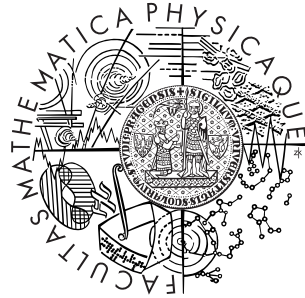


Univerzita Karlova v Praze  
Matematicko-fyzikální fakulta

## DIPLOMA THESIS



Tomáš Procházka

### **Two-photon decay of the Higgs boson: precise predictions of the standard model**

Ústav teoretické fyziky

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Chtěl bych poděkovat svému vedoucímu, prof. Hořejšímu, za trpělivost a za pomoc, kdykoliv jsem se dostal při výpočtech do potíží. Dále pak prof. Formánkovi za přednášky z kvantové teorie, díky kterým bylo aspoň částečné proniknutí do tohoto složitého oboru snazší. V neposlední řadě RNDr. Jířímu Novotnému za motivaci a za podporu.

Prohlašuji, že jsem svou diplomovou práci napsal samostatně a výhradně s použitím citovaných pramenů. Souhlasím se zapůjčováním práce.

V Praze dne 23. srpna 2007

Tomáš Procházka



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Název práce: Rozpad Higgsova bosonu na dva fotony: přesné předpovědi Standardního modelu

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Abstrakt: V předložené práci se zabýváme rozpadem Higgsova bosonu na dva fotony s důrazem na použití různých metod na výpočet Feynmanových diagramů. V první části podrobně počítáme více způsobů příspěvek smyčky s top kvarkem a  $W$ -bosonem. Dále se zabýváme chromodynamickými opravami ke hmotě kvarků a k interkaci kvarků s elektromagnetickým polem. V dalších kapitolách podrobně provádíme výpočet chromodynamických oprav ke smyčce s top kvarkem - nejprve rozvojem v malé hmotě Higgsova bosonu a poté přesně užitím identit získaných integrací per partes a diferenciálních rovnic. V závěru uvádíme přehled dalších výsledků popsanych v literatuře.

Klíčová slova: Feynmanovy diagramy, vícesmyčkové výpočty, Higgsův boson, polylogaritmy, QCD

Title: Two-photon decay of the Higgs boson: precise predictions of the Standard model

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Abstract: In the present work we are studying the Higgs boson decay into two photons with an emphasis on the use of various techniques used for the Feynman diagram calculations. In the first part we thoroughly evaluate the contribution of top quark and  $W$ -boson loop in various way. Then we consider the QCD corrections to quark masses and electromagnetic interaction. In next chapters we evaluate the QCD corrections to the top quark loop in detail - firstly by the expansion in a small Higgs mass and then exactly using the integration-by-parts identities and differential equations. Finally we make an overview of other results found in the literature.

Keywords: Feynman diagrams, multi-loop calculations, Higgs boson, polylogarithms, QCD



# Chapter 1

## Introduction

According to the present state of our knowledge, there are four forces acting between constituents of matter - the electromagnetic force, weak (nuclear) force, strong (nuclear) force and the gravitational force. The former two are described in a unified manner by the Glashow-Weinberg-Salam (GWS) model [24, 57, 49] and the strong force is described by the Quantum Chromodynamics [25, 47]. On the other hand, the gravitational force is described by the General Theory of Relativity, which has by now resisted attempts to be quantized.

The GWS model is Yang-Mills theory with  $SU(2) \times U(1)$  gauge group (corresponding to weak isospin and weak hypercharge) spontaneously broken down to the  $U(1)$  subgroup (electromagnetic gauge group) by the Higgs mechanism. As a result, the gauge bosons together with matter fields which were originally massless become massive (except for photon) and one physical scalar field is left corresponding to a spinless particle called the Higgs boson. Higgs boson is the only particle in the GWS model that has not been detected yet.

Most of the Higgs boson properties are fixed by the structure of GWS model, like the vacuum expectation value of the Higgs field or its coupling to fermions and gauge bosons. However, the mass of Higgs boson (together with self-couplings which depend on it) is left unspecified. There are theoretical upper bounds on Higgs mass coming from unitarity or requirement of applicability of perturbation theory with results of order 1 TeV.

The main decay mode of Higgs boson depends strongly on its mass. The region of masses  $m_W < m_H < 2m_W$  is usually called the intermediate mass region and regions above and below are called low-mass and high-mass regions. In the high-mass region the dominant decay is into  $W^+W^-$ ,  $ZZ$  or  $t\bar{t}$  pairs. The branching ratios in the range 100 – 200 GeV are shown in figure ??

In the present work we will study the decay rate of Higgs boson to two photons  $H \rightarrow \gamma\gamma$ . Although this process is very rare compared to the other decay modes of the Higgs boson (with branching ratio around  $10^{-3}$ ), it has clear signature and it is one of the most promising detection channels in the intermediate mass range. The  $\gamma\gamma$  mode will also allow precise measurement of Higgs mass and is sensitive to “new physics”. For instance, since the coupling of the Higgs boson to fermions whose masses are generated by Higgs mechanism is proportional to their masses, the decoupling theorem does not hold and heavy charged fermions do not decouple, so that they can give an important contribution to the  $\gamma\gamma$  decay width.

We will illustrate some methods used to calculate contributions to the decay rate  $H \rightarrow \gamma\gamma$ . The text is organized as follows: chapter 2 is devoted to one-loop calculations. It starts with a “textbook” calculation of charged fermion loop within the dimensional regularization. Since even in this simplest calculation we encounter infinities, calculation performed using the Pauli-Villars regularization follows to show a different way which however leads to the same result. Next section shows use of dispersion relations and analyticity properties to calculate the same amplitude. Finally, we calculate the amplitude in a “modern” way used later in two-loop calculation which consists of reduction of integrals to scalar amplitudes, using equations following from integration-by-parts identities to reduce number of independent integrals, and finally using differential equations to solve remaining integrals. Last part of chapter 2 gives results for one-loop amplitude for decay through the W-boson loop in unitary gauge and prepares ground for the two-loop calculations.

The next chapter shows calculation of QCD corrections to the fermion loop as an expansion in small Higgs mass and the following chapter shows how to calculate the two-loop amplitude exactly. Then we briefly review other computations appearing in the current literature on the subject.

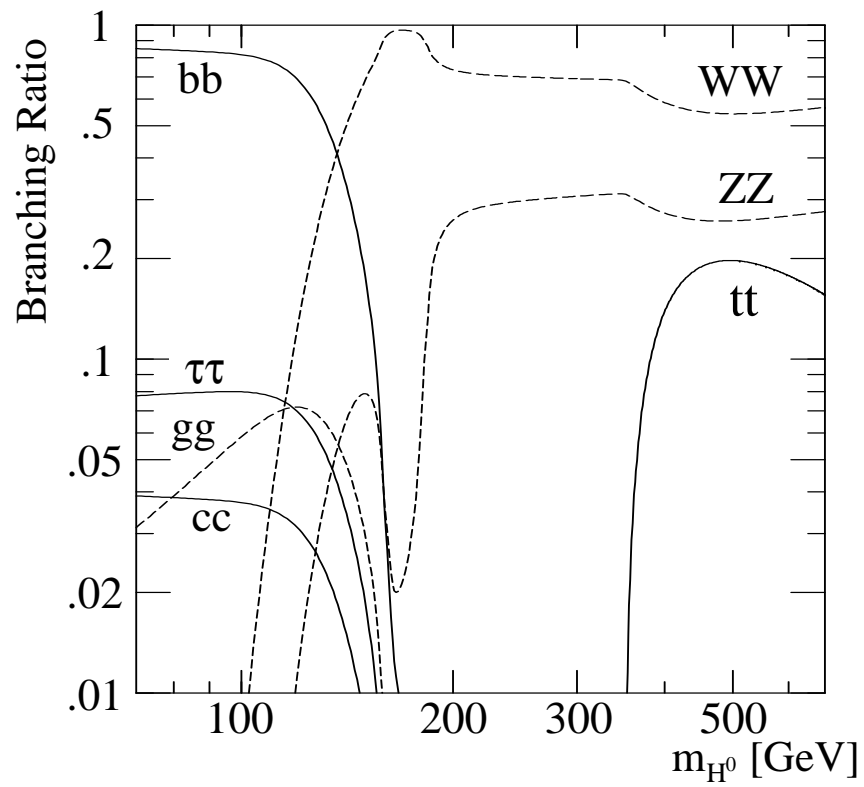


Figure 1.1: Higgs boson decay branching ratios, taken from the Particle Data Group

## Chapter 2

# Various one-loop calculations

### 2.1 Kinematics of $H \rightarrow \gamma\gamma$ and the general structure of invariant amplitude

Decaying Higgs boson is a massive spinless particle with spin 0 while the decay products are spin-1 massless photons. Since there is no preferred direction in the Higgs boson rest frame and momentum is conserved in the decay, produced photons propagate in opposite direction and their angular distribution is isotropic. Each photon is carrying half of the initial Higgs boson energy after the decay. Moreover, angular momentum conservation forces photons to have the same helicity. The only thing left that we can predict is the dependence of decay rate on the Higgs boson mass. According to general rules (see for instance [46]), the differential decay rate is given by

$$d\Gamma = \frac{1}{2m_H} |M|^2 \delta^4(P - p_1 - p_2) \frac{d^3p_1}{(2\pi)^3 2E_1} \frac{d^3p_2}{(2\pi)^3 2E_2} K$$

where  $M$  is the so-called invariant amplitude which contains all information about the specific process and is a Lorentz scalar,  $K$  is a combinatorial factor which is equal to  $\frac{1}{2}$  for two final state photons, and the remaining quantities have standard meaning. Since in our case the invariant amplitude is only a function of scalars (masses of Higgs boson and other particles), the differential decay rate can be integrated over “phase space” variables with the result

$$\Gamma = \frac{|M|^2}{32\pi m_H}$$

It remains to evaluate the invariant amplitude  $M$ .

Let  $p_1^\mu$  and  $p_2^\mu$  be four-momenta of outgoing photons and  $P^\mu = p_1^\mu + p_2^\mu$  the four-momentum of the incoming scalar Higgs particle. From the standard rules for constructing the invariant amplitude (LSZ formula), we expect it to be the Lorentz scalar depending linearly on two photon polarization vectors  $\epsilon_1$  and  $\epsilon_2$ , i.e.

$$M(p_1, \epsilon_1, p_2, \epsilon_2) = M_{\mu\nu}(p_1, p_2) \epsilon_1^{*\mu}(p_1) \epsilon_2^{*\nu}(p_2)$$

The vectors  $p_1$ ,  $p_2$  and  $P$  satisfy the on-mass-shell conditions  $p_1^2 = 0$ ,  $p_2^2 = 0$  and  $P^2 = (p_1 + p_2)^2 = m_H^2$ , where  $m_H$  denotes the Higgs boson mass. The tensor  $M_{\mu\nu}$  must satisfy several relations, including the Ward identity (related to the electromagnetic current conservation) for the outgoing photon lines, i.e.

$$\begin{aligned} M_{\mu\nu}(p_1, p_2) p_1^\mu &= 0 \\ M_{\mu\nu}(p_1, p_2) p_2^\nu &= 0 \end{aligned}$$

and also the symmetry relation (which follows from the fact that photons are bosons)

$$M_{\mu\nu}(p_1, p_2) = M_{\nu\mu}(p_2, p_1)$$

Writing the general tensor form of  $M^{\mu\nu}$  as

$$M^{\mu\nu} = A(p_1 \cdot p_2) g^{\mu\nu} + B p_1^\mu p_2^\nu + C p_1^\nu p_2^\mu + D p_1^\mu p_1^\nu + E p_2^\mu p_2^\nu + F \epsilon_{\mu\nu\rho\sigma} p_1^\rho p_2^\sigma \quad (2.1)$$

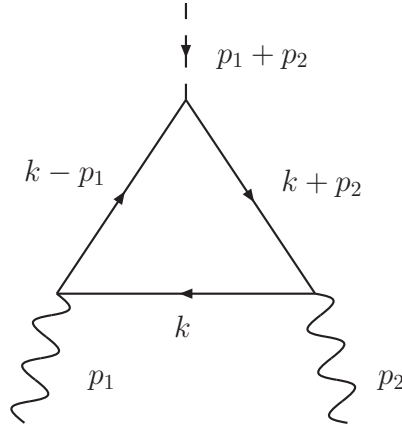


Figure 2.1: One loop fermion contribution to the amplitude

where the coefficients are scalar functions of invariants (in our case this is  $m_H$  together with masses of various particles which “mediate” the interaction), and using Ward identities, we obtain

$$M^{\mu\nu} = A[(p_1 \cdot p_2)g^{\mu\nu} - p_2^\mu p_1^\nu] + B p_1^\mu p_2^\nu + F \epsilon_{\mu\nu\rho\sigma} p_1^\rho p_2^\sigma \quad (2.2)$$

Note that not only the whole invariant amplitude but also the separate contributions from classes of graphs which are gauge invariant have this form. This remains true order by order when expanding invariant amplitude in powers of coupling constant, Planck constant (when expansion is done in the number of loops), or external momenta. Note that the formula (2.2) is valid only for on-shell photons and Higgs boson; we would get more complicated relations if we were interested in an amplitude for the off-shell particles. The coefficient  $F$  is zero for parity-conserving theories, which is our case of QED and QCD.

In the following, it will be useful to know the projections on the scalar amplitudes, that is, expressions for scalar coefficients in terms of  $M^{\mu\nu}$ . These are

$$\begin{aligned} A &= \frac{(p_1 p_2)g^{\mu\nu} - p_1^\mu p_2^\nu - p_2^\mu p_1^\nu}{(d-2)(p_1 p_2)^2} M_{\mu\nu} \\ B &= \frac{-(p_1 p_2)g^{\mu\nu} + p_1^\mu p_2^\nu + (d-1)p_2^\mu p_1^\nu}{(d-2)(p_1 p_2)^2} M_{\mu\nu} \\ C &= \frac{-(p_1 p_2)g^{\mu\nu} + (d-1)p_1^\mu p_2^\nu + p_2^\mu p_1^\nu}{(d-2)(p_1 p_2)^2} M_{\mu\nu} \end{aligned}$$

Here  $d$  is the dimension of space-time, writing it in this generality will be useful later. Note that we cannot write projection operators on  $D$  or  $E$  if we impose the condition  $p_1^2 = p_2^2 = 0$ .

The formula for decay rate written in terms of scalar amplitude is

$$\Gamma = \frac{m_H^3 |A|^2}{64\pi}$$

## 2.2 Direct evaluation of fermion loop

In this section we will evaluate the contribution of charged massive fermion loop to the amplitude directly according to the cook recipe described in QFT textbooks, for example [46]. According to the discussion at the end of previous section, we expect contribution from fermion loop to be in the form (2.2). Feynman diagram representing fermion contribution to the amplitude is shown in figure 2.1.

According to the Feynman rules (see section A.1), the contribution to the invariant amplitude reads

$$iM_f^{\mu\nu} = 2(-1) \frac{-igm}{2m_W} (ieQ)^2 i^3$$

$$\int \frac{d^4 k}{(2\pi)^4} \text{Tr} \left[ \frac{\not{k} - \not{p}_1 + m}{(k - p_1)^2 - m^2 + i\epsilon} \gamma^\mu \frac{\not{k} + m}{k^2 - m^2 + i\epsilon} \gamma^\nu \frac{\not{k} + \not{p}_2 + m}{(k + p_2)^2 - m^2 + i\epsilon} \right] \quad (2.3)$$

where  $m$  is the mass of circulating fermion and  $Q$  is its charge in units of positron charge. There are two Feynman diagrams contributing, differing in the direction of loop fermion charge flow (or, which is the same, in attachment of external photon lines). However, as can be shown using the  $\gamma$  matrix reversal property (A.2), both diagrams give the same contribution, which is the reason for including the factor of 2. Other way to see this is from the fact that the two amplitudes are related to each other by the exchange of  $p_1$  and  $p_2$  and  $\mu$  and  $\nu$ . But as seen from (2.2) this does not affect the scalar amplitudes. Factor  $(-1)$  comes from the fermion loop and  $i^3$  from fermion propagators. We are not summing over fermions or over colours in the case of quarks yet!

In the first step we simplify the numerator of this expression using the properties of  $\gamma$  matrices (see Section A.2). The result is:

$$\begin{aligned} \text{Tr} [(\not{k} - \not{p}_1 + m)\gamma^\mu(\not{k} + m)\gamma^\nu(\not{k} + \not{p}_2 + m)] = \\ m \cdot \text{Tr} \mathbb{1} [4k^\mu k^\nu - k^2 g^{\mu\nu} + 2k^\mu p_2^\nu - 2k^\nu p_1^\mu + p_1^\nu p_2^\mu - p_1^\mu p_2^\nu + (m^2 - p_1 p_2)g^{\mu\nu}] \end{aligned} \quad (2.4)$$

Note that we left the value of  $\text{Tr} \mathbb{1}$ , which is 4 in four-dimensional space-time, formally unspecified until later, to illustrate that the way of its extension to  $d$  dimensions is unimportant.

Next we use the Feynman parametrization to combine the denominators of equation (2.3) into a power of quadratic function in the loop momenta  $k$ . To do this, we use the formula (B.12) with the result (from now on we will not write explicitly the  $i\epsilon$  term which always occurs in the combination  $m^2 - i\epsilon$  and which gives the correct boundary conditions for propagators):

$$\begin{aligned} \frac{1}{[(k - p_1)^2 - m^2][(k + p_2)^2 - m^2][k^2 - m^2]} = 2 \int_0^1 dx \int_0^{1-x} dy \frac{1}{[k^2 + 2k \cdot (-xp_1 + yp_2) - m^2]^3} = \\ 2 \int_0^1 dx \int_0^{1-x} dy \frac{1}{[l^2 - m^2 - (xp_1 - yp_2)^2]^3} \end{aligned} \quad (2.5)$$

where  $l = k - xp_1 + yp_2$ .

Now we need to express the numerator using a new variable  $l$  instead of loop momentum  $k$ . After some algebraic manipulations we arrive at

$$\begin{aligned} \text{Tr} [(\not{k} - \not{p}_1 + m)\gamma^\mu(\not{k} + m)\gamma^\nu(\not{k} + \not{p}_2 + m)] = \\ m \cdot \text{Tr} \mathbb{1} [4l^\mu l^\nu - l^2 g^{\mu\nu} + g^{\mu\nu} [m^2 + (2xy - 1)p_1 \cdot p_2] + \\ 4xl^\mu p_1^\nu - 4yl^\nu p_2^\mu + (4x - 2)l^\nu p_1^\mu - (4y - 2)l^\mu p_2^\nu + g^{\mu\nu} (-2xl \cdot p_1 + 2yl \cdot p_2) + \\ p_1^\mu p_1^\nu (4x^2 - 2x) + p_2^\mu p_2^\nu (4y^2 - 2y) + p_1^\mu p_2^\nu (2x + 2y - 4xy - 1) + p_1^\nu p_2^\mu (1 - 4xy)] \end{aligned} \quad (2.6)$$

We can check that it has the symmetry  $p_1^\mu \leftrightarrow p_2^\nu$ ,  $x \leftrightarrow y$  except for terms that are linear in  $l$  and thus give no contribution after integration over  $l$ . Now we perform the integration over the shifted loop momentum  $l$ . The quadratic part of numerator is divergent and at the same time it seems to drop out if we use the symmetric integration formula

$$\int l^\mu l^\nu f(l^2, a) d^d l = \int \frac{l^2 g^{\mu\nu}}{d} f(l^2, a) d^d l \quad (2.7)$$

with  $a$  representing a dummy scalar variables. But this formula is valid only if the left hand side is finite for every  $\mu, \nu$ . If we do the naive calculation dropping the quadratic part, we arrive at amplitude which does not have the correct tensor structure (2.2). The reason for this is the fact that we are dealing with formally divergent integrals. We cannot use the formula (2.7) with  $d = 4$ , because the integral does not converge for this value of  $d$ . To proceed, we need to regularize integral somehow, to give it a correct mathematical meaning. In this section (and in the rest of the text if not otherwise stated), we will use the dimensional regularization. This is now a standard regularization method for Feynman diagram calculations, because it is relatively simple to handle and it preserves important symmetries like those that we used to derive (2.2). Calculating in dimensional regularization amounts to formally extending the parameter  $d$  to the neighbourhood of  $d = 4$  in the complex plane and looking at the behavior of amplitudes at  $d = 4$  only at the end of calculations. Divergent integrals manifest themselves as poles in  $d - 4$  and if this happens, we will need to use the renormalization procedure, as

will be seen later. But in this simple one-loop calculation the divergences will cancel, leaving a nonzero finite amplitude with the correct form (2.2).

So in  $d$  dimensions, we have formally

$$\begin{aligned}
\int \frac{d^d k}{(2\pi)^d} \text{Tr} \left[ \frac{\not{k} - \not{p}_1 + m}{(k - p_1)^2 - m^2 + i\epsilon} \gamma^\mu \frac{\not{k} + m}{k^2 - m^2 + i\epsilon} \gamma^\nu \frac{\not{k} + \not{p}_2 + m}{(k + p_2)^2 - m^2 + i\epsilon} \right] = \\
2 \int_0^1 dx \int_0^{1-x} dy \int \frac{d^d l}{(2\pi)^d} \frac{m \text{Tr} \mathbb{1} [4l^\mu l^\nu - l^2 g^{\mu\nu}]}{[l^2 - m^2 - (xp_1 - yp_2)^2]^3} + \\
2 \int_0^1 dx \int_0^{1-x} dy \int \frac{d^d l}{(2\pi)^d} \frac{m \text{Tr} \mathbb{1} [g^{\mu\nu} (m^2 + (2xy - 1)p_1 \cdot p_2)]}{[l^2 - m^2 - (xp_1 - yp_2)^2]^3} + \\
2 \int_0^1 dx \int_0^{1-x} dy \int \frac{d^d l}{(2\pi)^d} \frac{m \text{Tr} \mathbb{1} [p_1^\mu p_1^\nu (4x^2 - 2x) + p_2^\mu p_2^\nu (4y^2 - 2y)]}{[l^2 - m^2 - (xp_1 - yp_2)^2]^3} + \\
2 \int_0^1 dx \int_0^{1-x} dy \int \frac{d^d l}{(2\pi)^d} \frac{m \text{Tr} \mathbb{1} [p_1^\mu p_2^\nu (2x + 2y - 4xy - 1) + p_1^\nu p_2^\mu (1 - 4xy)]}{[l^2 - m^2 - (xp_1 - yp_2)^2]^3}
\end{aligned} \tag{2.8}$$

where we have shifted the integration variable  $k$  to  $l = k - xp_1 + yp_2$  and dropped the terms linear in  $l$  (which give zero contribution as can be seen by the substitution  $l \rightarrow -l$ ). To continue, we use the formula (B.13) along with (2.7) to calculate

$$\begin{aligned}
\int \frac{d^d l}{(2\pi)^d} \frac{1}{[l^2 - \Delta]^3} &= \frac{-iN}{2(4\pi)^2 m^2 \tilde{\Delta}^{1+\epsilon}} \\
\int \frac{d^d l}{(2\pi)^d} \frac{l^2}{[l^2 - \Delta]^3} &= \frac{iN}{(4\pi)^2} \frac{1}{\tilde{\Delta}^\epsilon} \frac{2 - \epsilon}{\epsilon} \\
\int \frac{d^d l}{(2\pi)^d} \frac{4l^\mu l^\nu - l^2 g^{\mu\nu}}{[l^2 - \Delta]^3} &= \frac{iN}{2(4\pi)^2 \tilde{\Delta}^\epsilon} g^{\mu\nu}
\end{aligned} \tag{2.9}$$

where we have defined  $d = 4 - 2\epsilon$  and  $\Delta = m^2 \tilde{\Delta} = m^2 + (xp_1 - yp_2)^2$ . The factor  $N$ , which will occur once for every loop, is

$$N = \frac{\Gamma(1 + \epsilon)(4\pi)^\epsilon}{(m^2)^\epsilon}$$

Many people like to expand it, which gives Euler-Mascheroni constant and other transcendental numbers. We will not do so, because it merely multiplies the result of every loop integration (as will be clearly seen in the Section 2.4) and is equal to 1 for  $\epsilon = 0$ . This means that whenever the sum of diagrams (including counter-terms) gives a convergent quantity, inclusion of  $N$  makes no difference. Moreover, appearance of  $N$  depends on convention used to define the  $d$ -dimensional integrals. We can modify the integration measure

$$\frac{d^d k}{(2\pi)^d}$$

by multiplying it by a factor which is equal to 1 for  $\epsilon = 0$  and differs in the neighbourhood of  $\epsilon = 0$ , and there are choices where this compensates the  $N$  factor. We may even include an arbitrary mass scale  $\mu$  which will change  $(m^2)^\epsilon$  to “dimensionally correct” value  $\left(\frac{\mu^2}{m^2}\right)^\epsilon$  but again, this is merely a convention and such inclusion of a scale  $\mu$  is as artificial as leaving  $N$  with  $(m^2)^{-\epsilon}$ . The inclusion of a scale  $\mu$  is useful when using various modifications of the dimensional renormalization, like the minimal subtraction. But in this work we will use the “physical” on-mass-shell renormalization scheme, so we can use  $N$  (without any redefinitions of measure).

We see that both integrals that we need are finite (they do not have a pole in  $\epsilon$ ), so we can put  $\epsilon = 0$ . For two-loop calculations however, we will need also the  $O(\epsilon)$  terms. This will be considered in Section 2.4 and in Section C.2.

The third term on the right hand side of (2.8) vanishes as can be clearly seen from

$$2 \int_0^1 dx \int_0^{1-x} dy \int \frac{d^d l}{(2\pi)^d} \frac{4x^2 - 2x}{(l^2 - m^2 + 2xyp_1p_2)^3} = 2 \int_0^1 dx \int_0^{1-x} dy \frac{-i}{(4\pi)^2} \frac{2x^2 - x}{m^2 - 2xyp_1p_2}$$

$$= \int_0^1 dx \frac{i}{(4\pi)^2} \frac{2x-1}{p_1 p_2} \ln \left[ 1 - 2x(1-x) \frac{p_1 p_2}{m^2} \right] = \int_{-\frac{1}{2}}^{+\frac{1}{2}} dx' \frac{i}{(4\pi)^2} \frac{2x'}{p_1 p_2} \ln \left[ 1 - \left( \frac{1}{2} + x' \right) \left( \frac{1}{2} - x' \right) \frac{p_1 p_2}{m^2} \right] = 0$$

where  $x' = x - \frac{1}{2}$ . The first and the second integral on the right hand side of (2.8) simplify to

$$2 \int_0^1 dx \int_0^{1-x} dy \frac{i}{(4\pi)^2} \frac{m \text{Tr} \mathbb{1} g^{\mu\nu}}{2} + 2 \int_0^1 dx \int_0^{1-x} dy \frac{-i}{(4\pi)^2} \frac{m \text{Tr} \mathbb{1} g^{\mu\nu} (m^2 + (2xy-1)p_1 \cdot p_2)}{2(m^2 + (xp_1 - yp_2)^2)} =$$

$$2 \int_0^1 dx \int_0^{1-x} dy \frac{-i}{(4\pi)^2} \frac{m \text{Tr} \mathbb{1} (4xy-1) g^{\mu\nu} p_1 \cdot p_2}{2(m^2 - 2xyp_1 \cdot p_2)} \quad (2.10)$$

Finally we are left with

$$\int \frac{d^d k}{(2\pi)^d} \text{Tr} \left[ \frac{\not{k} - \not{p}_1 + m}{(k-p_1)^2 - m^2 + i\epsilon} \gamma^\mu \frac{\not{k} + m}{k^2 - m^2 + i\epsilon} \gamma^\nu \frac{\not{k} + \not{p}_2 + m}{(k+p_2)^2 - m^2 + i\epsilon} \right] =$$

$$2 \int_0^1 dx \int_0^{1-x} dy \frac{-i}{(4\pi)^2} \frac{m \text{Tr} \mathbb{1} (4xy-1) g^{\mu\nu} p_1 \cdot p_2}{2(m^2 - 2xyp_1 \cdot p_2)}$$

$$+ 2 \int_0^1 dx \int_0^{1-x} dy \frac{-i}{(4\pi)^2} \frac{m \text{Tr} \mathbb{1} [p_1^\mu p_2^\nu (2x+2y-4xy-1) + p_1^\nu p_2^\mu (1-4xy)]}{2(m^2 - 2xyp_1 \cdot p_2)} \quad (2.11)$$

Comparing with (2.1) we see that

$$A_f = \frac{e^2 Q^2 g m}{m_W} \frac{4m}{(4\pi)^2} \int_0^1 dx \int_0^{1-x} dy \frac{4xy-1}{m^2 - 2xyp_1 \cdot p_2}$$

$$B_f = \frac{e^2 Q^2 g m}{m_W} \frac{4m}{(4\pi)^2} \int_0^1 dx \int_0^{1-x} dy \frac{2x+2y-4xy-1}{m^2 - 2xyp_1 \cdot p_2} \quad (2.12)$$

$$C_f = \frac{e^2 Q^2 g m}{m_W} \frac{4m}{(4\pi)^2} \int_0^1 dx \int_0^{1-x} dy \frac{1-4xy}{m^2 - 2xyp_1 \cdot p_2} = -A_f$$

where the subscript of  $A_f$  denotes the contribution of the fermion loop.

Integrals of this type are evaluated in Appendix C.1. The result can be written in the form

$$A_f = \frac{\alpha Q^2}{\pi v z} \left[ \frac{z-1}{2z} I_{-1}(z) - 1 \right] =$$

$$= \frac{\alpha Q^2}{\pi v z} \left[ \frac{(1+u)^2}{2(1-u)^2} H_{0,0}(u) - 1 \right] =$$

$$= \frac{\alpha Q^2}{4\pi} \sqrt{G_F \sqrt{2}} \cdot \frac{1}{z} \left[ \left( 1 - \frac{1}{z} \right) \left[ \ln \frac{\sqrt{z} + \sqrt{z-1}}{\sqrt{z} - \sqrt{z-1}} - i\pi \right]^2 - 4 \right] \quad \text{for } z > 1$$

$$B_f = \frac{\alpha Q^2}{\pi v z} \left[ 5 - 4 \sqrt{\frac{1-z}{z}} \text{arctg} \sqrt{\frac{z}{1-z}} - \left( 1 + \frac{1}{z} \right) \text{arctg}^2 \sqrt{\frac{z}{1-z}} \right]$$

$$= \frac{\alpha Q^2}{\pi v z} \left[ 5 + \frac{2(1+u)}{1-u} H_0(u) + \frac{1-6u+u^2}{2(1-u)^2} H_{0,0}(u) \right]$$

where

$$z = \frac{m_H^2}{4m^2}$$

and

$$u = \frac{\sqrt{1 - \frac{1}{z}} - 1}{\sqrt{1 - \frac{1}{z}} + 1}$$

is the variable which is very useful for expressing results (as we will see later) - see also Appendix B.1.2. For the definition of harmonic polylogarithms (which can be expressed in terms of elementary functions in this one-loop

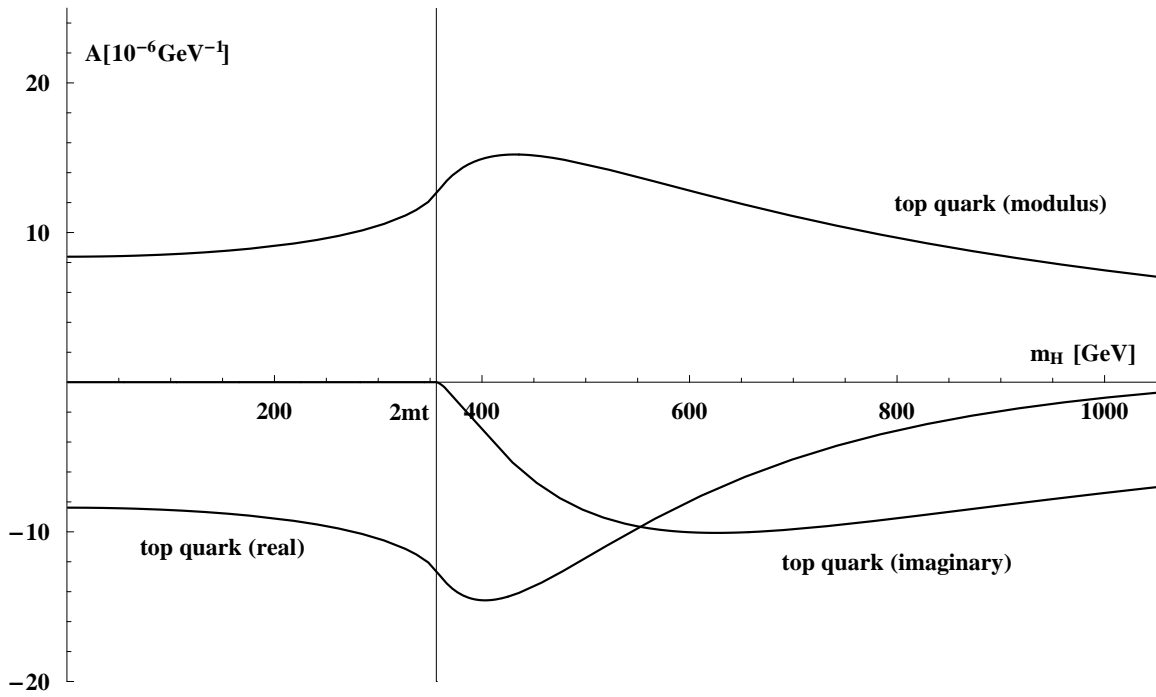


Figure 2.2: Contribution of top quark loop to the invariant amplitude

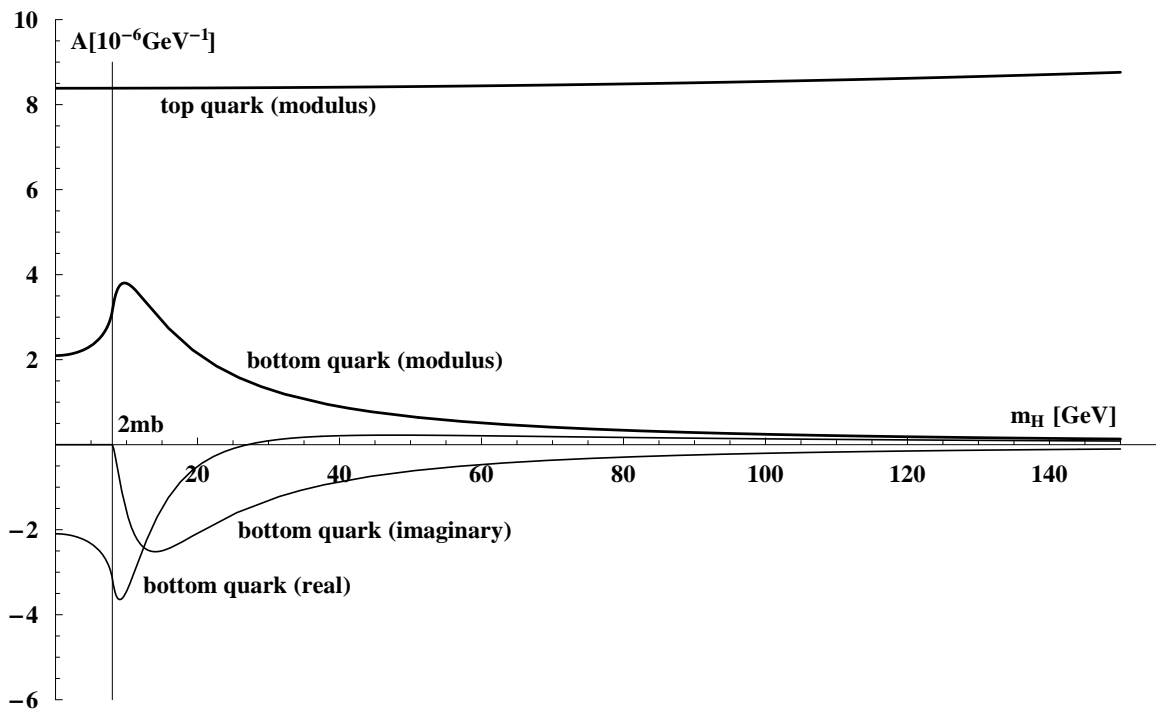


Figure 2.3: Comparison of bottom quark and top quark contribution to the invariant amplitude

case), see Appendix B.1.5. We used relations (A.1) to write the scalar amplitude in a form that exactly matches the result given in [35].

The dependence of the top quark contribution to the invariant amplitude is shown in the Figure 2.2. We can clearly see that the amplitude is real below the  $2m_t$  threshold and that even for massless Higgs boson the amplitude is nonzero (there is no decoupling - the Higgs boson “feels” all particles which have masses generated by the Higgs mechanism, no matter how heavy they happen to be!). Both real and imaginary parts approach zero for  $m_H \rightarrow \infty$ . Figure 2.3 compares the bottom and top quark contribution to the invariant amplitude. We see that the contribution of bottom quark loop to the process  $H \rightarrow \gamma\gamma$  for Higgs boson masses far from the  $2m_b$  threshold is only small. Even in the region of bottom quark masses around the threshold  $2m_b$  the contribution of bottom quark is smaller than the top quark contribution.

### 2.2.1 Pauli-Villars (invariant) regularization

As we have seen in the last section, the divergences are encountered even in the lowest order calculation. Since there is no direct coupling of Higgs boson to the electromagnetic field in the Standard Model (because photon is massless and also because of electromagnetic gauge invariance), the decay  $H \rightarrow \gamma\gamma$  cannot take place at the tree level. But this in turn means (since the GWS model is renormalizable) that there are no coupling constants which could absorb the possible divergent quantities resulting from the loop integration, so we must obtain a finite result for the one-loop integral independently of the choice of the regularization - there is no need to redefine any couplings at the one loop level. This was demonstrated in the previous section on the example of the dimensional regularization.

Since the dimensional regularization used in the previous section may at first sight seem artificial, it is useful to demonstrate the calculation with another regularization to see if they indeed give the same result. We have seen that the only scalar amplitude that contained a potential divergence was  $A$ . Now the electromagnetic gauge invariance of the amplitude gives a constraint on the scalar amplitudes that can be written as  $A = -C$ , and we know that  $C$  was given by a convergent integral. From this fact and the results of the previous paragraph it follows that any regularization which does not break the electromagnetic gauge invariance and which does not modify the values of convergent integrals must give the same (finite) result for  $A$  (equal to  $-C$ ).

Pauli-Villars regularization [45] is based on the observation that if we add to a loop integral a combination of integrals of the same form but with a heavy fermion circulating in the loop (which would formally vanish in the limit of infinite mass if the integral was convergent), it will under certain conditions give a convergent integral (which may diverge again if we take the limit of large mass and the original integral was divergent).

There is a little problem in our case since the fermion masses are generated by the Higgs mechanism, so the coupling of the Higgs boson is proportional to its mass. In order to use the Pauli-Villars procedure, we must consider this coupling as a constant which we do not modify. We can imagine calculating the contribution of loop with arbitrary scalar-fermion coupling constant and at the end substituting the specific value of coupling coming from the Higgs mechanism. On the other hand, the result of evaluating trace of three fermion propagators was proportional to the fermion mass (since this is a trace of odd number of  $\gamma$  matrices), and this factor must be modified for the procedure to make sense.

Integral giving a contribution to the scalar amplitude  $A$  is (see 2.8)

$$2 \int_0^1 dx \int_0^{1-x} dy \int \frac{d^d l}{(2\pi)^d} \frac{m \text{Tr} \mathbb{1} [4l^\mu l^\nu - l^2 g^{\mu\nu} + g^{\mu\nu}(m^2 - 2xyp_1p_2) + g^{\mu\nu}p_1p_2(4xy - 1)]}{[l^2 - m^2 + 2xyp_1p_2]^3}$$

We have written the numerator of integrand in a form such that the last term gives what we would expect from the electromagnetic gauge invariance. We would like to show that all other terms give zero. In the calculation involving the dimensional regularization we saw that up to  $O(\epsilon^0)$

$$\begin{aligned} \int \frac{d^d l}{(2\pi)^d} \frac{4l^\mu l^\nu - l^2 g^{\mu\nu}}{[l^2 - \Delta]^3} &\rightarrow \frac{ig^{\mu\nu}}{2(4\pi)^2} \\ \int \frac{d^d l}{(2\pi)^d} \frac{g^{\mu\nu} \Delta}{[l^2 - \Delta]^3} &\rightarrow -\frac{ig^{\mu\nu}}{2(4\pi)^2} \end{aligned}$$

so both contributions indeed cancelled themselves and we obtained gauge invariant result.

Let us now consider the Pauli-Villars regularization. We can put  $d = 4$  since we will work in 4 dimensions. The only divergent integral is

$$m \int \frac{d^4 l}{(2\pi)^4} \frac{4l^\mu l^\nu - l^2 g^{\mu\nu}}{[l^2 - \Delta]^3}$$

We consider the integral  $I(m)$  as a function of fermion mass  $m$  and make a linear combination

$$\sum_{i=0}^n C_i I(m_i)$$

with masses  $m_i$ ,  $i \geq 1$  that will be sent to infinity later. We will fix constants  $m_i$  and  $C_i$  later after we find the conditions that must be satisfied in order to get a finite result. If we had an absolutely convergent integral, we would expect that as  $m \rightarrow \infty$  the  $I(m) \rightarrow 0$  (very heavy particles are unable to propagate). To make the integral

$$\sum_i C_i m_i \int \frac{d^4 l}{(2\pi)^4} \frac{4l^\mu l^\nu - l^2 g^{\mu\nu}}{[l^2 - \Delta(m_i)]^3},$$

convergent, we impose the condition

$$\sum_i C_i m_i = 0$$

For convergent integral, we can use the symmetric integration formula (2.7) and we immediately see that

$$\int \frac{d^4 l}{(2\pi)^4} \sum_i C_i m_i \frac{4l^\mu l^\nu - l^2 g^{\mu\nu}}{[l^2 - \Delta(m_i)]^3} = 0$$

which is different from the case of dimensional regularization. But now also the second term is zero, because

$$\sum_i C_i m_i \int \frac{d^d l}{(2\pi)^d} \frac{g^{\mu\nu} \Delta(m_i)}{[l^2 - \Delta(m_i)]^3} = -\frac{ig^{\mu\nu}}{2(4\pi)^2} \sum_i C_i m_i = 0$$

We see that the integral becomes finite and gives a correct contribution after imposing only one condition,

$$\sum_i C_i m_i = 0.$$

This condition has a simple solution, for instance

$$\begin{aligned} m_0 &= m \\ C_0 &= 1 \\ m_1 &= \Lambda \\ C_1 &= -\frac{m}{\Lambda} \end{aligned}$$

The value of convergent integrals having a lower power of  $m$  in numerator than  $m^3$  (counting also the overall mass factor coming from the trace) is not changed in the limit  $\Lambda \rightarrow \infty$ , so we obtain exactly the same result as before. As explained earlier, this is due to the preservation of the electromagnetic gauge invariance by both regularizations. Values given to divergent integrals were different in both methods, but that is fine, since the values of these integrals by themselves have no physical meaning.

## 2.3 Evaluation of fermion loop using the dispersion relations

There is another method of evaluating the matrix element (2.3) based on analyticity properties and the so-called dispersion relations (for more details see [9, 30, 43, 46]). This method uses the fact that in some cases the evaluation of the imaginary part of invariant amplitude can be easier than evaluation of its real part. If we know in addition analytic properties of the invariant amplitude like the positions of cuts and poles and if the function decreases fast enough at infinity, the real part can be reconstructed from the imaginary part using the Cauchy's integral formula and we get the same analytic results as if we evaluated loop integrals using methods

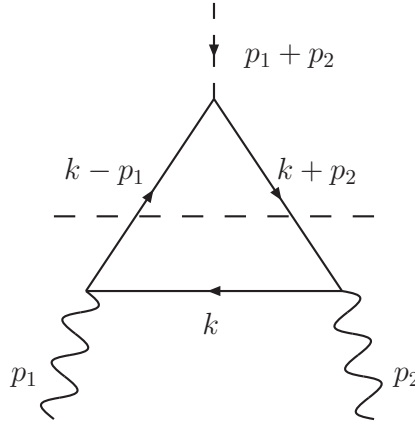


Figure 2.4: Fermion loop with cut propagators

of the previous section. Moreover, if we use this method for the one-loop integrals like the one we are dealing with, we do not encounter any infinities, because in the present case the two propagators are replaced by the delta functions and the remaining integrals are convergent (the possibly divergent quantities usually occur only as an infinite additive coefficients of a polynomial if the diagram was not convergent).

According to the Cutkosky's rules (for the simplest form of them see [46]), in order to get the discontinuity of amplitude  $M^{\mu\nu}$  defined as

$$\text{Disc } M^{\mu\nu}(z) \equiv M^{\mu\nu}(z + i\epsilon) - M^{\mu\nu}(z - i\epsilon) = 2i\Im M^{\mu\nu}(z + i\epsilon)$$

we must perform the cut as illustrated in Figure 2.4, This amounts to doing the substitutions

$$\begin{aligned} \frac{1}{(k - p_1)^2 - m^2 + i\epsilon} &\rightarrow -2\pi i \delta((k - p_1)^2 - m^2) \\ \frac{1}{(k + p_2)^2 - m^2 + i\epsilon} &\rightarrow -2\pi i \delta((k + p_2)^2 - m^2) \end{aligned}$$

in the original integral (2.3). It is usually said that we “put the virtual fermion on the mass-shell”. After doing these substitution, we obtain instead of the original amplitude  $M_{\mu\nu}$  the discontinuity of its imaginary part along the cut. To continue the evaluation of the integral, it is useful to choose a specific Lorentz frame in which the integrals become simpler. This is where the main drawback of this method shows up, which is a bit cumbersome integration over the remaining loop variables.

We choose the coordinates to be

$$\begin{aligned} p_1 &= (p, 0, 0, p) \\ p_2 &= (p, 0, 0, -p) \\ P &= p_1 + p_2 = (m_H = 2p, \vec{0}) \\ Q &= p_1 - p_2 = (0, 0, 0, 2p = m_H) \\ k &= (k_0, \vec{k}) \\ l &\equiv k + p_2 = (l_0, \vec{l}) \\ E_{\vec{l}} &\equiv \sqrt{m^2 + |\vec{l}|^2} \end{aligned}$$

Note that  $P^\mu$  is timelike and  $Q^\mu$  is spacelike. This will let us avoid some calculations. In order to satisfy the condition imposed by the delta functions, the Higgs boson must be heavier than the two fermions ( $m_H^2 \geq 4m^2$ ). We have

$$I = (-2\pi i)^2 \int_{k^0 > 0} \frac{d^4 k}{(2\pi)^4} \delta((k - p_1)^2 - m^2) \delta((k + p_2)^2 - m^2)$$

$$\begin{aligned}
&= (-2\pi i)^2 \int_{l^0 > 0} \frac{d^4 l}{(2\pi)^4} \delta(l^2 - m^2) \delta((l - P)^2 - m^2) \\
&= \frac{-1}{4\pi^2} \int \frac{d^3 \vec{l}}{2E_{\vec{l}}} \delta((l - P)^2 - m^2) \\
&= \frac{-1}{8\pi^2} \int |\vec{l}| dE_{\vec{l}} d\Omega_{\vec{l}} \delta(m_H^2 - 2m_H E_{\vec{l}}) \\
&= \frac{-1}{8\pi m_H} |\vec{l}| \int_{-1}^{+1} d\cos\theta = \frac{-1}{16\pi} \sqrt{1 - \frac{4m^2}{m_H^2}} \int_{-1}^{+1} d\cos\theta
\end{aligned}$$

where the delta functions impose the following restrictions:

$$\begin{aligned}
E_{\vec{l}} &= \frac{m_H}{2} \\
l_0 &= E_{\vec{l}} \\
|\vec{l}| &= \sqrt{E_{\vec{l}}^2 - m^2} = \frac{m_H}{2} \sqrt{1 - \frac{4m^2}{m_H^2}}
\end{aligned}$$

and  $\theta$  is the angle between  $\vec{l}$  and  $\vec{p}_2$ :

$$\vec{l} \cdot \vec{p}_2 = |\vec{l}| |\vec{p}_2| \cos\theta$$

These formulae remain valid even if the integrand contains any function of  $k$ , but we must express the vector  $k$  by means of vector  $l = k - p_2$  and apply the restrictions on  $l_0$  and  $|\vec{l}|$ . Now we express the denominator of the fermion propagator in our coordinate system:

$$k^2 - m^2 = \frac{m_H^2}{2} \left[ \sqrt{1 - \frac{4m^2}{m_H^2}} \cos\theta - 1 \right]$$

We have all ingredients to evaluate all three tensor integrals that we need. We denote

$$\{J, J^\mu, J^{\mu\nu}\} = \int_{k^0=0} \frac{d^4 k}{(2\pi)^4} (-2\pi i) \delta((k - p_1)^2 - m^2) (-2\pi i) \delta((k + p_2)^2 - m^2) \frac{\{1, k^\mu, k^\mu k^\nu\}}{k^2 - m^2}$$

The scalar integral is

$$J = \frac{-1}{16\pi} \int_{-1}^{+1} \frac{\sqrt{1 - \frac{4m^2}{m_H^2}} d\cos\theta}{\frac{m_H^2}{2} \left[ \sqrt{1 - \frac{4m^2}{m_H^2}} \cos\theta - 1 \right]} = \frac{1}{8\pi m_H^2} \ln \left[ \frac{\sqrt{z} + \sqrt{z-1}}{\sqrt{z} - \sqrt{z-1}} \right] = -\frac{1}{8\pi m_H^2} \ln(-u)$$

See how the variable  $u$  popped up. To calculate the vector integral  $J^\mu$ , we calculate its scalar projections  $J^\mu P_\mu$  and  $J^\mu Q_\mu$  and from them the value of  $J^\mu$ :

$$\begin{aligned}
J^\mu P_\mu &= 0 \\
J^\mu Q_\mu &= \frac{-1}{16\pi} \int_{-1}^{+1} \frac{\sqrt{1 - \frac{4m^2}{m_H^2}} \left[ \frac{m_H^2}{2} \left( \sqrt{1 - \frac{4m^2}{m_H^2}} \cos\theta - 1 \right) \right] d\cos\theta}{\frac{m_H^2}{2} \left[ \sqrt{1 - \frac{4m^2}{m_H^2}} \cos\theta - 1 \right]} \\
&= \frac{-1}{8\pi} \sqrt{1 - \frac{4m^2}{m_H^2}} = -\frac{1}{8\pi} \frac{1+u}{1-u} \\
J^\mu &= \frac{1}{8\pi m_H^2} \sqrt{1 - \frac{4m^2}{m_H^2}} Q^\mu = \frac{1}{8\pi m_H^2} \frac{1+u}{1-u} Q^\mu
\end{aligned}$$

Resulting  $J^\mu$  is proportional to  $Q^\mu$ , the result which could be expected. The integrand of  $J^\mu$  as a Lorentz vector is spacelike for every value of  $k^\mu$  allowed by the delta functions. It follows that also  $J^\mu$  is spacelike and thus proportional to  $Q^\mu$ .  $J^\mu$  also changes a sign when the role of  $p_1^\mu$  and  $p_2^\mu$  is exchanged, which we expected from the structure of the trace and the symmetry properties of  $M^{\mu\nu}$ .

