FACULTY OF MATHEMATICS AND PHYSICS
Charles University

DOCTORAL THESIS

Vít Kubelka

Filtering for Stochastic Evolution Equations

Department of Probability and Mathematical Statistics

Supervisor of the doctoral thesis: Prof. RNDr. Bohdan Maslowski, DrSc.
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Title: Filtering for Stochastic Evolution Equations

Author: Vít Kubelka

Department: Department of Probability and Mathematical Statistics

Supervisor: Prof. RNDr. Bohdan Maslowski, DrSc., Department of Probability and Mathematical Statistics

Abstract: Linear filtering problem for infinite-dimensional Gaussian processes is studied, the observation process being finite-dimensional. Integral equations for the filter and for covariance of the error are derived. General results are applied to linear SPDEs driven by Gauss-Volterra process observed at finitely many points of the domain and to delayed SPDEs driven by white noise. Subsequently, the continuous dependence of the filter and observation error on parameters which may be present both in the signal and the observation process is proved. These results are applied to signals governed by stochastic heat equations driven by distributed or pointwise fractional noise. The observation process may be a noisy observation of the signal at given points in the domain, the position of which may depend on the parameter.

Keywords: Kalman-Bucy filter, stochastic evolution equations, Gaussian processes, Hilbert spaces

Contents

Introduction

Let us consider a price of a stock on a market. This price consists of a trend and a noise. We would like to estimate the trend which is a random process that can be observed only through the price of the stock. It is a typical application of the filtering theory. In general, filtering theory is used in situations when we want to estimate a process which we can not fully observe and which is called signal. Instead of the signal we observe another process which is determined by the signal and a random noise. Such process is called observation process. Filtering theory has many applications in various scientific fields such as physics, biology and financial mathematics.

One of the fundamental parts of this theory is the situation when the signal is governed by a linear stochastic differential equation and the observation process depends on the signal also linearly. This model is so called Kalman-Bucy (KB) filter and was described for the first time by Kalman and Bucy [28]. Wiener process (white noise) is the source of randomness here for both the signal and the observation process, the two noises are mutually independent and only final-dimensional spaces are considered.

The aim of this Thesis is to study the linear filtering problem for infinite-dimensional Gaussian processes with finite-dimensional observation. Typically, the signal process may be governed by a linear stochastic partial differential equation (SPDE) driven by noise that is not white in time, like a Gaussian Volterra noise or, in particular, fractional Brownian motion (fBm). An abstract setting of a separable rigged Hilbert is considered to allow use of pointwise observations.

An analogous problem for finite-dimensional (or scalar) processes have been studied by Kleptsyna and Le Breton [30] for the case of general Gaussian process observed through a linear channel driven by a Wiener process. A rather general approach to filtering with fractional Brownian motion is presented in [29] and [31].

In infinite dimensions, a pioneering result belongs to Falb [22], where KB type result has been established. In this case, both signal and observation process live in a Hilbert space and are governed by linear evolution equation with a Q-Wiener process. We are not aware of any analogous result, except the work contained in this Thesis, in infinite dimensions for general Gaussian processes that would cover, for example, linear SPDEs driven by fractional noise (however, we would like to point out that a "dual" LQ control problem has been treated, for instance, in [21] and [20], while related statistical inference problems were addressed in numerous papers, like [42], [10] or [33]).

General theory derived in this Thesis covers also linear filtering of stochastic delayed evolution equations driven by a Gaussian noise. However, usually there is no explicit formula for the mean and covariance of these processes, which is essential for the application of the theory derived in this Thesis. We will deal with this problem in the case of stochastic evolution equation with a discrete delay in the drift, driven by Wiener process. To our best knowledge there are no other works for filtering stochastic delayed evolution equations. A similar problem in finite dimension is studied by different approach in [4]. Nonlinear filtering of stochastic delayed differential equations in finite dimension is addressed in [50].

The case with delayed observations is covered, for example, in [9] and [3], again in finite dimension.

The Thesis also solve a more general problem of continuous dependence of the filter on parameter which may be present both in signal and observation processes. Again, the abstract setting of a separable rigged Hilbert space is considered. So the signal is a Hilbert space-valued parameter-dependent Gaussian process and the observation is given by stochastic differential, the coefficients of which may also depend on the parameter. To our best knowledge, such problem was studied only by Týbl in finite-dimensional state space in unpublished work [49].

Author's publications related to this Thesis

V. Kubelka and B. Maslowski, *Filtering of Gaussian processes in Hilbert spaces*, Stochastics and Dynamics, 20(03):2050020, 2020.

In this paper linear filtering problem for infinite-dimensional Gaussian processes is studied, the observation process being finite-dimensional. Integral equations for the filter and for covariance of the error are derived. General results are applied to linear SPDEs driven by Gauss-Volterra process observed at finitely many points of the domain.

V. Kubelka and B. Maslowski, *Filtering for stochastic heat equation with fractional noise*, Proceedings of 21st European Young Statistician Meeting, Bernoulli Soc. Math. Stat. Probab., p. 25-29, Belgrade, 2019.

This paper summarizes results of the previous paper and extends previous applications to the case of a stochastic evolution equation driven by cylindrical fractional Brownian motion observed at finitely many points on an unbounded domain.

V. Kubelka, B. Maslowski and O. Týbl, *Parameter-dependent filtering of Gaussian processes in Hilbert spaces*, submitted to: Stochastic Analysis and Applications.

In this paper the same abstract setting as in the previous ones is considered and the continuous dependence of the filter and observation error on parameters which may be present both in the signal and the observation process is proved. The general results are applied to signals governed by stochastic heat equations driven by distributed or pointwise fractional noise. The observation process may be a noisy observation of the signal at given points in the domain, the position of which may depend on the parameter.

V. Kubelka and B. Maslowski, *Filtering of stochastic delayed differential equations in Hilbert spaces*, submitted to: Communications in Information and Systems.

This paper extends the results of the first paper to the case of non-centered Gaussian signals and applies the theory to a signal given by stochastic evolution equation with a discrete delay in the drift driven by Wiener process. The results are specified for a delayed SPDE driven by white noise.

Organisation of the Thesis

The Thesis is divided into three chapters.

The first chapter is divided in two sections. In Section 1.1 the linear filtering theory is introduced and some known results are recalled. In Section 1.2 some particular types of Gaussian processes are discussed. Basic definitions and properties are stated. Stochastic integral with respect to those processes is introduced and also the theory of stochastic evolution equations driven by them is recalled. These processes then play the role of the noise in stochastic evolution equations throughout the Thesis.

The second chapter covers the works by Kubelka and Maslowski [35], [36] and [34]. In Section 2.1 integral equations for the filter and for the covariance of the observation error on a rigged Hilbert space are derived. The observation process is assumed to be finite-dimensional. The uniqueness of the solutions is shown in Section 2.2.

In Section 2.3, the signal given by stochastic evolution equation driven by Wiener process is studied. It is shown that in that case the integral equations for the filter and for the covariance of the observation error simplifies to the infinite-dimensional analogue of standard KB filter.

The examples are given in Section 2.4. The general results are applied to stochastic parabolic equation perturbed by fBm and Gauss-Volterra noise, observed at finitely many points of the domain. In this case, comparing to the classical KB theorem, there are two major obstacles: the fact that the noise does not have independent increments and the need to apply the results to linear SPDEs with space-dependent noise and observation at specific points in the domain of the equation. While the first problem is treated similarly as in the finite-dimensional papers quoted above, the second one is overcome by posing the equation on a rigged Hilbert space. The larger space H (which is usually a Lebesgue space on the domain) is suitable for the definition of the noise term and the stochastic integral, while the smaller space V is contained in the space of continuous functions (for which values at given points are well defined). The case of a signal given by stochastic parabolic equation perturbed by Gauss-Volterra noise on a bounded domain is studied in 2.4.1 while the case of a signal given by stochastic parabolic equation perturbed by fBm noise on an unbounded domain is covered in 2.4.2.

Finally, in 2.5, the general results are applied to signal given by stochastic delayed evolution equations perturbed by Wiener process. Here the correlation between the signal increments is given by the delay in the drift instead of the correlated noise. The main obstacle here is to obtain the mean value of the signal and its covariances.

The third chapter covers the work Kubelka, Maslowski and Týbl [37] and deals with continuous dependence of the filter on a parameter. It is divided into four sections. in Section 3.1 the basic setting is explained in detail and the main filtering result from the second chapter is applied.

Section 3.2, which contains the heart of the proof of our main result, is devoted to continuous dependence of the covariance of the observation error. This mapping satisfies a nonlinear integral equation with non-Lipschitz right-hand side. Hence, it does not seem to be possible to proceed in a standard way by means of the Gronwall lemma. We use a method based on compactness of the family of solutions (which is proved by Arzela-Ascoli theorem for mappings taking values in operator spaces, utilizing so-called collective compactness of solutions and their adjoints, cf. [43]). Section 3.3 contains the main result of the chapter, the proof of continuous dependence of the filter.

In Section 3.4, these results are applied to the case of signal defined by linear SPDE driven by cylindrical fractional Brownian motion. Two examples of signal are then elaborated in more detail: The heat equation perturbed either with distributed or by pointwise fractional noise. Observations are finite-dimensional and the case of pointwise observation at some points in the domain, that my depend on the parameter, is also considered

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1. Review of the theory

The aim of this chapter is to introduce basic elements of the linear filtering theory and recall some definitions and known results for some particular types of Gaussian processes which then play the role of noise in stochastic evolution equations throughout the Thesis.

The chapter is divided in two sections. In Section 1.1 elements of linear filtering theory are introduced, an example is given and some known results are recalled.

In Section 1.2 the particular types of Gaussian processes are discussed. Basic definitions and properties are stated. Stochastic integral with respect to those processes is introduced and also the theory of stochastic evolution equations driven by them is recalled.

1.1 Continuous time linear filtering

First, let us recall the pioneering result in continuous time linear filtering theory which is by Kalman and Bucy [28].

Consider a stochastic basis $(\Omega, F, P, (F_t))$ and a real-valued random process $(\theta_t, t \geq 0)$ with evolution given by linear stochastic differential equation driven by an (F_t)-Wiener process $(\mathcal{W}_t, t \geq 0)$,

$$\mathrm{d}\theta_t = H(t)\theta_t \,\mathrm{d}t + \sigma(t)\,\mathrm{d}\mathcal{W}_t, \qquad \theta_0 = x, \quad x \in \mathbb{R}, \tag{1.1.1}$$

where H and σ are real-valued continuous functions and $\sigma(t) > 0$ for all $t \geq 0$. Further, suppose that the process θ can not be observed, but we can observe another real-valued random process $(\xi_t, t \geq 0)$ whose dynamics is given by process θ and some additional noise. More specifically,

$$\mathrm{d}\xi_t = A(t)\theta_t \,\mathrm{d}t + \sigma_1(t)\,\mathrm{d}W_t, \qquad \xi_0 = y, \quad y \in \mathbb{R}, \tag{1.1.2}$$

where A and σ_1 are real-valued continuous function, $\sigma_1(t) > 0$ for all $t \geq 0$ and $(W_t, t \geq 0)$ is a Wiener process independent to the Wiener process $\mathcal{W}$.

The unobservable process θ is called signal and the process ξ is called observation. We are interested in the optimal estimate of the signal based on the observation process which is defined as

$$\widehat{\theta}_t = \mathbb{E}[\theta_t | F_t^\xi],$$

where $(F_t^\xi)_{t\in[0,T]}$ is the filtration generated by the observation process ξ. The optimal estimate $\left(\widehat{\theta}_t, t \geq 0\right)$ is called filter and can be found using the following theorem.

