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report on the habilitation thesis

„Mathematical Analysis of Nonlinear Systems Describing Flows of Incompressible Fluids“

by Dr. Miroslav Bulíček

The habilitation thesis consists of 6 original papers and a summary describing the context and the methods in these papers. The presented results are impressive and constitute a nice piece of deep mathematical analysis.

The common topic of the papers is the existence and qualitative analysis of nonlinear partial differential equations describing the motion of fluids with complicated rheology. The papers [B1], [B2] and [B4] deal with a purely mechanical problem. The methods used therein can be generalized to treat thermomechanical problems in [B3] and [B6] as well as a problem from turbulence in [B5].

Dr. Bulíček published according to MathSciNet 73 papers within 15 years, which are cited 462 times from 295 authors. The number of publications is really impressive and shows the deep understanding of the underlying structure and the ability to use the developed techniques in different settings. The topics covered in these papers are very wide and include among others existence theory for elliptic, parabolic and hyperbolic problems, regularity theory of elliptic and parabolic problems, analysis of problems from applications as well as long time behaviour of flows. For me and many others Dr. Bulíček is one of the leading young personalities in nonlinear analysis worldwide.

In Chapter 1 the general setup of the thesis, namely the motion of homogeneous incompressible fluids, is presented. The main result of thesis is to show the existence of weak solutions for implicitly constituted fluids under Navier slip boundary conditions and to establish regularity properties in time for these solutions. Moreover, also the existence of weak solutions for the Kolmogorow model of turbulence is shown.

In Chapter 2, which explains the results and the techniques in [B1], [B2] and [B4], the mechanical setting is treated. First the advantages of using implicitly constituted models is explained. The mathematical setting to treat such models are maximal monotone ψ-graphs. Prominent examples for such models

are generalized Newtonian fluids and Herschel–Bulkley fluids. First the notion of weak solutions is given and then the literature concerning the existence theory is discussed. The main result in this setting is Theorem 2.1 showing the existence of weak solutions if $\psi(t) \sim t^q$, $q > \frac{2d}{d+2}$. This is a beautiful result including and generalizing all previous existence results for generalized Newtonian fluids and Herschel–Bulkley fluids in the case of Navier slip boundary conditions. The role of these boundary conditions is discussed and compared to Dirichlet boundary conditions, where additional difficulties would occur. It is stated that also the latter one can be treated with the developed methods. I think that it would be very nice if a paper in the setting of maximal monotone graphs under Dirichlet boundary conditions with details would be written. In fact, the special case of Dirichlet boundary conditions and Herschel–Bulkley fluids is treated in [ER2011], which appeared at the same time as [B4] and is not mentioned in the thesis. The proof of Theorem 2.1 is based on a generalisation of the Minty trick to maximal monotone ψ-graphs, which was not known before in this generality, the extension of the Lipschitz truncation method to general Young functions and the treatment of a Laplace problem with Neumann conditions in the Orlicz setting, which also was not known before.
After that the question of uniqueness and regularity is treated based on the results in [B4] and [B2]. After a discussion of the state of the art the new results for maximal monotone r-graphs with uniform monotonicity (2.52) are discussed. The main new result in [B4] is an improved time integrability of a weak solution in Theorem 2.3, which in turn implies uniqueness and regularity in Bochner-Nikolskii spaces. The used methods extend and improve the method used in [B2], where under an additional local Lipschitz property (2.53) for the explicitly given stress tensor better regularity locally in time in Bochner–Nikolskii spaces is proved in Theorem 2.4. These are really remarkable results, which are based on a clever iteration procedure for divided differences proved in [B2] and extended in [B4]. Based on this and [72] it is claimed that one can get locally in time existence of strong solutions for every $r \geq 11/5$. However, the result in [72] is local in space and as mentioned earlier available global in space regularity results are suboptimal. Thus, it is not clear what by existence of strong solution is meant, it seems to be localised both in time and in space.

In Chapter 3 the ideas from the previous chapter are extended to the treatment of heat conducting fluids. The additional difficulty consist in the just integrable right-hand side of the balance of internal energy. This can be circumvented when working with the total energy balance. The price to pay is the appearance of the pressure in a nontrivial way. Thus Dirichlet boundary condition are excluded from the analysis. The existence theory for explicite models for $r > \frac{3d}{d+2}$ is also due to Bulíček et al. [17], but is not part of the habilitation thesis. The regularity theory for such problems is however widely open. In [B3] and [B6] simplified models are treated and a first step in this direction is achieved. In [B3] the case of a Newtonian fluid with energy dependent viscosity in a space-periodic setting without convective term is treated. The by now classical result of L^2-maximal regularity for the heat equation and related systems is extended to the above setting under a structural condition, which allows for large oscillations in the viscosity. The key to this result is the choice of the „correct" quantity in the energy equation, namely $\sqrt{e}$ instead of e. In [B6] the two-dimensional case including the convective term under space-periodic boundary conditions is treated for a special constitutive relation with r-structure. Again a structural condition is needed to extend the results from [42] to the above setting. The regularity results in this chapter are technically demanding and need structural assumptions. In fact these structural conditions are not obvious and a deep inside is needed to find them.

In Chapter 4 the Kolmogorov model for turbulence is treated. This system has certain similarities with the topics treated in the previous chapters. In [B5] the system is reformulated in a way that the critical terms disappear. To this end again one has to find the „correct" quantities and formulate the problem for these quantities. For reglar solutions the two formulations are equivalent. Under appropriate assumptions the existence of global in time suitable weak solutions for large data is proved, which seems to be the first result at all in this direction. Again the integrability of the pressure is important and thus Navier boundary conditions are considered. The ideas used extend the techniques from [16], where Dr. Buíček is also a co-author.

In my opinion the thesis is very well written and presents beautiful results. The topics and papers included are chosen in a clever way among the variety of results of Dr. Bulíček such that they can presented in a unified and coherent manner. Both the papers and the thesis are carefully and well written with only few typos. The few points mentioned above are of minor nature and not essential. The thesis focuses on the main difficulties and ideas and presents in a clear way the solutions to overcome these problems.

Based on this thesis and the other contributions of Dr. Bulíček I **strongly recommend** to aprove the habilitation request of Dr. Bulíček.

Sincerely Yours,

[redacted]

Prof. Dr. Michael Růžička

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