_1^2$	$\beta_2^{(6)} = \tilde{\gamma}_1(1+2\rho_1+\rho_1^2)$	
$\beta_1^{(6)} = \gamma_0(1+\rho_1)$		
Seventh iteration		
$\beta_1^{(7)} = a-3+(a-4)\rho_1-2\rho_1^2$	$\beta_2^{(7)} = -2a+5+(-3a+7)\rho_1+(-a+3)\rho_1^2$	$\beta_3^{(7)} = 1+2\rho_1+\rho_1^2$
$N(\beta_1^{(7)}) = -1,\ Sgn(\beta_1^{(7)}) = (+,+,-)$	$N(\beta_2^{(7)}) = 2a^2-14a+23,\ Sgn(\beta_2^{(7)}) = (+,-,-)$	$N(\beta_3^{(7)}) = 1,\ Sgn(\beta_3^{(7)}) = (+,+,+)$
$\beta_1^{(7)} = \tilde{\gamma}_1(1+2\rho_1+\rho_1^2)$		$\beta_3^{(7)} = \gamma_{-1}\rho_1^2$
		$\beta_3^{(7)} = \gamma_0(1+\rho_1)$

There are interesting elements $\beta_3^{(7)} - h\beta_1^{(7)} - j\beta_2^{(7)}$, which generate all the elements γ_z. Indeed, for $j = 1$ and $-a+5 \le h \le 1$ we have:

$$\beta_3^{(7)} - h\beta_1^{(7)} - \beta_2^{(7)} = \gamma_{-h+2}(1+2\rho_1+\rho_1^2).$$

Eight iteration		
$\beta_1^{(8)} = -a^2+6a-10+(-a^2+6a-13)\rho_1+(a-7)\rho_1^2$	$\beta_2^{(8)} = -a^2+5a-5+(-a^2+6a-6)\rho_1+(2a-3)\rho_1^2$	$\beta_3^{(8)} = a-3+(a-4)\rho_1-2\rho_1^2$
$N(\beta_1^{(8)}) = 2a-7,\ Sgn(\beta_1^{(8)}) = (+,-,-)$	$N(\beta_2^{(8)}) = -a^2+5a-5,\ Sgn(\beta_2^{(8)}) = (+,-,+)$	$N(\beta_3^{(8)}) = -1,\ Sgn(\beta_3^{(8)}) = (+,+,-)$
		$\beta_3^{(8)} = \tilde{\gamma}_1(1+2\rho_1+\rho_1^2)$
Ninth iteration		
$\beta_1^{(9)} = -a^2+5a-5+(-a^2+6a-6)\rho_1+(2a-3)\rho_1^2$	$\beta_2^{(9)} = a^2-5a+7+(a^2-5a+9)\rho_1+(-a+5)\rho_1^2$	$\beta_3^{(9)} = -a^2+6a-10+(-a^2+6a-13)\rho_1+(a-7)\rho_1^2$
$N(\beta_1^{(9)}) = -a^2+5a-5,\ Sgn(\beta_1^{(9)}) = (+,-,+)$	$N(\beta_2^{(9)}) = 1,\ Sgn(\beta_2^{(9)}) = (+,+,+)$	$N(\beta_3^{(9)}) = 2a-7,\ Sgn(\beta_3^{(9)}) = (+,-,-)$
	$\beta_2^{(9)} = \gamma_0(a^2-4a+4+(a^2-4a+5)\rho_1+(-a+3)\rho_1^2)$	
	$\beta_2^{(9)} = \gamma_{-1}(-a+2+(-a+2)\rho_1+\rho_1^2)$	
Tenth iteration		
$\beta_1^{(10)} = a^2-5a+7+(a^2-5a+9)\rho_1+(-a+5)\rho_1^2$	$\beta_2^{(10)} = a-5-7\rho_1+(-a-4)\rho_1^2$	$\beta_3^{(10)} = -a^2+5a-5+(-a^2+6a-6)\rho_1+(2a-3)\rho_1^2$
$N(\beta_1^{(10)}) = 1,\ Sgn(\beta_1^{(10)}) = (+,+,+)$	$N(\beta_2^{(10)}) = -1,\ Sgn(\beta_2^{(10)}) = (+,+,-)$	$N(\beta_3^{(10)}) = -a^2+5a-5,\ Sgn(\beta_3^{(10)}) = (+,-,+)$
$\beta_1^{(10)} = \gamma_{-1}(-a+2+(-a+2)\rho_1+\rho_1^2)$	$\beta_2^{(10)} = \tilde{\gamma}_1(2+(a+3)\rho_1+(a+2)\rho_1^2)$	
$\beta_1^{(10)} = \gamma_0(a^2-4a+4+(a^2-4a+5)\rho_1+(-a+3)\rho_1^2)$		

So, we found out that all the elements γ_w are generated by the first or seventh iteration. (There could be some other iterations which generate all these elements, but we have not found it.) For both of this iterations, the first element of this iteration gives $\tilde{\gamma}_1$. Moreover, for each iteration, there exists a unit ϵ_1 (ϵ_7) such that $\beta_1^{(1)} = \tilde{\gamma}_1\epsilon_1$ ($\beta_1^{(7)} = \tilde{\gamma}_1\epsilon_7$) and $\beta_2^{(1)} - h\beta_1^{(1)} = \gamma_{-h}\epsilon_1$ ($\beta_3^{(1)} - h\beta_1^{(1)} - \beta_2^{(1)} = \gamma_{2-h}\epsilon_7$). Then there is at least one totally positive convergent in the first or seventh iteration. Let us notice that this condition can be satisfied also by some others iterations, for which we did not prove that they generate all elements γ_w.

(ii) Elements $\delta_{v,u}$

Let us denote by P the plane where the elements $\delta_{v,u}$ lie and by L the lattice which they generate on this plane. We will try to decide for which iteration the elements $\beta_2^{(i)} - h_1\beta_1^{(i)} - j_1\beta_3^{(i)}$ and $\beta_3^{(i)} - h_2\beta_1^{(i)} - j_2\beta_2^{(i)}$ generate this lattice, and then we examine for which h, j these elements generate exactly the elements $\delta_{v,u}$. We will proceed similarly to the previous case. First, we will discuss some specific iterations, which give us some interesting results, and then we show tables with results for all ten iterations.

When we look at the first iteration, which is $\beta^{(1)} = (-\rho_1, \rho_1^2, 1)$ with $N(\beta^{(1)}) = (-1, 1, 1)$, then we can see that $N(\beta_1^{(1)}) = N(\tilde{\delta}_{1,1})$ and $N(\beta_2^{(1)}) = N(\tilde{\delta}_{1,0})$. Then we can verify that

$$\beta_1^{(1)} = \tilde{\delta}_{1,1}\rho_2^2, \qquad \beta_2^{(1)} = \tilde{\delta}_{1,0}\rho_2^2.$$

It means that the elements $\beta_2^{(1)}$ and $\beta_2^{(1)}$ generate the basis of the lattice L.

And then by comparing norms we get that

$$N(\beta_3^{(1)} - h\beta_1^{(1)} - j\beta_2^{(1)}) = N(\delta_{a-1-h-j,a-1-h}),$$

and it is easy to verify the following equation:

$$\beta_3^{(i)} - h\beta_1^{(i)} - j\beta_2^{(i)} = \delta_{a-1-h-j,a-1-h}\rho_1^2.$$

Hence, when we take $2 < h < a-1$ and $2-h < j < 0$, then the elements $\beta_3^{(i)} - h\beta_1^{(i)} - j\beta_2^{(i)}$ generate the indecomposable elements $\delta_{v,u}$.

Similarly, in the third iteration, $N(\beta_1^{(3)}) = N(\tilde{\delta}_{-1,-1}), N(\beta_2^{(3)}) = N(\tilde{\delta}_{-1,0})$ and $N(\beta_3^{(3)}) = N(\delta_{0,0})$. We can verify that $\beta_1^{(3)} = \tilde{\delta}_{-1,-1}(\rho_1 + \rho_1^2), \beta_2^{(3)} = \tilde{\delta}_{-1,0}(\rho_1 + \rho_1^2)$ and $\beta_3^{(3)} = \delta_{0,0}(\rho_1 + \rho_1^2)$. Hence, the elements $\beta_1^{(3)}, \beta_2^{(3)}$ generate the basis of the lattice L, and the element $\beta_3^{(3)}$ is idecomposable. From this, we easily get that

$$N(\beta_3^{(3)} - h\beta_1^{(3)} - j\beta_2^{(3)}) = N(\delta_{h+j,j})$$

and

$$\beta_3^{(3)} - h\beta_1^{(3)} - j\beta_2^{(3)} = \delta_{h+j,j}(\rho_1 + \rho_1^2).$$

Therefore, when we fix $0 \leq j \leq a-3, 0 \leq h \leq a-3-j$, we get that these elements generate the indecomposable elements $\delta_{v,u}$.

For the fifth iteration, we have that $N(\beta_2^{(5)}) = N(\tilde{\delta}_{1,0}) = N(\tilde{\delta}_{-1,-1})$. From this we get that

$$\beta_2^{(5)} = \tilde{\delta}_{1,0}(1 + 2\rho + \rho^2)$$

and

$$\beta_2^{(5)} = \tilde{\delta}_{-1,-1}(1 + (1+2a)\rho_1 + (1+a)\rho_1^2).$$

Then, we can see that

$$\beta_3^{(3)} - 0\beta_1^{(3)} - j\beta_2^{(3)} = \delta_{j-1,j} \cdot (1 + (1+2a)\rho_1 + (1+a)\rho_1^2)$$

and

$$\beta_3^{(3)} - 2\beta_1^{(3)} - j\beta_2^{(3)} = \delta_{-j,2} \cdot (1 + 2\rho + \rho^2).$$

Hence, the elements $\beta_3^{(5)} - h\beta_1^{(5)} - j\beta_2^{(5)}$ generate at least two lines from the plane P but each of these lines is obtained after the multiplication by a different unit.

Similarly, as in the previous case, in the following tables, we can find the list of elements $\delta_{v,u}$, which are generated by convergents in some iterations. Again the list is probably incomplete but we found some iterations, which generate all elements $\delta_{v,u}$.

