ALEXANDER SLAVIK: "CLASSES OF MODULES ARISING IN ALGEBRAIC GEOMETRY" (PHD THESIS)

The thesis focuses on an important area of contemporary mathematics - that of representation theory of rings and schemes. The material is based on results from four published/accepted papers by the author and his collaborators: S. Estrada, M. Hrbek, L. Positselski, M. Prest, J. Stovicek.

- Chapter 2 is based on the paper "Quillen equivalent models for the derived category of flats and the resolution property" (joint with S. Estrada, *J. Aust. Math. Soc.*). It is known that for an affine scheme $X = Spec(R)$, the category $QCoh(X)$ of all quasi coherent sheaves on X is equivalent to the category $R - Mod$ of all R-modules. But, in general, $QCoh(X)$ lacks some of the properties that the category $R - Mod$ has. Notably, $QCoh(X)$ may not have enough projective objects. This has led to a search for replacements; Murfet solved the problem by using flat quasi coherent sheaves to define a generalization $(D(Flat(X)))$ of the homotopy category of the projective modules over a commutative ring. In the affine case $X = Spec(R)$, the homotopy category of projectives, $K(Proj(R))$, and the derived category of flats, $D(Flat(X))$, are triangle equivalent. However, from a homological point of view, flats are more complicated than the projectives. This is the reason for focusing on a refinement of the class of flat quasi coherent sheaves - the so called very flat quasi coherent sheaves. In the affine case $X = Spec(R)$, every very flat module has projective dimension at most 1; therefore the derived category of very flat modules, $D(\mathcal{VF}(R))$, and the homotopy category of projective modules, $K(Proj(R))$, are triangle equivalent in this case. So $D(\mathcal{VF}(R))$ and $D(Flat(R))$ are triangle equivalent in this case. It is a natural question to consider whether or not this (indirect) triangulated equivalence still holds over a non-affine scheme. The result is known to hold for semi-separated noetherian schemes of finite Krull dimension. In Chapter 2 of this thesis, a more general result is proved (Corollary 2.6.2): For any quasi-compact and semi-separated scheme X, the categories $D(Flat(X))$ and $D(\mathcal{VF}(X))$ are triangle equivalent.

The main result of Chapter 2 (Theorem 2.6.1) gives sufficient conditions on a subclass $\mathcal{A}_{qc}$ of that of the flat quasi-coherent sheaves in order to get a triangle equivalent category to the derived category of flats, $D(Flat(X))$. The result mentioned above (Corollary 2.6.2) follows from Theorem 2.6.1 for $\mathcal{A}$ being the class of very flat modules. It is also proved (Corollary 2.6.4) that if X is a scheme with enough infinite-dimensional vector bundles, then the categories $D(Flat(X))$ and $D(Vect(X))$ are triangle equivalent.

- The third chapter focuses on flat and quite flat modules. It is based on the paper "Countably generated flat modules are quite flat" (joint with M. Hrbek and L. Positselski, to appear in *J. Comm. Alg.*). The quite flat modules and the almost cotorsion modules were introduced by Positselski and Slavik. A module C is *almost cotorsion* if $Ext^1(S^{-1}R, C) = 0$ for all (at most) countable multiplicative subsets $S \subseteq R$. The quite flat modules are those in the left orthogonal class of the almost cotorsion modules. The main result of Chapter 3 (Theorem 3.2.4) is that

all countably generated flat modules over a commutative noetherian ring are quite flat. Then Theorem 3.3.8 shows that if R is a commutative noetherian ring whose spectrum has cardinality less than κ, where κ is a regular uncountable cardinal, then every flat module is a transfinite extension of $< \kappa$-generated flat modules. For $\kappa = \aleph_1$, this implies that every flat module is a transfinite extension of countably generated flat modules, therefore every flat is quite flat (Corollary 3.3.9). The result was known (it was proved by Positselski and Slavik in the paper where they introduced the quite flat modules), but Corollary 3.3.9 gives an alternate proof.
- Chapter 4 is based on the paper "Purity in categories of sheaves" (joint with M. Prest, *Math. Zeit.*) There are two notions of purity: the categorical one and the geometrical purity. We recall that a short exact sequence in $QCoh(X)$ is called categorically pure if it stays exact when applying the Hom functor from a finitely presented object. And, a short exact sequence in $QCoh(X)$, $0 \to \mathcal{A} \to B \to \mathcal{C} \to 0$, is called geometrically pure if it stays exact when applying the sheaf tensor product $-\bigotimes \mathcal{Y}$, for all $\mathcal{Y} \in \mathcal{O}_X - Mod$. These two notions of purity differ, in general. The main goal of Chapter 4 is to provide a better understanding of pure-injective objects with respect to both of these purities. More precisely, section 4.2 explores the relation between purity in the category $\mathcal{O}_X - Mod$ and purity in $QCoh(X)$. The pure exact sequences (both categorical and geometrical) are the same. However (as showed at the end of section 4.5), the pure-injectives in $QCoh(X)$ are quite different than those in $\mathcal{O}_X - Mod$. The main result of section 4.3 is the description of the geometric pure-injective objects in $\mathcal{O}_X - Mod$. Section 4.4 presents an example of the Ziegler spectrum of the category $\mathcal{O}_X - Mod$ over a local affine 1-dimensional scheme X. Section 4.5 shows that for a concentrated scheme X, categorical purity and geometric purity coincide if and only if X is affine. Section 4.6 is devoted to purity in the category of quasicoherent sheaves over a projective line. The goal is computing the Ziegler spectrum - both the points and the topology are described.

- Chapter 5 investigates the existence of flat generators and the connection with Matlis duality. It is based on the paper "On flat generators and Matlis duality" (joint with J. Stovicek, *Bull. Lond. Math. Soc.*, to appear). As already mentioned, it is known that, unless the scheme X is affine, the category $QCoh(X)$ does not usually have enough projective objects. This problem is often solved using flat objects. So a natural question to consider is: when does $QCoh(X)$ have enough flat objects? (or, equivalently, when does $QCoh(X)$ have a flat generator?) It is known that $QCoh(X)$ has enough flat objects when X is quasi compact and semiseparated. For quite some time, the hope was that the result extends to quasi compact quasi separated schemes. But the results in this thesis (section 5.2) show that the semiseparatedness is a necessary condition for the existence of enough flat quasi coherent sheaves.
The connection with Matlis duality is given by the main results of section 5.3. Theorem 5.3.10 and Corollary 5.3.13 show that for a quasi compact and quasi separated scheme X the following are equivalent: (1) the category $QCoh(X)$ has enough flat sheaves; (2) for any injective cogenerator δ of $QCoh(X)$ the contravariant internal hom functor $\mathcal{H}om^{qc}(-, \delta)$ is exact; (3) the scheme X is semiseparated.

The results are new and very nice, and the thesis is well written. In my opinion, this thesis is definitely suitable for a PhD degree.

Alina Iacob
Professor, Department of Mathematical Sciences
Georgia Southern University
Statesboro GA 30460-8093
Phone: (912) 478-5873
Fax: (912) 681-0654
aiacob@georgiasouthern.edu