Theorem 1.1.1. *The filter $\widehat{\theta}_t$ satisfies the stochastic differential equation*

$$\mathrm{d}\widehat{\theta}_t = H(t)\widehat{\theta}_t \,\mathrm{d}t + F(t)[\,\mathrm{d}\xi_t - A(t)\widehat{\theta}_t \,\mathrm{d}t]$$

with initial data $\widehat{\theta}_s = \mathbb{E}[\theta_s]$, $s \leq t$, *where* $F(t) = \Phi(t)A(t)/\sigma_1^2(t)$ *and* $\Phi(t)$ *is the variance of the error of the estimate* $\widehat{\theta}_t$ *and satisfies the differential equation*

$$\dot{\Phi}(t) = 2H(t)\Phi(t) - A^2(t)\Phi^2(t)/\sigma_1^2(t) + \sigma^2(t)$$

with initial condition $\Phi(s) = var(\theta_s)$.

Proof. See Kalman and Bucy [28]. □

For example, let us assume that we observe price of a stock on a market whose evolution is drawn on Figure 1.1 and we would like to estimate its slope which is hidden to us.

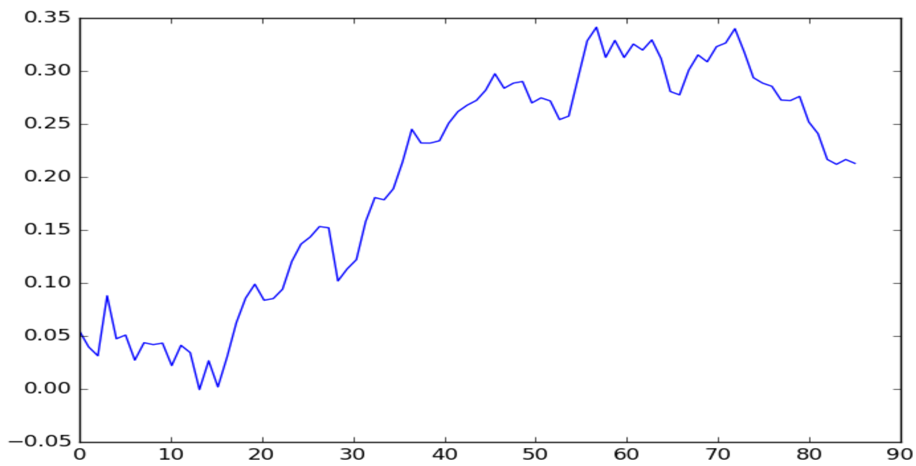

Figure 1.1: Stock price evolution in time. Source: Kaggle - Two Sigma Financial Modelling Challenge, stock id 816.

Further let us assume that the price of the stock $(\xi_t, t \geq 0)$ evolves according to the following model.

$$\begin{aligned} \mathrm{d}\theta_t &= -h\theta_t\,\mathrm{d}t + \sigma\,\mathrm{d}\mathcal{W}_t, & \theta_0 &= x, \\ \mathrm{d}\xi_t &= a\theta_t\,\mathrm{d}t + \sigma_1\,\mathrm{d}W_t, & \xi_0 &= y, \end{aligned} \tag{1.1.3}$$

where W and $\mathcal{W}$ are two independent Wiener processes and h, σ, a, σ_1 are positive. Now, we can apply Theorem 1.1.1 to obtain an estimate of the slope $(\theta_t, t \geq 0)$ which is the signal here whereas the stock price ξ is the observation process. The estimate (the filter) is (for the specific choice of parameters a, σ_1, h and σ found by a grid-search method) drawn on Figure 1.2.

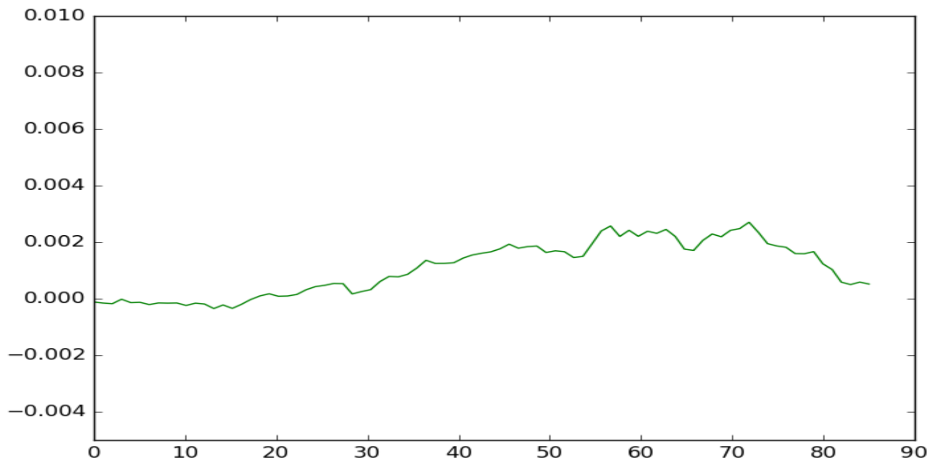

Figure 1.2: Slope of the stock price in Figure 1.1 modelled by Kalman-Bucy filter. Parameters of the model are: $a = 0.01$, $\sigma_1 = 5$, $h = 2$ and $\sigma = 20$.

Using stochastic calculus it is possible to find a unique solution to the stochastic differential equation (1.1.3) and, therefore, we can use the current observed stock price and the estimate of the current stock price slope to predict future values of the stock price.

A more general type of signal given by a finite-dimensional general Gaussian process is studied by [30]. More specifically, the signal $(\theta_t, t \geq 0)$ is considered to be an arbitrary R^d-valued mean-square continuous Gaussian process with the covariance function

$$K^\theta(t,s) = \mathbb{E}[(\theta_t - \mathbb{E}[\theta_t])(\theta_s - \mathbb{E}[\theta_s])^*]$$

and the observation process $(\xi_t, t \geq 0)$ is an R^d-valued process

$$\xi_t = \int_0^t A(t)\theta_s \, \mathrm{d}s + \int_0^t Q(t)\, \mathrm{d}W_t, \qquad \xi_0 = y, \quad y \in \mathbb{R}^d, \tag{1.1.4}$$

where $(A(t), t \geq 0)$ is a continuous $\mathbb{R}^{d\times d}$-valued function, $(Q(t), t \geq 0)$ is a continuous function with values in non-singular non-negative symmetric $d \times d$ matrices and $(W_t, t \geq 0)$ is a standard d-dimensional Wiener process independent of the signal θ. The filter $\left(\widehat{\theta}_t, t \geq 0\right)$ is then specified by Theorem 1.1.2.

Theorem 1.1.2. *For all $t \in [0,T]$ the filter is given as a solution to the stochastic integral equation*

$$\widehat{\theta}_t = E[\theta_t] + \int_0^t \Phi(t,s)A^T(s)Q^{-1}(s)\,\mathrm{d}\xi_s - \int_0^t \Phi(t,s)A^T(s)Q^{-1}(s)A(s)\widehat{\theta}_s\,\mathrm{d}s, \tag{1.1.5}$$

where

$$\Phi(t,s) = K^\theta(t,s) - \int_0^s \Phi(t,r)A^T(r)Q^{-1}(s)A(r)\Phi^T(s,r)\,\mathrm{d}r.r. \tag{1.1.6}$$

Moreover, for all $t \in [0,T]$, $\Phi(t,t)$ is the covariance of the estimation error at time $t \in [0,T]$, that is,

$$\Phi(t,t) = \mathbb{E}\left[(\theta_t - \widehat{\theta}_t)(\theta_t - \widehat{\theta}_t)^T\right] \tag{1.1.7}$$

holds.

Proof. See Kleptsyna and Le Breton [30]. □

A pioneering result on Kalman-Bucy filter in infinite-dimensional spaces belongs to Falb [22]. He deals with the case when both signal and observation proces live in a Hilbert space and each of them is governed by a linear evolution equation driven by a different Q-Wiener process. The two Q-Wiener processes are independent. Therefore, it is an infinite-dimensional analogy of the original Kalman-Bucy theorem (Theorem 1.1.1). For more detailed information see Corollary 7.9, Theorem 7.10 and Theorem 7.14 in [22].

To our best knowledge, there are no other works, except the work contained in this Thesis, in infinite dimensions for general Gaussian processes that would cover, for example, linear SPDEs driven by fractional noise.

1.2 Continuous time Gaussian processes

In the next chapters, we deal with signals given by infinite-dimensional continuous time Gaussian processes which are not necessarily Markovian. One such interesting case is, for example, stochastic delayed evolution equation driven by Wiener process which is discussed in Section 2.5. Another interesting examples arise from stochastic evolution equations driven by Gaussian fractional noises. Since the Gaussian fractional noises play the main role in most examples throughout the Thesis we now recall some basic definitions and facts from the theory of such processes and stochastic evolution equations driven by them. In Section 1.2.1 we study the α-regular Gauss-Volterra process. In Section 1.2.2 we deal with fractional Brownian motion (fBm) where we consider both the regular and the singular case. The regular case is a special case of the class of α-regular Gauss-Volterra processes. Finally, in Section 1.2.3 we slightly extend results from previous sections in the particular case of stochastic evolution equation driven by Wiener process.

Throughout this chapter we assume that $(U, \langle \cdot, \cdot \rangle_U)$ and $(V, \langle \cdot, \cdot \rangle_V)$ are real separable Hilbert spaces and we work on a stochastic basis $(\Omega, F, P, (F_t))$.

1.2.1 Gauss-Volterra processes

This Section mostly follows the works [12], [13], [11] and [14], see also [6].

First, we consider the scalar Volterra processes. We make use of the set

$$\Lambda = \{(t, s) \in [0, T]^2; 0 \le s \le t \le T\}$$

for a given time upper bound $T < \infty$.

Definition 1.2.1. *Let $\alpha \in \left(0, \frac{1}{2}\right)$ be given. By scalar Gauss-Volterra process (called also α-regular Volterra process) we understand a centred Gaussian process $\{b_t, t \in [0, T]\}$, $b_0 = 0$, the covariance of which takes the form*

$$\mathsf{E}[b_t b_s] = R(s, t) = \int_0^{s \wedge t} K(s, r) K(t, r) \, \mathrm{d}r, \quad s, t \ge 0,$$

where the kernel $(K(t,r),(t,r)\in\Lambda)$ *satisfies*

$$\begin{aligned} &(i) \quad K(0,0)=0,\\ &(ii) \quad K(\cdot,r)\in C^1([r,T]), \quad 0\le r\le T,\\ &(iii) \quad \left|\frac{\mathrm{d}K}{\mathrm{d}u}(u,r)\right| < C(u-r)^{\alpha-1}\left(\frac{u}{r}\right)^{\alpha}, (u,r)\in\Lambda. \end{aligned}$$

Remark 1.2.2. Note that in virtue of the Kolmogorov continuity criterion the process $(b_t, t\in[0,T])$ has an ϵ - Hölder version for $\epsilon<\alpha+\frac{1}{2}$ (cf. [13], Remark 2.1).

Remark 1.2.3. Considering the Gaussianity a scalar Gauss-Volterra process $\{b_t, t\in[0,T]\}$ can be represented as

$$b_t \overset{\text{law}}{=} \int_0^t K(t,r)\,\mathrm{d}W_r, \qquad t\in[0,T], \tag{1.2.1}$$

where $\{W_t, t\in[0,T]\}$ is the standard Wiener process.

Now, we recall the standard construction of stochastic integral with respect to a scalar Gauss-Volterra process. It is based on an Itô - type isometry for Volterra processes.