Let us proceed with  $J^{\mu\nu}$ . From the transformation properties under the Lorentz transformations, and since we have only two linearly independent four-vectors,  $P^\mu$  and  $Q^\mu$ , we can write

$$J^{\mu\nu} = C(p_1 \cdot p_2)g^{\mu\nu} + DQ^\mu Q^\nu + EP^\mu P^\nu + F(P^\mu Q^\nu + Q^\mu P^\nu) + G(P^\mu Q^\nu - Q^\mu P^\nu)$$

The coefficient functions are scalar functions of  $p_1$  and  $p_2$  and scalar invariants. However, we have  $p_1^2 = p_2^2 = 0$ , so the coefficient functions can depend only on  $m$  and  $m_H$ . Since  $J^{\mu\nu}$  is symmetric, the coefficient  $G$  must be zero. In addition, vector  $P^\mu$  is timelike and vectors  $Q^\mu$  and  $k^\mu$  are spacelike. Comparing the nonzero components of  $k^\mu k^\nu$  with components of  $J^{\mu\nu}$ , we obtain the relation  $F = 0$  and relation

$$C = -2E \quad (2.13)$$

Now we proceed with calculation of the projections. With our choice of vectors  $P^\mu$  and  $Q^\mu$ , the integration is simple and gives

$$\begin{aligned} J^{\mu\nu} g_{\mu\nu} &= \frac{m^2}{8\pi m_H^2} \ln \left[ \frac{\sqrt{z_q} + \sqrt{z_q - 1}}{\sqrt{z_q} - \sqrt{z_q - 1}} \right] - \frac{1}{8\pi} \sqrt{1 - \frac{4m^2}{m_H^2}} \\ J^{\mu\nu} Q_\mu Q_\nu &= \frac{m_H^2}{16\pi} \sqrt{1 - \frac{4m^2}{m_H^2}} \\ J^{\mu\nu} P_\mu P_\nu &= 0 \\ J^{\mu\nu} P_\mu Q_\nu &= 0 \end{aligned}$$

The last equation is also a consequence of  $F = G = 0$  and the orthogonality of  $P^\mu$  and  $Q^\nu$ . To reconstruct the tensor  $J^{\mu\nu}$  from its projections, we use the relations

$$\begin{aligned} 4(p_1 p_2)^2 C &= J^{\mu\nu} [-P_\mu P_\nu + Q_\mu Q_\nu + 2(p_1 p_2)g_{\mu\nu}] \\ 8(p_1 p_2)^2 D &= J^{\mu\nu} [-P_\mu P_\nu + 3Q_\mu Q_\nu + 2(p_1 p_2)g_{\mu\nu}] \\ 8(p_1 p_2)^2 E &= J^{\mu\nu} [3P_\mu P_\nu - Q_\mu Q_\nu - 2(p_1 p_2)g_{\mu\nu}] \end{aligned}$$

We can check that (2.13) is satisfied. Our final formula for  $J^{\mu\nu}$  reads

$$\begin{aligned} J^{\mu\nu} &= \frac{1}{16\pi m_H^2} \sqrt{1 - \frac{1}{z}(p_1^\mu p_1^\nu + p_2^\mu p_2^\nu - (p_1 p_2)g^{\mu\nu})} \\ &\quad + \frac{m^2}{8\pi m_H^4} \ln \left[ \frac{\sqrt{z} + \sqrt{z - 1}}{\sqrt{z} - \sqrt{z - 1}} \right] (-p_1^\mu p_2^\nu - p_2^\mu p_1^\nu + (p_1 p_2)g^{\mu\nu}) \end{aligned}$$

If we use the value of calculated integrals together with the equation (2.4), we arrive at

$$\begin{aligned} A|_{(x+i0)^{-1} \rightarrow -2\pi i\delta(x)} &= \frac{i\alpha Q^2}{vz} \left[ \left( \frac{1+u}{1-u} \right)^2 \ln(-u) \right] = \frac{i\alpha Q^2}{vz} \left[ \left( 1 - \frac{1}{z} \right) \ln \left( \frac{\sqrt{z} + \sqrt{z-1}}{\sqrt{z} - \sqrt{z-1}} \right) \right] \\ B|_{(x+i0)^{-1} \rightarrow -2\pi i\delta(x)} &= \frac{i\alpha Q^2}{vz} \left[ 4 \left( \frac{1+u}{1-u} \right) + \frac{1-6u+u^2}{(1-u)^2} \ln(-u) \right] \end{aligned}$$

Comparison of this result with the discontinuity of the imaginary part of result found in the previous section shows that

$$\begin{aligned} A|_{(x+i0)^{-1} \rightarrow -2\pi i\delta(x)} &= A(z+i\epsilon) - A(z-i\epsilon) = 2i\Im A \\ B|_{(x+i0)^{-1} \rightarrow -2\pi i\delta(x)} &= B(z+i\epsilon) - A(z-i\epsilon) = 2i\Im B \end{aligned}$$

To obtain the whole amplitude as an analytic function of  $z$ , we use the Cauchy's integral formula along the cut  $z > 1$  (we suppose that the amplitude falls off rapidly enough at the infinity, so that we can use the formula with no subtractions)

$$A(z) = \frac{1}{2\pi i} \oint \frac{A(t)dt}{t-z} = \frac{i}{2\pi} \int_1^\infty \frac{A(t+i\epsilon) - A(t-i\epsilon)}{t-z} dt = \frac{-2\alpha Q^2}{\pi v} I(z) = \frac{\alpha Q^2}{\pi v z} \left( \frac{(1+u)^2}{2(1-u)^2} H_{0,0}(u) - 1 \right)$$

In this formula we have used the result for integral  $I(z)$  defined by

$$I(z) = \frac{1}{4} \int_1^\infty \frac{1}{t} \left(1 - \frac{1}{t}\right) \ln \left( \frac{\sqrt{t} + \sqrt{t-1}}{\sqrt{t} - \sqrt{t-1}} \right) \frac{dt}{t-z}$$

which is evaluated in Appendix C.3; the result in the range  $0 < z < 1$  is

$$I(z) = \frac{1}{2z} + \frac{1}{2z} \left(1 - \frac{1}{z}\right) \left[ \operatorname{arctg} \sqrt{\frac{1-z}{z}} - \frac{\pi}{2} \right]^2 = \frac{1}{2z} + \frac{1}{2z} \left(1 - \frac{1}{z}\right) \operatorname{arctg}^2 \sqrt{\frac{z}{1-z}}$$

and this is equivalent to

$$I(z) = \frac{1}{2z} \left[ 1 - \frac{(1+u)^2}{2(1-u)^2} H_{0,0}(u) \right]$$

We see that the invariant amplitude is exactly what we obtained in the first section of this chapter. It must be said that in this way we have only obtained the expression for  $A(z)$  up to any “real” function of a complex variable, i.e. a function that satisfies the relation

$$f(z^*) = [f(z)]^*$$

The remaining information used to reconstruct the complete invariant amplitude must come from the other sources.

Generalization to more loops is possible and in fact there are multi-loop integrals that have been calculated with the method of dispersion relations [29], but there are complications with integration over three-particle (and even more-particle) phase space. Even the one-loop integration is comparable in difficulty with the integration over the Feynman parameters. In the next section, we will learn how to find algebraic relations between various integrals that will let us evaluate even the two-loop integrals with reasonable effort.

## 2.4 Evaluation of fermion loop through scalar amplitudes

As was demonstrated in the Section 2.2, there is a general recipe which lets us reduce any loop integral to the integral over the Feynman parameters. However, original loop integrals possess symmetries which are hidden after turning to the Feynman parameters. Thus it makes sense to exploit as many of these symmetries as possible before turning to the Feynman parameters. We will see later that these symmetries give rise to many relations among integrals which let us reduce the problem to evaluation of only a few integrals, called the Master integrals (MIs). We will work in the dimensional regularization as in Section 2.2. Our first step will be to project the tensorial amplitude  $M^{\mu\nu}$  on the scalar amplitudes. We denote

$$T = \frac{(d-2)(p_1 p_2)^2}{4m} \int \frac{d^d k}{(2\pi)^d} \frac{\operatorname{Tr} [(\not{k} - \not{p}_1 + m)\gamma^\mu(\not{k} + m)\gamma^\nu(\not{k} + \not{p}_2 + m)]}{[(k-p_1)^2 - m^2][(k+p_2)^2 - m^2][k^2 - m^2]}$$

where the factor in front of the integral sign was introduced for future convenience. Application of the projection operators evaluated earlier leads us to projections

$$\begin{aligned} T_A &= \int \frac{d^d k}{(2\pi)^d} \frac{[(6-d)(p_1 p_2)k^2 - 8(k p_1)(k p_2) + (d-2)m^2(p_1 p_2) - (d-2)(p_1 p_2)^2]}{[(k-p_1)^2 - m^2][(k+p_2)^2 - m^2][k^2 - m^2]} \\ T_B &= \int \frac{d^d k}{(2\pi)^d} \frac{[-4(p_1 p_2)k^2 + 4d(k p_1)(k p_2) - 2(d-2)(p_1 p_2)(k p_1 - k p_2) - (d-2)(p_1 p_2)^2]}{[(k-p_1)^2 - m^2][(k+p_2)^2 - m^2][k^2 - m^2]} \\ T_C &= \int \frac{d^d k}{(2\pi)^d} \frac{[-4(p_1 p_2)k^2 + 4d(k p_1)(k p_2) + (d-2)(p_1 p_2)^2]}{[(k-p_1)^2 - m^2][(k+p_2)^2 - m^2][k^2 - m^2]} \end{aligned}$$

### 2.4.1 Reduction of scalar integrals to Topo's - auxiliary diagram technique

Our case is simple enough, because we have as many independent scalars involving the loop momentum ( $k^2$ ,  $k p_1$ ,  $k p_2$ ) as there are terms in the denominator of integrand (“propagators”), which we will denote by

$$D_0 = k^2 - m^2$$

$$\begin{aligned} D_1 &= (k - p_1)^2 - m^2 \\ D_2 &= (k + p_2)^2 - m^2 \end{aligned}$$

These relations can be inverted (for  $p_1^2 = 0 = p_2^2$ )

$$\begin{aligned} k^2 &= D_0 + m^2 \\ kp_1 &= \frac{1}{2}D_0 - \frac{1}{2}D_1 \\ kp_2 &= -\frac{1}{2}D_0 + \frac{1}{2}D_2 \end{aligned}$$

and thus we can express the integrand as a linear combination (with constant coefficients) of the integrals of the form

$$\text{Topo}(a, b, c) = \int \frac{d^d k}{(2\pi)^d} \frac{1}{P_0^a P_1^b P_2^c}$$

with  $a, b, c$  integers. Explicitly,

$$\begin{aligned} T_A &= 2 \text{Topo}(1, 0, 0) - 2 \text{Topo}(0, 0, 1) - 2 \text{Topo}(0, 1, 0) + 2 \text{Topo}(-1, 1, 1) + \\ &\quad 2(6 - d)zm^2 \text{Topo}(0, 1, 1) + 8zm^4 \text{Topo}(1, 1, 1) + 4(2 - d)z^2m^4 \text{Topo}(1, 1, 1) \\ T_B &= -d \text{Topo}(1, 0, 0) + d \text{Topo}(0, 1, 0) + d \text{Topo}(0, 0, 1) - d \text{Topo}(-1, 1, 1) - \\ &\quad 4dzm^2 \text{Topo}(0, 1, 1) + 2(d - 2)zm^2 \text{Topo}(1, 1, 0) + 2(d - 2)zm^2 \text{Topo}(1, 0, 1) - \\ &\quad 8m^4z \text{Topo}(1, 1, 1) + 4(2 - d)z^2m^4 \text{Topo}(1, 1, 1) \\ T_C &= -d \text{Topo}(1, 0, 0) + d \text{Topo}(0, 1, 0) + d \text{Topo}(0, 0, 1) - d \text{Topo}(-1, 1, 1) - \\ &\quad 8zm^2 \text{Topo}(0, 1, 1) - 8zm^4 \text{Topo}(1, 1, 1) + 4(d - 2)z^2m^4 \text{Topo}(1, 1, 1) \end{aligned}$$

### 2.4.2 Integration by parts identities and reduction to Masters

Now it suffices to evaluate the scalar integrals  $\text{Topo}(a, b, c)$ . Our special kinematical relations imply that these integrals are symmetric in indices  $b$  and  $c$ ,

$$\text{Topo}(a, b, c) = \text{Topo}(a, c, b)$$

Moreover, when either  $b$  or  $c$  is zero, the integral must be independent of  $p_1$  and  $p_2$ , so we can put these equal to zero and we so have the relation

$$\text{Topo}(a, b, 0) = \text{Topo}(a, 0, b) = \text{Topo}(a + b, 0, 0)$$

More identities can be obtained by integrating by parts [53, 13]. We formally insert the relation

$$1 = \frac{1}{d} \frac{\partial}{\partial p^\mu} p^\mu$$

and integrate by parts, neglecting the ‘‘surface’’ terms. The result can again be expressed in terms of integrals  $\text{Topo}(a, b, c)$  with different values of  $a, b$  and  $c$ . It is convenient to introduce the following ‘‘operator’’ notation:

$$\begin{aligned} \mathbf{A} \text{Topo}(a, b, c) &= a \text{Topo}(a, b, c) \\ \mathbf{A}^+ \text{Topo}(a, b, c) &= \text{Topo}(a + 1, b, c) \\ \mathbf{A}^- \text{Topo}(a, b, c) &= \text{Topo}(a - 1, b, c) \end{aligned}$$

From which it follows

$$\begin{aligned} [\mathbf{A}, \mathbf{A}^+] &= \mathbf{A}^+ \\ [\mathbf{A}, \mathbf{A}^-] &= -\mathbf{A}^- \\ [\mathbf{A}^+, \mathbf{A}^-] &= 0 \\ \mathbf{A}^+ \mathbf{A}^- &= \mathbb{1} \end{aligned}$$

For our convenience (although it may be a little confusing), we will write

$$a\mathbf{A}^+$$

by which we mean

$$\mathbf{A}^+\mathbf{A}$$

that is, the indices  $a$ ,  $b$  and  $c$  should be substituted by the value of indices  $a$ ,  $b$  and  $c$  of integral  $\text{Topo}(a, b, c)$  to which the operator was applied. Whenever multiplying equations or doing other more complicated operator manipulations, one can return immediately to the correct notation by writing the indices  $a \dots$  to the right of the rising/lowering operators in the form of operators  $\mathbf{A} \dots$ . Last thing to be mentioned is the fact that the operator  $\mathbf{A}^+$  in IBP relations comes only in the combination with  $a$ :

$$a\mathbf{A}^+ \equiv \mathbf{A}^+\mathbf{A}$$

and the commutation relation of this operator with  $\mathbf{A}^-$  is

$$[a\mathbf{A}^+, \mathbf{A}^-] \equiv [\mathbf{A}^+\mathbf{A}, \mathbf{A}^-] = \mathbf{A}^+ [\mathbf{A}, \mathbf{A}^-] = -\mathbf{A}^+\mathbf{A}^- = -\mathbb{1}$$

that is, operators  $a\mathbf{A}^+$  and  $\mathbf{A}^-$  satisfy the commutation relations of an ordinary creation and annihilation operators.

After these remarks we can state the result of integration by parts in a compact form:

$$\begin{aligned} 0 &= (d - 2a - b - c) - b\mathbf{B}^+\mathbf{A}^- - c\mathbf{C}^+\mathbf{A}^- - 2m^2a\mathbf{A}^+ - 2m^2b\mathbf{B}^+ - 2m^2c\mathbf{C}^+ \\ 0 &= (b - a) + a\mathbf{A}^+\mathbf{B}^- + c\mathbf{C}^+\mathbf{B}^- - b\mathbf{B}^+\mathbf{A}^- - c\mathbf{C}^+\mathbf{A}^- - 4zm^2c\mathbf{C}^+ \\ 0 &= (c - a) + a\mathbf{A}^+\mathbf{C}^- + b\mathbf{B}^+\mathbf{C}^- - b\mathbf{B}^+\mathbf{A}^- - c\mathbf{C}^+\mathbf{A}^- - 4zm^2b\mathbf{B}^+ \end{aligned}$$

The last set of linear equations is obtained by exploiting the Lorentz-invariance properties [23]. The identity which follows for the three-point functions can be written as

$$(p_1^\mu p_2^\nu - p_2^\mu p_1^\nu) \sum_{n=1}^2 \left[ p_\mu^n \frac{\partial}{\partial p_n^\nu} - p_\nu^n \frac{\partial}{\partial p_n^\mu} \right] \text{Topo}(a, b, c) = 0$$

Note that the sum represents a generator of the Lorentz transformations acting on the four-vectors and this generator is contracted with the only antisymmetric tensor that we can write with only two four-vectors available (except for the tensor involving  $\epsilon^{\mu\nu\rho\sigma}$ , which however gives zero when contracted to the generator). Generalization to the four-point and higher functions is straightforward. Applied to our case, it yields

$$0 = (c - b) + b\mathbf{B}^+\mathbf{A}^- - c\mathbf{C}^+\mathbf{A}^-$$

This completes the list of identities that we have used to relate various  $\text{Topo}(a, b, c)$ 's (we did not use the relation coming from the derivative of the integral  $\text{Topo}(a, b, c)$  with respect to  $m^2$ , but this would only give us an identity which is linearly dependent on the other identities that we have used). We could try to solve these identities analytically, but this can turn out to be very difficult, especially in the two-loop case. Commonly used method is the Laporta's method [37, 36] which amounts to solving the subset of identities chosen according to certain rules. These rules are chosen in order to guarantee that we stay within the sector of integrals that are relevant for our purposes. We used the Maple implementation of Laporta's algorithm called AIR [7]. The general principle of operation is not very difficult - integrals  $\text{Topo}(n_1, n_2, n_3)$  are first ordered according to 3 criteria, which are (with decreasing priority):

- the number of "propagators" (the number of positive indices)

$$\mathcal{N}_{prop} = \sum_i \theta(n_i),$$

- the sum of positive indices

$$\mathcal{N}_+ = \sum_i \theta(n_i)(n_i - 1),$$

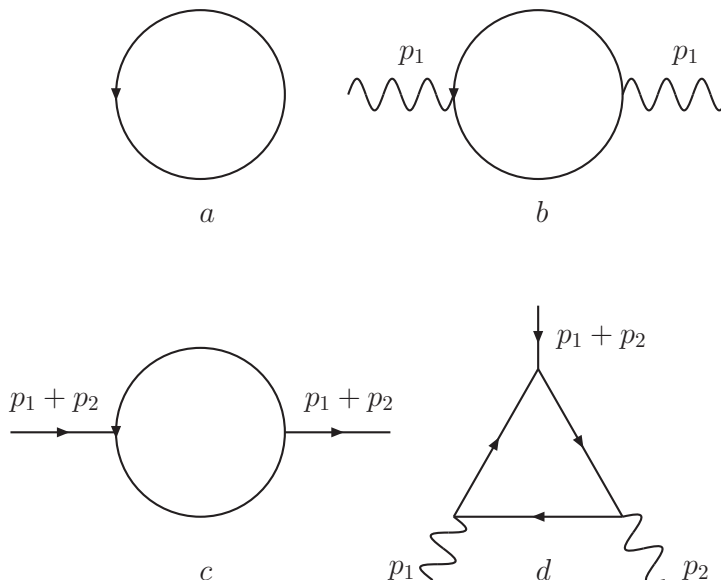


Figure 2.5: One-loop topologies (only the momentum flow is important)

- the sum of negative indices

$$\mathcal{N}_- = \sum_i \theta(-n_i)(-n_i)$$

After assigning priorities to integrals, we can “apply” the identities written in an operator form one by one to integrals, starting from the simplest one (with the lowest value of priority parameters  $\mathcal{N}_{prop}$ ,  $\mathcal{N}_+$  and  $\mathcal{N}_-$ ) and everytime eliminating the most difficult integral (with highest value of the priority parameters) encountered in the formula (this is a variation of Gauss elimination method for infinite set of variables).

After the reduction we are left with few integrals which cannot be expressed algebraically as combinations of simpler ones and these are called the Master integrals of the problem. We must evaluate them in another way. The choice of MIs is almost arbitrary, but one usually chooses them so that they have either the lowest possible value of priority parameters or such that the differential equations which will be used later to solve MIs become easier to solve. The number of Master integrals corresponding to each topology (see later) should not depend on their choice, but as far as we know there is no proof that they are really “irreducible” and there is yet no way how to predict their number before solving the system of equations.

Integrals  $\text{Topo}(a, b, c)$  with non-negative  $a$ ,  $b$  and  $c$  can be divided into groups (which we will call the topologies) according to the non-zero values of indices. We introduce a simplified notation which we will use to denote the various topologies: letter  $A$  will correspond to propagator  $D_0$ , letter  $B$  to propagator  $D_1$  and  $C$  will denote the propagator  $D_2$ . We will write a word specifying a topology with the letters in the word representing a nonzero (positive) power of the corresponding propagator. Then we can list all one-loop topologies (the symbol  $\propto$  stands for “up to an overall normalization factor”):

- 1 propagator topology - (A)(B), Figure 2.5a,  $z$ -independent, one MI ( $\text{Topo}(1, 0, 0) \propto I_A$ )
- 2 propagator topologies
  - type 1 - (AB), Figure 2.5b,  $z$ -independent, reducible to subtopologies
  - type 2 - (BC), Figure 2.5c, one MI ( $\text{Topo}(0, 1, 1) \propto I_B$ )
- 3 propagator topologies - (ABC), Figure 2.5d, one MI ( $\text{Topo}(1, 1, 1) \propto I_C$ )

Note that the integrals (A) and (B) belong to the same topology, because they differ only in the choice of the loop momentum. On the other hand, the topology (AB) is different from (A) (although the integrals are numerically the same since  $p_1^2 = p_2^2 = 0$ ). Each topology has various subtopologies, which can be obtained by contracting an internal line to a point.

As we will see later, when using the differential equations to find the values of MIs, we arrive at the homogeneous system of  $n$  ordinary linear differential equations, where  $n$  is the number of Master Integrals, and this can be in general difficult to solve. However, if we choose Master Integrals properly, this system becomes block-lower triangular (if we order integrals such that subtopologies come first). In general, the block on the diagonal corresponds to one topology. If we proceed from the simpler topologies to more complicated ones, we must solve the system of inhomogeneous ordinary linear differential equations with inhomogeneous terms coming from the subtopologies and the dimension of the system is determined by the number of Master integrals in the corresponding topology. In the case of the one-loop integrals the situation is very simple, because we have one Master integral for each topology, so it is enough to solve only 3 inhomogeneous first order ordinary linear differential equations.

Assigning a topology to integrals with some of indices negative is more complicated (of course before performing the reduction), but in the case of one-loop integrals the results are exactly as described in the previous paragraphs, if we count the negative indices as zeros.

The diagrammatical illustration of these topologies is given in figure 2.5. We can state the results of the reduction, that we will use in the following calculations:

$$\begin{aligned}
\text{Topo}(-1, 0, 0) &= 0 \\
\text{Topo}(0, 0, 0) &= 0 \\
\text{Topo}(2, 0, 0) &= \frac{d-2}{2m^2} \text{Topo}(1, 0, 0) \\
\text{Topo}(3, 0, 0) &= \frac{(d-2)(d-4)}{8m^4} \text{Topo}(1, 0, 0) \\
\text{Topo}(0, 1, -1) &= 4zm^2 \text{Topo}(1, 0, 0) \\
\text{Topo}(-1, 1, -1) &= \frac{8m^4z}{d} \text{Topo}(1, 0, 0) \\
\text{Topo}(1, 1, -1) &= (1 + (d-2)z) \text{Topo}(1, 0, 0) \\
\text{Topo}(1, 1, -2) &= \frac{8m^2z(3 + (d-2)z)}{3} \text{Topo}(1, 0, 0) \\
\text{Topo}(-1, 1, 1) &= \text{Topo}(1, 0, 0) - 2m^2z \text{Topo}(0, 1, 1) \\
\text{Topo}(-1, 1, 2) &= \frac{d-2}{4m^2(1-z)} \text{Topo}(1, 0, 0) + \frac{-1 + (d-2)z}{2(z-1)} \text{Topo}(0, 1, 1) \\
\text{Topo}(-2, 1, 1) &= \frac{-2zm^2d}{d-1} \text{Topo}(1, 0, 0) + \frac{4m^4z(zd-1)}{d-1} \text{Topo}(0, 1, 1) \\
\text{Topo}(0, 1, 2) &= \frac{d-2}{8m^4(z-1)} \text{Topo}(1, 0, 0) - \frac{d-3}{4m^2(z-1)} \text{Topo}(0, 1, 1) \\
\text{Topo}(1, 1, 2) &= \frac{(d-2)(z(d-4) - (d-3))}{16m^6z(z-1)} \text{Topo}(1, 0, 0) + \frac{(d-3)}{8m^4z(z-1)} \text{Topo}(0, 1, 1) \\
\text{Topo}(2, 1, 1) &= -\frac{(d-2)(d-3)}{8m^6z} \text{Topo}(1, 0, 0) + \frac{d-3}{4m^4z} \text{Topo}(0, 1, 1) + \frac{d-4}{2m^2} \text{Topo}(1, 1, 1)
\end{aligned}$$

### 2.4.3 Master Integral $\text{Topo}(1, 0, 0)$ and differential equations

After having described the topologies, it remains to evaluate the Master integrals. We could use the Feynman parameters as in Section 2.2, but to illustrate the method that will be used later in the two-loop calculations, we will show how to calculate them using the differential equations. But since the differential equations satisfied by the integrals are homogeneous, we must compute one integral in a different way. We choose the first integral, which does not depend on  $z$  and is trivial

$$\text{Topo}(1, 0, 0) \equiv \frac{iNm^2}{(4\pi)^2} I_A = \frac{iNm^2}{(4\pi)^2 \epsilon(1-\epsilon)} = \frac{iNm^2}{(4\pi)^2} \left[ \frac{1}{\epsilon} + 1 + \epsilon + \epsilon^2 + \epsilon^3 + \dots \right]$$

(we will need higher orders in  $\epsilon$  later in two-loop MIs evaluation). To derive the differential equation satisfied by MIs, we differentiate the integral  $\text{Topo}(a, b, c)$  with respect to the external momentum  $p_1$  and contract the result with  $p_1$ :

$$\begin{aligned}
z \frac{\partial}{\partial z} \text{Topo}(a, b, c) &= p_1^\mu \frac{\partial}{\partial p_1^\mu} \text{Topo}(a, b, c) = \int \frac{d^d k}{(2\pi)^d} \frac{2bkp_1}{D_0^a D_1^{b+1} D_2^c} = b \int \frac{d^d k}{(2\pi)^d} \frac{D_0 - D_1}{D_0^a D_1^{b+1} D_2^c} \\
&= b \text{Topo}(a-1, b+1, c) - b \text{Topo}(a, b, c) = c \text{Topo}(a-1, b, c+1) - c \text{Topo}(a, b, c)
\end{aligned}$$

The last equality follows from the symmetry in  $p_1$  and  $p_2$  (which is expressed also by the Lorentz invariance identity). Note that since the differentiation with respect to the external momenta is not compatible with imposing the on-shell conditions, we must be careful when performing these operations. We should perform the differentiation before imposing constraints  $p_1^2 = 0$  and treat  $\text{Topo}(a, b, c)$  as function of all three independent scalar invariants ( $p_1^2$ ,  $p_2^2$  and  $p_1 \cdot p_2$ ) and only after this is done we should impose the on-shell conditions.

#### 2.4.4 Master Integral $\text{Topo}(0, 1, 1)$

Application of the previous identity to  $\text{Topo}(0, 1, 1)$  yields

$$z \frac{\partial}{\partial z} \text{Topo}(0, 1, 1) = \text{Topo}(-1, 1, 2) - \text{Topo}(0, 1, 1) = \frac{d-2}{4m^2(1-z)} \text{Topo}(1, 0, 0) - \frac{1+(d-4)z}{2(1-z)} \text{Topo}(0, 1, 1)$$

This is the first order ordinary inhomogeneous linear differential equation which we can solve by the variation of parameters (we know that the solution to this type of differential equation can be always given in terms of integrals). This equation can be solved either exactly in  $d$  dimensions in terms of the hypergeometric function, or iteratively by an expansion in  $\epsilon$  around  $d = 4$ . The ‘‘exact’’  $d$ -dimensional result is

$$\text{Topo}(0, 1, 1) = \frac{iN}{(4\pi)^2 \epsilon} {}_2F_1 \left( \epsilon, 1; \frac{3}{2}; z \right)$$

This formula is nice and compact, but it is not easy to perform the  $\epsilon$ -expansion, so we will proceed indirectly via the expansion of the differential equation. Since  $\text{Topo}(0, 1, 1)$  is one-loop integral, its  $\epsilon$ -expansion starts at order  $O(\epsilon^{-1})$ . Moreover, every term of the differential equation is proportional to (a derivative of) the one-loop integral, so we can factor out the overall coefficient

$$\frac{iN}{(4\pi)^2}$$

that will occur once for every loop. As it turns out, a convenient variable for expressing the results is the variable  $u$  described in more detail in Section B.1.2. After changing the variable to  $u$ , the differential equation transforms to

$$I'_B(u) = \left( \frac{2}{1-u^2} + \frac{\epsilon(1-u)}{u(1+u)} \right) I_B(u) - \frac{2}{\epsilon(1-u^2)}$$

with

$$\text{Topo}(0, 1, 1) \equiv \frac{iN}{(4\pi)^2} I_B(u) = \frac{iN}{(4\pi)^2} \sum_{j=-1}^{\infty} \epsilon^j I_{Bj}(z)$$

(the explicit expression for  $I_A$  was used). We see that the equation to order  $O(\epsilon^{-1})$  reads

$$I'_{B,-1} = \frac{2}{1-u^2} I_{B,-1} - \frac{2}{1-u^2}$$

We will solve this equation (and all differential equations that will follow) for the non-physical values of  $z < 0$  which correspond to  $u \in (0; 1)$ . In this way we avoid cuts or singularities of the solution and the general solution can be obtained in an obvious way by an analytic continuation to any region. All equations that we will encounter will be inhomogeneous first order ordinary linear differential equations, i.e. equations of the form

$$f'(u) = a(u)f(u) + b(u)$$

These equations can be solved in a well known way (for more information see for example [28]). We first solve the homogeneous equation

$$f'(u) = a(u)f(u)$$

by separation of variables and obtain

$$f(u) = C \exp \left( \int^u a(t) dt \right)$$

then we use the variation of constants to find the solution of inhomogeneous equation that can be written in the form

$$f(u) = \left[ C + \int^u \frac{b(t)dt}{\exp\left(\int^t a(s)ds\right)} \right] \exp\left(\int^u a(t)dt\right)$$

In this way we find the solution of homogeneous part of the equation

$$\exp\left(\int \frac{2}{1-u^2}\right) = C_1 \frac{1+u}{1-u}$$

and then the solution to the inhomogeneous equation

$$C_1 \frac{1+u}{1-u} \left( \int \frac{1-u}{C_1(1+u)} \frac{-2}{1-u^2} du + C_2 \right),$$

which can be written as

$$\frac{2 + C_3(1+u)}{1-u}$$

The unknown constant (which is a result of using the differential equations approach) could be in principle solved by comparing this expression with the value of integral for any specific value of  $z$ . However, since the solutions of homogeneous equations turn out to be in general singular at  $u = 1$  (which corresponds to  $z = 0$ ), where there is no reason for the singular behaviour, the constants can usually be fixed by imposing the finiteness condition at  $u = 1$ . In our case this is equivalent to

$$C_3 = -1$$

and the final result for  $I_{B,-1}$  is

$$I_{B,-1} = 1.$$

Next obvious step is to consider order  $O(\epsilon^0)$  of the equation. We substitute the value of  $I_{B,-1}$  and continue similarly. Note that the homogeneous equation is the same for all orders in  $\epsilon$ , so we have to solve it only once for each Master Integral. In this way we arrive at

$$I_{B,0} = 2 + \frac{1+u}{1-u} \text{H}(0; u),$$

where we have introduced the harmonic polylogarithm (see Appendix B.1.5)

$$\text{H}(0; x) \equiv \ln(x)$$

We see from the character of the homogeneous solution and the particular solutions order by order that the expression for the solution can be found to any desired order in the form of integral of linear combination of functions

$$R(u) \prod_j H_j(u)$$

with  $R(u)$  a rational function with poles at  $u = 0$ ,  $u = 1$  or  $u = -1$  and  $\prod_j H_j(u)$  a product of harmonic polylogarithms. If we use the shuffle-algebra product expansion property of polylogarithms (i.e. that any product of harmonic polylogarithms can be expressed as a linear combination of harmonic polylogarithms), and if we express  $R(u)$  in the form of partial fractions, we are left with linear combination of rational functions and integrals of

$$\frac{1}{u^a} \text{H}(u) \quad \text{or} \quad \frac{1}{(1-u)^a} \text{H}(u) \quad \text{or} \quad \frac{1}{(1+u)^a} \text{H}(u)$$

with  $a$  a positive integer. By performing the integration by parts, we can reduce the value of  $a$  to a lower one and repeating this, we arrive at integrals of functions of type

$$\frac{1}{u} \text{H}(u) \quad \text{or} \quad \frac{1}{1-u} \text{H}(u) \quad \text{or} \quad \frac{1}{1+u} \text{H}(u)$$

Now from the definition of harmonic polylogarithms it immediately follows, that this is another harmonic polylogarithm with higher weight. The algorithm described above can be programmed for example in Mathematica

and thus in principle we can solve the equation to any desired order in  $\epsilon$ . Note however, that the results become more and more complicated as we proceed to higher orders. In this way we obtain

$$I_{B,1} = \frac{1+u}{1-u} \left\{ -4 - \zeta(2) + 2\mathbf{H}(0; u) + \mathbf{H}(0, 0; u) - 2\mathbf{H}(-1, 0; u) \right\} + \frac{8}{1-u}$$

$$I_{B,2} = \frac{1+u}{1-u} \left\{ -8 - 2\zeta(2) - 2\zeta(3) + (4 - \zeta(2))\mathbf{H}(0; u) + 2\zeta(2)\mathbf{H}(-1; u) + 2\mathbf{H}(0, 0; u) \right. \\ \left. - 4\mathbf{H}(-1, 0; u) + \mathbf{H}(0, 0, 0; u) - 2\mathbf{H}(0, -1, 0; u) - 2\mathbf{H}(-1, 0, 0; u) + 4\mathbf{H}(-1, -1, 0; u) \right\} + \frac{16}{1+u}$$

and

$$I_{B,3} = \frac{1+u}{1-u} \left\{ -4\zeta(2) - \frac{9}{4}\zeta(4) - 16 - 4\zeta(3) + (8 - 2\zeta(2) - 2\zeta(3))\mathbf{H}(0; u) + 4(\zeta(2) + \zeta(3))\mathbf{H}(-1; u) \right. \\ + (4 - \zeta(2))\mathbf{H}(0, 0; u) + 2(\zeta(2) - 4)\mathbf{H}(-1, 0; u) + 2\zeta(2)\mathbf{H}(0, -1; u) - 2\zeta(2)\mathbf{H}(-1, -1; u) + 2\mathbf{H}(0, 0, 0; u) \\ - 4\mathbf{H}(-1, 0, 0; u) + 4\mathbf{H}(0, -1, 0; u) + 8\mathbf{H}(-1, -1, 0; u) + \mathbf{H}(0, 0, 0, 0; u) - 2\mathbf{H}(-1, 0, 0, 0; u) - 2\mathbf{H}(0, 0, -1, 0; u) \\ \left. + 4\mathbf{H}(-1, 0, -1, 0; u) - 2\mathbf{H}(0, -1, 0, 0; u) + 4\mathbf{H}(-1, -1, 0, 0; u) + 4\mathbf{H}(0, -1, -1, 0; u) - 8\mathbf{H}(-1, -1, -1, 0; u) \right\} + \frac{32}{1-u}$$

### 2.4.5 Master Integral $\text{Topo}(1, 1, 1)$

The third one loop master integral is the integral  $\text{Topo}(1, 1, 1)$ . We proceed similarly as in the previous section, this time, however, we are dealing with a convergent integral. We denote

$$\text{Topo}(1, 1, 1) = \frac{iN}{(4\pi)^2 m^2} I_C = \frac{iN}{(4\pi)^2 m^2} \sum_{j=0}^{\infty} \epsilon^j I_{C,j}$$

and the differential equation is now

$$z \frac{\partial}{\partial z} \text{Topo}(1, 1, 1) = -\text{Topo}(1, 1, 1) + \frac{2-d}{8(1-z)} \text{Topo}(1, 0, 0) + \frac{d-3}{4(1-z)} \text{Topo}(0, 1, 1)$$

We can check, that order  $\epsilon^{-1}$  equation is satisfied. First few nonzero coefficients of  $I_C$  are

$$I_{C,0} = \frac{-u}{(1-u)^2} \mathbf{H}(0, 0; u)$$

$$I_{C,1} = \frac{u}{(1-u)^2} \left\{ \zeta(2)\mathbf{H}(0; u) + 3\zeta(3) - \mathbf{H}(0, 0, 0; u) + 2\mathbf{H}(0, -1, 0; u) \right\}$$

and

$$I_{C,2} = \frac{u}{(1-u)^2} \left\{ \frac{5}{4}\zeta(4) + 2\zeta(3)\mathbf{H}(0; u) + \zeta(2)\mathbf{H}(0, 0; u) - 2\zeta(2)\mathbf{H}(0, -1; u) - \right. \\ \left. \mathbf{H}(0, 0, 0, 0; u) + 2\mathbf{H}(0, 0, -1, 0; u) + 2\mathbf{H}(0, -1, 0, 0) - 4\mathbf{H}(0, -1, -1, 0; u) \right\}$$

The result for general  $d$  can also be written down, so for reference

$$I_C = -\frac{1}{2} {}_3F_2 \left( 1, 1, 1 + \epsilon; 2, \frac{3}{2}; z \right).$$

### 2.4.6 Results for scalar integrals

For one loop calculations, the necessary integrals to order  $O(\epsilon^0)$  are

$$\text{Topo}(-1, 0, 0) = 0$$

$$\begin{aligned}
\text{Topo}(-2, 1, 1) &= \frac{iNm^4}{(4\pi)^2} \left[ \frac{-4z}{\epsilon} + \frac{16z^2}{3\epsilon} - \frac{20z}{3} + \frac{104z^2}{9} + \left( \frac{8z}{3} - \frac{32z^2}{3} \right) \sqrt{\frac{1-z}{z}} \arctg \sqrt{\frac{z}{1-z}} \right] \\
\text{Topo}(-1, 1, -1) &= \frac{iNm^6}{(4\pi)^2} \left[ \frac{2z}{\epsilon} + 3z \right] \\
\text{Topo}(-1, 1, 1) &= \frac{iNm^2}{(4\pi)^2} \left[ \frac{1}{\epsilon} - \frac{2}{\epsilon}z + 1 - 4z + 4z \sqrt{\frac{1-z}{z}} \arctg \sqrt{\frac{z}{1-z}} \right] \\
\text{Topo}(0, 0, 0) &= 0 \\
\text{Topo}(0, 1, -1) &= \frac{iNm^4}{(4\pi)^2} \left[ \frac{4z}{\epsilon} + 4z \right] \\
\text{Topo}(0, 1, 1) &= \frac{iN}{(4\pi)^2} \left[ \frac{1}{\epsilon} + 2 - 2 \sqrt{\frac{1-z}{z}} \arctg \sqrt{\frac{z}{1-z}} \right] \\
\text{Topo}(1, 0, 0) &= \frac{iNm^2}{(4\pi)^2} \left[ \frac{1}{\epsilon} + 1 \right] \\
\text{Topo}(1, 1, -2) &= \frac{iNm^4}{(4\pi)^2} \left[ \frac{8z}{\epsilon} + \frac{16z^2}{3\epsilon} + 8z \right] \\
\text{Topo}(1, 1, -1) &= \frac{iNm^2}{(4\pi)^2} \left[ \frac{1}{\epsilon} + \frac{2z}{\epsilon} + 1 \right] \\
\text{Topo}(1, 1, 1) &= \frac{iN}{(4\pi)^2 m^2} \left[ -\frac{1}{2z} \arctg^2 \sqrt{\frac{z}{1-z}} \right] \\
\text{Topo}(2, 0, 0) &= \frac{iN}{(4\pi)^2} \frac{1}{\epsilon}
\end{aligned}$$

Some of integrals in the list will be needed in  $W$ -boson loop evaluation. Substituting these results to equations for  $T_A$ ,  $T_B$  and  $T_C$ , we arrive at the same results as before:

$$\begin{aligned}
A &= \frac{\alpha Q^2 N}{(d-2)\pi v z} [(d-4)I_B + 2((d-2)z-2)I_C] \\
B &= \frac{\alpha Q^2 N}{(d-2)\pi v z} [-(d-2)^2 I_A + dI_B + 2((d-2)z+2)I_C]
\end{aligned}$$

In the following, it will be useful to have a formula for scalar amplitude  $A$  valid for arbitrary dimension  $d$ . We just substitute the  $d$ -dimensional value of Master integrals to the previous result and obtain immediately

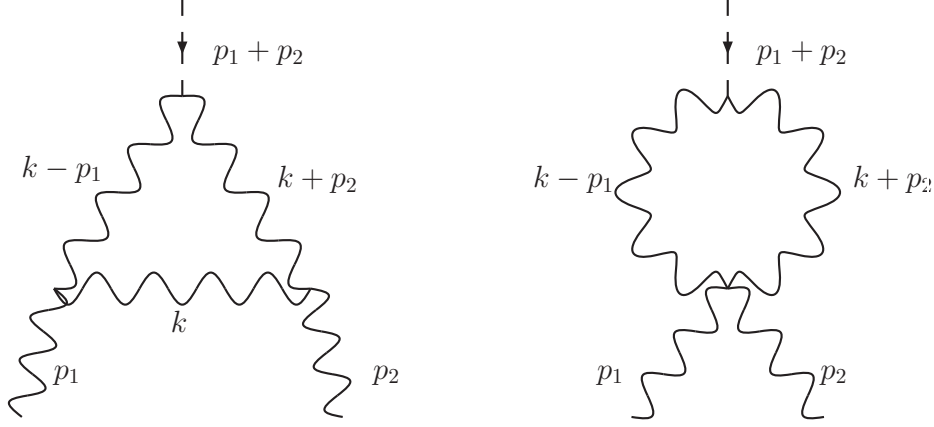
$$A = \frac{-\alpha Q^2 N}{\pi v z (1-\epsilon)} \left[ {}_2F_1 \left( 1, \epsilon; \frac{3}{2}; z \right) + (z - z\epsilon - 1) {}_3F_2 \left( 1, 1, 1 + \epsilon; 2, \frac{3}{2}; z \right) \right]$$

We will also make use of the expression for amplitude  $A$  valid to order  $O(\epsilon^1)$ . Again, this is easily done:

$$\begin{aligned}
A = \frac{\alpha Q^2 N}{\pi v z} \left[ -1 + \frac{(1+u)^2}{2(1-u)^2} H_{0,0}(u) + \epsilon \left( -3 - \frac{1+u}{1-u} H_0(u) - \frac{(1+u)^2 \zeta(2)}{2(1-u)^2} H_0(u) + \right. \right. \\
\left. \left. \frac{2u}{(1-u)^2} H_{0,0}(u) - \frac{3(1+u)^2}{2(1-u)^2} \zeta(3) + \frac{(1+u)^2}{2(1-u)^2} H_{0,0,0}(u) - \frac{(1+u)^2}{(1-u)^2} H_{0,-1,0}(u) \right) \right]
\end{aligned}$$

## 2.5 W-boson loop, scalar projection, U-gauge

After having evaluated the fermion contribution, we can now consider the gauge boson contribution to the invariant amplitude. There are two possible choices of the gauge-fixing parameter  $\xi$ , for which the calculation simplifies, one is  $\xi \rightarrow \infty$  (so-called unitary, U-gauge) in which the unphysical Goldstone boson fields disappear, so there is a smaller number of contributing diagrams. The second choice is  $\xi = 1$  (so-called 't Hooft gauge) in which the unphysical Goldstone boson fields have mass equal to the corresponding gauge boson, but non-diagonal term in propagators disappears, so the ‘‘numerator algebra’’ becomes simpler. Original calculations [19, 50] employed the second choice, but with help of computer the first choice is easier.

Figure 2.6: W-boson diagrams contributing to  $H \rightarrow \gamma\gamma$ 

The relevant diagrams are depicted in Figure 2.6. According to the Feynman rules (see Appendix A.1), we have

$$\begin{aligned}
iM_{\mu\nu}^{W1} &= -gm_W e^2 \int \frac{d^d k}{(2\pi)^d} \frac{(2g_{\mu\nu}g_{\kappa\lambda} - g_{\mu\kappa}g_{\nu\lambda} - g_{\mu\lambda}g_{\nu\kappa})}{[(k-p_1)^2 - m_W^2][(k+p_2)^2 - m_W^2]} \\
&\quad \left[ g^{\lambda\sigma} - \frac{(k+p_2)^\lambda(k+p_2)^\sigma}{m_W^2} \right] \left[ g_\sigma^\kappa - \frac{(k-p_1)^\kappa(k-p_1)_\sigma}{m_W^2} \right] \\
iM_{\mu\nu}^{W2} &= gm_W e^2 \int \frac{d^d k}{(2\pi)^d} \frac{V_{\mu\kappa\alpha}(-p_1, p_1-k, k)V_{\nu\beta\lambda}(-p_2, -k, k+p_2)}{[k^2 - m_W^2][(k-p_1)^2 - m_W^2][(k+p_2)^2 - m_W^2]} \\
&\quad \left[ g^{\lambda\sigma} - \frac{(k+p_2)^\lambda(k+p_2)^\sigma}{m_W^2} \right] \left[ g_\sigma^\kappa - \frac{(k-p_1)^\kappa(k-p_1)_\sigma}{m_W^2} \right] \left[ g^{\alpha\beta} - \frac{k^\alpha k^\beta}{m_W^2} \right] \\
iM_{\mu\nu}^{W3} &= gm_W e^2 \int \frac{d^d k}{(2\pi)^d} \frac{V_{\nu\kappa\alpha}(-p_2, p_2-k, k)V_{\mu\beta\lambda}(-p_1, -k, k+p_1)}{[k^2 - m_W^2][(k-p_2)^2 - m_W^2][(k+p_1)^2 - m_W^2]} \\
&\quad \left[ g^{\lambda\sigma} - \frac{(k+p_1)^\lambda(k+p_1)^\sigma}{m_W^2} \right] \left[ g_\sigma^\kappa - \frac{(k-p_2)^\kappa(k-p_2)_\sigma}{m_W^2} \right] \left[ g^{\alpha\beta} - \frac{k^\alpha k^\beta}{m_W^2} \right]
\end{aligned}$$

( $M_{\mu\nu}^{W3}$  is obtained from  $M_{\mu\nu}^{W2}$  by the exchange  $p_1^\mu \leftrightarrow p_2^\nu$ ; on the other hand, symmetrization of  $M_{\mu\nu}^{W1}$  is already included in the usual Feynman rules for  $WWAA$  vertex). We may proceed as in the direct calculation of fermion loop or using the projections to scalar amplitudes. Relevant integrals encountered during the evaluation of integral are listed at the end of Section 2.4 or in Appendix C.1. Resulting scalar amplitudes expressed in terms of Master integrals in  $d$ -dimensions are for first diagram

$$\begin{aligned}
A_1 &= \frac{g\alpha N}{4\pi z_W m_W} \left[ \frac{2zd(2d-3) + (d-2)}{2d(d-1)} I_A + \frac{-2(2d-3)z^2 + (4d-5)z - (d^2-3d+3)}{d-1} I_B \right] \\
B_1 &= \frac{g\alpha N}{4\pi z_W m_W} \left[ -\frac{(d-2)(2z+1)}{4(d-1)} I_A + \frac{2(d-2)z^2 + (4-3d)z + 1}{2(d-1)} I_B \right] \\
C_1 &= \frac{g\alpha N}{4\pi z_W m_W} \left[ -\frac{(d-2)(2z+1)}{4(d-1)} I_A + \frac{2(d-2)z^2 + (4-3d)z + 1}{2(d-1)} I_B \right]
\end{aligned}$$

and for the second one

$$\begin{aligned}
A_2 &= \frac{g\alpha N}{4\pi z_W m_W} \left[ \frac{-2zd(2d-3) - (d-2)}{4d(d-1)} I_A + \frac{-4z(2d-5) + 2(d-1)}{d-2} I_C + \right. \\
&\quad \left. \left( \frac{2d-3}{d-1} z^2 - \frac{6d^2-23d+18}{2(d^2-3d+2)} z + \frac{d^2-2}{2(d^2-3d+2)} \right) I_B \right]
\end{aligned}$$

$$\begin{aligned}
B_2 &= \frac{g\alpha N}{4\pi z_W m_W} \left[ \left( \frac{11d-28}{6} z^2 + \frac{4d^2-11d+6}{4(d-1)} z + \frac{4d^3-16d^2+21d-10}{8(d-1)} \right) I_A + \right. \\
&\quad \left( z^3 - \frac{4d-5}{2(d-1)} z^2 - \frac{d^2+6d-8}{4(d^2-3d+2)} z + \frac{-2d^3+4d^2-3d+2}{4(d^2-3d+2)} \right) I_B + \\
&\quad \left. \frac{-4(d-2)z^3 + 2(d-2)z^2 - (d^2-3d+6)z - 2(d-1)}{d-2} I_C \right] \\
C_2 &= \frac{g\alpha N}{4\pi z_W m_W} \left[ \frac{(2z+1)(d-2)}{8(d-1)} I_A + \frac{4(2d-5)z-2(d-1)}{d-2} I_C + \right. \\
&\quad \left. \left( -\frac{d-2}{2(d-1)} z^2 + \frac{7d^2-30d+24}{4(d^2-3d+2)} z + \frac{2d^3-12d^2+17d-6}{4(d^2-3d+2)} \right) I_B \right]
\end{aligned}$$

The final result for the amplitude  $A_W$  is

$$A_W = A_1 + 2A_2 = \frac{g\alpha N}{4\pi z_W m_W} \left[ -\frac{(d-4)(d+2z-1)}{d-2} I_B + \frac{4(2z(5-2d) + (d-1))}{d-2} I_C \right]$$

which, after substituting the values of Master integrals to order  $O(\epsilon^0)$ , gives finally

$$A_W = \frac{\alpha}{2\pi v z_W} \left[ 3 + 2z_W + \left( -3 + \frac{3}{2z_W} \right) H_{0,0}(u_W) \right]$$

The contribution of three-propagator diagram is already symmetric with respect to the exchange  $p_1^\mu \leftrightarrow p_2^\mu$  (since the resulting scalar projections can depend only on the combination  $p_1 \cdot p_2$ ), so we just count it twice. We see that the individual diagrams are not finite nor gauge invariant, however, their sum is gauge invariant and the physical amplitudes ( $A$  and  $C$ ) are also finite. The amplitude  $B_W$  is UV-divergent, but gives no contribution for physical amplitude, so there is no need to worry.

Note also that the contribution of single diagram gives more complicated results than the total gauge-invariant physical contribution. The splitting of the invariant amplitude into two or more diagrams is not physical. If we use  $R_\xi$ -gauge instead of U-gauge, we would have more diagrams, including the diagrams with propagating unphysical Goldstone boson. These which have  $\xi$ -dependent masses and interactions and thus the individual diagrams depend on the parameter  $\xi$ . It is the total sum of all diagrams (with the same orders of coupling constants in which we expand the amplitude) that gives the physical, gauge-invariant result, which is  $\xi$ -independent and has in general a simpler form than the intermediate results.

The dependence of the  $W$  boson contribution to the invariant amplitude is shown in the Figure 2.7. As in the case of top loop, the amplitude is real below the  $2m_W$  threshold, and also has a nonzero limit for  $m_H \rightarrow 0$ . The contribution of  $W$  boson loop has opposite sign compared to the top quark contribution, so they interfere destructively (see Figure 2.8 and 2.9). Around  $m_H = 650 \text{ GeV}$  the sum of both amplitudes is almost zero. This is clearly seen in Figure 2.9. The corresponding decay width is shown in Figure 5.7.

