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MASTER THESIS

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Characterization of functions with zero traces via the distance function

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Study programme: Mathematics
Study branch: Mathematical Analysis

Prague 2019

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I thank my supervisor Aleš Nekvinda for his thorough and comprehensive guidance and for devising a perfect topic of my master thesis, which will certainly affect my future mathematical specialization. I would like to thank as well the consultant Luboš Pick for many good advice and the constant support during my studies. I thank also other teachers for sharing their professional knowledge and for helpfulness, and other employees of faculty for their help. And finally, I would like to thank very much my parents and family for their love and enormous support.

Title: Characterization of functions with zero traces via the distance function

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Abstract: Consider a domain $\Omega \subset \mathbb{R}^N$ with Lipschitz boundary and let $d(x) = \operatorname{dist}(x, \partial\Omega)$. It is well known for $p \in (1, \infty)$ that $u \in W_0^{1,p}(\Omega)$ if and only if $u/d \in L^p(\Omega)$ and $\nabla u \in L^p(\Omega)$. Recently a new characterization appeared: it was proved that $u \in W_0^{1,p}(\Omega)$ if and only if $u/d \in L^1(\Omega)$ and $\nabla u \in L^p(\Omega)$. In the author's bachelor thesis the condition $u/d \in L^1(\Omega)$ was weakened to the condition $u/d \in L^{1,p}(\Omega)$, but only in the case $N = 1$. In this master thesis we prove that for $N \geq 1$, $p \in (1, \infty)$ and $q \in [1, \infty)$ we have $u \in W_0^{1,p}(\Omega)$ if and only if $u/d \in L^{1,q}(\Omega)$ and $\nabla u \in L^p(\Omega)$. Moreover, we present a counterexample to this equivalence in the case $q = \infty$.

Keywords: Sobolev spaces, Lorentz spaces, zero traces, Lipschitz domain, distance function.

Contents

Introduction

The theory of Sobolev spaces is widely used in the modern theory of partial differential equations, where the solution is very often described as an element of such spaces. For a solution of the Dirichlet problem under some conditions on the boundary of domain the spaces $W^{1,p}$ and $W_0^{1,p}$ are of crucial importance. The space $W_0^{1,p}$ is classically defined as a closure of smooth functions with a compact support in $W^{1,p}$.

This definition is somewhat theoretical. There are efforts to describe such space in another way, which is possibly more practical for purposes of modeling of such functions. It is proved in [2, Theorem V.3.4] that for certain regular domains $\Omega \subset \mathbb{R}^N$ and $p \in (1, \infty)$ the following equivalence holds:

$$u \in W_0^{1,p}(\Omega) \qquad \text{if and only if} \qquad \frac{u}{d} \in L^p(\Omega) \text{ and } \nabla u \in L^p(\Omega),$$

where the function $d(x)$ is defined as a distance of an element x from the boundary of the domain Ω.

This result was improved several times. In [6] it was shown that taking weak Lebesgue spaces in the condition for the function $\frac{u}{d}$ suffices to get the same conclusion. Namely, for $p \in (1, \infty)$,

$$u \in W_0^{1,p}(\Omega) \qquad \text{if and only if} \qquad \frac{u}{d} \in L^{p,\infty}(\Omega) \text{ and } \nabla u \in L^p(\Omega).$$

In [3], the assumption was further relaxed. The space $L^{p,\infty}$ was replaced with a bigger space, namely L^1, in other words,

$$u \in W_0^{1,p}(\Omega) \qquad \text{if and only if} \qquad \frac{u}{d} \in L^1(\Omega) \text{ and } \nabla u \in L^p(\Omega).$$

This result was extended in [4] to Sobolev spaces of higher order. More precisely, $u \in W_0^{k,p}(\Omega)$ if and only if $\frac{u}{d^k} \in L^1(\Omega)$ and $|D^k u| \in L^p(\Omega)$, where $D^k u$ denotes the vector of all weak derivatives of order k.

The goal of this thesis is to prove the same conclusion under even weaker condition for the space of function $\frac{u}{d}$. Namely, let $\Omega \subset \mathbb{R}^N$ be a domain with Lipschitz boundary. Then, for $p \in (1, \infty)$ and $q \in [1, \infty)$,

$$u \in W_0^{1,p}(\Omega) \qquad \text{if and only if} \qquad \frac{u}{d} \in L^{1,q}(\Omega) \text{ and } \nabla u \in L^p(\Omega).$$

We would like to point out that this condition on $\frac{u}{d}$ can not be weakened to $\frac{u}{d} \in L^{1,\infty}$. We will give a counterexample.

1. Preliminaries

The purpose of this chapter is to give a survey of concepts and results from real and functional analysis, which are in the close relationship with the topic of this thesis and are used in the proofs. Also we present some notation. Almost all this background material can be found in various monographs and articles that will be cited.

1.1 Basic background results from real analysis and measure theory

Before we turn to function spaces, we need to mention some fundamental results from the measure theory and the theory on continuous functions.

Notation 1.1. We denote the n-dimensional Lebesgue measure as λ^N, $N \in \mathbb{N}$. For one-dimensional Lebesgue measure we also write $|\cdot|$.

Let $(\mathscr{R}, \mu)$ be a σ-finite measure space. Let us denote by $\mathscr{M}(\mathscr{R}, \mu)$ the set of all μ-measurable functions from $\mathscr{R}$ to $[-\infty, \infty]$, by $\mathscr{M}_0(\mathscr{R}, \mu)$ the set of all functions from $\mathscr{M}(\mathscr{R}, \mu)$ that are finite μ-almost everywhere (we briefly write μ-a.e.) and by $\mathscr{M}_+(\mathscr{R}, \mu)$ we denote the subset of $\mathscr{M}_0(\mathscr{R}, \mu)$ consisting of nonnegative functions.

We shall write $A \approx B$ if there exist positive constants c_1 and c_2 independent of appropriate quantities involved in A and B such that $c_1 A \leq B \leq c_2 A$.

Theorem 1.2. ([7, Lemma 5.7.1]) *Let E be a λ^N-measurable subset of $\mathbb{R}^N$ and $f\colon E \to \mathbb{R}^N$ be a Lipschitz function with a constant of Lipschitz continuity K (i.e. $|f(x) - f(y)| \leq K\,|x - y|$ for each $x, y \in E$). Then $\lambda^N(f(E)) \leq K^N \lambda(E)$.*

Theorem 1.3. ([8, 30.3 Rademacher's Theorem]) *Let f be a Lipschitz function on an open set $G \subset \mathbb{R}^N$. Then f is differentiable λ^N-almost everywhere in G.*

Theorem 1.4. ([8, Corollary 23.5]) *Let f be an absolutely continuous function on $[a, b] \subset \mathbb{R}$ (we write $f \in AC[a, b]$ or shortly $f \in AC$). Then $\frac{df}{dt}$ exists almost everywhere in $[a, b]$, $\frac{df}{dt} \in L^1([a, b])$ and*

$$f(b) - f(a) = \int_a^b \frac{d}{dt} f(t)\, dt. \tag{1.1}$$

Definition 1.5 (weak derivative)**.** ([13]) Let $\Omega \subset \mathbb{R}^N$ be open set and let $\alpha = (\alpha_1, \ldots, \alpha_N) \subset (\mathbb{N} \cup \{0\})^N$ be a multiindex. Let $u, v^\alpha \in L^1_{\text{loc}}(\Omega)$. We say that v^α is a *weak derivative* of u with respect to α if for every test function $\varphi \in C_0^\infty(\Omega)$ we have

$$\int_\Omega u(x) D^\alpha \varphi(x)\, dx = (-1)^{|\alpha|} \int_\Omega v^\alpha(x) \varphi(x)\, dx,$$

where $|\alpha| = \sum_{i=1}^N \alpha_i$.

Theorem 1.6 (partition of unity)**.** ([13]) *Let $\Omega \subset \mathbb{R}^N$ be a bounded set and let $\{G_i\}_{i=1}^k$ be a system of open sets in $\mathbb{R}^N$ such that $\overline{\Omega} \subset \bigcup_{i=1}^k G_i$. Then there exist non-negative functions $\phi_i \in C_0^\infty(G_i)$, $i = 1, \ldots, k$, such that*

$$\|\phi_i\|_{C(\overline{\Omega})} \leq 1 \quad \textit{and} \quad \sum_{i=1}^k \phi_i(x) = 1 \textit{ for each } x \in \overline{\Omega}.$$

Definition 1.7 (continuous embedding)**.** ([12, Definition 1.15.5]) Let X, Y be two quasinormed linear spaces and let $X \subset Y$. We define the identity operator Id from X into Y as the operator which maps every element $u \in X$ onto itself: $\mathrm{Id}(u) = u$, regarded as an element of Y. We say that the space X is *continuously embedded* into the space Y if the identity operator is continuous, that is, if there exists a constant $c > 0$ such that

$$\|u\|_Y \leq c\, \|u\|_X \quad \text{for every } u \in X.$$

We denote this fact as $X \hookrightarrow Y$.

Let us recall one useful inequality.

Lemma 1.8. *Let $p \in [1, \infty)$. Then for each $a, b \in \mathbb{R}$ we have*

$$|a + b|^p \leq 2^{p-1}(|a|^p + |b|^p).$$

1.2 Lebesgue and Lorentz spaces

In this section we shall define several fundamental spaces, present some of their basic properties and specify certain relations between them.

Definition 1.9 (Lebesgue spaces)**.** ([1, Chapter 1]) Let $1 \leq p \leq \infty$. The collection $L^p(\mathscr{R}) = L^p(\mathscr{R}, \mu)$ of all functions $f \in \mathscr{M}(\mathscr{R}, \mu)$ such that $\|f\|_{L^p(\mathscr{R})} < \infty$, where

$$\|f\|_{L^p(\mathscr{R})} = \begin{cases} \left(\int_{\mathscr{R}} |f|^p d\mu\right)^{\frac{1}{p}}, & 1 \leq p < \infty, \\ \operatorname{ess\,sup}_{\mathscr{R}} |f|, & p = \infty, \end{cases}$$

is called the *Lebesgue space*.

As pointed out in [1], Lebesgue spaces are a pivotal example of the so-called *Banach function spaces*.

Definition 1.10. ([1, Definition 1.1]) We say that a function $\varrho : \mathscr{M}_+(\mathscr{R}, \mu) \to [0, \infty]$ is a *Banach function norm* if, for all f, g and $\{f_n\}_{n=1}^{\infty}$ in $\mathscr{M}_+(\mathscr{R}, \mu)$, for every $\lambda \geq 0$ and for all μ-measurable subsets E of $\mathscr{R}$, the following five properties are satisfied:

(P1) $\varrho(f) = 0 \Leftrightarrow f = 0$ μ-a.e.; $\varrho(\lambda f) = \lambda \varrho(f)$; $\varrho(f + g) \leq \varrho(f) + \varrho(g)$;
(P2) $0 \leq g \leq f$ μ-a.e. in $\mathscr{R} \Rightarrow \varrho(g) \leq \varrho(f)$;
(P3) $0 \leq f_n \nearrow f$ μ-a.e. in $\mathscr{R} \Rightarrow \varrho(f_n) \nearrow \varrho(f)$;
(P4) $\mu(E) < \infty \Rightarrow \varrho(\chi_E) < \infty$;
(P5) $\mu(E) < \infty \Rightarrow \int_E f\, d\mu \leq C_E \varrho(f)$ for some constant $C_E \in (0, \infty)$ possibly depending on E and ϱ but independent of f.

Definition 1.11. Let ϱ be a Banach function norm. We then say that the set $X = X(\varrho)$ of those functions in $\mathscr{M}(\mathscr{R}, \mu)$ for which $\varrho(|f|) < \infty$ is a *Banach function space*. For each $f \in X$ we then define

$$\|f\|_X = \varrho(|f|).$$

Let us denote the *Hölder conjugate exponent* p' to exponent $p \in [1,\infty]$ by

$$p' = \begin{cases} \infty, & p = 1, \\ \frac{p}{p-1}, & p \in (1,\infty), \\ 1, & p = \infty. \end{cases}$$

Theorem 1.12 (Hölder inequality)**.** ([12, Theorem 3.1.6 and Remark 3.10.5]) *Let $1 \le p \le \infty$, $f \in L^p(\mathscr{R})$ and $g \in L^{p'}(\mathscr{R})$. Then $fg \in L^1(\mathscr{R})$ and*

$$\|fg\|_{L^1(\mathscr{R})} \le \|f\|_{L^p(\mathscr{R})} \|g\|_{L^{p'}(\mathscr{R})}.$$

The following theorem is an easy consequence of the Hölder inequality.

Theorem 1.13. *Let $\mathscr{R}$ be a set having finite measure and let $1 \le p_2 < p_1 \le \infty$. Then*

$$L^{p_1}(\mathscr{R}) \hookrightarrow L^{p_2}(\mathscr{R})$$

with a constant of the embedding equal to $\mu(\mathscr{R})^{\frac{1}{p_2}-\frac{1}{p_1}}$.