Consider a linear space $\mathcal{E}_T$ of the V-valued deterministic step functions on interval $[0,T]$, i.e.

$$\mathcal{E}_T = \Bigg\{\varphi:[0,T]\to V, \varphi=\sum_{j=1}^{n-1}\varphi_j\mathbf{1}_{[t_j,t_{j+1})}+\varphi_n\mathbf{1}_{[t_n,t_{n+1}]}, \tag{1.2.2}$$

$$\varphi_j\in V, j\in 1,\dots,n, 0=t_1<t_2<\dots<t_{n+1}=T, n\in\mathbb{N}\Big\}. \tag{1.2.3}$$

For a scalar Gauss-Volterra process b define the linear operator $i:\mathcal{E}_T\to L^2(\Omega,V)$ as

$$i(\varphi)=i\left(\sum_{j=1}^{n-1}\varphi_j\mathbf{1}_{[t_j,t_{j+1})}+\varphi_n\mathbf{1}_{[t_n,t_{n+1}]}\right)=\sum_{j=1}^{n}\varphi_j\Big(b_{t_{j+1}}-b_{t_j}\Big).$$

The definition of the scalar Gauss-Volterra process allows us to define operator $\mathcal{K}^*:\mathcal{E}_T\to L^2([0,T],V)$ as

$$(\mathcal{K}^*\varphi)(r)=\int_r^T\varphi(u)\frac{\partial K}{\partial u}(u,r)\,\mathrm{d}u, \quad \varphi\in\mathcal{E}_T, \quad r\in[0,T].$$

Set

$$\langle\varphi,\zeta\rangle_{\mathcal{D}_T} := \langle\mathcal{K}^*\varphi,\mathcal{K}^*\zeta\rangle_{L^2([0,T],V)} \tag{1.2.4}$$

for $\varphi,\zeta\in\mathcal{E}_T$. For simplicity we assume that the operator $\mathcal{K}^*$ is injective and, therefore, (1.2.4) defines an inner product on $\mathcal{E}_T$. Otherwise, we switch to the quotient space $\tilde{\mathcal{E}_T}=\mathcal{E}_T/Ker\mathcal{K}^*$ and we lift the operator $\mathcal{K}^*$ to this quotient space.

Set $K(t,r)=0$ whenever $r>t$ then we can compute

$$\begin{aligned}
||i(\varphi)||^2_{L^2(\Omega,V)} &= \mathsf{E}\left\|\sum_{j=1}^n \varphi_j\Big(b_{t_{j+1}}-b_{t_j}\Big)\right\|^2_V \\
&= \sum_{j=1}^n\sum_{k=1}^n \langle\varphi_j,\varphi_k\rangle_V\, \mathsf{E}\,\Big(b_{t_{j+1}}-b_{t_j}\Big)\Big(b_{t_{k+1}}-b_{t_k}\Big) \\
&= \sum_{j=1}^n\sum_{k=1}^n \langle\varphi_j,\varphi_k\rangle_V R(t_{j+1},t_{k+1})-R(t_{j+1},t_k)-R(t_j,t_{k+1})+R(t_j,t_k) \\
&= \sum_{j=1}^n\sum_{k=1}^n \langle\varphi_j,\varphi_k\rangle_V \int_0^T (K(t_{j+1},r)-K(t_j,r))(K(t_{k+1},r)-K(t_k,r))\,\mathrm{d}r \\
&= \int_0^T \left\|\sum_{j=1}^n \varphi_j(K(t_{j+1},r)-K(t_j,r))\right\|^2_V \mathrm{d}r \\
&= ||\mathcal{K}^*\varphi||^2_{L^2([0,T],V)}\,. \qquad (1.2.5)
\end{aligned}$$

From (1.2.4) and (1.2.5) we have

$$||i(\varphi)||_{L^2(\Omega,V)} = ||\mathcal{K}^*\varphi||_{L^2([0,T],V)} = ||\varphi||_{\mathcal{D}_T}\,, \qquad (1.2.6)$$

where $||\cdot||_{\mathcal{D}_T}=\sqrt{\langle\cdot,\cdot\rangle_{\mathcal{D}_T}}$. Let $\mathcal{D}_T$ stand for the completion of $\mathcal{E}_T$ with respect to the norm $||\cdot||_{\mathcal{D}_T}$ and let the corresponding induced extension of the continuous linear operator $\mathcal{K}^*$ still be denoted by $\mathcal{K}^*$. Then the linear operator i can be extended to a linear isometry between $\mathcal{D}_T$ and a closed subspace of $L^2(\Omega,V)$. This linear isometry is again denoted by i.

The stochastic integral of an V-valued deterministic function $\varphi\in\mathcal{D}_T$ with respect to the scalar Gauss-Volterra process b is the random variable $i(\varphi)$.

Remark 1.2.4. Note that this construction of the stochastic integral holds in general for scalar Volterra processes. The Gaussianity is not needed. Also the conditions (*ii*) and (*iii*) in Definition 1.2.1 can be relaxed (cf. [12]).

Remark 1.2.5. Using the Gaussianity, as a generalization of the representation (1.2.1) we have

$$i(\varphi)\overset{\text{law}}{=}\int_0^T (K^*\varphi)(r)\,\mathrm{d}W_r,$$

where $\{W_t, t\in[0,T]\}$ is the standard Wiener process.

The following embedding result for the space of integrable functions $\mathcal{D}_T$ can be useful.

Proposition 1.2.6. *There exists a finite constant $c>0$ such that*

$$||\varphi||^2_{\mathcal{D}_T} \le c\Big(||\varphi||^2_{L^2\left([0,T],|K(s+,s)|^2\,\mathrm{d}s,H\right)} + ||\varphi||^2_{L^{\frac{2}{1+2\alpha}}([0,T],H)}\Big),$$

where

$$K(s+,s)=\lim_{t\to s+}K(t,s).$$

Proof. See Proposition 2.9 in [12]. □

As a consequence of the Proposition 1.2.6 we have the following corollary.

Corollary 1.2.7. *The space* $L^{\frac{2}{1+2\alpha}}([0,T],H)$ *is continuously embedded into the space of integrable functions* $\mathcal{D}_T$.

The usual notation of the integral $i(\varphi)$, $\varphi \in \mathcal{D}_{a,b}$, where $\mathcal{D}_{a,b}$, $0 \le a \le b < \infty$, is the space of all integrable functions on an interval $[a,b]$, is

$$i(\varphi) = \int_a^b \varphi(r)\, \mathrm{d}b_r.$$

Now, let us show some examples of scalar Gauss-Volterra processes.

Example 1.2.8. An important example of α - regular Gauss - Volterra process is the fractional Brownian motion (fBm) with the Hurst parameter $h > \dfrac{1}{2}$. It is a centred Gaussian process $\{b_t^h, t \ge 0\}$, $b_0^h = 0$, with the covariance function

$$\mathsf{E}\,[b_t^h b_s^h] = R^h(t,s) = \frac{1}{2}\Big(s^{2h} + t^{2h} - |t-s|^{2h}\Big), \qquad t, s \ge 0. \tag{1.2.7}$$

In this case $\alpha = h - \dfrac{1}{2}$ and

$$K(t,s) = K^h(t,s) = C_h \int_s^t \left(\frac{u}{s}\right)^{h-\frac{1}{2}} (u-s)^{h-\frac{3}{2}}\, \mathrm{d}u \mathbf{1}_{(0,t]}(s), \tag{1.2.8}$$

where C_h is an appropriate constant. The fBm is studied in detail in Section 1.2.2 where the singular case, i.e. $h < \dfrac{1}{2}$, is also discussed.

If $h = \frac{1}{2}$ the kernel takes the form $K(t,s) = \mathbf{1}_{(0,t]}(s)$ and the fBm is strictly regular for any $\alpha \in \Big(0, \frac{1}{2}\Big)$. This case corresponds to the standard Wiener process and it follows from Corollary 1.2.7 that $L^2([0,T],H)$ is continuously embedded into $\mathcal{D}_T$.

The singular case, $0 < h < \dfrac{1}{2}$, is not an α - regular Gauss-Volterra process, but it is still a process of Volterra type, see [18].

Example 1.2.9. Another example is the Liouville fractional Brownian motion (LfBm), which is a centred continuous Gaussian process $\Big\{b_t^{lh}, t \in [0,T]\Big\}$, $b_0^{lh} = 0$, with the kernel

$$K(t,s) = K^h(t,s) = \frac{1}{\Gamma\Big(h + \frac{1}{2}\Big)} (t-s)^{h-\frac{1}{2}} \mathbf{1}_{(0,t]}(s), \tag{1.2.9}$$

where Γ is the Gamma function and $h \in \Big(\frac{1}{2}, 1\Big)$. See, for example, [8] for a more detailed treatment.

Example 1.2.10. Another example arises when parameter $h = h(t)$ is a function of time. In the case of fBm we obtain multifractional Brownian motion which is studied, for example, in [5] and [25].

For simplicity, we will show such modification for the kernel of LfBm. Consider a kernel of the form

$$K(t,s) = (t-s)^{h(t)-\frac{1}{2}} \mathbf{1}_{(0,t]}(s), \quad t, s \in [0,T], \tag{1.2.10}$$

where $h : [0,1] \to \mathbb{R}$ is a function satisfying

1. $h \in C^1(0,1)$
2. there exists an $\epsilon > 0$ such that $h(t) \in \left[\frac{1}{2} + \epsilon, 1\right)$ for all $t \in [0,T]$
3. there exists a constant C_ϵ such that

$$|h'(t)| \leq C_\epsilon \min_{u \in (0,t)} \left[\left(\frac{t}{u}\right)^\epsilon \frac{1}{|log(t-u)|\,(t-u)} \right]$$

for all $t \in [0,T]$.

A centred continuous Gaussian process with the kernel given by (1.2.10) satisfies conditions in Definition 1.2.1 and, therefore, it is a scalar Gauss-Volterra process (for the verification of the appropriate conditions see [12], Example 2.14).

Now, we define infinte-dimensional Gauss-Volterra processes.

Definition 1.2.11. *A cylindrical Gauss-Volterra process $\{B_t, t \in [0,T]\}$ on U is defined by the formal series*

$$B_t = \sum_{n=1}^{\infty} b_n(t) e_n \tag{1.2.11}$$

where $\{e_n, n \in \mathbb{N}\}$ is an orthonormal basis in U and $\{b_n(t), t \in [0,T]\}_{n \in \mathbb{N}}$ is a sequence of independent scalar α-regular Gauss-Volterra processes with the same kernel K.

Remark 1.2.12. The series (1.2.11) does not converge in the space U but defines the system of scalar processes $\{B(x), x \in U\}$,

$$B_t(x) = \sum_{n=1}^{\infty} \langle e_n, x \rangle \beta_n(t).$$

Nevertheless, if the Hilbert space U is embedded into another Hilbert space U' and the embedding is Hilbert-Schmidt then the series converges in U'.