## 2.6 One-loop diagram with counter-term insertion

Before closing this chapter, we will prepare for the two-loop calculations of QCD corrections to the quark loop, which we will calculate in the next chapters. Since QCD corrections apply only to quarks, starting with this moment we will calculate all integrals for quarks and count all 3 colours from now on. In this section we will evaluate the contribution of diagrams shown in Figure 2.10. Here the cross on propagator represents a mass-insertion, coming from the Lagrangian (as will be explained later)

$$\mathcal{L}_{\delta_m} = -\delta_m \bar{\psi} \psi$$

which has the Feynman rule simply

$$-i\delta_m$$

This means that for instance the diagram 2.10c can be obtained from the one-loop diagram 2.1 by performing the substitution

$$\frac{\not{k} + m}{k^2 - m^2 + i\epsilon} \rightarrow \delta_m \left( \frac{\not{k} + m}{k^2 - m^2 + i\epsilon} \right)^2$$

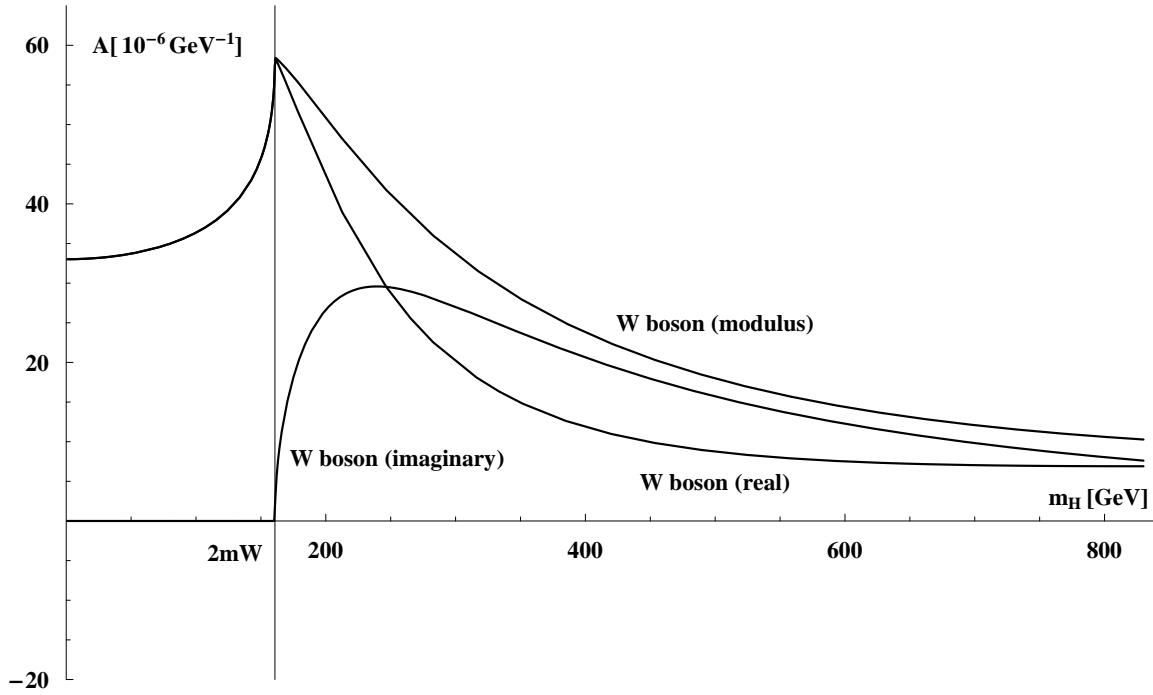


Figure 2.7: Real part, imaginary part and modulus of contribution of W boson loop to the invariant amplitude

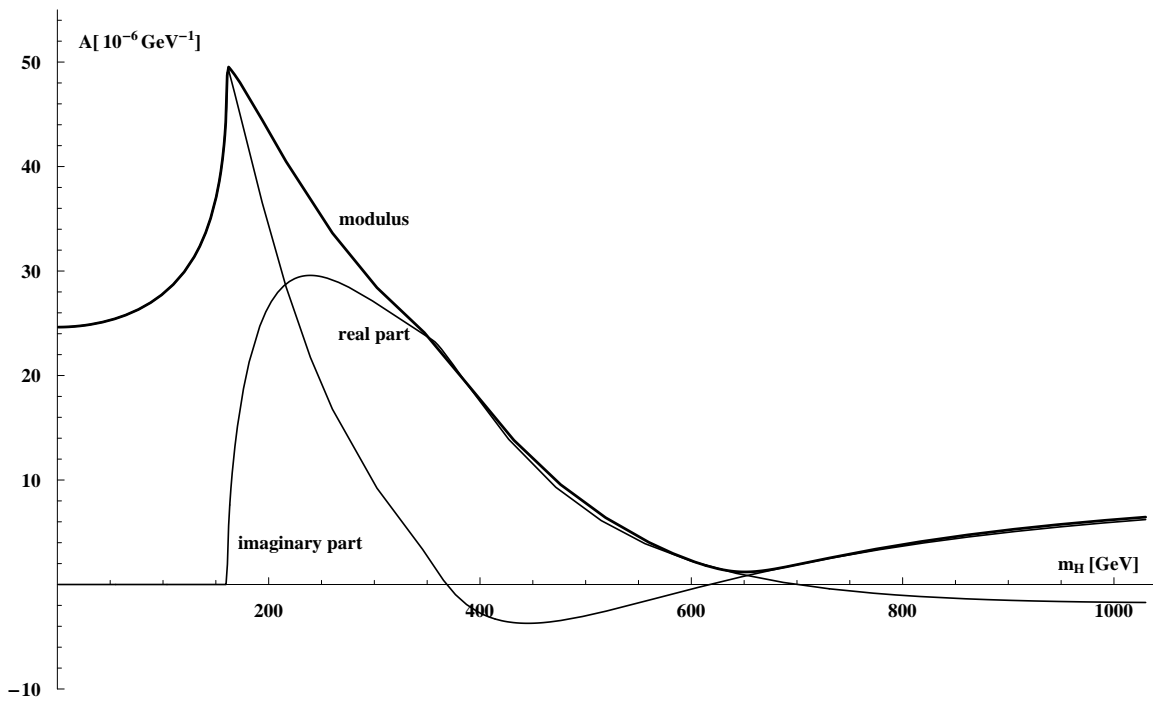


Figure 2.8: Real part, imaginary part and modulus of complete invariant amplitude evaluated to one loop order

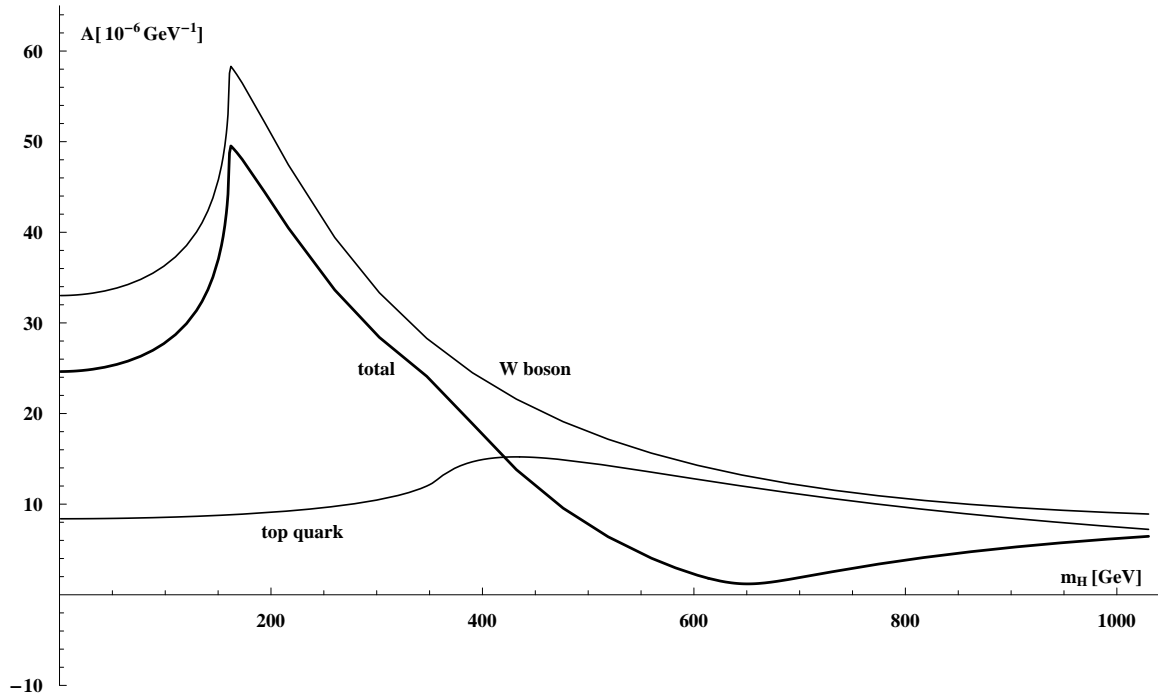


Figure 2.9: Modulus of top quark, W boson and total one loop contribution to invariant amplitude

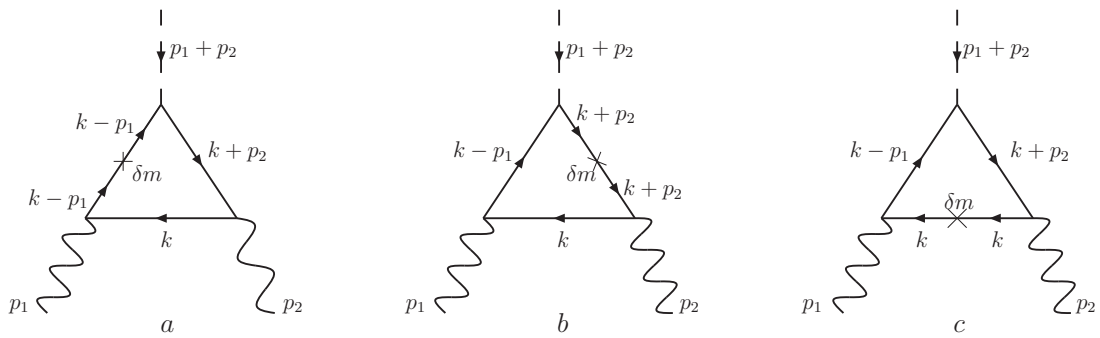


Figure 2.10: Fermion loop with mass counter-term insertion

and similarly for the other diagrams. We proceed as in the previous sections and obtain for the amplitude (as noted earlier, we sum over all 3 colours, which is represented by the colour factor  $N_C$ )

$$\begin{aligned}
A_{a,b} &= \frac{-\alpha Q^2 N N_C \delta_m}{\pi v z} \frac{\delta_m}{m} \left[ -\frac{(d-2)((d-3)z-2)}{4z} I_A - \frac{(d-4)z+2(d-2)}{2(d-2)z} I_B + \frac{2}{(d-2)} I_C \right] \\
B_{a,b} &= \frac{-\alpha Q^2 N N_C \delta_m}{\pi v z} \frac{\delta_m}{m} \left[ \frac{(d-2)(z-3)}{4z} I_A + \frac{-2z+3(d-2)}{2(d-2)z} I_B + \frac{2(d-3)}{(d-2)} I_C \right] \\
C_{a,b} &= \frac{-\alpha Q^2 N N_C \delta_m}{\pi v z} \frac{\delta_m}{m} \left[ \frac{(d-2)((d-3)z-3)}{4z} I_A + \frac{-2z+3(d-2)}{2(d-2)z} I_B - \frac{2}{d-2} I_C \right] \\
A_c &= \frac{-\alpha Q^2 N N_C \delta_m}{\pi v z} \frac{\delta_m}{m} \left[ \frac{(d-2)((d-2)z-2)}{2z} I_A - \frac{(d-3)z-2}{z} I_B - 2((d-3)z-2) I_C \right] \\
B_c &= \frac{-\alpha Q^2 N N_C \delta_m}{\pi v z} \frac{\delta_m}{m} \left[ \frac{(d-2)(2(d-3)z+3)}{2z} I_A - \frac{(d-2)z+3}{z} I_B - 2((d-3)z+4) I_C \right] \\
C_c &= \frac{-\alpha Q^2 N N_C \delta_m}{\pi v z} \frac{\delta_m}{m} \left[ -\frac{(d-2)((d-2)z-3)}{2z} I_A + \frac{(d-2)z-3}{z} I_B + 2((d-3)z-2) I_C \right]
\end{aligned}$$

The sum of all 3 diagrams should be gauge invariant, as we will see later, and indeed, the sum gives

$$A_{CTq} = -C_{CTq} = -\frac{\alpha Q^2 N N_C \delta_m}{\pi v z} \frac{\delta_m}{m} \left[ \frac{d-2}{2} I_A - \frac{d^2-4d+2}{d-2} I_B - 2 \frac{z(d^2-5d+6)-2(d-1)}{d-2} I_C \right]$$

which has an  $\epsilon$ -expansion to  $O(\epsilon)$

$$\begin{aligned}
A_{CTq} = & -\frac{\alpha Q^2 N N_C \delta_m}{\pi v z} \frac{\delta_m}{m} \left[ 1 - \frac{1+u}{1-u} H_0(u) - \frac{1+10u+u^2}{2(1-u)^2} H_{0,0}(u) + \epsilon \left( 3 + \frac{1+u}{1-u} \zeta(2) + \right. \right. \\
& \frac{3(1+10u+u^2)}{2(1-u)^2} \zeta(3) + \frac{1+u}{1-u} H_0(u) + \frac{1+10u+u^2}{2(1-u)^2} \zeta(2) H_0(u) + \frac{2u(u-2)}{(1-u)^2} H_{0,0}(u) + \\
& \left. \left. \frac{2(1+u)}{1-u} H_{-1,0}(u) - \frac{1+10u+u^2}{2(1-u)^2} H_{0,0,0}(u) + \frac{1+10u+u^2}{(1-u)^2} H_{0,-1,0}(u) \right) \right]
\end{aligned}$$

We can arrive at the same result for the sum of diagrams from the result of fermion loop by noticing that

$$\frac{d}{dm} \left( \frac{1}{\not{p} - m} \right) = \left( \frac{1}{\not{p} - m} \right)^2$$

which is valid for any matrix  $\not{p}$  independent of the parameter  $m$ . In this way we can relate the sum of one-loop integrals with  $\delta_m$  insertions to one-loop integral itself:

$$A_a + A_b + A_c = \delta_m m \frac{\partial}{\partial m} \left( \frac{1}{m} A \right)$$

(factors  $m$  are included because we do not want the derivative to act on quark mass  $m$  coming from the Higgs-quark coupling). It is necessary to apply the derivative also to  $N$  factor (whose  $m$  dependence is not written explicitly)! This relation of amplitudes makes it clear why the sum of three diagrams with  $\delta_m$  insertion was gauge invariant. This is no coincidence, since we know that the mass term in Lagrangian is gauge invariant and  $\delta_m$  interaction is in fact a mass term (one can check the consistency by calculating the propagator with an arbitrary number of  $\delta_m$  insertions and if one sums the resulting geometric series, one arrives at the propagator with mass shifted by  $\delta_m$ ).

## 2.7 Renormalization - introduction

Up to this moment we have not encountered any kind of divergence that would persist after performing the regularization procedure. But this is only a lucky coincidence, more precisely a consequence of the fact that we are performing a one-loop calculation of process in a renormalizable field theory that has no tree-level contribution.

Mass parameters and coupling constants that are in Lagrangian do not necessarily represent the properties of the physical one-particle states. It is so on the tree level, but the contributions of diagrams with loops imply non-trivial relations between parameters of Lagrangian and physical masses and charges. Although this seems quite natural, the problem is that when evaluating these loop correction, we encounter formally divergent integrals, even in the simple theories like  $\phi^4$  or QED in 4 dimensions.

There are more types of divergences that one must deal with when calculating the contributions of the Feynman diagrams, including the divergences of integrals coming from the the region of soft loop momenta (IR) or so-called UV divergences coming from regions of hard loop momenta. Different types of divergences have different causes, and UV divergences that we are interested in are connected with a small-distance behaviour of the theory. For more details see any QFT textbook ([9, 30, 46, 55, 56])

In order to manipulate with infinities, we must parametrize them in some way (this is called regularization). Although we have seen an example of Pauli-Villars regularization in Section 2.2.1, we will use the dimensional regularization in the rest of this text. If we had a pseudoscalar particle instead of the Higgs boson, we would have to be more careful when using the dimensional regularization, because the generalization of  $\epsilon_{\mu\nu\rho\sigma}$  and anticommuting  $\gamma_5$  to  $d$  dimensions can be problematic.

When calculating amplitudes according to the “bare perturbation theory”, the results expressed as functions of parameters of the bare Lagrangian (for example masses or the coupling constants) and the regularization parameter (cutoff or the dimension) are divergent in the limit of cutoff sent to infinity (or dimension sent to 4). However, if the theory is renormalizable, we obtain a finite results if we express the bare parameters as functions of some set of physical quantities and then express all the remaining physical quantities in terms of these few quantities. In practical calculations we can avoid this procedure by reorganizing the perturbation series according to the “renormalized perturbation theory”. We split the bare Lagrangian into two parts, one part containing the physical quantities representing the values of measurements as if calculated at tree level. The second part of Lagrangian (containing so-called counter-terms) is modified order by order of the perturbation theory to fix the physical quantities specified in the first part of Lagrangian. Conditions connecting the parameters of Lagrangian with the values of the measurements are called the renormalization conditions. These conditions give a meaning to the parameters appearing in the first part of the Lagrangian. Counter-terms give (order by order) connection between the physical parameters and the parameters of bare Lagrangian.

We start with the bare Lagrangian expressed in terms of the bare fields (i.e. fields that do not necessary have the residue of one-particle pole equal to 1). The part of the bare Lagrangian which is relevant for our calculations is

$$\begin{aligned} \mathcal{L} = & -\frac{1}{4}F_{B\mu\nu}F_B^{\mu\nu} - \frac{1}{4}G_{B\mu\nu}^a G_{Ba}^{\mu\nu} + \bar{\Psi}_B(i\not{\partial} - m_B)\Psi_B \\ & + \bar{\Psi}_B(Qe_B \not{A}_B + g_B \tau_a \not{B}_B^a)\Psi_B - \frac{m_B}{v} H_B \bar{\Psi}_B \Psi_B + \dots \end{aligned}$$

where  $\Psi$  represents a quark field,  $A$  is a vector potential of electromagnetic field,  $B$  a gluon field,  $H$  Higgs boson field and

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu \\ G_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu \end{aligned}$$

are “field strengths” corresponding to the photon and the gluon fields (not containing the gluon self-interactions);  $g_B$  is a (bare) QCD coupling constant and  $v$  is the vacuum expectation value of Higgs field as defined in Standard Model of electroweak interactions. As announced, we split the bare Lagrangian into two parts, first containing the “physical” fields and coupling constants (as will be defined by the renormalization conditions later) and the second containing a (local) counter-terms, which are chosen to keep the physical parameters fixed as specified by the renormalization conditions. As a first step, we rescale the bare fields so that the resulting fields (called the renormalized fields) have a wave function normalization (residue of the one-particle pole) equal to 1. This will compensate the field strength renormalization factors appearing in the LSZ formula, so we will no longer need to evaluate the  $Z$ -factors in order to calculate the S-matrix elements. We define

$$\begin{aligned} \Psi_B &= \sqrt{Z_q} \Psi_R \\ A_B &= \sqrt{Z_A} A_R \\ H_B &= \sqrt{Z_H} H_R \\ B_B &= \sqrt{Z_B} B_R \end{aligned}$$

$$\begin{aligned} eZ_e &= e_B Z_q \sqrt{Z_A} \\ gZ_g &= g_B Z_q \sqrt{Z_B} \end{aligned}$$

and express the relevant part of Lagrangian in the form

$$\begin{aligned} \mathcal{L} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}G_{\mu\nu}^a G_a^{\mu\nu} + \bar{\Psi}_R(i\cancel{\partial} - m)\Psi_R \\ &\quad + eQ\bar{\Psi}_R \cancel{A}_R \Psi_R + g\bar{\Psi}_R \tau_a \cancel{B}_R^a \Psi_R - \frac{m}{v}H_R\bar{\Psi}_R\Psi_R + \mathcal{L}_{CT} + \dots \\ \mathcal{L}_{CT} &= -\frac{1}{4}(Z_A - 1)F_{R\mu\nu}F_R^{\mu\nu} - \frac{1}{4}(Z_B - 1)G_{R\mu\nu}G_R^{\mu\nu} + (Z_q - 1)\bar{\Psi}_R(i\cancel{\partial})\Psi_R \\ &\quad - (Z_q m_B - m)\bar{\Psi}_R\Psi_R + (Z_e - 1)eQ\bar{\Psi}_R \cancel{A}_R \Psi_R \\ &\quad + (Z_g - 1)g\bar{\Psi}_R \vec{\tau} \cdot \vec{\cancel{B}}_R \Psi_R + \left(\frac{m}{v} - \sqrt{Z_H}Z_q \frac{m_B}{v_B}\right)H_R\bar{\Psi}_R\Psi_R \end{aligned}$$

After introducing a notation

$$\begin{aligned} \delta_m &= Z_q(m_B - m) \\ \delta_H &= Z_q \frac{m_B}{m} \left( \frac{v\sqrt{Z_H}}{v_B} - 1 \right) \\ \delta Z_A &= Z_A - 1 \\ \delta Z_B &= Z_B - 1 \\ \delta Z_q &= Z_q - 1 \\ \delta Z_e &= Z_e - 1 \\ \delta Z_g &= Z_g - 1 \end{aligned}$$

and suppressing index R for renormalized fields (renormalized fields will be used from now on), the expression for counter-term Lagrangian can be recast to the form

$$\begin{aligned} \mathcal{L}_{CT} &= -\frac{1}{4}\delta Z_A F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}\delta Z_B G_{\mu\nu}^a G_a^{\mu\nu} + \delta Z_q \bar{\Psi}(i\cancel{\partial} - m)\Psi - \delta_m \bar{\Psi}\Psi \\ &\quad + \delta Z_e eQ\bar{\Psi} \cancel{A}\Psi + \delta Z_g \bar{\Psi} \vec{\tau} \cdot \vec{\cancel{B}}\Psi - \delta Z_q \frac{m}{v}H\bar{\Psi}\Psi - \frac{\delta_m}{m} \frac{m}{v}H\bar{\Psi}\Psi - \delta_H \frac{m}{v}H\bar{\Psi}\Psi \end{aligned} \quad (2.14)$$

Of these counter-terms, only  $\delta_m$ ,  $Z_q$  and  $Z_e$  will be needed for evaluating contribution of QCD corrections to the quark loop. Other counter-terms either do not receive a correction of desired order of couplings ( $\delta_H$ ,  $\delta Z_A$ ) or cannot be inserted into the one-loop integral ( $\delta Z_B$  and  $\delta Z_g$ ) such that the whole contribution is still order that we are looking for. The question of renormalization of  $v$  is more subtle. According to [18], Higgs boson carries no QCD colour, so the combination  $v+H$  in Lagrangian, which gives rise both to Higgs-quark interaction and quark mass, must be renormalized as a whole. Thus counter-term for Higgs-quark Yukawa coupling is fixed by the quark field strength normalization and quark self-energy counter-terms. As a consequence, Higgs-quark vertex function has nonzero value for  $q^2 = 0$  (zero transferred momentum) unlike for example the electromagnetic vertex, where one loop renormalization amounts to a subtraction of the value at zero transferred momentum.

Now we will impose the renormalization conditions that will fix all of the counter-terms that we have introduced. The divergent parts of counter-terms are fixed by the condition of finiteness of results, while the finite parts depend on our choice of the "renormalization scheme". The renormalization conditions are used to remove the freedom in choice of the splitting of finite parts between two parts of the Lagrangian. We will work in the "on-mass-shell" renormalization scheme, which corresponds to the following renormalization conditions (see Figure 2.11 for definitions; the definition of  $\Gamma$  is not entirely correct (see [46]), it should be the amputated three-point Green function with possibly off-shell fermions - the definition shown on Figure 2.11 is for simplicity and the difference is not important for our purposes since the renormalization conditions are for on-shell leptons where both definitions match):

$$\begin{aligned} \Sigma(\not{p} = m) &= 0 \\ \left. \frac{d}{d\not{p}}\Sigma(\not{p}) \right|_{\not{p}=m} &= 0 \end{aligned}$$

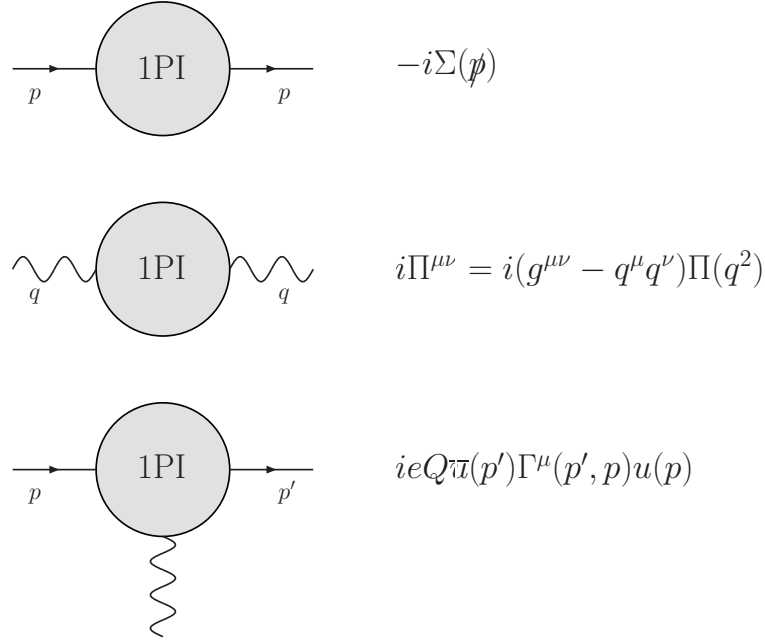


Figure 2.11: Definition of form-factors which are used in renormalization conditions

$$\begin{aligned}\Pi(q^2 = 0) &= 0 \\ \bar{u}(p)\Gamma(p' = p)^\mu u(p) &= \bar{u}(p)\gamma^\mu u(p)\end{aligned}$$

The first condition keeps the quark mass  $m$  in the first part of Lagrangian fixed and equal to the “physical quark mass” (if there were free quarks), the second condition keeps the quark wave function normalization constant equal to 1. The third condition keeps residue of the photon propagator equal to 1 (we do not need to apply this condition since, as we said,  $Z_A$  does not receive any corrections relevant to our process). The last condition fixes the quark-photon coupling constant  $e$ , by specifying the value of  $Z_e$  order by order. For more details see for instance [46].

## 2.8 Evaluation of counter-terms

In this section we will evaluate the basic one-loop diagrams which will occur as subdiagrams of QCD corrections to the quark loop. They represent the lowest order QCD correction to the quark propagator and to interaction of quarks with electromagnetic field and with Higgs boson. Calculations like in this section are described in more detail in every textbook dealing with QED [9, 30, 46, 55].

### 2.8.1 Quark self-energy

To find out the value of  $\delta_m$  we need to calculate the 1PI quark two-point function (quark propagator, self-energy) to order  $O(g^2)$ . We will work in the Feynman-'t Hooft gauge (see Section A.1,  $\xi = 1$ ) and regularize the infrared divergences by giving the photon a (small) mass  $\mu$ . According to the Feynman rules, the contribution of the diagram shown in Figure 2.12 reads

$$-i\Sigma_{2g}(p)\delta_{ij} = -g^2 \sum_{a,r} T_{jr}^a T_{ri}^a \int \frac{d^d k}{(2\pi)^d} \frac{\gamma_\mu(\not{p} - \not{k} + m)\gamma^\mu}{[(p-k)^2 - m^2 + i\epsilon][k^2 - \mu^2 + i\epsilon]}$$

where  $a$ ,  $i$  and  $j$  are the colour indices and the summation takes place over the gluon colour index  $a$  and the intermediate quark colour index  $n$ . We write  $\delta_{ij}$  on the left-hand side because the QCD corrections to quark propagator are diagonal in the colour indices. Using the Feynman parametrization, gamma matrix formula

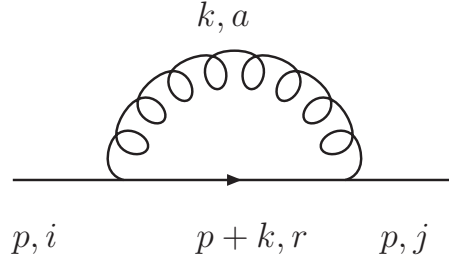


Figure 2.12: One loop fermion self-energy diagram

(A.3) and after the integration over loop momenta, we obtain

$$\begin{aligned} -i\Sigma_{2g}(p) &= -g^2 C_F \int \frac{d^d l}{(2\pi)^d} \int_0^1 dx \frac{dm + (2-d)(1-x)\not{p} - (2-d)l}{[l^2 - x^2 p^2 + x p^2 - x m^2 - y \mu^2 + i\epsilon]^2} \\ &= -ig^2 C_F \frac{1}{(4\pi)^{\frac{d}{2}}} \Gamma\left(2 - \frac{d}{2}\right) \int_0^1 dx \frac{dm + (2-d)(1-x)\not{p}}{(x m^2 - x(1-x)p^2 + (1-x)\mu^2)^{2-\frac{d}{2}}} \end{aligned}$$

with  $C_F = C_2(S) = \frac{4}{3}$ . Contribution of counter-terms according to (2.14) is

$$-i\Sigma_{2CT}(p) = i\delta Z_q(\not{p} - m) - i\delta m,$$

so the complete inverse propagator (amputated two-point Green function) to order  $O(g^2)$  is

$$-i(\not{p} - m - \Sigma_{2g}(p) - \Sigma_{2CT}(p) + \dots)$$

The first renormalization condition fixes the value of  $\delta m$  (note that for  $\epsilon < 1$  and  $\not{p} = m$  the integral is IR convergent and we can set  $\mu^2 = 0$ ):

$$0 = -i\delta m - \frac{i\alpha_S m C_F}{4\pi} \Gamma(\epsilon) \left(\frac{4\pi}{m^2}\right)^\epsilon \int_0^1 dx \frac{2 + (2-2\epsilon)x}{x^{2\epsilon}} = -i\delta m - \frac{i\alpha_S m N C_F}{4\pi\epsilon} \left(\frac{3-2\epsilon}{1-2\epsilon}\right)$$

This gives

$$\frac{\delta m}{m} = -\frac{\alpha_S N C_F}{4\pi\epsilon} \left(\frac{3-2\epsilon}{1-2\epsilon}\right) = -\frac{3\alpha_S N C_F}{4\pi} \left(\frac{1}{\epsilon} + \frac{4}{3}\right) + O(\epsilon)$$

in accordance with [14] (opposite sign is result of opposite sign in definition of  $\delta m$ ). On the other hand, the wave function renormalization  $\delta Z_q$  is IR-divergent and is equal to

$$\delta Z_q = \frac{\alpha_S N C_F}{2\pi\epsilon} \int_0^1 dx (1-x) \left[ \frac{-1+\epsilon}{\left(x^2 + (1-x)\frac{\mu^2}{m^2}\right)^\epsilon} + 2\epsilon x \frac{1+x-\epsilon x}{\left(x^2 + (1-x)\frac{\mu^2}{m^2}\right)^{\epsilon+1}} \right]$$

We do not need better-looking expression for this counter-term, for as we will see, this counter-term cancels against a similar contribution of the counter-term  $Z_e$ .

## 2.8.2 Quark-photon vertex

Now we are going to calculate the one-loop QCD correction to the electromagnetic vertex function to show that electromagnetic vertex renormalization constant and quark field strength renormalization constant have the same value. This follows from Ward identities, but we will check it explicitly. Corresponding Feynman diagram is shown in Figure 2.13 ( $i, j$  and  $a$  are the colour indices). According to the Feynman rules given in Appendix A.1, the contribution to the amputated three-point vertex Green function reads

$$ieQ\Gamma_{2g}^\mu(p, p')\delta_{ji} = ieQ(-ig^2) \left(\sum_{r,b} T_{jr}^b T_{ri}^b\right) \int \frac{d^d k}{(2\pi)^d} \frac{\gamma_\nu(\not{p}' + \not{k} + m)\gamma^\mu(\not{p} + \not{k} + m)\gamma^\nu}{[k^2 - \mu^2][(p' + k)^2 - m^2][(p + k)^2 - m^2]}.$$

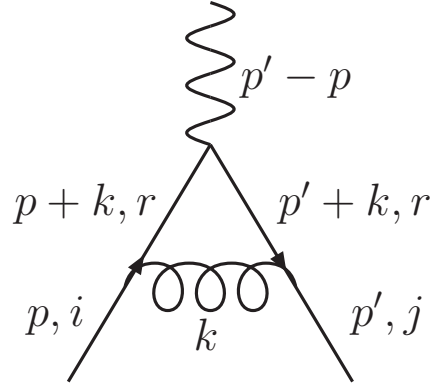


Figure 2.13: One-loop QCD correction to quark-photon vertex

Again, IR divergences are regularized by the introduction of (a fictive) photon mass  $\mu$  and  $i\epsilon$  terms are not written explicitly. We are still working in the Feynman-'t Hooft gauge. Factoring out the colour matrix  $\delta_{ji}$  on the left-hand side will be justified after calculating the term with colour indices in the right-hand side. As usual, we perform Feynman parametrization and integration over loop momentum. Our choice of Feynman parameters is

$$\frac{1}{[k^2 - \mu^2][(p' + k)^2 - m^2][(p + k)^2 - m^2]} = \int_0^1 dx \int_0^{1-x} dy \frac{2}{[l^2 - \Delta]^3}$$

where

$$\begin{aligned} l^\nu &= k^\nu + xp^\nu + yp'^\nu \\ k^\nu &= l^\nu - xp^\nu - yp'^\nu \\ \Delta &= (x + y)m^2 + (1 - x - y)\mu^2 + x(x - 1)p^2 + y(y - 1)p'^2 + 2xy p \cdot p' \end{aligned}$$

Expressing the numerator of the integrand using  $l$  instead of  $k$ , performing a contraction with help of identities (A.2) and dropping odd terms in  $l$  gives

$$\begin{aligned} \gamma_\nu(\not{p}' + \not{k} + m)\gamma_\mu(\not{p} + \not{k} + m)\gamma^\nu &\rightarrow (2 - d)l\gamma_\mu l + \not{p}'\gamma_\mu\not{p}(6 - d)(1 - x - y) + (d - 4)m(\gamma_\mu\not{p} + \not{p}'\gamma_\mu) \\ &+ p_\mu [4(1 - 2x)m + 2(4 - d)xm + 2(d - 2)x(1 - x)\not{p} - 4(1 - x)(1 - y)\not{p}' + 2(4 - d)xy\not{p}'] \\ &+ p'_\mu [4(1 - 2y)m + 2(4 - d)ym + 2(d - 2)y(1 - y)\not{p}' - 4(1 - x)(1 - y)\not{p} + 2(4 - d)xy\not{p}] \\ &+ \gamma_\mu [(2 - d)x(1 - x)p^2 + (2 - d)y(1 - y)p'^2 + (2 - d)m^2 + 4(1 - x)(1 - y)pp' + 2(d - 4)xypp'] \end{aligned}$$

Note that we have put  $\not{p}'$  to the left and  $\not{p}$  to the right for future convenience. The integration over loop momenta gives

$$\begin{aligned} 2 \int \frac{d^d l}{(2\pi)^d} \frac{1}{(l^2 - \Delta)^3} &= \frac{-iN}{(4\pi)^2 m^2 \tilde{\Delta}^{1+\epsilon}} \\ 2 \int \frac{d^d l}{(2\pi)^d} \frac{\frac{2-d}{d} l^2}{(l^2 - \Delta)^3} &= \frac{-iN}{(4\pi)^2 \tilde{\Delta}^\epsilon} \left( \frac{1}{\epsilon} - 1 \right) \end{aligned}$$

(valid for any  $\epsilon$ ) with the notation  $\Delta = m^2 \tilde{\Delta}$ . According to the renormalization conditions, we will need the expression for  $\Gamma$  only for the on-shell quarks and sandwiched between quark bispinors. This means that we can write effectively

$$\begin{aligned} \not{p}u(p) &= mu(p) \\ \bar{u}(p')\not{p}' &= \bar{u}(p')m \\ p^2 &= m^2 \\ p'^2 &= m^2 \end{aligned}$$

everywhere (we will denote this by  $\simeq$ ). We have

$$\begin{aligned} \Gamma_{2g}^\mu(p, p') &\simeq \frac{2\alpha_S C_F N}{4\pi\epsilon} (1-\epsilon)^2 \gamma_\mu \int_0^1 dx \int_0^{1-x} dy \tilde{\Delta}^{-\epsilon} \\ &\quad - \frac{2\alpha_S C_F N}{4\pi} \int_0^1 dx \int_0^{1-x} dy \frac{1}{\tilde{\Delta}^{1+\epsilon}} \left[ \frac{p^\mu + p'^\mu}{m} ((x+y)(1-x-y) + \epsilon(x+y)^2) \right. \\ &\quad \left. \gamma_\mu \left( (x+y)^2 - 4(x+y) + 2 - \epsilon(x+y)^2 - \frac{q^2}{m^2} (1-x)(1-y) + \epsilon xy \frac{q^2}{m^2} \right) \right] \end{aligned}$$

with

$$\begin{aligned} \Delta &= m^2 \tilde{\Delta} = (x+y)^2 m^2 - xyq^2 + (1-x-y)\mu^2 \\ q_\mu &= p'_\mu - p_\mu \end{aligned}$$

The term proportional to  $q^\mu$  was dropped because it is multiplied by the expression that is odd with respect to the exchange  $x \leftrightarrow y$  and vanishes after the integration. The resulting expression is usually written in another form. We use the Gordon identity (A.2) to trade  $p^\mu + p'^\mu$  for  $\Sigma^{\mu\nu}$  and obtain

$$\begin{aligned} \Gamma_{2g}^\mu(p, p') &\simeq \frac{2\alpha_S C_F N}{4\pi} \int_0^1 dx \int_0^{1-x} dy \tilde{\Delta}^{-1-\epsilon} \left[ i\Sigma^{\mu\nu} \frac{q_\nu}{m} ((x+y)(1-x-y) + \epsilon(x+y)^2) \right. \\ &\quad \left. + \gamma^\mu \left( (x+y)^2 + 2(x+y) - 2 - \epsilon(x+y)^2 + \frac{q^2}{m^2} (1-x)(1-y) - \epsilon xy \frac{q^2}{m^2} + \frac{(1-\epsilon)^2 \tilde{\Delta}}{\epsilon} \right) \right] \end{aligned}$$

To apply the renormalization conditions, we need this integral for  $p = p'$  ( $q = 0$ ). For  $q = 0$  the integral depends only on the combination  $x + y$  so it is reasonable to return to the variable  $z = 1 - x - y$  in original. This gives

$$\Gamma_{2g}^\mu(p, p) \simeq \frac{2\alpha_S C_F N}{4\pi} \int_0^1 dz (1-z) \left[ \gamma^\mu \frac{z^2 - 4z + 1 - \epsilon(1-z)^2}{\left( (1-z)^2 + z \frac{\mu^2}{m^2} \right)^{1+\epsilon}} + \gamma^\mu \left( (1-z)^2 + z \frac{\mu^2}{m^2} \right)^{-\epsilon} \frac{(1-\epsilon)^2}{\epsilon} \right]$$

Therefore, in order to satisfy the renormalization conditions, we must set

$$\delta Z_e = -\frac{2\alpha_S C_F N}{4\pi\epsilon} \int_0^1 dz (1-z) \left[ \epsilon \frac{z^2 - 4z + 1 - \epsilon(1-z)^2}{\left( (1-z)^2 + z \frac{\mu^2}{m^2} \right)^{1+\epsilon}} + \left( (1-z)^2 + z \frac{\mu^2}{m^2} \right)^{-\epsilon} (1-\epsilon)^2 \right]$$

in correspondence with [46]. Now we can easily check that  $\delta Z_q = \delta Z_e$ . After an obvious change of variables in the second integral, we have

$$\begin{aligned} \delta Z_q &\simeq \int_0^1 dx (1-x) [(-1+\epsilon)D^{-\epsilon} + 2\epsilon x(1+x-\epsilon x)D^{-1-\epsilon}] \\ \delta Z_e &\simeq -\int_0^1 dx x [(1-\epsilon)^2 D^{-\epsilon} + \epsilon(-2+2x+x^2-\epsilon x^2)D^{-1-\epsilon}] \end{aligned}$$

with

$$D = x^2 - (1-x) \frac{\mu^2}{m^2}$$

and  $\simeq$  sign means that we omitted the same proportionality factors in both equations. Subtracting both equations gives

$$\begin{aligned} \delta Z_q - \delta Z_e &\simeq (1-\epsilon) \int_0^1 dx [(-1+2x-\epsilon x)D^{-\epsilon} + \epsilon D^{-1-\epsilon}(2x^2-x^3)] \\ &= (1-\epsilon) \int_0^1 dx \frac{d}{dx} [x(1-x)D^{-\epsilon}] = 0 \end{aligned}$$

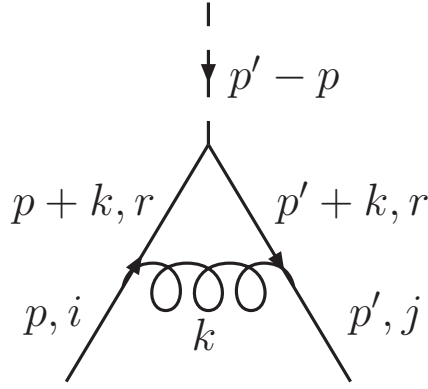


Figure 2.14: One-loop QCD correction to Higgs-quark vertex

since

$$(1-x) \frac{d}{dx} D^{-\epsilon} = \epsilon D^{-\epsilon} + \epsilon D^{-1-\epsilon} x(x-2)$$

Note that the relation between these two counter-terms (and the validity of Ward identity to this order of  $g^2$ ) can be expected directly from the Feynman integral, before introducing the Feynman parametrization which makes expressions somewhat obscure, since by shifting the loop momentum we have formally

$$\Gamma_{2g}^\mu(p, p) = -ig^2 C_F \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 - \mu^2 + i\epsilon} \gamma^\nu \frac{1}{\not{p} + \not{k} - m + i\epsilon} \gamma^\mu \frac{1}{\not{p} + \not{k} - m + i\epsilon} \gamma^\nu$$

while

$$\frac{d}{dp_\mu} \Sigma_{2g}(p) = +ig^2 C_F \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 - \mu^2 + i\epsilon} \gamma^\nu \frac{1}{\not{p} + \not{k} - m + i\epsilon} \gamma^\mu \frac{1}{\not{p} + \not{k} - m + i\epsilon} \gamma^\nu$$

### 2.8.3 Higgs-quark vertex

Last one-loop integral that we will evaluate in this section is the QCD correction to the quark-Higgs vertex. Calculation is very similar to the quark-photon vertex. We will not need it for two-loop calculations in the next chapter, but it is useful to check explicitly that the renormalization of  $v$  is not necessary and that the amplitude is made finite to this order by considering only  $\delta_m$  and  $\delta Z_q$ .

The contribution of diagram (2.14) is

$$\left(-i \frac{m}{v}\right) \delta_{ji} V_{H2g} = \left(-i \frac{m}{v}\right) (-ig^2) \sum_{a,r} T_{jr}^a T_{ri}^a \int \frac{d^d k}{(2\pi)^d} \frac{\gamma_\nu (\not{p}' + \not{k} + m) (\not{p} + \not{k} + m) \gamma^\nu}{[k^2 - \mu^2 + i\epsilon] [(p' + k)^2 - m^2 + i\epsilon] [(p + k)^2 - m^2 + i\epsilon]}$$

The denominator is exactly the same as in our previous calculation, so we use the same expressions for  $\Delta$  and the shifted loop momentum  $l$ . After dropping the terms linear in  $l$  we obtain

$$\begin{aligned} \gamma_\nu (\not{p}' + \not{k} + m) (\not{p} + \not{k} + m) \gamma^\nu \rightarrow \\ dl^2 + dm^2 - dx(1-x)p^2 - dy(1-y)p'^2 + 2pp'(2-2x-2y+dx) \\ + m(d-2)(2x-1)\not{p} + m(d-2)(2y-1)\not{p}' + \not{p}'\not{p}(1-x-y)(d-4) \end{aligned}$$

As before, we restrict to the case of the on-mass-shell quarks with  $V_{2g}$  sandwiched between the quark bispinors. This gives

$$V_{H2g} \simeq \frac{2\alpha_S C_F N}{4\pi\epsilon} \int_0^1 dx \int_0^{1-x} dy \tilde{\Delta}^{-1-\epsilon} \left[ (2-\epsilon)^2 \tilde{\Delta} + \right.$$

$$\epsilon \left( 2z(1-z) - 2 + \epsilon(1-z)^2 + \frac{q^2}{m^2}(1-x-y+2xy-\epsilon xy) \right) \Big]$$

with  $z = 1 - x - y$ . It only remains to check that the counter-terms  $\delta_m$  and  $\delta Z_q$  will indeed cancel the divergent part of this integral. Their contribution to the vertex factor  $V_H$  is according to (2.14) simply

$$V_{HCT} \simeq \frac{\delta_m}{m} + \delta Z_q$$

Now the divergent parts of these expressions are easily seen to be

$$\begin{aligned} \frac{\delta_m}{m} &\rightarrow -\frac{3\alpha_S C_F}{4\pi\epsilon} + \text{reg.} \\ Z_q &\rightarrow -\frac{\alpha_S C_F}{4\pi\epsilon} + \text{reg.} \\ V_{H2g} &\rightarrow +\frac{4\alpha_S C_F}{4\pi\epsilon} + \text{reg.} \end{aligned}$$

which illustrates again the subtle mechanism of compensation of divergences in Standard Model.

## 2.9 Counter-term contribution to two-loop QCD corrections

In the last section of this chapter, we will make use of the results of the previous sections and give the total contribution of counter-term diagrams to the QCD corrections to  $H \rightarrow \gamma\gamma$ . We include it in this chapter because all necessary integrals one-loop integrals and have been calculated with tools of this chapter.

Figure 2.15 shows all counter-term diagrams relevant for the calculation of the leading order QCD corrections to the quark loop (every diagram in figure represents in fact 2 diagrams, differing in the direction of the fermion number flow. Note that only 3 counter-terms contribute to  $O(g^2)$  corrections to our process, namely  $\delta_m$ ,  $\delta Z_e$  and  $\delta Z_q$ .  $\delta_H$  counter-term does not contribute, because neither vacuum expectation value of Higgs field  $v$  nor Higgs field wave function are renormalized to order  $O(g^2)$ . We have no photon propagator in diagram, so the renormalization of the photon wave function has no effect to our order. Finally the renormalization of the quark-gluon vertex and the gluon wave function are of order  $O(g^2)$  so the diagrams in Figure 2.15 are all that we need to include in our calculation. Note that there is a correspondence between the counter-term diagrams and two-loop diagrams with internal loop contracted to the point.

Diagrams in Figure 2.15 split naturally into two groups, namely the diagrams containing the field strength renormalization and the electric charge renormalization constants ( $\delta Z_q$  and  $\delta Z_e$ ), and the diagrams containing the mass counter-term  $\delta_m$ . We will show that the sum of diagrams in the first group gives zero. The crucial observation is that the Ward identities (following from the electromagnetic gauge invariance) imply the relation

$$Z_e = Z_q$$

to all orders [30, 46, 55]. For completeness we have checked this to  $O(g^2)$  explicitly in the previous section. According to the Feynman rules following from the counter-term Lagrangian (2.14), the contribution of the first set of diagrams is:

$$\begin{aligned} M_{CT,ghi}^{\mu\nu} &= -3\delta Z_q M_f^{\mu\nu} \\ M_{CT,m}^{\mu\nu} &= \delta Z_q M_f^{\mu\nu} \\ M_{CT,no}^{\mu\nu} &= 2\delta Z_e M_f^{\mu\nu}, \end{aligned}$$

where  $M_f$  is the one-loop amplitude. Note that this is where the ‘‘wise’’ choice of  $\delta_m$  multiplying  $\bar{\Psi}\Psi$  and  $Z_q$  multiplying  $\bar{\Psi}(i\not{\partial} - m)\Psi$  and not only  $\bar{\Psi}i\not{\partial}\Psi$  pays off (in contrast with [46]). We see also that this cancellation takes place everytime we have one fermion loop with a similar interactions as in our case (i.e. interaction carrying one power of  $Z_e$  or  $Z_q$ , for instance the interaction with electromagnetic field, gluon field, Higgs boson), independently of the number of interactions.

