Math ICJ

Institut Camille Jordan
UMR 5208 du CNRS

Frank Wagner
Full Professor
Institut Camille Jordan
Université Claude Bernard Lyon 1
wagner@math.univ-lyon1.fr

Lyon, 12 February 2020

Review of the Habilitation Thesis of Jan Hubička

The habilitation thesis of Jan Hubička consists of nine scientific papers, six of which have already appeared in scientific journals (including one book chapter), two have been submitted, and the ninth one is available in preprint form. Of these nine papers, eight are in collaboration with Jaroslav Nešetřil (Charles University), three with David Evans (Imperial College London) and four with Matěj Konečný (Charles University); only one paper has other co-authors. They are preceeded by a forty-page introduction, giving a very clear and comprehensible survey of the domain, the background, and the results obtained.

The work presented in this thesis constitutes a major advance in Nešetřil's classifiaction programme for Ramsey classes, and, in a parallel development, of the corresponding classification programme for EPPA classes (to be defined below). In fact, one important contribution of Hubička is the fact of having unified the treatment of these two disjoint but analogous cases.

Recall that given a countable language $\mathcal{L}$, a Fraïssé class $\mathcal{K}$ is a class of finite $\mathcal{L}$-structures closed under isomorphism and substructure (hereditary), and which has the joint embedding (JEP) and amalgamation (AP) properties. A fundamental theorem of Fraïssé states that given a Fraïssé class $\mathcal{K}$, there is a unique countable ultrahomogeneous structure $\mathfrak{M}_{\mathcal{K}}$, its *Fraïssé limit*, whose age (i.e. the collection of finite substructures) is precisely $\mathcal{K}$. Thus, properties of the Fraïssé class can be studied by means of its Fraïssé limit and vice versa. We say that $\mathcal{K}$ is a *Ramsey class*, or has the *Ramsey property*, if for every $A, B \in \mathcal{K}$ there is $C \in \mathcal{K}$ such that for every colouring of all embeddings of A into C with two (or finitely many) colours there is an embedding f of B into C such that all embeddings of A into $f(B)$ have the same colour. This generalizes the classical Ramsey theorem which states that the class of all finite linear orders is Ramsey. The class $\mathcal{K}$ has *EPPA* (extension property for partial automorphisms), or the *Hrushovski property*, if for every $A \in \mathcal{K}$ there is $B \in \mathcal{K}$ containing A such that every partial automorphism of A extends to an automorphism of B; the structure B is called an *EPPA witness* for A. Finally, $\mathcal{K}$ has *coherent* EPPA if for all $A \in \mathcal{K}$ there is an EPPA witness B and a map $\phi \mapsto \hat{\phi}$ from the partial automorphisms of A to the automorphisms of B such that whenever the range of ϕ equals the domain of ψ, then $\widehat{\psi} \circ \widehat{\phi} = \widehat{\psi \circ \phi}$.

Note that under JEP both the Ramsey and EPPA properties imply AP. Thus Fraïssé

Université Claude Bernard Lyon 1

Bât. Braconnier – 43, bd. du 11 Novembre
F-69622 Villeurbanne Cedex, France
Tél. : +33(0)4724 32941 Télécopie : +33(0)4724 31687

classes form a natural set-up for their analysis. Moreover, there are deep connections to topological dynamics, on one hand through the Kechris-Pestov-Todorčević correspondence between the Ramsey property and extreme amenability of the automorphism group of the Fraïssé limit, on the other hand via the Kechris-Rosendal theorem that EPPA implies amenability of the automorphism group.

Now the automorphism group of every Ramsey structure (i.e. Fraïssé limit of a Ramsey class) stabilizes some linear order. On the other hand, linearly odered finite structures have no automorphisms. Thus, Ramsey and EPPA classes are disjoint (except for trivial examples).

The principal results of Jan Hubička can be found in papers [HN19] for the Ramsey property and [HKN19b] for EPPA classes, where he shows that:

- For any countable language $\mathcal{L}$ with an order, the class of all finite linearly ordered $\mathcal{L}$-structures is Ramsey.
- For any finite language $\mathcal{L}$ with only unary function symbols, the class of all finite $\mathcal{L}$-structures has coherent EPPA.