Let us construct a stochastic integral with respect to a cylindrical Gauss-Volterra process B defined by (1.2.11). Set

$$I(F) := \sum_{n=1}^{\infty} \int_0^T F e_n \, \mathrm{d}b^{(n)}, \tag{1.2.12}$$

where $F : [0,T] \to \mathcal{L}(U,V)$ is a deterministic operator-valued function such that

$$I(F) \text{ converges in } L^2(\Omega, V) \text{ and } Fe_n \in \mathcal{D}_T \text{ for all } n \in N. \tag{1.2.13}$$

Consider an operator $F : [0,T] \to \mathcal{L}(U,V)$ and let us find a condition when it is integrable. Define the operator $P : U \to L^2([0,T], V)$ as

$$P(u) = K^*(F(\cdot)u), \quad u \in U$$

and assume that P is Hilbert-Schmidt. Denoting partial sums

$$S_N = \sum_{n=1}^{N} \int_0^T F(r) e_n \, \mathrm{d}b_r^{(n)}$$

we have

$$
\begin{aligned}
\mathsf{E}\, ||S_N - S_M||_V^2 &= \mathsf{E} \left\| \sum_{n=M+1}^{N} \int_0^T F(r) e_n \, \mathrm{d}b_r^{(n)} \right\|_V^2 \\
&= \sum_{n=M+1}^{N} \int_0^T ||(K^* F(\cdot) e_n)(r)||_V^2 \, \mathrm{d}r \\
&= \sum_{n=M+1}^{N} ||P e_n||^2_{L^2([0,T],V)}
\end{aligned}
$$

for all $N, M \in \mathbb{N}$, $N > M$. We used the isometry (1.2.6) and the independence of scalar Gauss-Volterra processes $\left\{ b^{(n)}, n \in N \right\}$. Using the Hilbert-Schmidt property of the operator P the last sum tends to zero as $M, N \to \infty$ and, therefore, the series $(S_N, N \in \mathbb{N})$ is Cauchy in $L^2(\Omega, V)$. The limit of this sequence is the stochastic integral of the operator F with respect to the cylindrical Gauss-Volterra process B and will be denoted as $I(F)$. The operator P is determined by the operator F and, therefore, we say that the operator F is integrable if and only if the corresponding operator P is Hilbert-Schmidt.

Note that the stochasic integral does not depend on the choice of the basis. Also notice that the Gaussianity is not needed for the construction of the integral. As in the case of the scalar Gauss-Volterra processes we will use notation

$$
I(F) = \int_a^b F(s) \, \mathrm{d}B_s
$$

for the integral on an interval $[a, b]$.

We have a similar embedding result for the space of integrable operators as in the scalar case. It is shown in the following proposition together with some important properties of the integral.

Proposition 1.2.13. *If*

$$
F \in L^2\Big([0,T], |K(s+,s)|^2 \, \mathrm{d}s, \mathcal{L}_2(H,V)\Big) \cap L^{\frac{2}{1+2\alpha}}([0,T], \mathcal{L}_2(H,V))
$$

then the stochastic integral $\int F \, \mathrm{d}B$ exists.

Furthermore, the process $\left(\int_0^t F(s) \, \mathrm{d}B_s, t \in [0,T] \right)$ is mean-square continuous and admits a version with measurable sample path.

Proof. See Proposition 3.3 in [12]. □

Throughout the Thesis we usually work in infinite-dimensional spaces. Therefore, we speak about Gauss-Volterra process instead of cylindrical Gauss-Volterra process if there is no danger of confusion.

Now, let us discuss stochastic evolution equations driven by Gauss-Volterra processes.

Consider a linear stochastic evolution equation of the form

$$
\mathrm{d}\theta_t = \mathcal{A}\theta_s \, \mathrm{d}s + G \, \mathrm{d}B_t, \quad t \in [0,T], \quad \theta_0 = x, \tag{1.2.14}
$$

where $\mathcal{A} : Dom(\mathcal{A}) \to V,\ Dom(\mathcal{A}) \subset V$ is an infinitesimal generator of a strongly continuous semigroup $(S(t))_{t \geq 0}$ in V, $G \in \mathcal{L}(U,V)$, $\{B_t, t \in [0,T]\}$ is a cylindrical Gauss-Volterra process on U and $x \in V$.

Definition 1.2.14. *A V-valued process $\{\theta_t, t \in [0,T]\}$ defined as*

$$\theta_t = S(t)x + \int_0^t S(t-r)G \, \mathrm{d}B_r, \quad t \in [0,T]$$

is said to be a mild solution of the stochastic evolution equation (1.2.14).

Remark 1.2.15. Note that the process $\{\theta_t, t \in [0,T]\}$ is a mild solution of the stochastic evolution equation (1.2.14) if and only if the process

$$\tilde{\theta} = \{\theta_t - S(t)x, t \in [0,T]\}$$

is a mild solution of the same stochastic evolution equation but with initial value $0 \in V$. Therefore, for simplicity we assume that $x = 0$ for the rest of the section.

Denote $f(r) := |K(r+,r)|^2$, $r \in [0,T)$ and $f(r) := 0$ otherwise. The following theorem states the conditions for the existence of a mild solution.

Proposition 1.2.16. *If $S(t)G \in \mathcal{L}_2(U,V)$ for all $t \in (0,T)$ and there exists $T_0 \in (0,T]$ such that we have*

$$\int_0^{T_0} f(t-r) \, ||S(r)G||^2_{\mathcal{L}_2(U,V)} \, \mathrm{d}r + \int_0^{T_0} ||S(r)G||^{\frac{2}{1+2\alpha}}_{\mathcal{L}_2(U,V)} \, \mathrm{d}r < \infty$$

holds for all $t \in [0,T]$ then θ_t is a well defined V-valued random variable for all $t \in [0,T]$.

Proof. See Proposition 4.1 in [12]. □

Next Proposition states the conditions under which the mild solution θ is mean-square right continuous and admits a version with measurable sample path.

Proposition 1.2.17. *Assume that $S(t)G \in \mathcal{L}_2(U,V)$ for all $t \in (0,T]$.*

(1) Suppose that function $f(r) := |K(r+,r)|^2$ is bounded on $[0,T)$. If there exists a $T_0 \in (0,T]$ such that

$$S(\cdot)G \in L^2([0,T_0], \mathcal{L}_2(U,V)),$$

then θ is mean-square right continuous and admits a version with measurable sample path.

(2) Suppose that function f is identical to zero on $[0,T)$. If there exists a $T_0 \in (0,T]$ such that

$$S(\cdot)G \in L^{\frac{2}{1+2\alpha}}([0,T_0], \mathcal{L}_2(U,V)),$$

Then θ is mean-square right continuous and admits a version with measurable sample path.

Proof. See Corollary 4.3 in [12]. □

Note that in Proposition 1.2.17 the standard integration theory for cylindrical processes is covered by (1) in the case of Wiener process and by (2) for fBm with $h > \frac{1}{2}$.

If the semigroup S is analytical then there exists $\lambda \in \mathbb{R}$ such that the operator $(\lambda I - \mathcal{A})$ is strictly positive. Therefore, for $\delta \geq 0$, we can define the Hilbert space

$$V_\delta = Dom((\lambda I - \mathcal{A})^\delta)$$

equipped with the graph norm topology (the spaces V_δ are independent of the choice of regular value λ, so λ is supposed to be fixed here). For the theory of analytical semigroups see, for example, [45].

In the next chapters we will use following Theorem which is a particular case of Corollary 4.1. in [13].

Theorem 1.2.18. *Assume $S(u)G \in \mathcal{L}_2(U, V)$, $u \in [0, T]$, and let there exist a $\gamma \in [0, \alpha + 1/2)$ such that*

$$||S(u)G||_{\mathcal{L}_2(U,V)} \leq cu^{-\gamma}, \quad u \in [0, T] \tag{1.2.15}$$

for a constant $c > 0$. Then $\{\theta_t, t \in [0, T]\}$ has a continuous version in the space V_δ for

$$0 \leq \delta < \alpha + \frac{1}{2} - \gamma. \tag{1.2.16}$$

If $G \in \mathcal{L}_2(U, V)$ (which corresponds to the case when the driving noise in (2.4.3) may be represented by a genuine V-valued Gauss-Volterra process), the condition (1.2.15) is satisfied with $\gamma = 0$ and (1.2.16) reads $0 \leq \delta < \alpha + 1/2$.

Proof. See Corollary 4.1. in [13]. □

1.2.2 Fractional Brownian motion

Fractional Brownian motion (fBm) is a natural generalization of the well known Wiener process. It is a centred Gaussian process with continuous trajectories whose increments, in contrary to Wiener process, need not be independent. It was first introduced by Kolmogorov [32] and further studied, for example, by Mandelbrot [40] and Hurst [26]. In his work dealing with long-term capacity of reservoirs along the Nile river, Hurst demonstrated the usefulness of this process. Since then fBm was used in many different areas, for example, telecommunications and financial mathematics.

This section mostly follow the works [19], [44] and [33], see also [48].

Definition 1.2.19. *By scalar fractional Brownian motion with parameter $h \in (0, 1)$ we understand a continuous centred Gaussian process $\{b_t^h, t \geq 0\}$, $b_0^h = 0$, the covariance of which takes the form*

$$E[b_t^h b_s^h] = R^h(t, s) = \frac{1}{2}\Big(s^{2h} + t^{2h} - |t - s|^{2h}\Big), \qquad t, s \geq 0. \tag{1.2.17}$$

It is easy to see that the process has stationary increments and for $h = \frac{1}{2}$ we obtain Wiener process.

As it was pointed out in previous Section in Example 1.2.8, if $h \geq \frac{1}{2}$ it is a scalar Gauss-Volterra process. The case $h > \frac{1}{2}$ is usually called regular and the case $h < \frac{1}{2}$ is usually called singular.

Note that the function (1.2.17) is homogeneous of order $2h$ which yields so-called self-similarity property of fBm, i.e. processes

$$\Big\{\alpha^h b_t^h, t \in [0,T]\Big\}, \qquad \Big\{b_{\alpha t}^h, t \in [0,T]\Big\}$$

have the same distribution for any $\alpha > 0$.

The sample path of fBm are almost nowhere differentiable but by Kolmogorov-Chentsov theorem almost all trajectories are locally Hölder continuous of order strictly less then h. Also note that fBm is semimartingale only if $h = \frac{1}{2}$ (cf. [7]).

We can see that for any $t \geq 0$ and $n \in \mathbb{N}$ the covariance $\rho_h(n)$ of two increments $b_t^h - b_{t-1}^h$ and $b_{t+n}^h - b_{t+n-1}^h$ is

$$\rho_h(n) = \frac{1}{2}\Big((n+1)^{2h} + (n-1)^{2h} - 2n^{2h}\Big), \tag{1.2.18}$$

which is zero only for $h = \frac{1}{2}$. It further follows from (1.2.18) that in the regular case, i.e. $h > \frac{1}{2}$, the increments are positively correlated and

$$\sum_{n=1}^{\infty} \rho_h(n) = \infty. \tag{1.2.19}$$

On the other hand in the singular case, i.e. $h < \frac{1}{2}$, the increments are negatively correlated and

$$\sum_{n=1}^{\infty} \rho_h(n) < \infty.$$

The property (1.2.19) is usually called long-range dependence.

The fBm can be represented as an integral of a suitable kernel with respect to the Wiener process. More specifically we have

$$b_t^h = \int_0^t K_h(t,s)\,\mathrm{d}W_s, \tag{1.2.20}$$

where $(W_t, t \geq 0)$ is a standard Wiener process and

$$K_h(t,s) = C_h(t-s)^{h-\frac{1}{2}} + C_h\Big(\frac{1}{2} - h\Big)\int_s^t (u-s)^{h-\frac{3}{2}}\bigg(1 - \Big(\frac{s}{u}\Big)^{\frac{1}{2}-h}\bigg)\,\mathrm{d}u \tag{1.2.21}$$

for $0 \leq s \leq t$ and

$$C_h = \left[\frac{2h\Gamma\big(h+\frac{1}{2}\big)\Gamma\big(\frac{3}{2}-h\big)}{\Gamma(2-2h)}\right]^{\frac{1}{2}}.$$

If $h > \frac{1}{2}$ then the kernel K can be simplified to (1.2.8) and we have an (α-regular) Gauss-Volterra process. For more information on the integral representation, see [2] and [1].

Now, we will discuss the construction of stochastic integral with respect to the fBm. The construction holds for both the regular and the singular case. We are mainly interested in the singular case, i.e. $h < \frac{1}{2}$, as the cases for $h \geq \frac{1}{2}$ are covered by the theory of scalar Gauss-Volterra processes in Section 1.2.1.