The contribution of diagrams containing  $\delta_m$  is non-zero and is needed to compensate the divergences of two-loop integrals. The simplest diagram is the diagram 2.15p - its contribution is related to the one-loop integral by only rescaling the size of coupling of the quark-Higgs vertex:

$$M_{CT,p}^{\mu\nu} = \frac{\delta_m}{m} M_f^{\mu\nu}$$

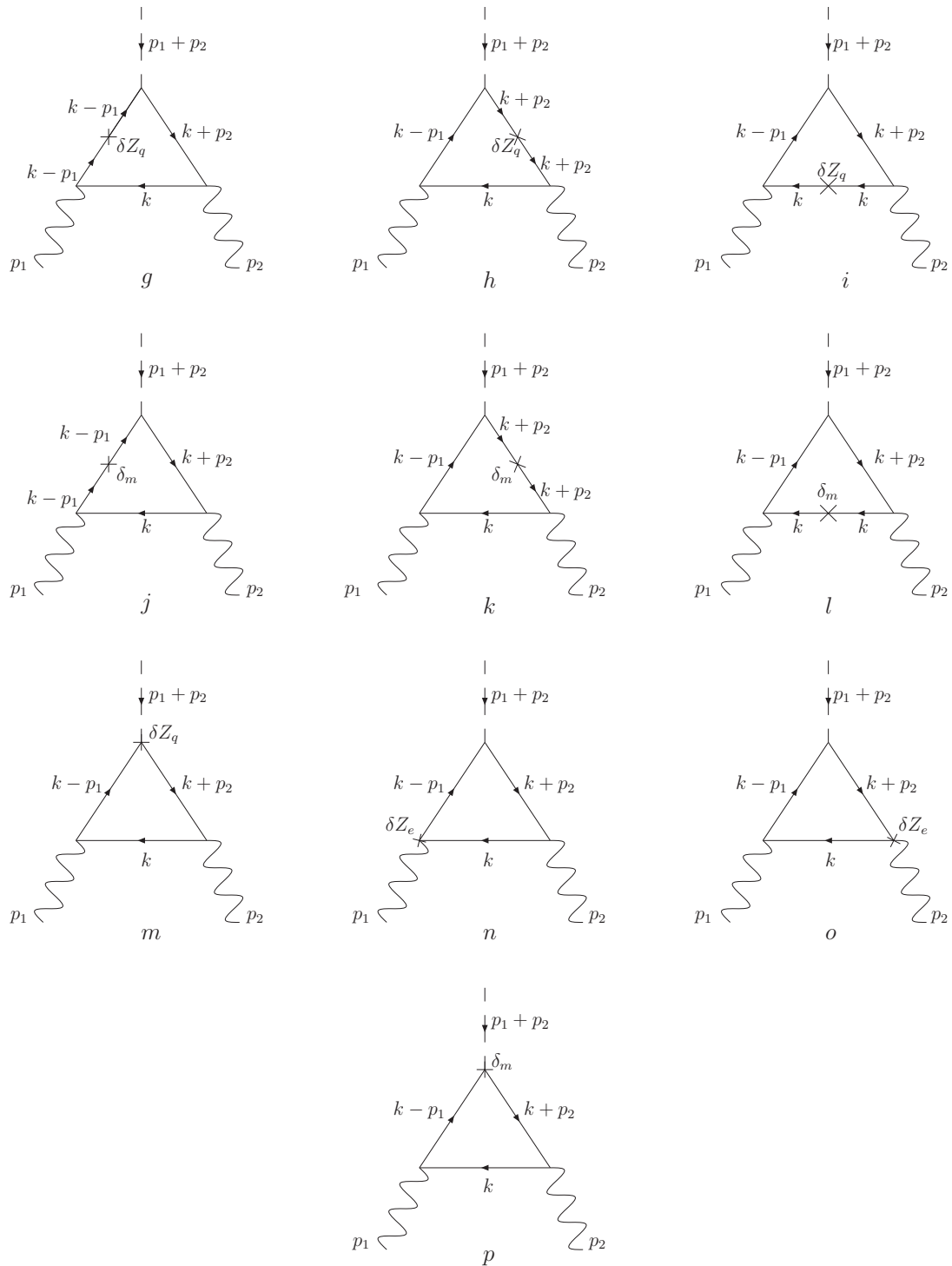


Figure 2.15: Counter-term diagrams contributing to QCD corrections

The contribution of the remaining diagrams (2.15j-1) was evaluated in the Section 2.6. The total contribution of all counter-term diagrams can be written compactly as

$$M_{CT}^{\mu\nu} = M_{CT,j}^{\mu\nu} + M_{CT,k}^{\mu\nu} + M_{CT,l}^{\mu\nu} + M_{CT,p}^{\mu\nu} = \delta_m m \frac{\partial}{\partial m} \left( \frac{1}{m} M_f^{\mu\nu} \right) + \frac{\delta_m}{m} M_f^{\mu\nu} = \frac{\delta_m}{m} \left( m \frac{\partial}{\partial m} \right) M_f^{\mu\nu} \quad (2.15)$$

or in other words, we need only to apply the Euler's operator to the one-loop result. Since as we have seen, the counter-term  $\delta_m$  has  $\frac{1}{\epsilon}$  pole, we need the one-loop result evaluated to order  $O(\epsilon)$ . As a result, we obtain

$$A_{CT} = \frac{\alpha Q^2 N N_C}{\pi v z} \frac{\delta_m}{m} \left[ -2 + \frac{1+u}{1-u} H_0(u) + \frac{1+6u+u^2}{(1-u)^2} H_{0,0}(u) + \epsilon \left( -6 - \frac{1+u}{1-u} \zeta(2) - \frac{3(1+6u+u^2)}{(1-u)^2} \zeta(3) - \frac{2(1+u)}{1-u} H_0(u) - \frac{1+6u+u^2}{(1-u)^2} \zeta(2) H_0(u) - \frac{2u(u-3)}{(1-u)^2} H_{0,0}(u) - \frac{2(1+u)}{1-u} H_{-1,0}(u) + \frac{1+6u+u^2}{(1-u)^2} H_{0,0,0}(u) - \frac{2(1+6u+u^2)}{(1-u)^2} H_{0,-1,0}(u) \right) \right]$$

Note that this expression is gauge invariant. Finally, substituting the result for  $\delta_m$  gives the final result of last 4 sections:

$$A_{CT} = \frac{\alpha \alpha_S Q^2 N^2 N_C C_F}{\pi^2 v z} \left[ \frac{1}{\epsilon} \left( \frac{3}{2} - \frac{3(1+u)}{4(1-u)} H_0(u) - \frac{3(1+6u+u^2)}{4(1-u)^2} H_{0,0}(u) \right) + \frac{13}{2} + \frac{3(1+u)}{4(1-u)} \zeta(2) + \frac{9(1+6u+u^2)}{4(1-u)^2} \zeta(3) + \frac{1+u}{2(1-u)} H_0(u) + \frac{3(1+6u+u^2)}{4(1-u)^2} \zeta(2) H_0(u) + \frac{-2-21u+u^2}{2(1-u)^2} H_{0,0}(u) + \frac{3(1+u)}{2(1-u)} H_{-1,0}(u) - \frac{3(1+6u+u^2)}{4(1-u)^2} H_{0,0,0}(u) + \frac{3(1+6u+u^2)}{2(1-u)^2} H_{0,-1,0}(u) \right] \quad (2.16)$$



## Chapter 3

# Two-loop QCD corrections, small momentum expansion

In this section we will use the method of expansion of invariant amplitude in the small Higgs boson mass (large quark mass) to evaluate the leading order QCD corrections to the quark loop. This will apply mainly to the top quark loop, for which the parameter  $z$  is up to about  $\frac{1}{4}$  in the intermediate mass range. Evaluation of the QCD corrections to order  $O(z^0)$  has been done in [59], while the subleading  $O(z)$  contribution was considered in [14]. We will follow this article in this chapter and we will also describe how to proceed to higher orders.

In order to check if the expansion around  $z = 0$  can give a useful approximation to the exact result, we can look at the one-loop amplitude for which we have know the exact result and compare it with an expansion in  $z$  around  $z = 0$  to see, how fast the series converges. The expanded invariant amplitude is

$$A_f = -\frac{\alpha Q^2 N_C}{\pi v} \left[ \frac{2}{3} + \frac{7}{45}z + \frac{4}{63}z^2 + \frac{52}{1575}z^3 + \frac{1024}{51975}z^4 + \frac{2432}{189189}z^5 \right] + O(z^6)$$

Since, as we know, there is a singularity at  $z = 1$  (a branching point), the expansion will not be reliable near  $z = 1$ . The plot of exact invariant amplitude and its  $O(z)$ ,  $O(z^3)$  and  $O(z^5)$  expansion is given in Figure 3.1 (with arbitrary units on the  $y$ -axis). Notice that all coefficient of the Taylor expansion of the one-loop invariant amplitude have the same sign, so the sequence of partial sums is monotonic.

Our case is simple enough in that there are no “cuts” of two-loop diagrams, which would cut across the massless propagators. From the analytic properties of the Feynman diagrams (for details see again [9, 30, 43, 46]) it then follows, that there is no branching going at  $z = 0$  and that the asymptotic expansion of the invariant amplitude at this point does not differ from its ordinary Taylor expansion. For more information about the small momentum expansions, see for instance [20].

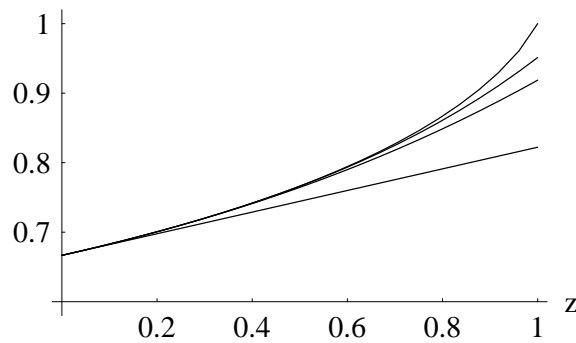
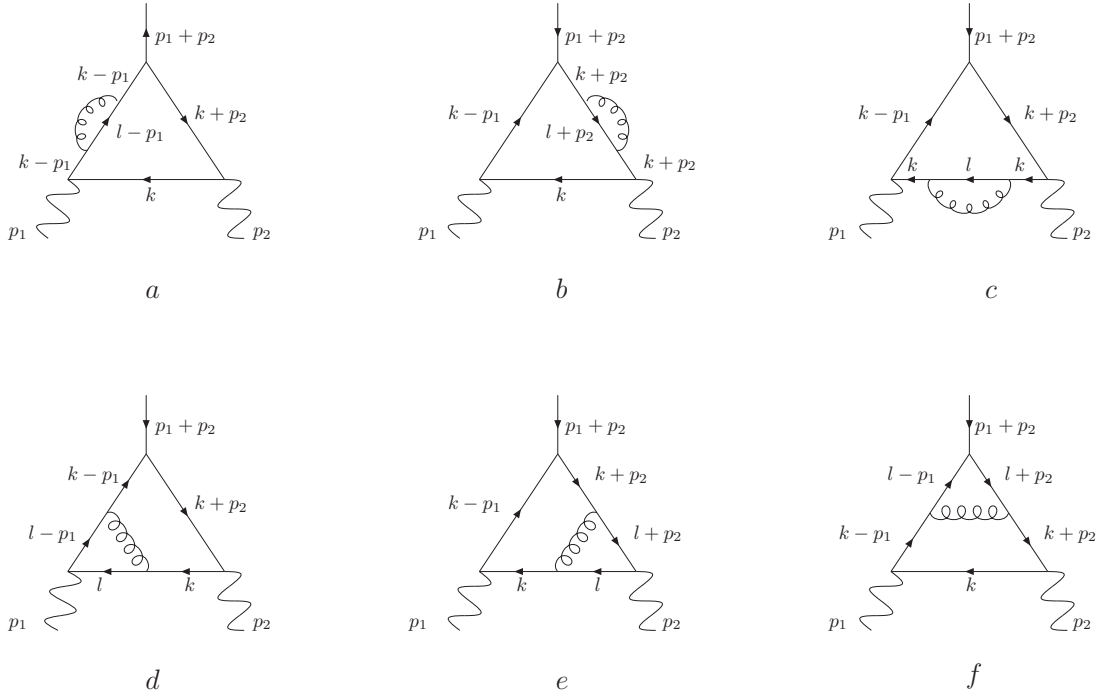


Figure 3.1: Comparison of exact one-loop amplitude and its Taylor expansion for  $z \in (0; 1)$ , with Taylor expansions evaluated to order  $O(z)$ ,  $O(z^3)$  and  $O(z^5)$ ; all coefficients of Taylor expansion have the same sign

Figure 3.2: Two-loop QCD corrections to the process  $H \rightarrow \gamma\gamma$ 

### 3.1 Expansion in external momentum

The diagrams contributing to QCD corrections to the quark loop are shown in Figure 3.2 and the corresponding counter-term diagrams in Figure 2.15. The counter-term diagrams were evaluated in the previous chapter and the result expanded around  $z = 0$  is

$$A_{CT} = \frac{\alpha\alpha_S Q^2 N^2 N_C C_F}{\pi^2 v} \left[ -1 - z \left( \frac{7}{30\epsilon} + \frac{7}{9} \right) - z^2 \left( \frac{4}{21\epsilon} + \frac{40}{63} \right) - z^3 \left( \frac{26}{175\epsilon} + \frac{13}{25} \right) \right] + O(z^4, \epsilon)$$

Note that Mathematica 5.1 has problems with expanding this function around  $z = 0$ . One must be careful and expand to higher order in  $z$  to check if lower-order coefficients are “stable”. Alternative way around is to use the exact  $d$ -dimensional result which involves only the hypergeometric functions which can be easily expanded in  $z$  and perform the  $\epsilon$ -expansion after performing the  $z$ -expansion.

In this chapter we will evaluate the contribution of diagrams in Figure 3.2 by performing a systematic expansion in  $z$  by the technique of Hoogeveen [27]. According to the Feynman rules, the contributions of these diagrams are

$$iM_a^{\mu\nu} = \frac{2ig^2e^2Q^2N_C C_F m}{v} \int \frac{d^d k}{(2\pi)^d} \frac{d^d l}{(2\pi)^d} \frac{\text{Tr} [\gamma^\mu (\not{k} + m) \gamma^\nu (\not{k} + \not{p}_2 + m) (\not{k} - \not{p}_1 + m) \gamma_\sigma (\not{l} - \not{p}_1 + m) \gamma^\sigma (\not{k} - \not{p}_1 + m)]}{[k^2 - m^2][(k + p_2)^2 - m^2][(k - p_1)^2 - m^2]^2[(l - p_1)^2 - m^2][(k - l)^2]}$$

$$iM_b^{\mu\nu} = \frac{2ig^2e^2Q^2N_C C_F m}{v} \int \frac{d^d k}{(2\pi)^d} \frac{d^d l}{(2\pi)^d} \frac{\text{Tr} [\gamma^\mu (\not{k} + m) \gamma^\nu (\not{k} + \not{p}_2 + m) \gamma_\sigma (\not{k} + \not{p}_2 + m) \gamma^\sigma (\not{k} + \not{p}_2 + m) (\not{k} - \not{p}_1 + m)]}{[k^2 - m^2][(k + p_2)^2 - m^2]^2[(k - p_1)^2 - m^2][(l + p_2)^2 - m^2][(k - l)^2]}$$

$$\begin{aligned}
iM_c^{\mu\nu} &= \frac{2ig^2e^2Q^2N_C C_F m}{v} \int \frac{d^d k}{(2\pi)^d} \frac{d^d l}{(2\pi)^d} \frac{\text{Tr} [\gamma^\mu (\not{k} + m) \gamma_\sigma (\not{l} + m) \gamma^\sigma (\not{k} + m) \gamma^\nu (\not{k} + \not{p}_2 + m) (\not{k} - \not{p}_1 + m)]}{[(k-l)^2][k^2 - m^2]^2[(k+p_2)^2 - m^2][(k-p_1)^2 - m^2][l^2 - m^2]} \\
iM_d^{\mu\nu} &= \frac{2ig^2e^2Q^2N_C C_F m}{v} \int \frac{d^d k}{(2\pi)^d} \frac{d^d l}{(2\pi)^d} \frac{\text{Tr} [\gamma^\mu (\not{l} + m) \gamma_\sigma (\not{k} + m) \gamma^\nu (\not{k} + \not{p}_2 + m) (\not{k} - \not{p}_1 + m) \gamma^\sigma (\not{l} - \not{p}_1 + m)]}{[k^2 - m^2][(k-p_1)^2 - m^2][(k+p_2)^2 - m^2][l^2 - m^2][(l-p_1)^2 - m^2][(k-l)^2]} \\
iM_e^{\mu\nu} &= \frac{2ig^2e^2Q^2N_C C_F m}{v} \int \frac{d^d k}{(2\pi)^d} \frac{d^d l}{(2\pi)^d} \frac{\text{Tr} [\gamma^\mu (\not{k} + m) \gamma_\sigma (\not{l} + m) \gamma^\nu (\not{l} + \not{p}_2 + m) \gamma^\sigma (\not{k} + \not{p}_2 + m) (\not{k} - \not{p}_1 + m)]}{[k^2 - m^2][(k-p_1)^2 - m^2][(k+p_2)^2 - m^2][l^2 - m^2][(l+p_2)^2 - m^2][(k-l)^2]} \\
iM_f^{\mu\nu} &= \frac{2ig^2e^2Q^2N_C C_F m}{v} \int \frac{d^d k}{(2\pi)^d} \frac{d^d l}{(2\pi)^d} \frac{\text{Tr} [\gamma^\mu (\not{k} + m) \gamma^\nu (\not{k} + \not{p}_2 + m) \gamma_\sigma (\not{l} + \not{p}_2 + m) (\not{l} - \not{p}_1 + m) \gamma^\sigma (\not{k} - \not{p}_1 + m)]}{[k^2 - m^2][(k+p_2)^2 - m^2][(k-p_1)^2 - m^2][(l+p_2)^2 - m^2][(l-p_1)^2 - m^2][(k-l)^2]}
\end{aligned}$$

Note that there is only one fermion loop corresponding to one factor  $(-1)$ .

We will show the general method on the example of integral (3.2c). We see that the most difficult part of the integrand are the denominators. Evaluation is simplified if we expand the propagators containing the external momenta, i.e. we make use of the relations

$$\begin{aligned}
\frac{1}{(k+p_2)^2 - m^2} &= \frac{1}{k^2 - m^2} \sum_{n=0}^{\infty} \left( \frac{-2k \cdot p_2}{k^2 - m^2} \right)^n \\
\frac{1}{(k-p_1)^2 - m^2} &= \frac{1}{k^2 - m^2} \sum_{n=0}^{\infty} \left( \frac{2k \cdot p_1}{k^2 - m^2} \right)^n
\end{aligned}$$

and expand the propagators to any desired order. In order to simplify the algebraic manipulations it is also useful to use the projection operators from Section 2.1. In this way we arrive at a linear combination of integrals of the form

$$\int \frac{d^d k}{(2\pi)^d} \frac{d^d l}{(2\pi)^d} \frac{(k \cdot l)^a (k \cdot p_n)^{b_n} (l \cdot p_n)^{c_n}}{(k^2 - m^2)^i (l^2 - m^2)^j ((k-l)^2 - \mu^2)}$$

Using the symmetries of integrals explained in more detail in Section 3.2 it is possible to reduce these integrals to integrals of the form

$$B_{kl} = \int \frac{d^d p}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} \frac{1}{(p^2 - m^2)^k (q^2 - m^2)^l ((p-q)^2 - \mu^2)}$$

Evaluation of these integrals will be considered in Section 3.3 and in Section 3.4. Using those results and using any computer program for algebraic manipulations, the contribution of diagrams on Figure 3.2 to invariant amplitude to order  $O(z)$  can be written as

$$\begin{aligned}
A_a = A_b &= -\frac{\alpha\alpha_S N^2 Q^2}{\pi^2 v} \left[ -\frac{3}{4\epsilon^2 z} + \frac{3}{8\epsilon z} + \frac{1}{48z} - \frac{59}{30\epsilon} + \frac{7}{90} - \frac{277z}{315\epsilon} - \frac{12821z}{9450} \right] \\
A_c &= -\frac{\alpha\alpha_S N^2 Q^2}{\pi^2 v} \left[ \frac{3}{2\epsilon^2 z} - \frac{3}{4\epsilon z} + \frac{35}{24z} - \frac{1}{15\epsilon} - \frac{436}{135} - \frac{34z}{315\epsilon} - \frac{6719z}{9450} \right] \\
A_d = A_e &= -\frac{\alpha\alpha_S N^2 Q^2}{\pi^2 v} \left[ -\frac{3}{2z} + \frac{2}{3\epsilon} + \frac{17}{27} + \frac{7z}{45\epsilon} + \frac{1151z}{4050} \right]
\end{aligned}$$

$$\begin{aligned}
A_f &= -\frac{\alpha\alpha_S N^2 Q^2}{\pi^2 v} \left[ \frac{3}{2z} + \frac{8}{3\epsilon} - \frac{113}{27} + \frac{28z}{45\epsilon} + \frac{6287z}{4050} \right] \\
C_a = C_b &= -\frac{\alpha\alpha_S N^2 Q^2}{\pi^2 v} \left[ \frac{41}{30\epsilon} + \frac{47}{90} + \frac{223z}{315\epsilon} + \frac{5848z}{4725} \right] \\
C_c &= -\frac{\alpha\alpha_S N^2 Q^2}{\pi^2 v} \left[ \frac{19}{15\epsilon} + \frac{194}{135} + \frac{142z}{315\epsilon} + \frac{1132z}{1575} \right] \\
C_d = C_e &= -\frac{\alpha\alpha_S N^2 Q^2}{\pi^2 v} \left[ -\frac{2}{3\epsilon} - \frac{1}{27} - \frac{7z}{45\epsilon} - \frac{109z}{2025} \right] \\
C_f &= -\frac{\alpha\alpha_S N^2 Q^2}{\pi^2 v} \left[ -\frac{8}{3\epsilon} + \frac{97}{27} - \frac{28z}{45\epsilon} - \frac{722z}{405} \right]
\end{aligned}$$

Summing the amplitudes of all diagrams and corresponding counter-term diagrams results in a formula

$$A_{QCD} = -C_{QCD} = \frac{\alpha\alpha_S Q^2 N_C C_F}{\pi^2 v} \left[ \frac{1}{2} - \frac{61}{135} z \right] + O(z^2)$$

Note that the equivalence of  $A$  and  $C$  factors following from the electromagnetic gauge invariance and the cancellation of divergent terms helped us check the correctness of the result. Needless to say, this result agrees completely with [14] and with that of [59] after an obvious change of notation and normalization.

## 3.2 Tensor integrals

In this section we will consider relations that are generalization of a simple formula 2.7. After expanding the propagators containing the external momentum in the denominator, we arrive at the expression which involves various combinations of the scalar products of external and loop momenta in the numerator and only loop momenta in the denominator, e.g.

$$\int \frac{d^d k}{(2\pi)^d} \frac{d^d l}{(2\pi)^d} \frac{(p_1 k)(p_1 l)^2 (p_2 k)(p_2 l)^2}{(k^2 - m^2)^r (l^2 - m^2)^s (k - l)^2}$$

This looks similar to the terms in integrand before the expansion in small external momenta, however, we have much simpler denominators (not depending on any external vector). When this happens, it is possible to express the integral as a linear combination of scalar integrals with no external momentum in the numerator.

Results of performing the integration of tensor integrals containing only the loop momenta cannot depend on any four-vector, since this would break the Lorentz-invariance. The only two tensors that we have in game are the metric tensor and the Levi-Civita tensor (since both of these tensors have even number of indices, we see that the integration of any tensor object independent of the external momenta and having the odd number of indices is zero; this also follows from the symmetry under inversion/parity operation). Moreover, if integrand contains only four-vectors that transform as vectors (as is of course the case of the loop momenta), we know that the result does not contain the Levi-Civita tensor. If the integrand has any symmetry under the permutation of its Lorentz indices, the result must also possess this symmetry. These simple considerations lead us to strong restrictions on the form of the integral.

When we factor out external momenta, we see that we are dealing with integrals of the form

$$\int \frac{d^d k}{(2\pi)^d} \frac{d^d l}{(2\pi)^d} \frac{k^\alpha k^\beta l^\mu l^\nu l^\rho l^\sigma}{(k^2 - m^2)^r (l^2 - m^2)^s (k - l)^2}$$

We see from the transformation properties under the Lorentz transformations that this must be equal to a linear combination of tensor products of the metric tensor with various indices. Moreover, integrand is symmetric under the exchange of  $(\alpha, \beta)$  and  $(\mu, \nu, \rho, \sigma)$ . This means that we must look for tensor products of metric tensor with symmetry dictated by the symmetry of integral. Contracting with tensors of this form gives a system of linear equations for unknown coefficients of the linear combination and after solving it we obtain integral expressed as a combination of constant tensors with coefficients that are scalar integrals that will be evaluated in the next section. Finally, contracting both sides with the external momenta gives the desired expression for integral in terms of the scalar integrals. In this section we will develop a systematic procedure for finding out values of various contractions.

Any tensor (made of tensor products of metric tensors) that is symmetric in two groups of indices,  $A = (\alpha_1, \alpha_2, \dots)$  and  $B = (\beta_1, \beta_2, \dots)$  is a linear combination of tensors obtained by symmetrization in indices  $A$  and  $B$  of basic monomials

$$g^{\alpha_1\alpha_2} \dots g^{\beta_1\beta_2} \dots g^{\alpha_n\beta_m} \dots$$

and these in turn depend only on the number of  $AA$  pairs ( $n_{AA}$ ), the number of  $BB$  pairs ( $n_{BB}$ ) and the  $AB$ 's paired together ( $n_{AB}$ ). This lets us introduce the diagrams which form a basis of a vector space of tensors which have the desired symmetry:

$$\begin{aligned} \begin{array}{|c|} \hline A \\ \hline A \\ \hline \end{array} &= g^{\alpha_1\alpha_2} \\ \begin{array}{|c|c|} \hline A & B \\ \hline A & B \\ \hline \end{array} &= g^{\alpha_1\alpha_2} g^{\beta_1\beta_2} \\ \begin{array}{|c|c|} \hline A & A \\ \hline B & B \\ \hline \end{array} &= g^{\alpha_1\beta_1} g^{\alpha_2\beta_2} + g^{\alpha_1\beta_2} g^{\alpha_2\beta_1} \\ \begin{array}{|c|c|c|} \hline A & B & B \\ \hline A & B & B \\ \hline \end{array} &= g^{\alpha_1\alpha_2} g^{\beta_1\beta_2} g^{\beta_3\beta_4} + g^{\alpha_1\alpha_2} g^{\beta_1\beta_3} g^{\beta_2\beta_4} + g^{\alpha_1\alpha_2} g^{\beta_1\beta_4} g^{\beta_2\beta_3} \\ \begin{array}{|c|c|c|} \hline A & A & B \\ \hline B & B & B \\ \hline \end{array} &= g^{\alpha_1\beta_1} g^{\alpha_2\beta_2} g^{\beta_3\beta_4} + \dots (12 \text{ terms}) \\ \begin{array}{|c|c|c|} \hline A & A & A \\ \hline B & B & B \\ \hline \end{array} &= g^{\alpha_1\beta_1} g^{\alpha_2\beta_2} g^{\alpha_3\beta_3} + \dots (6 \text{ terms}) \\ \begin{array}{|c|c|c|} \hline A & A & B \\ \hline A & B & B \\ \hline \end{array} &= g^{\alpha_1\alpha_2} g^{\alpha_3\beta_1} g^{\beta_2\beta_3} + \dots (9 \text{ terms}) \end{aligned}$$

That is, every column represents one metric tensor  $g^{\alpha\beta}$  and all terms on the right hand side are independent and have coefficient 1. We write  $AA$  pairs to the left, followed by  $AB$  pairs and finally  $BB$  pairs come to the right. We see immediately that for the fixed number of indices of the first group ( $n_A$ ) and fixed number of indices of the second group ( $n_B$ ) every base tensor is given if we specify either the number of  $AA$  pairs ( $n_{AA}$ ),  $BB$  pairs ( $n_{BB}$ ) or  $AB$  pairs ( $n_{AB}$ ). We choose the last choice to preserve the symmetry in  $A$  and  $B$ . The following relations are immediate consequence of these definitions:

$$\begin{aligned} n_A &= 2n_{AA} + n_{AB} \\ n_B &= 2n_{BB} + n_{AB} \end{aligned}$$

The dimension of the vector space of symmetric tensors is given by the number of values that  $n_{AB}$  can have. If (without loss of the generality)  $n_A > n_B$  then  $(n_A - n_B)$  (which is even since  $(n_A + n_B)$  is)  $A$ 's must be paired together and the remaining  $n_A - n_B$   $A$ 's can be paired either with another  $A$  or with  $B$ . That is, for given  $n_A$  and  $n_B$  the number of basis vectors is given by

$$\begin{aligned} \frac{\min(n_A, n_B) + 2}{2} & \text{for } \min(n_A, n_B) \text{ even} \\ \frac{\min(n_A, n_B) + 1}{2} & \text{for } \min(n_A, n_B) \text{ odd} \end{aligned}$$

The number of terms in the expression for a tensor represented by a table is

$$\text{Length} \left( \begin{array}{|c|} \hline \cdot \\ \hline \end{array} \right) = \frac{n_A! n_B!}{n_{AA}! n_{AB}! n_{BB}! 2^{n_{AA} + n_{BB}}}$$

That is, we take all permutations of  $n_A$  and  $n_B$  indices and divide it by the number of tensors that are dependent ( $n_{AA} 2^{n_{AA}}$  represents the number of permutations of indices among  $AA$  pairs that do not change the basis vector etc.). We could also proceed by writing a recurrence relations and solving them:

$$\begin{aligned} \text{Length} \left( \begin{array}{|c|c|c|} \hline A & \cdot & \cdot \\ \hline A & \cdot & \cdot \\ \hline \end{array} \right) &= \frac{n_A(n_A - 1)}{2n_{AA}} \text{Length} \left( \begin{array}{|c|c|} \hline \cdot & \cdot \\ \hline \cdot & \cdot \\ \hline \end{array} \right) \\ \text{Length} \left( \begin{array}{|c|c|c|} \hline \cdot & A & \cdot \\ \hline \cdot & B & \cdot \\ \hline \end{array} \right) &= \frac{n_A n_B}{n_{AB}} \text{Length} \left( \begin{array}{|c|c|} \hline \cdot & \cdot \\ \hline \cdot & \cdot \\ \hline \end{array} \right) \end{aligned}$$

$$\text{Length} \left( \begin{array}{|c|c|c|} \hline \cdot & \cdot & B \\ \hline \cdot & \cdot & B \\ \hline \end{array} \right) = \frac{n_B(n_B - 1)}{2n_{BB}} \text{Length} \left( \begin{array}{|c|c|} \hline \cdot & \cdot \\ \hline \cdot & \cdot \\ \hline \end{array} \right)$$

Now we have everything ready to calculate the contractions of symmetric tensors. We first derive a formula for contraction of tensor in two indices. We fix two  $A$  indices. In every term they can occur either as indices of the same metric tensor (in the same  $AA$  pair), in a different  $AA$  pairs, or one of them in  $AA$  pair and the other in  $AB$  pair or finally both in  $AB$  pairs. If we denote the fixed indices by boldface (and the symbol of table with bold indices represent all terms of tensor corresponding to the table that have fixed indices as specified), we have diagrammatically

$$\begin{aligned} \text{Tr}_{AA} \begin{array}{|c|c|} \hline \mathbf{A} & \cdot \\ \hline \mathbf{A} & \cdot \\ \hline \end{array} & \rightarrow d \begin{array}{|c|} \hline \cdot \\ \hline \cdot \\ \hline \end{array} \\ \text{Tr}_{AA} \begin{array}{|c|c|c|} \hline \mathbf{A} & \mathbf{A} & \cdot \\ \hline A & A & \cdot \\ \hline \end{array} & \rightarrow 2(n_{AA} - 1) \begin{array}{|c|c|} \hline A & \cdot \\ \hline A & \cdot \\ \hline \end{array} \\ \text{Tr}_{AA} \begin{array}{|c|c|c|} \hline \mathbf{A} & \mathbf{A} & \cdot \\ \hline A & B & \cdot \\ \hline \end{array} & \rightarrow 2n_{AB} \begin{array}{|c|c|} \hline A & \cdot \\ \hline B & \cdot \\ \hline \end{array} \\ \text{Tr}_{AA} \begin{array}{|c|c|c|} \hline \mathbf{A} & \mathbf{A} & \cdot \\ \hline B & B & \cdot \\ \hline \end{array} & \rightarrow 2(n_{BB} + 1) \begin{array}{|c|c|} \hline B & \cdot \\ \hline B & \cdot \\ \hline \end{array} \end{aligned}$$

The coefficients are chosen such that the both sides have the same number of the terms (if we count a term with coefficient  $a \in \mathbb{N}$  as  $a$  terms), for instance the left hand side of the third line has

$$\text{Length} \left( \begin{array}{|c|c|c|} \hline \mathbf{A} & \mathbf{A} & \cdot \\ \hline A & B & \cdot \\ \hline \end{array} \right) = 2n_B(n_A - 2) \text{Length} \left( \begin{array}{|c|} \hline \cdot \\ \hline \cdot \\ \hline \end{array} \right) = 2n_B(n_A - 2) \frac{(n_A - 3)!(n_B - 1)!}{(n_{AA} - 1)!(n_{AB} - 1)!n_{BB}!2^{n_{AA}-1+n_{BB}}}$$

(the factor 2 comes from the permutation of two fixed indices) while the tensor on the right hand side has

$$\text{Length} \left( \begin{array}{|c|c|} \hline A & \cdot \\ \hline B & \cdot \\ \hline \end{array} \right) = \frac{(n_A - 2)!n_B!}{(n_{AA} - 1)!n_{AB}!n_{BB}!2^{n_{AA}-1+n_{BB}}}$$

Comparing both expressions, we see that the tensor on the left hand side gives rise to  $2n_{AB}$  times the tensor on the right hand side. Other expressions are similar. Factor  $d$  in the first line denotes the dimension which comes from the contraction  $g_{\alpha\beta}g^{\alpha\beta}$ . This is the only term which produces the dimension  $d$  after a contraction. If we sum up these four cases, we obtain the formula

$$\text{Tr}_{AA} \begin{array}{|c|} \hline \cdot \\ \hline \cdot \\ \hline \end{array} = [d + 2n_{AA} + 2n_{AB} - 2] \left[ \begin{array}{|c|c|} \hline A & \cdot \\ \hline A & \cdot \\ \hline \end{array} \rightarrow \begin{array}{|c|} \hline \cdot \\ \hline \cdot \\ \hline \end{array} \right] + [2n_{BB} + 2] \left[ \begin{array}{|c|c|c|} \hline A & A & \cdot \\ \hline B & B & \cdot \\ \hline \end{array} \rightarrow \begin{array}{|c|c|} \hline B & \cdot \\ \hline B & \cdot \\ \hline \end{array} \right]$$

Although the diagrammatical representation has its advantages, we will also denote the symmetric basis tensors by a shorter notation

$$G(n_A, n_B; n_{AB})$$

The previous identity reads

$$\text{Tr}_{AA} G(n_A, n_B; n_{AB}) = (d + 2n_{AA} + 2n_{AB} - 2)G(n_A - 2, n_B; n_{AB}) + (2n_{BB} + 2)G(n_A - 2, n_B; n_{AB} - 2)$$

Note that the equation applies only if  $n_A > 1$ , first term only if  $n_{AA} \geq 1$  (which equivalent to  $n_A - 2 \geq n_{AB}$ ) and the second term if  $n_{AB} \geq 2$ . All of these conditions are automatically satisfied if we define

$$\begin{aligned} G(n_A, n_B; n_{AB}) &= 0 && \text{if } n_{AB} < 0 \\ G(n_A, n_B; n_{AB}) &= 0 && \text{if } n_A < n_{AB} \text{ or } n_B < n_{AB} \end{aligned}$$

(which automatically implies zero if  $n_A < 0$  or  $n_B < 0$ ). The trace over  $B$  indices is given by the symmetric formula

$$\text{Tr}_{BB} G(n_A, n_B; n_{AB}) = (d + 2n_{AB} + 2n_{BB} - 2)G(n_A, n_B - 2; n_{AB}) + (2n_{AA} + 2)G(n_A, n_B - 2; n_{AB} - 2)$$

and formula for trace over one  $A$  index and one  $B$  index can be derived in a similar way and is

$$\text{Tr}_{AB} G(n_A, n_B; n_{AB}) = (d + 2n_{AA} + n_{AB} + 2n_{BB} - 1)G(n_A - 1, n_B - 1; n_{AB} - 1) +$$

$$(n_{AB} + 1)G(n_A - 1, n_B - 1; n_{AB} + 1)$$

or diagrammatically

$$\text{Tr}_{AB} \begin{array}{|c|} \hline \cdot \\ \hline \end{array} = [d + 2n_{AA} + n_{AB} + 2n_{BB} - 1] \begin{array}{|c|} \hline A \cdot \\ \hline B \cdot \\ \hline \end{array} \rightarrow \begin{array}{|c|} \hline \cdot \\ \hline \end{array} + [n_{AB} + 1] \begin{array}{|c|} \hline A B \cdot \\ \hline A B \cdot \\ \hline \end{array} \rightarrow \begin{array}{|c|} \hline A \cdot \\ \hline B \cdot \\ \hline \end{array}$$

The same remarks as before apply, that is, if we define  $G(n_A, n_B; n_{AB})$  to be zero whenever any coefficient is negative or if  $n_A < n_{AB}$  or  $n_B < n_{AB}$ , then the formula remains valid for any allowed values of parameters on the left hand side.

Now we will write the relations for the complete trace in all indices. We will define a product of tables with the same  $n_A$  and  $n_B$  to be their contraction. Since the tensor  $G(n_A, n_B; n_{AB})$  is already symmetric in  $A$ 's and  $B$ 's, its contraction with any two tensors which differ only by the permutation of  $A$ 's and  $B$ 's gives the same result. This lets us immediately write the recurrence relation

$$C(n_A, n_B; n_1, n_2) = \frac{n_A(n_A - 1)}{n_A - n_1} [(d + n_A + n_2 - 2)C(n_A - 2, n_B; n_1, n_2) + (n_B - n_2 + 2)C(n_A - 2, n_B; n_1, n_2 - 2)]$$

where we used a shorthand

$$C(n_A, n_B; n_1, n_2) = G(n_A, n_B; n_1) \cdot G(n_A, n_B; n_2)$$

This formula cannot be used when  $n_A = n_1$ , in which case there is a zero denominator. The coefficient comes from the difference in the number of terms of  $G(n_A, n_B; n_1)$  and  $G(n_A, n_B; n_1 - 1)$ . Similarly

$$C(n_A, n_B; n_1, n_2) = \frac{n_A n_B}{n_1} [(d + n_A + n_B - n_2 - 1)C(n_A - 1, n_B - 1; n_1 - 1, n_2 - 1) + (n_2 + 1)C(n_A - 1, n_B - 1; n_1 - 1, n_2 + 1)]$$

Contraction of the table with a product of loop momenta which has the same symmetry as the table is trivial (each term gives the same value):

$$G(n_A, n_B, n_{AB}) k^{\alpha_1} \dots k^{\alpha_{n_A}} l^{\beta_1} \dots l^{\beta_{n_B}} = (k^2)^{n_{AA}} (l^2)^{n_{BB}} (k \cdot l)^{n_{AB}} \text{Length}(G(n_A, n_B, n_{AB}))$$

Now when we can multiply tables together and with loop momenta, all that remains to do is to know how to contract the external momenta with the tables. We specialize to our case of interest, which is two external null momenta ( $p_1^2 = 0 = p_2^2$ ). We denote the contraction of  $G(n_A, n_B, n_{AB})$  with  $n_1^A$   $p_1$ -vectors with index from the group  $A$ ,  $n_1^B$   $p_1$ -vectors with index from group  $B$  etc. by

$$E(n_A, n_B, n_{AB}; n_1^A, n_2^A, n_1^B, n_2^B)$$

If we fix one vector  $p_1^A$ , it can be paired in  $n_2^A$  ways with vector  $p_2^A$  if  $n_{AA} \neq 0$  or with  $n_2^B$  vectors  $p_2^B$  if  $n_{AB} \neq 0$ . This leads to the relation

$$E(n_A, n_B, n_{AB}; n_1^A, n_2^A, n_1^B, n_2^B) = (p_1 \cdot p_2) n_2^A E(n_A - 2, n_B, n_{AB}; n_1^A - 1, n_2^A - 1, n_1^B, n_2^B) + (p_1 p_2) n_2^B E(n_A - 1, n_B - 1, n_{AB} - 1; n_1^A - 1, n_2^A, n_1^B, n_2^B - 1)$$

valid as long as  $p_1^A \neq 0$ . If we use it with  $p_1^A$  zero and define the contractions with negative coefficients to be zero, we get incorrect results (right hand side would be zero while left hand side can be nonzero). Similar relation holds true when we exchange  $A$ 's and  $B$ 's or  $n_1$ 's with  $n_2$ 's.

As an example, we show how to obtain the formula

$$\int \frac{d^d k}{(2\pi)^d} \frac{d^d l}{(2\pi)^d} \frac{(p_1 k)(p_1 l)^2 (p_2 k)(p_2 l)^2}{(k^2 - m^2)^r (l^2 - m^2)^s (k - l)^2} = (p_1 p_2)^3 \int \frac{d^d k}{(2\pi)^d} \frac{d^d l}{(2\pi)^d} \frac{2(d+1)k^2 (l^2)^2 + 4(d-2)l^2 (kl)^2}{(k^2 - m^2)^r (l^2 - m^2)^s (k - l)^2}$$

We first factor our the external momenta and obtain

$$p_1^{\alpha_1} p_2^{\alpha_2} p_1^{\beta_1} p_1^{\beta_2} p_2^{\beta_3} p_2^{\beta_4} \int \frac{d^d k}{(2\pi)^d} \frac{d^d l}{(2\pi)^d} \frac{k_{\alpha_1} k_{\alpha_2} l_{\beta_1} l_{\beta_2} l_{\beta_3} l_{\beta_4}}{(k^2 - m^2)^r (l^2 - m^2)^s (k - l)^2}$$

Integral on the right hand side is symmetric in  $(\alpha_1, \alpha_2)$  and  $(\beta_1, \beta_2, \beta_3, \beta_4)$  so it must be linear combination of  $G(2, 4, n_{AB})$ 's:

$$\int \frac{d^d k}{(2\pi)^d} \frac{d^d l}{(2\pi)^d} \frac{k_{\alpha_1} k_{\alpha_2} l_{\beta_1} l_{\beta_2} l_{\beta_3} l_{\beta_4}}{(k^2 - m^2)^r (l^2 - m^2)^s (k - l)^2} = K \begin{array}{|c|c|c|} \hline A & B & B \\ \hline A & B & B \\ \hline \end{array} + L \begin{array}{|c|c|c|} \hline A & A & B \\ \hline B & B & B \\ \hline \end{array}$$

Contracting with the external momenta, we have

$$\int \frac{d^d k}{(2\pi)^d} \frac{d^d l}{(2\pi)^d} \frac{(p_1 k)(p_1 l)^2 (p_2 k)(p_2 l)^2}{(k^2 - m^2)^r (l^2 - m^2)^s (k - l)^2} = KE(2, 4, 0; 1, 1, 2, 2) + LE(2, 4, 2; 1, 1, 2, 2)$$

To find the value of coefficients  $K$  and  $L$  (which of course depend on the parameters in the integral), we contract both sides of the tensorial equation with both linearly independent vectors and arrive at the system of the linear equations:

$$\begin{aligned} 3 \int \frac{d^d k}{(2\pi)^d} \frac{d^d l}{(2\pi)^d} \frac{k^2 (l^2)^2}{(k^2 - m^2)^r (l^2 - m^2)^s (k - l)^2} &= KC(2, 4; 0, 0) + LC(2, 4; 2, 0) \\ 12 \int \frac{d^d k}{(2\pi)^d} \frac{d^d l}{(2\pi)^d} \frac{l^2 (kl)^2}{(k^2 - m^2)^r (l^2 - m^2)^s (k - l)^2} &= KC(2, 4; 0, 2) + LC(2, 4; 2, 2) \end{aligned}$$

where we substituted

$$\begin{aligned} \text{Length} \left( \begin{array}{|c|c|c|} \hline A & B & B \\ \hline A & B & B \\ \hline \end{array} \right) &= 3 \\ \text{Length} \left( \begin{array}{|c|c|c|} \hline A & A & B \\ \hline B & B & B \\ \hline \end{array} \right) &= 12 \end{aligned}$$

for the lengths. Using the recurrence relations, we easily arrive at

$$\begin{aligned} C(4, 2; 0, 0) &= 3d^2(d+2) \\ C(4, 2; 2, 0) &= 12d(d+2) \\ C(4, 2; 2, 2) &= 12d(d+2)(d+3) \end{aligned}$$

Solving the resulting system of equations for  $K$  and  $L$  thus gives

$$\begin{aligned} K &= \int \frac{d^d k}{(2\pi)^d} \frac{d^d l}{(2\pi)^d} \frac{1}{(k^2 - m^2)^r (l^2 - m^2)^s (k - l)^2} \frac{(d+3)k^2(l^2)^2 - 4l^2(kl)^2}{d(d+2)(d+4)(d-1)} \\ L &= \int \frac{d^d k}{(2\pi)^d} \frac{d^d l}{(2\pi)^d} \frac{1}{(k^2 - m^2)^r (l^2 - m^2)^s (k - l)^2} \frac{dl^2(kl)^2 - k^2(l^2)^2}{d(d+2)(d+4)(d-1)} \end{aligned}$$

Using the recurrence relations again,

$$\begin{aligned} E(2, 4, 0; 1, 1, 2, 2) &= 2 \\ E(2, 4, 2; 1, 1, 2, 2) &= 4 \end{aligned}$$

and the advertised result immediately follows.

Finally, we list some integrals encountered in the evaluation of QCD corrections by the small momentum expansion to order  $O(z)$ .

$$\begin{aligned} k^A k^A &\rightarrow \frac{k^2}{d} G(2, 0, 0) \\ k^A l^A &\rightarrow \frac{(kl)}{d} G(1, 1, 1) \\ k^A k^A k^A k^A &\rightarrow \frac{(k^2)^2}{d^2 + 2d} G(4, 0, 0) \\ k^A k^A k^A l^B &\rightarrow \frac{k^2(kl)}{d^2 + 2d} G(3, 1, 1) \\ k^A k^A l^B l^B &\rightarrow \left[ \frac{(d+1)k^2 l^2 - 2(kl)^2}{d(d+2)(d-1)} \right] G(2, 2, 0) + \left[ \frac{d(kl)^2 - k^2 l^2}{d(d+2)(d-1)} \right] G(2, 2, 2) \end{aligned}$$

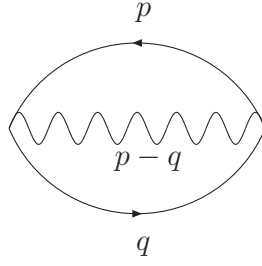


Figure 3.3: Bubble integral used in external momentum expansions

$$\begin{aligned}
k^A k^A k^A k^A k^A k^A &\rightarrow \frac{(k^2)^3}{d^3 + 6d^2 + 8d} G(6, 0, 0) \\
k^A k^A k^A k^A k^A l^B &\rightarrow \frac{(k^2)^2 kl}{d^3 + 6d^2 + 8d} G(5, 1, 1) \\
k^A k^A l^B l^B l^B l^B &\rightarrow \left[ \frac{(d+3)k^2(l^2)^2 - 4(kl)^2 l^2}{d(d+2)(d+4)(d-1)} \right] G(2, 4, 0) + \left[ \frac{dl^2(kl)^2 - k^2(l^2)^2}{d(d+2)(d+4)(d-1)} \right] G(2, 4, 2) \\
k^A k^A k^A l^B l^B l^B &\rightarrow \left[ \frac{(d+2)(kl)^3 - 3k^2 l^2 (kl)}{d(d+2)(d+4)(d-1)} \right] G(3, 3, 3) + \left[ \frac{-2(kl)^3 + (d+1)k^2 l^2 (kl)}{d(d+2)(d+4)(d-1)} \right] G(3, 3, 1)
\end{aligned}$$

### 3.3 Evaluation of the vacuum bubble master integrals

In this section we will evaluate the vacuum bubble integrals (we will proceed following [27] and [54]). The integrals that we need are

$$B_{kl}(M^2, m^2) = \int \frac{d^d p d^d q}{(2\pi)^{2d}} \frac{1}{(p^2 - M^2)^k (q^2 - M^2)^l ((p - q)^2 - m^2)}$$

Differentiation with respect to  $m^2$  gives the a more general integral

$$\frac{1}{n!} \frac{\partial^n}{\partial (m^2)^n} B_{kl} = \int \frac{d^d p d^d q}{(2\pi)^{2d}} \frac{1}{(p^2 - M^2)^k (q^2 - M^2)^l ((p - q)^2 - m^2)^{n+1}}$$

Integration by parts using the identity (summation over the Lorentz indices takes place as usual)

$$\frac{d}{dp^\mu} p^\mu = d$$

results in a recurrence relation satisfied by  $B_{kl}$

$$2kM^2 B(k+1, l) = (d - 2k - 1 - m^2 \frac{\partial}{\partial m^2}) B(k, l) - \frac{\partial}{\partial M^2} B(k-1, l) + \frac{\partial}{\partial m^2} B(k, l-1)$$

We see that in order to evaluate the integrals  $B_{kl}$  it is enough to calculate  $B_{11}$  and a products of the one-loop integrals  $B_{k0}$ . Note that although in our case we have  $m^2 = 0$ , the introduction of non-zero mass  $m^2$  is necessary in order to arrive at these recurrence relations. For another way of evaluating the necessary integrals, see Section 3.4.

We will not evaluate  $B_{11}$  directly, but will use the Euler relation for homogeneous functions in the form

$$(d-3)B_{11} = \left( m^2 \frac{\partial}{\partial m^2} + M^2 \frac{\partial}{\partial M^2} \right) B_{11} = m^2 I(m^2, M^2, M^2) + 2M^2 I(M^2, M^2, m^2)$$

where

$$I(m_0, m_1, m_2) = \int \frac{d^d p d^d q}{(2\pi)^{2d}} \frac{1}{[p^2 - m_0^2]^2 [q^2 - m_1^2] [(p - q)^2 - m_2^2]}$$

This integral was the most difficult integral that we encountered. It is evidently symmetric under the exchange of  $m_1$  and  $m_2$ . Before performing the integration over the loop momenta, we factor out the mass scale  $m_0^2$ , introducing a dimensionless parameters

$$a = \frac{m_1^2}{m_0^2}$$

$$b = \frac{m_2^2}{m_0^2}$$

Next step is to perform the Wick rotation, which is easier before introducing the Feynman parameters. We introduce the first Feynman parameter that puts together parameters  $a$  and  $b$  (to preserve the manifest symmetry), integrate over  $q$  and then introduce the second Feynman parameter which lets us perform integration over  $p$ . The resulting expression reads

$$I = -\frac{(m_0^2)^{d-4}}{(4\pi)^d} \Gamma(4-d) \int_0^1 dx \int_0^1 dy \frac{y(1-y)^{1-\frac{d}{2}}}{(x(1-x))^{2-\frac{d}{2}} (y+(1-y)\mu^2)^{4-d}}$$

where

$$\mu^2 = \frac{ax + b(1-x)}{x(1-x)}$$

This integral is evaluated in Appendix C.4.