First iteration		
$\beta_1^{(1)} = -\rho_1,\ Sgn(\beta_1^{(1)}) = (+,+,-)$	$\beta_2^{(1)} = \rho_1^2,\ Sgn(\beta_2^{(1)}) = (+,+,+)$	$\beta_3^{(1)} = 1,\ Sgn(\beta_3^{(1)}) = (+,+,+)$
$N(\beta_1^{(1)}) = -1$	$N(\beta_2^{(1)}) = 1$	$N(\beta_3^{(1)}) = 1$
$\beta_1^{(1)} = \tilde{\delta}_{1,1}\rho_1^2$	$\beta_2^{(1)} = \tilde{\delta}_{1,0}\rho_1^2$	

The elements $\beta_2^{(1)} - h\beta_1^{(1)} - j\beta_3^{(1)}$ generate a line in a plane P and we have that:

$$\beta_2^{(1)} - h\beta_1^{(1)} + h\beta_3^{(1)} = \delta_{-h,-h}(\rho + \rho^2).$$

And the elements $\beta_3^{(1)} - h\beta_1^{(1)} - j\beta_2^{(1)}$ generate all the elements $\delta_{v,u}$, for $2 < h < a-1$ and $2 - h < j < 0$ we have that

$$\beta_3^{(1)} - h\beta_1^{(1)} - j\beta_2^{(1)} = \delta_{a-1-h-j,a-1-h}\rho^2.$$

Second iteration		
$\beta_1^{(2)} = \rho_1^2$	$\beta_2^{(2)} = 1 + \rho_1$	$\beta_3^{(2)} = -\rho_1$
$N(\beta_1^{(2)}) = 1,\ Sgn(\beta_1^{(2)}) = (+,+,+)$	$N(\beta_2^{(2)}) = 1,\ Sgn(\beta_2^{(2)}) = (+,+,+)$	$N(\beta_3^{(2)}) = -1,\ Sgn(\beta_3^{(2)}) = (+,+,-)$
$\beta_1^{(2)} = \tilde{\delta}_{1,0}\rho_1^2$		$\beta_3^{(2)} = \tilde{\delta}_{1,1}\rho_1^2$

The elements $\beta_2^{(2)} - h\beta_1^{(2)} - j\beta_3^{(2)}$ generate all the elements $\delta_{v,u}$, for $1 \leq j \leq a-2$ and $1 - j \leq h \leq 0$ we have:

$$\beta_2^{(2)} - h\beta_1^{(2)} - j\beta_3^{(2)} = \delta_{a-2-h-j,a-2-j}\rho^2.$$

Third iteration		
$\beta_1^{(3)} = 1 + \rho_1$	$\beta_2^{(3)} = -\rho_1 - \rho_1^2$	$\beta_3^{(3)} = \rho_1^2$
$N(\beta_1^{(3)}) = 1,\ Sgn(\beta_1^{(3)}) = (+,+,+)$	$N(\beta_2^{(3)}) = -1,\ Sgn(\beta_2^{(3)}) = (+,+,-)$	$N(\beta_3^{(3)}) = 1,\ Sgn(\beta_3^{(3)}) = (+,+,+)$
$\beta_1^{(3)} = \tilde{\delta}_{-1,-1}(\rho_1 + \rho_1^2)$	$\beta_2^{(3)} = \tilde{\delta}_{-1,0}(\rho_1 + \rho_1^2)$	$\beta_3^{(3)} = \delta_{0,0}(\rho_1 + \rho_1^2)$

There are interesting elements $\beta_3^{(3)} - h\beta_1^{(3)} - j\beta_2^{(3)}$; they generate all the elements $\delta_{v,u}$. For $0 \leq j \leq a-3, 0 \leq h \leq a-3-j$ we have:

$$\beta_3^{(3)} - h\beta_1^{(3)} - j\beta_2^{(3)} = \delta_{-1+h+j,h}(\rho + \rho^2).$$

Fourth iteration		
$\beta_1^{(4)} = -\rho_1 - \rho_1^2$	$\beta_2^{(4)} = -a + 3 + (-a+3)\rho_1 + \rho_1^2$	$\beta_3^{(4)} = 1 + \rho_1$
$N(\beta_1^{(4)}) = -1,\ Sgn(\beta_1^{(4)}) = (+,+,-)$	$N(\beta_2^{(4)}) = -2a + 7,\ Sgn(\beta_2^{(4)}) = (+,-,+)$	$N(\beta_3^{(4)}) = 1,\ Sgn(\beta_3^{(4)}) = (+,+,+)$
$\beta_1^{(4)} = \tilde{\delta}_{1,1}(1 + a\rho + a\rho^2)$		$\beta_3^{(4)} = \tilde{\delta}_{-1,-1}(\rho + \rho^2)$

In this case, the elements $\beta_3^{(4)} - h\beta_1^{(4)} - j\beta_2^{(4)}$ generate at least three different line at the plane P. We can see, that

$$\beta_3^{(4)} - h\beta_1^{(4)} - (2-h)\beta_2^{(4)} = \delta_{1,2-h}(1 + 2\rho + \rho^2),$$

$$\beta_3^{(4)} - h\beta_1^{(4)} + (h-1)\beta_2^{(4)} = \delta_{h-1,0}(3 - a + (4-a)\rho + 2\rho^2),$$

$$\beta_3^{(4)} - h\beta_1^{(4)} = \delta_{a-1-h,a-1-h}(1 + a\rho + a\rho^2).$$

Fifth iteration		
$\beta_1^{(5)} = -a + 3 + (-a+3)\rho_1 + \rho_1^2$	$\beta_2^{(5)} = 1 + 2\rho_1 + \rho_1^2$	$\beta_3^{(5)} = -\rho_1 - \rho_1^2$
$N(\beta_1^{(5)}) = -2a + 7,\ Sgn(\beta_1^{(5)}) = (+,-,+)$	$N(\beta_2^{(5)}) = 1,\ Sgn(\beta_2^{(5)}) = (+,+,+)$	$N(\beta_3^{(5)}) = -1,\ Sgn(\beta_3^{(5)}) = (+,+,-)$
	$\beta_2^{(5)} = \tilde{\delta}_{-1,-1}(1+(1+a)\rho_1+(1+a)\rho_1^2)$	$\beta_3^{(5)} = \delta_{-1,0}(1+(1+a)\rho_1+(1+a)\rho_1^2)$
	$\beta_2^{(5)} = \tilde{\delta}_{1,0}(1 + 2\rho_1 + \rho_1^2)$	

For the fifth iteration, for the elements $\beta_2^{(5)} - h\beta_1^{(5)} - j\beta_3^{(5)}$, we can see that

$$\beta_2^{(5)} - h\beta_1^{(5)} - (-h+1)\beta_3^{(5)} = \delta_{1,h}(1 + 2\rho_1 + \rho_1^2)$$

and

$$\beta_2^{(5)} - h\beta_1^{(5)} + h\beta_3^{(5)} = \delta_{-h,0}(-a + 3 + (-a+4)\rho_1 + 2\rho_1^2).$$

Then for the elements $\beta_3^{(5)} - h\beta_1^{(5)} - j\beta_2^{(5)}$, we have that

$$\beta_3^{(3)} - j\beta_2^{(3)} = \delta_{j-1,j}(1 + (1+2a)\rho_1 + (1+a)\rho_1^2)$$

and

$$\beta_3^{(3)} - 2\beta_1^{(3)} - j\beta_2^{(3)} = \delta_{-j,2}(1 + 2\rho + \rho^2).$$

Sixth iteration		
$\beta_1^{(6)} = 1 + 2\rho_1 + \rho_1^2$	$\beta_2^{(6)} = a - 3 + (a-4)\rho_1 - 2\rho_1^2$	$\beta_3^{(6)} = -a + 3 + (-a+3)\rho_1 + \rho_1^2$
$N(\beta_1^{(6)}) = 1,\ Sgn(\beta_1^{(6)}) = (+,+,+)$	$N(\beta_2^{(6)}) = -1,\ Sgn(\beta_2^{(6)}) = (+,+,-)$	$N(\beta_3^{(6)}) = -2a + 7,\ Sgn(\beta_3^{(6)}) = (+,-,+)$
$\beta_1^{(6)} = \tilde{\delta}_{1,0}(1 + 2\rho_1 + \rho_1^2)$	$\beta_2^{(6)} = \tilde{\delta}_{0,1}(1 + 2\rho_1 + \rho_1^2)$	$\beta_3^{(6)} = \delta_{1,0}(2 - a + (2-a)\rho_1 + \rho_1^2)$
$\beta_1^{(6)} = \tilde{\delta}_{-1,-1}(1+(1+a)\rho_1+(1+a)\rho_1^2)$	$\beta_2^{(6)} = \tilde{\delta}_{1,1}(2 + (3+a)\rho_1 + (2+a)\rho_1^2)$	

For the elements $\beta_2^{(6)} - h\beta_1^{(6)} - j\beta_3^{(6)}$ we can get that

$$\beta_2^{(6)} - h\beta_1^{(6)} - 1\beta_3^{(6)} = \delta_{-h,2}(1 + 2\rho_1 + \rho_1^2),$$

$$\beta_2^{(6)} - h\beta_1^{(6)} + 1\beta_3^{(6)} = \delta_{h-1,h}(1 + (1+a)\rho_1 + (1+a)\rho_1^2).$$

Hence, these elements generate two line at the plane P. Then, for elements $\beta_3^{(6)} - h\beta_1^{(6)} - j\beta_2^{(6)}$ we get, that

$$\beta_3^{(6)} - h\beta_1^{(6)} - h\beta_2^{(6)} = \delta_{1+h,0}(2 - a + (2-a)\rho_1 + \rho_1^2)$$

and

$$\beta_3^{(6)} - a\beta_1^{(6)} - j\beta_2^{(6)} = \delta_{-4-j,-2-j}(2 + (3+a)\rho_1 + (2+a)\rho_1^2),$$

so these elements also generate two lines at the plane P.

Seventh iteration		
$\beta_1^{(7)} = a - 3 + (a-4)\rho_1 - 2\rho_1^2$	$\beta_2^{(7)} = -2a + 5 + (-3a+7)\rho_1 + (-a+3)\rho_1^2$	$\beta_3^{(7)} = 1 + 2\rho_1 + \rho_1^2$
$N(\beta_1^{(7)}) = -1,\ Sgn(\beta_1^{(7)}) = (+,+,-)$	$N(\beta_2^{(7)}) = 2a^2 - 14a + 23,\ Sgn(\beta_2^{(7)}) = (+,-,-)$	$N(\beta_3^{(7)}) = 1,\ Sgn(\beta_3^{(7)}) = (+,+,+)$
$\beta_1^{(7)} = \tilde{\delta}_{-1,0}(3 - a + (4-a)\rho_1 + 2\rho_1^2)$		$\beta_3^{(7)} = \delta_{0,0}(3 - a + (4-a)\rho_1 + 2\rho_1^2)$
$\beta_1^{(7)} = \tilde{\delta}_{0,1}(1 + 2\rho_1 + \rho_1^2)$		$\beta_3^{(7)} = \tilde{\delta}_{1,0}(1 + 2\rho_1 + \rho_1^2)$
$\beta_1^{(7)} = \tilde{\delta}_{1,1}(2 + (3+a)\rho_1 + (2+a)\rho_1^2)$		

For the seventh iteration we get, that

$$\beta_2^{(7)} - h\beta_1^{(7)} - 2\beta_3^{(7)} = \delta_{-h-4,-h-2}(2 + (3+a)\rho_1 + (2+a)\rho_1^2),$$

$$\beta_2^{(7)} - h\beta_1^{(7)} + 1\beta_3^{(7)} = \delta_{-a+h-4,1}(3 - a + (4-a)\rho_1 + 2\rho_1^2),$$

$$\beta_3^{(7)} - h\beta_1^{(7)} - 2\beta_2^{(7)} = \delta_{-h+5,-h+4}(2 + (3+a)\rho_1 + (2+a)\rho_1^2),$$

$$\beta_3^{(7)} - h\beta_1^{(7)} = \delta_{h,0}(3 - a + (4-a)\rho_1 + 2\rho_1^2).$$

Hence, the elements $\beta_2^{(7)} - h\beta_1^{(7)} - j\beta_3^{(7)}$ and $\beta_3^{(7)} - h\beta_1^{(7)} - j\beta_2^{(7)}$ generate some lines from the plane P.