Theorem 1.14 (Hardy inequality)**.** ([7, Theorem 6.8.7]) *Let $a, b \in \mathbb{R}$, $a < b$, $u \in L^p(a,b)$ and let $p \in (1,\infty)$. Then*

$$\int_a^b \left(\frac{1}{t-a} \int_a^t |u(s)| \, ds \right)^p dt \le \left(\frac{p}{p-1} \right)^p \int_a^b |u(x)|^p \, dx,$$
$$\int_a^b \left(\frac{1}{b-t} \int_t^b |u(s)| \, ds \right)^p dt \le \left(\frac{p}{p-1} \right)^p \int_a^b |u(x)|^p \, dx.$$

Now we turn to Lorentz spaces, which have a crucial importance for the main theorem of this thesis. We start with the definition of the nonincreasing rearrangement.

Definition 1.15. ([12, Definition 7.1.6]) Let $f \in \mathscr{M}_0(\mathscr{R},\mu)$. Then the function $f^*\colon [0,\infty) \to [0,\infty)$ defined by

$$f^*(t) = \inf\{\lambda > 0 \colon \mu(\{x \in \mathscr{R} \colon |f(x)| > \lambda\}) \le t\}, \quad t \in [0,\infty),$$

is called a *nonincreasing rearrangement* of f.

Let us recall some properties of nonincreasing rearrangement.

Properties 1.16. ([1, Proposition 2.1.7]) *Let $f, g \in \mathscr{M}_0(\mathscr{R},\mu)$. Then f^* is a nonnegative, nonincreasing, right-continuous function on $[0,\infty)$ such that*

$$\text{if } |g| \le |f| \ \mu\text{-a.e. on } \mathscr{R}, \text{ then } g^*(t) \le f^*(t), \quad t \in [0,\mu(R)),$$
$$(af)^* = |a|\, f^*,$$
$$(|f|^\alpha)^* = (f^*)^\alpha, \quad \alpha > 0.$$

Definition 1.17 (Lorentz spaces)**.** ([12, Definition 8.1.1]) Let $1 \le p, q \le \infty$. The collection $L^{p,q}(\mathscr{R}) = L^{p,q}(\mathscr{R},\mu)$ of all functions $f \in \mathscr{M}_0(\mathscr{R},\mu)$ such that $\|f\|_{L^{p,q}(\mathscr{R})} < \infty$, where

$$\|f\|_{L^{p,q}(\mathscr{R})} = \begin{cases} \left(\int_0^\infty [t^{\frac{1}{p}} f^*(t)]^q \frac{dt}{t} \right)^{\frac{1}{q}}, & 1 \le q < \infty, \\ \sup_{0<t<\infty} t^{\frac{1}{p}} f^*(t), & q = \infty, \end{cases}$$

is called the *Lorentz space*.

Remark 1.18. The functional $\|\cdot\|_{L^{p,q}(\mathscr{R})}$ is not always a norm on $\mathscr{M}(\mathscr{R},\mu)$, but it is at least a quasinorm (i.e. the triangle inequality is satisfied with a multiplicative constant, more precisely, for each $u, v \in L^{p,q}(\mathscr{R})$ we have $\|u+v\|_{L^{p,q}(\mathscr{R})} \leq c(\|u\|_{L^{p,q}(\mathscr{R})} + \|v\|_{L^{p,q}(\mathscr{R})})$ for some positive constant c). However, in cases

$$(p = q = 1) \quad \text{or} \quad (1 < p < \infty \text{ and } 1 \leq q \leq \infty) \quad \text{or} \quad (p = q = \infty),$$

the functional $\|\cdot\|_{L^{p,q}(\mathscr{R})}$ is equivalent to a norm of a Banach function space and, consequently, it has fine properties of Banach function spaces. In cases

$$(1 \leq q \leq p < \infty) \quad \text{or} \quad (p = q = \infty)$$

the functional $\|\cdot\|_{L^{p,q}(\mathscr{R})}$ is a norm.

Recall that there does not exist any norm equivalent to $\|\cdot\|_{L^{1,q}}$, $q > 1$.

We will use in our text also the following equivalent description of Lorentz (quasi)norm.

Remark 1.19. (Lorentz norm via distribution) ([9, Proposition 3.6]) The functional $\|\cdot\|_{L^{p,q}(\mathscr{R})}$ can be equivalently rewritten as

$$\|f\|_{L^{p,q}(\mathscr{R})} = p^{\frac{1}{q}} \left\| \lambda^{1-\frac{1}{q}} \mu(\{x \in \mathscr{R} : |f(x)| > \lambda\})^{\frac{1}{p}} \right\|_{L^q(0,\infty)}.$$

At the end of this section we will present the relations between Lebesgue and Lorentz spaces.

Theorem 1.20. ([1, Proposition 4.2]) *Suppose that $p, q, r \in [1, \infty]$ and*

$$1 \leq q \leq r \leq \infty.$$

Then

$$L^{p,q}(\mathscr{R}) \hookrightarrow L^{p,r}(\mathscr{R})$$

with a constant of embedding equal to $\left(\frac{p}{q}\right)^{\frac{1}{q}-\frac{1}{r}}$.

Embeddings between $L^{p,q}(\mathscr{R})$ spaces, where p is varying, are similar to embeddings between $L^p(\mathscr{R})$ spaces and they do not depend on the second parametr q. Thus, let $\mathscr{R}$ be a set of finite measure and

$$1 \leq p_2 < p_1 \leq \infty \quad \textit{and} \quad 1 \leq q, s \leq \infty.$$

Then

$$L^{p_1,q}(\mathscr{R}) \hookrightarrow L^{p_2,s}(\mathscr{R}).$$

Let us point out that each of the spaces $L^{1,q}$ with $q > 1$ is essentially larger than L^1, hence the main result of this thesis considerably improves the known ones.

1.3 Sobolev spaces

Another class of spaces of crucial importance in the topic of this thesis is that of the Sobolev spaces. Let us focus on its definition and properties, which we will use in this thesis.

Definition 1.21 (Sobolev spaces)**.** ([7, 5.4.1]) Let $\Omega \subset \mathbb{R}^N$ be an open set, let m be a nonnegative integer and $1 \leq p \leq \infty$. Set

$$W^{m,p}(\Omega) := \{u \in L^p(\Omega)\colon D^\alpha u \in L^p(\Omega) \text{ for } 0 \leq |\alpha| \leq m\},$$

where we denote by $\alpha = (\alpha_1, \ldots, \alpha_N)$ a multiindex and by $D^\alpha u$ a weak derivative of u with respect to α. The set $W^{m,p}(\Omega)$ is called the *Sobolev space.* We define the functional $\|\cdot\|_{W^{m,p}(\Omega)}$ as follows:

$$\|u\|_{W^{m,p}(\Omega)} = \begin{cases} \left(\sum_{0\leq|\alpha|\leq m} \|D^\alpha u\|^p_{L^p(\Omega)}\right)^{\frac{1}{p}}, & 1 \leq p < \infty, \\ \max_{0\leq|\alpha|\leq m} \|D^\alpha u\|_{L^\infty(\Omega)}, & p = \infty, \end{cases}$$

for every function u for which the right-hand side is defined.

We define the set $W^{m,p}_0(\Omega)$ as the closure of $C^\infty_0(\Omega)$ in the space $W^{m,p}(\Omega)$.

Remark 1.22. The sets $W^{m,p}(\Omega)$ and $W^{m,p}_0(\Omega)$ equipped with the functional $\|\cdot\|_{W^{m,p}(\Omega)}$ are normed linear (and moreover Banach) spaces.

Theorem 1.23 (Beppo-Levi)**.** ([10, Theorem 5.3]) *Let us denote $Q = (a_1, b_1) \times \cdots \times (a_N, b_N)$ a bounded N-dimensional interval, $Q_i = Q \cap \{x_i = 0\}$ and π_i the orthogonal projection of Q on Q_i. Let $u \in W^{1,1}(Q)$. Then there exists $\overline{u}$, which equals to u almost everywhere, with the following properties:*

(BL1) *For each $i \in \{1, ..., N\}$ and λ^{N-1}-almost every $y \in Q_i$, the function $u_y\colon t \to \overline{u}(y + t\boldsymbol{e}_i)$ is absolutely continuous on (a_i, b_i)*

(BL2) *For each $i \in \{1, ..., N\}$, the function $g_i\colon x \to u'_{\pi_i(x)}(x_i)$ is a weak derivative of u with respect to the i-th variable.*

Definition 1.24. We say that $\Omega \subset \mathbb{R}^N$ is a *domain* if it is open, bounded and connected.

Now, let us introduce a Lipschitz domain, similarly to [7].

Definition 1.25 (Lipschitz domain)**.** Let $\Omega \in \mathbb{R}^N$ be a domain. We say that Ω is a domain with Lipschitz boundary, eventually a Lipschitz domain, if there exist $\alpha, \beta \in (0, \infty)$ and $M \in \mathbb{N}$ systems of Cartesian coordinates and Lipschitz functions $a_r, r = 1, \ldots, M$, such that

- for r-th system we denote $x = (x_{r_1}, \ldots, x_{r_N}) := (x'_r, x_{r_N})$ and

$$\Delta_r = \left\{x'_r \in \mathbb{R}^{N-1}, |x_{r_i}| < \alpha, i = 1, \ldots, N-1\right\},$$

- $a_r\colon \Delta_r \longrightarrow \mathbb{R}$ and if we denote by R_r a rotational and translational mapping from r-th system of Cartesian coordinates to global system of Cartesian coordinates, then for each $x \in \partial\Omega$ there exists $r \in \{1, \ldots, M\}$ and x'_r such that $x = R_r(x'_r, a_r(x'_r))$,

- if we define

$$\begin{aligned}
V_r^+ &:= \left\{(x_r', x_{r_N}) \in \mathbb{R}^N : x_r' \in \Delta_r, a_r(x_r') < x_{r_N} < a_r(x_r') + \beta\right\},\\
V_r^- &:= \left\{(x_r', x_{r_N}) \in \mathbb{R}^N : x_r' \in \Delta_r, a_r(x_r') - \beta < x_{r_N} < a_r(x_r')\right\},\\
\Lambda_r &:= \left\{(x_r', x_{r_N}) \in \mathbb{R}^N : x_r' \in \Delta_r, a_r(x_r') = x_{r_N}\right\},\\
V_r &:= V_r^+ \cup V_r^- \cup \Lambda_r,
\end{aligned}$$

  then $R_r(V_r^+) \subset \Omega$, $R_r(V_r^-) \subset \mathbb{R}^N \setminus \overline{\Omega}$ and $R_r(\Lambda_r) \subset \partial\Omega$.

Remark 1.26. From the definition of the Lipschitz domain we have

$$\partial\Omega = \bigcup_{r=1}^{M} R_r(\Lambda_r) \subset \bigcup_{r=1}^{M} R_r(V_r).$$

Hence $\{R_r(V_r)\}_{r=1}^M$ is an open covering of $\partial\Omega$.

Note that, in the case $N = 1$, an open and bounded interval can be considered as a Lipschitz domain.

Theorem 1.27. ([11, Section 1.1.11]) *Let Ω be a Lipschitz domain and $p \in [1,\infty)$. Then $W^{1,p}(\Omega)$ coincides with the set*

$$\{u \in L^1_{\mathrm{loc}}(\Omega) \colon D^\alpha u \in L^p(\Omega) \text{ for } |\alpha| = 1\}.$$

Theorem 1.28. ([7, Section 6.4]) *Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain and $p \in [1,\infty)$. We define the continuous linear operator $T \colon C^\infty(\overline{\Omega}) \to C(\partial\Omega)$ by*

$$Tu := u|_{\partial\Omega}.$$

The operator T is called the trace operator. *There exists the unique extension of the operator T such that*

$$T \colon W^{1,p}(\Omega) \to L^q(\partial\Omega)$$

is continuous for each

$$q \in \begin{cases} \left[1, \frac{np-p}{n-p}\right] & \text{if } p < n,\\ [1,\infty) & \text{if } p = n,\\ [1,\infty] & \text{if } p > n. \end{cases}$$

Recall that the set $C^\infty(\overline{\Omega})$ is dense in $W^{1,p}(\Omega)$, $p \in [1,\infty)$, if Ω is a Lipschitz domain.

