Report on the Doctoral Thesis
"Additive Combinatorics and Number Theory"

by

Jaroslav Hancl

Department of Applied Mathematics,
Charles University,
Prague, Czech Republic

1. Overview

Enumeration problems are important part of Mathematics. Counting partitions of integers with some given properties were already mentioned in the 16th century in letters of Leibniz and Bernoulli; later results of Euler, Hardy and Ramanujan are still part of graduate courses in Combinatorics. This area is still an active part of combinatorics and the thesis contributes to its theory. Other parts of the thesis studies related problems on ordered graphs and hypergraphs.

The thesis consists of three chapters on enumeration problems: Chapter 1 is on Integer Partitions, Chapter 2 is on Ordered Graphs, and Chapter 3 is on Partial Ordered Sets. The first two chapters are based one-one submitted papers, which are published or to be published in leading research journals of the area, and the final chapter contains new results which potentially could be part of an additional paper.

Chapter 1: The topics of this function is estimating the growth function of some given partition families. About 100 years ago a celebrated result of Hardy and Ramanujan (see Theore 1.4) gave an asymptotically sharp formula for the number of partitions of n. Not much later, Schur (see Theorem 1.6) determined the asymptotic of the number of partitions of n which used only parts from a given finite family S. The first main result of the chapter (Theorem 1.8) provides a complementary result, it estimates the number of partitions where none of the parts is a member of S.

Families of partition which satisfy some natural hereditary property are called *ideal*. One would think that the growth function of an ideal is a nice function. However, somewhat surprisingly, (though in the literature there are some similar phenomena), some examples are constructed showing that it is not necessarily the case (see Theorem 1.12). Here, I think it would be necessary to discuss related results of J. Balogh, Bollobás and Weinreich, The penultimate rate of growth for graph properties, European J. Comb, 22, 277–289. There are additional results about independent partitions, see for example Theorem 1.19.

The proofs seem to be highly non-trivial. Probably, this Chapter is the most interesting part of the thesis.

Chapter 2: This Chapter studies the growth rate of ordered graphs and hypergraphs. The pedigree of the problems are coming from the famous Stanley-Wilf conjecture on permutation, which was solved by combined results of Klazar and Marcus and Tardos, who were building on a fundamental paper of Füredi and P. Hajnal. The latter paper should be added to the references and discussion.

Chapter 2 provides a detailed analyzis of possible growth functions, such as Theorem 2.11 claims jump from constant to linear function, and Theorem 2.12 which shows jump from a polynomial to an exponential function. A nice technical part of the proof is the introduction of three-dimensional crossing matrices. The proofs are technical and long, I have not had a chance to read the details.

Chapter 3: Here a general theory is built, originated from families of permutations, hence the notion of Wilf equivalence. Two famlies of discrete structures are *Wilf equivalent* if the growth function of families avoiding them are the same. Note that Euler classical result (he had many, one of them) is proving Wilf equivalence in the integer partition problems, certainly he did not use the name "Wilf" at that time. In this section mainly posets and 'subword' posets are discussed. Two methods for generating Wilf equivalent pairs of sets were presented: the automorphism method, and the Cohen-Remmel method.

Summary: The thesis contains many interesting enumeration results, in two different parts of combinatorics. The proof methods are highly non-trivial; it shows that the candidate can use deep mathematical methods, and also there are plenty of new ideas. The thesis is in English, which is not the native language of the candidate. Though the grammar is not fantastic, it is reasonably well-written, contains very few typos only, see the extra page for minor comments.

The thesis is a fitting basis for awarding the degree of "Doctor of Philosophy".

Sincerely yours,
July 20, 2020, Champaign IL, USA

Professor József Balogh
University of Illinois
at Urbana-Champaign, USA

Minor comments:

In Preface: “you can find there results” $\rightarrow$ One can find results in the thesis
”focus is concentrated” $\rightarrow$ focus is on OR is concentrated
”Besides latter types”: what does it mean?
Chapter 1:
”Benoulli”: Bernoulli ?
”to introducing” $\rightarrow$ to introduce
Page 24: ”Balogh, Bollobás and Morris [12, 11] proved an astonishingly precise characterization of growth for ordered graphs. It describes possible growth in the ”lower” region below... They were inspired by Kaiser and Klazar [41] who proved similar result for permutations.” I wonder how to get such conclusion? The first two authors (Balogh and Bollobás) with Weinreich have four earlier papers on related results.
”Scheinerman and Zito [61] were the first to investigate”: My understanding that Alekseev [2] was about the same time, independently.