Eight iteration		
$\beta_1^{(8)} = -a^2+6a-10+(-a^2+6a-13)\rho_1+(a-7)\rho_1^2$	$\beta_2^{(8)} = -a^2+5a-5+(-a^2+6a-6)\rho_1+(2a-3)\rho_1^2$	$\beta_3^{(8)} = a-3+(a-4)\rho_1-2\rho_1^2$
$N(\beta_1^{(8)}) = 2a-7,\ Sgn(\beta_1^{(8)}) = (+,-,-)$	$N(\beta_2^{(8)}) = -a^2+5a-5,\ Sgn(\beta_2^{(8)}) = (+,-,+)$	$N(\beta_3^{(8)}) = -1,\ Sgn(\beta_3^{(8)}) = (+,+,-)$
$\beta_1^{(8)} = \delta_{0,1}(-a+5+7\rho_1+(a+4)\rho_1^2)$	$\beta_2^{(8)} = \delta_{1,1}(-a+5+7\rho_1+(a+4)\rho_1^2)$	$\beta_3^{(8)} = \delta_{-1,0}(-a+5+7\rho_1+(a+4)\rho_1^2)$

Here we can see that the elements $\beta_2^{(8)} - h\beta_1^{(8)} - j\beta_3^{(8)}$ and $\beta_3^{(8)} - h\beta_1^{(8)} - j\beta_2^{(8)}$ generate some lines at P:

$$\beta_2^{(8)} - h\beta_1^{(8)} + h\beta_3^{(8)} = \delta_{1-h,1-h}(-a+5+7\rho_1+(a+4)\rho_1^2)$$

and

$$\beta_3^{(8)} - h\beta_1^{(8)} + h\beta_2^{(8)} = \delta_{h-1,0}(-a+5+7\rho_1+(a+4)\rho_1^2).$$

Ninth iteration		
$\beta_1^{(9)} = -a^2+5a-5+(-a^2+6a-6)\rho_1+(2a-3)\rho_1^2$	$\beta_2^{(9)} = a^2-5a+7+(a^2-5a+9)\rho_1+(-a+5)\rho_1^2$	$\beta_3^{(9)} = -a^2+6a-10+(-a^2+6a-13)\rho_1+(a-7)\rho_1^2$
$N(\beta_1^{(9)}) = -a^2+5a-5,\ Sgn(\beta_1^{(9)}) = (+,-,+)$	$N(\beta_2^{(9)}) = 1,\ Sgn(\beta_2^{(9)}) = (+,+,+)$	$N(\beta_3^{(9)}) = 2a-7,\ Sgn(\beta_3^{(9)}) = (+,-,-)$
$\beta_1^{(9)} = \delta_{1,1}(-a+5+7\rho_1+(a+4)\rho_1^2)$	$\beta_2^{(9)} = \tilde{\delta}_{-1,-1}(-a+5+7\rho_1+(a+4)\rho_1^2)$	$\beta_3^{(9)} = \delta_{0,1}(-a+5+7\rho_1+(a+4)\rho_1^2)$

Here we get that:

$$\beta_3^{(9)} - j\beta_2^{(9)} = \delta_{j,1+j}(-a+5+7\rho_1+(a+4)\rho_1^2).$$

Tenth iteration		
$\beta_1^{(10)} = a^2-5a+7+(a^2-5a+9)\rho_1+(-a+5)\rho_1^2$	$\beta_2^{(10)} = a-5-7\rho_1+(-a-4)\rho_1^2$	$\beta_3^{(10)} = -a^2+5a-5+(-a^2+6a-6)\rho_1+(2a-3)\rho_1^2$
$N(\beta_1^{(10)}) = 1,\ Sgn(\beta_1^{(10)}) = (+,+,+)$	$N(\beta_2^{(10)}) = -1,\ Sgn(\beta_2^{(10)}) = (+,+,-)$	$N(\beta_3^{(10)}) = -a^2+5a-5,\ Sgn(\beta_3^{(10)}) = (+,-,+)$
$\beta_1^{(10)} = \tilde{\delta}_{-1,-1}(-a+5+7\rho_1+(a+4)\rho_1^2)$	$\beta_2^{(10)} = \tilde{\delta}_{-1,0}(-a+5+7\rho_1+(a+4)\rho_1^2)$	$\beta_3^{(10)} = \delta_{1,1}(-a+5+7\rho_1+(a+4)\rho_1^2)$

For the tenth iteration we get that elements $\beta_3^{(10)} - h\beta_1^{(10)} - j\beta_2^{(10)}$ generate all the elements $\delta_{v,u}$. For the $-1 \le j \le a-4$ and $0 \le h \le a-4-j$ we get that

$$\beta_3^{(10)} - h\beta_1^{(10)} - j\beta_2^{(10)} = \delta_{1+h+j,1+h}(-a+5+7\rho_1+(a+4)\rho_1^2).$$

Therefore, the first, second, third, and tenth iteration generate all the elements $\delta_{v,u}$, actually, they generate the whole lattice L. For all of these iterations, it holds that two of the convergents form a basis of lattice L. Moreover, for each iteration, there is just one unit ϵ such that two convergents in the iteration multiplied by ϵ are equal to the basis of the lattice L. Moreover, for the third and tenth iterations the third convergent multiplied by the same unit ϵ lie in the plane P. Hence, maybe it is significant whether two of the convergents (after multiplying by the same unit ϵ) generate a basis of the lattice L.

In the sixth iteration, it also holds that the first two convergents (multiplied by the same unit) generate the basis of lattice L. However, this iteration does not generate the whole lattice L. In this case, the third convergent (multiplied by the same unit) does not lie in this plane, and this convergent has a large norm. (The similar elements in the first and second iteration have small norms.)

In summary, it seems that the most important thing is if two of the convergents (after multiplying by the same unit ϵ) generate a basis of the lattice L. Then, when the third convergent (after multiplying by the same unit ϵ) lie in the lattice L, the iteration always generates the elements $\delta_{v,u}$. But if the third convergent (after multiplying by the same unit ϵ) does not lie at the lattice L, then the iteration does not have to generate all these elements. This may depend on the value of the norm of this element.

(iii) Elements ζ_z

The elements ζ_z lie in a line as the elements γ_w, hence we proceed as in this case. We look at the same elements and by comparing of norms we get the following tables, where for each iteration, we can find a list of the elements that this iteration generates. The list of these elements is probably incomplete, and we wrote just the elements $\beta_2^{(i)} - h\beta_1^{(i)} - j\beta_3^{(i)}, \beta_3^{(i)} - h\beta_1^{(i)} - j\beta_2^{(i)}$, for which we proved that they generate all the elements ζ_z.

First iteration		
$\beta_1^{(1)} = -\rho_1$	$\beta_2^{(1)} = \rho_1^2$	$\beta_3^{(1)} = 1$
$N(\beta_1^{(1)}) = -1,\ Sgn(\beta_1^{(1)}) = (+,+,-)$	$N(\beta_2^{(1)}) = 1,\ Sgn(\beta_2^{(1)}) = (+,+,+)$	$N(\beta_3^{(1)}) = 1,\ Sgn(\beta_3^{(1)}) = (+,+,+)$
$\beta_1^{(1)} = \tilde{\zeta}_{-1}(1 + a\rho_1 - \rho_1^2)$	$\beta_2^{(1)} = \tilde{\zeta}_1(-1 + (1-a)\rho_1 + \rho_1^2)$	$\beta_3^{(1)} = \tilde{\zeta}_1(-a\rho_1 + \rho_1^2)$

We can see that the elements $\beta_2^{(1)} - h\beta_1^{(1)} - j\beta_3^{(1)}$ generate all elements ζ_z. For $4 \leq h \leq a$ and $j = 2 - h$ we get

$$\beta_2^{(1)} - h\beta_1^{(1)} - (2-h)\beta_3^{(1)} = \zeta_{h-4}.$$

Second iteration		
$\beta_1^{(2)} = \rho_1^2$	$\beta_2^{(2)} = 1 + \rho_1$	$\beta_3^{(2)} = -\rho_1$
$N(\beta_1^{(2)}) = 1,\ Sgn(\beta_1^{(2)}) = (+,+,+)$	$N(\beta_2^{(2)}) = 1,\ Sgn(\beta_2^{(2)}) = (+,+,+)$	$N(\beta_3^{(2)}) = -1,\ Sgn(\beta_3^{(2)}) = (+,+,-)$
$\beta_1^{(2)} = \tilde{\zeta}_1(-1 + (1-a)\rho_1 + \rho_1^2)$	$\beta_2^{(2)} = \tilde{\zeta}_1$	$\beta_3^{(2)} = \tilde{\zeta}_{-1}(1 + a\rho_1 - \rho_1^2)$
Third iteration		
$\beta_1^{(3)} = 1 + \rho_1$	$\beta_2^{(3)} = -\rho_1 - \rho_1^2$	$\beta_3^{(3)} = \rho_1^2$
$N(\beta_1^{(3)}) = 1,\ Sgn(\beta_1^{(3)}) = (+,+,+)$	$N(\beta_2^{(3)}) = -1,\ Sgn(\beta_2^{(3)}) = (+,+,-)$	$N(\beta_3^{(3)}) = 1,\ Sgn(\beta_3^{(3)}) = (+,+,+)$
$\beta_1^{(3)} = \tilde{\zeta}_1$	$\beta_2^{(3)} = \tilde{\zeta}_{-1}\rho_1$	$\beta_3^{(3)} = \tilde{\zeta}_1(-1 + (1-a)\rho_1 + \rho_1^2)$

Here, the elements $\beta_3^{(3)} - h\beta_1^{(3)} - j\beta_2^{(3)}$ generate all the elements ζ_z. For $j = a - i$ and $-a + 1 \leq h \leq -3$ we get

$$\beta_3^{(3)} - h\beta_1^{(3)} - (a-h)\beta_2^{(3)} = \zeta_{-h-3}(1 + \rho_1).$$

Fourth iteration		
$\beta_1^{(4)} = -\rho_1 - \rho_1^2$	$\beta_2^{(4)} = -a + 3 + (-a+3)\rho_1 + \rho_1^2$	$\beta_3^{(4)} = 1 + \rho_1$
$N(\beta_1^{(4)}) = -1,\ Sgn(\beta_1^{(4)}) = (+,+,-)$	$N(\beta_2^{(4)}) = -2a + 7,\ Sgn(\beta_2^{(4)}) = (+,-,+)$	$N(\beta_3^{(4)}) = 1,\ Sgn(\beta_3^{(4)}) = (+,+,+)$
$\beta_1^{(4)} = \tilde{\zeta}_{-1}\rho_1$		$\beta_3^{(4)} = \tilde{\zeta}_1$
Fifth iteration		
$\beta_1^{(5)} = -a + 3 + (-a+3)\rho_1 + \rho_1^2$	$\beta_2^{(5)} = 1 + 2\rho_1 + \rho_1^2$	$\beta_3^{(5)} = -\rho_1 - \rho_1^2$
$N(\beta_1^{(5)}) = -2a + 7,\ Sgn(\beta_1^{(5)}) = (+,-,+)$	$N(\beta_2^{(5)}) = 1,\ Sgn(\beta_2^{(5)}) = (+,+,+)$	$N(\beta_3^{(5)}) = -1,\ Sgn(\beta_3^{(5)}) = (+,+,-)$
	$\beta_2^{(5)} = \tilde{\zeta}_1(1 + \rho_1)$	$\beta_3^{(5)} = \tilde{\zeta}_{-1}\rho_1$
Sixth iteration		
$\beta_1^{(6)} = 1 + 2\rho_1 + \rho_1^2$	$\beta_2^{(6)} = a - 3 + (a-4)\rho_1 - 2\rho_1^2$	$\beta_3^{(6)} = -a + 3 + (-a+3)\rho_1 + \rho_1^2$
$N(\beta_1^{(6)}) = 1,\ Sgn(\beta_1^{(6)}) = (+,+,+)$	$N(\beta_2^{(6)}) = -1,\ Sgn(\beta_2^{(6)}) = (+,+,-)$	$N(\beta_3^{(6)}) = -2a + 7,\ Sgn(\beta_3^{(6)}) = (+,-,+)$
$\beta_1^{(6)} = \tilde{\zeta}_1(1 + \rho_1)$	$\beta_2^{(6)} = \tilde{\zeta}_{-1}(2 - a + (2-a)\rho_1 + \rho_1^2)$	

It is easy to see that

$$\beta_3^{(6)} - h\beta_1^{(6)} - (h+1)\beta_2^{(6)} = \zeta_h(1 - a + (1-a)\rho_1 + \rho_1^2).$$

Hence, for $0 \leq h \leq a - 4$ and $j = h + 1$ the elements $\beta_3^{(6)} - h\beta_1^{(6)} - j\beta_2^{(6)}$ generate all the elements ζ_z.