Lemma 1.29. *Let $u \in W^{1,p}(\Omega)$, $p \in [1,\infty)$, and let $\{u_n\}_{n=1}^\infty$ be a sequence of Lipschitz functions on $\overline{\Omega}$ such that $u_n \to u$ in $W^{1,p}(\Omega)$. Then $Tu_n \to Tu$ in $L^p(\partial\Omega)$.*

Proof. Let $v_n \in \mathcal{C}^\infty(\overline{\Omega})$, $v_n \to u$ in $W^{1,p}(\Omega)$. Clearly, $\|u_n - v_n\|_{W^{1,p}(\Omega)} \to 0$. The operator $T \colon W^{1,p}(\Omega) \to L^p(\partial\Omega)$ is linear and continuous, and so $\|Tu_n - Tv_n\|_{L^p(\partial\Omega)} \to 0$. Consequently,

$$\|Tu_n - Tu\|_{L^p(\partial\Omega)} \le \|Tu_n - Tv_n\|_{L^p(\partial\Omega)} + \|Tv_n - Tu\|_{L^p(\partial\Omega)} \to 0,$$

which finishes the proof. $\square$

Theorem 1.30. ([7, Theorem 6.6.4]) *Let $\Omega \subset \mathbb{R}^N$ be a Lipschitz domain. Then*

$$W_0^{1,p}(\Omega) = \{u \in W^{1,p}(\Omega), Tu = 0 \text{ a.e. in } \partial\Omega\}.$$

Theorem 1.31 (Poincaré-Friedrichs inequality)**.** ([13]) *Let $\Omega \subset \mathbb{R}^N$ be a Lipschitz domain and $p \in [1, \infty)$. Let $\Gamma \subset \partial\Omega$ of positive $(N-1)$-dimensional measure. Then there exist positive constants c_1 and c_2 such that for every $u \in W^{1,p}(\Omega)$ we have*

$$c_1 \|u\|_{W^{1,p}(\Omega)} \leq \left(\|\nabla u\|^p_{L^p(\Omega)} + \int_\Gamma |Tu|^p \, dS \right)^{\frac{1}{p}} \leq c_2 \|u\|_{W^{1,p}(\Omega)} .$$

1.4 The trace lemma

In this section we would like to introduce a certain lemma which will be used later in the proofs. It is probably known, however we present its proof, for the sake of completeness.

Lemma 1.32. *Let us denote $Q_N = (0,1)^N$, $N \in \mathbb{N}$. Let $u \in W^{1,p}(Q_N)$. Then there exists $M \subset Q_{N-1}$ such that*

$$\lambda^{N-1}(Q_{N-1} \setminus M) = 0 \quad \text{and} \quad Tu(x', 0) = \lim_{t \to 0+} u(x', t) \text{ for each } x' \in M.$$

We will need the following auxiliary lemma.

Lemma 1.33. *Assume $p \in [1, \infty)$. Then the inequality*

$$a + a^{1-p} \geq (p-1)^{1/p}$$

holds for each $a > 0$.

Proof. The assertion for the case $p = 1$ holds trivially. Let $p > 1$. Set $f(a) = a + a^{1-p}$. Then $f'(a) = 1 + (1-p)a^{-p}$ and $f'(a) = 0$ if and only if $a = (p-1)^{1/p}$. It can be easily verified that it is a point of global minimum. Thus

$$f(a) \geq (p-1)^{1/p} + (p-1)^{(1-p)/p} \geq (p-1)^{1/p}$$

for each $a > 0$, which completes the proof. $\square$

Lemma 1.34. *Let $u \in AC(0,1)$. Then there exists $\lim_{t \to 0+} u(t)$.*

Proof. Let us assume the contrary. Thus there exist $\{t_n\}$ and $\{\overline{t_n}\}$ approaching zero such that $u(t_n) \to c \in \mathbb{R}$ and $u(\overline{t_n}) \to d \in \mathbb{R}$, $c \neq d$. Let us take $\varepsilon = \frac{|c-d|}{2}$. Fix $\delta > 0$ arbitrary. We find $n_0 \in \mathbb{N}$ such that for each $n, m > n_0$ we have $|u(t_n) - c| \leq \frac{\varepsilon}{2}$, $|u(\overline{t_m}) - d| \leq \frac{\varepsilon}{2}$ and $|t_n - \overline{t_m}| < \delta$. By the triangle inequality we have

$$\begin{aligned} |c - d| &= |c - u(t_n) + u(t_n) - u(\overline{t_m}) + u(\overline{t_m}) - d| \\ &\leq |c - u(t_n)| + |u(t_n) - u(\overline{t_m})| + |u(\overline{t_m}) - d| , \end{aligned}$$

thus

$$\varepsilon \leq |u(t_n) - u(\overline{t_m})| ,$$

which contradicts the absolute continuity of u. $\square$

Lemma 1.35. *Let $u \in AC(0,1)$, $\delta > 0$ and $|\lim_{t\to 0+} u(t)| \geq \delta$. Then*

$$\int_0^1 (|u(t)|^p + |u'(t)|^p)dt \geq (\delta/2)^p \min(1, (p-1)^{1/p}).$$

Proof. We can assume that $\lim_{t\to 0+} u(t) \geq \delta$, otherwise we take $-u$. Set

$$a = \inf\{t \in (0,1); u(t) \leq \frac{\delta}{2}\}.$$

Clearly, $a > 0$. If the set from the definition of a is empty, then we have

$$\int_0^1 (|u(t)|^p + |u'(t)|^p)dt \geq \int_0^1 |u(t)|^p dt \geq (\delta/2)^p,$$

and the assertion follows.

If $a < 1$ then, by the Hölder inequality and the properties of AC functions, we have

$$a^{1-p} \left|\lim_{t\to 0+} u(t) - u(a)\right|^p \leq a^{1-p} \left(\int_0^a |u'(s)|ds\right)^p \leq \int_0^a |u'(s)|^p ds,$$

which gives, together with Lemma 1.33,

$$\begin{aligned}
&\int_0^1 (|u(t)|^p + |u'(t)|^p)dt \geq \int_0^a (|u(t)|^p + |u'(t)|^p)dt \\
&\geq a(\delta/2)^p + a^{1-p} \left|\lim_{t\to 0+} u(t) - u(a)\right|^p \geq a(\delta/2)^p + a^{1-p}(\delta/2)^p \\
&= (\delta/2)^p(a + a^{1-p}) \geq (\delta/2)^p (p-1)^{1/p},
\end{aligned}$$

establishing the claim. □

Proof of Lemma 1.32. Let $u \in W^{1,p}(Q_N)$. By Theorem 1.23, there exists a set $M \subset Q_{N-1}$ for which $t \mapsto u(x', t)$ is AC on $(0,1)$, $x' \in M$, and $\lambda(Q_{N-1} \setminus M) = 0$. Denote

$$f(x') = \lim_{t\to 0_+} u(x', t),$$

which exists for each $x' \in M$ due to Lemma 1.34. Assume that our assertion does not hold. Then there exist $\alpha > 0$ and $A \subset M$ with $\lambda^{N-1}(A) > 0$ such that

$$|Tu(x', 0) - f(x')| \geq \alpha, \quad x' \in A.$$

Let $u_n \in \mathcal{C}^\infty(\overline{Q_N})$ be such that $u_n \to u$ in $W^{1,p}(Q_N)$. Then $Tu_n \to Tu$ in $L^p(Q_{N-1})$. Find $n_0 \in \mathbb{N}$ such that for each $n > n_0$ we have

$$\|u_n - u\|^p_{W^{1,p}(Q_N)} < \frac{\lambda^{N-1}(A)}{2} \left(\frac{\alpha}{4}\right)^p \min(1, (p-1)^{1/p}) \tag{1.2}$$

and

$$\int_{Q_{N-1}} |Tu_n(x', 0) - Tu(x', 0)|^p dx' \leq \frac{\alpha^p \lambda^{N-1}(A)}{2^{p+1}}.$$

Fix $m > n_0$. Set

$$B = \left\{ x \in Q_{N-1}; |Tu_m(x',0) - Tu(x',0)| \geq \frac{\alpha}{2} \right\}.$$

Then

$$\frac{\alpha^p \lambda^{N-1}(A)}{2^{p+1}} \geq \int_B |Tu_m(x',0) - Tu(x',0)|^p dx' \geq \lambda^{N-1}(B) \left(\frac{\alpha}{2}\right)^p$$

and thus

$$\frac{\lambda^{N-1}(A)}{2} \geq \lambda^{N-1}(B), \quad \text{and} \quad \lambda^{N-1}(Q_{N-1} \setminus B) \geq 1 - \frac{\lambda^{N-1}(A)}{2}.$$

So,

$$\lambda^{N-1}((Q_{N-1} \setminus B) \cap A) \geq \frac{\lambda^{N-1}(A)}{2}.$$

Moreover, for $x' \in (Q_{N-1} \setminus B) \cap A$ we have

$$\begin{aligned} \alpha &\leq |Tu(x',0) - f(x')| \leq |Tu(x',0) - Tu_m(x',0)| + |Tu_m(x',0) - f(x')| \\ &\leq \frac{\alpha}{2} + |Tu_m(x',0) - f(x')| \end{aligned}$$

and

$$|Tu_m(x',0) - f(x')| = |u_m(x',0) - f(x')| = |\lim_{t\to 0+} (u_m(x',t) - u(x',t))|,$$

thus

$$\frac{\alpha}{2} \leq |\lim_{t\to 0+} (u_m(x',t) - u(x',t))|.$$

Applying Lemma 1.35 to the function $u_m - u$, we obtain

$$\begin{aligned} &\|u_m - u\|^p_{W^{1,p}(Q_N)} \\ &\quad \geq \int_{(Q_{N-1}\setminus B)\cap A} \left(\int_0^1 (|u_m(x',t) - u(x',t)|^p + |u_m'(x',t) - u'(x',t)|^p)\, dt \right) dx' \\ &\quad \geq \lambda^{N-1}((Q_{N-1} \setminus B) \cap A) \left(\frac{\alpha}{4}\right)^p \min(1, (p-1)^{1/p}) \\ &\quad \geq \frac{\lambda^{N-1}(A)}{2} \left(\frac{\alpha}{4}\right)^p \min(1, (p-1)^{1/p}) \end{aligned}$$

which is a contradiction with (1.2). $\square$

2. Formulation of the problem

In this chapter we show the background of our problem of characterization of functions with zero traces from Sobolev spaces using the distance function from the boundary. We will present a summary of known results, definitions of appropriate function spaces and the formulation of the main result.

2.1 Survey of known results

We start with the distance function.

Definition 2.1 (distance function from the boundary)**.** Let $\Omega \subset \mathbb{R}^N$ be a nonempty open and bounded set. We define the function $d\colon \Omega \longrightarrow (0,\infty)$ as $d(x) = \operatorname{dist}(x, \partial\Omega)$.

Now we turn to some historical facts in the research of a characterization of functions vanishing at the boundary using the distance function. The first result on this topic can be found in the book [2] by D. E. Edmunds and W. D. Evans published in 1987. It is based on results of D. J. Harris, C. Kenig, J. Kadlec and A. Kufner [5].

Theorem 2.2. *[2, Theorem V.3.4 and Remark V.3.5] Let Ω be a nonempty open subset of $\mathbb{R}^N$ such that $\Omega \neq \mathbb{R}^N$. Let $p \in [1,\infty)$ and $m \in \mathbb{N}$. Then if $u \in W^{m,p}(\Omega)$ and $\frac{u}{d^m} \in L^p(\Omega)$, it follows that $u \in W_0^{m,p}(\Omega)$.*

If moreover $p \in (1,\infty)$ and Ω is bounded with suitably regular boundary (e.g. Lipschitz), then $u \in W_0^{m,p}(\Omega)$ implies $\frac{u}{d^m} \in L^p(\Omega)$.

Ten years later, the paper [6] of J. Kinnunen and O. Martio was published, in which the condition for $u \in W_0^{1,p}$ was weakened.

Theorem 2.3. *[6, Theorem 3.13] Let Ω be an open set and suppose that $u \in W^{1,p}(\Omega)$, $p \in (1,\infty)$. Then $\frac{u}{d} \in L^{p,\infty}(\Omega)$ implies $u \in W_0^{1,p}(\Omega)$.*

Note that for a suitably regular domain Ω we have also, by embeddings of Lorentz spaces and the second part of Theorem 2.2, that $u \in W_0^{1,p}(\Omega)$ implies $\frac{u}{d} \in L^{p,\infty}(\Omega)$.

However, even this result was later improved. In 2017, the paper [3] by A. Nekvinda and D. E. Edmunds was published, where the assumption was further relaxed. Here some condition for regularity of domain in both inclusions is needed. A Lipschitz domain is an example of domain with such regularity.

Theorem 2.4. *[3, Theorem 5.5] Let $\Omega \subset \mathbb{R}^N$ be bounded and regular, $p \in (1,\infty)$. Then $u \in W_0^{1,p}(\Omega)$ if and only if $\nabla u \in L^p(\Omega)$ and $\frac{u}{d} \in L^1(\Omega)$.*

Note that the original result in [3] was formulated for variable exponent p. This result was one year later extended by the same authors to Sobolev spaces of higher order. As a consequence of [4, Theorem 6.1] we obtain the following theorem.