We will need the expression for  $B_{kl}(M^2, m^2)$  only to  $O(1)$  in  $\left(\frac{m^2}{M^2}\right)$ , because in our case  $m^2$  is either zero or if we were more careful, the IR regulator  $\mu^2$  (gluon mass). Looking at the recurrence relation among  $B_{kl}$  we see that in order to arrive at  $B_{kl}$  to order  $O(1)$ , we need to know  $B_{k-1,l}$  to the same order and  $B_{k-1,l-1}$  and  $B_{k-2,l}$  to order  $O(m^2/M^2)$ . Continuing this argument we see that the necessary order in  $m^2$  to which we must calculate integrals depends only on the sum  $k+l$ . Since we encountered integrals up to  $k+l=9$ , and we want to know their value to order  $O(1)$ , we must start with  $B_{11}$  at least to order  $O(m^6/M^6)$ . Substituting our results and expanding in  $m^2/M^2$  yields (again, the expansion in Mathematica 5.1 was unstable)

$$B_{11} = \frac{-M^2 N^2}{2(4\pi)^4} \left[ \frac{2}{\epsilon^2} + \frac{6}{\epsilon} + 14 + R \left( \frac{1}{\epsilon^2} + \frac{3}{\epsilon} - \frac{2}{\epsilon} \ln R + \ln^2 R - 2 \ln R - 1 \right) \right. \\ \left. + R^2 \left( \frac{8}{9} - \frac{\ln R}{3} \right) + R^3 \left( \frac{31}{450} - \frac{\ln R}{30} \right) + R^4 \left( \frac{389}{44100} - \frac{\ln R}{210} \right) + R^5 \left( \frac{1097}{793800} - \frac{\ln R}{1260} \right) \right. \\ \left. + R^6 \left( \frac{23819}{96049800} - \frac{\ln R}{6930} \right) + R^7 \left( \frac{293141}{6492966480} - \frac{\ln R}{36036} \right) + O(R^8 \ln R) \right]$$

with

$$R = \frac{m^2}{M^2}$$

Now it is simple to verify that

$$B_{10} = -\frac{M^4 N^2 R}{2(4\pi)^4} \left( \frac{2}{\epsilon^2} + \frac{4}{\epsilon} + 6 - \frac{2 \ln R}{\epsilon} + \ln^2 R - 4 \ln R \right)$$

$$B_{20} = -\frac{M^2 R}{2(4\pi)^4} \left( \frac{2}{\epsilon^2} + \frac{2}{\epsilon} + 2 - \frac{2 \ln R}{\epsilon} + \ln^2 R - 2 \ln R \right)$$

$$B_{30} = -\frac{N^2 R}{2(4\pi)^4} \left( -\frac{1}{\epsilon} - 1 + \ln R \right)$$

$$B_{40} = -\frac{N^2 R}{2(4\pi)^4 M^2} \left( \frac{1}{3\epsilon} + \frac{2}{3} - \frac{1}{3} \ln R \right)$$

$$B_{50} = -\frac{N^2 R}{2(4\pi)^4 M^4} \left( -\frac{1}{6\epsilon} - \frac{5}{12} + \frac{1}{6} \ln R \right)$$

$$B_{60} = -\frac{N^2 R}{2(4\pi)^4 M^6} \left( \frac{1}{10\epsilon} + \frac{17}{60} - \frac{1}{10} \ln R \right)$$

$$\begin{aligned}
B_{70} &= -\frac{N^2 R}{2(4\pi)^4 M^8} \left( -\frac{1}{15\epsilon} - \frac{37}{180} + \frac{1}{15} \ln R \right) \\
B_{80} &= -\frac{N^2 R}{2(4\pi)^4 M^{10}} \left( \frac{1}{21\epsilon} + \frac{197}{1260} - \frac{1}{21} \ln R \right)
\end{aligned}$$

Using the IBP relations we evaluate the remaining integrals that were needed throughout the evaluation of the QCD corrections. Although the most of these integrals are given in [27], there are some typographical mistakes, so for completeness here is the list of  $B_{kl}$ :

$$\begin{aligned}
B_{21} &= \frac{N^2}{2(4\pi)^4} \left[ -\frac{1}{\epsilon^2} - \frac{1}{\epsilon} - 1 + 2R + R \ln R + \frac{5R^2}{18} + \frac{R^2}{6} \ln R + \frac{47R^3}{900} + \frac{R^3}{30} \ln R \right] \\
B_{31} &= -\frac{N^2}{2(4\pi)^4 M^2} \left[ -\frac{1}{\epsilon} + \frac{7R}{18} + \frac{R}{3} \ln R + \frac{37R^2}{300} + \frac{R^2}{10} \ln R \right] \\
B_{41} &= +\frac{N^2}{2(4\pi)^4 M^4} \left[ -\frac{1}{3\epsilon} - \frac{1}{2} + \frac{5R}{36} + \frac{R}{6} \ln R + \frac{59R^2}{900} - \frac{R^2}{15} \ln R \right] \\
B_{51} &= -\frac{N^2}{2(4\pi)^4 M^6} \left[ -\frac{1}{6\epsilon} - \frac{13}{36} + \frac{13R}{200} + \frac{R}{10} \ln R \right] \\
B_{61} &= +\frac{N^2}{2(4\pi)^4 M^8} \left[ -\frac{1}{10\epsilon} - \frac{31}{120} + \frac{8R}{225} + \frac{R}{15} \ln R \right] \\
B_{71} &= -\frac{N^2}{2(4\pi)^4 M^{10}} \left[ -\frac{1}{15\epsilon} - \frac{173}{900} \right] \\
B_{81} &= +\frac{N^2}{2(4\pi)^4 M^{12}} \left[ -\frac{1}{21\epsilon} - \frac{187}{1260} \right] \\
B_{22} &= -\frac{1}{2(4\pi)^4 M^2} \left[ -2 + \frac{2R}{9} + \frac{R}{3} \ln R + \frac{32R^2}{225} + \frac{2R^2}{15} \ln R \right] \\
B_{32} &= +\frac{1}{2(4\pi)^4 M^4} \left[ -\frac{1}{2} + \frac{R}{36} + \frac{R}{6} \ln R + \frac{11R^2}{150} + \frac{R^2}{10} \ln R \right] \\
B_{42} &= -\frac{1}{2(4\pi)^4 M^6} \left[ -\frac{2}{9} - \frac{R}{100} + \frac{R}{10} \ln R \right] \\
B_{52} &= +\frac{1}{2(4\pi)^4 M^8} \left[ -\frac{1}{8} - \frac{4R}{225} + \frac{R}{15} \ln R \right] \\
B_{62} &= -\frac{1}{2(4\pi)^4 M^6} \left[ -\frac{2}{25} \right] \\
B_{33} &= -\frac{1}{2(4\pi)^4 M^6} \left[ -\frac{1}{6} - \frac{2R}{75} + \frac{R}{10} \ln R \right] \\
B_{43} &= +\frac{1}{2(4\pi)^4 M^8} \left[ -\frac{1}{12} - \frac{31R}{900} + \frac{R}{15} \ln R \right] \\
B_{53} &= -\frac{1}{2(4\pi)^4 M^{10}} \left[ -\frac{1}{20} \right] \\
B_{63} &= +\frac{1}{2(4\pi)^4 M^{12}} \left[ -\frac{1}{30} \right] \\
B_{44} &= -\frac{1}{2(4\pi)^4 M^{10}} \left[ -\frac{2}{45} \right] \\
B_{54} &= +\frac{1}{2(4\pi)^4 M^{12}} \left[ -\frac{1}{36} \right]
\end{aligned}$$

The simple form of the singular integrals (with  $\frac{1}{\epsilon}$  poles) when compared to [27] is consequence of not expanding the  $N$  factors, which would otherwise produce many  $\gamma$  or  $\zeta(n)$  factors.

### 3.4 Evaluation of vacuum bubble master integrals with massless propagator

In this section we will show a different way how to arrive at results used in the preceding sections. The reason for doing so is that in the previous calculation we did not take the full advantage of having a massless photon. Looking at our results, we see that we can in fact put  $m^2 = 0$  in all of the integrals (correspondingly to a massless photon). However, the application of the recurrence relation relied heavily on having  $m^2 \neq 0$ , and in order to calculate integrals  $B_{kl}$  with higher values of  $k$  and  $l$  we needed expansion of  $B_{11}$  to high orders in  $m^2/M^2$ . As we will see in this section, all  $B_{kl}$  with one massless propagator and the other propagators having the same mass can be derived from  $B_{11}(M^2, m^2 = 0)$ . Moreover, since the major complication in the case  $m^2 \neq 0$  was the presence of two mass scales, which manifested itself in quite complicated integration over the Feynman parameters, this alternative way will be much more straightforward and simple.

As in the previous section, we will use the integration-by-parts relations. We must choose a family of integrals among which we will try to find as many relations as necessary to solve them. After putting  $m^2 = 0$ , we have no simple possibility to relate the integral with propagator  $(p - q)^2$  to the integral with higher powers of this propagator (in previous section this connection was given by the differentiation with respect to  $m^2$ ). This leads us to considering whole family

$$B_{abc}(M^2) = \int \frac{d^d p}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} \frac{1}{(p^2 - M^2)^a (q^2 - M^2)^b (p - q)^{2c}} \quad (3.1)$$

By rescaling the momenta, we see that the integral is homogeneous function of the only parameter with dimension of mass that is in the problem, that is

$$B_{abc}(M^2) = (M^2)^{d-a-b-c} B_{abc}(1)$$

So our goal is to determine just the value of constants  $B_{abc}(1)$ . We see that on one hand

$$M^2 \frac{\partial}{\partial M^2} B_{abc} = (d - a - b - c) B_{abc}$$

and on the other hand the partial differentiation with respect to  $M^2$  in (3.1) yields

$$\frac{\partial}{\partial M^2} B_{abc} = a B_{a+1,b,c} + b B_{a,b+1,c}$$

Using again the operator notation for integrals with shifted values of parameters, we have

$$\frac{\partial}{\partial M^2} = \mathbf{A}^+ \mathbf{A} + \mathbf{B}^+ \mathbf{B} \equiv a \mathbf{A}^+ + b \mathbf{B}^+ \quad (3.2)$$

The other set of relations is obtained by applying the integration-by-parts identities. By using the identities that are formally written as

$$\begin{aligned} \frac{\partial}{\partial p^\mu} p^\mu &= d \\ \frac{\partial}{\partial p^\mu} q^\mu &= 0 \end{aligned}$$

we obtain the relations

$$\begin{aligned} 2aM^2 \mathbf{A}^+ &= (d - 2a - c) - c \mathbf{A}^- \mathbf{C}^+ + c \mathbf{B}^- \mathbf{C}^+ \\ 2aM^2 \mathbf{A}^+ &= (c - a) + a \mathbf{A}^+ \mathbf{C}^- - a \mathbf{A}^+ \mathbf{B}^- + c \mathbf{B}^- \mathbf{C}^+ - c \mathbf{A}^- \mathbf{C}^+ \end{aligned}$$

On the left hand side we have written the terms that increase the total sum  $a + b + c$  while the operators on the right hand side keep it at the same value. The difference of these two equations and the equation obtained by interchanging the role of  $a$  and  $b$  which are symmetric are

$$\begin{aligned} 0 &= (a + 2c - d) + a \mathbf{A}^+ \mathbf{C}^- - a \mathbf{A}^+ \mathbf{B}^- \\ 0 &= (b + 2c - d) + b \mathbf{B}^+ \mathbf{C}^- - b \mathbf{A}^- \mathbf{B}^+ \end{aligned}$$

Multiplying the first equation by  $b\mathbf{B}^+$  and the second by  $a\mathbf{C}^+$  and subtracting yields

$$(2b + 2c - d)a\mathbf{A}^+ = (2a + 2c - d)b\mathbf{B}^+$$

That is, since our  $d = 4 - 2\epsilon$  which is not integer (at least for small  $\epsilon$ ), we have for  $b \neq 0$

$$\mathbf{B}^+ = \frac{a(2b + 2c - d)}{b(2a + 2c - d)}\mathbf{A}^+$$

which combined with equation (3.2) yields

$$a\mathbf{A}^+ = \frac{2a + 2c - d}{2a + 2b + 4c - 2d} \frac{\partial}{\partial M^2}$$

Symmetrically

$$b\mathbf{B}^+ = \frac{2b + 2c - d}{2a + 2b + 4c - 2d} \frac{\partial}{\partial M^2}$$

For completeness we will derive formula for  $\mathbf{C}^+$  although we will not need it. Since  $B_{abc}$  is a polynomial, we can use the homogeneity relation to eliminate the differentiation with respect to  $M^2$ :

$$\begin{aligned} \mathbf{A}^+ &= \frac{(2a + 2c - d)(d - a - b - c)}{M^2 a(2a + 2b + 4c - 2d)} \\ \mathbf{B}^+ &= \frac{(2b + 2c - d)(d - a - b - c)}{M^2 b(2a + 2b + 4c - 2d)} \end{aligned}$$

These relations are easily inverted

$$\begin{aligned} \mathbf{A}^- &= \frac{M^2(a - 1)(2a + 2b + 4c - 2d - 2)}{(2a + 2c - d - 2)(d - a - b - c + 1)} \\ \mathbf{B}^- &= \frac{M^2(b - 1)(2a + 2b + 4c - 2d - 2)}{(2b + 2c - d - 2)(d - a - b - c + 1)} \end{aligned}$$

and substituting to the previous equations gives

$$\mathbf{C}^+ = \frac{2(a + b + c + 1 - d)(a + c - \frac{d}{2} - 1)(b + c - \frac{d}{2} - 1)}{M^2(a + b + 2c - d)(2c - d)}$$

Note that these relations strongly resemble the contiguous relations among the hypergeometric functions. This is no coincidence, since as we know, many loop integrals can be expressed in terms of the hypergeometric functions (and we will see the exact  $d$ -dimensional expression for the integrals considered in this section in the next chapter).

These relations can be used to relate all integrals of our family to the products of the one loop integrals (when one of  $a, b, c$  is zero) and one two-loop integral (we choose  $B_{111}$ ) - this is the Master integral of our problem. After the Wick rotation and appropriate shift of the loop momenta, introduction of Feynman parameters and integration similar to the massive case we obtain

$$\begin{aligned} B_{111}(M^2) &= \int \frac{d^d p_E}{(2\pi)^d} \frac{d^d q_E}{(2\pi)^d} \frac{1}{(p_E^2 + M^2)((p_E - q_E)^2 + M^2)q_E^2} \\ &= \frac{\Gamma(3 - d)(M^2)^{d-3}}{(4\pi)^d} \int_0^1 (x(1-x))^{1-\frac{d}{2}} dx \int_0^1 y^{\frac{d}{2}-2} dy \\ &= \frac{M^2 N^2}{(4\pi)^4 \epsilon^2 (1 - \epsilon)(-1 + 2\epsilon)} = -\frac{M^2 N^2}{(4\pi)^4 \epsilon^2} [1 + 3\epsilon + 7\epsilon^2] + O(\epsilon) \end{aligned}$$

In this way we reproduce precisely the result of the previous section, but without any complicated integration. Except for the last expression everything is valid in any number of dimensions, which is another advantage of this derivation. Note that the exact  $d$ -dimensional result for  $B_{abc}$  will be given in Section 4.3.1 of the next chapter. It can be evaluated directly using the Feynman parametrization.



## Chapter 4

# Two-loop QCD corrections, exact

In this chapter we will evaluate the contribution of the leading order QCD corrections to the quark loop exactly for  $d = 4$ , without any restriction on the Higgs boson mass. QCD corrections to  $H \rightarrow \gamma\gamma$  have been first numerically analyzed in the intermediate mass range in [17] by numerical integration of the Feynman parameter integrals. Both [17] and [59, 14] followed in the previous chapter considered only the invariant amplitude below the  $2m_t$  threshold, where it is real. Authors of [42, 29] improved the situation by considering the amplitude in the whole range of Higgs masses. They first evaluated analytically the imaginary part of the amplitude and then numerically integrated the dispersion integral (the explicit expression for the imaginary part of amplitude is given in [29]). Expressions for the invariant amplitude in the form of (very complicated!) one-dimensional integrals were given in [51]. The first explicit result written only in terms of the polylogarithms (without any integration) appeared almost ten years later, in [21]. Results of this article were used later [26] to find an “analytic” expression for other integrals considered in [51]. The method that was used was the “Expansion&Inversion”. The authors made an “ansatz” inspired by results in [21] and used Taylor expansion of integrals in [51] to find the value of free parameters on the ansatz. Note that the order of expansion was around  $z^{100}$  and integers appearing in the numerator and denominator of coefficients were of order  $10^{180}$ .

Many intermediate results and details of calculation of processes  $H \rightarrow \gamma\gamma$  and  $gg \rightarrow H$  were given later in articles [4, 6]. We will proceed similarly to those articles (with the calculation being analogous to the calculation performed in the section 2.4).

### 4.1 Reduction of amplitude to scalar integrals

The relevant diagrams and amplitudes have been shown in Section 3.1. After applying the projection operators to the scalar amplitudes, we want to express all scalars in the numerator of integrand in terms of the propagators. In the one-loop case there were 3 independent scalars and 3 propagators, but in general we can have more scalars than there are propagators, so not all scalars can be eliminated. There are solutions that are widely used, the “auxiliary diagram scheme” and the “shift scheme” [1]. We will use the auxiliary diagram scheme, which in our case amounts to extending the set of propagators by one so that all scalars can be expressed in terms of them. We see that in our case diagrams a-c have only 5 independent propagators and diagrams d-f 6 propagators, while the number of independent scalars is 7. But if we take all 7 propagators that occur in any of diagrams, we can eliminate all the scalars. We choose the propagators

$$\begin{aligned} D_1 &= (k-l)^2 \\ D_2 &= k^2 - m^2 \\ D_3 &= (k-p_1)^2 - m^2 \\ D_4 &= (k+p_2)^2 - m^2 \\ D_5 &= l^2 - m^2 \\ D_6 &= (l-p_1)^2 - m^2 \\ D_7 &= (l+p_2)^2 - m^2 \end{aligned}$$

which gives

$$k^2 = D_2 + m^2$$

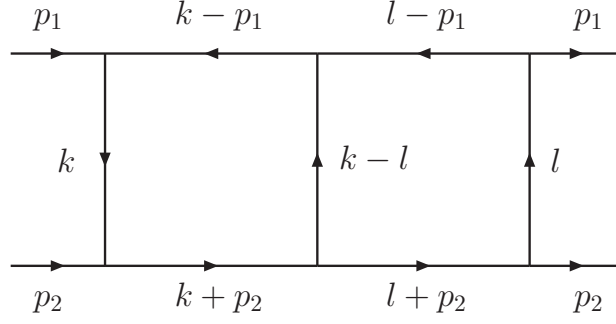


Figure 4.1: Two loop auxiliary diagram containing all possible propagators

$$\begin{aligned}
kp_1 &= -\frac{1}{2}D_3 + \frac{1}{2}D_2 \\
kp_2 &= \frac{1}{2}D_4 - \frac{1}{2}D_2 \\
l^2 &= D_5 + m^2 \\
lp_1 &= -\frac{1}{2}D_6 + \frac{1}{2}D_5 \\
lp_2 &= \frac{1}{2}D_7 - \frac{1}{2}D_5 \\
kl &= -\frac{1}{2}D_1 + \frac{1}{2}D_2 + \frac{1}{2}D_5 + m^2
\end{aligned}$$

The diagram 4.1 (the auxiliary diagram) representing a momentum routing has all of these propagators. After eliminating all the scalars, we have an expression for the amplitude as a linear combination of (about 180) integrals of the form

$$\text{Topo}(n_1, n_2, n_3, n_4, n_5, n_6, n_7) \equiv \int \frac{d^d k}{(2\pi)^d} \frac{d^d l}{(2\pi)^d} \frac{1}{D_1^{n_1} D_2^{n_2} D_3^{n_3} D_4^{n_4} D_5^{n_5} D_6^{n_6} D_7^{n_7}}$$

with  $n_i$  integers (also negative and zero).

## 4.2 IBP and LI identities

The next step will be to use the symmetries of these integrals together with integration-by-parts relations and Lorentz-invariance identities to reduce the number of independent integrals. Lorentz-invariance identity is

$$0 = (-n_3 - n_6 + n_4 + n_7) + n_3 \mathbf{2}^- \mathbf{3}^+ + n_6 \mathbf{5}^- \mathbf{6}^+ - n_4 \mathbf{2}^- \mathbf{4}^+ - n_7 \mathbf{5}^- \mathbf{7}^+$$

and 8 IBP identities are

$$\begin{aligned}
0 &= (d - 2n_2 - n_3 - n_4 - n_1) - 2n_2 m^2 \mathbf{2}^+ - n_3 \mathbf{2}^- \mathbf{3}^+ - 2n_3 m^2 \mathbf{3}^+ - n_4 \mathbf{2}^- \mathbf{4}^+ \\
&\quad - 2m^2 n_4 \mathbf{4}^+ - n_1 \mathbf{2}^- \mathbf{1}^+ + n_1 \mathbf{5}^- \mathbf{1}^+ \\
0 &= (-n_2 + n_1) + n_2 \mathbf{1}^- \mathbf{2}^+ - n_2 \mathbf{5}^- \mathbf{2}^+ - 2m^2 n_2 \mathbf{2}^+ - n_3 \mathbf{2}^- \mathbf{3}^+ + n_3 \mathbf{1}^- \mathbf{3}^+ - n_3 \mathbf{6}^- \mathbf{3}^+ \\
&\quad - 2m^2 n_3 \mathbf{3}^+ - n_4 \mathbf{2}^- \mathbf{4}^+ + n_4 \mathbf{1}^- \mathbf{4}^+ - n_4 \mathbf{7}^- \mathbf{4}^+ - 2m^2 n_4 \mathbf{4}^+ - n_1 \mathbf{2}^- \mathbf{1}^+ + n_1 \mathbf{5}^- \mathbf{1}^+ \\
0 &= (-n_2 + n_3) + n_2 \mathbf{2}^+ \mathbf{3}^- - n_3 \mathbf{2}^- \mathbf{3}^+ - 2n_4 p_1 p_2 \mathbf{4}^+ - n_4 \mathbf{2}^- \mathbf{4}^+ + n_4 \mathbf{3}^- \mathbf{4}^+ \\
&\quad - n_1 \mathbf{2}^- \mathbf{1}^+ + n_1 \mathbf{3}^- \mathbf{1}^+ + n_1 \mathbf{5}^- \mathbf{1}^+ - n_1 \mathbf{6}^- \mathbf{1}^+ \\
0 &= (n_2 - n_4) - n_2 \mathbf{4}^- \mathbf{2}^+ + 2p_1 p_2 n_3 \mathbf{3}^+ + n_3 \mathbf{2}^- \mathbf{3}^+ - n_3 \mathbf{4}^- \mathbf{3}^+ + n_4 \mathbf{2}^- \mathbf{4}^+ \\
&\quad + n_1 \mathbf{2}^- \mathbf{1}^+ - n_1 \mathbf{4}^- \mathbf{1}^+ - n_1 \mathbf{5}^- \mathbf{1}^+ + n_1 \mathbf{7}^- \mathbf{1}^+ \\
0 &= (n_1 - n_5) + n_1 \mathbf{2}^- \mathbf{1}^+ - n_1 \mathbf{5}^- \mathbf{1}^+ - n_5 \mathbf{2}^- \mathbf{5}^+ + n_5 \mathbf{1}^- \mathbf{5}^+ - 2m^2 n_5 \mathbf{5}^+ - 2m^2 n_6 \mathbf{6}^+ \\
&\quad - 2m^2 n_7 \mathbf{7}^+ - n_6 \mathbf{3}^- \mathbf{6}^+ + n_6 \mathbf{1}^- \mathbf{6}^+ - n_6 \mathbf{5}^- \mathbf{6}^+ - n_7 \mathbf{4}^- \mathbf{7}^+ + n_7 \mathbf{1}^- \mathbf{7}^+ - n_7 \mathbf{5}^- \mathbf{7}^+
\end{aligned}$$

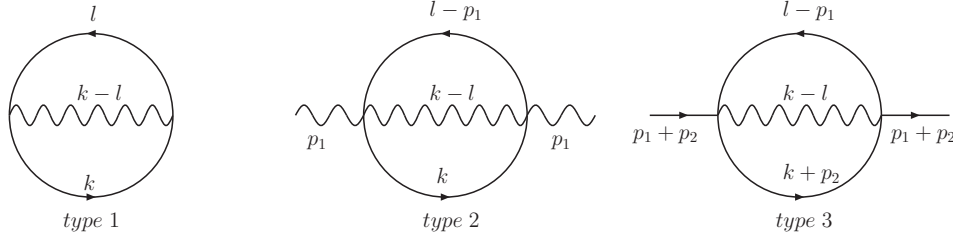


Figure 4.2: Three propagator two-loop topologies

$$\begin{aligned}
0 &= (d - n_1 - 2n_5 - n_6 - n_7) + n_1 \mathbf{2}^- \mathbf{1}^+ - n_1 \mathbf{5}^- \mathbf{1}^+ - 2n_5 m^2 \mathbf{5}^+ - n_6 \mathbf{5}^- \mathbf{6}^+ \\
&\quad - 2n_6 m^2 \mathbf{6}^+ - n_7 \mathbf{5}^- \mathbf{7}^+ - 2n_7 m^2 \mathbf{7}^+ \\
0 &= (-n_5 + n_6) + n_1 \mathbf{2}^- \mathbf{1}^+ - n_1 \mathbf{3}^- \mathbf{1}^+ - n_1 \mathbf{5}^- \mathbf{1}^+ + n_1 \mathbf{6}^- \mathbf{1}^+ + n_5 \mathbf{6}^- \mathbf{5}^+ - n_6 \mathbf{5}^- \mathbf{6}^+ \\
&\quad - 2p_1 p_2 n_7 \mathbf{7}^+ - n_7 \mathbf{5}^- \mathbf{7}^+ + n_7 \mathbf{6}^- \mathbf{7}^+ \\
0 &= (n_5 - n_7) - n_1 \mathbf{2}^- \mathbf{1}^+ + n_1 \mathbf{4}^- \mathbf{1}^+ + n_1 \mathbf{5}^- \mathbf{1}^+ - n_1 \mathbf{7}^- \mathbf{1}^+ - n_5 \mathbf{7}^- \mathbf{5}^+ + 2p_1 p_2 n_6 \mathbf{6}^+ \\
&\quad + n_6 \mathbf{5}^- \mathbf{6}^+ - n_6 \mathbf{7}^- \mathbf{6}^+ + n_7 \mathbf{5}^- \mathbf{7}^+
\end{aligned}$$

We have also the discrete symmetries like the symmetry  $k \leftrightarrow l$

$$\text{Topo}(n_1, n_2, n_3, n_4, n_5, n_6, n_7) = \text{Topo}(n_1, n_5, n_6, n_7, n_2, n_3, n_4)$$

and the symmetry  $p_1 \leftrightarrow p_2$

$$\text{Topo}(n_1, n_2, n_3, n_4, n_5, n_6, n_7) = \text{Topo}(n_1, n_2, n_4, n_3, n_5, n_7, n_6)$$

and also simplifications to one-loop integrals

$$\text{Topo}(0, n_2, n_3, n_4, n_5, n_6, n_7) = \text{Topo}(n_2, n_3, n_4) \text{Topo}(n_5, n_6, n_7)$$

(which can be easily generalized for any  $n_1 \leq 0$ ) or

$$\text{Topo}(n_1, n_2, n_3, n_4, 0, 0, 0) = \text{Topo}(n_2, n_3, n_4) \text{Topo}(n_1, 0, 0)$$

(with generalization to the case  $n_5 \leq 0$  and  $l_6 \leq 0$  and  $l_7 \leq 0$ ) and similarly for the first loop momentum. Finally, when  $p_1$  or  $p_2$  does not occur in the integral, we have relations like

$$\text{Topo}(n_1, n_2, n_3, 0, n_5, n_6, 0) = \text{Topo}(n_1, n_2 + n_3, 0, 0, n_5 + n_6, 0, 0)$$

and generalization of this can also be written down for any case when  $n_4 \leq 0$  and  $n_7 \leq 0$  and similarly for  $p_2$ .

Although all these equations form a system of linear equation, its general solution is probably very complicated. We will again use the Laporta's method and solve these equations for a large number of specific values of  $n_i$  using the computer program (AIR). As a result of reduction, we obtain all necessary scalar integrals expressed in terms of 3 one-loop integrals and 9 genuine two-loop integrals (which are not products of one loop integrals). As in the case of the one-loop integrals, we can divide the scalar integrals  $\text{Topo}(n_1, n_2, n_3, n_4, n_5, n_6, n_7)$  with non-negative  $n_i$ 's into groups according to which propagators are present (topologies).

We introduce a simplified notation for describing groups of integrals with the same set of non-zero propagators. Letter  $A$  corresponds to propagators  $D_2$  and  $D_5$ , letter  $B$  to propagators  $D_3$  and  $D_6$  and letter  $C$  to  $D_4$  and  $D_7$ . Propagator  $D_1$  is always present for genuine two-loop integrals. Finally, propagators  $D_2$ ,  $D_3$  and  $D_4$  correspond to one loop momentum and  $D_5$ ,  $D_6$  and  $D_7$  to the one, so we write them as one word. For example, symbol  $(AC, BC)$  is a shorthand for integrals which have positive power of propagators  $(D_1, D_2, D_3, D_6, D_7)$  or  $(D_1, D_2, D_4, D_6, D_7)$  or  $(D_1, D_3, D_4, D_5, D_6)$  or  $(D_1, D_3, D_4, D_5, D_6)$ . These four groups are connected by a discrete symmetry transformation and symbol  $(AC, BC)$  describes whole family of integrals of this type. After introducing this notation, we can list all genuine two-loop topologies:

- 3 propagator topologies (Figure 4.2)

- type 1 (A,A)(B,B) -  $z$ -independent, reducible to its subtopologies
- type 2 (A,B) -  $z$ -independent, reducible to its subtopologies
- type 3 (B,C) - 2 Master Integrals
- 4 propagator topologies (Figure 4.3)
  - type 1 (A,AB)(B,AB) -  $z$ -independent, reducible to its subtopologies
  - type 2 (B,BC) - reducible to its subtopologies
  - type 3 (B,AC) - 1 Master Integral
  - type 4 (A,BC) - 3 Master Integrals
- 5 propagator topologies (Figure 4.4)
  - type 1 (AB,AB) -  $z$ -independent, reducible to its subtopologies
  - type 2 (BC,BC) - reducible to its subtopologies
  - type 3 (A,ABC) - reducible to its subtopologies
  - type 4 (B,ABC) - reducible to its subtopologies
  - type 5 (AB,AC) - 1 Master Integral
  - type 6 (AB,BC) - 2 Master Integrals
- 6 propagator topologies (Figure 4.5)
  - type 1 (BC,ABC) - reducible to its subtopologies
  - type 2 (AB,ABC) - reducible to its subtopologies
- 7 propagator topologies (Figure 4.1)
  - type 1 (ABC,ABC) - auxiliary diagram

Note that (A,A) and (B,B) are related only through a shift of the loop momentum, so they do not in fact represent a different topologies. The same holds true for (A,AB) and (B,AB). On the other hand, (A,A) and (A,B) represent a different momentum flow, so we can consider them as two different topologies, although the value of integrals which they represent is the same, because  $p_1$  is light-like momentum and value of integral depends only on its square, so the result is the same as if there were no incoming momentum  $p_1$ .

### 4.3 Differential equations for Master Integrals

Now all that remains to do in order to calculate the amplitude is to evaluate the 9 Master Integrals that are left after the reduction. As in the one-loop case, we will follow the differential equations approach. The differential equation satisfied by any  $\text{Topo}(n_1, n_2, n_3, n_4, n_5, n_6, n_7)$  is symbolically

$$\begin{aligned}
z \frac{\partial}{\partial z} &= -n_3 - n_6 + n_3 \mathbf{2}^- \mathbf{3}^+ + n_6 \mathbf{5}^- \mathbf{6}^+ \\
&= -n_4 - n_7 + n_4 \mathbf{2}^- \mathbf{4}^+ + n_7 \mathbf{5}^- \mathbf{7}^+
\end{aligned} \tag{4.1}$$

(the equivalence of these two expression is exactly the Lorentz-invariance identity). Applying this formal identity to the Master Integrals and expressing integrals on the right-hand side in terms of Master Integrals and already calculated one-loop Master Integrals gives inhomogeneous system of first order ordinary differential equations. General solution of such system can be very complicated, but fortunately in our case the system is block diagonal with respect to various topologies. At this point it is important to choose Master Integrals (the base of topology) in such way, that the system of equations becomes as simple as possible. Optimal choice would be to make the system upper triangular. In next sections, we will see that this is in fact possible.

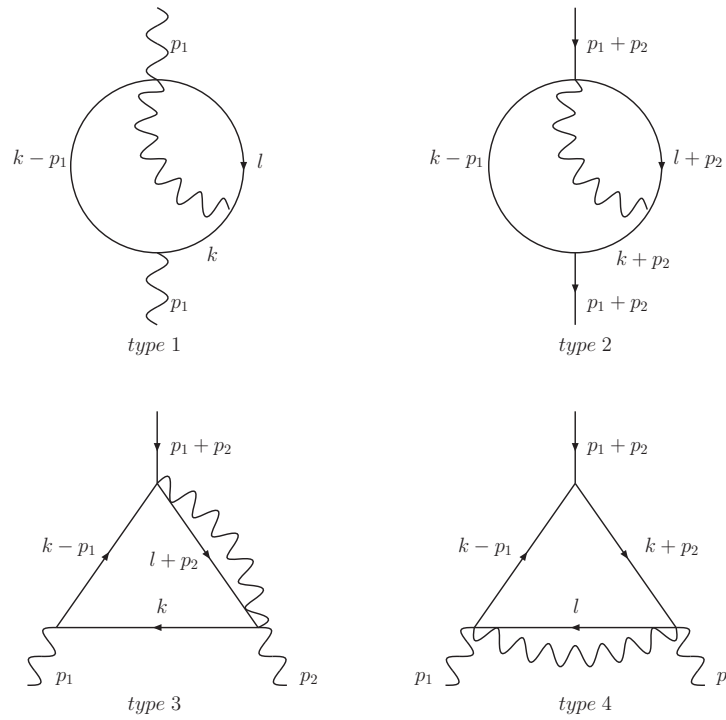


Figure 4.3: Four propagator two-loop topologies

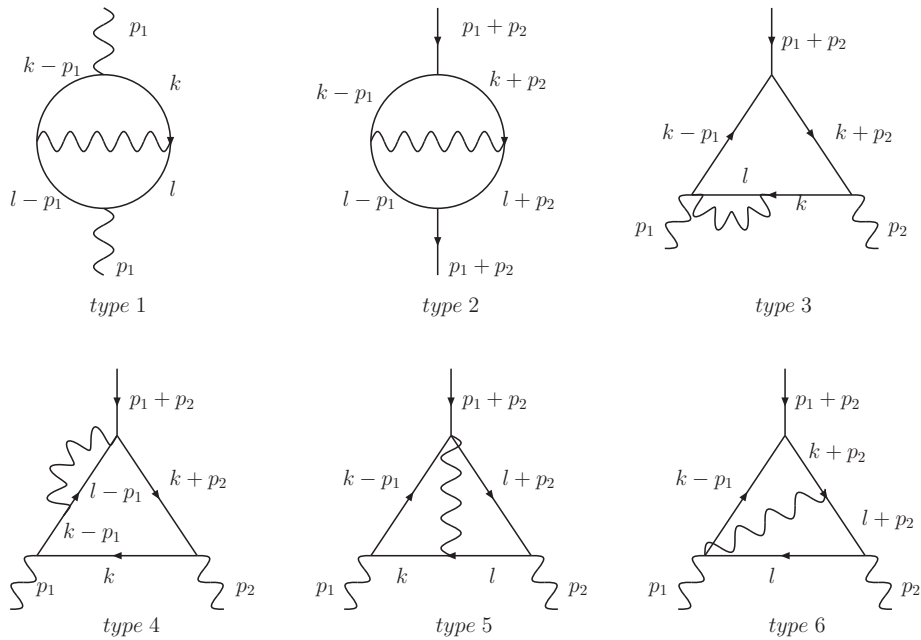


Figure 4.4: Five propagator two-loop topologies

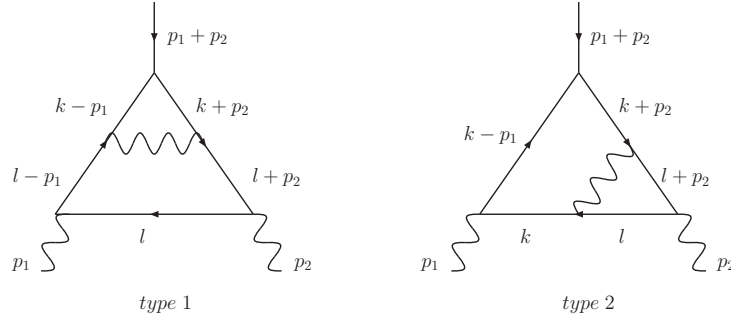


Figure 4.5: Six propagator two-loop topologies

### 4.3.1 Three-propagator, type 3 topology

As already mentioned in the list of topologies, this topology has 2 Master Integrals. Before turning to differential equations, we list the exact result valid for any positive power of  $a$ ,  $b$  and  $c$  and any dimension  $d$ :

$$\text{Topo}(c, 0, a, 0, 0, 0, b) = \frac{(-1)^{a+b+c+1} \Gamma(a+b+c-d) \Gamma\left(\frac{d}{2}-c\right) B\left(a+c-\frac{d}{2}, b+c-\frac{d}{2}\right)}{\Gamma(a) \Gamma(b) \Gamma\left(\frac{d}{2}\right) (m^2)^{a+b+c-d} (4\pi)^d} \cdot {}_4F_3\left(\begin{matrix} a+b+c-d, c, a+c-\frac{d}{2}, b+c-\frac{d}{2} \\ \frac{d}{2}, \frac{a}{2}+\frac{b}{2}+c-\frac{d}{2}, \frac{a}{2}+\frac{b}{2}+c-\frac{d}{2}+\frac{1}{2} \end{matrix}; z\right) \quad (4.2)$$

This can be derived by application of Feynman parameters and expressing the resulting integrals in terms of the hypergeometric functions. Although this formula is not very illuminating, it can be used to check if we have correct results for  $d \rightarrow 4$  expressions. In addition, the singularities of this function are clearly factorized out, unlike in the integral representation form. In the case  $a = b = c = 1$  this formula simplifies to

$$\text{Topo}(1, 0, 1, 0, 0, 0, 1) = -\frac{N^2 m^2 {}_3F_2\left(1, \epsilon, 2\epsilon-1; 2-\epsilon, \epsilon+\frac{1}{2}; z\right)}{(4\pi)^4 \epsilon^2 (1-3\epsilon+2\epsilon^2)}$$

The formula to which formula (4.2) reduces for  $z = 0$  (the right hand side remains the same except for the hypergeometric functions which becomes 1) is useful for two reasons. First reason is that this is precisely the massless vacuum bubble integral evaluated in the Section 3.4, which was needed in order to evaluate the integrals coming from the small momentum expansion. We evaluated it in two ways, one way was using the complicated integration of massive vacuum bubble integral according to [27, 54], and the second way was the application of integration-by-parts identities and a direct evaluation of the massless bubble integral using the Feynman parametrization.

The second reason why the formula (4.2) is useful is the fact that after solving the differential equations for Master integrals we need to fix the integration constant. Usually this can be done simply by requiring the finiteness of the result at  $z = 0$ , because diagrams that we will evaluate do not have singularities at  $z = 0$ , but the solutions of homogeneous equations are in general singular at this point. But the “correct” way for fixing the integration constant is by comparison of the solution with an unknown integration constant with the original Master integral at one point. The simplest choice is  $z = 0$  because for this point the relation

$$\text{Topo}(n_1, n_2, n_3, n_4, n_5, n_6, n_7; z = 0) = \text{Topo}(n_1, n_2 + n_3 + n_4, 0, 0, n_5 + n_6 + n_7)$$

holds as been shown in the previous section.

Let us proceed to  $\epsilon$ -expansion using the differential equations. Since there are two Master integrals in this topology, the situation is more complicated then in the one-loop integrals considered in Section 2.4. The set of two differential equations has the general form

$$\begin{aligned} f_1'(\epsilon, u) &= a_{11}(\epsilon, u) f_1(\epsilon, u) + a_{12}(\epsilon, u) f_2(\epsilon, u) + b_1(\epsilon, u) \\ f_2'(\epsilon, u) &= a_{21}(\epsilon, u) f_1(\epsilon, u) + a_{22}(\epsilon, u) f_2(\epsilon, u) + b_2(\epsilon, u) \end{aligned}$$

The system of first order ordinary linear differential equations is in general difficult to solve, so we will try to solve it by expanding it in  $\epsilon$ . Functions  $a_{ij}$  and  $b_i$  in all equations that we will encounter are polynomials in

$\epsilon$  and  $u$  and thus have a convergent Laurent series expansion in  $\epsilon$  with finite number of terms in the principal part. We expect the same behaviour for solutions  $f_j$ , with principal part consisting potentially of simple and double poles in  $\epsilon$  (because every loop integration can produce only a pole  $\frac{1}{\epsilon}$ ). Under these assumptions, we can write

$$\begin{aligned} f_i(\epsilon, u) &= \sum_{k=-2}^{\infty} \epsilon^k f_i^k(u) \\ a_{ij}(\epsilon, u) &= \sum_{k=o_{ij}}^{\infty} \epsilon^k a_{ij}^k(u) \\ b_i(\epsilon, u) &= \sum_{k=o_i}^{\infty} \epsilon^k b_i^k(u) \end{aligned}$$

(by this we mean that the leading-order coefficient of  $a_{ij}$  and  $b_i$  is non-zero for the definition of  $o_{ij}$  and  $o_i$  to make sense) and the order  $\epsilon^n$  of equations becomes

$$\begin{aligned} f_1^{n'}(u) &= \sum_m a_{11}^m(u) f_1^{n-m}(u) + \sum_m a_{12}^m(u) f_2^{n-m}(u) + \sum_m b_1^m(u) \\ f_2^{n'}(u) &= \sum_m a_{21}^m(u) f_1^{n-m}(u) + \sum_m a_{22}^m(u) f_2^{n-m}(u) + \sum_m b_2^m(u) \end{aligned}$$

Suppose that we know the solution  $f_i^m$  for  $m < n$  (as explained, this is true for example for  $n = -2$ ). We see that if we satisfy all the conditions

$$\begin{aligned} o_{11} &\geq 0 \\ o_{22} &\geq 0 \\ o_{12} &\geq 0 \\ o_{21} &\geq 1 \end{aligned}$$

then the system of equations becomes triangular and we may immediately solve it using a quadrature (note that this does not at all depend on the value of  $o_i$ ). The same holds true if we exchange the role of  $o_{12}$  and  $o_{21}$ . But this is only a sufficient condition. One can check that multiplying the function  $f_1$  by  $\epsilon^k$  is equivalent to multiplying the off-diagonal terms  $a_{12}$  by  $\epsilon^{-k}$  and  $a_{21}$  by  $\epsilon^k$  (and an unimportant change in  $b_i$ ), so we can modify the conditions to

$$\begin{aligned} o_{11} &\geq 0 \\ o_{22} &\geq 0 \\ o_{12} + o_{21} &\geq 1 \end{aligned} \tag{4.3}$$

Now comes the question of choice of Master integrals. We can try to make a choice of them in such way that the conditions (4.3) are satisfied. Using the equations relating the various integrals  $\text{Topo}(c, 0, a, 0, 0, b)$  for this topology (which we know as a result of the reduction), we can write the system of the differential equations for any pair of two independent integrals of this topology. We have restricted to integrals satisfying

$$\begin{aligned} \mathcal{N}_+ &< 3 \\ \mathcal{N}_- &= 0 \end{aligned}$$

and found these possible choices of Master integrals such that the system satisfies the conditions (4.3):

- one of integrals is  $\text{Topo}(1, 0, 1, 0, 0, 0, 1)$ ,  $\text{Topo}(1, 0, 2, 0, 0, 0, 1)$  or  $\text{Topo}(1, 0, 2, 0, 0, 0, 2)$  and the other one is  $\text{Topo}(2, 0, 1, 0, 0, 0, 1)$ ,  $\text{Topo}(2, 0, 2, 0, 0, 0, 1)$  or  $\text{Topo}(1, 0, 3, 0, 0, 0, 1)$  (this gives 9 possible choices of Master integrals)
- integrals  $\text{Topo}(2, 0, 2, 0, 0, 0, 1)$  and  $\text{Topo}(1, 0, 2, 0, 0, 0, 2)$  (the ‘‘exceptional’’ choice of Master integrals)

With the exceptional choice the system of equations becomes not only triangular, but also relatively simple and this choice was employed [6]. Authors of [10] have chosen a different set of MI's corresponding to  $\text{Topo}(1, 0, 1, 0, 0, 0, 1)$  and (one half of) the combination

$$\text{Topo}(1, 0, 1, 0, 0, -1, 1) - \text{Topo}(1, 0, 0) \text{Topo}(1, 0, 0)$$

Note that adding to the Master integral a combination of integrals coming from the subtopology (which we already know) should not change the homogeneous differential equation (this holds if the reduction to Master integrals is complete, in a sense that no new non-trivial relations come up from the fact that we may combine the equation (4.1) with the results of reduction, i.e. choose an integral, apply the formula (4.1) and express result in terms of Master integrals, or first express the integral in terms of Master integrals, differentiate the expression with help of the formula (4.1) and express results in terms of Master integrals). But that this is true has not been to our knowledge proved yet.

For the choice of [10] the system of differential equations is relatively simple, but it is not triangular. It is possible to solve such system by transforming it to a second order ordinary linear differential equation, but we will not do that and instead we choose the set

$$\begin{aligned} \text{Topo}(1, 0, 2, 0, 0, 0, 2) &\equiv \left( \frac{iN}{(4\pi)^2} \right)^2 \frac{1}{m^2} f_1(\epsilon, u) \equiv \left( \frac{iN}{(4\pi)^2} \right)^2 \frac{1}{m^2} \sum_{j=0}^{\infty} \epsilon^j f_1^j(u) \\ \text{Topo}(2, 0, 2, 0, 0, 0, 1) &\equiv \left( \frac{iN}{(4\pi)^2} \right)^2 \frac{1}{m^2} f_2(\epsilon, u) \equiv \left( \frac{iN}{(4\pi)^2} \right)^2 \frac{1}{m^2} \sum_{j=-1}^{\infty} \epsilon^j f_2^j(u) \end{aligned}$$

In terms of these Master integrals the system of differential equations reads

$$\begin{aligned} f_1'(u) &= \frac{1+u}{u(1-u)} [f_1(u) + 2\epsilon f_2(u)] \\ f_2'(u) &= \frac{1}{u(1-u^2)} [-(u + \epsilon(1-4u+u^2))f_1(u) + (1+u^2 + 3\epsilon(1-u)^2)f_2(u)] - \frac{1}{\epsilon(1-u^2)} \end{aligned}$$

Its solution proceeds in the same way as we solved the one-loop integrals of Section 2.4. The result:

$$\begin{aligned} f_2^{-1}(u) &= \frac{-u}{1-u^2} H_0(u) \\ f_1^0(u) &= \frac{-2u}{(1-u)^2} H_{0,0}(u) \\ f_2^0(u) &= \frac{-u}{1-u^2} \left[ -\zeta(2) - \frac{2}{1-u} H_{0,0}(u) + 5H_{0,0}(u) + 2H_{1,0}(u) - 6H_{-1,0}(u) \right] \\ f_1^1(u) &= \frac{u}{(1-u)^2} [6\zeta(3) + 2\zeta(2)H_0(u) - 6H_{0,0,0}(u) + 4H_{1,0,0}(u) - 4H_{0,1,0}(u) + 12H_{0,-1,0}(u)] \\ f_2^1(u) &= \frac{-u}{1-u^2} \left[ -14\zeta(2) + \frac{1}{1-u} \left( 6\zeta(3) - 6H_{0,0,0}(u) + 4H_{1,0,0}(u) - 4H_{0,1,0}(u) \right. \right. \\ &\quad \left. \left. + 12H_{0,-1,0}(u) + 2\zeta(2)H_0(u) \right) + 13H_{0,0,0}(u) + 10H_{0,1,0}(u) - 30H_{0,-1,0}(u) \right. \\ &\quad \left. - 24H_{-1,0,0}(u) + 6H_{1,0,0}(u) + 4H_{1,1,0}(u) - 12H_{1,-1,0}(u) + 36H_{-1,-1,0}(u) \right. \\ &\quad \left. - 12H_{-1,1,0}(u) - 5\zeta(2)H_0(u) + 6\zeta(2)H_{-1}(u) - 2\zeta(2)H_1(u) \right] \\ f_1^2(u) &= \frac{u}{(1-u)^2} \left[ \frac{13}{2}\zeta(4) - 14H_{0,0,0,0}(u) - 20H_{0,1,0,0}(u) - 12H_{0,0,1,0}(u) \right. \\ &\quad \left. + 36H_{0,0,-1,0}(u) + 12H_{1,0,0,0}(u) - 8H_{1,1,0,0}(u) + 8H_{1,0,1,0}(u) \right. \\ &\quad \left. - 24H_{1,0,-1,0}(u) + 48H_{0,-1,0,0}(u) - 8H_{0,1,1,0}(u) + 24H_{0,1,-1,0}(u) \right. \\ &\quad \left. - 72H_{0,-1,-1,0}(u) + 24H_{0,-1,1,0}(u) + 6\zeta(2)H_{0,0}(u) - 12\zeta(2)H_{0,-1}(u) \right. \\ &\quad \left. + 4\zeta(2)H_{0,1}(u) - 4\zeta(2)H_{1,0}(u) - 12\zeta(3)H_1(u) + 16\zeta(3)H_0(u) \right] \end{aligned}$$

$$\begin{aligned}
f_2^2(u) = & \frac{-u}{1-u^2} \left[ \frac{1}{1-u} \left( -14H_{0,0,0,0}(u) - 20H_{0,1,0,0}(u) - 12H_{0,0,1,0}(u) \right. \right. \\
& + 12H_{1,0,0,0}(u) - 8H_{0,1,1,0}(u) - 8H_{1,1,0,0}(u) + 8H_{1,0,1,0}(u) \\
& + 36H_{0,0,-1,0}(u) + 48H_{0,-1,0,0}(u) + 24H_{0,1,-1,0}(u) + 24H_{0,-1,1,0}(u) \\
& - 24H_{1,0,-1,0}(u) - 72H_{0,-1,-1,0}(u) + 6\zeta(2)H_{0,0}(u) - 12\zeta(2)H_{0,-1}(u) \\
& + 4\zeta(2)H_{0,1}(u) - 4\zeta(2)H_{1,0}(u) - 12\zeta(3)H_1(u) + 16\zeta(3)H_0(u) \\
& \left. \left. + \frac{13}{2}\zeta(4) \right) + 29H_{0,0,0,0}(u) + 26H_{0,0,1,0}(u) + 54H_{0,1,0,0}(u) \right. \\
& + 20H_{0,1,1,0}(u) + 14H_{1,0,0,0}(u) + 12H_{1,0,1,0}(u) + 20H_{1,1,0,0}(u) \\
& + 8H_{1,1,1,0}(u) - 78H_{0,0,-1,0}(u) - 120H_{0,-1,0,0}(u) - 60H_{-1,0,0,0}(u) \\
& - 60H_{0,1,-1,0}(u) - 60H_{0,-1,1,0}(u) - 36H_{1,0,-1,0}(u) - 48H_{1,-1,0,0}(u) \\
& - 48H_{-1,0,1,0}(u) - 48H_{-1,1,0,0}(u) - 24H_{1,1,-1,0}(u) - 24H_{1,-1,1,0}(u) \\
& - 24H_{-1,1,1,0}(u) + 180H_{0,-1,-1,0}(u) + 72H_{1,-1,-1,0}(u) \\
& + 144H_{-1,0,-1,0}(u) + 72H_{-1,1,-1,0}(u) + 144H_{-1,-1,0,0}(u) \\
& + 72H_{-1,-1,1,0}(u) - 216H_{-1,-1,-1,0}(u) - 13\zeta(2)H_{0,0}(u) \\
& + 30\zeta(2)H_{0,-1}(u) - 10\zeta(2)H_{0,1}(u) - 6\zeta(2)H_{1,0}(u) + 12\zeta(2)H_{1,-1}(u) \\
& - 4\zeta(2)H_{1,1}(u) + 24\zeta(2)H_{-1,0}(u) - 36\zeta(2)H_{-1,-1}(u) \\
& + 12\zeta(2)H_{-1,1}(u) - 34\zeta(3)H_0(u) - 16\zeta(3)H_1(u) \\
& \left. \left. + 66\zeta(3)H_{-1}(u) - \frac{61}{4}\zeta(4) \right] \right]
\end{aligned}$$

### 4.3.2 Four-propagator, type 3 topology

Since this topology has only one Master integral, we can choose

$$\text{Topo}(1, 1, 0, 1, 0, 1, 0) \equiv \left( \frac{iN}{(4\pi)^2} \right)^2 f_3(\epsilon, u) \equiv \left( \frac{iN}{(4\pi)^2} \right)^2 \sum_{j=-2}^{\infty} \epsilon^j f_3^j(u)$$

The solution can be almost immediately written down

$$\begin{aligned}
f_3^{-2} &= \frac{1}{2} \\
f_3^{-1} &= \frac{1}{2} \\
f_3^0 &= -\frac{5}{2} - H_{0,0}(u) + \frac{u}{(1-u)^2} \left( 2H_{0,0,0,0}(u) + 4H_{1,0,0,0}(u) - 4\zeta(3) \right) - 2\frac{1+u}{1-u} H_0(u) \\
f_3^1 &= -\frac{35}{2} + 6H_{0,-1,0}(u) - 2H_{0,1,0}(u) + \zeta(2)H_0(u) + \frac{u}{(1-u)^2} \left( 4\zeta(4) - 12H_{0,0,-1,0}(u) \right. \\
& + 4H_{0,0,1,0}(u) - 24H_{1,0,-1,0}(u) + 8H_{1,0,1,0}(u) - 4H_{0,1,0,0}(u) + 6H_{0,0,0,0}(u) \\
& + 12H_{1,0,0,0}(u) - 8H_{1,1,0,0}(u) - 12\zeta(3)H_1(u) - 2\zeta(2)H_{0,0}(u) - 4\zeta(2)H_{1,0}(u) \\
& \left. - 6\zeta(3)H_0(u) \right) + \frac{1+u}{1-u} \left( 2\zeta(2) + 12H_{-1,0}(u) - 4H_{1,0}(u) - 12H_0(u) \right) \\
& + \frac{3u^2 - 10u + 3}{(1-u)^2} \zeta(3) + \frac{9u^2 + 2u - 7}{(1-u)^2} H_{0,0}(u) - \frac{3u^2 - 8u + 3}{(1-u)^2} H_{0,0,0,0}(u) \\
& + 2\frac{1+u^2}{(1-u)^2} H_{1,0,0,0}(u)
\end{aligned}$$

### 4.3.3 Four-propagator, type 4 topology

This topology is the most complicated topology encountered, because it has three Master integrals. But we will see that after choosing a proper set of Master integrals the system of differential equations will be triangular to

order  $O(\epsilon^0)$ , so the solution is not more difficult than in the case of topologies with one master integral. Note that in the literature [4, 6, 11] another set of Master integrals is considered,

- $\text{Topo}(1, 0, 1, 1, 1, 0, 0)$ ,
- $\frac{1}{2} \text{Topo}(1, -1, 1, 1, 1, 0, 0) - \frac{1}{2} \text{Topo}(1, 0, 0) \text{Topo}(0, 1, 1)$
- $\text{Topo}(1, 0, 1, 1, 3, 0, 0)$ ,

but for this set the system of differential equations does not become triangular, it only splits to order  $O(\epsilon^0)$  into 2 + 1 equations. The remaining system of 2 differential equations is solved by transforming it into a one second-order ordinary linear differential equation which is then solved.