Seventh iteration		
$\beta_1^{(7)} = a - 3 + (a-4)\rho_1 - 2\rho_1^2$	$\beta_2^{(7)} = -2a + 5 + (-3a+7)\rho_1 + (-a+3)\rho_1^2$	$\beta_3^{(7)} = 1 + 2\rho_1 + \rho_1^2$
$N(\beta_1^{(7)}) = -1,\ Sgn(\beta_1^{(7)}) = (+,+,-)$	$N(\beta_2^{(7)}) = 2a^2 - 14a + 23,\ Sgn(\beta_2^{(7)}) = (+,-,-)$	$N(\beta_3^{(7)}) = 1,\ Sgn(\beta_3^{(7)}) = (+,+,+)$
$\beta_1^{(7)} = \tilde{\zeta}_{-1}(2 - a + (2-a)\rho_1 + \rho_1^2)$		$\beta_3^{(7)} = \tilde{\zeta}_1(1 + \rho_1)$

We get that:

$$\beta_2^{(7)} - h\beta_1^{(7)} - (h+1)\beta_3^{(7)} = \zeta_{a+h-1}(1 - a + (1-a)\rho_1 + \rho_1^2).$$

Thus, the elements $\beta_2^{(7)} - h\beta_1^{(7)} - j\beta_3^{(7)}$, for $1 \leq h \leq a-3$ and $j = -a+h+1$, generate all the elements ζ_z.

Eight iteration		
$\beta_1^{(8)} = -a^2 + 6a - 10 + (-a^2 + 6a - 13)\rho_1 + (a-7)\rho_1^2$	$\beta_2^{(8)} = -a^2 + 5a - 5 + (-a^2 + 6a - 6)\rho_1 + (2a-3)\rho_1^2$	$\beta_3^{(8)} = a - 3 + (a-4)\rho_1 - 2\rho_1^2$
$N(\beta_1^{(8)}) = 2a - 7,\ Sgn(\beta_1^{(8)}) = (+,-,-)$	$N(\beta_2^{(8)}) = -a^2 + 5a - 5,\ Sgn(\beta_2^{(8)}) = (+,-,+)$	$N(\beta_3^{(8)}) = -1,\ Sgn(\beta_3^{(8)}) = (+,+,-)$
		$\beta_3^{(8)} = \tilde{\zeta}_{-1}(2 - a + (2-a)\rho_1 + \rho_1^2)$

In this iteration, the elements $\beta_2^{(8)} - h\beta_1^{(8)} - j\beta_3^{(8)}$ generate all the elements ζ_z. For $3 \leq h \leq a-2$ and $j = -h+2$ we have:

$$\beta_2^{(8)} - h\beta_1^{(8)} - (-h+2)\beta_3^{(8)} = \zeta_{h-3}(4 - 4a + a^2 + (5 - 4a + a^2)\rho_1 + (3-a)\rho_1^2).$$

Ninth iteration		
$\beta_1^{(9)} = -a^2 + 5a - 5 + (-a^2 + 6a - 6)\rho_1 + (2a-3)\rho_1^2$	$\beta_2^{(9)} = a^2 - 5a + 7 + (a^2 - 5a + 9)\rho_1 + (-a+5)\rho_1^2$	$\beta_3^{(9)} = -a^2 + 6a - 10 + (-a^2 + 6a - 13)\rho_1 + (a-7)\rho_1^2$
$N(\beta_1^{(9)}) = -a^2 + 5a - 5,\ Sgn(\beta_1^{(9)}) = (+,-,+)$	$N(\beta_2^{(9)}) = 1,\ Sgn(\beta_2^{(9)}) = (+,+,+)$	$N(\beta_3^{(9)}) = 2a - 7,\ Sgn(\beta_3^{(9)}) = (+,-,-)$
	$\beta_2^{(9)} = \tilde{\zeta}_1(4-4a+a^2+(5-4a+a^2)\rho_1 + (3-a)\rho_1^2)$	
Tenth iteration		
$\beta_1^{(10)} = a^2 - 5a + 7 + (a^2 - 5a + 9)\rho_1 + (-a+5)\rho_1^2$	$\beta_2^{(10)} = a - 5 - 7\rho_1 + (-a-4)\rho_1^2$	$\beta_3^{(10)} = -a^2 + 5a - 5 + (-a^2 + 6a - 6)\rho_1 + (2a-3)\rho_1^2$
$N(\beta_1^{(10)}) = 1,\ Sgn(\beta_1^{(10)}) = (+,+,+)$	$N(\beta_2^{(10)}) = -1,\ Sgn(\beta_2^{(10)}) = (+,+,-)$	$N(\beta_3^{(10)}) = -a^2+5a-5,\ Sgn(\beta_3^{(10)}) = (+,-,+)$
$\beta_1^{(10)} = \tilde{\zeta}_1(4-4a+a^2+(5-4a+a^2)\rho_1 + (3-a)\rho_1^2)$	$\beta_2^{(10)} = \tilde{\zeta}_{-1}(3 - a + (4-a)\rho_1 + 2\rho_1^2)$	

We can see that

$$\beta_3^{(10)} - h\beta_1^{(10)} - (a-h-1)\beta_2^{(10)} = \zeta_{-h-4}(7 - 5a + a^2 + (9 - 5a + a^2)\rho_1 + (5-a)\rho_1^2).$$

Hence, the elements $\beta_3^{(10)} - h\beta_1^{(10)} - j\beta_2^{(10)}$, for $-a \leq h \leq -4$ and $j = a-h-1$, generate all the elements ζ_z.

Therefore, we found out that first, third, sixth, seventh, eight and tenth iteration generate the elements ζ_z. Unlike previous cases the unit do not correspond, i.e. in each of this iteration at least one convergent generates the element $\tilde{\zeta}_1$ or $\tilde{\zeta}_{-1}$, but the unit which we multiplied these elements is different than unit by which we multiplied elements $\beta_2^{(i)} - h\beta_1^{(i)} - j\beta_3^{(i)}$ or $\beta_3^{(i)} - h\beta_1^{(i)} - j\beta_2^{(i)}$.

Thus, for the first root, we obtain that some iterations generate some indecomposable elements. Moreover, we proved that the first iteration generates all the indecomposable elements. We show that usually (except for the element ζ_z) in the iterations which generates some indecomposable elements the convergents generate the elements $\tilde{\gamma}_l, \tilde{\delta}_{n,m}, \tilde{\zeta}_k$. It seems that there is some connection between these two assertions. Also, when we look at the norms of the convergents, it seems that it is better when they are smaller.

6.2.2 Second root

For the second root, we proceed in the same way. Here it is enough to examine just the first five iterations because the hJPA expansion of the triple $(1, |\rho_2|, \rho_2^2)$ is periodic with preperiod length 3 and period length 3.

(i) Elements γ_w

The following tables are, as in the previous case, the list of those elements γ_w which are generated by this expansion.

First iteration		
$\beta_1^{(1)} = -\rho_2$	$\beta_2^{(1)} = \rho_2^2$	$\beta_3^{(1)} = 1$
$N(\beta_1^{(1)}) = -1,\ Sgn(\beta_1^{(1)}) = (+,+,-)$	$N(\beta_2^{(1)}) = 1,\ Sgn(\beta_2^{(1)}) = (+,+,+)$	$N(\beta_3^{(1)}) = 1,\ Sgn(\beta_3^{(1)}) = (+,+,+)$
$\beta_1^{(1)} = \tilde{\gamma}_1(-1 + (1-a)\rho_2 + \rho_2^2)$	$\beta_2^{(1)} = \gamma_0(-1 + (1-a)\rho_2 + \rho_2^2)$	$\beta_3^{(1)} = \gamma_0(-a\rho_2 + \rho_2^2)$

In this case, the elements $\beta_2^{(1)} - h\beta_1^{(1)} - j\beta_3^{(1)}$ generate all the elements γ_w. For $j = 0$ and $-a+3 \leq h \leq -1$ we get:

$$\beta_2^{(1)} - h\beta_1^{(1)} = \gamma_{-h}(-1 + (1-a)\rho_2 + \rho_2^2).$$

(This is exactly the same as the first iteration of the hJPA expansion of the $(1, |\rho_1|, \rho_1^2)$.)

Second iteration		
$\beta_1^{(2)} = \rho_2^2$	$\beta_2^{(2)} = 1 + (a-2)\rho_2$	$\beta_3^{(2)} = -\rho_2$
$N(\beta_1^{(2)}) = 1,\ Sgn(\beta_1^{(2)}) = (+,+,+)$	$N(\beta_2^{(2)}) = -a^2 + 5a - 5,\ Sgn(\beta_2^{(2)}) = (-,+,+)$	$N(\beta_3^{(2)}) = -1,\ Sgn(\beta_3^{(2)}) = (+,+,-)$
$\beta_1^{(2)} = \gamma_0(-1 + (1-a)\rho_2 + \rho_2^2)$		$\beta_3^{(2)} = \tilde{\gamma}_1(-1 + (1-a)\rho_2 + \rho_2^2)$
Third iteration		
$\beta_1^{(3)} = 1 + (-2+a)\rho_2 + (-a+2)\rho_2^2$	$\beta_2^{(3)} = -\rho_2 + (-a+2)\rho_2^2$	$\beta_3^{(3)} = \rho_2^2$
$N(\beta_1^{(3)}) = 2a^2 - 12a + 17,\ Sgn(\beta_1^{(3)}) = (-,+,-)$	$N(\beta_2^{(3)}) = a^2 - 5a + 5,\ Sgn(\beta_2^{(3)}) = (-,+,-)$	$N(\beta_3^{(3)}) = 1,\ Sgn(\beta_3^{(3)}) = (+,+,+)$
		$\beta_3^{(3)} = \gamma_0(-1 + (1-a)\rho_1 + \rho_1^2)$
Fourth iteration		
$\beta_1^{(4)} = -1 + (-a+1)\rho_2$	$\beta_2^{(4)} = -1 + (-a+2)\rho_2 + (a-1)\rho_2^2$	$\beta_3^{(4)} = 1 + (-2+a)\rho_2 + (-a+2)\rho_2^2$
$N(\beta_1^{(4)}) = -1,\ Sgn(\beta_1^{(4)}) = (+,+,-)$	$N(\beta_2^{(4)}) = 2a - 1,\ Sgn(\beta_2^{(4)}) = (+,+,+)$	$N(\beta_3^{(4)}) = 2a^2 - 12a + 17,\ Sgn(\beta_3^{(4)}) = (-,+,-)$
$\beta_1^{(4)} = \tilde{\gamma}_1(2 - a + (-1 + 3a - a^2)\rho_2 + (-2+a)\rho_2^2)$	$\beta_2^{(4)} = \gamma_1(2 - a + (-1 + 3a - a^2)\rho_2 + (-2+a)\rho_2^2)$	

Here we get that

$$\beta_2^{(4)} - h\beta_1^{(4)} = \gamma_{1-h}(2 - a + (-1 + 3a - a^2)\rho_2 + (-2+a)\rho_2^2).$$

Hence, for $j = 0$ and $-a+4 \leq h \leq 1$, the elements $\beta_2^{(4)} - h\beta_1^{(4)} - j\beta_3^{(4)}$ generate all elements γ_w.

