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March 22, 2021

Evaluation of the Doctoral Thesis

Numerical Analysis of Problems in Time-Dependent Domains

by

RNDr. Monika Balázsová

Problem and classifcation

In the thesis of RNDr. Monika Balázsová the numerical analysis of a family of discontinuous Galerkin approximation schemes in space and time for problems in time-dependent domains is presented. In these methods the arbitrary Lagrangian-Eulerian technique is used to capture the evolving domains. By introducing new concepts and techniques of error estimation for approximation schemes on time-dependent domains, rigorous stability and error estimates are proved for a nonlinear model problem. Moreover, the results of numerical simulations, based on these approximation schemes, are presented for problems of fluid-structure interaction. A benchmark problem and flow induced vocal folds vibrations are studied numerically. Thereby, the performance and accuracy properties of the proposed numerical approach are demonstrated.

The numerical simulation of multi-physics problems on evolving domains is subject to intensive current research in many branches of natural sciences and technology. Such problems offer an abundance of technical and biomedical applications. Consequently, quantitative methods, based on numerical simulations, are desirable in analyzing experimental data and designing theories based on mathematical concepts. Time-dependent domains add a further facet of considerable complexity on the design, analysis and implementation of discretization schemes for partial differential equations. Thus, the thesis of Monika Balázsová addresses a field of high scientific interest and active current research. The thesis yields a substantial contribution and progress to the ability to analyze numerically multi-physics systems on time-dependent domains. In this thesis, an innovative

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family of discretization schemes is proposed, their rigorous numerical analysis is given and, finally, their performance properties are illustrated by simulations such that a holistic framework for the considered class of problems is provided.

Contents

The thesis of Monika Balázsová starts in the Section *Introduction* with a sketch of the physical problem to be studied in the sequel and a review of literature and available methods for the numerical approximation of partial differential equations on time-dependent domains. In Section 1 the mathematical model of compressible viscous flow is introduced for further motivation of this work and the numerical investigations presented in Section 5. In Section 2 a nonlinear model problem of convection-diffusion type, to be studied in the numerical analysis of the proposed family of discretization schemes, is introduced. Here, I would like to mention that it is a common practice to restrict the numerical analysis to simplified prototype problems that mimic the mathematical structure of more sophisticated multi-physics systems. Further, all assumptions made about the data and coefficient functions as well the definition of the discrete function spaces and forms are given. The arbitrary Lagrangian-Eulerian method (ALE), transforming the problem from the physical domain to a time-independent fixed reference domain for the numerical approximation, is introduced further. The ALE transformation is applied locally, i.e. on each subinterval of the temporal mesh. Finally, a family of space-time discontinuous Galerkin methods (STDGM) that are combined with the ALE technique for the treatment of evolving domains are proposed in Definition 1. Section 1 and 2 yield a clearly written and mathematically consistent statement of the considered problem and the suggested new approach ALE–STDGM.

In Section 3 a stability analysis is presented for ALE–STDGM. The overall result of unconditional stability, that represents one of the key properties for robustness and applicability in practice of the numerical method, is summarized in Theorem 5. Due to the presence of time-dependent domains various new auxiliary results are proved firstly. An essential ingredient in the proof of stability is the application of the discrete characteristic function for controlling some quantity. Here, the concept of the discrete characteristic function is generalized to time-dependent domains and the ALE framework. The proof of stability of ALE–STDGM is characterized by highly technical and complex estimates that are presented with high reliability. In this context, I would like to stress the careful treatment of all constants arising in the estimates. This requires high effort, but illustrates nicely the impact of model, geometry and discretization parameters on the final stability result.

In Section 4 an error analysis is presented for ALE–STDGM. The overall estimate in term of powers of the space and time mesh sizes h and τ is summarized in Theorem 12. Again the concept of the discrete characteristic function is exploited to derive the desired estimate. Even though standard techniques of error estimation (like splitting the error $e = u - U$ into a projection error η and discretization error ξ and making use of the discrete Gronwall lemma) are applied, the evolving domain and ALE technique demand for technical refinements and

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generalizations of results that are available for time-independent domains. This extension strongly complicates the error analysis. Similar to Section 3, the error analysis is also done with high accurateness. Again, the constants of the error estimates are treated carefully and specified with respect to their dependence on problem and discretization parameters which is a challenging task.