Theorem 2.5. *Let $\Omega \subset \mathbb{R}^N$ be bounded and regular, $p \in (1,\infty)$. Then $u \in W_0^{m,p}(\Omega)$ if and only if $D^\alpha u \in L^p(\Omega)$, $|\alpha| = m$, and $\frac{u}{d^m} \in L^1(\Omega)$.*

2.2 The main result

We will show that the above-mentioned results can be further improved. Our goal is to prove that we can impose a yet weaker condition on $\frac{u}{d}$ which will still preserve the property $u \in W_0^{1,p}(\Omega)$. Namely, we will show that in fact it is enough to require that $\frac{u}{d} \in L^{1,q}$, $q \in [1, \infty)$. We recall that $L^{1,q}$ is an essentially larger space than L^1.

Definition 2.6. Let $\Omega \subset \mathbb{R}^N$ be an open and bounded set, let $u \in \mathscr{M}_0(\Omega, \lambda^N)$ be a function and $p, q \in [1, \infty]$. Let us denote $\tilde{u} = \frac{u}{d}$. The function u is an element of the set $W_d(L^{1,q}, L^p)(\Omega)$ if it satisfies

$$\|\tilde{u}\|_{L^{1,q}(\Omega)} + \|\nabla u\|_{L^p(\Omega)} < \infty.$$

We define the functional $\|\cdot\|_{W_d(L^{1,q},L^p)(\Omega)}$ as

$$\|u\|_{W_d(L^{1,q},L^p)(\Omega)} = \|\tilde{u}\|_{L^{1,q}(\Omega)} + \|\nabla u\|_{L^p(\Omega)}.$$

Convention 2.7. We will often write $W_d(L^{1,q}, L^p)$ instead of $W_d(L^{1,q}, L^p)(\Omega)$ if no confusion can arise.

Theorem 2.8. *Let $\Omega \in \mathbb{R}^N$ be open and bounded. The structure $W_d(L^{1,q}, L^p)$ is a linear space and the functional $\|\cdot\|_{W_d(L^{1,q},L^p)}$ is a quasinorm on $W_d(L^{1,q}, L^p)$.*

Proof. Let us take $u, v \in \mathscr{M}_0(\Omega, \lambda^N)$. We have

$$\begin{aligned}
\|u+v\|_{W_d(L^{1,q},L^p)} &= \left\|\frac{u+v}{d}\right\|_{L^{1,q}(\Omega)} + \|\nabla(u+v)\|_{L^p(\Omega)} \\
&\leq C\left(\left\|\frac{u}{d}\right\|_{L^{1,q}(\Omega)} + \left\|\frac{v}{d}\right\|_{L^{1,q}(\Omega)}\right) + \|\nabla u\|_{L^p(\Omega)} + \|\nabla v\|_{L^p(\Omega)} \\
&\leq C\left(\|u\|_{W_d(L^{1,q},L^p)} + \|v\|_{W_d(L^{1,q},L^p)}\right),
\end{aligned}$$

where $C \geq 1$ is the constant of quasi-subadditivity of quasinorm $\|\cdot\|_{L^{1,q}(\Omega)}$. Additionally, for some $c \in \mathbb{R}$, we have

$$\begin{aligned}
\|cu\|_{W_d(L^{1,q},L^p)} &= \left\|\frac{cu}{d}\right\|_{L^{1,q}(\Omega)} + \|\nabla(cu)\|_{L^p(\Omega)} \\
&= |c|\left\|\frac{u}{d}\right\|_{L^{1,q}(\Omega)} + |c|\,\|\nabla u\|_{L^p(\Omega)} \\
&= |c|\,\|u\|_{W_d(L^{1,q},L^p)}.
\end{aligned}$$

Finally, by properties of $\|\cdot\|_{L^{1,q}(\Omega)}$ and $\|\cdot\|_{L^p(\Omega)}$, we have $\|u\|_{W_d(L^{1,q},L^p)} \geq 0$ and $\|u\|_{W_d(L^{1,q},L^p)} = 0$ if and only if $u = 0$ almost everywhere. This completes the proof. $\square$

Theorem 2.9 (main theorem)**.** *Let $\Omega \subset \mathbb{R}^N$, $N \in \mathbb{N}$, be a Lipschitz domain, $p \in (1, \infty)$ and $q \in [1, \infty)$. Then*

$$W_0^{1,p}(\Omega) = W_d(L^{1,q}, L^p)(\Omega)$$

and the norm $\|\cdot\|_{W^{1,p}}$ is equivalent to the quasinorm $\|\cdot\|_{W_d(L^{1,q},L^p)}$.

In other words, the function $u \in \mathscr{M}_0(\Omega, \mu)$ satisfies

$$\left\| \frac{u}{d} \right\|_{L^{1,q}(\Omega)} < \infty \qquad \text{and} \qquad \|\nabla u\|_{L^p(\Omega)} < \infty$$

if and only if

$$u \in W_0^{1,p}(\Omega).$$

Moreover, there exist $C_1, C_2 \in (0, \infty)$ such that

$$C_1 \|u\|_{W^{1,p}(\Omega)} \leq \left\| \frac{u}{d} \right\|_{L^{1,q}(\Omega)} + \|\nabla u\|_{L^p(\Omega)} \leq C_2 \|u\|_{W^{1,p}(\Omega)}.$$

The proof of this theorem is given in following chapters.

3. Auxiliary result on a cube

In the proof of the main theorem for a Lipschitz domain we will use a classical method used for a proof of embeddings between Sobolev spaces or for investigation of weak solutions of PDE's. The method is based on the partition of unity and on characterization of each small part of the boundary of domain. A crucial moment is to know the result for each such part. This is the core of the proof and it is given in this chapter.

3.1 Definition of used objects

From now on, let us denote $Q_N = (0,1)^N$.

Definition 3.1 (distance function from a part of the boundary)**.** We define a function $d_N : Q_N \longrightarrow (0,\infty)$ by $d_N(x) = \operatorname{dist}(x, \{y \in \mathbb{R}^N, y_N = 0\}) = x_N$.

Definition 3.2. Let $u \in \mathscr{M}_0(Q_N, \lambda^N)$ be a function. Let us denote $\tilde{u} = \frac{u}{d_N}$ in this chapter. The function u is an element of the set $T^{q,p}$ if it satisfies

$$\|\tilde{u}\|_{L^{1,q}(Q_N)} + \|\nabla u\|_{L^p(Q_N)} < \infty.$$

We define the functional $\|\cdot\|_{T^{q,p}}$ as

$$\|u\|_{T^{q,p}} = \|\tilde{u}\|_{L^{1,q}(Q_N)} + \|\nabla u\|_{L^p(Q_N)} .$$

Properties 3.3. *The structure $T^{q,p}$ is a linear space and the functional $\|\cdot\|_{T^{q,p}}$ is a quasinorm of $T^{q,p}$.*

Proof. The proof is analogous to that of Theorem 2.8. □

3.2 Proof of a local statement

Let us moreover denote $Q_0 = \{0\}$ and λ^0 the Dirac measure δ_0. With such notation the next two proofs will work even in the case $N = 1$.

Lemma 3.4. *Let $u \in W^{1,p}(Q_N)$ and $p \geq 1$. Let us denote $P = [0,1]^{N-1} \times \{x_N = 0\}$. Suppose that for every $\varepsilon > 0$ and every $\delta > 0$ there exists the set $M \subset P$ such that $\lambda^{N-1}(M) > 1 - \delta$ and $\int_M |Tu(x)|^p \, d\lambda^{N-1}(x) < \varepsilon$. Then $Tu(x) = 0$ λ^{N-1}-a.e. in P.*

Proof. Let us assume that there exist $A \subset P, \lambda^{N-1}(A) > 0$, and $a > 0$ such that $|Tu(x)| > a$ for each $x \in A$ for a contradiction. Let us take $\varepsilon > 0$ and $\delta > 0$ such that

$$\varepsilon < \frac{1}{4} a^p \lambda^{N-1}(A) \qquad \text{and} \qquad \delta < \frac{1}{2}\lambda^{N-1}(A).$$

We take M from the assumption. Then we have $\lambda^{N-1}(M) > 1 - \delta > 1 - \frac{1}{2}\lambda^{N-1}(A)$.

Since $\lambda^{N-1}(A) + \lambda^{N-1}(M) = \lambda^{N-1}(A \cap M) + \lambda^{N-1}(A \cup M)$ and $\lambda^{N-1}(A \cup M) \leq 1$, we get

$$\frac{\lambda^{N-1}(A)}{2} < \lambda^{N-1}(A \cap M).$$

Therefore

$$\begin{aligned}\varepsilon &> \int_M |Tu(x)|^p \, d\lambda^{N-1}(x) \geq \int_{A\cap M} |Tu(x)|^p \, d\lambda^{N-1}(x) \geq a^p \lambda^{N-1}(A \cup M) \\ &> \frac{1}{2} a^p \lambda^{N-1}(A) > 2\varepsilon,\end{aligned}$$

which is a contradiction. $\square$

Theorem 3.5. *Let $p, q \in [1, \infty)$, $N \in \mathbb{N}$. Let u be a function from $T^{q,p}$. Then u is an element of the Sobolev space $W^{1,p}(Q_N)$ with zero trace on the set*

$$\left\{x \in [0,1]^N, x_N = 0\right\} \subset \partial Q_N,$$

i.e. $Tu(x) = 0$ almost everywhere on this set.

Proof. By assumptions we have $\|\nabla u\|_{L^p(Q_N)} < \infty$, thus by Theorem 1.27 we get $u \in W^{1,p}(Q_N)$.

We now turn to the proof of the second part. Let us take arbitrary $\varepsilon > 0$ and $\delta \in (0, 1)$. Then, using properties of real numbers, we take $k \in \mathbb{N}$, $k \geq 3$, such that $\frac{1}{k-1} < \delta \leq \frac{1}{k-2}$. Let us denote

$$I_{k,n} = \left[\left(\frac{1}{k}\right)^{n+1}, \left(\frac{1}{k}\right)^{n}\right].$$

Note that $\lambda^1(I_{k,n}) = \left(\frac{1}{k}\right)^n \frac{k-1}{k}$. Then

$$\sum_{n=0}^{\infty} \int_{I_{k,n}} t^{q-1} \tilde{u}^*(t)^q \, dt = \int_0^1 t^{q-1} \tilde{u}^*(t)^q \, dt < \infty$$

by the assumptions. Denoting

$$a_{k,n}^q = \int_{I_{k,n}} t^{q-1} \tilde{u}^*(t)^q \, dt$$

we have $\sum_{n=0}^{\infty} a_{k,n}^q < \infty$, which immediately implies $\lim_{n\to\infty} a_{k,n} = 0$.

We claim that there exist constants $c_{k,n} \in \left[\left(\frac{1}{k-1}\right)^{q-1}, \left(\frac{k}{k-1}\right)^{q-1}\right]$ such that

$$a_{k,n}^q = c_{k,n} |I_{k,n}|^{q-1} \int_{I_{k,n}} \tilde{u}^*(t)^q \, dt.$$

Indeed,

$$\begin{aligned}a_{k,n}^q &= \int_{(\frac{1}{k})^{n+1}}^{(\frac{1}{k})^n} t^{q-1} \tilde{u}^*(t)^q \, dt \\ &\leq \left(\frac{1}{k^n}\right)^{q-1} \int_{I_{k,n}} \tilde{u}^*(t)^q \, dt = |I_{k,n}|^{q-1} \left(\frac{k}{k-1}\right)^{q-1} \int_{I_{k,n}} \tilde{u}^*(t)^q \, dt\end{aligned}$$

and

$$a_{k,n}^q \geq \left(\frac{1}{k^{n+1}}\right)^{q-1} \int_{I_{k,n}} \tilde{u}^*(t)^q \, dt = |I_{k,n}|^{q-1} \left(\frac{1}{k-1}\right)^{q-1} \int_{I_{k,n}} \tilde{u}^*(t)^q \, dt.$$

Let us denote

$$P_{k,n} = \frac{a_{k,n}^q}{c_{k,n}|I_{k,n}|^q} = \frac{1}{|I_{k,n}|}\int_{I_{k,n}} (|\tilde{u}|^q)^*(t)\,dt,$$

where we use $(\tilde{u}^*)^q = (|\tilde{u}|^q)^*$, and define

$$M_{k,n} = \left\{x \in Q_{N-1} \times I_{k,n}, |\tilde{u}(x)|^q \leq P_{k,n+1}\right\}.$$

Assume for the time being that

$$\lambda^N(\{x \in Q_{N-1} \times I_{k,n} \colon |\tilde{u}(x)|^q > P_{k,n+1}\}) > \frac{1}{k-1}|I_{k,n}| = \frac{1}{k^{n+1}}. \tag{3.1}$$

Since $\{x \in Q_{N-1} \times I_{k,n} \colon |\tilde{u}(x)|^q > P_{k,n+1} + \omega\} \nearrow \{x \in Q_{N-1} \times I_{k,n} \colon |\tilde{u}(x)|^q > P_{k,n+1}\}$ for $\omega \to 0_+$, there exists $\omega_0 > 0$ such that

$$\lambda^N(\{x \in Q_{N-1} \times I_{k,n} \colon |\tilde{u}(x)|^q > P_{k,n+1} + \omega_0\}) > \frac{1}{k^{n+1}}. \tag{3.2}$$