Fifth iteration		
$\beta_1^{(5)} = \rho_2 + (a-1)\rho_2^2$	$\beta_2^{(5)} = a - 3 + (a^2 - 4a + 2)\rho_2 + (-a+2)\rho_2^2$	$\beta_3^{(5)} = -1 + (-a+1)\rho_2$
$N(\beta_1^{(5)}) = 1,\ Sgn(\beta_1^{(5)}) = (+,+,+)$	$N(\beta_2^{(5)}) = 2a - 7,\ Sgn(\beta_2^{(5)}) = (-,+,-)$	$N(\beta_3^{(5)}) = -1,\ Sgn(\beta_3^{(5)}) = (+,+,-)$
$\gamma_0(2 - a + (-1 + 3a - a^2)\rho_2 + (-2+a)\rho_2^2)$		$\tilde{\gamma}_1(2 - a + (-1 + 3a - a^2)\rho_2 + (-2+a)\rho_2^2)$

We can see that all the elements γ_w are generated in the first and fourth iteration (where the first iteration is the same as the first iteration for the first root). Again, the convergents in this iteration generate elements $\tilde{\gamma}_1$, and in both iteration, there is some totally positive convergent. Actually, in both of these iterations, the elements γ_w can be expressed as $\beta_2^{(i)} - h\beta_1^{(i)} - j\beta_3^{(i)}$ where $j = 0$, and the totally positive element is $\beta_2^{(i)}$.

(ii) Elements $\delta_{v,u}$

In the following tables, we can find a list of those elements, which are generated by convergents in some iterations. Again this list is incomplete, but we found that almost all iterations generated all the elements $\delta_{v,u}$.

First iteration		
$\beta_1^{(1)} = -\rho_2$	$\beta_2^{(1)} = \rho_2^2$	$\beta_3^{(1)} = 1$
$N(\beta_1^{(1)}) = -1,\ Sgn(\beta_1^{(1)}) = (+,+,-)$	$N(\beta_2^{(1)}) = 1,\ Sgn(\beta_2^{(1)}) = (+,+,+)$	$N(\beta_3^{(1)}) = 1,\ Sgn(\beta_3^{(1)}) = (+,+,+)$
$\beta_1^{(1)} = \tilde{\delta}_{1,1}\rho_2^2$	$\beta_2^{(1)} = \tilde{\delta}_{1,0}\rho_2^2$	$\beta_3^{(1)} = \delta_{-2,-1}(\rho_2 + \rho_2^2)$
	$\beta_2^{(1)} = \delta_{0,0}(\rho_2 + \rho_2^2)$	

Here, we get that the elements $\beta_3^{(1)} - h\beta_1^{(1)} - j\beta_2^{(1)}$ generate all elements $\delta_{v,u}$. Indeed for $2 \leq h \leq a+1$ and $2-h \leq j \leq 0$ we get:

$$\beta_3^{(1)} - h\beta_1^{(1)} - j\beta_2^{(1)} = \delta_{a-1-h-j,a-1-h}\rho_2^2.$$

For the elements $\beta_2^{(1)} - h\beta_1^{(1)} - j\beta_3^{(1)}$ we can see that

$$\beta_2^{(1)} - h\beta_1^{(1)} + h\beta_3^{(1)} = \delta_{-h,-h}(\rho_2 + \rho_2^2).$$

Second iteration		
$\beta_1^{(2)} = \rho_2^2$	$\beta_2^{(2)} = 1 + (a-2)\rho_2$	$\beta_3^{(2)} = -\rho_2$
$N(\beta_1^{(2)}) = 1,\ Sgn(\beta_1^{(2)}) = (+,+,+)$	$N(\beta_2^{(2)}) = -a^2 + 5a - 5,\ Sgn(\beta_2^{(2)}) = (-,+,+)$	$N(\beta_3^{(2)}) = -1,\ Sgn(\beta_3^{(2)}) = (+,+,-)$
$\beta_1^{(2)} = \tilde{\delta}_{1,0}\rho_2^2$	$\beta_2^{(2)} = \delta_{1,1}\rho_2^2$	$\beta_3^{(2)} = \tilde{\delta}_{1,1}\rho_2^2$
$\beta_1^{(2)} = \tilde{\delta}_{-1,-1}(1 + a\rho_2 + (a-1)\rho_2^2)$		

It holds that

$$\beta_2^{(2)} - h\beta_1^{(2)} - j\beta_3^{(2)} = \delta_{1-h-j,1-j}\rho_2^2.$$

Hence the elements $\beta_2^{(2)} - h\beta_1^{(2)} - j\beta_3^{(2)}$ generate all elements $\delta_{v,u}$. Then, for elements $\beta_3^{(2)} - h\beta_1^{(2)} - j\beta_2^{(2)}$ we get that:

$$\beta_3^{(2)} - h\beta_1^{(2)} + 1\beta_2^{(2)} = \delta_{2-h,2}\rho_2^2$$

and

$$\beta_3^{(2)} - h\beta_1^{(2)} - 1\beta_2^{(2)} = \delta_{-1+h,h}(1 + a\rho_2 + (-1+a)\rho_2^2).$$

Third iteration		
$\beta_1^{(3)} = 1 + (-2+a)\rho_2 + (-a+2)\rho_2^2$	$\beta_2^{(3)} = -\rho_2 + (-a+2)\rho_2^2$	$\beta_3^{(3)} = \rho_2^2$
$N(\beta_1^{(3)}) = 2a^2 - 12a + 17,\ Sgn(\beta_1^{(3)}) = (-,+,-)$	$N(\beta_2^{(3)}) = a^2 - 5a + 5,\ Sgn(\beta_2^{(3)}) = (-,+,-)$	$N(\beta_3^{(3)}) = 1,\ Sgn(\beta_3^{(3)}) = (+,+,+)$
		$\beta_3^{(3)} = \tilde{\delta}_{-1,-1}(1 + a\rho_2 + (a-1)\rho_2^2)$

Here we found just that

$$\beta_2^{(3)} - \beta_1^{(3)} - j\beta_3^{(3)} = \delta_{-1+j,j}(1 + a\rho_2 + (-1+a)\rho_2^2).$$

Fourth iteration		
$\beta_1^{(4)} = -1 + (-a+1)\rho_2$	$\beta_2^{(4)} = -1 + (-a+2)\rho_2 + (a-1)\rho_2^2$	$\beta_3^{(4)} = 1 + (-2+a)\rho_2 + (-a+2)\rho_2^2$
$N(\beta_1^{(4)}) = -1,\ Sgn(\beta_1^{(4)}) = (+,+,-)$	$N(\beta_2^{(4)}) = 2a - 1,\ Sgn(\beta_2^{(4)}) = (+,+,+)$	$N(\beta_3^{(4)}) = 2a^2 - 12a + 17,\ Sgn(\beta_3^{(4)}) = (-,+,-)$
$\beta_1^{(4)} = \tilde{\delta}_{1,1}(\rho_2 + (-1+a)\rho_2^2)$	$\beta_2^{(4)} = \tilde{\delta}_{2,1}(\rho_2 + (-1+a)\rho_2^2)$	

The elements $\beta_3^{(4)} - h\beta_1^{(4)} - j\beta_2^{(4)}$ generate all elements $\delta_{v,u}$. We get that

$$\beta_3^{(4)} - h\beta_1^{(4)} - j\beta_2^{(4)} = \delta_{a-4-h-2j,a-3-h-j}(\rho_2 + (-1+a)\rho_2^2).$$

And the elements $\beta_2^{(4)} - h\beta_1^{(4)} - j\beta_3^{(4)}$ generate a line in P:

$$\beta_2^{(4)} - h\beta_1^{(4)} + \beta_3^{(4)} = \delta_{a-2-h,a-2-h}(\rho_2 + (-1+a)\rho_2^2).$$

Fifth iteration		
$\beta_1^{(5)} = \rho_2 + (a-1)\rho_2^2$	$\beta_2^{(5)} = a-3+(a^2-4a+2)\rho_2+(-a+2)\rho_2^2$	$\beta_3^{(5)} = -1 + (-a+1)\rho_2$
$N(\beta_1^{(5)}) = 1,\ Sgn(\beta_1^{(5)}) = (+,+,+)$	$N(\beta_2^{(5)}) = 2a - 7,\ Sgn(\beta_2^{(5)}) = (-,+,-)$	$N(\beta_3^{(5)}) = -1,\ Sgn(\beta_3^{(5)}) = (+,+,-)$
$\beta_1^{(5)} = \tilde{\delta}_{1,0}(\rho_2 + (-1+a)\rho_2^2)$	$\beta_2^{(5)} = \delta_{0,1}(\rho_2 + (-1+a)\rho_2^2)$	$\beta_3^{(5)} = \tilde{\delta}_{1,1}(\rho_2 + (-1+a)\rho_2^2)$
$\beta_1^{(5)} = \tilde{\delta}_{-1,-1}(-1 + a + (-a+a^2)\rho_2 + (2-2a+a^2)\rho_2^2)$		

Here we get that

$$\beta_2^{(5)} - h\beta_1^{(5)} - j\beta_3^{(5)} = \delta_{-h-j,1-j}(\rho_2 + (-1+a)\rho_2^2).$$

Hence, for $4-a \leq j \leq 1$ and $1 \leq h \leq a-3+j$, the elements $\beta_2^{(5)} - h\beta_1^{(5)} - j\beta_3^{(5)}$ generate all the elements $\delta_{u,v}$. And also we can see that the elements $\beta_3^{(5)} - h\beta_1^{(5)} - j\beta_2^{(5)}$ generate at P at least two lines:

$$\beta_3^{(5)} - h\beta_1^{(5)} + \beta_2^{(5)} = \delta_{1-h,2}(\rho_2 + (-1+a)\rho_2^2)$$

and

$$\beta_3^{(5)} - h\beta_1^{(5)} - \beta_2^{(5)} = \delta_{-2+h,-1+h}(-1+a+(-a+a^2)\rho_2 + (2-2a+a^2)\rho_2^2).$$

We found out that all the iterations, except the third iteration, give all the elements $\delta_{v,u}$. But in the third iteration, the first and second convergent have a large norm. Similarly, as in the previous case, the convergents in all iterations (except the third) generated a basis of lattice L. And again, there is a correspondence between the unit which we need (i.e. in every iteration the two convergents which generate a basis of L lie in the same plane containing indecomposable elements).