Section 5 is devoted to the numerical validation and application of the proposed class of ALE discontinuous Galerkin methods. This is done for a system of fluid-structure interaction (FSI) coupling fluid mechanics with elastodynamics. The numerical simulation of FSI has become an important and highly active field of research over the last two decades with prominent fields of applications. Firstly, the model of compressible viscous flow is recalled and models of linear and nonlinear elasticity are introduced. Further, the coupling conditions of the either subproblems of FSI are presented. Then, the discretization of the flow problem by the suggested ALE–STDGM is carefully derived. For the equations of elastodynamics space-time discontinuous Galerkin approximations are applied as well, such that a uniform discretization framework is obtained for the overall multi-physics system which is advantageous for analyses and implementational issues. The discrete subproblems of fluid flow and elasticity are solved iteratively by a strong coupling approach. To validate the suggested technique, a nonlinear benchmark problem of the literature (cf. [75] of the thesis) is studied. For a sequence of successively refined temporal meshes and different polynomial degrees in time results that are obtained by the ALE–STDGM approach are presented and carefully compared with the results of the literature. A good agreement with results of the literature is shown which confirms the accuracy and efficiency of the used approach to FSI and, thus, of the ALE space-time discontinuous Galerkin method. Moreover, numerical results for neo-Hookean material are presented. In this case, reference values are not available. Finally, flow induced vocal folds vibrations in a simplified two-dimensional human vocal tract are analyzed. Results of simulations that are performed for the linear and nonlinear models of elasticity, introduced before, are carefully compared with each other. The conclusion of the simulations is that nonlinear models of elasticity are required to capture physical realism. Section 5 yields a convincing demonstration of the performance and reliability properties of the proposed ALE space-time discontinuous Galerkin approach. For highly challenging problems, a numerical validation of the proposed methods is presented. The results of the numerical computations are illustrated very clearly and verifiably.

The thesis ends with a summary and some conclusions.

Points of minor criticism

I have the following points of criticism that, however, do not reduce the high scientific standard and value of the thesis.

- In Section *Introduction* a more general review of approaches for evolving domains could have been given. Only ALE type methods are focused. Recently, approaches based on combined Cut–FEM and ghost penalty

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approaches have been studied intensively in the literature as well.

- In same cases, definitions or auxiliaries are introduced in the (sub-)sections of the numerical analysis, i.e., Section 3 and 4. This minors the readability of the thesis. All definitions could have been done more strictly in an extra section, or in subsections of Section 2, respectively.
- In Subsection 4.2.1 and 4.2.3 abstract error estimates are presented for $q = 1$ and $q \geq 2$. It's hard to see why the case $q = 1$ is studied separately and not within the framework of Subsection 4.2.3. An additional remark regarding the necessity of this separation in the error estimation would have been helpful. In my reading, I did not find any explicit explanation.
- In Subsection 5.4.1, only piecewise polynomials are used for the approximation in space ($s = 1$). In particular in the case of piecewise quadratic polynomials in time ($q^* = 2$), also the usage of polynomials of higher order in space would have been of interest and a logical consequence.

Summary appraisal

In this thesis RNDr. Monika Balázsová proposed a new family of numerical methods for a challenging problem of multi-physics and high interest in practice. By rigorous error estimates for a model problem and numerical simulations for the multi-physics system a profound validation of the schemes is provided. All proofs are characterized by high accurateness. The simulations illustrate the expected performance properties of the numerical methods impressively. By this thesis RNDr. Balázsová has shown her scientific qualification and ability for creative scientific work. She has demonstrated her expertise to solve complex problems by innovative methods and new mathematical tools on highest scientific level.

I recommend the Faculty of Mathematics and Physics to accept this work as a doctoral thesis and to continue the doctoral examination with the defence.

Yours sincerely

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