Then, for $\nu \in (0, |I_{k,n+1}|)$,

$$\begin{aligned}
&\inf\left\{\xi > 0 \colon \lambda^N(\{x \in Q_N \colon |\tilde{u}(x)|^q > \xi\}) \leq \frac{1}{k^{n+1}} - \nu\right\} \\
&\quad = (|\tilde{u}|^q)^*\left(\frac{1}{k^{n+1}} - \nu\right) \geq (|\tilde{u}|^q \chi_{Q_{N-1}\times I_{k,n}})^*\left(\frac{1}{k^{n+1}} - \nu\right) \\
&\quad = \inf\left\{\xi > 0 \colon \lambda^N(\{x \in Q_{N-1} \times I_{k,n} \colon |\tilde{u}(x)|^q > \xi\}) \leq \frac{1}{k^{n+1}} - \nu\right\} := D.
\end{aligned}$$

If $D < P_{k,n+1} + \omega_0$, then there exists ξ_0 such that $D \leq \xi_0 < P_{k,n+1} + \omega_0$ with

$$\begin{aligned}
&\lambda^N(\{x \in Q_{N-1} \times I_{k,n} \colon |\tilde{u}(x)|^q > P_{k,n+1} + \omega_0\}) \\
&\qquad \leq \lambda^N(\{x \in Q_{N-1} \times I_{k,n} \colon |\tilde{u}(x)|^q > \xi_0\}) \leq \frac{1}{k^{n+1}} - \nu,
\end{aligned}$$

which contradicts (3.2). Thus $D \geq P_{k,n+1} + \omega_0$. Then for each $\nu \in (0, |I_{k,n+1}|)$ we have proved

$$(|\tilde{u}|^q)^*\left(\left(\frac{1}{k}\right)^{n+1} - \nu\right) \geq P_{k,n+1} + \omega_0 > P_{k,n+1}.$$

Therefore the function $(|\tilde{u}|^q)^*$ is strictly greater then $P_{k,n+1}$ inside of the interval $I_{k,n+1}$ and

$$P_{k,n+1} = \frac{1}{|I_{k,n+1}|}\int_{I_{k,n+1}} (|\tilde{u}|^q)^*(t)\,dt > \frac{|I_{k,n+1}|}{|I_{k,n+1}|}P_{k,n+1} = P_{k,n+1},$$

which is contradiction. Hence inequality (3.1) does not hold. This proves

$$\begin{aligned}
\lambda^N(M_{k,n}) &= |I_{k,n}| - \lambda^N(\{x \in Q_{N-1} \times I_{k,n} \colon |\tilde{u}(x)|^q > P_{k,n+1}\}) \\
&\geq |I_{k,n}| - \frac{1}{k-1}|I_{k,n}| = \frac{k-2}{k-1}|I_{k,n}|.
\end{aligned}$$

From the Fubini theorem we deduce that there exists $t_{k,n} \in I_{k,n}$ such that

$$\lambda^{N-1}(\{x \in Q_N, x_N = t_{k,n}\} \cap M_{k,n}) \geq \frac{k-2}{k-1} > 1 - \delta. \tag{3.3}$$

Moreover, by the definition of $M_{k,n}$, we get, for $x \in M_{k,n}$,

$$\frac{|u(x)|}{d_N(x)} \leq \frac{a_{k,(n+1)}}{c_{k,(n+1)}^{\frac{1}{q}}|I_{k,(n+1)}|} \leq \frac{a_{k,(n+1)}}{\left(\frac{1}{k-1}\right)^{\frac{q-1}{q}}|I_{k,(n+1)}|}$$

and, since $d_N(x) = x_N \in \left[\left(\frac{1}{k}\right)^{n+1}, \left(\frac{1}{k}\right)^{n}\right]$, we obtain, denoting $c_k = \frac{k^2}{(k-1)^{\frac{1}{q}}}$,

$$|u(x)| \leq c_k a_{k,(n+1)}. \tag{3.4}$$

Thanks to $u \in W^{1,p}(Q_N)$, we have by the Beppo-Levi theorem that u is absolutely continuous on almost every line parallel to one of the axes and classical derivatives of such AC function coincide with weak derivatives. Let $(x', 0) \in Q_{N-1} \times \{x_N = 0\}$ be such that $u_{(x',0)} \colon t \longrightarrow u((x', 0) + te^N)$ is absolutely continuous on $t \in (0, 1)$. Due to Lemma 1.32, we can extend it continuously for almost every $x' \in Q_{N-1}$ with the value $Tu((x', 0))$ for $t = 0$. Consequently we can use the Newton-Leibniz formula (1.1)

$$u((x', t_{k,n})) - Tu((x', 0)) = \int_0^{t_{k,n}} \frac{\partial}{\partial x_N} u((x', \zeta))\, d\zeta \tag{3.5}$$

for almost every $x' \in Q_{N-1}$. Therefore

$$|Tu((x', 0))| \leq \int_0^{t_{k,n}} \left|\frac{\partial}{\partial x_N} u((x', \zeta))\right| d\zeta + |u((x', t_{k,n}))|.$$

Raising this to p and using the Hölder inequality, we get

$$\begin{aligned} |Tu((x', 0))|^p &\leq 2^{p-1}\left(\left(\int_0^{t_{k,n}} \left|\frac{\partial}{\partial x_N} u((x', \zeta))\right| d\zeta\right)^p + |u((x', t_{k,n}))|^p\right) \\ &\leq 2^{p-1}\left((t_{k,n})^{p-1} \int_0^{t_{k,n}} \left|\frac{\partial}{\partial x_N} u((x', \zeta))\right|^p d\zeta + |u((x', t_{k,n}))|^p\right). \end{aligned} \tag{3.6}$$

Denote $P = [0, 1]^{N-1} \times \{x_N = 0\}$ and let $\Pi_P \colon Q_N \longrightarrow P$ be the orthogonal projection. Let $B \subset Q_N$. Then we denote $\pi(B) \subset (0, 1)^{N-1}$ such that $\pi(B) \times \{x_N = 0\} = \Pi_P(B)$.

Let us integrate (3.6) over the set $\pi(\{x \in Q_N, x_N = t_{k,n}\} \cap M_{k,n})$ (we write just π for brevity). More precisely, we integrate over that subset of π on which (3.5) holds. We recall that this set has the same measure as π. Thus

$$\begin{aligned} \int_\pi |Tu((x', 0))|^p\, dx' \leq\, & 2^{p-1}(t_{k,n})^{p-1} \int_\pi \int_0^{t_{k,n}} \left|\frac{\partial}{\partial x_N} u((x', \zeta))\right|^p d\zeta\, dx' \\ &+ 2^{p-1} \int_\pi |u((x', t_{k,n}))|^p\, dx'. \end{aligned}$$

By an elementary computation,

$$\int_\pi \int_0^{t_{k,n}} \left|\frac{\partial}{\partial x_N} u((x', \zeta))\right|^p d\zeta\, dx' \leq \int_{(0,1)^{N-1}} \int_0^{t_{k,n}} \left|\frac{\partial}{\partial x_N} u((x', \zeta))\right|^p d\zeta\, dx'$$

and

$$\int_{(0,1)^{N-1}} \int_0^{t_{k,n}} \left| \frac{\partial}{\partial x_N} u((x',\zeta)) \right|^p \, d\zeta \, dx'$$
$$\leq \int_{Q_N} |\nabla u(x)|^p \, dx = \|\nabla u\|^p_{L^p(Q_N)} < \infty. \tag{3.7}$$

By (3.4), we have

$$\int_{\pi(\{x \in Q_N, x_N = t_{k,n}\} \cap M_{k,n})} |u((x', t_{k,n}))|^p \, dx' \leq a^p_{k,(n+1)} c^p_k.$$

Thus,

$$\int_{\Pi_P(\{x \in Q_N, x_N = t_{k,n}\} \cap M_{k,n})} |Tu(x)|^p \, d\lambda^{N-1}(x)$$
$$\leq 2^{p-1} (t_{k,n})^{p-1} \int_{(0,1)^{N-1}} \int_0^{t_{k,n}} \left| \frac{\partial}{\partial x_N} u((x',\zeta)) \right|^p \, d\zeta \, dx' + 2^{p-1} a^p_{k,(n+1)} c^p_k.$$

Since $\lim_{n\to\infty} t_{k,n} = 0$, using (3.7), the continuity of the Lebesgue integral and the fact that $\lim_{n\to\infty} a_{k,(n+1)} = 0$, we get that to our $\varepsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that

$$\int_{\Pi_P(\{x \in Q_N, x_N = t_{k,n_0}\} \cap M_{k,n_0})} |Tu(x)|^p \, d\lambda^{N-1}(x) < \varepsilon$$

and, by (3.3),

$$\lambda^{N-1}(\Pi_P(\{x \in Q_N, x_N = t_{k,n_0}\} \cap M_{k,n_0})) = \lambda^{N-1}(\{x \in Q_N, x_N = t_{k,n_0}\} \cap M_{k,n_0})$$
$$> 1 - \delta.$$

Finally, applying Lemma 3.4 to ε, δ and sets $\Pi_P(\{x \in Q_N, x_N = t_{k,n_0}\} \cap M_{k,n_0})$, we complete the proof. □

We note that natural analogues of vast majority of the steps in the proof make good sense also in the case $N = 1$. In the exceptional circumstances when this is not the case, it turns out that they are either not needed and thus can be skipped, or they may be slightly modified.

The proof is almost complete for the case $N = 1$. It suffices to apply the method presented in [14, Chapter 6] using reflection and smoothing to the former embedding, and applying the Hardy inequality to the latter one. It is also possible to easily modify the method which is presented below for higher dimensions to the dimension one.

4. Main result for a domain having a Lipschitz boundary

In this chapter we extend our result from a cube to a Lipschitz domain. We can assume $N \geq 2$. First we prove an embedding between spaces $W_0^{1,p}(\Omega)$ and $W_d(L^{1,q}, L^p)(\Omega)$, which is new and fundamental for this thesis. We apply the results proved in the previous chapter. To this end we need to describe the boundary using Lipschitz functions and to transform parts of the domain by a bilipschitz mapping. We prove some results about such mapping. This chapter includes also the reverse embedding of spaces, which completes the proof.

4.1 Lipschitz domain and bilipschitz mapping

First let us mention a quite interesting theorem, which appeared in the course of the proof of our statement. For the proof it is not strictly necessary, but it is of independent interest.

Let us denote by $d_{\partial\Omega} \colon \Omega \longrightarrow (0, \infty)$ the distance function from the boundary of the domain Ω.

Theorem 4.1. *Let $q \in [1, \infty]$, G and Ω be domains in $\mathbb{R}^N$ and $B \colon \mathbb{R}^N \to \mathbb{R}^N$ be a bilipschitz mapping such that $B(G) = \Omega$. Then, for $u \in \mathscr{M}_0(\Omega, \lambda^N)$, we have $\frac{u(x)}{d_{\partial\Omega}(x)} \in L^{1,q}(\Omega)$ if and only if $\frac{u(B(y))}{d_{\partial G}(y)} \in L^{1,q}(G)$ and there exist positive constants C_1 and C_2 depending on B such that*

$$C_1 \left\| \frac{u(x)}{d_{\partial\Omega}(x)} \right\|_{L^{1,q}(\Omega)} \leq \left\| \frac{u(B(y))}{d_{\partial G}(y)} \right\|_{L^{1,q}(G)} \leq C_2 \left\| \frac{u(x)}{d_{\partial\Omega}(x)} \right\|_{L^{1,q}(\Omega)} .$$

Proof. We assume that B is bilipschitz, therefore it suffices to show just one implication. Let us suppose that $\frac{u(x)}{d_{\partial\Omega}(x)} \in L^{1,q}(\Omega)$ and let L be a constant of B, i.e. $|B(y_1) - B(y_2)| \leq L|y_1 - y_2|$ for every $y_1, y_2 \in G$ and $|B^{-1}(x_1) - B^{-1}(x_2)| \leq L|x_1 - x_2|$ for every $x_1, x_2 \in \Omega$.