Both essentially not only unify all previously known results, mainly by Nešetřil-Rödl but also [HN18a] for Ramsey classes, and by Hrushovski, Herwig, and [EHN17, EHN19] for EPPA classes, but also add numerous new structures to the list. They thus constitute a definitive solution for the unrestricted problem, and a major advance for the subject.

Concerning restricted classes, we need some more definitions. A homomorphism $A \to B$ is a homomorphism-embedding if its restriction to every *irreducible* substructure of A (i.e. which does not decompose as a free amalgam of proper substructures) is an embedding. An amalgamation $f_i : A \to B_i$ and $g_i : B_i \to C$ for $i = 1, 2$ is *strong* if $g_1(B_1) \cap g_2(B_2) = g_1(f_1(A)) = g_2(f_2(A))$. Given a class $\mathcal{K}$, a subclass $\mathcal{K}_0$ of irreducible structures is *locally finite* if for every $A \in \mathcal{K}_0$ and $B_0 \in \mathcal{K}$ there is a finite integer n such that every $\mathcal{L}$-structure B has a homomorphism-embedded image $B' \in \mathcal{K}_0$ provided

1. every irreducible substructure of B can be embedded into A,
2. there is a homomorphism-embedding of B to B_0, and
3. every substructure of B on at most n vertices homomorphism-embeds into a structure in $\mathcal{K}$.

$\mathcal{K}_0$ is locally finite *automorphism preserving* if B' can always be chosen to contain B as (possibly not fully induced) substructure and such that there is a homomorphism $f : \mathrm{Aut}(A) \to \mathrm{Aut}(B')$ such that $f(\sigma)$ extends σ for all $\sigma \in \mathrm{Aut}(A)$.

Then [HN19, HKN19b]:

- Let $\mathcal{L}$ be a countable language, $\mathcal{K}$ be a Ramsey class of irreducible finite $\mathcal{L}$-structures, and $\mathcal{K}_0$ a hereditary locally finite subclass with strong amalgamation. Then $\mathcal{K}_0$ is Ramsey.
- Let $\mathcal{L}$ be finite language with only unary function symbols, $\mathcal{K}$ be an EPPA class of irreducible finite $\mathcal{L}$-structures, and $\mathcal{K}_0$ a hereditary locally finite automorphism-preserving subclass with strong amalgamation. Then $\mathcal{K}_0$ has EPPA. Moreover, if $\mathcal{K}$

is coherent, so is $\mathcal{K}_0$.

These two theorems yield the Ramsey property or EPPA for numerous examples old and new, such as free amalgamation classes, partial orders (Ramsey only—they cannot have EPPA), and metric spaces (e.g. [HKN19a]). They describe a common and general framework how to obtain Ramsey or EPPA results. However, there are examples where this approach does not seem to apply, such as EPPA for the class of all finite groups, or for the class of all skew bilinear forms, and others where a possible application is work in progress.

It should be further noted that the results concerning EPPA are phrased in a more general framework, which allows arity-preserving permutations of the language (permorphisms), and multi-valued functions.

A negative result is given in [EHN19], where an ω-categorical structure $\mathfrak{M}$ is constructed by Hrushovki amalgamation which has no *canonical* Ramsey expansion, but a non-canonical one, and which is not EPPA, but has an EPPA expansion.

In my opinion, the results assembled in Jan Hubička's habilitation thesis form a major contribution to combinatorics, more precisely to (generalized) Ramsey theory. The provide a definite answer in the unrestricted contect to an important open problem (or two: whether a class is Ramsey or has EPPA), and a unified a highly general approach in the restricted context. The methods used build on earlier work of Nešetřil and others, but constitute a significant amelioration, simplifying, unifying and generalizing previous approaches. Moreover, the paper [EHN19] in appendix E also delineates the boundaries by giving a counter-example to both properties. I recommend acceptance of this thesis for the habilitation process.

[redacted]