The construction follows a similar pattern as the case of scalar Gauss-Volterra processes.

Consider the linear space $\mathcal{E}_T$ of the V-valued deterministic step functions on interval $[0,T]$ defined in (1.2.2) and define the operator $\mathcal{K}^* : \mathcal{E}_T \to L^2([0,T],V)$ as

$$(\mathcal{K}^*\varphi)(r) = \varphi(r)K_h(T,r) + \int_r^T (\varphi(u) - \varphi(r))\frac{\partial K}{\partial u}(u,r)\,\mathrm{d}u, \quad \varphi \in \mathcal{E}_T, \quad r \in [0,T].$$

For a scalar fBm $\left\{b_t^h, t \geq 0\right\}$ with the Hurst parameter $h \in (0,1)$ denote a linear operator $i : \mathcal{E}_T \to L^2(\Omega, H)$ as

$$i(\varphi) = i\left(\sum_{j=1}^{n-1} \varphi_j \mathbf{1}_{[t_j,t_{j+1})} + \varphi_n \mathbf{1}_{[t_n,t_{n+1}]}\right) = \sum_{j=1}^{n} \varphi_j \left(b_{t_{j+1}}^h - b_{t_j}^h\right).$$

In the same way as in the case of Gauss-Volterra processes we can define a scalar product on $\mathcal{E}_T$ by

$$\langle \varphi, \zeta \rangle_{\mathcal{D}_T} := \langle \mathcal{K}^*\varphi, \mathcal{K}^*\zeta \rangle_{L^2([0,T],V)} \tag{1.2.22}$$

for $\varphi, \zeta \in \mathcal{E}_T$ and it is easy to show that

$$||i(\varphi)||_{L^2(\Omega,V)}^2 = ||\mathcal{K}^*\varphi||_{L^2([0,T],V)}^2 . \tag{1.2.23}$$

Therefore, using (1.2.22) and (1.2.23) we have

$$||i(\varphi)||_{L^2(\Omega,V)} = ||\mathcal{K}^*\varphi||_{L^2([0,T],V)} = ||\varphi||_{\mathcal{D}_T} , \tag{1.2.24}$$

where $||\cdot||_{\mathcal{D}_T} = \sqrt{\langle \cdot, \cdot \rangle_{\mathcal{D}_T}}$. As in the case of Gauss-Volterra processes using (1.2.24) let $\mathcal{D}_T$ stand for the completion of $\mathcal{E}_T$ with respect to the norm $||\cdot||_{\mathcal{D}_T}$ and extend the linear operator i into a linear isometry between $\mathcal{D}_T$ and a closed subspace of $L^2(\Omega, V)$.

The stochastic integral of a V-valued deterministic function $\varphi \in \mathcal{D}_T$ with respect to the scalar fBm b^h is the random variable $i(\varphi)$ which will be usually denoted as

$$i(\varphi) = \int_0^T \varphi(r)\,\mathrm{d}b_r^h.$$

In the regular case, i.e. $h > \frac{1}{2}$, we have the continuous embedding

$$L^p([0,T],V) \hookrightarrow \mathcal{D}_T \tag{1.2.25}$$

for any $p > \frac{1}{h}$. It follows directly from the Collorary 1.2.7 (in fact, (1.2.25) holds also with $p = \frac{1}{h}$).

In the singular case, i.e. $h < \frac{1}{2}$, the space of integrable functions is smaller (compared to the regular case). We have, for example, the embeddings

$$H^1([0,T],V) \subset \mathcal{D}_T$$

and

$$C^\beta([0,T],V) \subset \mathcal{D}_T, \quad \beta > \frac{1}{2} - h$$

(cf. Lemma 5.20 in [24]).

If $h < \frac{1}{2}$, the (extended) linear operator K^* can be described by the fractional derivative as

$$K^*\varphi(t) = C_h t^{\frac{1}{2}-h} D_{T-}^{\frac{1}{2}-h}\Big(u_{h-\frac{1}{2}}\varphi\Big)$$

with its domain being

$$\mathcal{D}_T = I_{T-}^{\frac{1}{2}-h}\Big(L^2([0,T],V)\Big),$$

where $\Big(u_{h-\frac{1}{2}}\varphi\Big)(s) = s^{h-\frac{1}{2}}\varphi(s)$ (see Proposition 6 in [2]). Some basic definitions from theory of fractional calculus are summarized in Appendix A.

Now, let us define infinte-dimensional fBm. In Chapter 3 we will make use of the two-sided cylindrical fractional Brownian motion.

Definition 1.2.20. *By the two-sided cylindrical fractional Brownian motion with parameter $h \in (0,1)$ on the Hilbert space U we understand process $\{B_t^h, t \in \mathbb{R}\}$ defined by the formal series*

$$B_t^h = \sum_{n=1}^{\infty} b_n^h(t) e_n \tag{1.2.26}$$

where $\{e_n, n \in \mathbb{N}\}$ is an orthonormal basis in U and $\{b_n^h(t), t \in R\}_{n\in\mathbb{N}}$ is a sequence of independent two-sided scalar fBm with the same parameter h. The two-sided scalar fBm is defined by the Definition 1.2.19 only with $t \in \mathbb{R}$.

Remark 1.2.21. Except the real time domain the definition is analogous to the cylindrical Gauss-Volterra processes studied in Section 1.2.1. For example, the series (1.2.26) does not converge in the space U but defines the system of scalar processes $\{B(x), x \in U\}$,

$$B_t^h(x) = \sum_{n=1}^{\infty} \langle e_n, x\rangle b_n^h(t).$$

Most of the time we will work with only one-sided cylindrical fBm, i.e. for $t \geq 0$, and we will usually omit the word cylindrical when it is clear from the context.

Stochastic integral with respect to a cylindrical fBm is, even for $h < \frac{1}{2}$, defined similarly as the stochastic integral with respect to a cylindrical Gauss-Volterra process.

Definition 1.2.22. *A stochastic integral with respect to a cylindrical fBm given by (1.2.26) is defined as*

$$I(F) := \sum_{n=1}^{\infty} \int_0^T F e_n \, \mathrm{d}b_n^h, \tag{1.2.27}$$

where $F : [0,T] \to \mathcal{L}(U,V)$ is a deterministic operator-valued function such that

$$I(F) \text{ converges in } L^2(\Omega, V) \text{ and } Fe_n \in \mathcal{D}_T \text{ for all } n \in N. \tag{1.2.28}$$

We say that an operator $F : [0,T] \to \mathcal{L}(U,V)$ is integrable if it satisfies (1.2.28).

The next theorem shows that the set of integrable operators is also consistent with the case of Gauss-Volterra processes.

Proposition 1.2.23. *Consider $F : [0,T] \to \mathcal{L}(U,V)$ such that $Fx \in \mathcal{D}_T$ for each $x \in U$ and define an operator $P : U \to L^2([0,T],V)$ as*

$$P(u) = K^*(F(\cdot)u), \quad u \in U$$

If P is a Hilbert-Schmidt operator, then the stochastic integral (1.2.27) is a well defined V-valued random variable.

Proof. See Proposition 11.3 in [44]. □

Now, let us discuss stochastic evolution equations driven by fBm. We will see that most of the results, which holds for the regular case and are special cases of results from the previous chapter dealing with Gauss-Volterra processes, hold analogously even for singular fBm. We will allow more general noises, for example a point-wise noise and we will use the two-sided fBm to obtain a strictly stationary solution to a stochastic evolution equation driven by fBm.

Consider a linear stochastic differential equation of the form

$$\mathrm{d}\theta_t = \mathcal{A}\theta_s \,\mathrm{d}s + G \,\mathrm{d}B_t^h, \quad t \in [0,T], \quad \theta_0 = x, \tag{1.2.29}$$

where $\mathcal{A} : Dom(\mathcal{A}) \to V$, $Dom(\mathcal{A}) \subset V$ is an infinitesimal generator of a strongly continuous analytic semigroup $(S(t))_{t\geq 0}$ in V, $\{B_t^h, t \in [0,T]\}$ is a cylindrical fBm with Hurst parameter $h \in (0,1)$ and $x \in L^2(\Omega, V)$. The noise term satisfies

$$G : U \supset Dom(G) \to (Dom(A^*))' \quad \text{and} \quad (\lambda I - \mathcal{A})^{\varepsilon - 1} G \in \mathcal{L}(U,V) \tag{1.2.30}$$

for a given $\varepsilon \in (0,1]$, where $(Dom(A^*))'$ is the dual with respect to the topology of V and $\lambda > 0$ is large enough so that the operator $\lambda I - \mathcal{A}$ is strictly positive.

Remark 1.2.24. If in condition (1.2.30) $\varepsilon = 1$, i.e. $G \in \mathcal{L}(U,V)$, the semigroup $(S(t))_{t\geq 0}$ does not have to be analytic which is the most usual case in standard examples. Nevertheless, for example, in the case of SPDE with pointwise fractional noise ε must be chosen strictly less then one. For more general results, see [41].

A mild solution to the equation (1.2.29) is a V-valued process $\{\theta_t, t \in [0,T]\}$ defined as

$$\theta_t = S(t)x + \int_0^t S(t-r)G \,\mathrm{d}B_r^h, \quad t \geq 0.$$

If $h \geq \frac{1}{2}$ the sufficient conditions for existence of the mild solution are given, for example, by Proposition 1.2.16. The following proposition holds for $h \in (0,1)$ and more general noises given by condition (1.2.30).

Proposition 1.2.25. *Assume that there exists a $\gamma \in [0,h)$ such that*

$$||S(u)G||_{\mathcal{L}_2(U,V)} \leq cu^{-\gamma}, \quad u \in [0,T] \tag{1.2.31}$$

for a constant $c > 0$. Then $\{\theta_t, t \in [0,T]\}$ is a well-defined V-valued process with continuous path.

Proof. See [33], Proposition 2.1. □

The following theorem is an analogy of Theorem 1.2.18 and shows that we can obtain a mild solution in a more regular space.

Theorem 1.2.26. *Assume $G \in \mathcal{L}(U, V)$, $S(u)G \in \mathcal{L}_2(U, V)$, $u \in [0, T]$ and that condition (1.2.31) holds. Then $\{\theta_t, t \in [0, T]\}$ has a continuous version in the space V_δ for*

$$0 \leq \delta < h - \gamma. \tag{1.2.32}$$

If $G \in \mathcal{L}_2(U, V)$ the condition (1.2.31) is satisfied with $\gamma = 0$ and (1.2.32) reads $0 \leq \delta < h$.

Proof. The case when $h \geq \frac{1}{2}$ follows from Theorem 1.2.18. The singular case, i.e. $h < \frac{1}{2}$ follows from Theorem 11.11 and Corollary 11.12 in [44]. □

The two-sided fBm can be utilized to obtain a strictly stationary solution.

Theorem 1.2.27. *Assume that condition (1.2.31) holds and, moreover,*

$$||S(t)||_{\mathcal{L}(V)} \leq Me^{-\rho t},$$

holds for all $t > 0$ and some constants $M > 0$, $\rho > 0$. Then there exists a strictly stationary continuous mild solution of the stochastic evolution equation (1.2.29), i.e. there exists an initial value $\tilde{x} \in L^2(V)$ such that the mild solution

$$\theta_t = S(t)\tilde{x} + \int_0^t S(t-r)G \, \mathrm{d}B_r^h, \quad t \geq 0,$$

is a strictly stationary Gaussian process with continuous path.

Proof. See [33], Proposition 2.2. □

1.2.3 Wiener process

Now, we discuss stochastic evolution equations with respect to the (cylindrical) Wiener process which is a centred Gaussian process with continuous trajectories and independent increments that can be obtained as a special example of (cylindrical) fBm when the Hurst parameter h is set to $\frac{1}{2}$. It is also an example of Gauss-Volterra process with kernel given as $K(t, s) = \mathbf{1}_{(0,t]}(s)$. Therefore, all the definitions and results from the previous sections can be applied to Wiener process.