(iii) Elements ζ_z

In the following tables, we can find a list of those elements, which are generated by the iterations of this expansion.

First iteration		
$\beta_1^{(1)} = -\rho_2$	$\beta_2^{(1)} = \rho_2^2$	$\beta_3^{(1)} = 1$
$N(\beta_1^{(1)}) = -1,\ Sgn(\beta_1^{(1)}) = (+,+,-)$	$N(\beta_2^{(1)}) = 1,\ Sgn(\beta_2^{(1)}) = (+,+,+)$	$N(\beta_3^{(1)}) = 1,\ Sgn(\beta_3^{(1)}) = (+,+,+)$
$\beta_1^{(1)} = \tilde{\zeta}_{-1}(1+a\rho_1-\rho_1^2)$	$\beta_2^{(1)} = \tilde{\zeta}_1(-1+(1-a)\rho_1+\rho_1^2)$	$\beta_3^{(1)} = \tilde{\zeta}_1(-a\rho_1+\rho_1^2)$

The first iteration generate all elements ζ_z. For $4 \leq h \leq a$ and $j = 2-h$ we get that

$$\beta_2^{(1)} - h\beta_1^{(1)} - (2-h)\beta_3^{(1)} = \zeta_{h-4}.$$

Second iteration		
$\beta_1^{(2)} = \rho_2^2$	$\beta_2^{(2)} = 1+(a-2)\rho_2$	$\beta_3^{(2)} = -\rho_2$
$N(\beta_1^{(2)}) = 1,\ Sgn(\beta_1^{(2)}) = (+,+,+)$	$N(\beta_2^{(2)}) = -a^2+5a-5,\ Sgn(\beta_2^{(2)}) = (-,+,+)$	$N(\beta_3^{(2)}) = -1,\ Sgn(\beta_3^{(2)}) = (+,+,-)$
$\beta_1^{(2)} = \tilde{\zeta}_1(-1+(1-a)\rho_1+\rho_1^2)$		$\beta_3^{(2)} = \tilde{\zeta}_{-1}(1+a\rho_1-\rho_1^2)$
Third iteration		
$\beta_1^{(3)} = 1+(-2+a)\rho_2+(-a+2)\rho_2^2$	$\beta_2^{(3)} = -\rho_2+(-a+2)\rho_2^2$	$\beta_3^{(3)} = \rho_2^2$
$N(\beta_1^{(3)}) = 2a^2-12a+17,\ Sgn(\beta_1^{(3)}) = (-,+,-)$	$N(\beta_2^{(3)}) = a^2-5a+5,\ Sgn(\beta_2^{(3)}) = (-,+,-)$	$N(\beta_3^{(3)}) = 1,\ Sgn(\beta_3^{(3)}) = (+,+,+)$
		$\beta_3^{(3)} = \tilde{\zeta}_1(-1+(1-a)\rho_1+\rho_1^2)$
Fourth iteration		
$\beta_1^{(4)} = -1+(-a+1)\rho_2$	$\beta_2^{(4)} = -1+(-a+2)\rho_2+(a-1)\rho_2^2$	$\beta_3^{(4)} = 1+(-2+a)\rho_2+(-a+2)\rho_2^2$
$N(\beta_1^{(4)}) = -1,\ Sgn(\beta_1^{(4)}) = (+,+,-)$	$N(\beta_2^{(4)}) = 2a-1,\ Sgn(\beta_2^{(4)}) = (+,+,+)$	$N(\beta_3^{(4)}) = 2a^2-12a+17,\ Sgn(\beta_3^{(4)}) = (-,+,-)$
$\beta_1^{(4)} = \tilde{\zeta}_{-1}(-1+a+(-2a+a^2)\rho_1+(2-a)\rho_1^2)$		
Fifth iteration		
$\beta_1^{(5)} = \rho_2+(a-1)\rho_2^2$	$\beta_2^{(5)} = a-3+(a^2-4a+2)\rho_2+(-a+2)\rho_2^2$	$\beta_3^{(5)} = -1+(-a+1)\rho_2$
$N(\beta_1^{(5)}) = 1,\ Sgn(\beta_1^{(5)}) = (+,+,+)$	$N(\beta_2^{(5)}) = 2a-7,\ Sgn(\beta_2^{(5)}) = (-,+,-)$	$N(\beta_3^{(5)}) = -1,\ Sgn(\beta_3^{(5)}) = (+,+,-)$
$\beta_1^{(5)} = \tilde{\zeta}_1(2-a+(-1+3a-a^2)\rho_1+(-2+a)\rho_1^2)$		$\beta_2^{(5)} = \tilde{\zeta}_{-1}(-1+a+(-2a+a^2)\rho_1+(2-a)\rho_1^2)$

We proved that the first iteration generates all these elements. In these iterations, the convergents generate the element $\tilde{\zeta}_1$ and there is a correspondence between the unit by which we multiplied the convergents and the elements

$\beta_2^{(1)} - h\beta_1^{(1)} - j\beta_3^{(1)}$. For the other iterations, we did not prove if they generate all the elements ζ_z. However, if some other iterations generate all elements ζ_z, there is probably not correspondence between these units, because we focus on these cases and we were not successful.

In summary, the iterations in the hJPA expansion of the triple $(1, |\rho_2|, \rho_2^2)$ generated all the indecomposable elements. Again, the first iteration generated all indecomposable elements, but the first iteration is exactly the same as the first iteration in the previous case. It seems that for iteration which generated some indecomposable elements it is important if the convergents are equal to some elements $\tilde{\gamma}_l, \tilde{\delta}_{m,n}$ or $\tilde{\zeta}_k$ (up to multiplication by units).

6.2.3 Third root

For the last root and the hJPA expansion of the sequence $(1, |\rho_3|, \rho_3^2)$, it is enough to examine the first two iterations because the other iterations generate the same indecomposable elements as the second iteration.

(i) Elements γ_w

For the elements γ_w, we found out that both of these iterations generate all these elements. In both of these iterations, the convergents generated $\tilde{\gamma}_{-1}$ and in the second iteration, we used the same unit $(1 + 2a + a^2 + (-1 - a + 2a^2 + a^3)\rho_3 + (-2a - a^2)\rho_3^2)$ for the multiplication of the convergent and the elements $\beta_2^{(1)} - h\beta_1^{(1)} - j\beta_3^{(1)}$. However, in the first iteration there is not a correspondence between units. It can be because none of the indices h, j is constant in the elements $\beta_3^{(1)} - h\beta_1^{(1)} - j\beta_2^{(1)}$, which generated elements γ_w. In the following tables we can find a list of elements γ_w, which are generated by the iterations of this expansion.

First iteration		
$\beta_1^{(1)} = -a + \rho_3$	$\beta_2^{(1)} = -a^2 + \rho_3^2$	$\beta_3^{(1)} = 1$
$N(\beta_1^{(1)}) = 1,\ Sgn(\beta_1^{(1)}) = (-,-,+)$	$N(\beta_2^{(1)}) = 1 - 2a + 2a^3,\ Sgn(\beta_2^{(1)}) = (-,-,+)$	$N(\beta_3^{(1)}) = 1,\ Sgn(\beta_3^{(1)}) = (+,+,+)$
$\beta_1^{(1)} = \tilde{\gamma}_{-1}(-1 - a + (1 - a - a^2)\rho_3 + (1 + a)\rho_3^2)$		$\beta_3^{(1)} = \gamma_{-1}(1 + a + (-1 + a^2)\rho_3 - a\rho_3^2)$

We get that the elements $\beta_3^{(1)} - h\beta_1^{(1)} - j\beta_2^{(1)}$ generate all the elements γ_w. For $2 \leq j \leq a - 2$ and $h = -aj$ we get:

$$\beta_3^{(1)} + aj\beta_1^{(1)} - j\beta_2^{(1)} = \gamma_{-1+j}(1 + a + (-1 + a^2)\rho_3 - a\rho_3^2).$$

Second iteration		
$\beta_1^{(2)} = a^2 - 2a\rho_3 + \rho_3^2$	$\beta_2^{(2)} = 1 + a^2 + a^3 + (-a - a^2)\rho_3$	$\beta_3^{(2)} = -a + \rho_3$
$N(\beta_1^{(2)}) = 1,\ Sgn(\beta_1^{(2)}) = (+,+,+)$	$N(\beta_2^{(2)}) = 1 + a + 3a^2 + 2a^3,\ Sgn(\beta_2^{(2)}) = (+,+,+)$	$N(\beta_3^{(2)}) = 1,\ Sgn(\beta_3^{(2)}) = (-,-,+)$
$\beta_1^{(2)} = \tilde{\gamma}_{-1}(1 + 2a + a^2 + (-1 - a + 2a^2 + a^3)\rho_3 + (-2a - a^2)\rho_3^2)$		$\beta_3^{(2)} = \gamma_0(1 + (a + a^2)\rho_3 + (-1 - a)\rho_3^2)$

We can see that the elements $\beta_2^{(1)} - h\beta_1^{(1)} - j\beta_3^{(1)}$ generate all the elements γ_w. For $2 + a \leq h \leq 2a - 2$ and $j = 0$ we get:

$$\beta_2^{(1)} - h\beta_1^{(1)} = \gamma_{-1-a+h}(1 + 2a + a^2 + (-1 - a + 2a^2 + a^3)\rho_3 + (-2a - a^2)\rho_3^2).$$

(ii) Elements $\delta_{v,u}$

We did not find out if some iterations of this expansion generate all the elements $\delta_{v,u}$. It is probably related to the fact that in both iterations, the second

convergent has a large norm. And in each iteration, the convergents generated a basis of lattice L, but they are not multiplied by the same unit. In the following tables, we can find a list of elements $\delta_{v,u}$, which are generated by the iterations of this expansion.

First iteration		
$\beta_1^{(1)} = -a + \rho_3$	$\beta_2^{(1)} = -a^2 + \rho_3^2$	$\beta_3^{(1)} = 1$
$N(\beta_1^{(1)}) = 1,\ Sgn(\beta_1^{(1)}) = (-,-,+)$	$N(\beta_2^{(1)}) = 1 - 2a + 2a^3,\ Sgn(\beta_2^{(1)}) = (-,-,+)$	$N(\beta_3^{(1)}) = 1,\ Sgn(\beta_3^{(1)}) = (+,+,+)$
$\beta_1^{(1)} = \delta_{-2,-1}$		$\beta_3^{(1)} = \tilde{\delta}_{-1,-1}(\rho_3)$

The elements $\beta_2^{(1)} - h\beta_1^{(1)} - j\beta_3^{(1)}$ generate a line at P:

$$\beta_2^{(1)} - a\beta_1^{(1)} - j\beta_3^{(1)} = \delta_{-2+j,-1+j}\rho_3.$$

Second iteration		
$\beta_1^{(2)} = a^2 - 2a\rho_3 + \rho_3^2$	$\beta_2^{(2)} = 1 + a^2 + a^3 + (-a - a^2)\rho_3$	$\beta_3^{(2)} = -a + \rho_3$
$N(\beta_1^{(2)}) = 1,\ Sgn(\beta_1^{(2)}) = (+,+,+)$	$N(\beta_2^{(2)}) = 1 + a + 3a^2 + 2a^3,\ Sgn(\beta_2^{(2)}) = (+,+,+)$	$N(\beta_3^{(2)}) = 1,\ Sgn(\beta_3^{(2)}) = (-,-,+)$
$\beta_1^{(2)} = \tilde{\delta}_{0,-1}(1 + 2a + a^2 + (-1 - a + 2a^2 + a^3)\rho_3 + (-2a - a^2)\rho_3^2)$		$\beta_2^{(2)} = \delta_{-2,-1}$

(iii) Elements ζ_z

For these elements, we again did not find out if that some iteration generate all these elements ζ_z. We just found some specific combinations of i, h, j such that an iteration generated one isolated element $\zeta_{\tilde{z}}$, but it not so interesting. In the following tables, we can find a list of elements $\tilde{\zeta}_k$, which are generated by the iterations of this expansion.