Choose $x \in \Omega$ arbitrarily and find $y \in G$ such that $B(y) = x$. Further we define a function $\tilde{d}_{\partial\Omega} \colon \Omega \to (0, \infty)$ such that

$$\tilde{d}_{\partial\Omega}(x) = \inf_{\tilde{y} \in \partial G, |y - \tilde{y}| = d_{\partial G}(y)} |x - B(\tilde{y})| .$$

Note that for each $\tilde{y} \in \partial G$ we have $B\tilde{y} \in \partial\Omega$ since B is bilipschitz. Thus we have

$$\tilde{d}_{\partial\Omega}(x) = \inf_{\tilde{y} \in \partial G, |y - \tilde{y}| = d_{\partial G}(y)} |B(y) - B(\tilde{y})| \leq L d_{\partial G}(y),$$

which implies

$$\frac{u(B(y))}{d_{\partial G}(y)} \leq L \frac{u(B(y))}{\tilde{d}_{\partial\Omega}(B(y))}.$$

Using the properties of the nonincreasing rearrangement, we get

$$\left(\frac{u(B(y))}{d_{\partial G}(y)} \right)^* \leq L \left(\frac{u(B(y))}{\tilde{d}_{\partial\Omega}(B(y))} \right)^*,$$

and, consequently, by the property of the norm $\|\cdot\|_{L^q(\Omega)}$,

$$\left\| \frac{u(B(y))}{d_{\partial G}(y)} \right\|_{L^{1,q}(G)} \leq L \left\| \frac{u(B(y))}{\tilde{d}_{\partial\Omega}(B(y))} \right\|_{L^{1,q}(G)}. \tag{4.1}$$

Further, we obtain trivially $\tilde{d}_{\partial\Omega}(x) \geq d_{\partial\Omega}(x)$ and therefore, similarly to (4.1),

$$\left\| \frac{u(x)}{\tilde{d}_{\partial\Omega}(x)} \right\|_{L^{1,q}(\Omega)} \leq \left\| \frac{u(x)}{d_{\partial\Omega}(x)} \right\|_{L^{1,q}(\Omega)}. \tag{4.2}$$

Let us denote

$$\begin{aligned} \widetilde{m}_\xi &:= \left\{ x \in \Omega, \left| \frac{u(x)}{\tilde{d}_{\partial\Omega}(x)} \right| > \xi \right\}, \\ m_\xi &:= \left\{ y \in G, \left| \frac{u(B(y))}{\tilde{d}_{\partial\Omega}(B(y))} \right| > \xi \right\}. \end{aligned}$$

Then, by 1.2,

$$\lambda^N(m_\xi) = \lambda^N(B^{-1}(\widetilde{m}_\xi)) \leq L^N \lambda^N(\widetilde{m}_\xi). \tag{4.3}$$

Altogether, we get,

$$\begin{aligned} \left\| \frac{u(B(y))}{d_{\partial G}(y)} \right\|_{L^{1,q}(G)} &\overset{(4.1)}{\leq} L \left\| \frac{u(B(y))}{\tilde{d}_{\partial\Omega}(B(y))} \right\|_{L^{1,q}(G)} = L \left(\int_0^\infty \lambda^N \left(m_\xi\right)^q \xi^{q-1} \, d\xi \right)^{\frac{1}{q}} \\ &\overset{(4.3)}{\leq} L^{N+1} \left(\int_0^\infty \lambda^N \left(\widetilde{m}_\xi\right)^q \xi^{q-1} \, d\xi \right)^{\frac{1}{q}} = L^{N+1} \left\| \frac{u(x)}{\tilde{d}_{\partial\Omega}(x)} \right\|_{L^{1,q}(\Omega)} \\ &\overset{(4.2)}{\leq} L^{N+1} \left\| \frac{u(x)}{d_{\partial\Omega}(x)} \right\|_{L^{1,q}(\Omega)} < \infty, \end{aligned}$$

where we use the equivalent description of the Lorentz norm by level sets (Remark 1.19). We obtain $\frac{u(B(y))}{d_{\partial G}(y)} \in L^{1,q}(G)$ and the required inequality. The proof is complete. $\square$

Now we define the operator of linearisation of the boundary. We use the notation from Definition 1.25. Moreover, we define new variables $(y', y_N) \in (0,1)^{N-1} \times (-1,1)$ and the function $B_r \colon (0,1)^{N-1} \times (-1,1) \longrightarrow V_r$ such that

$$\begin{aligned} x'_r &= 2\alpha \left(y' - \left(\frac{1}{2}\right)' \right), \\ x_{r_N} &= a_r \left(2\alpha \left(y' - \left(\frac{1}{2}\right)' \right) \right) + \beta y_N, \end{aligned} \tag{4.4}$$

where we denote by $(N-1)$-vector $(\frac{1}{2})' = (\frac{1}{2}, \ldots, \frac{1}{2})$. This mapping is a bijection, thus we have the inverse function $B_r^{-1} \colon V_r \longrightarrow (0,1)^{N-1} \times (-1,1)$ such that

$$\begin{aligned} y' &= \frac{x'_r}{2\alpha} + \left(\frac{1}{2}\right)', \\ y_N &= \frac{x_{r_N}}{\beta} - \frac{a_r(x'_r)}{\beta}. \end{aligned}$$

Note that we have a bijection $B_r \colon (0,1)^N \longrightarrow V_r^+$.

Let us prove that B_r is bilipschitz and preserves Sobolev spaces.

Theorem 4.2. *Let a_r be a Lipschitz function on $\overline{\Delta_r}$ and let the mappings B_r and B_r^{-1} be defined as above. Then B_r and B_r^{-1} are Lipschitz functions as well, consequently B_r is a bilipschitz mapping.*

Moreover, there exist positive constants C_1 and C_2, depending just on a_r, α, β and the dimension, such that, for every $u \in W^{1,p}(V_r^+)$, $p \in [1, \infty)$, one has $U := u \circ B_r \in W^{1,p}(Q_N)$ and

$$C_1 \|u\| W^{1,p}(V_r^+) \leq \|U\| W^{1,p}(Q_N) \leq C_2 \|u\| W^{1,p}(V_r^+). \tag{4.5}$$

Proof. Let us focus on the Lipschitz continuity. We assume that K is the constant of the Lipschitz continuity of function a_r. Let $y^1, y^2 \in (0,1)^{N-1} \times (-1,1)$. Then

$$\begin{aligned}
\left|B_r(y^1) - B_r(y^2)\right| &= \left|\begin{pmatrix} 2\alpha((y^1)' - (\frac{1}{2})') - 2\alpha((y^2)' - (\frac{1}{2})') \\ a_r(2\alpha((y^1)' - (\frac{1}{2})')) + \beta y_N^1 - a_r(2\alpha((y^2)' - (\frac{1}{2})')) - \beta y_N^2 \end{pmatrix}\right| \\
&= \left|\begin{pmatrix} 2\alpha((y^1)' - (y^2)') \\ a_r(2\alpha((y^1)' - (\frac{1}{2})')) - a_r(2\alpha((y^2)' - (\frac{1}{2})')) + \beta(y_N^1 - y_N^2) \end{pmatrix}\right| \\
&\leq \left|\begin{pmatrix} 2\alpha((y^1)' - (y^2)') \\ \beta(y_N^1 - y_N^2) \end{pmatrix}\right| + K\left|2\alpha((y^1)' - (y^2)')\right| \\
&\leq (\max\{2\alpha, \beta\} + 2K\alpha)\left|y^1 - y^2\right|.
\end{aligned}$$

Analogously, let $x^1, x^2 \in V_r$. Then

$$\begin{aligned}
\left|B_r^{-1}(x^1) - B_r^{-1}(x^2)\right| &= \left|\begin{pmatrix} \frac{1}{2\alpha}(x^1)' + (\frac{1}{2})' - \frac{1}{2\alpha}(x^2)' - (\frac{1}{2})' \\ \frac{x_N^1}{\beta} - \frac{a_r((x^1)')}{\beta} - \frac{x_N^2}{\beta} + \frac{a_r((x^2)')}{\beta} \end{pmatrix}\right| \\
&= \left|\begin{pmatrix} \frac{1}{2\alpha}((x^1)' - (x^2)') \\ \frac{1}{\beta}(x_N^1 - x_N^2) - \frac{1}{\beta}(a_r((x^1)') - a_r((x^2)')) \end{pmatrix}\right| \\
&\leq \left|\begin{pmatrix} \frac{1}{2\alpha}((x^1)' - (x^2)') \\ \frac{1}{\beta}(x_N^1 - x_N^2) \end{pmatrix}\right| + \frac{1}{\beta}\left|\begin{pmatrix} 0 \\ a_r((x^1)') - a_r((x^2)') \end{pmatrix}\right| \\
&\leq \left(\max\left\{\frac{1}{2\alpha}, \frac{1}{\beta}\right\} + \frac{1}{\beta}K\right)\left|x^1 - x^2\right|,
\end{aligned}$$

which completes the proof of the fact that B_r is bilipschitz.

Let us turn to embeddings of Sobolev spaces. We proceed similarly to [13]. By the Rademacher theorem we have that a_r is differentiable almost everywhere on Δ_r. Let us denote by

$$(a_r)_i(x') := \frac{\partial a_r(x')}{\partial x_i}.$$

We have $|(a_r)_i(x')| \leq K$. Let us denote

$$G_{ij}(y) = \frac{\partial (B_r)_i(y)}{\partial y_j} = \begin{cases} 2\alpha\delta_{ij}, & i = 1, \ldots, N-1, \\ 2\alpha(a_r)_j(2\alpha(y' - \frac{1}{2})), & i = N, j = 1, \ldots, N-1, \\ \beta, & i = j = N \end{cases}$$

and

$$G_{ij}^{-1}(x) = \frac{\partial (B_r^{-1})_i(x)}{\partial x_j} = \begin{cases} \frac{1}{2\alpha}\delta_{ij}, & i = 1, \ldots, N-1, \\ -\frac{1}{\beta}(a_r)_j(x'), & i = N, j = 1, \ldots, N-1, \\ \frac{1}{\beta}, & i = j = N \end{cases}$$

the derivatives of B_r and B_r^{-1}, respectively, which also exist almost everywhere on Q_N, respectively on V_r^+. Thus we have the Jacobians

$$J_B(y) = \det(G_{ij}(y)) = 2^{N-1}\alpha^{N-1}\beta,$$
$$J_{B^{-1}}(x) = \det(G_{ij}^{-1}(x)) = 2^{1-N}\alpha^{1-N}\beta^{-1}.$$

Now, due to the fact that B_r is bilipschitz and u is measurable, U is also measurable and we use the Change of Variable Theorem ([8, 34.18]). We obtain

$$\begin{aligned}\|U\|_{L^p(Q_N)}^p &= \int_{Q_N} |U(y)|^p \, dy = \int_{Q_N} |u(B_r(y))|^p \, dy = \int_{V_r^+} |u(x)|^p \, J_{B_r^{-1}}(x)\, dx \\ &= 2^{1-N}\alpha^{1-N}\beta^{-1} \|u\|_{L^p(V_r^+)}^p .\end{aligned}$$

Thanks to the chain rule for weak derivatives (see for example [13]) we have

$$\frac{\partial U}{\partial y_i}(y) = \sum_{j=1}^{N} \frac{\partial u}{\partial x_j}(B_r(y))G_{ji}(y)$$

almost everywhere in Q_N. We use the Change of Variable Theorem again and we get

$$\begin{aligned}\left\|\frac{\partial U}{\partial y_i}\right\|_{L^p(Q_N)}^p &= \int_{Q_N} \left|\frac{\partial U}{\partial y_i}(y)\right|^p dy = \int_{Q_N} \left|\sum_{j=1}^{N} \frac{\partial u}{\partial x_j}(B_r(y))G_{ji}(y)\right|^p dy \\ &\le N^{p-1} \sum_{j=1}^{N} \int_{Q_N} \left|\frac{\partial u}{\partial x_j}(B_r(y))\right|^p |G_{ji}(y)|^p \, dy \\ &\le N^{p-1} \sum_{j=1}^{N} \|G_{ji}(y)\|_\infty^p \int_{Q_N} \left|\frac{\partial u}{\partial x_j}(B_r(y))\right|^p dy \\ &= N^{p-1} \sum_{j=1}^{N} \|G_{ji}(y)\|_\infty^p \int_{V_r^+} \left|\frac{\partial u}{\partial x_j}(x)\right|^p J_{B_r^{-1}}(x)\, dx \\ &= N^{p-1}2^{1-N}\alpha^{1-N}\beta^{-1} \sum_{j=1}^{N} \|G_{ji}(y)\|_\infty^p \int_{V_r^+} \left|\frac{\partial u}{\partial x_j}(x)\right|^p dx \\ &= C^p(\alpha,\beta,N,a_r) \|u\|_{W^{1,p}(V_r^+)}^p .\end{aligned}$$

Altogether we obtain

$$\|U\|_{W^{1,p}(Q_N)} = \left(\|U\|_{L^p(Q_N)}^p + \sum_{i=1}^{N} \left\|\frac{\partial U}{\partial y_i}\right\|_{L^p(Q_N)}^p\right)^{\frac{1}{p}} \le C_2 \|u\|_{W^{1,p}(V_r^+)} , \tag{4.6}$$

which yields the second inequality in (4.5). The first inequality comes analogously. □

We will now present a particular version of a known general trace theorem specified for the use of bilipschitz functions.