In this section we only briefly discuss slightly more general type of stochastic evolution equations containing an additional deterministic function in the drift which is not covered in previous sections and we will explicitly state the covariance of the mild solution. These results will be useful in following chapters.

Consider a stochastic evolution equation of the form

$$\begin{aligned} \mathrm{d}\theta_t &= (\mathcal{A}_0\theta_t + f(t)) \, \mathrm{d}t + G \, \mathrm{d}\mathcal{W}_t, \qquad t \in [0, T], \\ \theta_t &= x, \qquad x \in L^2(\Omega, V), \end{aligned} \tag{1.2.33}$$

where $\mathcal{A}_0$ is an infinitesimal generator of a strongly continuous semigroup $(S(t))_{t \in \mathbb{R}_+}$ in V, f is a deterministic V-valued function such that $f \in L^2([0, T], V)$,

$G \in \mathcal{L}(V)$ and $\{\mathcal{W}_t, t \in [0,T]\}$ is an V - valued standard cylindrical Wiener process.

To obtain a mild solution, it is enough to apply condition (ii) from Proposition 1.2.16 which simplifies to

$$S(t)G \in \mathcal{L}_2(H),\ t \in [0,T] \quad \text{and} \quad \int_0^{T_0} \mid S(r)G \mid^2_{\mathcal{L}_2(H)} \,\mathrm{d}r < \infty$$

for some $T_0 > 0$. Then the mild solution to (1.2.33) is a unique mean - square continuous V-valued random process $\{\theta_t, t \in [0,T]\}$ given as

$$\theta_t = S(t)x + \int_0^t S(t-r)f(r)\,\mathrm{d}r + \int_0^t S(t-r)G\,\mathrm{d}\mathcal{W}_r.$$

The mean value of this process is given as

$$\mathsf{E}\left[\theta_t\right] = S(t)x + \int_0^t S(t-r)f(r)\,\mathrm{d}r, \quad t \in [0,T]$$

and the covariance of the process $\rho(t,s) = \mathsf{E}\left[(\theta_t - \mathsf{E}\left[\theta_t\right]) \circ (\theta_s - \mathsf{E}\left[\theta_s\right])\right]$ satisfies

$$\rho(t,s) = S(t)\rho(0,0)S^*(t) + \int_0^s S(t-v)QS^*(s-v)\,\mathrm{d}v, \quad Q = GG^*.$$

For more details see, for example, [16] and [17].

2. Linear filtering for infinite - dimensional Gaussian processes

The aim of this chapter is to find an appropriate form of solution to the linear filtering problem for infinite-dimensional Gaussian processes with finite-dimensional observation and to give some examples. Typically, the signal process may be governed by a linear SPDE driven by noise that is not white in time, like a Gaussian Volterra noise or, in particular, fractional Brownian motion.

The chapter is divided into five sections. In Section 2.1 the problem is posed and the main result (Theorem 2.1.1) is stated and proved. It is shown that the filter and observation error satisfy certain integral equations. In Section 2.2 uniqueness of solutions to the (nonlinear) integral equation for the error covariance (Theorem 2.2.1) is shown.

In Section 2.3 it is demonstrated that if the signal process is governed by linear stochastic evolution equation driven by a standard cylindrical Wiener process, our result stated in Theorem 2.1.1 reduces to an infinite-dimensional analogue of the classical Kalman-Bucy Theorem (Theorem 2.3.1).

Then, in Section 2.4 the general results are applied to signal governed by stochastic evolution equations driven by fractional noise. In 2.4.1 the general results are applied to $2m$-th order stochastic parabolic equation driven by Gauss-Voltera noise on a bounded domain. In Corollary 2.4.1 the main results are specified to the case of pointwise observation of the signal. Another application is stated in 2.4.2, where the general results are applied to stochastic heat equation driven by fBm on an unbounded domain again with pointwise observation of the signal.

Finally, in Section 2.5, the general results are applied to signal given by stochastic delayed evolution equation perturbed by Wiener process. In Theorem 2.5.1 the mean value and covariances of the signal, which are necessary for application of Theorem 2.1.1, are specified.

2.1 Solution to the filtering problem

Let (H, V) be a rigged separable Hilbert space, where $H = (H, \langle\cdot,\cdot\rangle_H, \|\cdot\|_H)$ and $V = (V, \langle\cdot,\cdot\rangle_V, \|\cdot\|_V)$ are separable Hilbert spaces such that $V \subset H$, V is dense in H and identifying H with the dual H^* the embeddings

$$V \hookrightarrow H = H^* \hookrightarrow V^*$$

are continuous and dense. The duality pairing between V and V^* is defined by the inner product on H, that is $\langle u, v\rangle_{V,V^*} = \langle u, v\rangle_H$ for $u \in V \subset H$ and $v \in H \subset V^*$.

For arbitrary $x, y \in V$ we define tensor product $x \circ y \in \mathcal{L}(V^*, V)$, $(x \circ y)v = x\langle y,v\rangle_{V,V^*}$, $v \in V^*$.

Let us consider stochastic basis $(\Omega, F, P, (F_t))$ and the signal $\theta = \{\theta_t, t \in [0,T]\}$ that is a Gaussian mean - square continuous measurable process in V. Let $\xi = \{\xi_t, t \in [0,T]\}$ denote an $\mathbb{R}^n$ - valued observation process given as

$$\xi_t = \int_0^t A(s)\theta_s \,\mathrm{d}s + W_t, \tag{2.1.1}$$

where $(A(s))_{s\in[0,T]}$ is a family of linear operators $V \to \mathbb{R}^n$ such that mapping $t \to A(t)$ is strongly measurable and $\|A(t)\|_{\mathcal{L}(V,\mathbb{R}^n)} \leq K$, $t \in [0,T]$ for some $K < \infty$. Here $W = \{W_t, t \in [0,T]\}$ is a standard $\mathbb{R}^n$ - valued Wiener process independent of the signal θ.

Further, assume that for each $t \in [0,T]$ operator $A(t)$ can be decomposed into functionals $A_1(t), \dots, A_n(t) \in V^*$ such that

$$A(t)b = (\langle b, A_1(t)\rangle_{V,V^*}, \dots, \langle b, A_n(t)\rangle_{V,V^*})^T$$

for all $b \in V$. Note that the dual operator $A^*(t)$: $\mathbb{R}^n \to V^*$ then satisfies $A^*(t)z = \sum_{i=1}^n z_i A_i(t)$ for all $z \in \mathbb{R}^n$.

We are dealing with the optimal filter $\widehat{\theta}_t$, which is defined as

$$\widehat{\theta}_t = \mathbb{E}[\theta_t | F_t^\xi],$$

where $(F_t^\xi)_{t\in[0,T]}$ is the filtration generated by the observation process ξ.

Set $K^\theta(t,s) = \mathbb{E}[(\theta_t - \mathbb{E}[\theta_t]) \circ (\theta_s - \mathbb{E}[\theta_s])]$, $t, s \in [0,T]$. Notice that the mean-square continuity of the process $\{\theta_t, t \in [0,T]\}$ implies that the mapping $K^\theta : [0,T]^2 \to \mathcal{L}(V^*, V)$ is strongly continuous and bounded.

Theorem 2.1.1. *Let $\Lambda = \{(t,s) \in [0,T]^2; 0 \leq s \leq t \leq T\}$. The filter $\widehat{\theta}$ satisfies the stochastic integral equation*

$$\widehat{\theta}_t = E[\theta_t] + \int_0^t \Phi(t,s)A^*(s)\,\mathrm{d}\xi_s - \int_0^t \Phi(t,s)A^*(s)A(s)\widehat{\theta}_s\,\mathrm{d}s, \quad t \in [0,T], \quad (2.1.2)$$

where operator Φ: $\Lambda \to \mathcal{L}(V^*, V)$ defined as $\Phi(t,s) = \mathbb{E}[\theta_t \circ (\theta_s - \widehat{\theta}_s)]$ for all $(s,t) \in \Lambda$ is strongly continuous and satisfies the integral equation

$$\Phi(t,s) = K^\theta(t,s) - \sum_{j=1}^n \int_0^s (\Phi(t,r)A_j(r)) \circ (\Phi(s,r)A_j(r))\,\mathrm{d}r. \quad (2.1.3)$$

Moreover, for all $t \in [0,T]$, $\Phi(t,t)$ is the covariance of the estimation error at time $t \in [0,T]$, that is,

$$\Phi(t,t) = \mathbb{E}\left[(\theta_t - \widehat{\theta}_t) \circ (\theta_t - \widehat{\theta}_t)\right] \quad (2.1.4)$$

holds.

In the proof of theorem (2.1.1) the following lemma will be useful.

Lemma 2.1.2. *Let us consider an $\mathbb{R}^n$ - valued process of diffusion type $\xi = \{\xi_t, t \geq 0\}$ on probability space $(\Omega, F, P, (F_t))$ with the differential*

$$\mathrm{d}\xi_t = a_t\,\mathrm{d}t + \mathrm{d}W_t, \quad \xi_0 = 0, \quad (2.1.5)$$

where $a = \{a_t, t \geq 0\}$ is an (F_t^ξ) - progresively measurable $\mathbb{R}^n$ - valued process and $W = \{W_t, t \geq 0\}$ is a standard (F_t) - Wiener process in $\mathbb{R}^n$. If

$$\int_0^T \mathbb{E}|\, a_s \,|^2\,\mathrm{d}s < \infty \quad (2.1.6)$$

for every $T < \infty$, then any one - dimensional (F_t^ξ) - martingale X, forming together with (ξ, W) a Gaussian system can be represented in the form

$$X_t = \mathbb{E}X_0 + \sum_{j=1}^n \int_0^t f_j(s)\,\mathrm{d}W_s^j, \quad (2.1.7)$$

where $f_1, \dots, f_n$ are deterministic square integrable measurable functions.

Proof. Note that by (2.1.5) W is also (F_t^ξ) - Wiener process.

According to Theorem 3.1 in [27] there is a representation

$$X_t = \mathbb{E}X_0 + \sum_{j=1}^{n} \int_0^t f_j(s)\, \mathrm{d}W_s^j, \tag{2.1.8}$$

where $f = (f_1, \ldots, f_n)$ is (F_t^ξ) - progresively measurable $\mathbb{R}^n$ - valued process and

$$\int_0^T \mathbb{E}|\ f(s)\ |^2\, \mathrm{d}s < \infty \tag{2.1.9}$$

for every $T < \infty$.

Using Gaussianity of the process (X, ξ, W) and Theorem on Normal Correlation (cf. Theorem 13.1 in [39]) it can be shown, analogously to the proof of Theorem 5.21 in [38], that for all $j = 1, \ldots, n$ and for all $0 \leq s \leq t < \infty$

$$\mathbb{E}\left[(X_t - X_s)\left(W_t^j - W_s^j\right) \mid F_s^\xi\right] = \mathbb{E}\left[(X_t - X_s)\left(W_t^j - W_s^j\right)\right] \tag{2.1.10}$$

holds.