Our choice of Master integrals

$$\begin{aligned} \text{Topo}(1, 1, 0, 0, 0, 1, 1) &\equiv \left( \frac{iN}{(4\pi)^2} \right)^2 f_4(\epsilon, u) \equiv \left( \frac{iN}{(4\pi)^2} \right)^2 \sum_{j=-2}^{\infty} \epsilon^j f_4^j(u) \\ \text{Topo}(1, 1, 0, 0, 0, 1, 2) &\equiv \left( \frac{iN}{(4\pi)^2} \right)^2 \frac{1}{m^2} f_5(\epsilon, u) \equiv \left( \frac{iN}{(4\pi)^2} \right)^2 \frac{1}{m^2} \sum_{j=-1}^{\infty} \epsilon^j f_5^j(u) \\ \text{Topo}(1, 2, 0, 0, 0, 1, 1) &\equiv \left( \frac{iN}{(4\pi)^2} \right)^2 \frac{1}{m^2} f_6(\epsilon, u) \equiv \left( \frac{iN}{(4\pi)^2} \right)^2 \frac{1}{m^2} \sum_{j=0}^{\infty} \epsilon^j f_6^j(u) \end{aligned}$$

leads to the system of the differential equations

$$\begin{aligned} f_4' &= \frac{2\epsilon(1+u)}{u(1-u)} f_4 + \frac{2(1+u)}{u(1-u)} f_5 + \frac{1+u}{u(1-u)} f_6 \\ f_5' &= \frac{2\epsilon(1-3\epsilon)}{1-u^2} f_4 + \frac{1+u^2-4\epsilon u}{u(1-u^2)} f_5 - \frac{4\epsilon}{1-u^2} f_6 \\ &\quad + \frac{2u}{\epsilon^2(1-u)(1+u)^3} - \frac{2u+\epsilon(1-u^2)}{\epsilon(1-u)(1+u)^3} f_B + \frac{u}{\epsilon(1+u)^2} f_B' \\ f_6' &= \frac{(1-2\epsilon)(1+u)}{u(1-u)} f_5 + \frac{1+u}{u(1-u)} f_6 + \frac{1}{\epsilon^2(1-u^2)} - \frac{1-2\epsilon}{\epsilon(1-u^2)} f_B \end{aligned}$$

which is easily solved order by order in  $\epsilon$  because it is diagonal to order  $O(\epsilon^0)$ . At each order, we solve it for  $f_5$  first, then use it to find  $f_6$  and finally we use both of them to find a solution for  $f_4$ . Then we can go on to next order in  $\epsilon$ . The results are (although they may look difficult, they were found just by an application of procedure described in Section 2.4.4):

$$\begin{aligned} f_4^{-2}(u) &= \frac{1}{2} \\ f_4^{-1}(u) &= \frac{5}{2} + \frac{1+u}{1-u} H_0(u) \\ f_4^0(u) &= \frac{19}{2} + \frac{u}{(1-u)^2} \left( 4\zeta(3) + 2H_{0,1,0}(u) + H_{0,0,0}(u) + 2\zeta(2)H_0(u) \right) \\ &\quad + \frac{1+u}{1-u} \left( 5H_0(u) - 2H_{-1,0}(u) + H_{1,0}(u) \right) + \frac{u+2}{1-u} H_{0,0}(u) \\ f_4^1(u) &= \frac{65}{2} - \frac{u^2-9u+4}{(1-u)^2} \zeta(3) + \frac{u}{(1-u)^2} \left( -11\zeta(4) + 2H_{0,0,1,0}(u) - 2H_{0,-1,0,0}(u) \right. \\ &\quad - 4H_{0,-1,1,0}(u) - 2H_{0,0,-1,0}(u) - 4H_{1,0,1,0}(u) - 4H_{0,1,-1,0}(u) + 4H_{0,1,0,0}(u) \\ &\quad + 4H_{0,1,1,0}(u) + 3H_{0,0,0,0}(u) - 2H_{1,0,0,0}(u) - 4\zeta(2)H_{0,-1}(u) + 2\zeta(2)H_{0,1}(u) \\ &\quad \left. - 3\zeta(3)H_0(u) - 8\zeta(3)H_1(u) + \zeta(2)H_{0,0}(u) - 4\zeta(2)H_{1,0}(u) \right) \\ &\quad + \frac{1+u}{1-u} \left( -10H_{-1,0}(u) + 4H_{-1,-1,0}(u) - 3H_{-1,0,0}(u) - 2H_{-1,1,0}(u) \right) \end{aligned}$$

$$\begin{aligned}
& -2H_{1,-1,0}(u) + 2H_{1,1,0}(u) + 19H_0(u) + \zeta(2)H_1(u) + 5H_{1,0}(u) \Big) \\
& + \frac{u+3}{1-u}H_{1,0,0}(u) + \frac{u+2}{1-u} \left( -2H_{0,-1,0}(u) + 5H_{0,0}(u) \right) \\
& - 2\frac{u^2-u-1}{(1-u)^2}H_{0,1,0}(u) - \frac{2u^2+u-4}{(1-u)^2}H_{0,0,0}(u) - \frac{u(u-3)}{(1-u)^2}\zeta(2)H_0(u)
\end{aligned}$$

$$f_5^{-1}(u) = \frac{u}{1-u^2}H_0(u)$$

$$f_5^0(u) = \frac{u}{1-u^2} \left( \zeta(2) + 2H_0(u) - 2H_{-1,0}(u) + 2H_{0,0}(u) + 2H_{1,0}(u) \right)$$

$$\begin{aligned}
f_5^1(u) = & \frac{u}{1-u^2} \left( 2\zeta(2) + 4H_{1,1,0}(u) + 4H_{1,0,0}(u) - 4H_{1,-1,0}(u) - 4H_{-1,1,0}(u) \right. \\
& - 4H_{-1,0,0}(u) + 4H_{-1,-1,0}(u) + 4H_{1,0}(u) + 4H_{0,0}(u) - 4H_{-1,0}(u) \\
& \left. - 4H_{0,-1,0}(u) + 2\zeta(2)H_1(u) - 2\zeta(2)H_{-1}(u) + 4H_0(u) \right) - \frac{u(3u+5)}{(1-u)^2(1+u)}\zeta(3) \\
& - \frac{2u(3u-2)}{(1-u)^2(1+u)}H_{0,0,0}(u) - \frac{2u(3u-1)}{(1-u)^2(1+u)}H_{0,1,0}(u) - \frac{4u^2}{(1-u)^2(1+u)}\zeta(2)H_0(u)
\end{aligned}$$

$$\begin{aligned}
f_5^2(u) = & \frac{u}{1-u^2} \left( 4\zeta(2) + 6\zeta(3) - \frac{47}{2}\zeta(4) + 8H_{1,1,1,0}(u) + 8H_{1,1,0,0}(u) - 8H_{1,1,-1,0}(u) - 8H_{1,-1,1,0}(u) \right. \\
& - 8H_{1,-1,0,0}(u) + 8H_{1,-1,-1,0}(u) - 8H_{-1,1,1,0}(u) - 8H_{-1,1,0,0}(u) + 8H_{-1,1,-1,0}(u) \\
& - 10H_{-1,0,0,0}(u) + 8H_{-1,-1,1,0}(u) + 8H_{-1,-1,0,0}(u) - 8H_{-1,-1,-1,0}(u) \\
& + 8H_{1,1,0}(u) + 8H_{1,0,0}(u) - 8H_{1,-1,0}(u) - 8H_{1,0,-1,0}(u) - 8H_{-1,0,1,0}(u) \\
& - 8H_{-1,1,0} - 8H_{-1,0,0}(0) + 8H_{-1,-1,0}(u) + 8H_{-1,0,-1,0}(u) + 8H_{0,-1,-1,0}(u) \\
& + 4\zeta(2)H_{1,1}(u) - 4\zeta(2)H_{1,-1}(u) - 4\zeta(2)H_{-1,1}(u) - 8H_{-1,0}(u) - 4\zeta(2)H_{-1,0}(u) \\
& - 8H_{0,-1,0}(u) + 4\zeta(2)H_{-1,-1}(u) - 4\zeta(2)H_{-1}(u) + 2\zeta(3)H_{-1}(u) + 8H_{0,0}(u) \\
& \left. + 8H_{1,0}(u) + 4\zeta(2)H_1(u) + 8H_0(u) \right) + \frac{u}{(1+u)(1-u)^2} \left( 22\zeta(4) - 16\zeta(3) \right. \\
& - 4(2u-3)H_{1,0,0,0}(u) - 2(7u-4)H_{0,0,0,0}(u) - 4(3u-1)H_{0,1,1,0}(u) \\
& - 2(7u-3)H_{0,1,0,0}(u) + 4(3u-1)H_{0,1,-1,0}(u) - 4(u-3)H_{1,0,1,0}(u) \\
& - 4(3u-2)H_{0,0,0}(u) + 4(3u-1)H_{0,-1,1,0}(u) + 2(5u-3)H_{0,-1,0,0}(u) \\
& - 4(2u-1)H_{0,0,1,0}(u) - 4(3u-1)H_{0,1,0}(u) + 4(3u-2)H_{0,0,-1,0}(u) \\
& - 2(3u-1)\zeta(2)H_{0,1}(u) + 8u\zeta(2)H_{0,-1}(u) - 2u\zeta(2)H_{0,0}(u) + 8\zeta(2)H_{1,0}(u) \\
& \left. + 2(5u+3)\zeta(3)H_1(u) - 8u\zeta(2)H_0(u) + 2(7u-4)\zeta(3)H_0(u) \right)
\end{aligned}$$

$$f_6^0(u) = \frac{u}{(1-u)^2} \left( 4\zeta(3) + 2\zeta(2)H_0(u) + 2H_{0,1,0}(u) + H_{0,0,0}(u) \right)$$

$$\begin{aligned}
f_6^1(u) = & \frac{u}{(1-u)^2} \left( -11\zeta(4) - 8\zeta(3)H_1(u) - 3\zeta(3)H_0(u) - 2H_{1,0,0,0}(u) + 3H_{0,0,0,0}(u) \right. \\
& + 4H_{0,1,1,0}(u) + 4H_{0,1,0,0}(u) - 4H_{0,1,-1,0}(u) - 4H_{1,0,1,0}(u) - 4H_{0,-1,1,0}(u) \\
& - 2H_{0,-1,0,0}(u) + 2H_{0,0,1,0}(u) - 4\zeta(2)H_{1,0}(u) + \zeta(2)H_{0,0}(u) - 2H_{0,0,-1,0}(u) \\
& \left. + 2\zeta(2)H_{0,1}(u) - 4\zeta(2)H_{0,-1}(u) \right)
\end{aligned}$$

#### 4.3.4 Five-propagator, type 5 topology

This topology has one Master integral. There is no Master integral corresponding to this topology given in [6], but they give a result for a reducible six-propagator topology having this five-propagator, type 5 topology as

its subtopology. On the contrary, [4] gives the result for this topology with the (simplest) choice of the Master integral

$$\text{Topo}(1, 1, 1, 0, 1, 0, 1) \equiv \left( \frac{iN}{(4\pi)^2} \right)^2 \frac{1}{m^2} f_7(\epsilon, u) \equiv \left( \frac{iN}{(4\pi)^2} \right)^2 \frac{1}{m^2} \sum_{j=0}^{\infty} \epsilon^j f_7^j(u)$$

The solution of the differential equation is again straightforward and results in

$$f_7^0(u) = \frac{u}{(1-u)^2} \left( -4H_{0,1,0,0} - 2H_{0,0,0,0} - 4H_{1,0,0,0} - 8H_{1,1,0,0} + 6\zeta(4) + 4\zeta(3)H_0 + 8\zeta(3)H_1 \right)$$

Be careful, [4] gives the incorrect sign of this result.

### 4.3.5 Five-propagator, type 6 topology

The last topology that we will consider has two Master integrals. It is relatively easy to choose Master integrals such that the system of differential equations is triangular to  $O(\epsilon)$ . We chose

$$\begin{aligned} \text{Topo}(1, 0, 1, 1, 1, 0, 1) &\equiv \left( \frac{iN}{(4\pi)^2} \right)^2 \frac{1}{m^2} f_8(\epsilon, u) \equiv \left( \frac{iN}{(4\pi)^2} \right)^2 \frac{1}{m^2} \sum_{j=0}^{\infty} \epsilon^j f_8^j(u) \\ \text{Topo}(1, 0, 1, 2, 1, 0, 1) &\equiv \left( \frac{iN}{(4\pi)^2} \right)^2 \frac{1}{m^4} f_9(\epsilon, u) \equiv \left( \frac{iN}{(4\pi)^2} \right)^2 \frac{1}{m^4} \sum_{j=0}^{\infty} \epsilon^j f_9^j(u) \end{aligned}$$

and in terms of these the solution is

$$\begin{aligned} f_8^0(u) &= \frac{u}{(1-u)^2} \left( -4H_{1,1,0,0} - 3H_{1,0,0,0} - 2H_{0,1,0,0} - 2H_{1,0,1,0} - H_{0,0,1,0} \right. \\ &\quad \left. - 2\zeta(2)H_{1,0} - \zeta(2)H_{0,0} - \frac{5}{2}\zeta(4) \right) \\ f_9^0(u) &= \frac{-3u^2}{(1-u^2)(1-u)^2} H_{0,0,0} \\ f_9^1(u) &= \frac{u^2}{(1-u^2)(1-u)^2} \left( 11\zeta(3)H_0 + 9\zeta(4) - 9H_{0,0,0,0} + 6H_{-1,0,0,0} + 14H_{0,1,0,0} \right. \\ &\quad \left. - 4H_{0,0,1,0} + 7\zeta(2)H_{0,0} + 22H_{0,0,-1,0} \right) \end{aligned}$$

This completes the list of all Master integrals that we needed. Authors of [6] choose a different set of Master integrals, namely  $\text{Topo}(1, 0, 1, 1, 1, 0, 1)$  together with  $\text{Topo}(1, 0, 1, 1, 1, 1, 1)$  which in our classification belongs to another (6-propagator) topology, but a reducible one having this topology as a subtopology, so this choice of Master integrals is legitimate. Moreover, the system of differential equations for this choice of Master integrals is again diagonal, and so we can easily solve it. In [4] another choice of Master integrals is employed, which is equivalent to  $\text{Topo}(1, 0, 1, 1, 1, 0, 1)$  and  $\text{Topo}(1, -1, 1, 1, 1, 0)$  (the difference is only in the subtopology terms).

## 4.4 Results

After we have performed the reduction of scalar integrals to the Master integrals and found their expansion in  $\epsilon$  to a sufficient order, we may use these results to find the expression for the invariant amplitude. The two-loop diagrams of Figure 3.2 give

$$\begin{aligned} A_{QCD2} &= -\frac{2\alpha\alpha_S Q^2 N_C C_F N^2}{v(4\pi)^2 4z^2} \left[ \frac{1}{\epsilon} \left( -12 \frac{(1-u)^2}{u} + 6 \frac{1-u^2}{u} H_0 + \frac{6(1+6u+u^2)}{u} H_{0,0} \right) \right. \\ &\quad \left. + \frac{1+u+u^2+u^3}{u(1-u)} \left( -18\zeta(4) - 2H_{0,0,0,0} - 28H_{0,1,0,0} + 16H_{0,-1,0,0} + 8H_{0,0,1,0} \right. \right. \\ &\quad \left. \left. - 32H_{0,0,-1,0} - 8\zeta(2)H_{0,0} - 16\zeta(3)H_0 \right) + \frac{1-u^2}{u} \left( -6\zeta(2) - 12H_{-1,0} - 16H_0 \right) \right] \end{aligned}$$

$$\begin{aligned}
& -\frac{72(1-u)^2}{u} + \frac{1}{u} \left( -4(7u^2 + 26u + 7)H_{0,-1,0} + 4(1+u)^2 H_{0,1,0} \right. \\
& -4(5u^2 - 6u + 5)H_{1,0,0} - \frac{4(11u^2 - 20u - 3)}{(1-u)} H_{0,0,0} - 4(u^2 - 27u - 2)H_{0,0} \\
& \left. -2(5u^2 + 22u + 5)\zeta(2)H_0 - (14 + 164u + 14u^2)\zeta(3) \right)
\end{aligned}$$

We see that the the divergent part is the same (up to a sign) as in the contribution of the counter-term diagrams (2.16) which can serve as a check of correctness of the calculations. Adding the both results gives

$$\begin{aligned}
A_{QCD} &= \frac{\alpha\alpha_S Q^2 N_C C_F N^2}{\pi^2 v z} \left[ \frac{1+u+u^2+u^3}{(1-u)^3} \left( -\frac{9}{4}\zeta(4) - \frac{7}{2}H_{0,1,0,0}(u) + H_{0,0,1,0}(u) \right. \right. \\
& -\frac{1}{4}H_{0,0,0,0}(u) - 4H_{0,0,-1,0}(u) + 2H_{0,-1,0,0}(u) - \zeta(2)H_{0,0}(u) - 2\zeta(3)H_0(u) \left. \right) \\
& + \frac{1-14u+u^2}{2(1-u)^2}\zeta(3) + \frac{(1+u)^2}{(1-u)^2} \left( -2H_{0,-1,0}(u) + \frac{1}{2}H_{0,1,0}(u) - \frac{1}{2}\zeta(2)H_0(u) \right) \\
& + \frac{3u}{(1-u)^2}H_{0,0}(u) + \frac{3+25u-7u^2+3u^3}{4(1-u)^3}H_{0,0,0}(u) - \frac{5-6u+5u^2}{2(1-u)^2}H_{1,0,0}(u) \\
& \left. - \frac{3(1+u)}{2(1-u)}\zeta(3)H_0(u) - \frac{5}{2} \right]
\end{aligned}$$

Note that only reduced a set of special functions appears in the final result, which lets us express this result in terms of only polylogarithms

$$\begin{aligned}
A_{QCD} &= \frac{\alpha\alpha_S Q^2 N_C C_F N^2}{\pi^2 v z} \left[ \frac{1+u+u^2+u^3}{(1-u)^3} \left( -\frac{1}{96}\ln^4(u) - \ln^2(u)\text{Li}_2(-u) - 18\text{Li}_4(-u) \right. \right. \\
& -\frac{7}{4}\ln^2(u)\text{Li}_2(u) + 8\ln(u)\text{Li}_3(-u) + 8\ln(u)\text{Li}_3(u) - 2\zeta(3)\ln(u) - \frac{27}{2}\text{Li}_4(u) \\
& \left. -\frac{9}{4}\zeta(4) - \frac{1}{2}\zeta(2)\ln^2(u) \right) + \frac{(1+u)^2}{(1-u)^2} \left( 2\ln(u)\text{Li}_2(-u) - 4\text{Li}_3(-u) - \frac{1}{2}\zeta(2)\ln(u) \right) \\
& + \frac{5-6u+5u^2}{4(1-u)^2}\ln^2(u)\ln(1-u) + \frac{3u}{2(1-u)^2}\ln^2(u) + \frac{3+25u-7u^2+3u^3}{24(1-u)^3}\ln^3(u) \\
& + \frac{1-14u+u^2}{2(1-u)^2}\zeta(3) - \frac{7-2u+7u^2}{2(1-u)^2}\text{Li}_3(u) - \frac{3(1+u)}{2(1-u)}\ln(u) \\
& \left. + \frac{3u^2-2u+3}{(1-u)^2}\ln(u)\text{Li}_2(u) - \frac{5}{2} \right]
\end{aligned}$$

Comparison of the one loop top quark contribution to the two loop QCD corrected result is shown in Figure 4.6. We see that the two-loop diagrams can lead to important corrections above the  $2m_t$  threshold. Below the threshold (and thus also in the intermediate mass range) the size of the corrections is smaller as can be seen also in Figure 5.6.

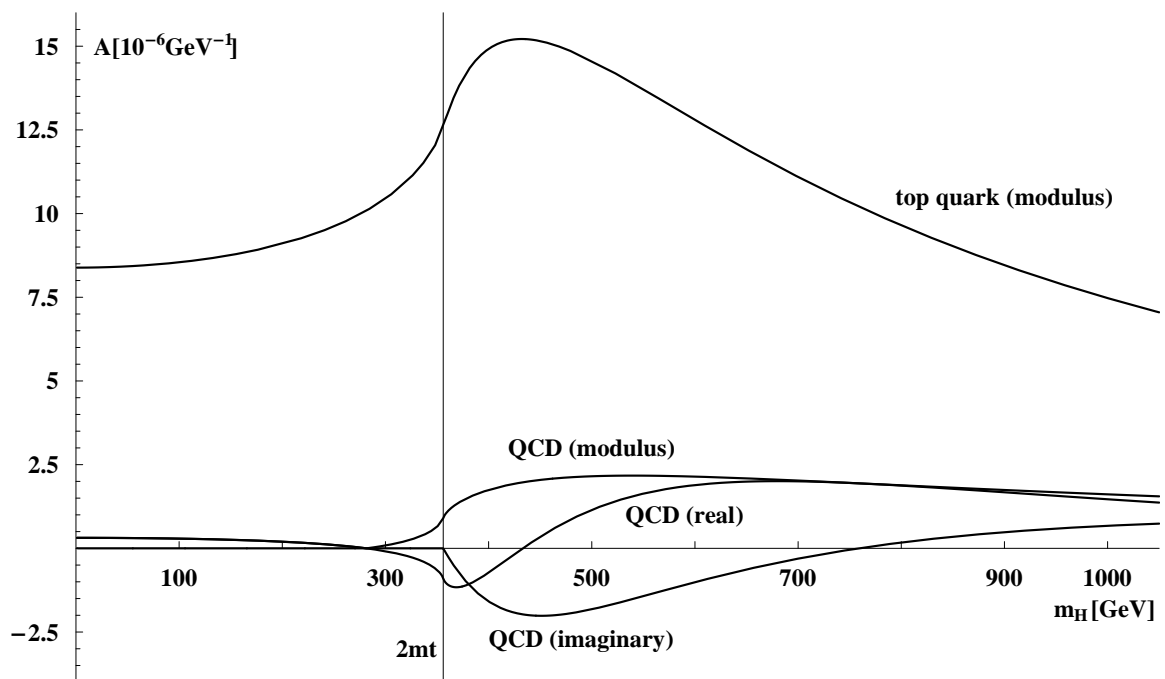


Figure 4.6: Comparison of leading order QCD correction to top quark loop with the one loop result

# Chapter 5

## Overview of various computations

### 5.1 QCD corrections, small momentum expansion and three loops

Author of [52] has used a small momentum expansion to evaluate the  $O(\alpha_S^2 z_q^2)$  correction. He first considers the lower-order correction using the low-energy theorem for Higgs boson. The basic idea is that the coupling of Higgs boson to quarks is proportional to their mass and is given by the substitution

$$m_q \rightarrow m_q \left( 1 + \frac{H}{v} \right)$$

so that in the limit of light Higgs boson (zero external momentum) we can obtain the result by differentiating the quark contribution to photon two-point function with respect to the virtual particle mass (this is similar to the way in which we calculated the contribution of counter-terms inserted in the fermion loop). Then he derives the same result by the use of so-called Fock-Schwinger gauge for the external photons. Finally, he directly expands the Feynman integrals in an external momentum (typical diagrams are shown in Figure 5.1) and arrives at the result

$$A_{QCD3} = -\frac{\alpha\alpha_S^2 Q^2 N_C}{\pi^3 v} \left[ -\frac{31}{36} + z \left( -\frac{19531913}{933120} + \frac{14}{135}\zeta(2) + \frac{28}{135}\zeta(2)\ln(2) + \frac{821063}{41472}\zeta(3) \right) \right. \\ \left. + z^2 \left( -\frac{56709666623}{3919104000} + \frac{16}{189}\zeta(2) + \frac{32}{567}\zeta(2)\ln(2) + \frac{72438107}{4976640}\zeta(3) \right) \right]$$

written in the on-shell renormalization scheme. The series converges much better if results are expressed in terms of  $\overline{MS}$  mass, and these results are also given in [52].

### 5.2 Non-Sudakov type double-logarithms

In the limit of small quark mass the behaviour of  $A_f$  and  $A_{QCD}$  is

$$A_f = \frac{\alpha Q^2 N_C}{\pi v} \frac{4m_q^2}{m_H^2} \left( \frac{1}{4} \ln^2 \frac{m_q^2}{m_H^2} \right) + O \left( \frac{m_q^2}{m_H^2} \ln \frac{m_q^2}{m_H^2} \right)$$

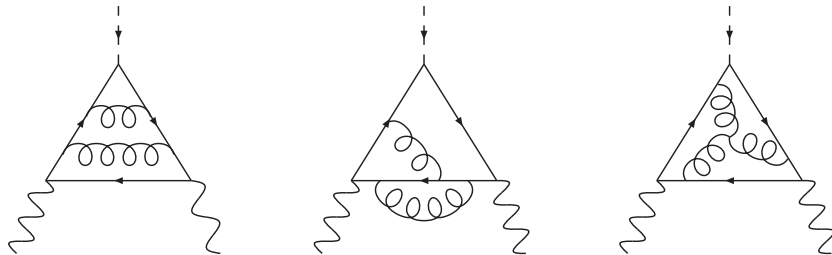


Figure 5.1: Typical diagrams contributing to next-to-leading order QCD corrections to quark loop

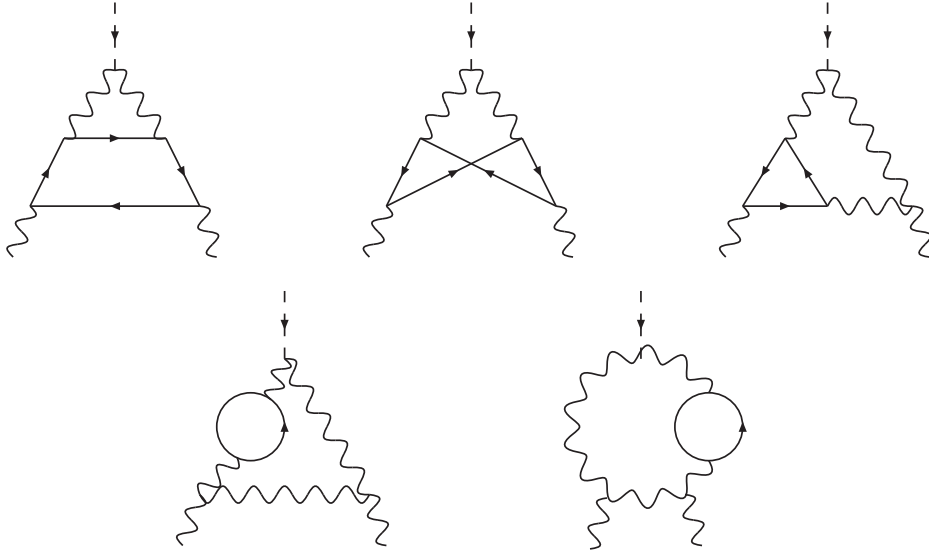


Figure 5.2: Diagrams contributing to light fermion electroweak corrections

$$A_{\text{QCD}} = \frac{\alpha Q^2 N_C}{\pi v} \frac{4m_q^2}{m_H^2} \left( \frac{-\alpha_S C_F}{2\pi} \right) \left( \frac{1}{48} \ln^4 \frac{m_q^2}{m_H^2} \right) + O \left( \frac{m_q^2}{m_H^2} \ln^3 \frac{m_q^2}{m_H^2} \right)$$

so if the similar behaviour appears in higher orders of expansion, effectively are not not expanding in  $\alpha_S$  but in

$$\alpha_S \ln^2 \frac{m_q^2}{m_H^2}$$

Diagrams where these double logarithms originate can be identified and leading double logarithms can be extracted. This has been done (also to next-to-leading order) in [?, 5] and after summation of contribution of whole family of these diagrams, they obtain to leading order

$$A_{\text{resum}} = A_f \sum_{n=0}^{\infty} \frac{2\Gamma(n+1)}{\Gamma(2n+3)} \left( -\frac{\alpha_S C_F}{2\pi} \ln^2 \frac{m_q^2}{m_H^2} \right)^n = A_f {}_2F_2 \left( 1, 1; 2, \frac{3}{2}; -\frac{\alpha_S C_F}{8\pi} \ln^2 \frac{m_q^2}{m_H^2} \right)$$

The running of the QCD coupling in the bottom quark contribution is considered in [41] with conclusion that the effective scale is given roughly by

$$\alpha_S(10m_b^2)$$

### 5.3 Light-fermion contribution

We have seen in the one-loop calculation that the light fermions do not give a significant contribution to the amplitude, since their coupling to the Higgs boson is proportional to their mass. But this coupling can be avoided if one considers the two-loop diagrams with the Higgs boson coupled to the gauge bosons and these coupled to the light quarks like in the Figure 5.2. Since we sum over the generations, their contribution is not necessary irrelevant. The authors of [3, 2] proceeded similarly like we did in the evaluation of QCD corrections - they projected the invariant amplitude to the scalar amplitudes, reduced the integrals to a linear combination of integrals analogous to our Topo's, used the IBP identities to find the relations among them and finally evaluated the remaining integrals with the help of the differential equations. They employed the Background Field Method quantization, which reduced the number of the diagrams. The biggest technical difference between the QCD corrections and the light fermion contribution is the presence of three thresholds in  $m_H^2$ , at  $m_H^2 = 0$ ,  $m_H^2 = m_W^2$  and  $m_H^2 = 4m_W^2$  in the latter case. We see that the additional non-zero threshold does not let us express the results in terms of HPLs. Authors introduce an extended set of harmonic polylogarithms, which they call the

Generalized Harmonic Polylogarithms (GHPLs). Their result is

$$A_{\text{lf}} = \frac{\alpha}{2\pi v} \frac{\alpha}{8\pi \sin^2 \theta_W z_W} \left[ 3f_2(0, 4z_W) + \frac{3}{\cos^4 \theta_W} \left( \frac{1}{2} - 2\sin^2 \theta_W + 4\sin^4 \theta_W \right) f_1(4z_W) \right. \\ \left. + 2N_C f_2 \left( -\frac{2}{9}, 4z_W \right) + \frac{2N_C}{\cos^4 \theta_W} \left( \frac{11}{36} - \frac{19}{27} \sin^2 \theta_W + \frac{70}{81} \sin^4 \theta_W \right) f_1(4z_W) \right]$$

with

$$x f_1(x) = -4x + 2(x-1)H_{-1}(-x) - 2H_{0,-1}(-x) + 2(x-3)H_{0,0,-1}(-x) \\ + 2(x-2)H_{0,-r,-r}(-x) - 3(x-2)H_{-r,-r,-1}(-x) - \sqrt{1 - \frac{4}{x}} \left[ 2xH_{-r}(-x) \right. \\ \left. - 3(x-2)H_{-4,-r,-1}(-x) + 2(x-2)H_{-r,0,-1}(-x) + 2(x-2)H_{-r,-r,-r}(-x) \right]$$

and

$$x f_2(q, x) = -8x(1+q) + 4(1+q)(x-1)H_{-1}(-x) - 2(x+2q)H_{0,-1}(-x) \\ - \frac{2}{3}(5x-12)H_{-r,-r}(-x) - 6(x(1+q) - 3 - 2q)H_{-r,-r,-1}(-x) \\ + 2(1+2q) \left[ (x-2)H_{0,-r,-r}(-x) + (x-3)H_{0,0,-1}(-x) \right] \\ - \sqrt{1 - \frac{4}{x}} \left[ 2x(1+2q)H_{-r}(-x) - 6q(x-2)H_{-4,-r,-1}(-x) \right. \\ \left. + 4q(x-2)H_{-r,0,-1}(-x) + 4q(x-2)H_{-r,-r,-r}(-x) \right] \\ + \frac{6(x-2)^2}{\sqrt{x(x-4)}} H_{-r,-1}(-x)$$

There is a singularity at  $x = \frac{1}{4}$  corresponding to 2  $W$ -boson threshold and this singularity in fact signals an improper treatment of the  $W$ -boson propagator. Authors regularize the singularity by performing a substitution

$$m_W \rightarrow m_W - i \frac{\Gamma_W}{2}$$

in the square root of the last factor of  $f_2(q, x)$  and check for the independence of the amplitude on the regulator.

## 5.4 Top-quark induced electroweak corrections

Electroweak corrections to  $H \rightarrow \gamma\gamma$  involving a virtual top quark have been considered for the first time in [39]. There the authors evaluated the  $O(\frac{m_t^2}{v})$  corrections in the heavy top quark limit and considered the diagrams with internal exchange of Higgs and the unphysical Goldstone bosons (for some examples see Figure 5.3). Later the authors of [22, 33] used the asymptotic-expansion technique and obtained the expansion of full  $O(\frac{m_t^2}{v^2})$  result in  $z_W$  to order  $O(z_W^4)$ , which corresponds to the formal hierarchy  $2m_t \gg 2m_W \gg m_H$ , and arrived at the result that did not match the previous result (the result of [22, 33] was later confirmed by [15], who considered the complete set of electro-weak corrections). The authors projected the tensorial amplitude to the scalar amplitudes  $A$  and  $C$  and calculated separately both of them, to check if they were the same (up to a sign). They worked in the  $R_\xi$  gauge and checked if the  $\xi$  parameter drops out in the final result. The bottom quark was considered to be massless and the CKM matrix was taken to be diagonal. Ultraviolet divergences were regularized by the dimensional regularization and the on-shell renormalization scheme was adopted. The number of diagrams was of order 1000, some of them are in Figure 5.3 and Figure 5.4. Due to the masslessness of the bottom quarks, the asymptotic expansion yielded non-trivial terms beyond the terms obtained by the ordinary Taylor expansion. Their result is

$$A_{\text{EW}t} = -\frac{\alpha}{\pi v} \frac{m_t^2}{16\pi^2 v^2} \left[ \frac{367}{108} + \frac{11}{18} z_W + \frac{19}{63} z_W^2 + \frac{58}{315} z_W^3 + \frac{1312}{10395} z_W^4 \right]$$

Electroweak corrections to  $H \rightarrow \gamma\gamma$  in the heavy top quark limit have been considered also in [16].

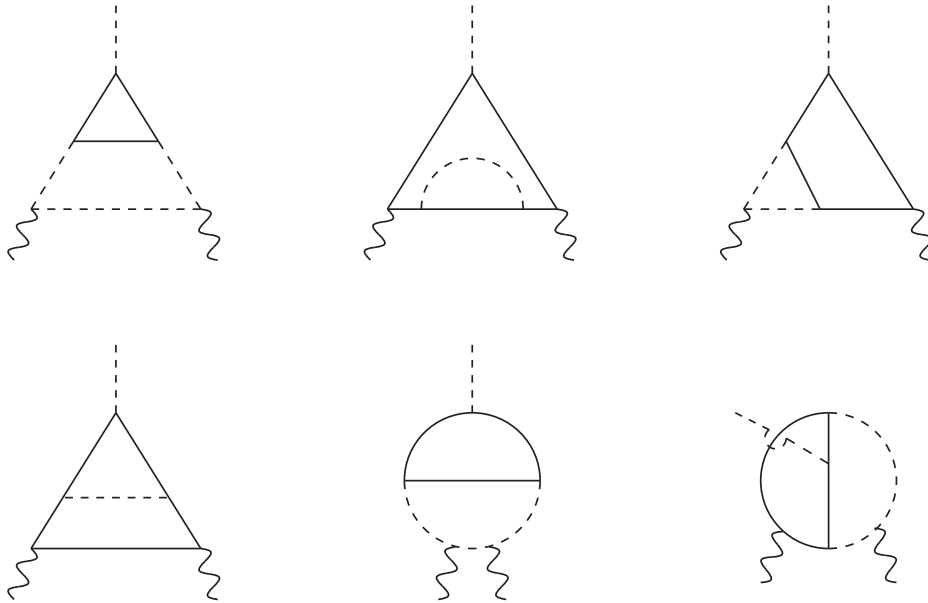


Figure 5.3: Some of diagrams contributing to the top-induced electroweak corrections

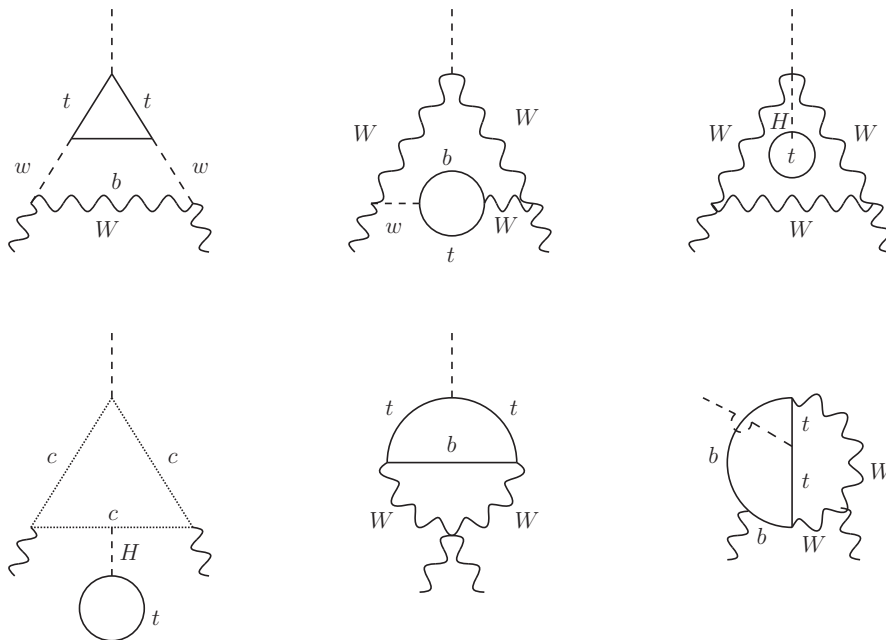


Figure 5.4: Some of diagrams contributing to higher order top-induced electroweak corrections

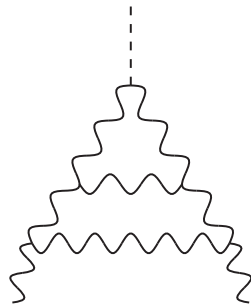


Figure 5.5: Sample diagram contributing to the pure bosonic electroweak corrections

## 5.5 Electroweak corrections in the heavy Higgs boson limit

In this text we have mainly focused on the Higgs boson in the intermediate mass region. For heavy Higgs boson the two-loop electroweak corrections of order  $O(G_F m_H^2)$  have been considered in [34]. Authors used the Equivalence theorem which let them evaluate the high energy contribution of  $W$  and  $Z$  boson from the contribution of  $w^\pm$  and  $z$  the unphysical Goldstone bosons. The loop integrals have been evaluated using the dispersion relations. The corrections are  $\simeq 30\%$  for Higgs boson with  $m_H = 1\text{TeV}$ .

## 5.6 Pure bosonic electroweak corrections and the complete set of corrections

Finally the complete set of electroweak corrections in the intermediate Higgs boson range has been calculated in [15] using the expansion in a small Higgs boson mass. This covers both the light fermion contribution, the electroweak corrections involving the third generation quarks, and the pure bosonic contributions. The first two groups have been already discussed. Sample diagram of the third group is in the Figure 5.5. Before evaluating the diagrams, authors argue from the structure of cuts of diagrams, why the asymptotic expansion of the diagrams gives the same result as an ordinary Taylor expansion in variable  $\frac{m_H^2}{4m_W^2}$ . As in the evaluation of the light fermion contribution, the Background Field Method of quantization has been used. On-shell renormalization for physical particles has been used and around 1700 diagrams have been evaluated. Also the gauge parameter  $\xi$  was renormalized in order to expand the off-shell amplitude. The analytical results turned out to be complicated, so only a numerical results for coefficients was given. The convergence near  $2W$ -threshold was improved by using a Pade approximant. The final result which contains both the top-quark induced electroweak corrections and the light-fermion contribution discussed earlier reads

$$A_{\text{EW}} = A_{\text{lep}} + A_{\text{lq}} + A_{\text{q3}} + A_{\text{YM}}$$

where  $A_{\text{lep}}$  stands for the contribution of leptons,  $A_{\text{lq}}$  for the light quarks,  $A_{\text{q3}}$  the third generation quarks and  $A_{\text{YM}}$  the pure gauge boson contribution

$$\begin{aligned} A_{\text{lep}} &= \frac{\alpha^2}{8\pi^2 v \sin^2 \theta_W} \left( \frac{3}{4z_W} C_W^l f_2(0, 4z_W) + \frac{3}{4z_Z} C_Z^l f_1(4z_Z) \right) \\ A_{\text{lq}} &= \frac{\alpha^2}{8\pi^2 v \sin^2 \theta_W} \left( \frac{1}{2z_W} C_W^q f_2\left(-\frac{2}{9}, 4z_W\right) + \frac{1}{2z_Z} C_Z^q f_1(4z_Z) \right) \\ A_{\text{q3}} &= \frac{\alpha^2}{8\pi^2 v \sin^2 \theta_W} \left( \frac{1}{4z_Z} C_Z^b f_1(4z_Z) + \sum_{j=0}^3 c_j^t z_W^j - K_W Q_t^2 N_C \left( \frac{4}{3} + \frac{14}{45} z_t + \frac{8}{63} z_t^2 + \frac{104}{1575} z_t^3 \right) \right) \\ A_{\text{YM}} &= \frac{\alpha^2}{8\pi^2 v \sin^2 \theta_W} \left( \sum_{j=0}^3 c_j^W z_W^j + K_W \left( 7 + \frac{22}{15} z_W + \frac{76}{105} z_W^2 + \frac{232}{525} z_W^3 \right) \right) \end{aligned}$$

with

$$\begin{aligned}
K_W &= \frac{13 - 2\pi\sqrt{3}}{4} z_W + \frac{(3 + 4 \cos^2 \theta_W) \ln(\cos^2 \theta_W)}{8 \sin^2 \theta_W} + \frac{3z_W \ln(4z_W)}{8z_W - 2} + \frac{5z_Z + 3}{16z_Z \cos^2 \theta_W} + \frac{5z_W + 3}{8z_W} \\
&\quad - 3 \left( \frac{\sqrt{\frac{1}{z_W} - 1}}{2} + \frac{\frac{1}{8z_W^2}}{\sqrt{\frac{1}{z_W} - 1}} \right) \operatorname{arctg} \frac{1}{\sqrt{\frac{1}{z_W} - 1}} - \frac{3}{2 \cos^2 \theta_W} \left( \frac{\sqrt{\frac{1}{z_Z} - 1}}{2} + \frac{\frac{1}{8z_Z^2}}{\sqrt{\frac{1}{z_Z} - 1}} \right) \operatorname{arctg} \frac{1}{\sqrt{\frac{1}{z_Z} - 1}} \\
K_t &= \frac{N_C z_W}{z_t} \left[ -\frac{1}{4z_t} - \frac{1}{8} + \left( \frac{1}{4z_t} + \frac{1}{2} \right) \sqrt{\frac{1}{z_t} - 1} \operatorname{arctg} \frac{1}{\sqrt{\frac{1}{z_t} - 1}} \right] \\
C_Z^q &= \frac{4N_C}{\cos^2 \theta_W} [Q_u^2((z_-^u)^2 + (z_+^u)^2) + Q_d^2((z_-^d)^2 + (z_+^d)^2)] \\
C_Z^l &= \frac{4}{\cos^2 \theta_W} [(z_-^l)^2 + (z_+^l)^2] \\
C_Z^b &= \frac{4N_C}{\cos^2 \theta_W} Q_d^2((z_-^d)^2 + (z_+^d)^2) \\
C_W^q &= 2N_C \\
C_W^l &= 2 \\
z_+^i &= T_3 - Q_i \sin^2 \theta_W \\
z_-^i &= -Q_i \sin^2 \theta_W
\end{aligned}$$

and the numerical coefficients

$$\begin{aligned}
c_0^t &= -54.4 + 6.07z_W \\
c_1^t &= -13.3 + 3.02z_W \\
c_2^t &= -7.00 + 1.84z_W \\
c_3^t &= -4.35 + 1.18z_W \\
c_0^W &= 16.3 - 1.72z_W \\
c_1^W &= 25.7 - 2.64z_W \\
c_2^W &= 15.5 - 2.05z_W \\
c_3^W &= 10.2 - 1.46z_W
\end{aligned}$$

Note that there is a different power of  $\cos \theta_W$  in the light fermion contribution in  $C$  factors then in the previous section.

The summary of these corrections in the intermediate mass region is shown in Figure 5.6. Quantity  $\delta$  is defined as

$$\Gamma_{2l} = \Gamma_{1l}(1 + \delta)$$

with  $\Gamma_{1l}$  and  $\Gamma_{2l}$  decay width of  $H \rightarrow \gamma\gamma$  evaluated to one and two loops. These results are confirmed by a numerical analysis done recently by [44], who also consider the size of corrections near  $2m_W$  threshold. Their approach is independent of any expansion except for an expansion in the SM coupling constants.

The total one-loop decay rate is shown in Figure 5.7. The combination of QCD corrections and EW corrections gives the correction to the one-loop result that ranges between  $\pm 1.5\%$  in the intermediate mass range, so it does not manifest itself in the logarithmic plot. Such small corrections are result of the compensation of QCD and the electroweak corrections. Above the  $2m_W$  threshold, both corrections become positive leading to an overall effect of  $\simeq 4\%$ .

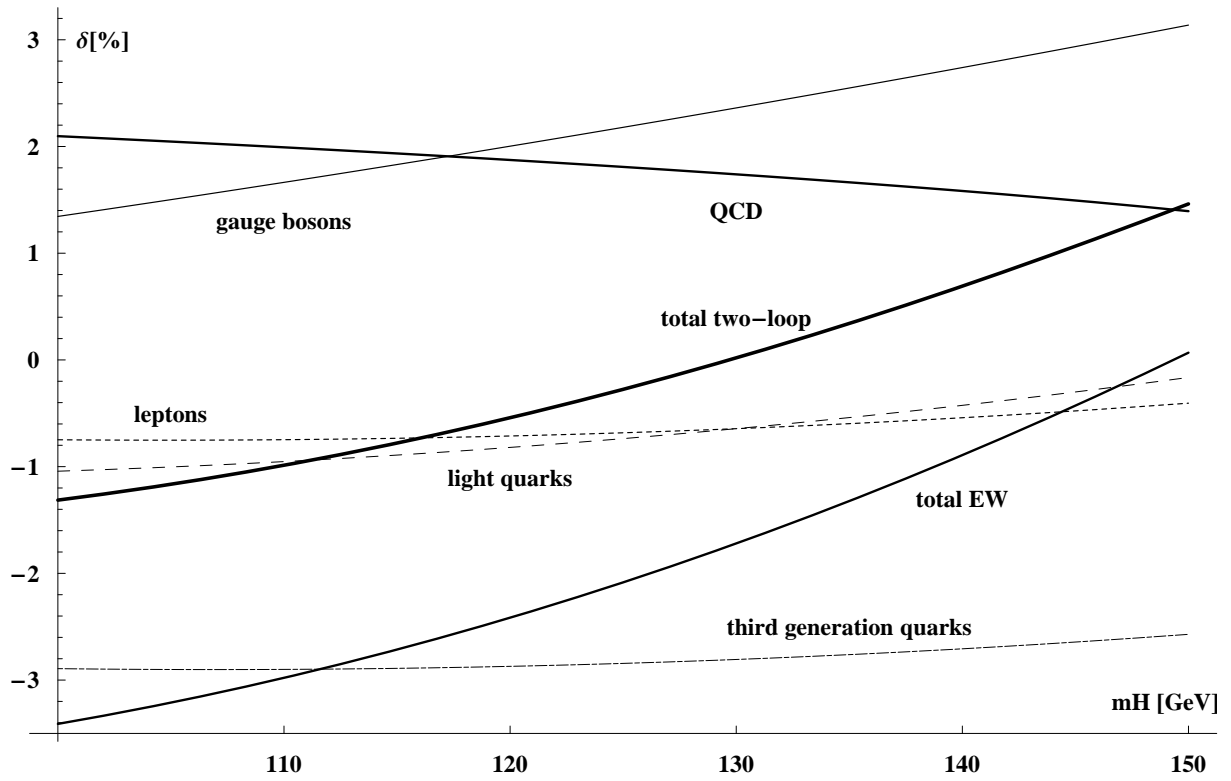


Figure 5.6: Summary of two-loop corrections to  $H \rightarrow \gamma\gamma$

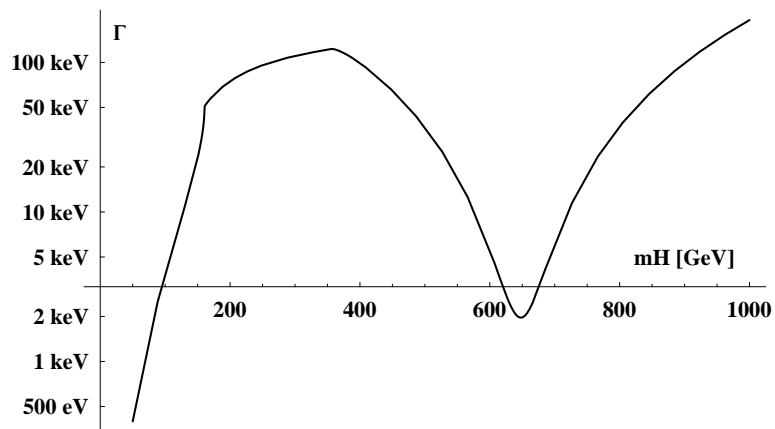


Figure 5.7: Decay width  $H \rightarrow \gamma\gamma$  calculated at one-loop level



# Chapter 6

## Conclusion

We have studied the two photon decay of the Higgs boson. Precise knowledge of the decay width predicted by the Standard Model is necessary for the comparison with the experimental data. Although rare, this process is important because it allows a precise measurement of the Higgs boson mass. It is also sensitive to “new physics” beyond the SM. We have seen that the heavy fermions which have masses generated by the Higgs mechanism do not decouple and thus give a significant contribution to the invariant amplitude even in the limit  $\frac{m_H}{m_f} \rightarrow 0$ .