First iteration		
$\beta_1^{(1)} = -a + \rho_3$	$\beta_2^{(1)} = -a^2 + \rho_3^2$	$\beta_3^{(1)} = 1$
$N(\beta_1^{(1)}) = 1,\ Sgn(\beta_1^{(1)}) = (-,-,+)$	$N(\beta_2^{(1)}) = 1 - 2a + 2a^3,\ Sgn(\beta_2^{(1)}) = (-,-,+)$	$N(\beta_3^{(1)}) = 1,\ Sgn(\beta_3^{(1)}) = (+,+,+)$
$\beta_1^{(1)} = \tilde{\zeta}_1(1 + (a + a^2)\rho_3 + (-1 - a)\rho_3^2)$		$\beta_3^{(1)} = \tilde{\zeta}_1(-a\rho_3 + \rho_3^2)$
Second iteration		
$\beta_1^{(2)} = a^2 - 2a\rho_3 + \rho_3^2$	$\beta_2^{(2)} = 1 + a^2 + a^3 + (-a - a^2)\rho_3$	$\beta_3^{(2)} = -a + \rho_3$
$N(\beta_1^{(2)}) = 1,\ Sgn(\beta_1^{(2)}) = (+,+,+)$	$N(\beta_2^{(2)}) = 1 + a + 3a^2 + 2a^3,\ Sgn(\beta_2^{(2)}) = (+,+,+)$	$N(\beta_3^{(2)}) = 1,\ Sgn(\beta_3^{(2)}) = (-,-,+)$
$\beta_1^{(2)} = \tilde{\zeta}_1(-1 - 2a + (1 - a - 2a^2 - a^3)\rho_3 + (1 + 2a + a^2)\rho_3^2)$		$\beta_3^{(2)} = \tilde{\zeta}_1(1 + (a + a^2)\rho_3 + (-1 - a)\rho_3^2)$

In summary, this expansion does not generate all the indecomposable elements. This is probably related to the fact that the norms in this expansion are large. The norms of the second convergent in both iterations are cubic polynomials in a. Hence, they are a lot larger than the norms of any indecomposable elements (so they are not indecomposable elements).

6.3 Summary

We have seen that some iterations of the expansions generated some indecomposable elements. The first two expansion generate all the indecomposable elements. We did not find if the third expansion generates all indecomposable elements but this expansion has some coefficients larger norm (norm which is cubic in a), hence we think that this expansion does not generate all indecomposable elements. Moreover, we found some rules when the iteration can generate the

indecomposable elements. Here, we will formulate some general results for the expansions of the triples of elements from the fields $\mathbb{Q}(\rho_i)$.

6.3.1 Elements γ_w

In the previous section we have seen that in some iterations, the elements $\beta_2^{(i)} - h_1\beta_1^{(i)} - j_1\beta_3^{(i)}$ and $\beta_3^{(i)} - h_2\beta_1^{(i)} - j_2\beta_2^{(i)}$ generated all elements γ_w. We have noticed that in the iteration which generated all the elements γ_w, often some convergent generates the element $\tilde{\gamma}_1$ or $\tilde{\gamma}_{-1}$. Moreover, in most of the cases, at least one of these elements is totally positive. From this we can get the following conjecture. We state it just for elements $\beta_2^{(i)} - h\beta_1^{(i)} - j\beta_3^{(i)}$, but we can formulate a similar conjecture for elements $\beta_3^{(i)} - h\beta_1^{(i)} - j\beta_2^{(i)}$.

Conjecture 1. *There exist a nonzero polynomial $g(x) \in \mathbb{Z}[x]$ with the following properties: Let us fix a $a \in \mathbb{N}, a \geq 5$. Let $(\beta_1^{(0)}, \beta_2^{(0)}, \beta_3^{(0)})$ be a basis of the field $\mathbb{Q}(\rho)$, where ρ is one of the roots of a polynomial $x^3 - (a-1)x^2 - ax - 1$. And let $\langle \beta^{(\nu)} \rangle_{\nu \geq 0}$ be the hJPA expansion of a vector $(\beta_1^{(0)}, \beta_2^{(0)}, \beta_3^{(0)})$. Then whenever for some iteration i, there exist a unit ϵ and $b, c \in \mathbb{Z}$ such that*

$$b\beta_1^{(i)} + c\beta_3^{(i)} = \tilde{\gamma}_1\epsilon, \qquad |N(\beta_2^{(i)})| \leq g(a) \qquad \text{and} \qquad Sgn(\beta_2^{(i)}\epsilon) = (+,+,+),$$

then for every $1 \leq w \leq a-3$, there exist $h, j \in \mathbb{Z}$ such that

$$(\beta_2^{(i)} - h\beta_1^{(i)} - j\beta_3^{(i)})\epsilon = \gamma_w.$$

Equivalently, for these h, j it holds that the elements $\beta_2^{(i)} - h\beta_1^{(i)} - j\beta_3^{(i)}$ generate all the elements γ_w.

From the examples, it seems that g could be a constant or linear polynomial. But it does not seem that g is a quadratic polynomial because the norm of the elements γ_w is linear in a.

The assumption $b\beta_1^{(i)} + c\beta_3^{(i)} = \tilde{\gamma}_1\epsilon$ leads to the Thue equations

$$b\frac{\beta_1^{(i)}}{\tilde{\gamma}_1} + c\frac{\beta_3^{(i)}}{\tilde{\gamma}_1} = \tilde{\gamma}_1\epsilon$$

and

$$N\left(b\frac{\beta_1^{(i)}}{\tilde{\gamma}_1} + c\frac{\beta_3^{(i)}}{\tilde{\gamma}_1}\right) = \pm 1,$$

which have, for concrete a, finitely many solutions [19]. However, we do not know how the solutions for a general a look like, hence this assumptions are hard to verify for some iteration in the expansion. Also, even if we know that the assumptions of conjecture hold, it is hard to find such indices h, j that the elements $\beta_2^{(i)} - h\beta_1^{(i)} - j\beta_3^{(i)}$ generate the elements γ_w, because this problem again leads to the Thue equation. Firstly, we have to find some d, e such that

$$(\beta_2^{(i)} - d\beta_1^{(i)} - e\beta_3^{(i)})\epsilon = \gamma_0$$

and

$$N(\beta_2^{(i)} - d\beta_1^{(i)} - e\beta_3^{(i)}) = 1.$$

We can use any γ_w, but we use γ_0 because it is a unit. When we find such indices d, e, it is clear that for every $1 \leq w \leq a-3$ the following equation holds:

$$\left(\beta_2^{(i)} - (d+wb)\beta_1^{(i)} - (e+wc)\beta_3^{(i)})\right)\epsilon = \gamma_w.$$

The assumption $Sgn(\beta_2^{(i)}\epsilon) = (+,+,+)$ is maybe unnecessary, but it holds in every iterations for which we found out that it generates all the elements γ_w. Also these elements are the only totally positive indecomposable elements and it makes sense that the signature is important.

We can easily see that the almost opposite implication holds.

Theorem 19. *Let $\langle\beta^{(\nu)}\rangle_{\nu\geq 0}$ be the hJPA expansion of a vector $(\beta_1^{(0)}, \beta_2^{(0)}, \beta_3^{(0)}) \in \mathbb{Q}(\rho)^3$. Let us assume that elements $\beta_2^{(i)} - h\beta_1^{(i)} - j\beta_3^{(i)}$ generate all the elements γ_w, i.e. there exists unit ϵ such that for every $1 \leq w \leq a-3$*

$$\gamma_w\epsilon \in \left\{\beta_2^{(i)} - h\beta_1^{(i)} - j\beta_3^{(i)}, h, j \in \mathbb{Z}.\right\}$$

Then there exist $b, c \in \mathbb{Z}$ such that $b\beta_1^{(i)} + c\beta_3^{(i)} = \tilde{\gamma}_1\epsilon$.

Proof. From assumptions we know that there have to exist $h, j, h', j' \in \mathbb{Z}$ such that

$$\gamma_1\epsilon = \beta_2^{(i)} - h\beta_1^{(i)} - j\beta_3^{(i)}$$

and

$$\gamma_2\epsilon = \beta_2^{(i)} - h'\beta_1^{(i)} - j'\beta_3^{(i)}.$$

From this we can get that

$$\begin{aligned}\tilde{\gamma}_1\epsilon &= (\gamma_2 - \gamma_1)\epsilon = (\beta_2^{(i)} - h'\beta_1^{(i)} - j'\beta_3^{(i)}) - (\beta_2^{(i)} - h\beta_1^{(i)} - j\beta_3^{(i)}) \\ &= (h-h')\beta_1^{(i)} + (j-j')\beta_3^{(i)}.\end{aligned}$$

Hence, the theorem holds for $b = (h-h')$ and $c = (j-j')$. □

6.3.2 Elements $\delta_{v,u}$

In the previous section, we saw some iterations which generate elements $\delta_{v,u}$. It seems that the most important thing here is whether two of the convergents (after multiplying by the same unit ϵ) generate a basis of the lattice L. Then, when the third convergent has a small norm, the iteration generates the indecomposable elements $\delta_{v,u}$. From this we can get the following conjectures, we again write just the conjecture for $\beta_2^{(i)} - h\beta_1^{(i)} - j\beta_3^{(i)}$.

Conjecture 2. *There exist a nonzero polynomial $g(x) \in \mathbb{Z}[x]$ with the following properties: Let us fix a $a \in \mathbb{N}, a \geq 5$. Let $(\beta_1^{(0)}, \beta_2^{(0)}, \beta_3^{(0)})$ be a basis of the field $\mathbb{Q}(\rho)$, where ρ is one of the roots of a polynomial $x^3 - (a-1)x^2 - ax - 1$. And let $\langle\beta^{(\nu)}\rangle_{\nu\geq 0}$ be the hJPA expansion of a vector $(\beta_1^{(0)}, \beta_2^{(0)}, \beta_3^{(0)})$. Then whenever for some iteration i, there exist a unit ϵ such that*

$$\beta_1^{(i)} = \tilde{\delta}_{n,m}\epsilon, \qquad \beta_3^{(i)} = \tilde{\delta}_{n',m'}\epsilon \qquad \text{and} \qquad |N(\beta_2^{(i)})| \leq g(a),$$

where the pair $\tilde{\delta}_{n,m}, \tilde{\delta}_{n',m'}$ are a basis of lattice L, then for every $1 \leq v \leq a-3$, $0 \leq u \leq v$, there exist $h, j \in \mathbb{Z}$ such that

$$(\beta_2^{(i)} - h\beta_1^{(i)} - j\beta_3^{(i)})\epsilon = \delta_{v,u}.$$

Equivalently, for these h, j it holds that the elements $\beta_2^{(i)} - h\beta_1^{(i)} - j\beta_3^{(i)}$ generate all the elements $\delta_{v,u}$.