Furter, using Itô formula and the fact that mixed variation $\langle W^j, W^k\rangle_t = 0$ for $j \neq k$ and $\langle W^j, W^k\rangle_t = t$ for $j = k$ we have

$$(X_t - X_s)\left(W_t^j - W_s^j\right) \tag{2.1.11}$$

$$= \int_s^t (X_r - X_s)\ \mathrm{d}W_r^j + \sum_{k=1}^{n} \int_s^t \left(W_r^j - W_s^j\right) f_k(r)\, \mathrm{d}W_r^k + \int_s^t f_j(r)\, \mathrm{d}r. \tag{2.1.12}$$

Now, using (2.1.10), (2.1.11) and martingale property of Itô integral for all $j = 1, \ldots, n$ we get

$$\mathbb{E}\left[\int_s^t f_j(r)\, \mathrm{d}r\right] = \mathbb{E}\left[\int_s^t f_j(r)\, \mathrm{d}r \mid F_s^\xi\right].$$

Taking into account (2.1.9) we can use Fubini's theorem to obtain

$$\int_s^t \mathbb{E}\left[f_j(r)\right]\ \mathrm{d}r = \int_s^t \mathbb{E}\left[f_j(r) \mid F_s^\xi\right]\ \mathrm{d}r \tag{2.1.13}$$

for all $j = 1, \ldots, n$.

We can use (2.1.13) piecewise on a sequence of decompositions

$$\{0 = t_0^{(n)} < \cdots < t_n^{(n)} = t, n \in \mathbb{N}\}$$

of interval $[0, t]$ such that $\max_{i=1,\ldots n} \mid t_i^{(n)} - t_{i-1}^{(n)} \mid \xrightarrow{n\to\infty} 0$ to obtain

$$\int_0^t \mathbb{E}\left[f_j(r)\right]\ \mathrm{d}r = \int_0^t f_{j,n}(r)\, \mathrm{d}r, \quad j = 1, \ldots, n,$$

where $f_{j,n}(r) = \mathbb{E}\left[f_j(r) \mid F_{t_i^{(n)}}^\xi\right]$, $t_i^{(n)} \leq r < t_{i+1}^{(n)}$.

Analogously to the proof of Theorem 5.21 in [38], using continuity of the filtration $(F_s^\xi,\ 0 < s < t)$ which follows from Theorem 5.19 in [38] and the uniform integrability of $\{f_{j,n}, n \in \mathbb{N}\}$ we get

$$\mathsf{E}\ \left|\ \int_s^t \{\mathsf{E}\ [f_j(r)] - f_j(r)\}\, \mathrm{d}r\ \right| \xrightarrow{n\to\infty} 0, \quad j = 1, \ldots, n.$$

From this, for all $j = 1, \ldots, n$ and each $t,\ 0 \le t \le T$, we have

$$\int_s^t \mathbb{E}\left[f_j(r)\right] \mathrm{d}r = \int_s^t f_j(r)\,\mathrm{d}r$$

and, therefore, for all $j = 1, \ldots, n$ and for almost all $t,\ 0 \le t \le T$

$$f_j(t) = \mathbb{E}\left[f_j(t)\right]. \tag{2.1.14}$$

Equality (2.1.14) together with (2.1.8) proves the representation (2.1.7). □

Now, let us prove of theorem (2.1.1).

Proof. According to Lemma 2.2 in [27] process $\{\tilde{W}_t, t \in [0,T]\}$ defined as

$$\tilde{W}_t = \xi_t - \int_0^t \mathbb{E}[A(r)\theta_r \mid F_t^\xi]\,\mathrm{d}r = \xi_t - \int_0^t A(r)\widehat{\theta}_r\,\mathrm{d}r. \tag{2.1.15}$$

is $\mathbb{R}^n$ - valued (F_t^ξ) - standard Wiener process called innovation process. The formula (2.1.15) reads

$$\mathrm{d}\xi_t = A(t)\widehat{\theta}_t\,\mathrm{d}t + \mathrm{d}\tilde{W}_t, \quad \xi_0 = 0. \tag{2.1.16}$$

Define the square integrable V - valued process $M^s = \{M_t^s, t \in [0,T]\}$ as

$$M_t^s = \mathbb{E}[\theta_s | F_t^\xi] \tag{2.1.17}$$

for all $s \in [0,T]$. Note that $\widehat{\theta}_t = M_t^t$ and the proces M^s is (F_t^ξ) - martingale.

Let $\{e_i, i \in I\}$ be an orthonormal basis on V. Using Lemma 2.1.2

$$M_t^s = \sum_{i\in I} e_i M_{i,t}^s = \sum_{i\in I} e_i \left(\langle \mathsf{E}\left[\theta_s\right], e_i \rangle_H + \sum_{j=1}^n \int_0^t F_{i,j}^s(r)\,\mathrm{d}\tilde{W}_r^j \right) \tag{2.1.18}$$

where $F_{i,1}^s, \ldots, F_{i,n}^s$ are deterministic square integrable measurable functions, holds for every $s \in [0,T]$ and every $i \in I$. Using Itô isometry

$$\begin{aligned}
\infty > \mathbb{E}\|M_t^s\|_V^2 &= \sum_{i\in I} \mathbb{E}(M_{i,t}^s)^2 = \sum_{i\in I} \mathbb{E}\left[\langle \mathsf{E}\left[\theta_s\right], e_i \rangle_H + \sum_{j=1}^n \int_0^t F_{i,j}^s(r)\,\mathrm{d}\tilde{W}_r^j \right]^2 \\
&= \sum_{i\in I} \left(\langle \mathsf{E}\left[\theta_s\right], e_i \rangle_H\right)^2 + \sum_{j=1}^n \int_0^t \left(F_{i,j}^s(r)\right)^2 \mathrm{d}r \\
&= ||\mathsf{E}\left[\theta_s\right]||_{L^2(V)} + \sum_{j=1}^n \int_0^t \sum_{i\in I} \left(F_{i,j}^s(r)\right)^2 \mathrm{d}r = ||\mathsf{E}\left[\theta_s\right]||_{L^2(V)} + \sum_{j=1}^n \|F_j^s\|_{L^2(V)},
\end{aligned}$$

where $F^s(r) = (F_1^s(r), \ldots, F_n^s(r))$ is square integrable V^n - valued deterministic function such that $F_j^s(r) = \sum_{i\in I} e_i F_{i,j}^s(r)$ for all $j = 1, \ldots, n$.

Therefore, swaping the sums and the integral in (2.1.18) we finally obtain

$$M_t^s = \mathsf{E}\ [\theta_s] + \int_0^t F^s(r)\,\mathrm{d}\tilde{W}_r = \mathsf{E}\ [\theta_s] + \sum_{j=1}^n \int_0^t F_j^s(r)\,\mathrm{d}\tilde{W}_r^j. \tag{2.1.19}$$

Now, for every $s \in [0,T]$, we can consider an arbitrary square integrable measurable V^n - valued deterministic function $f^s(r) = (f_1^s(r), \dots, f_n^s(r))$ to show

$$\mathbb{E}\left[(\theta_s - M_t^s) \circ \left(\int_0^t f^s(r)\,\mathrm{d}\tilde{W}_r\right)\right] = 0. \tag{2.1.20}$$

Indeed, using (2.1.17) for arbitrary $v \in V^*$ we have

$$\mathbb{E}\left[(\theta_s - M_t^s) \circ \left(\int_0^t f^s(r)\,\mathrm{d}\tilde{W}_r\right)\right] v = \mathbb{E}\left[(\theta_s - M_t^s)\,\langle \int_0^t f^s(r)\,\mathrm{d}\tilde{W}_r, v\rangle_{V,V^*}\right]$$
$$= \mathbb{E}\left[\mathbb{E}\left[(\theta_s - M_t^s)\,\langle \int_0^t f^s(r)\,\mathrm{d}\tilde{W}_r, v\rangle_{V,V^*} \mid F_t^\xi\right]\right]$$
$$= \mathbb{E}\left[\left(\mathbb{E}\left[\theta_s \mid F_t^\xi\right] - M_t^s\right)\langle \int_0^t f^s(r)\,\mathrm{d}\tilde{W}_r, v\rangle_{V,V^*}\right] = 0.$$

Further, we show that

$$\mathbb{E}\left[M_t^s \circ \left(\int_0^t f^s(r)\,\mathrm{d}\tilde{W}_r\right)\right] = \sum_{j=1}^n \int_0^t F_j^s(r) \circ f_j^s(r)\,\mathrm{d}r. \tag{2.1.21}$$

Note that to prove equality of two arbitrary operators $P, Q \in \mathcal{L}(V^*, V)$ it is sufficient to show $\langle Pv, b\rangle_V = \langle Qv, b\rangle_V$ for all elements $v \in V^*$, $b \in V$. We have that

$$\langle \mathbb{E}\left[M_t^s \circ \left(\int_0^t f^s(r)\,\mathrm{d}\tilde{W}_r\right)\right] v, b\rangle_V = \langle \mathbb{E}\left[M_t^s \langle \int_0^t f^s(r)\,\mathrm{d}\tilde{W}_r, v\rangle_{V,V^*}\right], b\rangle_V$$
$$= \mathbb{E}\left[\langle M_t^s \langle \int_0^t f^s(r)\,\mathrm{d}\tilde{W}_r, v\rangle_{V,V^*}, b\rangle_V\right] = \mathbb{E}\left[\langle \int_0^t f^s(r)\,\mathrm{d}\tilde{W}_r, v\rangle_{V,V^*}\langle M_t^s, b\rangle_V\right]$$
$$= \mathbb{E}\left[\langle \int_0^t f^s(r)\,\mathrm{d}\tilde{W}_r, v\rangle_{V,V^*}\langle \mathsf{E}\,[\theta_s] + \int_0^t F^s(r)\,\mathrm{d}\tilde{W}_r, b\rangle_V\right]$$
$$= \mathbb{E}\left[\langle \sum_{j=1}^n \int_0^t f_j^s(r)\,\mathrm{d}\tilde{W}_r^j, v\rangle_{V,V^*}\langle \mathsf{E}\,[\theta_s] + \sum_{j=1}^n \int_0^t F_j^s(r)\,\mathrm{d}\tilde{W}_r^j, b\rangle_V\right]$$
$$= \mathbb{E}\left[\sum_{j=1}^n \int_0^t \langle f_j^s(r), v\rangle_{V,V^*}\,\mathrm{d}\tilde{W}_r^j\left(\langle \mathsf{E}\,[\theta_s], b\rangle_V + \sum_{j=1}^n \int_0^t \langle F_j^s(r), b\rangle_V\,\mathrm{d}\tilde{W}_r^j\right)\right].$$

Using Itô isometry and the fact that mixed variation $\langle \tilde{W}^i, \tilde{W}^j\rangle = 0,\ i \neq j$ we obtain

$$\mathbb{E}\left[\sum_{j=1}^n \int_0^t \langle f_j^s(r), v\rangle_{V,V^*}\,\mathrm{d}\tilde{W}_r^j \sum_{j=1}^n \int_0^t \langle F_j^s(r), b\rangle_V\,\mathrm{d}\tilde{W}_r^j\right]$$
$$= \sum_{j=1}^n \int_0^t \langle f_j^s(r), v\rangle_{V,V^*}\langle F_j^s(r), b\rangle_V\,\mathrm{d}r = \sum_{j=1}^n \langle \left(\int_0^t F_j^s(r) \circ f_j^s(r)\,\mathrm{d}r\right) v, b\rangle_V$$
$$= \langle \left(\sum_{j=1}^n \int_0^t F_j^s(r) \circ f_j^s(r)\,\mathrm{d}r\right) v, b\rangle_V$$

which concludes the proof of equality (2.1.21).