Precise determination of the  $H \rightarrow \gamma\gamma$  decay width requires an inclusion of the small corrections to the invariant amplitude. The leading order contribution to the  $H \rightarrow \gamma\gamma$  comes from the  $W$ -boson and top quark loops. All other corrections (which are relevant for Higgs boson masses compatible with the present experimental input) are from the two-loop integrals. It is well known that the evaluation of the two-loop integrals is very complicated. This work focused on illustration of some methods that are used when evaluating these integrals. Last 30 years have shown various ways, how to attack the problem. Since these techniques are in general not explained in the standard textbooks, we tried to make the calculations as detailed as possible. Although all results that we obtained have been found in the literature already, the analytic results for  $H \rightarrow \gamma\gamma$  are still quite recent, and we calculated them independently.

In the second chapter we described the one loop calculations in detail. The fermion loop integral has been evaluated in three different ways, first by using the standard Feynman parametrization explained in every textbook in the QFT, then by calculating the imaginary part and using the dispersion relations. The last way was to using the integration-by-parts identities and the differential equation approach. This modern approach has many advantages and is very useful when used for the two-loop calculations. One of advantages is the fact that once we evaluate one class of diagrams using this approach, it is relatively simple to use these results for related diagrams. This is what has been presented in the rest of the first chapter - the results of the fermion loop integration were used to write the contribution of  $W$ -boson loop and one-loop counter-term diagrams that had to be evaluated in order to make the two-loop results finite.

The next chapter tried to evaluate the two-loop diagrams for the leading order QCD corrections to quark loop by a small momentum expansion. The method of asymptotic expansions is frequently used, because the computations are much simpler when compared to the exact evaluation of diagrams, but at the same time often physically relevant. At the beginning of the chapter the result found in the literature is derived and then we described, how to proceed to higher orders. Although this simple analysis is not world-shaking, it is difficult to find the explicit expressions in the literature. Last part of the chapter focuses on the evaluation of the difficult two-loop vacuum bubble integrals. We have tried to make the exposition as detailed as possible, because this integral was the most difficult integral that was encountered, mainly because it involves three mass scales (it depends on two arbitrary parameters).

The fourth chapter considered the exact analytic evaluation of the QCD corrections. This was the most interesting part of the work, and surprisingly it was not too difficult, because the combination of IBP identities together with the differential approach reduces the problem essentially to an algebraic problem, which can be solved on the computer. This is another of the advantages mentioned above.

The last chapter gave a brief summary of other calculation that have been considered by various authors. Figure 5.6 shows the summary of various corrections. We see that in the intermediate mass regions the correction to the one-loop result  $\delta$  ranges between  $\pm 1.5\%$ . The one-loop result for the decay width is shown in Figure 5.7.



# Appendix A

## Reference formulae

### A.1 Feynman rules

In this chapter we will list the Feynman rules that were used throughout the calculations. For a comprehensive list of Feynman rules of the Standard Model in  $R_\xi$  gauge see [12]. The Feynman rules are shown in Figure A.1. In the case of WWA vertex all the momenta flow into the vertex. The incoming  $W_\mu^+$  with momentum  $k_1$  is equivalent to the outgoing  $W_\mu^-$  with momentum  $-k_1$ . Vertex functions  $V_{\mu\nu\rho}$  and  $S_{\mu\nu,\rho\sigma}$  are defined as

$$\begin{aligned} V_{\mu\nu\rho}(k_1, k_2, k_3) &= (k_1 - k_2)_\rho g_{\mu\nu} + (k_2 - k_3)_\mu g_{\nu\rho} + (k_3 - k_1)_\nu g_{\rho\mu} \\ S_{\mu\nu,\rho\sigma} &= 2g_{\mu\nu}g_{\rho\sigma} - g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho} \end{aligned}$$

Matrices  $T$  are standard (halves of)  $\mathfrak{su}(3)$  Gell-Mann matrices. For a comparison with the results in literature, following formulae are useful:

$$\begin{aligned} \frac{1}{v} &= \frac{g_W}{2m_W} \\ \frac{G_F}{\sqrt{2}} &= \frac{g_W^2}{8m_W^2} \\ \alpha &= \frac{e^2}{4\pi} \\ \alpha_S &= \frac{g_S^2}{4\pi} \end{aligned} \tag{A.1}$$

where we tried to distinguish between the weak isospin coupling constant  $g_W$  and the QCD coupling constant  $g_S$ .

### A.2 Gamma and colour matrices

Throughout the calculations, we used the following formulae for  $d$ -dimensional  $\gamma$  matrices

$$\begin{aligned} \gamma^\alpha\gamma^\beta + \gamma^\beta\gamma^\alpha &= 2g^{\alpha\beta} \\ \gamma_\mu\gamma^\mu &= d\mathbb{1} \\ \gamma_\mu\gamma_\alpha\gamma^\mu &= (2-d)\gamma_\alpha \\ \gamma_\mu\gamma_\alpha\gamma_\beta\gamma^\mu &= 4g_{\alpha\beta} + (d-4)\gamma_\alpha\gamma_\beta \\ \gamma_\mu\gamma_\alpha\gamma_\beta\gamma_\gamma\gamma^\mu &= -2\gamma_\gamma\gamma_\beta\gamma_\alpha + (4-d)\gamma_\alpha\gamma_\beta\gamma_\gamma \\ \text{Tr}(\gamma_\alpha\gamma_\beta) &= \text{Tr}\mathbb{1}g_{\alpha\beta} \\ \text{Tr}(\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta) &= \text{Tr}\mathbb{1}(g^{\alpha\beta}g^{\gamma\delta} - g^{\alpha\gamma}g^{\beta\delta} + g^{\alpha\delta}g^{\beta\gamma}) \\ \text{Tr}(\gamma^\alpha\gamma^\beta\dots\gamma^\omega) &= \text{Tr}(\gamma^\omega\dots\gamma^\beta\gamma^\alpha) \end{aligned} \tag{A.2}$$

Gordon identity was used to bring the electromagnetic vertex to the standard form

$$\bar{u}(p')\gamma_\mu u(p) = \bar{u}(p') \left[ \frac{p_\mu + p'_\mu}{2m} + \frac{i}{2m} \Sigma^{\mu\nu} (p'_\nu - p_\nu) \right] u(p)$$

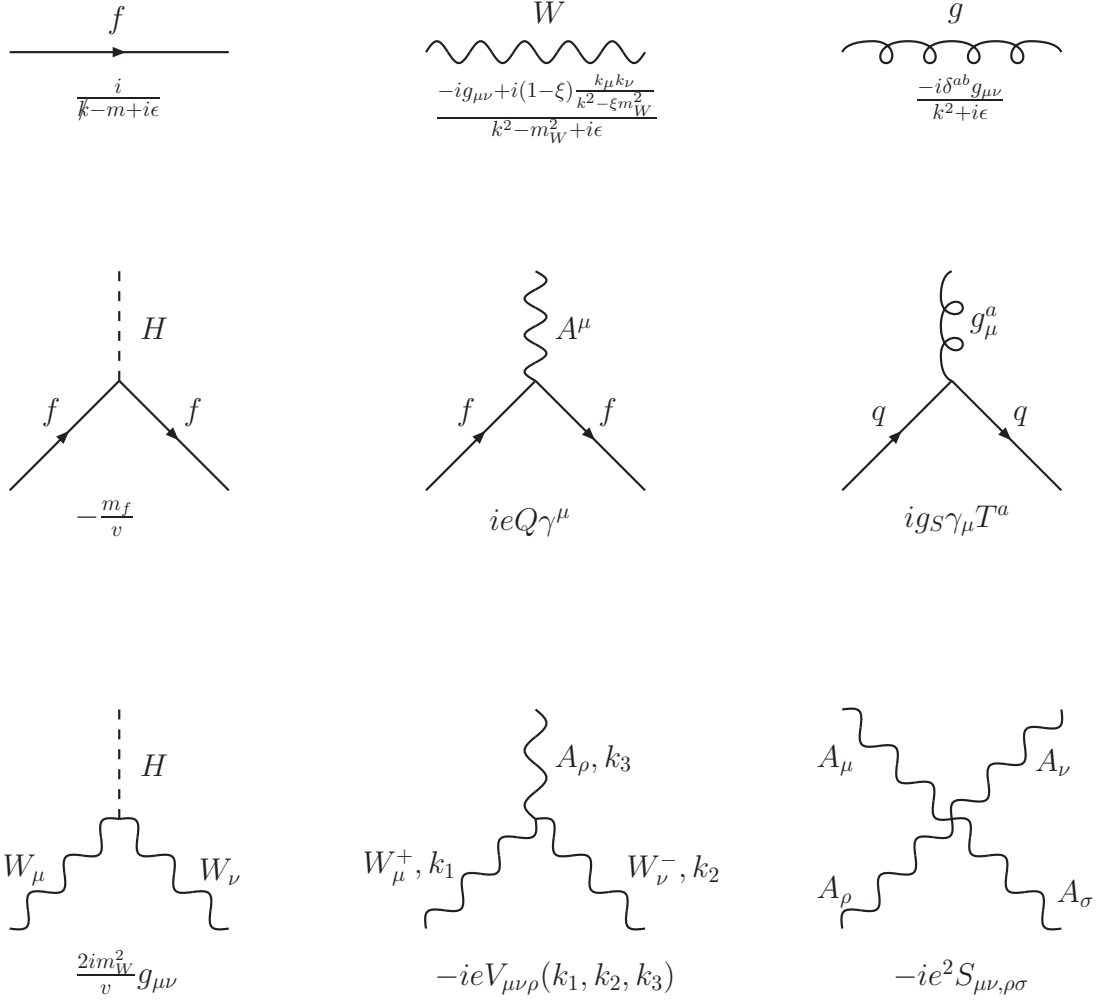


Figure A.1: Feynman rules

where

$$\begin{aligned}\Sigma^{\mu\nu} &= \frac{i}{2}[\gamma^\mu, \gamma^\nu] \\ (\not{p} - m)u(p) &= 0 \\ \bar{u}(p)(\not{p} - m) &= 0\end{aligned}$$

Let us derive some formulae useful for manipulations with  $\mathfrak{su}(3)$  colour matrices. For any matrix representation  $T^a$  of Lie algebra we define the structure constants  $f_{abc}$  as

$$[T^a, T^b] = i \sum_c f_{abc} T^c$$

Note that  $i$  is inserted because we want the matrices  $T^a$  to be Hermitian with real structure constants  $f_{abc}$ . Jacobi identity

$$[T^a, [T^b, T^c]] + [T^b, [T^c, T^a]] + [T^c, [T^a, T^b]] = 0.$$

implies that there exists a representation of algebra called the adjoint representation defined by

$$(T_A^a)_{ij} = -i f_{aij}$$

For a compact simple Lie algebra like  $\mathfrak{su}(3)$  there always exists a basis with totally antisymmetric structure constants. With this information, it is easy to show that matrix  $\sum_a T^a T^a$  commutes with all elements of the algebra and the Schur lemma tells us that in any irreducible representation this must be proportional to the identity matrix:

$$\sum_a T_R^a T_R^a = C_2(R) \mathbb{1} \quad (\text{A.3})$$

Operator on the left-hand side is so-called quadratic Casimir operator. Applying this to the adjoint representation, which is irreducible in the case of the simple Lie algebras, we immediately obtain a useful formula

$$\sum_{cd} f_{acd} f_{bcd} = C_2(A) \delta_{ab}$$

Using this formula, we have

$$\sum_a T_R^a T_R^b T_R^a = \left( C_2(R) - \frac{1}{2} C_2(A) \right) T_R^b \quad (\text{A.4})$$

which was needed in the evaluation of QCD correction to the quark electromagnetic vertex. In the case of standard representation of  $\mathfrak{su}(3)$  defined by

$$T_S^a = \frac{1}{2} \lambda^a$$

where  $\lambda^a$  are the eight Gell-Mann matrices, normalized so that  $\text{Tr}(\lambda^a \lambda^b) = 2\delta_{ab}$  we have

$$\begin{aligned}C_2(A) &= 3 \\ C_2(S) &= \frac{4}{3}\end{aligned}$$



# Appendix B

## Mathematical appendices

### B.1 Functions

In the following sections we will list some formulae that were used frequently. For more information about special functions see [8, 58].

#### B.1.1 Källen form

In this section we will examine Källen quadratic form defined by

$$\lambda^2(m_1^2, m_2^2, m_3^2) = m_1^2 + m_2^2 + m_3^2 - 2m_1m_2 - 2m_1m_3 - 2m_2m_3$$

We see that it is a completely symmetric homogeneous second order polynomial of 3 variables. We factor out one mass, defining

$$a = \frac{m_1^2}{m_3^2}$$
$$b = \frac{m_2^2}{m_3^2}$$

and obtain

$$\lambda^2(m_1^2, m_2^2, m_3^2) = m_3^2(1 + a^2 + b^2 - 2a - 2b - 2ab)$$

We are interested mainly in its behavior when the parameters are physical masses, that is nonnegative real numbers. If we hold the parameter  $a$  fixed we see that the quadratic form has zero points at

$$b = 1 + a \pm \sqrt{a}$$

and the form itself is negative for

$$b \in (1 + a - 2\sqrt{a}; 1 + a + 2\sqrt{a})$$

and positive for  $b < 1 + a - 2\sqrt{a}$  or  $b > 1 + a + 2\sqrt{a}$ , that is, there is a region of negative  $\lambda^2$  which has center at  $1 + a$  and a width  $4\sqrt{a}$ . At  $a = 1$  the lower endpoint becomes zero, so there is no lower region of  $b$  where  $\lambda^2$  would be positive. When  $a$  moves from this point, the lower region “opens” and it is always possible to choose small enough  $b$  for which  $\lambda^2 > 0$ . We can see this in another way if we introduce parameters that are symmetric with respect to  $a$  and  $b$ :

$$x = \frac{a + b}{2}$$
$$y = (a - b)$$
$$2a = 2x + y$$
$$2b = 2x - y$$

When expressed in terms of  $x$  and  $y$ ,  $\lambda^2$  reads

$$\lambda^2 = 1 - 4x + y^2$$

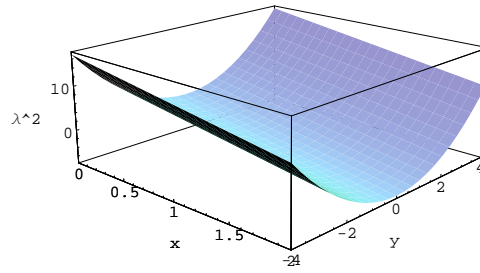


Figure B.1: Plot of Kallen quadratic form in symmetric variables

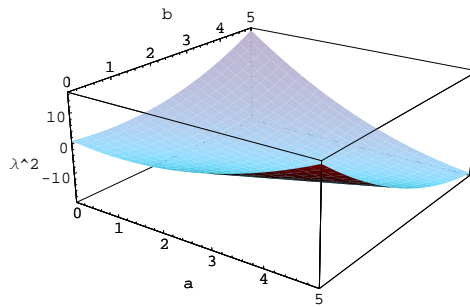


Figure B.2: Plot of Kallen quadratic form in standard variables

and we see clearly that given a mass difference  $y$  we can always find mean mass for which  $\lambda^2$  is negative. When  $a$  is given, the minimal value of  $\lambda^2$  is at point  $b = a + 1$  and the function has no local maxima (global maxima are at  $b = 0$  with value  $(a - 1)^2$  and  $b \rightarrow \infty$  where  $\lambda^2$  diverges).

### B.1.2 Conformal mapping of the cut complex plane to the unit disk

In this section we will study the conformal mapping given by

$$u = f(z) = \frac{\sqrt{1 - \frac{1}{z}} - 1}{\sqrt{1 - \frac{1}{z}} + 1}$$

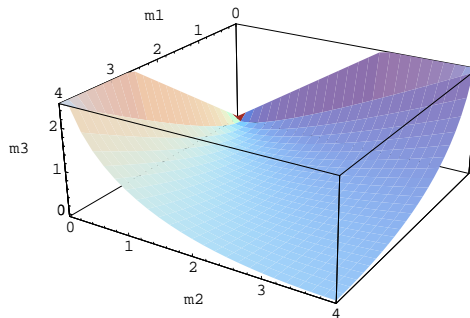


Figure B.3: Value of  $m_3$  of lower root of  $\lambda^2$

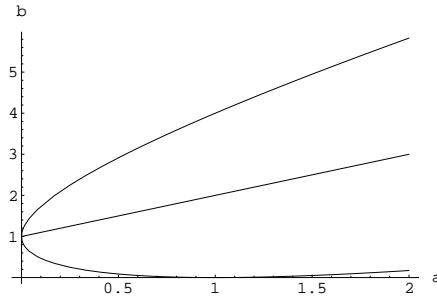


Figure B.4: Illustration of region of parameters with negative  $\lambda^2$  for fixed  $a$

where by square root we mean its principal value (as usual). The function is defined for all  $z \in \mathbb{C} \setminus [0; 1]$ . We see that it is a composition of mappings

$$\begin{aligned} z_1 &= f_1(z) = \frac{1}{z} \\ z_2 &= f_2(z_1) = 1 - z_1 \\ z_3 &= f_3(z_2) = \sqrt{z_2} \\ u &= f_4(z_3) = \frac{z_3 - 1}{z_3 + 1} \end{aligned}$$

with  $f_1$  an inversion,  $f_2$  a translation and a reflection,  $f_3$  a square root and  $f_4$  a linear fractional transformation (Cayley transform). Both  $f_1$  and  $f_2$  are defined in the whole complex plane, one-to-one, and they are their own inverses. The mapping  $f_4$  is one-to-one mapping of the Riemann sphere ( $\mathbb{C} \cup \{\infty\}$ ) with inverse given by

$$z_3 = f_4^{-1}(u) = \frac{1 + u}{1 - u}$$

Point  $z_3 = x + iy$  is mapped by  $f_3$  to

$$u = f_3(x + iy) = \frac{(x^2 + y^2 - 1) + 2iy}{(x + 1)^2 + y^2}$$

Finally, the mapping  $f_3$ , is not defined for  $z_2 < 0$  which corresponds to the cut  $z \in [0; 1]$  and its image is the open right half-plane  $\{z_3 : \Re z_3 > 0\}$ .

Let us look where does the neighbourhood of the cut map ( $\epsilon$  infinitesimal)

$$[0; 1] \pm i\epsilon \xrightarrow{\frac{1}{z}} [1; \infty] \mp i\epsilon \xrightarrow{1-z_1} [-\infty; 0] \pm i\epsilon \xrightarrow{\sqrt{z_2}} \pm i[0; \infty] + \epsilon \xrightarrow{\frac{z_3-1}{z_3+1}} C_\epsilon^\pm$$

with

$$\begin{aligned} C_\epsilon^+ &= \{u : u = (1 - \epsilon)e^{i\phi}, \phi \in [0; \pi]\} \\ C_\epsilon^- &= \{u : u = (1 - \epsilon)e^{i\phi}, \phi \in [-\pi; 0]\} \end{aligned}$$

the upper and the lower semicircle of the unit disk with center at 0. It is easy to check that  $f_4$  maps the imaginary axis onto the unit circle (preserving a sign of imaginary part) and the right (left) open half-plane inside (outside) of the unit circle. Similarly the unit circle is mapped onto the imaginary axis and the open unit disk onto the open left half-plane. The complement of the closed unit disk is mapped onto the open right half-plane.

Putting all together, the picture is as follows:  $f(z)$  maps the Riemann sphere without  $[0; 1]$  onto the open unit disk with the line above the cut mapped onto the upper semicircle and line below the cut mapped onto the lower semicircle. The Sign of the imaginary part of numbers is preserved and the special points include

$$\begin{aligned} f(0) &= 1 \\ f(1) &= -1 \end{aligned}$$

$$\begin{aligned} f(\infty) &= 0 \\ f\left(\frac{1}{2} + i\epsilon\right) &= i \\ f\left(\frac{1}{2} - i\epsilon\right) &= -i \end{aligned}$$

Since the mapping is one-to-one and onto the unit disk, it has the inverse mapping given by

$$z = f^{-1}(u) = \frac{(1-u)^2}{-4u}$$

If we extend this mapping to the whole complex plane, it maps points conjugate by the circle inversion to the same point:

$$f^{-1}(u) = f^{-1}\left(\frac{1}{u}\right)$$

Finally, we will use these expressions frequently to transform the differential equation to the variable  $u$ :

$$\begin{aligned} \frac{du}{dz} &= \frac{4u^2}{1-u^2} \\ 1-z &= \frac{(1+u)^2}{4u} \\ z \frac{\partial}{\partial z} &= -\frac{u(1-u)}{1+u} \frac{\partial}{\partial u} \\ \sqrt{1-\frac{1}{z}} &= \frac{1+u}{1-u} \end{aligned}$$

### B.1.3 Euler gamma and beta functions, digamma and polygamma functions

Gamma function can be defined for  $\Re z > 0$  as

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt = 2 \int_0^\infty t^{2z-1} e^{-t^2} dt$$

Using the integration by parts we see that for the real values it satisfies the relation  $\Gamma(z+1) = z\Gamma(z)$  and this lets us analytically continue this function to the whole complex plane except for  $z \in \{0, -1, -2, \dots\}$  where the function has simple poles (note that this implies that  $\infty$  is a limiting point of poles, so it is very singular).

From the relations  $\Gamma(x+1) = x\Gamma(x)$ ,  $\Gamma(1) = 1$  and  $\Gamma'(1) = -\gamma$  it follows that

$$\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma + O(\epsilon) \tag{B.1}$$

$$\Gamma(n+\epsilon) = \Gamma(n) + \epsilon\Gamma(n) \left(1 + \frac{1}{2} + \dots + \frac{1}{n-1} - \gamma\right) + O(\epsilon^2) \tag{B.2}$$

$$\Gamma(-n+\epsilon) = \frac{(-1)^n}{n!} \left[\frac{1}{\epsilon} + 1 + \frac{1}{2} + \dots + \frac{1}{n-1} + \frac{1}{n} - \gamma\right] + O(\epsilon) \tag{B.3}$$

Beta function can be defined by the integral

$$B(p, q) = \int_0^1 x^{p-1} (1-x)^{q-1} dx$$

which is valid as long as  $\Re p > 0$  and  $\Re q > 0$ . It can be expressed using only the gamma function, but it is useful for performing several integrals. The relation between the Euler gamma and beta function is easily derived

$$\begin{aligned} \Gamma(p)\Gamma(q) &= 4 \int_0^\infty \int_0^\infty u^{2p-1} v^{2q-1} e^{-u^2-v^2} du dv = \\ &= 4 \int_0^\infty r^{2p+2q-1} e^{-r^2} dr \int_0^{\frac{\pi}{2}} (\cos \phi)^{2p-1} (\sin \phi)^{2q-1} d\phi \end{aligned}$$

$$= \Gamma(p+q)B(p, q)$$

so we get

$$B(p, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

There are several useful expression for the beta function which are obtained from its definition using the various substitutions

$$\begin{aligned} B(p, q) &= 2 \int_0^{\frac{\pi}{2}} (\sin x)^{2p-1} (\cos x)^{2q-1} dx \\ &= \int_0^{\infty} \frac{x^{p-1}}{(1+x)^{p+q}} dx \\ &= 2 \int_0^1 y^{2p-1} (1-y^2)^{q-1} dx \end{aligned}$$

(the second expression follows after the substitution  $t = \frac{x}{1+x}$ , third after the substitution  $x = y^2$ ).

Digamma function and polygamma functions are defined as

$$\begin{aligned} \Psi(z) &= \frac{d}{dz} \ln \Gamma(z) \\ \Psi^{(m)}(z) &= \left( \frac{d}{dz} \right)^{m+1} \ln \Gamma(z) \end{aligned}$$

which means that  $\Psi(z)$  is just a logarithmic derivative of  $\Gamma(z)$ . Values of all derivatives of the gamma function starting with the 2nd derivative at the point 1 can be expressed in terms of the zeta function values in natural numbers and powers of  $\gamma$ . This will let us express a first few coefficients in the Taylor expansion of the gamma function in the vicinity of  $z = 1$ . Our starting point will be the Euler's formula (see [58]) for the digamma function:

$$\Psi(z) = \int_0^{\infty} \left( \frac{e^{-t}}{t} - \frac{e^{-zt}}{1-e^{-t}} \right) dt$$

From this it follows that

$$\begin{aligned} \Psi^{(m)}(z) &= (-1)^{m+1} \int_0^{\infty} \frac{t^m e^{-zt}}{1-e^{-t}} dt \\ &= (-1)^{m+1} \sum_{k=0}^{\infty} \frac{\Gamma(m+1)}{(z+k)^{m+1}} \end{aligned}$$

where we used geometric the series to represent  $(1-e^{-t})^{-1}$  and expressed the integral using the definition of gamma function. Now it is easy to shift  $z \rightarrow z+1$  and use the generalized binomial formula and the definition of Riemann zeta function to get

$$\begin{aligned} \Psi^{(m)}(1+z) &= (-1)^{m+1} \sum_{k=0}^{\infty} \frac{\Gamma(m+1)}{(z+k+1)^{m+1}} \\ &= (-1)^{m+1} \Gamma(m+1) \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \binom{-m-1}{l} \frac{z^l}{(k+1)^{l+m+1}} \\ &= (-1)^{m+1} \sum_{l=0}^{\infty} z^l \frac{(-1)^l \Gamma(m+l+1)}{\Gamma(l+1)} \sum_{k=1}^{\infty} \frac{1}{k^{m+l+1}} \\ &= \sum_{l=0}^{\infty} z^l \frac{(-1)^{m+l+1} \Gamma(m+l+1)}{\Gamma(l+1)} \zeta(m+l+1) \end{aligned}$$

The nontrivial part of this calculation is to relate the derivatives of the gamma function to the derivatives of its logarithm. First few derivatives of gamma function are

$$\Gamma'' = \Psi'(1) + \Gamma'^2(1)$$

$$\Gamma^{(3)} = \Psi''(1) + 3\Gamma''(1)\Gamma'(1) - 2\Gamma'^3(1)$$

This allows us to write

$$\begin{aligned}\Gamma'(1) &= -\gamma \\ \Gamma''(1) &= \gamma^2 + \frac{\pi^2}{6} \\ \Gamma'''(1) &= -\gamma^3 - \gamma\frac{\pi^2}{2} - 2\zeta(3)\end{aligned}$$

Digamma function also allows us to sum up some series. From the representation

$$\psi(z) = -\frac{1}{z} - \gamma - \sum_{k=1}^{\infty} \left( \frac{1}{k+z} - \frac{1}{k} \right)$$

it follows that

$$\psi(1) = -\gamma$$

and

$$\psi\left(\frac{1}{2}\right) = -2 - \gamma - \lim_{t \rightarrow 1^-} \sum_{k=1}^{\infty} t^k \left( \frac{1}{k + \frac{1}{2}} - \frac{1}{k} \right) = -2 - \gamma - \lim_{t \rightarrow 1^-} \left( \frac{2}{\sqrt{t}} \operatorname{arctgh} \sqrt{t} - 2 + \ln(1-t) \right) = -\gamma - 2 \ln 2$$

Now it is simple to show that

$$\begin{aligned}\psi\left(\frac{1}{2} + n\right) &= -\gamma - 2 \ln 2 + 2 \sum_{k=1}^{\infty} \frac{1}{2k+1} \\ \psi(1+n) &= -\gamma + \sum_{k=1}^{\infty} \frac{1}{k} \\ \sum_{k=1}^n \frac{(-1)^k}{k} &= -\ln 2 + \frac{(-1)^n}{2} \left[ \psi\left(1 + \frac{n}{2}\right) - \psi\left(\frac{1}{2} - \frac{n}{2}\right) \right]\end{aligned}$$

Generalization of the Euler gamma and beta functions are the incomplete gamma and beta functions defined by the integral

$$\begin{aligned}\gamma(a, z) &= \int_0^z t^{a-1} e^{-t} dt \\ \Gamma(a, z) &= \int_z^{\infty} t^{a-1} e^{-t} dt \\ B(z; a, b) &= \int_0^z t^{a-1} (1-t)^{b-1} dt\end{aligned}$$

### B.1.4 Polylogarithms

Functions that occur very frequently in the evaluation of integrals over the Feynman parameters are polylogarithms and more general functions (harmonic polylogarithms, multiple logarithms). All of them are generalizations of the logarithm. In this section we will focus mainly on the dilogarithm, because it is simplest (nontrivial) function from this set and occurs most frequently. For more information about polylogarithms see [38].

We define the polylogarithm as

$$\operatorname{Li}_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n}$$

for  $n \in \mathbb{N}$ . This series converges in the unit disk centered at 0. Note that

$$\sum_{k=1}^{\infty} \frac{z^k}{k} = -\ln(1-z)$$

so our definition of polylogarithms is really a generalization of the logarithm. Let us consider  $\text{Li}_2$ , the dilogarithm. The differentiation of the series gives the following integral formula valid within the radius of convergence:

$$\text{Li}_2(z) = - \int_0^z \frac{\ln(1-z)}{z} dz = - \int_0^1 \frac{\ln(1-zt)}{t} dt$$

where we are integrating along any contour lying within the radius of convergence and having its endpoints at 0 and  $z$ . If we remove the line  $[1; \infty)$  from the complex plane and use the principal value of the logarithm, we see that we can use this formula as a definition of  $\text{Li}_2(z)$  for any complex value except for  $z \in [1; \infty)$ , if we integrate along any contour which has endpoints at 0 and at  $z$ . The resulting function can be called the principal value of  $\text{Li}_2(z)$  and it is a holomorphic function of  $z$  in the open set  $C - [1; \infty)$ . It is however useful in the following to define  $\text{Li}_2(1)$  as a limiting value at this point, although we will see that there is a branching point at this point. Before investigating further the analytic properties of the dilogarithm, let us derive some useful formulae. The first one follows from the differentiation of  $\text{Li}_2(-\frac{1}{z})$ :

$$\begin{aligned} \frac{d}{dz} \text{Li}_2\left(-\frac{1}{z}\right) &= \frac{\ln(1+z) - \ln(z)}{z} \\ \text{Li}_2\left(-\frac{1}{z}\right) &= \int^z \frac{\ln(1+t) - \ln(t)}{t} dt + C \\ &= -\text{Li}_2(-z) - \frac{1}{2} \ln^2(z) + 2 \text{Li}_2(-1) \end{aligned}$$

where the additive constant has been fixed by setting  $z = 1$ . This formula is valid for  $z > 0$  or  $\Im z \neq 0$ . Using the limiting value of this formula at  $z = -1$  (according to the Abel's theorem of convergence) we obtain the relation

$$\text{Li}_2(1) = 2 \text{Li}_2(-1) - \text{Li}_2(1) + \frac{\pi^2}{2}$$

Splitting the power series of  $\text{Li}_2(-1)$  to odd and even terms gives

$$\text{Li}_2(-1) = -\frac{1}{2} \text{Li}_2(1)$$

Combining these two relations gives

$$\begin{aligned} \text{Li}_2(1) &= \frac{\pi^2}{6} \\ \text{Li}_2(-1) &= -\frac{\pi^2}{12} \end{aligned}$$

and finally

$$\text{Li}_2\left(-\frac{1}{z}\right) + \text{Li}_2(-z) = -\frac{\pi^2}{6} - \frac{1}{2} \ln^2(z)$$

It will be useful in following to rewrite this formula for values of  $z = -x \mp i\epsilon$ ,  $x > 1$  above and below the cut:

$$\text{Li}_2(x \pm i\epsilon) = -\text{Li}_2\left(\frac{1}{x}\right) - \frac{1}{2} \ln^2(x) + \frac{\pi^2}{3} \pm i\pi \ln(x)$$

The next identity which is also very useful is obtained from the integral formula for  $\text{Li}_2(z)$  for  $z \in (0; 1)$  using the integration by parts:

$$\text{Li}_2(z) + \text{Li}_2(1-z) = \frac{\pi^2}{6} - \ln(1-z) \ln(z) \tag{B.4}$$

The uniqueness theorem guarantees that this formula is valid for all complex  $z$  except for cuts at  $z > 1$  or  $z < 0$ . The following identity can be proved as in the previous case by the differentiation and the integration:

$$\text{Li}_2\left(\frac{-z}{1-z}\right) = -\text{Li}_2(z) - \frac{1}{2} \ln^2(1-z)$$

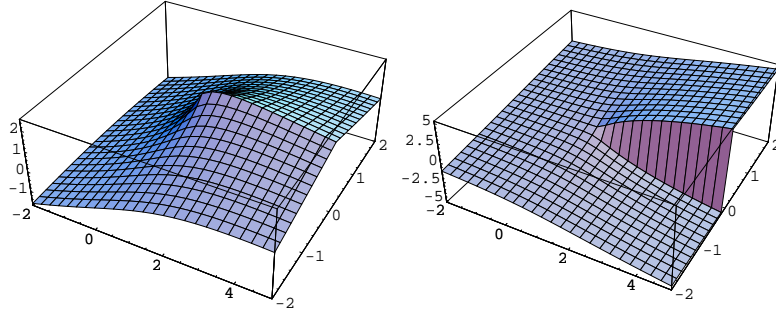


Figure B.5: Real and imaginary part of  $\text{Li}_2(z)$  for  $\Re z \in [-2; 5]$  and  $\Im z \in [-2; 2]$

The simplest of so called factorization identities can also be useful:

$$\frac{1}{2} \text{Li}_2(z^2) = \text{Li}_2(z) + \text{Li}_2(-z)$$

This formula is derived by writing for  $x \in (0; 1)$

$$\ln(1 - x^2) = \ln(1 - x) + \ln(1 + x)$$

integrating and using the analytic continuation (or directly from the series expansion of  $\text{Li}_2(x)$  at  $x = 0$  and the uniqueness theorem).

Let us finish this brief summary of the dilogarithm by the description of its analytic structure. From the integral expression for  $\text{Li}_2(z)$  we see that there is a branching point at  $z = 1$  and on all sheets except for the principal branch also an additional logarithmic singularity at  $z = 0$ . If we did not stay in the region of the principal branch, we would have to remove also the line  $[-\infty; 0]$  from the definition of  $\text{Li}_2(z)$ . From formula (B.1.4) we see that the discontinuity in the imaginary part of  $\text{Li}_2(z)$  when crossing the cut  $(1; \infty)$  at point  $x > 1$  in the counterclockwise direction is

$$\text{Li}_2(x + i\epsilon) - \text{Li}_2(x - i\epsilon) = 2\pi i \ln(x)$$

and in contrast to the discontinuity of imaginary part of logarithm depends on the point where we crossed the cut. The singularity at  $z = 0$  in other sheets of  $\text{Li}_2(z)$  is the usual logarithmic singularity.

Note that the discontinuity of the dilogarithm cannot be removed by subtracting a combination of the logarithms that have the same behaviour at the cut, in our case  $-\ln(z)\ln(1-z)$ , because this function has an additional cut in interval  $(-\infty; 0)$ . However, there is so-called Rogers L-function which adds one half of  $\ln(z)\ln(1-z)$ , that is

$$L(z) = \text{Li}_2(z) + \frac{1}{2} \ln(z)\ln(1-z),$$

and has two cuts which are symmetric with respect to the point  $z = \frac{1}{2}$ . Some dilogarithm identities have simpler form when expressed by means of this function, for example identity (B.4), which in the terms of Rogers L-function reads

$$L(z) + L(1-z) = \frac{\pi^2}{6}$$

and which is just a statement that  $L(z)$  is an odd function with respect to the point  $\frac{1}{2}$  except for the constant term. In fact, it is relatively easy to show (with help of formulas in the section about the digamma function) that the series expansion of the Rogers L-function at  $\frac{1}{2}$  is

$$L\left(\frac{1}{2} + x\right) = \frac{\pi^2}{12} + \frac{1}{2} \sum_{k=0}^{\infty} \frac{(2x)^{2k+1}}{2k+1} \left[ \psi(k+1) - \psi\left(k + \frac{1}{2}\right) \right]$$

In the course of the simplification of the integrals we needed the following complicated identity which is best expressed in terms of the Rogers L-function:

$$L(x) + L(y) = L(xy) + L\left(\frac{x(1-y)}{1-xy}\right) + L\left(\frac{y(1-x)}{1-xy}\right)$$

It is not evident in this form that this formula is useful, but performing the substitutions

$$\begin{aligned} a &= \frac{x(1-y)}{1-xy} \\ b &= \frac{y(1-x)}{1-xy} \\ x &= \frac{a}{1-b} \\ y &= \frac{b}{1-a} \end{aligned}$$

brings formula into the form

$$\mathrm{L}\left(\frac{a}{1-b}\frac{b}{1-a}\right) = \mathrm{L}\left(\frac{a}{1-b}\right) + \mathrm{L}\left(\frac{b}{1-a}\right) - \mathrm{L}(a) - \mathrm{L}(b) \quad (\text{B.5})$$

which is exactly what we needed. To summarize, these are the basic identities satisfied by the dilogarithm:

$$\mathrm{Li}_2(x) + \mathrm{Li}_2(-x) = \frac{1}{2}\mathrm{Li}_2(x^2) \quad (\text{B.6})$$

$$\mathrm{Li}_2(1-x) + \mathrm{Li}_2\left(1-\frac{1}{x}\right) = -\frac{1}{2}(\ln x)^2 \quad (\text{B.7})$$

$$\mathrm{Li}_2(x) + \mathrm{Li}_2(1-x) = \frac{\pi^2}{6} - \ln(x)\ln(1-x) \quad (\text{B.8})$$

$$\mathrm{Li}_2(-x) - \mathrm{Li}_2(1-x) + \frac{1}{2}\mathrm{Li}_2(1-x^2) = -\frac{\pi^2}{12} - \ln(x)\ln(1+x) \quad (\text{B.9})$$

### B.1.5 Harmonic polylogarithms

The generalization of the polylogarithms that is quite recent and very useful for the Feynman integral calculations are the harmonic polylogarithms [48]. They are connected with so-called harmonic sums which have also a broad use. The definition of the harmonic polylogarithms proceeds iteratively by the repeated integration. Every harmonic polylogarithm is indexed by a  $w$ -tuple of numbers which take the values  $\{-1, 0, +1\}$  ( $w$  is called the weight). We group the indices in a  $w$ -dimensional vector  $\vec{m}_w$  and define the basic set of functions

$$\begin{aligned} f(0, x) &= \frac{1}{x} \\ f(+1, x) &= \frac{1}{1-x} \\ f(-1, x) &= \frac{1}{1+x} \end{aligned}$$

We will also write  $|\vec{m}|$  instead of weight of  $\vec{m}$ . If we denote

$$\vec{m}_w = (a, \vec{m}_{w-1})$$

(i.e. we denote by  $a$  the leftmost index and by  $\vec{m}_{w-1}$  the remaining indices), we can define

$$\begin{aligned} \mathrm{H}(\vec{m}_w; x) &= \int_0^x f(a, t)\mathrm{H}(\vec{m}_{w-1}; t)dt \\ \mathrm{H}(-; x) &= 1 \end{aligned}$$

where by  $\mathrm{H}(-; x)$  we mean the polylogarithm with no indices. Harmonic polylogarithms of weight  $w$  with all indices zero are defined separately by

$$\mathrm{H}(\vec{0}_w; x) = \frac{1}{w!}\ln^w x$$

For example, the weight 1 harmonic polylogarithms are

$$\mathrm{H}(0; x) = \frac{1}{x}$$

$$\begin{aligned} \mathbf{H}(1; x) &= \frac{1}{1-x} \\ \mathbf{H}(-1; x) &= \frac{1}{1+x} \end{aligned}$$

From these definition it immediately follows the relation for the derivative

$$\frac{d}{dx} \mathbf{H}(\vec{m}_w; x) = f(a; x) \mathbf{H}(\vec{m}_{w-1}; x)$$

All weight 2 dilogarithms can be expressed in terms of the logarithms and Euler dilogarithm, for instance

$$\begin{aligned} \mathbf{H}(0, 1; x) &= \text{Li}_2(x) \\ \mathbf{H}(-1, 1; x) &= \text{Li}_2\left(\frac{x+1}{2}\right) - \ln 2 \ln(1+x) - \text{Li}_2\left(\frac{1}{2}\right) \end{aligned}$$

for other relations see [48]. Harmonic polylogarithms of weight 3 can be expressed in terms of polylogarithms of weight 3, but starting from weight 4 this is no longer true. This means that the set of harmonic polylogarithms is bigger than the set of polylogarithms (because all polylogarithms are also the harmonic polylogarithms).

It is useful to know how these functions behave in the vicinity of  $z = -1$ ,  $z = 0$  and  $z = 1$ . We will consider this later in more detail, but for now notice that HPL's vanish at  $x = 0$  if there is a nonzero index (for  $w$  zeros the behaviour is  $\frac{1}{w!} \ln^w x$  from the definition). Similarly HPL's are finite at  $z = 1$  if the leftmost index is not equal to 1. Otherwise the leading behaviour is  $\frac{1}{n!} (-\ln(1-x))^n$  where  $n$  is the number of leading 1's (this is not true exactly, 0's following 1's decrease  $n$  by one - see [48]).

By far the most important identities for our purpose are the product expansion identities. They can be written in the form

$$\mathbf{H}(\vec{p}; x) \mathbf{H}(\vec{q}; x) = \sum_{\vec{r}^*} \mathbf{H}(\vec{r}; x)$$

where  $*$  means that we sum over all permutations of  $|\vec{p}| + |\vec{q}|$  indices of  $\vec{p}$  and  $\vec{q}$  such that order of indices in  $\vec{r}$  coming from  $\vec{p}$  and these coming from  $\vec{q}$  is not changed. Note that these identities do not necessary hold at singular points, for details see again [48]. One of the simplest nontrivial examples of these identities is the formula

$$\begin{aligned} \mathbf{H}(a, b; x) \mathbf{H}(c, d; x) &= \mathbf{H}(a, b, c, d; x) + \mathbf{H}(a, c, b, d; x) + \mathbf{H}(a, c, d, b; x) \\ &\quad + \mathbf{H}(c, a, b, d; x) + \mathbf{H}(c, a, d, b; x) + \mathbf{H}(c, d, a, b; x) \end{aligned}$$

This kind of multiplication is used in the connection with the ‘‘shuffle algebra’’ (it reminds shuffling two decks of cards). These product identities can be used to isolate the logarithmic behaviour near the singular points so that the functions can be expanded in the power series.

For more information about the power series expansions, reduction of the harmonic polylogarithms to a smaller set of functions, relation to  $Z$ -sums and  $S$ -sums, generalization of the transformation identities of the polylogarithms to harmonic polylogarithms, evaluation of the harmonic polylogarithms at special points, Mellin transforms and proofs of all statements that appeared here, see [48]. For our purposes, the definition (and the formula for differentiation), product expansion and values of HPLs at special points and the most important properties of the harmonic polylogarithms. The Mathematica package HPL [40] was very useful for performing these operations.

Only the basic set of harmonic polylogarithms has been used in the calculation of QCD corrections, but an extended set has been used in the evaluation of light fermion contribution [3, 2]. Two more possible index values are added,  $-4$  and  $-r$  with the corresponding generating functions

$$\begin{aligned} f(-4; x) &= \frac{1}{4+x} \\ f(-r; x) &= \frac{1}{\sqrt{x(x+4)}} \end{aligned}$$

The extended set is called the Generalized Harmonic Polylogarithms (GHPLs) by authors.

## B.2 Hypergeometric summation

Great many of special functions occurring in the calculations are of the hypergeometric type. This includes the trigonometric functions, the Bessel functions, the complete elliptic integrals or the polylogarithms. Functions of this type also result in the Feynman integral calculations in the dimensional regularization in  $d$  dimensions. When any of these functions is expressed by the power series in a neighbourhood of a some point, the ratio of successive coefficients  $a_n/a_{n-1}$  is a rational function of  $n$ . This leads us to a definition of the hypergeometric series

$${}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; z) = \sum_{k=0}^{\infty} \frac{(a_1)_k \dots (a_p)_k}{(b_1)_k \dots (b_q)_k} \frac{z^k}{k!}$$

where the Pochhammer symbol is defined as

$$(a)_n = a(a+1) \dots (a+n-1)$$

for  $a \in \mathbb{C}$  and  $n \in \mathbb{N}$  and  $(a)_0 = 1$ . The series is convergent in the whole complex plane for  $p < q + 1$ , in the unit disk for  $p = q + 1$  and divergent except  $z = 0$  for  $p > q + 1$ . A hypergeometric function is obtained from the hypergeometric series by an analytic continuation. The following simple identities can be useful when putting a power series of a function to the form used in the definition of hypergeometric series:

$$\begin{aligned} \Gamma\left(k + \frac{1}{2}\right) &= \sqrt{\pi}(2k-1)!!2^{-k} = \frac{\sqrt{\pi}(2k)!}{2^{2k}k!} = \sqrt{\pi}\left(\frac{1}{2}\right)_k \\ (2k)!! &= 2^k k! = 2^k(1)_k \\ (2k-1)!! &= \frac{(2k)!}{(2k)!!} = \frac{(2k)!}{k!} 2^{-k} = 2^k \left(\frac{1}{2}\right)_k \\ \Gamma(2k+1) &= 2^{2k}(1)_k \left(\frac{1}{2}\right)_k \\ \Gamma(a-k) &= \frac{(-1)^k \Gamma(a)}{(1-a)_k} \\ \frac{1}{a+k} &= \frac{(a)_k}{a(a+1)_k} \\ (a)_{n-k} &= \frac{(a)_n}{(a+n-k)_k} = \frac{(-1)^k (a)_n}{(1-a-n)_k} \\ \binom{a}{k} &= \frac{\Gamma(a+1)}{\Gamma(a-k+1)k!} = \frac{(-1)^{k-1} a(1-a)_{k-1}}{k!} \end{aligned}$$

for  $k$  a positive integer and  $a$  any complex number for which formulas make sense. Let us give some examples of special functions that can be written in the form of the hypergeometric series.

$$\begin{aligned} \exp(z) &= {}_0F_0(z) \\ \frac{1}{(1-z)^a} &= {}_1F_0(a; z) \\ \sin(z) &= z {}_0F_1\left(-; \frac{3}{2}; -\frac{1}{4}z^2\right) \\ \cos(z) &= {}_0F_1\left(-; \frac{1}{2}; -\frac{1}{4}z^2\right) \\ J_a(2z) &= \frac{z^a}{\Gamma(a+1)} {}_0F_1(-; a+1; -z^2) \\ -\ln(1-z) &= z {}_2F_1(1, 1; 2; z) \\ \arcsin(z) &= z {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; z^2\right) \\ \operatorname{arctg}(z) &= z {}_2F_1\left(\frac{1}{2}, 1; \frac{3}{2}; -z^2\right) \\ \gamma(a; z) &= \frac{z^a}{a} {}_1F_1(a; a+1; -z) \\ B(z; a, b) &= \frac{z^a}{a} {}_2F_1(a, 1-b; a+1; z) \\ \operatorname{Li}_2(z) &= z {}_3F_2(1, 1, 1; 2, 2; z) \\ \operatorname{Li}_3(z) &= z {}_4F_3(1, 1, 1, 1; 2, 2, 2; z) \end{aligned}$$

We see on the example of the polylogarithms that although many functions can indeed be expressed using the hypergeometric functions, this does not always have to be the most convenient way. This list also illustrates that the hypergeometric functions are very general and that the properties of these functions strongly depend on the values of their parameters.

Most important formulae are the integral expressions for the hypergeometric functions

$$\begin{aligned} {}_1F_1(a; b; z) &= \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)} \int_0^1 e^{zt} t^{a-1} (1-t)^{b-a-1} dt \\ {}_2F_1(a, b; c; z) &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-zt)^{-a} dt \end{aligned}$$

Validity of these expressions can be checked easily by expanding the factor containing  $z$  and integrating the series term by term with help of the beta function.

The best understood hypergeometric functions are  ${}_1F_1$  (the confluent hypergeometric function) and  ${}_2F_1$  (the Gauss hypergeometric function). There are many hypergeometric transformation formulae, which relate hypergeometric function at  $z$  with a hypergeometric function at a different point. In the case of Gauss hypergeometric function, the most important are the linear fractional transformations, interchanging the points  $0, 1, \infty$  (sometimes called the relations between Kummer's solutions, because they are connected to the solutions of the hypergeometric equation at  $z = 0, z = 1, z = \infty$ ) and the quadratic transformations (which are connected to the conformal mapping considered in Section B.1.2). The simple example for transformation formula is the Kummer's relation for the confluent hypergeometric function

$$e^z {}_1F_1(a; b; -z) = {}_1F_1(b-a; b; z)$$

which follows easily from the integral formula for  ${}_1F_1(a; b; z)$  after change of variables  $t \leftrightarrow 1-t$ .