Lemma 4.3. *Let B_r be the bilipschitz function defined above. Then $(Tu) \circ B_r = T(u \circ B_r)$ λ^{N-1}-almost everywhere in the set $P = Q_{N-1} \times \{0\}$.*

Proof. Let $u \in W^{1,p}(V_r^+)$ and $\{u_n\}_{n=1}^\infty \subset \mathcal{C}^\infty(\overline{V_r^+})$ be a sequence such that $u_n \to u$ in $W^{1,p}(V_r^+)$. Using inequality (4.6) for the function $u_n - u$ we get

$$\|u_n \circ B_r - u \circ B_r\|_{W^{1,p}(Q_N)} \le C_2 \|u_n - u\|_{W^{1,p}(V_r^+)} \to 0$$

and similarly

$$\|u \circ B_r\|_{W^{1,p}(Q_N)} \le C_2 \|u\|_{W^{1,p}(V_r^+)} < \infty.$$

Using this and Lemma 1.29 for Lipschitz functions $\{u_n \circ B_r\}_{n=1}^\infty$ and $u \circ B_r$, we get

$$T(u_n \circ B_r) \to T(u \circ B_r) \text{ in } L^p(P).$$

Since u_n are smooth it makes sense to write $u_n(x)$ for $x \in \Lambda_r$ and $(u_n \circ B_r)(y)$ for $y \in P$. It is easy to see $(T(u_n \circ B_r))(y) = ((Tu_n) \circ B_r)(y)$, $y \in P$. But

$$(Tu_n) \circ B_r = T(u_n \circ B_r) \to T(u \circ B_r) \text{ in } L^p(P).$$

Thus, $T(u \circ B_r) = (Tu) \circ B_r$ almost everywhere in the set P. □

4.2 Proof of the main new implication

Theorem 4.4. *Let $\Omega \subset \mathbb{R}^N$ be a Lipschitz domain and $p, q \in [1, \infty)$. Then*

$$W_d(L^{1,q}, L^p)(\Omega) \subset W_0^{1,p}(\Omega)$$

and there exists a positive constant C such that for each $u \in W_d(L^{1,q}, L^p)$ we have

$$\|u\|_{W^{1,p}} \le C \|u\|_{W_d(L^{1,q},L^p)}.$$

Proof. First let us assume that $u \in W_0^{1,p}(\Omega)$. Then, by the Poincaré inequality (Theorem 1.31),

$$\|u\|_{W^{1,p}(\Omega)} = \|u\|_{W_0^{1,p}(\Omega)} \approx \|\nabla u\|_{L^p(\Omega)}.$$

Therefore,

$$\|u\|_{W_d(L^{1,q},L^p)(\Omega)} = \|g\|_{L^{1,q}(\Omega)} + \|\nabla u\|_{L^p(\Omega)} \ge \|\nabla u\|_{L^p(\Omega)} \ge C \|u\|_{W^{1,p}(\Omega)}.$$

Thus, it remains to prove

$$W_d(L^{1,q}, L^p)(\Omega) \subset W_0^{1,p}(\Omega).$$

Let $u \in W_d(L^{1,q}, L^p)(\Omega)$. Then $\nabla u \in L^p(\Omega)$ and thus, by Theorem 1.27, $u \in W^{1,p}(\Omega)$. We take an open covering $\{\Omega_r\}_{r=1}^{M+1}$ of Ω such that $\Omega_r = V_r, r = 1, \ldots, M$, from the definition of a Lipschitz domain and $\operatorname{dist}(\Omega_{M+1}, \mathbb{R}^N \setminus \Omega) > 0$. Note that we can take Ω_r such that for each $x \in \Omega_r$, $r = 1, \ldots, M$, inequalities $c_1 d(x) \le \operatorname{dist}(x, \partial\Omega \cap \Omega_r) \le c_2 d(x)$ hold with some positive constants c_1, c_2. We find the partition of unity $\{\phi_r\}_{r=1}^{M+1}$ such that $\phi_r \in C_0^\infty(\Omega_r)$ and for each $x \in \overline{\Omega}$ we have $\sum_{r=1}^{M+1} \phi_r(x) = 1$.

Let us denote $u_r = \phi_r \cdot u$ in Ω, $r = 1, \ldots, M+1$. Then $\operatorname{supp} u_r \subset \Omega_r$ and for each $x \in \Omega$ we have

$$u(x) = u(x) \sum_{r=1}^{M+1} \phi_r(x) = \sum_{r=1}^{M+1} u_r(x).$$

We show that $u_r \in W_d(L^{1,q}, L^p)(\Omega)$. We have $\frac{u}{d} \in L^{1,q}(\Omega)$ and

$$\left|\frac{\phi_r(x) \cdot u(x)}{d(x)}\right| \leq \left|\frac{u(x)}{d(x)}\right|,$$

thus, by properties of non-increasing rearrangement and norm $\|\cdot\|_{L^q(\Omega)}$,

$$\frac{u_r}{d} \in L^{1,q}(\Omega). \tag{4.7}$$

We also have $\nabla u \in L^p(\Omega)$. By the properties of the weak gradient and boundedness of norms $\|\phi_r\|_{C^1(\Omega)}$ and $\|u\|_{W^{1,p}(\Omega)}$, it follows that

$$\begin{aligned}\int_\Omega |\nabla(\phi_r \cdot u)(x)|^p \, dx &\leq 2^{p-1} \left(\int_\Omega |\nabla u(x) \cdot \phi_r(x)|^p \, dx + \int_\Omega |u(x) \cdot \nabla \phi_r(x)|^p \, dx\right) \\ &\leq 2^{p-1} \|\phi_r\|^p_{C^1(\Omega)} \left(\int_\Omega |u(x)|^p \, dx + \int_\Omega |\nabla u(x)|^p \, dx\right) < \infty\end{aligned}$$

and therefore $\nabla u_r \in L^p(\Omega)$ (and consequently $u_r \in W^{1,p}(\Omega)$). Altogether $u_r \in W_d(L^{1,q}, L^p)(\Omega)$, $r = 1, \dots, M+1$.

Now, we fix $r \in \{1, \dots, M\}$. We apply the operator R_r of rotation and translation from the definition of Lipschitz boundary of Ω such that $u_r \circ R_r$. Since this modification does not change any essential properties of the function u_r, we shall, with a slight abuse of notation, denote the new function, namely $u_r \circ R_r$, by u_r again. Now we work in local coordinates.

Let us denote $v_r(y) = u_r(B_r(y)), y \in Q_N$, where $B_r \colon Q_N \longrightarrow V_r^+$ is an operator of linearisation of the boundary defined in (4.4). From Theorem 4.2 and the fact that $u_r \in W^{1,p}(\Omega)$, we get $v_r = u_r \circ B_r \in W^{1,p}(Q_N)$ and, consequently,

$$\nabla v_r \in L^p(Q_N). \tag{4.8}$$

We claim that $\frac{v_r}{d_N} \in L^{1,q}(Q_N)$. Denote $\tilde{d}(x) = \operatorname{dist}(x, x_2)$, where $x = (x'_r, x_{r_N}) \in V_r^+$ and $x_2 = (x'_r, a(x'_r)) \in \partial\Omega$, which is unique for x. Thus

$$\tilde{d}(x) = |x_{r_N} - (x_2)_{r_N}| = |x_{r_N} - a(x'_r)|. \tag{4.9}$$

Note that for $x = B_r(y)$ is such x_2 an image in mapping B_r of $y_2 \in Q_{N-1} \times \{0\}$ that

$$d_N(y) = |y - y_2| = |y_N - (y_2)_N|. \tag{4.10}$$

Let us focus on relations between d and $\tilde{d}$. We have

$$d(x) \leq \tilde{d}(x)$$

thus

$$\frac{u_r(x)}{\tilde{d}(x)} \leq \frac{u_r(x)}{d(x)}$$

and therefore, by properties of nonincreasing rearrangement and the norm $\|\cdot\|_{L^q(\Omega)}$,

$$\left\|\frac{u_r(x)}{\tilde{d}(x)}\right\|_{L^{1,q}(\Omega)} \leq \left\|\frac{u_r(x)}{d(x)}\right\|_{L^{1,q}(\Omega)}. \tag{4.11}$$

We now turn to the reverse estimate. Due to the Lipschitz continuity of a_r, the angle between vectors $x_1 - x$ (where $x_1 \in \partial\Omega \cap V_r$ such that $d(x) = |x - x_1|$) and $x_2 - x$ is bounded with the upper bound depending on the constant K of Lipschitz continuity of a_r and using basic geometric thoughts and sine theorem we get that

$$\frac{\tilde{d}(x)}{d(x)} \leq \frac{1}{\cos(\operatorname{arctg} K)}$$

for each $x \in V_r^+$. Consequently,

$$d \approx \tilde{d}. \tag{4.12}$$

Now let us focus on relations with d_N. Thanks to (4.10), the definition of operator B_r and (4.9),

$$d_N(y) = |y_N - (y_2)_N| = \left| \frac{x_{r_N}}{\beta} - \frac{a_r(x_r')}{\beta} - \frac{(x_2)_{r_N}}{\beta} + \frac{a_r(x_r')}{\beta} \right| = \frac{\tilde{d}(x)}{\beta}. \tag{4.13}$$

Now let us denote

$$\begin{aligned} \widetilde{m}_\xi &:= \left\{ x \in V_r^+, \left| \frac{u_r(x)}{\tilde{d}(x)} \right| > \xi \right\}, \\ m_\xi &:= \left\{ y \in Q_N, \left| \frac{u_r(B_r(y))}{\tilde{d}(B_r(y))} \right| > \xi \right\}. \end{aligned}$$

Then, by 1.2,

$$\lambda^N(m_\xi) = \lambda^N(B_r^{-1}(\widetilde{m}_\xi)) \leq (Lip_{B_r^{-1}})^N \lambda^N(\widetilde{m}_\xi). \tag{4.14}$$

Altogether, we get

$$\begin{aligned} \left\| \frac{v_r(y)}{d_N(y)} \right\|_{L^{1,q}(Q_N)} &\overset{(4.13)}{=} \beta \left\| \frac{u_r(B_r(y))}{\tilde{d}(B_r(y))} \right\|_{L^{1,q}(Q_N)} = \beta \left(\int_0^\infty \lambda^N (m_\xi)^q \, \xi^{q-1} \, d\xi \right)^{\frac{1}{q}} \\ &\overset{(4.14)}{\leq} \beta (Lip_{B_r^-1})^N \left(\int_0^\infty \lambda^N (\widetilde{m}_\xi)^q \, \xi^{q-1} \, d\xi \right)^{\frac{1}{q}} \\ &= \beta (Lip_{B_r^-1})^N \left\| \frac{u_r(x)}{\tilde{d}(x)} \right\|_{L^{1,q}(V_r^+)} = \beta (Lip_{B_r^-1})^N \left\| \frac{u_r(x)}{\tilde{d}(x)} \right\|_{L^{1,q}(\Omega)} \\ &\overset{(4.11)}{\leq} \beta (Lip_{B_r^-1})^N \left\| \frac{u_r(x)}{d(x)} \right\|_{L^{1,q}(\Omega)} < \infty, \end{aligned} \tag{4.15}$$

where we used properties of Lorentz norm and (4.7). Consequently, $\frac{v_r}{d_N} \in L^{1,q}(Q_N)$ and, adding (4.8), $v_r \in T^{q,p}$.

By Theorem 3.5, v_r is an element of $W^{1,p}(Q_N)$ with zero trace on the set $\left\{ y \in [0,1]^N, y_N = 0 \right\}$, it means that $Tv_r(y) = 0$ almost everywhere in the set

$$Q_{N-1} \times \{y_N = 0\} = P.$$

We apply the mapping B_r again. We have

$$B_r(Q_{N-1} \times \{y_N = 0\}) = \Lambda_r$$

and, by Lemma 4.3, $Tu_r(x) = Tu_r(B_r(y)) = Tv_r(y)$ for almost every $y \in P$. Thus

$$\begin{aligned}\lambda^{N-1}(\{x' \in \Delta_r, |Tu_r(x', a_r(x'))| > 0\})& \\ &= \lambda^{N-1}(\{x' \in \Delta_r, (x', a_r(x')) \in B_r(\{y \in P, |Tv_r(y)| > 0\})\}) \\ &\le (Lip_{B_r})^{N-1}\lambda^{N-1}(\{y \in P, |Tv_r(y)| > 0\}) = 0,\end{aligned}$$

and therefore the function u_r, which is the element of $W^{1,p}(\Omega)$, has a zero trace on the set $\overline{\Lambda_r} \subset \partial\Omega$.

Finally, due to $\sum_{r=1}^{M+1} \phi_r(x) = 1$ for $x \in \partial\Omega$ and T is a linear operator, we have for each $x \in \partial\Omega$

$$\sum_{r=1}^{M+1} Tu_r(x) = T(\sum_{r=1}^{M+1} u_r(x)) = T(\sum_{r=1}^{M+1} u(x)\phi_r(x)) = Tu(x).$$

Thus, and due to $\partial\Omega = \bigcup_{r=1}^{M} \Lambda_r$ and $\operatorname{supp} Tu_r \subset \Lambda_r$,

$$\lambda^{N-1}(\{x \in \partial\Omega, |Tu(x)| > 0\}) \le \sum_{r=1}^{M} \lambda^{N-1}(\{x \in \Lambda_r, |Tu_r(x)| > 0\}) = 0,$$

and the function u has zero trace on $\partial\Omega$. By Theorem 1.30, $u \in W_0^{1,p}(\Omega)$. This completes the proof. □

4.3 Proof of the reverse implication

The reverse implication is far easier it can be derived from embeddings between Lorentz spaces and known results. However, for the sake of completeness, we shall present here an elementary proof based on the Hardy inequality.