Norms of the elements $\delta_{v,u}$ are quadratic polynomials in a. So, we think that g could be a constant or linear polynomial.

Here, if assumptions hold, is relatively easy to find the right h, j for some v, u the right h, j. Because if the assumptions (and the conjecture) hold, then we have the following equations:

$$\{\delta_{v,u}, v, u \in \mathbb{Z}\} = \left\{(\beta_2^{(i)} - h\beta_1^{(i)} - j\beta_3^{(i)})\epsilon, h, j \in \mathbb{Z}\right\}$$

Hence, for every h, j there exist v, u such that

$$(\beta_2^{(i)} - h\beta_1^{(i)} - j\beta_3^{(i)})\epsilon = \delta_{v,u}.$$

Especially, for $h, j = 0$ we get that there exist v, u such that

$$\beta_2^{(i)}\epsilon = \delta_{v,u}.$$

When we find solution of these equations, it is easy to determine right $h, j \in \mathbb{Z}$ for every $1 \leq v \leq a-3, 0 \leq u \leq v$ such that $(\beta_2^{(i)} - h\beta_1^{(i)} - j\beta_3^{(i)})\epsilon = \delta_{v,u}$. That follows from the fact that we know a basis of lattice L.

In the following theorem, we show that the almost opposite implication of the conjecture holds:

Theorem 20. *Let $\langle\beta^{(\nu)}\rangle_{\nu\geq 0}$ be the hJPA expansion of a vector $(\beta_1^{(0)}, \beta_2^{(0)}, \beta_3^{(0)}) \in \mathbb{Q}(\rho)^3$. Then assume that elements $\beta_2^{(i)} - h\beta_1^{(i)} - j\beta_3^{(i)}$ generate all elements $\delta_{v,u}$, i.e. that there exist unit ϵ such that for every $1 \leq v \leq a-3, 0 \leq u \leq v$,*

$$\delta_{v,u}\epsilon \in \left\{\beta_2^{(i)} - h\beta_1^{(i)} - j\beta_3^{(i)}, h, j \in \mathbb{Z}\right\}.$$

Then there exists a m, n, m', n' such that $\beta_1^{(i)} = \tilde{\delta}_{n,m}\epsilon$ and $\beta_3^{(i)} = \tilde{\delta}_{n',m'}\epsilon$, where the vectors $(n, m), (n', m')$ are linearly independent.

Proof. From the assumptions we know that there have to exist b, c, b', c', b'', c'' such that:

$$\beta_2^{(i)} - b\beta_1^{(i)} - c\beta_3^{(i)} = \delta_{1,0}\epsilon,$$
$$\beta_2^{(i)} - b'\beta_1^{(i)} - c'\beta_3^{(i)} = \delta_{2,0}\epsilon,$$
$$\beta_2^{(i)} - b''\beta_1^{(i)} - c''\beta_3^{(i)} = \delta_{1,1}\epsilon.$$

From this we get that:

$$\tilde{\delta}_{1,0}\epsilon \;=\; (\delta_{2,0} - \delta_{1,0})\epsilon \;=\; (b - b')\beta_1^{(i)} - (c - c')\beta_3^{(i)}$$

and

$$\tilde{\delta}_{0,1}\epsilon \;=\; (\delta_{1,1} - \delta_{1,0})\epsilon \;=\; (b - b'')\beta_1^{(i)} - (c - c'')\beta_3^{(i)}.$$

Hence, we get that $\beta_1^{(i)}, \beta_3^{(i)}$ generate a basis of lattice L. Therefore, there exist m, n, m', n' such that $\beta_1^{(i)} = \tilde{\delta}_{n,m}\epsilon$ and $\beta_3^{(i)} = \tilde{\delta}_{n',m'}\epsilon$, where the vectors $(n, m), (n', m')$ are linearly independent. □

6.3.3 Elements ζ_z

Elements ζ_z, similarly as the elements γ_w, lie on a line. And we saw that for iterations that generate elements ζ_z, we need similar conditions. We have noticed that in the iteration which generated all elements ζ_z often some convergent generates the element $\tilde{\zeta}_1$ or $\tilde{\zeta}_{-1}$. From this, we can get the following conjecture. We again state it just for elements $\beta_2^{(i)} - h\beta_1^{(i)} - j\beta_3^{(i)}$.

Conjecture 3. *There exist a nonzero polynomial $g(x) \in \mathbb{Z}[x]$ with the following properties: Let us fix a $a \in \mathbb{N}, a \geq 5$. Let $(\beta_1^{(0)}, \beta_2^{(0)}, \beta_3^{(0)})$ be a basis of the field $\mathbb{Q}(\rho)$, where ρ is one of the roots of a polynomial $x^3 - (a-1)x^2 - ax - 1$. And let $\langle \beta^{(\nu)} \rangle_{\nu \geq 0}$ be the hJPA expansion of a vector $(\beta_1^{(0)}, \beta_2^{(0)}, \beta_3^{(0)})$. Then whenever for some iteration i, there exist a unit ϵ and $b, c \in \mathbb{Z}$ such that*

$$b\beta_1^{(i)} + c\beta_3^{(i)} = \tilde{\zeta}_1\epsilon \qquad \text{and} \qquad |N(\beta_2^{(i)})| \leq g(a),$$

then for every $0 \leq z \leq a-4$, there exist $h, j \in \mathbb{Z}$ such that

$$(\beta_2^{(i)} - h\beta_1^{(i)} - j\beta_3^{(i)})\epsilon = \zeta_z.$$

Equivalently, for these h, j it holds that the elements $\beta_2^{(i)} - h\beta_1^{(i)} - j\beta_3^{(i)}$ generate all the elements ζ_z.

From the examples, it seems that g could be a constant or linear polynomial. And as the case for the elements γ_w, the assumption $b\beta_1^{(i)} + c\beta_3^{(i)} = \tilde{\zeta}_1\epsilon$ is hard to verify, because for this, we need to solve the Thue equation:

$$b\frac{\beta_1^{(i)}}{\tilde{\zeta}_1} + c\frac{\beta_3^{(i)}}{\tilde{\zeta}_1} = \tilde{\gamma}_1\epsilon.$$

Also, even if we know that the assumptions of conjecture hold, it is hard to find such h, j that the element $\beta_2^{(i)} - h\beta_1^{(i)} - j\beta_3^{(i)}$ generate the elements ζ_z, because this problem again leads to Thue equation.

And as in the previous case, we can easily see that the almost opposite implication holds:

Theorem 21. *Let $\langle \beta^{(\nu)} \rangle_{\nu \geq 0}$ be the hJPA expansion of a vector $(\beta_1^{(0)}, \beta_2^{(0)}, \beta_3^{(0)}) \in \mathbb{Q}(\rho)^3$. Then assume that elements $\beta_2^{(i)} - h\beta_1^{(i)} - j\beta_3^{(i)}$ generate all elements ζ_z, i.e. there exists unit ϵ such that for every $0 \leq z \leq a-4$*

$$\zeta_z\epsilon \in \left\{\beta_2^{(i)} - h\beta_1^{(i)} - j\beta_3^{(i)}, h, j \in \mathbb{Z}\right\}.$$

Then there exist $b, c \in \mathbb{Z}$ such that $b\beta_1^{(i)} + c\beta_3^{(i)} = \tilde{\zeta}_1\epsilon$.

Proof. The proof is the same as the proof of Theorem 19. □

Bibliography

[1] L. Bernstein. *The Jacobi-Perron algorithm: Its theory and application*, volume 207. Springer, 2006.

[2] V. Blomer and V. Kala. Number fields without n-ary universal quadratic forms. *Mathematical Proceedings of the Cambridge Philosophical Society*, 159(2):239–252, 2015.

[3] A. Dress and R. Scharlau. Indecomposable totally positive numbers in real quadratic orders. *Journal of Number Theory*, 14(3):292–306, 1982.

[4] E. Dubois. Algorithme de Jacobi-Perron dans un corps de séries formelles. *Séminaire Delange-Pisot-Poitou. Théorie des nombres*, 13(1):1–11, 1970.

[5] E. Dubois, A. Farhane, and R. Roux. Etude des interruptions dans l'algorithme de Jacobi–Perron. *Bulletin of The Australian Mathematical Society - BULL AUST MATH SOC*, 69, 04 2004.

[6] E. Dubois, A. Farhane, and R. Roux. The jacobi–perron algorithm and pisot numbers. *Acta Arithmetica*, 111:269–275, 2004.

[7] V. Ennola. Fundamental units in a family of cubic fields. *Journal de théorie des nombres de Bordeaux*, 16(3):569–575, 2004.

[8] V. Ennola. *Cubic number fields with exceptional units*. De Gruyter, 2011.

[9] H. Hasse and L. Bernstein. Einheitenberechnung mittels des Jacobi-Perronschen algorithmus. *Journal für die reine und angewandte Mathematik*, 1965(218):51–69, 1965.

[10] V. Kala. Universal quadratic forms and elements of small norm in real quadratic fields. *Bulletin of the Australian Mathematical Society*, 94(1):7–14, 2016.

[11] V. Kala, E. Sgallová, and M. Tinková. Algebraic application of Jacobi-Perron algorithm over famillies of cubic field, in preparation.

[12] V. Kala and M. Tinková. Universal quadratic forms, small norms and traces in families of number fields, `https://arxiv.org/abs/2005.12312`. 2020.

[13] C. Levesque and G. Rhin. A few classes of periodic continued fractions. *Utilitas Mathematica*, 30:79–107, 1986.

[14] Z. Masáková. Multidimensional continued fractions: Jacobi-Perron algorithm for simultaneous rational approximation of d real numbers. `https://people.fjfi.cvut.cz/masakzuz/prezentace_soubory/MultiDimContFrac.pdf`, 2008. Accessed: 2021-05-16.

[15] R. Paysant-Le Roux and E. Dubois. Développement périodique par l'algorithme de Jacobi-Perron et nombre de pisot-vijayaraghavan. *Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences, Série A*, 272:649–652, 01 1971.

[16] R. Paysant-Le Roux and E. Dubois. Une application des nombres de Pisot à l'algorithme de Jacobi-Perron. *Monatshefte für Mathematik*, 98(2):145–155, 1984.

[17] O. Perron. *Die Lehre von den Kettenbruchen*. B. G. Teubner, 1913.

[18] E. Thomas. Fundamental units for orders in certain cubic number fields. *J. Reine Angew. Math.*, 1979(310):33–55, 1979.

[19] A. Thue. Über Annäherungswerte algebraischer Zahlen. (135):284–305, 1909.

[20] T. Vaughan. A note on the Jacobi-Perron algorithm. *Pacific Journal of Mathematics*, 72(1):261–271, 1977.

[21] P. Voutier. Families of periodic Jacobi–Perron algorithms for all period lengths. *Journal of Number Theory*, 168:472–486, 2016.

[22] wikipedia.org. Geometrical properties of polynomial roots. `https://en.wikipedia.org/wiki/Geometrical_properties_of_polynomial_roots#Discs_containing_some_roots`. Accessed: 2021-05-16.

[23] wikipedia.org. Perron's irreducibility criterion. `https://en.wikipedia.org/wiki/Perron%27s_irreducibility_criterion`. Accessed: 2021-05-16.