Different and also very useful family of relations for the hypergeometric function are the relations between the hypergeometric functions which have parameters differing by an integer (these functions are called the contiguous functions). For example any three  ${}_2F_1$  series whose corresponding parameters differ by integers are linearly dependent (over the field of rational functions of parameters and  $z$ ). The basic 15 relations of this type were derived by Gauss, and look like

$$(a + (b - c + 1)z - 1) {}_2F_1(a, b; c; z) = (a - c) {}_2F_1(a - 1, b; c; z) + (c - 1)(1 - z) {}_2F_1(a, b; c - 1; z)$$

$$(c - 2a + (a - b)z) {}_2F_1(a, b; c; z) = (c - a) {}_2F_1(a - 1, b; c; z) - a(1 - z) {}_2F_1(a + 1, b; c; z)$$

Example of more complicated identity is the Clausen's formula for terminating well-poised 2-balanced  ${}_4F_3$  is

$${}_4F_3\left(a, b, \frac{1}{2} - a - b - n, -n; 1 - a - n, 1 - b - n, a + b + \frac{1}{2}; 1\right) = \frac{(2a)_n (a + b)_n (2b)_n}{(a)_n (2a + 2b)_n (b)_n}$$

We have

$$\begin{aligned} \arcsin^2(x) &= \sum_{k,l=0}^{\infty} \frac{x^{2k+1} x^{2l+1} (2k-1)!! (2l-1)!!}{(2k)!! (2l)!! (2k+1)(2l+1)} = \\ &= \sum_{n=0}^{\infty} x^{2n+2} \frac{(\frac{1}{2})_n (\frac{1}{2})_n}{(1)_n (\frac{3}{2})_n} \sum_{k=0}^n \frac{(\frac{1}{2})_k (\frac{1}{2})_k (-n)_k (-\frac{1}{2} - n)_k}{(1)_k (\frac{3}{2})_k (\frac{1}{2} - n)_k (\frac{1}{2} - n)_k} = \\ &= \sum_{n=0}^{\infty} x^{2n+2} \frac{(\frac{1}{2})_n (\frac{1}{2})_n}{(1)_n (\frac{3}{2})_n} {}_4F_3\left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2} - n, -n; \frac{1}{2} - n, \frac{1}{2} - n, \frac{3}{2}; 1\right) \\ &= \sum_{n=0}^{\infty} x^{2n+2} \frac{(1)_n (1)_n}{(2)_n (\frac{3}{2})_n} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{(2x)^{2n}}{n^2 \binom{2n}{n}} = x^2 {}_3F_2\left(1, 1, 1; 2, \frac{3}{2}; x^2\right) \end{aligned}$$

The Taylor expansion of  ${}_2F_1(a, b; c; z)$  at  $z = 0$  is just a definition of hypergeometric series, but sometimes it is necessary to expand this function in its parameters. This is often the case of applications to the Feynman integrals (which is why the study of expansions became important recently), where the dimension  $d$  of dimensional regularization appears in parameters and we want to know the Maclaurin expansion in  $\epsilon = \frac{d}{2} - 2$  of results around  $d = 4$ . For more information and references to the literature see [31, 32]. As a simple example, we are going to illustrate a simple expansion that was useful in the evaluation of the vacuum bubble master integrals in Section 3.3. We want to find the expansion of

$${}_2F_1(1, \epsilon_1; 1 + \epsilon_2; z)$$

to the second order in  $\epsilon$ . A straightforward way is to use the integral representation formula, which can be easily expanded in  $\epsilon_1$  and  $\epsilon_2$  and integrated term by term. But the integral representation formula does not hold for these values of parameters (integrals would not converge), so the first step would be to use the relations between contiguous functions to relate this function to  ${}_2F_1(2, \epsilon_1; 2 + \epsilon_2; z)$  (which has the convergent integral representation)

But we skip this long derivation by finding the expansion directly from the series representation of hypergeometric function. First note that

$$\begin{aligned} {}_2F_1(0, 1; 1 + \epsilon; z) &= 1 \\ {}_2F_1(\epsilon, 1; 1; z) &= {}_1F_0(\epsilon; z) = (1 - z)^{-\epsilon} \end{aligned}$$

This means that if we want to know the expansion of  ${}_2F_1(1, \epsilon_1; 1 + \epsilon_2; z)$  to  $O(\epsilon^2)$  it is enough to find

$$\left. \frac{\partial^2}{\partial \epsilon_1 \partial \epsilon_2} \right|_{\epsilon_1=0=\epsilon_2} {}_2F_1(1, \epsilon_1; 1 + \epsilon_2; z) = - \sum_{n=1}^{\infty} \frac{z^n (n-1)!}{n!} \sum_{l=1}^n \frac{1}{l} = -\text{Li}_2(z) - \frac{1}{2} \ln^2(1 - z)$$

where in the last step we used the following relations

$$\begin{aligned} \text{Li}_2(z) &= \sum_{n=1}^{\infty} \frac{z^n}{n^2} \\ \ln(1 - z)^2 &= \sum_{k,l=1}^{\infty} \frac{z^{k+l}}{kl} = \sum_{n=1}^{\infty} z^n \sum_{k=1}^{n-1} \frac{1}{k(n-k)} = \sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{k=1}^{n-1} \left( \frac{1}{k} + \frac{1}{n-k} \right) = 2 \sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{k=1}^{n-1} \frac{1}{k} \end{aligned}$$

Thus the final result reads

$${}_2F_1(1, \epsilon_1; 1 + \epsilon_2; z) = 1 - \epsilon_1 \ln(1 - z) + \frac{\epsilon_1^2}{2} \ln^2(1 - z) - \epsilon_1 \epsilon_2 \left( \text{Li}_2(z) + \frac{1}{2} \ln^2(1 - z) \right) + O(\epsilon^3) \quad (\text{B.10})$$

## B.3 Integrals

### B.3.1 Feynman and Schwinger parameter integrals

The basic formula for the Schwinger alpha parametrization of Feynman integrals is the following formula

$$\frac{1}{A^n} = \frac{1}{\Gamma(n)} \int_0^\infty du u^{n-1} e^{-uA} \quad (\text{B.11})$$

Notice that this is just an integral formula for the gamma function written in another form. The formula is valid as long as  $\Re A > 0$  and  $n \in \mathbb{N}$  (or even more generally  $\Re n > 0$ ). Although we did not use the Schwinger parametrization, it is useful in some calculations and it is more general than the following Feynman parametrization

$$\begin{aligned} \frac{1}{AB} &= \int_0^1 dx \frac{1}{[xA + (1-x)B]^2} \\ \frac{1}{ABC} &= 2 \int_0^1 dx \int_0^1 dy \int_0^1 dz \delta(x+y+z-1) \frac{1}{(Ax + By + Cz)^3} \end{aligned} \quad (\text{B.12})$$

and in general

$$\frac{1}{A_1^{m_1} A_2^{m_2} \dots A_n^{m_n}} = \int_0^1 dx_1 \dots dx_n \delta(\sum x_i - 1) \frac{\prod x_i^{m_i-1}}{(\sum x_i A_i)^{\sum m_i}} \frac{\Gamma(m_1 + \dots + m_n)}{\Gamma(m_1) \dots \Gamma(m_n)}$$

The proof uses the alpha parametrization integral and the introduction of scaling variable  $\lambda$

$$\begin{aligned} \frac{1}{\prod A_i^{m_i}} &= \frac{1}{\prod \Gamma(m_i)} \int_0^\infty \prod du_i \prod u_i^{m_i-1} e^{-\sum A_i u_i} \\ &= \frac{1}{\prod \Gamma(m_i)} \int_0^\infty \prod du_i u_i^{m_i-1} e^{-\sum A_i u_i} \int_0^\infty d\lambda \delta(\lambda - \sum u_i) \\ &= \frac{1}{\prod \Gamma(m_i)} \int_0^\infty \frac{d\lambda}{\lambda} \int_0^\infty \prod dx_i x_i^{m_i-1} \delta(1 - \sum x_i) \lambda^{\sum m_i} e^{-\lambda \sum A_i x_i} \\ &= \frac{1}{\prod \Gamma(m_i)} \int_0^\infty \prod dx_i x_i^{m_i-1} \delta(1 - \sum x_i) \frac{\Gamma(\sum m_i)}{(\sum A_i x_i)^{\sum m_i}} \\ &= \frac{\Gamma(\sum m_i)}{\prod \Gamma(m_i)} \int_0^1 \prod dx_i \delta(1 - \sum x_i) \frac{\prod x_i^{m_i-1}}{(\sum A_i x_i)^{\sum m_i}} \end{aligned}$$

### B.3.2 Gaussian and Fresnel integrals

Using the well known expression for the Gaussian integral

$$\int_0^\infty e^{-Ax^2} dx = \frac{1}{2} \sqrt{\frac{\pi}{A}}$$

(which is valid for  $\Re A > 0$ ) and considering the limiting case for which  $\Re A = 0$  but  $\Im A \neq 0$  we obtain (sometimes called Fresnel integrals)

$$\int_0^\infty e^{\pm iax^2} dx = \frac{1}{2} \sqrt{\pi} a e^{\pm i\frac{\pi}{4}}$$

These integrals do not exist as Lebesgue integrals, but they exist in the form of the generalized Lebesgue or Newton integral.

### B.3.3 Integration of radial functions

In this subsection we will derive some basic formulae that are used frequently for the integration of radial functions in  $d$  dimensions. Our starting point will be the formula for surface of  $d-1$ -dimensional sphere ( $d-1$ -dimensional from topological/measure point of view,  $d$ -dimensional from geometric point of view), that is,  $d-1$  dimensional measure of the set

$$S_{d-1} = \{x \in \mathbb{R}^d : x_1^2 + x_2^2 + \dots + x_d^2 = 1\}$$

To calculate this, we will use the well known trick with the Gaussian integration

$$\begin{aligned} \int_{\mathbb{R}^d} d^d x e^{-x^2} &= \int_0^\infty dr r^{d-1} \int d\Omega_{d-1} e^{-r^2} = \frac{1}{2} \int d\Omega_{d-1} \int_0^\infty e^{-t} t^{\frac{d}{2}-1} dt \\ &= \frac{1}{2} \Gamma\left(\frac{d}{2}\right) \int d\Omega_{d-1} = \left(\int_{-\infty}^{+\infty} e^{-x^2} dx\right)^d = \pi^{\frac{d}{2}} \end{aligned}$$

from which it follows that

$$\Omega_{d-1} = \int d\Omega_{d-1} = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}$$

We can check that for first few dimensions we get the correct results:

$$\begin{aligned} \Omega_0 &= 2 \\ \Omega_1 &= 2\pi \\ \Omega_2 &= 4\pi \\ \Omega_3 &= 2\pi^2 \end{aligned}$$

The meaning of 0-dimensional formula must be understood in the sense that there are 2 points at the distance 1 from the origin, and this is the 0-dimensional “volume” of a sphere (sphere in 1-dimensional space). We use it in the following formula which makes also a good sense for  $d = 1$ :

$$\int_{\mathbb{R}^d} f(\vec{x}) d^d \vec{x} = \Omega_{d-1} \int_0^\infty \hat{f}(r) dr$$

for functions which are integrable and satisfy (the radial functions)

$$f(\vec{x}) = \hat{f}(|\vec{x}|)$$

It is interesting to check that the surface area of sphere of unit radius in 7 dimensions is numerically the largest and with increasing number of dimensions it decreases rapidly (but we are comparing numbers of different “dimension”!)

Now we can derive our master formula for the one-loop momentum integrations:

$$I = \int \frac{d^d l}{(2\pi)^d} \frac{(l^2)^r}{(l^2 - \Delta + i\epsilon)^s} \quad (\text{B.13})$$

$$= i(-1)^{r-s} \int \frac{d^d l_E}{(2\pi)^d} \frac{(l_E^2)^r}{(l_E^2 + \Delta)^s} \quad (\text{B.14})$$

$$= i(-1)^{r-s} \Omega_d \int_0^\infty \frac{l_E^{d-1} dl_E}{(2\pi)^d} \frac{l_E^{2r}}{(l_E^2 + \Delta)^s} \quad (\text{B.15})$$

$$= i(-1)^{r-s} \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} \frac{\Delta^{r-s+\frac{d}{2}}}{2(2\pi)^d} \int_0^\infty \frac{x^{r+\frac{d}{2}-1}}{(1+x)^s} dx \quad (\text{B.16})$$

$$= i \frac{(-1)^{r-s}}{(4\pi)^{\frac{d}{2}}} \Delta^{r+\frac{d}{2}-s} \frac{\Gamma(r+\frac{d}{2})\Gamma(s-r-\frac{d}{2})}{\Gamma(\frac{d}{2})\Gamma(s)} \quad (\text{B.17})$$

we used the Wick rotation [30, 46, 55], the radial integration and the definition of the beta function. For parameters for which the formula is not valid mathematically this is usually taken as the definition of integrals.



# Appendix C

## Some specific integrals

### C.1 Evaluation of integrals from fermion loop

In this section we will evaluate the integrals  $A$  and  $B$  encountered in the fermion loop calculation. We start with

$$\int_0^1 dx \int_0^{1-x} \frac{1-4xy}{1-i\epsilon-4xyz}$$

where

$$z = \frac{p_1 \cdot p_2}{2m^2} = \frac{m_H^2}{4m^2}$$

In the intermediate mass region we have  $z > 1$  for light quarks and leptons and  $0 < z < 1$  for the top quark. The integral is an analytic function of variable  $z$  in the whole complex plane except for a cut on the real axis for  $z > 1$ . The  $i\epsilon$  prescription tells us to take the values from the upper half plane in this interval. The integration over  $y$  is elementary and leads to

$$\int_0^1 dx \int_0^{1-x} \frac{1-4xy}{1-i\epsilon-4xyz} = \int_0^1 dx \left[ \frac{1-x}{z} + \frac{1-z}{4xz^2} \ln(1-4(z+i\epsilon)x(1-x)) \right] = \frac{1}{2z} + \frac{1-z}{4z^2} I_{-1}(z+i\epsilon) \quad (\text{C.1})$$

where

$$I_{-1}(z) = \int_0^1 \frac{dx}{x} \ln(1-4zx(1-x))$$

This function considered as an analytic function of variable  $z$  has a branch cut on the real axis in interval  $z \in (1; \infty)$ . Firstly we specialize to  $z \in (-\infty; 1)$ . In this interval, the argument of logarithm is positive and nonzero for all  $x \in (0; 1)$ . We have

$$I'_{-1}(z) = -4 \int_0^1 \frac{1-x}{1-4zx(1-x)} dx = \begin{cases} -\frac{2}{z} \sqrt{\frac{z}{1-z}} \operatorname{arctg} \sqrt{\frac{z}{1-z}} & \text{for } z \in (0; 1) \\ -\frac{1}{z} \sqrt{\frac{-z}{1-z}} \ln \frac{\sqrt{1-z}-\sqrt{-z}}{\sqrt{1-z}+\sqrt{-z}} & \text{for } z < 0 \end{cases}$$

and the integration gives for  $z \in (0; 1)$

$$I_{-1}(z) = -2 \int \frac{1}{z} \sqrt{\frac{z}{1-z}} \operatorname{arctg} \sqrt{\frac{z}{1-z}} dz = -4 \int \frac{\operatorname{arctg} y}{1+y^2} dy = -2 \left[ \operatorname{arctg} \sqrt{\frac{z}{1-z}} \right]^2 \quad (\text{C.2})$$

We have performed the substitution  $y = \sqrt{\frac{z}{1-z}}$  and the integration constant is zero as can be checked easily; similarly for  $z \in (-\infty; 0)$

$$I_{-1}(z) = \int \frac{-1}{z} \ln \left[ \frac{\sqrt{1-z}-\sqrt{-z}}{\sqrt{1-z}+\sqrt{-z}} \right] dz = \frac{1}{2} \ln^2 \left( \frac{\sqrt{1-z}-\sqrt{-z}}{\sqrt{1-z}+\sqrt{-z}} \right)$$

It can also be checked that the analytical continuation of one of results gives the second result and vice versa. To get the result for values  $z \in (1; +\infty)$ , we must use the  $i\epsilon$  prescription (otherwise the argument of the logarithm

would be negative). The integral expression for  $I'_{-1}(z)$  is still valid and the integration gives

$$I'_{-1}(z \pm i\epsilon) = -4 \int_0^1 \frac{1-x}{1-4x(1-x)(z \pm i\epsilon)} dx = \frac{1}{z} \sqrt{\frac{z}{z-1}} \left[ \ln \frac{\sqrt{z} + \sqrt{z-1}}{\sqrt{z} - \sqrt{z-1}} \mp i\pi \right] \quad (\text{C.3})$$

Final integration gives the result valid for  $z \in (1; +\infty)$

$$I_{-1}(z \pm i\epsilon) = \int \frac{1}{z} \sqrt{\frac{z}{z-1}} \left[ \ln \frac{\sqrt{z} + \sqrt{z-1}}{\sqrt{z} - \sqrt{z-1}} \mp i\pi \right] dz = \frac{1}{2} \left[ \ln \left( \frac{\sqrt{z} + \sqrt{z-1}}{\sqrt{z} - \sqrt{z-1}} \right) \mp i\pi \right]^2 \quad (\text{C.4})$$

Our integral  $I_{-1}$  can also be expressed quickly using dilogarithms. We will restrict variable  $z$  to range  $z \in (-\infty; 0)$ . In this case the argument of logarithm has 2 real roots, one negative and one greater than 1:

$$x_{\pm} = \frac{1}{2} \pm \frac{1}{2} \sqrt{1 - \frac{1}{z}}$$

so the argument of logarithm is always positive for  $x \in [0; 1]$ . We have

$$\begin{aligned} I_{-1}(z) &= \int_0^1 \frac{dx}{x} \ln(1 - 4zx(1-x)) = \int_0^1 \frac{dx}{x} \ln(-4z(x-x_-)(x_+ - x)) \\ &= \int_0^1 \frac{dx}{x} \left[ \ln \left( 1 - \frac{x}{x_+} \right) + \ln \left( 1 - \frac{x}{x_-} \right) + \ln(4zx_-x_+) \right] \\ &= -\text{Li}_2 \left( \frac{1}{x_+} \right) - \text{Li}_2 \left( \frac{1}{x_-} \right) \\ &= -\text{Li}_2 \left[ 2z \left( 1 + \sqrt{1 - \frac{1}{z}} \right) \right] - \text{Li}_2 \left[ 2z \left( 1 - \sqrt{1 - \frac{1}{z}} \right) \right] \\ &= \frac{1}{2} \ln \left( \frac{\sqrt{1-z} - \sqrt{-z}}{\sqrt{1-z} + \sqrt{-z}} \right)^2 \end{aligned}$$

where we used the fact that  $4zx_-x_+ = 1$  and the last line follows from the identity (B.7) with

$$x = \frac{\sqrt{1-z} - \sqrt{-z}}{\sqrt{1-z} + \sqrt{-z}} = \frac{\sqrt{z-1} - \sqrt{z}}{\sqrt{z-1} + \sqrt{z}}$$

Integral for  $I_{-1}(z)$  evaluated in this section is a special case of class of integrals encountered throughout the evaluation of the one-loop integrals which have particle of one mass circulating in the loop. We can use these integrals to evaluate the factor  $B_f$  and also in the evaluation of the gauge boson loop contribution to  $H \rightarrow \gamma\gamma$  decay rate. We define

$$I_n(z) = \int_0^1 x^n \ln(1 - 4zx(1-x)) dx$$

and restrict  $z$  to range  $z \in (0; 1)$  for a moment. For other values of  $z$ , we can again perform analytic continuation of our results or use directly results from special case  $n = -1$ , because as we will see, value of  $I_n$  can be expressed in terms of functions already encountered. Integrals that will be needed in sequel are

$$\begin{aligned} I_{-1} &= -2 \text{arctg}^2 \sqrt{\frac{z}{1-z}} = H_{0,0}(u) \\ I_0 &= -2 + 2 \sqrt{\frac{1-z}{z}} \text{arctg} \sqrt{\frac{z}{1-z}} = -2 - \frac{1+u}{1-u} H_0(u) \\ I_1 &= -1 + \sqrt{\frac{1-z}{z}} \text{arctg} \sqrt{\frac{z}{1-z}} = -1 - \frac{1+u}{2(1-u)} H_0(u) \\ I_2 &= \frac{1}{6z} - \frac{13}{18} + \left( -\frac{1}{6z} + \frac{2}{3} \right) \sqrt{\frac{1-z}{z}} \text{arctg} \sqrt{\frac{z}{1-z}} \\ I_3 &= \frac{1}{4z} - \frac{7}{12} + \left( -\frac{1}{4z} + \frac{1}{2} \right) \sqrt{\frac{1-z}{z}} \text{arctg} \sqrt{\frac{z}{1-z}} \end{aligned}$$

$$\begin{aligned}
I_4 &= -\frac{1}{40z^2} + \frac{37}{120z} - \frac{149}{300} + \left( \frac{1}{40z^2} - \frac{3}{10z} + \frac{2}{5} \right) \sqrt{\frac{1-z}{z}} \operatorname{arctg} \sqrt{\frac{z}{1-z}} \\
I_5 &= -\frac{1}{16z^2} + \frac{17}{48z} - \frac{157}{360} + \left( \frac{1}{16z^2} - \frac{1}{3z} + \frac{1}{3} \right) \sqrt{\frac{1-z}{z}} \operatorname{arctg} \sqrt{\frac{z}{1-z}}
\end{aligned}$$

Note that for the signs of the square root to be valid we must keep  $z$  in the range  $0 < z < 1$ . Although these integrals can be integrated easily by the integration by parts and integrating resulting partial fractions, it will be useful to derive them in a systematic way. Exponential generating functional of  $I_n(z)$  can be expressed in closed form using the exponential integral function (Ei), but it would not be more transparent than directly calculating integrals  $I_n(z)$ . As a first step, we shift the monomial  $x^n$  to  $(x - \frac{1}{2})^n$ , since the integral is symmetric around this point, and define a new family of integrals:

$$I_n(z) = \int_0^1 x^n \ln(1 - 4Lx(1-x)) dx = \int_0^1 \left( x - \frac{1}{2} + \frac{1}{2} \right)^n \ln(1 - 4Lx(1-x)) dx = \sum_{k=0}^n \binom{n}{k} 2^{k-n} J_k(z)$$

where

$$J_n(z) = \int_0^1 \left( x - \frac{1}{2} \right)^n \ln(1 - 4Lx(1-x)) dx$$

Now by integration by parts and changing the integration variable

$$J_n(z) = \frac{-8L}{n+1} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{x^{n+2}}{1-z+4zx^2} dx = \frac{-2}{n+1} \left( \sqrt{\frac{1-z}{4L}} \right)^{n+1} \int_{-\sqrt{\frac{z}{1-z}}}^{\sqrt{\frac{z}{1-z}}} \frac{y^{n+2} dy}{1+y^2}$$

Integral of the form

$$K_n(b) = \int_{-b}^b \frac{x^n}{1+x^2} dx$$

which is zero for odd  $n$  satisfies the recurrence relation

$$\begin{aligned}
K_{2n}(b) &= -K_{2n-2}(b) + \frac{2}{2n-1} b^{2n-1} \\
K_0(b) &= 2 \operatorname{arctg}(b)
\end{aligned}$$

which gives

$$\begin{aligned}
K_{2n}(b) &= 2(-1)^n \left[ \operatorname{arctg}(b) - \sum_{k=0}^{n-1} \frac{(-1)^k b^{2k-1}}{2k+1} \right] = \frac{2b^{2n+1}}{2n+1} {}_2F_1 \left( 1, n + \frac{1}{2}; n + \frac{3}{2}; -b^2 \right) \\
&= 2(-1)^n \sum_{k=n}^{\infty} \frac{(-1)^k b^{2k+1}}{2k+1}
\end{aligned}$$

Note that the sum in the first line is exactly the first  $n$  terms of series for  $\operatorname{arctg}(b)$ . Finally the expression for  $J_n(z)$  reads

$$J_{2n}(z) = \frac{(-1)^n}{(2n+1)2^{2n-1}} \left( \sqrt{\frac{1-z}{z}} \right)^{2n+1} \left[ \operatorname{arctg} \left( \sqrt{\frac{z}{1-z}} \right) - \sum_{k=0}^n \frac{(-1)^k}{2k+1} \left( \sqrt{\frac{z}{1-z}} \right)^{2k+1} \right]$$

## C.2 Exact $d$ -dimensional result for fermion loop

In this section we will show how to evaluate the one-loop fermion amplitude exactly in  $d$ -dimensions, which then lets us evaluate the counter-term contributions needed in two-loop calculations in way which is independent on Section 2.4. We will see that the result will be function which can be easily expanded in  $z = 0$ . Our starting point will be the expression for scalar amplitude  $A$  (2.12), but without omitted  $\tilde{\Delta}^{-\epsilon}$  and  $N$  (see (2.9))

$$A_{\text{ex}} = \frac{2\alpha Q^2 N}{\pi v} \int_0^1 dx \int_0^{1-x} dy \frac{4xy - 1}{(1 - 4xy)^{1+\epsilon}}$$

After the simplification of the integrand and performing the integration over  $y$ , we have

$$A_{\text{ex}} = \frac{2\alpha Q^2 N}{\pi v} \frac{1}{4z^2} \left[ \frac{1}{1-\epsilon} \int_0^1 \frac{dx}{x} \left( (1-4x(1-x)z)^{1-\epsilon} - 1 \right) + \frac{1-z}{\epsilon} \int_0^1 \frac{dx}{x} \left( (1-4x(1-x)z)^{-\epsilon} - 1 \right) \right]$$

We see that we need to consider only the integral

$$\begin{aligned} I(z, a) &= \int_0^1 \frac{dx}{x} [(1-4x(1-x)z)^a - 1] = \int_{-\frac{1}{2}}^{+\frac{1}{2}} \frac{dx}{x + \frac{1}{2}} [(1-z+4zx^2)^a - 1] \\ &= \frac{1}{2} \int_{-\frac{1}{2}}^{+\frac{1}{2}} dx \left( \frac{1}{x + \frac{1}{2}} - \frac{1}{x - \frac{1}{2}} \right) [(1-z+4zx^2)^a - 1] \\ &= -4 \int_0^{\frac{1}{2}} dx \frac{(1-z+4zx^2)^a - 1}{4x^2 - 1} = 2 \int_0^1 \frac{(1-(1-x^2)z)^a - 1}{1-x^2} \\ &= -2 \sum_{k=1}^{\infty} \binom{a}{k} z^k \int_0^1 (x^2-1)^{k-1} dx \end{aligned}$$

Now

$$\int_0^1 dx (x^2-1)^n = \frac{1}{2} (-1)^n B\left(\frac{1}{2}, n+1\right) = \frac{1}{2} (-1)^n \frac{\sqrt{\pi} \Gamma(n+1)}{\Gamma(n+\frac{3}{2})} = (-1)^n \sum_{k=0}^n \frac{(-1)^k}{2k+1} \binom{n}{k}$$

so we have

$$I(z, a) = \sum_{k=1}^{\infty} (-z)^k \binom{a}{k} \frac{\sqrt{\pi} \Gamma(k)}{\Gamma(k+\frac{1}{2})} = (-2az) \sum_{k=0}^{\infty} \frac{(1)_k (1)_k (1+a)_k}{(1)_k (2)_k (\frac{3}{2})_k} (-z)^k = -2az {}_3F_2\left(1, 1, 1-a; 2, \frac{3}{2}; z\right)$$

After substituting this expression in expression for  $A_\epsilon$ , we finally obtain

$$A_{\text{ex}} = \frac{-\alpha Q^2 N}{\pi v z} \left[ {}_3F_2\left(1, 1, \epsilon; 2, \frac{3}{2}; z\right) + (z-1) {}_3F_2\left(1, 1, 1+\epsilon; 2, \frac{3}{2}; z\right) \right]$$

Although this expression is not very illuminating, it can be easily expanded in  $z$ . Moreover we can check (with the help of some contiguous hypergeometric relations) that it matches the exact result calculated in the Section 2.4.

### C.3 Dispersion integral

In section 2.3 we calculated the imaginary part of the amplitude. The whole amplitude is given by the dispersion integral

$$I(z) = \frac{1}{4} \int_1^\infty \frac{1}{t} \left(1 - \frac{1}{t}\right) \ln \left( \frac{\sqrt{t} + \sqrt{t-1}}{\sqrt{t} - \sqrt{t-1}} \right) \frac{dt}{t-z}$$

In this section we will show one way how such a function can be integrated. We consider the case  $0 < z < 1$ . Result for other values of  $z$  follows from the analytic continuation. We perform the substitution  $t = \text{ch}^2 v$  to simplify the logarithm. We obtain

$$I(z) = \int_0^\infty \frac{1}{\text{ch } v} \left(1 - \frac{1}{\text{ch}^2 v}\right) \frac{v \text{sh } v dv}{\text{ch}^2 v - z}$$

To continue (we are going to integrate by parts to get rid of  $u$  in numerator of integrand), we calculate following integral (after substituting for  $\text{ch } v$  it is integral of rational function)

$$J(z, v) = \int \frac{1}{\text{ch } v} \left(1 - \frac{1}{\text{ch}^2 v}\right) \frac{\text{sh } v dv}{\text{ch}^2 v - z} = \frac{-1}{z} \left(1 - \frac{1}{z}\right) \left( \ln \text{ch } v - \frac{\ln(\text{ch}^2 v - z)}{2} \right) - \frac{1}{2z \text{ch}^2 v}$$

Now we perform integration by parts in integral for  $I(z)$ . The boundary terms are zero, so

$$I(z) = \int_0^\infty \left[ \frac{1}{z} \left(1 - \frac{1}{z}\right) \left( \ln \text{ch } v - \frac{\ln(\text{ch}^2 v - z)}{2} \right) + \frac{1}{2z \text{ch}^2 v} \right] dv$$

$$= \frac{1}{2z} \int_0^\infty \frac{dv}{\text{ch}^2 v} + \frac{1}{z} \left(1 - \frac{1}{z}\right) \int_0^\infty \left( \ln \text{ch } v - \frac{1}{2} \ln(\text{ch } v - \sqrt{z}) - \frac{1}{2} \ln(\text{ch } v + \sqrt{z}) \right) dv$$

Using the definition of dilogarithm in appendix, we find (dropping the unimportant integration constants)

$$\begin{aligned} \int \ln(1 + e^{ax}) dx &= -\frac{1}{a} \text{Li}_2(-e^{ax}) \\ \int \ln(e^{\pm x} + z) dx &= x \ln z \mp \text{Li}_2\left(\frac{-e^{\pm x}}{z}\right) \\ \int \ln \text{ch } x dx &= \frac{x^2}{2} - x \ln 2 + \frac{1}{2} \text{Li}_2(-e^{-2x}) \\ \int \ln(\text{ch } x + z) dx &= \frac{-x^2}{2} - x \ln 2 - \text{Li}_2\left(\frac{-e^x}{z + \sqrt{z^2 - 1}}\right) - \text{Li}_2\left(\frac{-e^x}{z - \sqrt{z^2 - 1}}\right) \\ &= \frac{x^2}{2} - x \ln 2 + \text{Li}_2\left(\frac{-e^{-x}}{z + \sqrt{z^2 - 1}}\right) + \text{Li}_2\left(\frac{-e^{-x}}{z - \sqrt{z^2 - 1}}\right) \end{aligned}$$

Substituting these formulas and using the relations among dilogarithms we finally arrive at

$$\int_0^\infty \ln \frac{\text{ch } v}{\text{ch}^2 v - z} dv = -\frac{1}{4} \left[ \frac{\pi^2}{2} + \ln^2(\sqrt{z} + \sqrt{z-1}) + \ln^2(-\sqrt{z} - \sqrt{z-1}) \right] = \frac{1}{2} \left[ \text{arctg} \sqrt{\frac{1-z}{z}} - \frac{\pi}{2} \right]^2$$

so the formula for  $I(z)$  reads

$$\begin{aligned} I(z) &= \frac{1}{2z} + \frac{1}{2z} \left(1 - \frac{1}{z}\right) \left[ \text{arctg} \sqrt{\frac{1-z}{z}} - \frac{\pi}{2} \right]^2 = \frac{1}{2z} + \frac{1}{2z} \left(1 - \frac{1}{z}\right) \text{arctg}^2 \sqrt{\frac{z}{1-z}} \\ &= \frac{1}{2z} \left[ 1 - \frac{(1+u)^2}{2(1-u)^2} \text{H}_{0,0}(u) \right] \end{aligned}$$

Note that we could probably use the proper variable  $u$  from the beginning and according to [9] use the substitution

$$t = \frac{1}{1-x^2}$$

which is related to the integration variable  $v$  by

$$x = \text{tgh } v$$

## C.4 Integral encountered in the evaluation of vacuum bubble integrals

In this section we will show how to integrate the integral encountered in Section 3.3.

$$I = -\frac{(m_0^2)^{d-4}}{(4\pi)^d} \Gamma(4-d) \int_0^1 dx \int_0^1 dy \frac{y(1-y)^{1-\frac{d}{2}}}{(x(1-x))^{2-\frac{d}{2}} (y+(1-y)\mu^2)^{4-d}}$$

where

$$\mu^2 = \frac{ax + b(1-x)}{x(1-x)}$$

Comparing the integral over  $y$  with the integral representation of Gauss hypergeometric function (B.2) we see almost immediately, that

$$I = -\frac{(m_0^2)^{-2\epsilon}}{(4\pi)^{4-2\epsilon}} \frac{\Gamma(2\epsilon)\Gamma(\epsilon)}{\Gamma(2+\epsilon)} \int_0^1 dx (\mu^2)^{-2\epsilon} (x(1-x))^{-\epsilon} {}_2F_1\left(2, 2\epsilon; 2+\epsilon; 1 - \frac{1}{\mu^2}\right)$$

We could expand the integrand in  $\epsilon = 2 - \frac{d}{2}$  first and then integrate the resulting integrals, but expanding  ${}_2F_1$  is simpler once we know the expansion of  ${}_2F_1$  in parameters. Using the relations between contiguous hypergeometric functions, namely

$${}_2F_1(2\epsilon, 2; 2 + \epsilon; z) = \frac{\epsilon + 1}{z(\epsilon - 1)} [\epsilon + (-z - \epsilon + 2\epsilon z){}_2F_1(1, 2\epsilon; 1 + \epsilon; z)]$$

and formula (B.10), gives

$$I = -\frac{(m_0^2)^{-2\epsilon} \Gamma(2\epsilon)\Gamma(\epsilon)}{(4\pi)^{4-2\epsilon} \Gamma(2 + \epsilon)} \int_0^1 dx \left[ 1 - \epsilon \ln(x(1-x)) - 2\epsilon^2 + \frac{\epsilon^2}{2} \ln^2(x(1-x)) \right. \\ \left. + 2\epsilon^2 \operatorname{Li}_2(1 - \mu^2) - 2\epsilon^2 \frac{\mu^2}{1 - \mu^2} \ln \mu^2 \right]$$

The integration of terms on the first line is elementary, so all that is left is the integration of the two terms on the second line. Integration of the second term is straightforward (although lengthy) and the result brought to a symmetric form and expressed in terms of dilogarithms is

$$-\int_0^1 dx \frac{\mu^2}{1 - \mu^2} \ln \mu^2 = \left( \frac{-\lambda^2 - a - b + 1}{2\lambda} + \frac{a - b}{2} \right) \\ \left[ \operatorname{Li}_2 \left( \frac{\lambda + 1 + a - b}{\lambda + 1 - a + b} \right) - \operatorname{Li}_2 \left( \frac{\lambda - 1 + a - b}{\lambda - 1 - a + b} \right) + \operatorname{Li}_2 \left( \frac{\lambda - 1 - a + b}{\lambda + 1 - a + b} \right) - \operatorname{Li}_2 \left( \frac{\lambda + 1 - a + b}{\lambda - 1 - a + b} \right) \right. \\ \left. + \ln \left( \frac{\lambda + 1 + a - b}{\lambda + 1 - a + b} \right) \ln \left( 1 - \frac{\lambda + 1 + a - b}{\lambda + 1 - a + b} \right) - \ln \left( \frac{\lambda - 1 + a - b}{\lambda - 1 - a + b} \right) \ln \left( 1 - \frac{\lambda - 1 + a - b}{\lambda - 1 - a + b} \right) \right] \\ + (\lambda \leftrightarrow -\lambda)$$

where  $\lambda$  is shorthand for

$$\lambda = \sqrt{\lambda^2(a, b, 1)}$$

with Källén quadratic form

$$\lambda^2(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2yz - 2zx$$

The most difficult part of the integration is the integral

$$\int_0^1 dx \operatorname{Li}_2(1 - \mu^2) = -\int_0^1 \frac{dy}{y} \int_0^1 dx \ln \left( 1 - y \frac{x(1-x) - ax - b(1-x)}{x(1-x)} \right)$$

We first perform an integration over the variable  $x$  which yields

$$\int_0^1 \operatorname{Li}_2(1 - \mu^2) dx = -\frac{\pi^2}{6} - \int_0^1 \frac{dy}{2y(1-y)} \left[ -4(1-y) \ln 2 \right. \\ \left. + (1 + y(a - b - 1) + S) \ln(1 + y(a - b - 1) + S) - (-1 - y(a - b - 1) + S) \ln(-1 - y(a - b - 1) + S) \right. \\ \left. + (1 + y(b - a - 1) + S) \ln(1 + y(b - a - 1) + S) - (-1 - y(b - a - 1) + S) \ln(-1 - y(b - a - 1) + S) \right]$$

with

$$S = \sqrt{\lambda^2 y^2 + 2y(a + b - 1) + 1}$$

Notice the manifest symmetry in  $a$  and  $b$ . Now we apply the Euler substitution

$$\sqrt{1 + 2\alpha y + \lambda^2 y^2} = 1 - yz$$

which implies

$$z = \frac{1 - S}{y}$$

and

$$y = \frac{2(z + \alpha)}{z^2 - \lambda^2}$$

The Jacobian of this transformation is

$$\frac{dy}{dz} = \frac{2}{z^2 - \lambda^2} - \frac{4z(z + \alpha)}{(z^2 - \lambda^2)^2}$$

so after the substitution and a shift of the integration variables we obtain

$$\begin{aligned} \int_0^1 dx \operatorname{Li}_2(1 - \mu^2) &= \frac{\pi^2}{6} - \int_{\sqrt{(a-b)^2}}^{a+b} dz \left( \frac{1}{a+b-z} + \frac{1}{z-a+b} + \frac{1}{z+a-b} \right) \frac{1}{(z-1+\lambda)(z-1-\lambda)} \\ &\quad \left[ -2z(a+b-z) \ln(a+b-z) - (z+a-b)(z-a+b) \ln((z+a-b)(z-a+b)) \right. \\ &\quad \left. + 2a(z-a+b) \ln 2a + 2b(z+a-b) \ln 2b \right] \end{aligned}$$

Now all integrals can be expressed in a straightforward way in terms of dilogarithms

$$\begin{aligned} \int_0^1 -\operatorname{Li}_2(1 - \mu^2) dx &= \\ &\left\{ \frac{a-b}{2} \left[ \operatorname{Li}_2 \left( \frac{\lambda+1+a-b}{\lambda+1-a+b} \right) - \operatorname{Li}_2 \left( \frac{\lambda-1+a-b}{\lambda-1-a+b} \right) \right] + \frac{a-b-\lambda}{2} \left[ \operatorname{Li}_2 \left( \frac{\lambda-1-a+b}{\lambda+1-a+b} \right) - \operatorname{Li}_2 \left( \frac{\lambda+1-a+b}{\lambda-1-a+b} \right) \right] \right\} \\ &\quad + \{\lambda \leftrightarrow -\lambda\} \\ &\quad + \frac{1}{2} \ln(a) \ln(b) + \frac{a-b}{2} \ln \frac{a}{b} \ln((a-b)^2) - \frac{1}{4} (a-b) \ln \frac{a}{b} \ln ab \\ &+ \frac{a-b}{4} \left[ \ln^2 \left( \frac{\lambda-1+a-b}{\lambda+1-a+b} \right) - \ln^2 \left( \frac{\lambda+1+a-b}{\lambda-1-a+b} \right) \right] - \frac{\lambda}{4} \left[ \ln \frac{a}{b} \ln \left( \frac{\lambda+1+a-b}{\lambda-1-a+b} \right) + \ln \frac{a}{b} \ln \left( \frac{\lambda-1+a-b}{\lambda+1-a+b} \right) \right] \end{aligned}$$

which is valid as long as  $a > 0$  and  $b \in (a+1-2\sqrt{a}; a+1+2\sqrt{a})$ , the boundary terms given by the zeros of  $\lambda^2(1, a, b)$ . Adding this to the previous result, we obtain the final result of integration (valid for  $\lambda^2 < 0$ )

$$\begin{aligned} \int_0^1 \left( \operatorname{Li}_2(1 - \mu^2) - \frac{\mu^2}{1 - \mu^2} \ln(\mu^2) \right) dx &= -\frac{1}{2} \ln a \ln b + \frac{1-a-b}{\lambda} \left[ \operatorname{Li}_2 \left( \frac{\lambda-1-a+b}{\lambda+1-a+b} \right) - \operatorname{Li}_2 \left( \frac{\lambda+1+a-b}{\lambda-1-a+b} \right) \right] \\ &+ \frac{1}{4} \ln^2 \left( \frac{\lambda+1-a+b}{\lambda+1-a+b} \right) - \frac{1}{4} \ln^2 \left( \frac{\lambda-1+a-b}{\lambda-1-a+b} \right) + \frac{1}{4} \ln^2 \left( -\frac{\lambda+1-a+b}{\lambda-1-a+b} \right) - \frac{1}{4} \ln^2 \left( -\frac{\lambda+1+a-b}{\lambda-1-a+b} \right) \end{aligned}$$

which is again valid for  $a > 0$  and  $b \in (a+1-2\sqrt{a}; a+1+2\sqrt{a})$ .

Let us add a few words about bringing the integrals to the form shown above. When performing the integration we often split the integrand in an ‘‘artificial’’ way which is comfortable for the integration, but which brings in many new terms which cancel between various parts of integral. Functions that we obtained can be written in many different forms and it can be difficult to see which terms add up to zero. To see these cancellations in more a systematic way, it is useful to choose a smaller set of expressions and try to express all the expressions with their help. We first specialize to values of  $a, b$  which satisfy  $\sqrt{(a-b)^2} = a-b$ . This breaks the symmetry of the result with respect to the exchange of  $a \leftrightarrow b$ , but we can restore it at any time by using this equation again. Then we are left with expressions containing

$$\begin{aligned} (\lambda+1-a-b) &= \frac{(\lambda+1+a-b)(\lambda+1-a+b)}{2} \\ -4a &= (\lambda+1+a-b)(\lambda-1-a+b) \\ -4b &= (\lambda+1-a+b)(\lambda-1+a-b) \\ \frac{(a-b)^2 - (a+b) + \lambda(a-b)}{-2a} &= \frac{\lambda-1+a-b}{\lambda-1-a+b} \\ \frac{(a-b)^2 - (a+b) - \lambda(a-b)}{-2b} &= \frac{\lambda-1-a+b}{\lambda-1+a-b} \end{aligned}$$

This suggests trying to express everything using just

$$\lambda+1+a-b$$

$$\begin{aligned} \lambda + 1 - a + b \\ \lambda - 1 + a - b \\ \lambda - 1 - a + b \end{aligned}$$

Next we start to simplify the dilogarithms. We can use the one-variable identities for the dilogarithm to reduce their number. Note that all of these identities relate the dilogarithm at one point to the dilogarithm at another point which is related to the first point by a linear fractional transformation in the complex plane which fixes points  $\{0, 1, \infty\}$  (modulo logarithmic terms). These logarithmic terms are the reason why we leave the simplification of logarithms at the end. If we are unlucky (as was our case), there remains one complicated dilogarithm, in our calculation it was

$$\text{Li}_2 \left( \frac{(\lambda + 1 + a - b)(\lambda - 1 - a + b)}{(\lambda + 1 - a + b)(\lambda - 1 + a - b)} \right)$$

This one cannot be expressed using simpler dilogarithms using only one-variable dilogarithmic identities. But formula (B.5) saves the situation. Remaining thing to be done is the simplification of the logarithms which is straightforward due to strong logarithm identities available.

# Bibliography

- [1] U. Aglietti and R. Bonciani. Master integrals with 2 and 3 massive propagators for the 2-loop electroweak form factor: Planar case. *Nucl. Phys.*, B698:277–318, 2004.
- [2] U. Aglietti, R. Bonciani, G. Degrassi, and A. Vicini. Master integrals for the two-loop light fermion contributions to  $gg \rightarrow H$  and  $H \rightarrow \gamma\gamma$ . *Phys. Lett.*, B600:57–64, 2004.
- [3] U. Aglietti, R. Bonciani, G. Degrassi, and A. Vicini. Two-loop light fermion contribution to Higgs production and decays. *Phys. Lett.*, B595:432–441, 2004.
- [4] U. Aglietti, R. Bonciani, G. Degrassi, and A. Vicini. Analytic results for virtual QCD corrections to Higgs production and decay. *JHEP*, 01:021, 2007.
- [5] R. Akhoury, H. Wang, and Oleg I. Yakovlev. On the resummation of large qcd logarithms in  $h \rightarrow \gamma\gamma$ . *Phys. Rev.*, D64:113008, 2001.
- [6] Charalampos Anastasiou, Stefan Beerli, Stefan Bucherer, Alejandro Daleo, and Zoltan Kunszt. Two-loop amplitudes and master integrals for the production of a Higgs boson via a massive quark and a scalar-quark loop. *JHEP*, 01:082, 2007.
- [7] Charalampos Anastasiou and Achilleas Lazopoulos. Automatic integral reduction for higher order perturbative calculations. *JHEP*, 07:046, 2004.
- [8] G.E. Andrews, R. Askey, and R. Roy. Special functions. Cambridge University Press (2001) 620 p.
- [9] V.B. Berestetsky, E.M. Lifshitz, and L.P. Pitaevsky. Quantum electrodynamics. Oxford, Uk: Pergamon (1982) 652 P. (Course Of Theoretical Physics, 4).
- [10] R. Bonciani, P. Mastrolia, and E. Remiddi. Vertex diagrams for the QED form factors at the 2-loop level. *Nucl. Phys.*, B661:289–343, 2003.
- [11] R. Bonciani, P. Mastrolia, and E. Remiddi. Master integrals for the 2-loop QCD virtual corrections to the forward-backward asymmetry. *Nucl. Phys.*, B690:138–176, 2004.
- [12] T. P. Cheng and L. F. Li. Gauge theory of elementary particle physics. Oxford, Uk: Clarendon (1984) 536 P. (Oxford Science Publications).
- [13] K. G. Chetyrkin and F. V. Tkachov. Integration by parts: The algorithm to calculate  $\beta$  functions in 4 loops. *Nucl. Phys.*, B192:159–204, 1981.
- [14] S. Dawson and R. P. Kauffman. QCD corrections to  $H \rightarrow \gamma\gamma$ . *Phys. Rev.*, D47:1264–1267, 1993.
- [15] Giuseppe Degrassi and Fabio Maltoni. Two-loop electroweak corrections to the Higgs-boson decay  $H \rightarrow \gamma\gamma$ . *Nucl. Phys.*, B724:183–196, 2005.
- [16] A. Djouadi, P. Gambino, and Bernd A. Kniehl. Two-loop electroweak heavy-fermion corrections to Higgs-boson production and decay. *Nucl. Phys.*, B523:17–39, 1998.
- [17] A. Djouadi, M. Spira, J. J. van der Bij, and P. M. Zerwas. QCD corrections to  $\gamma\gamma$  decays of Higgs particles in the intermediate mass range. *Phys. Lett.*, B257:187–190, 1991.

- [18] Manuel Drees and Ken-ichi Hikasa. Note on QCD corrections to hadronic Higgs decay. *Phys. Lett.*, B240:455, 1990.
- [19] John R. Ellis, Mary K. Gaillard, and Dimitri V. Nanopoulos. A phenomenological profile of the Higgs boson. *Nucl. Phys.*, B106:292, 1976.
- [20] J. Fleischer and O. V. Tarasov. Calculation of *feynman* diagrams from their small momentum expansion. *Z. Phys.*, C64:413–426, 1994.
- [21] J. Fleischer, O. V. Tarasov, and V. O. Tarasov. Analytical result for the two-loop QCD correction to the decay  $H \rightarrow 2\gamma$ . *Phys. Lett.*, B584:294–297, 2004.
- [22] Frank Fugel, Bernd A. Kniehl, and Matthias Steinhauser. Two-loop electroweak correction of  $O(G_F M_t^2)$  to the Higgs-boson decay into photons. *Nucl. Phys.*, B702:333–345, 2004.
- [23] T. Gehrmann and E. Remiddi. Differential equations for two-loop four-point functions. *Nucl. Phys.*, B580:485–518, 2000.
- [24] S. L. Glashow. Partial symmetries of weak interactions. *Nucl. Phys.*, 22:579–588, 1961.
- [25] D. J. Gross and Frank Wilczek. Ultraviolet behavior of non-abelian gauge theories. *Phys. Rev. Lett.*, 30:1343–1346, 1973.
- [26] Robert Harlander and Philipp Kant. Higgs production and decay: Analytic results at next-to-leading order QCD. *JHEP*, 12:015, 2005.
- [27] F. Hoogeveen. The influence of a heavy fermion doublet on Higgs boson production via the gluon fusion mechanism. *Nucl. Phys.*, B259:19, 1985.
- [28] E.L. Ince. Ordinary differential equations. Dover Publications (1956) 558 p.
- [29] M. Inoue, R. Najima, T. Oka, and J. Saito. QCD corrections to two photon decay of the Higgs boson and its reverse process. *Mod. Phys. Lett.*, A9:1189–1194, 1994.
- [30] C. Itzykson and J.B. Zuber. Quantum field theory. New York, Usa: McGraw-Hill (1980) 705 p. (International Series In Pure and Applied Physics).
- [31] M. Yu. Kalmykov, B. F. L. Ward, and S. A. Yost. Multiple (inverse) binomial sums of arbitrary weight and depth and the all-order epsilon-expansion of generalized hypergeometric functions with one half-integer value of parameter. 2007.
- [32] M. Yu. Kalmykov, B. F. L. Ward, and S. A. Yost. On the all-order epsilon-expansion of generalized hypergeometric functions with integer values of parameters. 2007.
- [33] Bernd A. Kniehl. Dominant two-loop electroweak correction to  $H \rightarrow \gamma\gamma$ . 2005.
- [34] J. G. Korner, K. Melnikov, and Oleg I. Yakovlev. Two loop  $O(G_F m_H^2)$  radiative corrections to the Higgs decay width  $H \rightarrow \gamma\gamma$  for large Higgs boson masses. *Phys. Rev.*, D53:3737–3745, 1996.
- [35] M.I. Kotsky and Oleg I. Yakovlev. Resummation of double logarithms in the process  $H \rightarrow \gamma\gamma$ . *Phys. Lett.*, B418:335–344, 1998.
- [36] S. Laporta. High-precision calculation of multi-loop *feynman* integrals by difference equations. *Int. J. Mod. Phys.*, A15:5087–5159, 2000.
- [37] S. Laporta and E. Remiddi. The analytical value of the electron (g-2) at order  $\alpha^3$  in QED. *Phys. Lett.*, B379:283–291, 1996.
- [38] L. Lewin. Polylogarithms and associated functions. Elsevier Science Ltd. (1981) 359 p.
- [39] Yi Liao and Xiao-yuan Li.  $O(\alpha^2 G_F m_t^2)$  contributions to  $H \rightarrow \gamma\gamma$ . *Phys. Lett.*, B396:225–230, 1997.
- [40] D Maitre. HPL, a mathematica implementation of the harmonic polylogarithms. *Comput. Phys. Commun.*, 174:222–240, 2006.

- [41] Michael Melles. Renormalization group effects in the process  $h \rightarrow \gamma\gamma$ . *Phys. Rev.*, D60:075009, 1999.
- [42] K. Melnikov and Oleg I. Yakovlev.  $H \rightarrow \gamma\gamma$  decay: QCD radiative correction. *Phys. Lett.*, B312:179–183, 1993.
- [43] Noboru Nakanishi. Graph theory and *feynman* integrals. Gordon and Breach, Science Publishers, Inc. (1971) 224 p.
- [44] Giampiero Passarino, Christian Sturm, and Sandro Uccirati. Complete two-loop corrections to  $H \rightarrow \gamma\gamma$ . 2007.
- [45] W. Pauli and F. Villars. On the invariant regularization in relativistic quantum theory. *Rev. Mod. Phys.*, 21:434–444, 1949.
- [46] Michael E. Peskin and D. V. Schroeder. An introduction to quantum field theory. Reading, USA: Addison-Wesley (1995) 842 p.
- [47] H. David Politzer. Reliable perturbative results for strong interactions? *Phys. Rev. Lett.*, 30:1346–1349, 1973.
- [48] E. Remiddi and J. A. M. Vermaseren. Harmonic polylogarithms. *Int. J. Mod. Phys.*, A15:725–754, 2000.
- [49] Abdus Salam. Weak and electromagnetic interactions. Originally printed in \*Svartholm: Elementary Particle Theory, Proceedings Of The Nobel Symposium Held 1968 At Lerum, Sweden\*, Stockholm 1968, 367-377.
- [50] Mikhail A. Shifman, A. I. Vainshtein, M. B. Voloshin, and Valentin I. Zakharov. Low-energy theorems for Higgs boson couplings to photons. *Sov. J. Nucl. Phys.*, 30:711–716, 1979.
- [51] M. Spira, A. Djouadi, D. Graudenz, and P. M. Zerwas. Higgs boson production at the LHC. *Nucl. Phys.*, B453:17–82, 1995.
- [52] Matthias Steinhauser. Corrections of  $O(\alpha_s^2)$  to the decay of an intermediate-mass Higgs boson into two photons. 1996.
- [53] F. V. Tkachov. A theorem on analytical calculability of four loop renormalization group functions. *Phys. Lett.*, B100:65–68, 1981.
- [54] J. van der Bij and M. J. G. Veltman. Two loop large Higgs mass correction to the  $\rho$  parameter. *Nucl. Phys.*, B231:205, 1984.
- [55] Steven Weinberg. The quantum theory of fields. vol. 1: Foundations. Cambridge, UK: Univ. Pr. (1995) 609 p.
- [56] Steven Weinberg. The quantum theory of fields. vol. 2: Modern applications. Cambridge, UK: Univ. Pr. (1996) 489 p.
- [57] Steven Weinberg. A model of leptons. *Phys. Rev. Lett.*, 19:1264–1266, 1967.
- [58] E.T. Whittaker and G.N. Watson. A course of modern analysis. Cambridge University Press (1996) 608 p. (Cambridge Mathematical Library).
- [59] Han-Qing Zheng and Dan-Di Wu. First order QCD corrections to the decay of the Higgs boson into two photons. *Phys. Rev.*, D42:3760–3763, 1990.