REPORT ON THESIS BY ALEXANDER SLÁVIK
Classes of modules arising in algebraic geometry

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The thesis under review consists of an introduction, and four research chapters, all of which contain interesting original results. The first research chapter discusses, given a scheme X, conditions under which a subcategory $\mathcal{A}$ of the category of flat quasi-coherent sheaves give rise to an equivalence of triangulated categories

$$\mathsf{D}(\mathcal{A}) \cong \mathsf{D}(\mathrm{Flat}(X)) \tag{1}$$

In fact, an even stronger result is proved, showing that under some natural conditions, the category of unbounded cochain complexes of objects from $\mathcal{A}$ has an abelian model category structure, and the equivalence (1) is induced by a Quillen equivalence. Several interesting instances of this equivalence are given. In particular, it is shown that for any quasi-compact and semi-separated scheme X, the equivalence (1) holds for $\mathcal{A}$ being the category of very flat quasi-coherent sheaves. This is a far reaching generalization of a recent theorem of Positselski. Another interesting instance of this equivalence is for X being a quasi-compact semi-separated scheme which satisfies the resolution property. In that case, this equivalence holds for $\mathcal{A}$ being the category of infinite dimensional vector bundles over X. The next chapter is a chapter in commutative algebra. It shows that if R is a commutative noetherian ring, and F is a flat module which is countably generated, then F is a direct summand of a transfinite extension of modules of the form $S^{-1}R$, where S is a countable multiplicative subset of R. Such a result can be seen, in some sense, as an improvement of the Govorov-Lazard Theorem, by obtaining a better description of flat modules of these type. Another similar result is obtained for commutative noetherian rings whose prime spectrum has a cardinality bounded by an uncountable regular cardinal. Chapter 4 discusses different notions of purity for the categories of sheaves of modules and quasi-coherent sheaves over a quasicompact quasiseparated scheme. It is shown that for such a scheme X, the geometric pure-injective objects in the category of sheaves of modules over X are exactly skyscraper sheaves whose stalk is an indecomposable pure-injective module. It is further shown that the set of indecomposable geometric pure-injective quasi-coherent sheaves is a closed quasicompact subset of the Ziegler spectrum of the quasi coherent sheaves on X. The final chapter 5 discusses the question

of when, for a quasicompact quasiseparated scheme X, does the category of quasi-coherent sheaves over X has a flat generator. The interesting main result of this chapter shows that this is the case if and only if the scheme X is also semiseparated. It is further shown that this is equivalent to the fact that for any injective object $\mathcal{E}$ of the category of quasi-coherent sheaves over X, the functor $\mathcal{H}om^{\mathrm{qc}}(-,\mathcal{E})$ is exact; equivalently, that the functor $\mathcal{H}om^{\mathrm{qc}}(-,\mathcal{E})$ is exact for some injective cogenerator $\mathcal{E}$ of the category of quasi-coherent sheaves over X.

The proofs seem to be correct and well written. Each chapter consists of an introduction, giving the background to each subject, and explaining the main ideas of the chapter. In my opinion, the results of this thesis are highly original and important. The different models for the derived category of flat quasicoherent sheaves over a scheme are an important powerful tool, which should have various applications in the study of schemes in algebraic geometry. The characterization of countably generated flat modules is a useful powerful result, and the new characterization of semiseparated schemes suggests that certain future research directions in the homological algebra over schemes should concentrate on the semiseparated case, showing real limitations for the ability to develop such a general theory over quasiseparated schemes which are not semiseparated.

To summarize, this thesis clearly demonstrates the author ability to produce creative scientific work, and I recommend that it will be accepted as a PhD Thesis.

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