Theorem 4.5. *Let $\Omega \subset \mathbb{R}^N$ be a Lipschitz domain and $p \in (1, \infty), q \in [1, \infty)$. Then*

$$W_0^{1,p}(\Omega) \subset W_d(L^{1,q}, L^p)(\Omega)$$

and there exists a positive constant, depending on Ω, p and q, such that for each $u \in W_0^{1,p}(\Omega)$ we have

$$\|u\|_{W_d(L^{1,q},L^p)} \le C\,\|u\|_{W^{1,p}(\Omega)}\,.$$

Proof. We will use the same notation, description of the boundary, partition of unity and mappings B_r as in the proof of Theorem 4.4. Here we assume that $u \in W_0^{1,p}(\Omega)$, thus $u_r \in W^{1,p}(\Omega)$ and therefore, due to Lemma 4.2, $v_r \in W^{1,p}(Q_N)$ and

$$\|v_r\|_{W^{1,p}(Q_N)} \le C_{r,2}\,\|u_r\|_{W^{1,p}(\Omega)} \le C_{r,2}\,\|u\|_{W^{1,p}(\Omega)}\,.$$

Since $u \in W_0^{1,p}(\Omega)$ and due to Theorem 1.23 and Lemma 1.32 we have for almost every $y' \in Q_{N-1}$ that $t \to v_r(y', t)$ is AC(0,1) and $\lim_{t\to 0_+} v_r(y', t) = 0$. For such y' we apply the Hardy inequality to the function

$$t \to \frac{\partial v_r}{\partial y_N}(y', t),\ t \in (0, 1).$$

This function is in $L^p((0,1))$ by the Fubini theorem and due to our assumptions. Note that $p>1$. This gives

$$\int_0^1 \left|\frac{v_r}{d_N}(y',t)\right|^p dt = \int_0^1 \left|\frac{1}{t}\int_0^t \frac{\partial v_r}{\partial y_N}(y',s)\,ds\right|^p dt \leq \int_0^1 \left(\frac{1}{t}\int_0^t \left|\frac{\partial v_r}{\partial y_N}(y',s)\right| ds\right)^p dt$$
$$\leq \left(\frac{p}{p-1}\right)^p \int_0^1 \left|\frac{\partial v_r}{\partial y_N}(y',t)\right|^p dt. \tag{4.16}$$

Thus, and using embeddings of Lebesgue and Lorentz spaces, we get

$$\begin{aligned}
\left\|\frac{v_r}{d_N}\right\|_{L^{1,q}(Q_N)} &\leq \left\|\frac{v_r}{d_N}\right\|_{L^1(Q_N)} \leq \left\|\frac{v_r}{d_N}\right\|_{L^p(Q_N)} = \left(\int_{Q_{N-1}}\int_0^1 \left|\frac{v_r}{d_N}(y',t)\right|^p dt\,dy'\right)^{\frac{1}{p}} \\
&\overset{(4.16)}{\leq} \left(\frac{p}{p-1}\right)\left(\int_{Q_{N-1}}\int_0^1 \left|\frac{\partial v_r}{\partial y_N}(y',t)\right|^p dt\,dy'\right)^{\frac{1}{p}} \\
&= \left(\frac{p}{p-1}\right)\left(\int_{Q_N} \left|\frac{\partial v_r}{\partial y_N}(y)\right|^p dy\right)^{\frac{1}{p}} \\
&\leq \left(\frac{p}{p-1}\right)\left(\int_{Q_N} |v_r|^p\,dy + \int_{Q_N}\left|\frac{\partial v_r}{\partial y_1}\right|^p dy + \cdots + \int_{Q_N}\left|\frac{\partial v_r}{\partial y_N}\right|^p dy\right)^{\frac{1}{p}} \\
&= \left(\frac{p}{p-1}\right)\|v_r\|_{W^{1,p}(Q_N)}.
\end{aligned}$$

Similarly to the preceding proof (see in particular (4.13), (4.12) and (4.15)) we obtain for each $r=1,\dots,M$ that

$$\left\|\frac{u_r}{d}\right\|_{L^{1,q}(\Omega)} \leq C_{r,1}\left\|\frac{v_r}{d_N}\right\|_{L^{1,q}(Q_N)},$$

where $C_{r,1}$ depends on the Lipschitz continuity of B_r and $\partial\Omega$. Thus we get

$$\begin{aligned}
\left\|\frac{u_r}{d}\right\|_{L^{1,q}(\Omega)} &\leq C_{r,1}\left\|\frac{v_r}{d_N}\right\|_{L^{1,q}(Q_N)} \leq C_{r,1}\left(\frac{p}{p-1}\right)\|v_r\|_{W^{1,p}(Q_N)} \\
&\leq C_{r,1}\left(\frac{p}{p-1}\right)C_{r,2}\|u\|_{W^{1,p}(\Omega)} = C_r\|u\|_{W^{1,p}(\Omega)},
\end{aligned}$$

where we denote $C_r = C_{r,1}\left(\frac{p}{p-1}\right)C_{r,2}$, $r=1,\dots,M$. We also have, by embeddings between Lebesgue and Lorentz spaces,

$$\begin{aligned}
\left\|\frac{u_{M+1}}{d}\right\|_{L^{1,q}(\Omega)} &\leq \frac{1}{\operatorname{dist}(\Omega_{M+1},\mathbb{R}^N\setminus\Omega)}\|u_{M+1}\|_{L^{1,q}(\Omega)} \\
&\leq \frac{1}{\operatorname{dist}(\Omega_{M+1},\mathbb{R}^N\setminus\Omega)}\lambda^N(\Omega)^{1-\frac{1}{p}}\|u_{M+1}\|_{L^p(\Omega)} \\
&\leq C_{M+1}\|u\|_{W^{1,p}(\Omega)},
\end{aligned}$$

where $C_{M+1} := \lambda^N(\Omega)^{1-\frac{1}{p}}/\operatorname{dist}(\Omega_{M+1},\mathbb{R}^N\setminus\Omega)$. Altogether we have, denoting C_{quasi} the constant of quasi-subadditivity of the quasinorm $\|\cdot\|_{L^{1,q}(\Omega)}$,

$$\left\|\frac{u}{d}\right\|_{L^{1,q}(\Omega)} = \left\|\frac{\sum_{r=1}^{M+1}u_r}{d}\right\|_{L^{1,q}(\Omega)} \leq C_{quasi}^M\sum_{r=1}^{M+1}\left\|\frac{u_r}{d}\right\|_{L^{1,q}(\Omega)} \leq C\|u\|_{W^{1,p}(\Omega)},$$

where $C = C_{quasi}^M \sum_{r=1}^{M+1} C_r$ depends on the shape and measure of Ω and p, q. Finally,

$$\|u\|_{W_d(L^{1,q},L^p)} = \left\| \frac{u}{d} \right\|_{L^{1,q}(\Omega)} + \|\nabla u\|_{L^p(\Omega)} \leq (C+1) \|u\|_{W^{1,p}(\Omega)},$$

which completes the proof. $\square$

5. Conclusion

Our main goal was to prove Theorem 2.9. It can now be achieved thanks to Theorem 4.4 and Theorem 4.5.

Proof of Theorem 2.9. Let $N \in \mathbb{N}$, $p \in (1,\infty)$, $q \in [1,\infty)$ and $\Omega \subset \mathbb{R}^N$ be a Lipschitz domain. From Theorem 4.4 it follows that

$$W_d(L^{1,q},L^p)(\Omega) \subset W_0^{1,p}(\Omega)$$

and there exists a positive constant C_1, depending on Ω, p and q, such that for each $u \in W_d(L^{1,q},L^p)(\Omega)$ we have

$$\|u\|_{W^{1,p}(\Omega)} \leq C_1 \|u\|_{W_d(L^{1,q},L^p)}\,.$$

Further we proved that from Theorem 4.5 it follows that

$$W_0^{1,p}(\Omega) \subset W_d(L^{1,q},L^p)(\Omega),$$

and there exists a positive constant C_2, depending on Ω, p and q, such that for each $u \in W_0^{1,p}(\Omega)$ we have

$$\|u\|_{W_d(L^{1,q},L^p)} \leq C_2 \|u\|_{W^{1,p}(\Omega)}\,.$$

Composing these inclusions and estimates we obtain

$$W_d(L^{1,q},L^p)(\Omega) = W_0^{1,p}(\Omega)$$

and for each u from this set we have

$$\|u\|_{W^{1,p}(\Omega)} \approx \|u\|_{W_d(L^{1,q},L^p)}\,,$$

which establishes our claim. □

We shall finish wish some concluding remarks.

Remark 5.1. It is worth noticing that it follows from our results that the quasi-norm $\|u\|_{W_d(L^{1,q},L^p)}$ is equivalent to a norm.

Remark 5.2. It is conceivable that a result similar to the main theorem of this thesis could be obtained for more general domains than for those with the Lipschitz boundary. It is also likely that such extension could be obtained using ideas similar to those developed in this thesis.

Remark 5.3. Let Ω be a Lipschitz domain, $p \in (1,\infty)$, $q \in [1,\infty]$. In [14], a question was posed about optimality in the sense whether $u \in W_0^{1,p}(a,b)$ if and only if $\frac{u(x)}{d(x)} \in L^{1,q}(a,b)$ and $u'(x) \in L^p(a,b)$, where (a,b) was an open interval. In [14] we answered this question for some ranges of parameters, however the question remained partly open.

Our new results answer the question not only for an interval in one dimension, but also for a general Lipschitz domain in the Euclidean space of arbitrary dimension. We have the positive answer for each $q \in [1,\infty)$ from Theorem 2.9.

Now we shall point out that the answer is negative when $q = \infty$. This can be demonstrated using the following example.

Set $\Omega = (0,1)^N$, $N \in \mathbb{N}$, and $u(x) = 1$ for each $x \in \Omega$. The graph of $d(x)$ is a "pyramid" with vertex in $(\frac{1}{2}, \dots, \frac{1}{2})$, where $d(x)$ attains the value $\frac{1}{2}$. For example in one dimension we have

$$d(x) = \begin{cases} x, & x \in (0, \frac{1}{2}], \\ 1 - x, & x \in (\frac{1}{2}, 1). \end{cases}$$

Let us compute the distribution function $\xi \mapsto \lambda^N(\{x \in \Omega; \frac{1}{d(x)} > \xi\})$. It is clearly equal to one for $\xi \in [0,2]$, thus it suffices to compute it for $\xi > 2$. We obtain

$$\begin{aligned} \lambda^N(\{x \in \Omega; \frac{1}{d(x)} > \xi\}) &= \lambda^N(\{x \in \Omega; d(x) < 1/\xi\}) \\ &= \lambda^N(\Omega \setminus \{x \in \Omega; d(x) \geq 1/\xi\}) \\ &= 1 - \left(1 - \frac{2}{\xi}\right)^N, \end{aligned}$$

where $(1 - \frac{2}{\xi})^N$ is the volume of a cube, whose distance from the boundary of Ω is $\frac{1}{\xi}$.

Thus, and by the definition of the Lorentz norm via distribution function (see Remark 1.19), we get

$$\left\| \frac{1}{d} \right\|_{L^{1,\infty}(\Omega)} = \sup_{\xi > 0} \xi \lambda^N(\{x \in \Omega; \frac{1}{d(x)} > \xi\}) = \max \left\{ 2, \sup_{\xi > 2} \xi \left(1 - (1 - 2/\xi)^N\right) \right\}.$$

Changing variables $1 - \frac{2}{\xi} = s$, $s \in (0,1)$, we obtain

$$\sup_{\xi > 2} \xi \left(1 - (1 - 2/\xi)^N\right) = \sup_{s \in (0,1)} \frac{2}{1-s}(1 - s^N) = \sup_{s \in (0,1)} 2(1 + s + \dots + s^{N-1}) = 2N,$$

and, consequently, $\frac{u}{d}$ in $L^{1,\infty}(\Omega)$. Moreover, due to $\nabla u = 0$, we have $u \in W_d(L^{1,\infty}, L^p)$. Further $u \in W^{1,p}(\Omega)$ and therefore, by the Beppo-Levi theorem, u is absolutely continuous on almost every line parallel to the any of the axes in $(0,1)^N$, so it follows from Lemma 1.32 that $Tu = 1$ a.e., which contradicts $u \in W_0^{1,p}(\Omega)$, as follows from Theorem 1.30.

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