Next, we show equality

$$\mathbb{E}\left[\theta_s \circ \left(\int_0^t f^s(r)\,\mathrm{d}\tilde{W}_r\right)\right] = \sum_{j=1}^n \int_0^t \left(\mathbb{E}\left[\theta_s \circ (\theta_r - \widehat{\theta}_r)\right] A_j(r)\right) \circ f_j^s(r)\,\mathrm{d}r. \tag{2.1.22}$$

Using (2.1.1), (2.1.15) and independence of θ and W we have

$$\mathbb{E}\left[\theta_s \circ \left(\int_0^t f^s(r)\,\mathrm{d}\tilde{W}_r\right)\right] = \mathbb{E}\left[\theta_s \circ \left(\int_0^t f^s(r)\,\mathrm{d}\xi_r - \int_0^t f^s(r)A(r)\widehat{\theta}_r\,\mathrm{d}r\right)\right]$$
$$= \mathbb{E}\left[\theta_s \circ \left(\int_0^t \sum_{j=1}^n f_j^s(r)\langle\theta_r, A_j(r)\rangle_{V,V^*}\,\mathrm{d}r \right.\right.$$
$$\left.\left. + \int_0^t f^s(r)\,\mathrm{d}W_r - \int_0^t \sum_{j=1}^n f_j^s(r)\langle\widehat{\theta}_r, A_j(r)\rangle_{V,V^*}\,\mathrm{d}r\right)\right]$$
$$= \mathbb{E}\left[\theta_s \circ \left(\int_0^t \sum_{j=1}^n f_j^s(r)\langle\theta_r - \widehat{\theta}_r, A_j(r)\rangle_{V,V^*}\,\mathrm{d}r\right)\right]$$
$$= \sum_{j=1}^n \int_0^t \mathbb{E}\left[\theta_s \circ \left(f_j^s(r)\langle\theta_r - \widehat{\theta}_r, A_j(r)\rangle_{V,V^*}\right)\right]\,\mathrm{d}r.$$

To complete the proof of equality (2.1.22) it is sufficient to show

$$\theta_s \circ \left(f_j^s\langle(r)\theta_r - \widehat{\theta}_r, A_j(r)\rangle_{V,V^*}\right) = \left[\left(\theta_s \circ (\theta_r - \widehat{\theta}_r)\right) A_j(r)\right] \circ f_j^s(r), \quad j = 1, \ldots, n. \tag{2.1.23}$$

By the definition of tensor product we obtain

$$\langle\theta_s \circ \left(f_j^s(r)\langle\theta_r - \widehat{\theta}_r, A_j(r)\rangle_{V,V^*}\right)v, b\rangle_V = \langle\theta_s\langle f_j^s(r)\langle\theta_r - \widehat{\theta}_r, A_j(r)\rangle_{V,V^*}, v\rangle_{V,V^*}, b\rangle_V$$
$$= \langle\theta_r - \widehat{\theta}_r, A_j(r)\rangle_{V,V^*}\langle f_j^s(r), v\rangle_{V,V^*}\langle\theta_s, b\rangle_V$$

for all $v \in V^*$, $b \in V$. Similarly we get

$$\langle\left(\left[\left(\theta_s \circ (\theta_r - \widehat{\theta}_r)\right) A_j(r)\right] \circ f_j^s(r)\right) v, b\rangle_V = \langle\left(\theta_s \circ (\theta_r - \widehat{\theta}_r)\right) A_j(r)\langle f_j^s(r), v\rangle_{V,V^*}, b\rangle_V$$
$$= \langle f_j^s(r), v\rangle_{V,V^*}\langle\theta_s\langle\theta_r - \widehat{\theta}_r, A_j(r)\rangle_{V,V^*}, b\rangle_V$$
$$= \langle\theta_r - \widehat{\theta}_r, A_j(r)\rangle_{V,V^*}\langle f_j^s(r), v\rangle_{V,V^*}\langle\theta_s, b\rangle_V.$$

Which proves equality (2.1.23) and, therefore, completes the proof of (2.1.22).

Combining equalities (2.1.20), (2.1.21) and (2.1.22) we obtain

$$\sum_{j=1}^n \int_0^t F_j^s(r) \circ f_j^s(r)\,\mathrm{d}r = \sum_{j=1}^n \int_0^t \left(\mathbb{E}\left[\theta_s \circ (\theta_r - \widehat{\theta}_r)\right] A_j(r)\right) \circ f_j^s(r)\,\mathrm{d}r. \tag{2.1.24}$$

The formula (2.1.24) holds for any arbitrary square integrable V^n - valued deterministic function $f^s(r) = (f_1^s(r), \ldots, f_n^s(r))$ hence

$$F_j^s(r) = \mathbb{E}\left[\theta_s \circ (\theta_r - \widehat{\theta}_r)\right] A_j(r) = \Phi(s,r)A_j(r) \tag{2.1.25}$$

for all $s \in [0,T]$ and for almost all $r \in [0,T]$.

From (2.1.19) and (2.1.25), by the choice $s = t$ and taking into account (2.1.15) we have

$$\widehat{\theta}_t = M_t^t = \mathsf{E}\,[\theta_t] + \sum_{j=1}^n \int_0^t F_j^t(r)\,\mathrm{d}\tilde{W}_r^j = \mathsf{E}\,[\theta_t] + \sum_{j=1}^n \int_0^t \Phi(t,r)A_j(r)\,\mathrm{d}\tilde{W}_r^j$$
$$= \mathsf{E}\,[\theta_t] + \sum_{j=1}^n \int_0^t \Phi(t,r)A_j(r)\,\mathrm{d}\xi_r^j - \sum_{j=1}^n \int_0^t \Phi(t,r)A_j(r)\langle\widehat{\theta}_r, A_j(r)\rangle_{V,V^*}\,\mathrm{d}r$$
$$= \mathsf{E}\,[\theta_t] + \int_0^t \Phi(t,s)A^*(s)\,\mathrm{d}\xi_s - \int_0^t \Phi(t,s)A^*(s)A(s)\widehat{\theta}_s\,\mathrm{d}s$$

which concludes the proof of stochastic integral equation (2.1.2).

Let us verify the formula (2.1.3). Notice that

$$\Phi(t,s) = \mathbb{E}\left[\theta_t \circ (\theta_s - \widehat{\theta}_s)\right] = \mathbb{E}\left[\theta_t \circ \theta_s\right] - \mathbb{E}\left[\theta_t \circ \widehat{\theta}_s\right]. \tag{2.1.26}$$

Using the above proved representation of $\widehat{\theta}_s$, (2.1.1), (2.1.15) and independence of θ and W, we have

$$\begin{aligned}
&\mathbb{E}\left[\theta_t \circ \widehat{\theta}_s\right] \\
&= \mathbb{E}\left[\theta_t \circ \left(\mathsf{E}\left[\theta_t s\right] + \int_0^s \Phi(s,r)A^*(r)\,\mathrm{d}\xi_r - \int_0^s \Phi(s,r)A^*(r)A(r)\widehat{\theta}_r\,\mathrm{d}r\right)\right] \\
&= \mathbb{E}\left[\theta_t \circ \left(\mathsf{E}\left[\theta_s\right] + \int_0^s \Phi(s,r)A^*(r)A(r)\theta_r\,\mathrm{d}r\right.\right. \\
&\qquad\left.\left. + \int_0^s \Phi(s,r)A^*(r)\,\mathrm{d}W_r - \int_0^s \Phi(s,r)A^*(r)A(r)\widehat{\theta}_r\,\mathrm{d}r\right)\right] \\
&= \mathsf{E}\left[\theta_t\right] \circ E[\theta_s] + \mathbb{E}\left[\theta_t \circ \left(\int_0^s \Phi(s,r)A^*(r)A(r)(\theta_r - \widehat{\theta}_r)\,\mathrm{d}r\right)\right] \\
&= \mathsf{E}\left[\theta_t\right] \circ E[\theta_s] + \int_0^s \mathbb{E}\left[\theta_t \circ \left(\Phi(s,r)A^*(r)A(r)(\theta_r - \widehat{\theta}_r)\right)\right]\,\mathrm{d}r \\
&= \mathsf{E}\left[\theta_t\right] \circ E[\theta_s] + \int_0^s \mathbb{E}\left[\sum_{j=1}^n \theta_t \circ \left[\Phi(s,r)A_j(r)\langle\theta_r - \widehat{\theta}_r, A_j(r)\rangle_{V,V^*}\right]\right]\,\mathrm{d}r.
\end{aligned} \tag{2.1.27}$$

For arbitrary $v \in V^*$, $b \in V$ we have that

$$\begin{aligned}
&\langle\mathbb{E}\left[\sum_{j=1}^n \theta_t \circ \left[\Phi(s,r)A_j(r)\langle\theta_r - \widehat{\theta}_r, A_j(r)\rangle_{V,V^*}\right]\right] v, b\rangle_V \\
&= \mathbb{E}\left[\sum_{j=1}^n \langle\theta_r - \widehat{\theta}_r, A_j(r)\rangle_{V,V^*}\langle\Phi(s,r)A_j(r), v\rangle_{V,V^*}\langle\theta_t, b\rangle_V\right] \\
&= \mathbb{E}\left[\sum_{j=1}^n \langle\theta_t\langle\theta_r - \widehat{\theta}_r, A_j(r)\rangle_{V,V^*}, b\rangle_V \langle\Phi(s,r)A_j(r), v\rangle_{V,V^*}\right] \\
&= \langle\sum_{j=1}^n \mathbb{E}\left[\theta_t \circ \theta_r - \widehat{\theta}_r\right] A_j(r)\langle\Phi(s,r)A_j(r), v\rangle_{V,V^*}, b\rangle_V \\
&= \langle\sum_{j=1}^n \left[\Phi(t,r)A_j(r) \circ \Phi(s,r)A_j(r)\right] v, b\rangle_V.
\end{aligned}$$

By (2.1.26) and (2.1.27) it follows that

$$\Phi(t,s) = \mathsf{E}\left[\theta_t \circ \theta_s\right] - \mathsf{E}\left[\theta_t\right] \circ \mathsf{E}\left[\theta_s\right] - \sum_{j=1}^n \int_0^s \Phi(t,r)A_j(r) \circ \Phi(s,r)A_j(r)\,\mathrm{d}r, \tag{2.1.28}$$

This, together with (2.1.26), proves the formula (2.1.3).

By (2.1.26) the mapping $t \mapsto \Phi(t,s)$ is strongly continuous on $[0,T]$ uniformly with respect to $s \in [0,t]$ hence the representation (2.1.28) implies the mean-square continuity of the proces $\hat{\theta}$ in V. By (2.1.28) and (2.1.26) the mapping Φ: $\Lambda \to \mathcal{L}(V^*, V)$ is strongly continuous.

It remains to prove equality (2.1.4). For every $v \in V^*$ it holds

$$\mathbb{E}\left[\widehat{\theta}_t \circ (\theta_t - \widehat{\theta}_t)\right] v = \mathbb{E}\left[\mathbb{E}\left[\widehat{\theta}_t \langle \theta_t - \widehat{\theta}_t, v\rangle_{V,V^*} \mid F_t^\xi\right]\right]$$
$$= \mathbb{E}\left[\widehat{\theta}_t \left(\langle \mathbb{E}\left[\theta_t \mid F_t^\xi\right], v\rangle_{V,V^*} - \langle \widehat{\theta}_t, v\rangle_{V,V^*}\right)\right] = 0,$$

therefore, we obtain

$$\mathbb{E}\left[(\theta_t - \widehat{\theta}_t) \circ (\theta_t - \widehat{\theta}_t)\right] = \Phi(t,t) + \mathbb{E}\left[\widehat{\theta}_t \circ (\theta_t - \widehat{\theta}_t)\right] = \Phi(t,t)$$

which completes the proof of Theorem 2.1.1. □

Remark 2.1.3. Note that if the signal $\theta = \{\theta_t, t \in [0,T]\}$ takes its values in the Hilbert space H, the family $(A(s))_{s\in[0,T]}$ of observation operators $H \to \mathbb{R}^n$ can be characterised by n functionals from the dual space of H and no embedding is needed. In this case the equation (2.1.3) can be expressed using adjoint operators A^* and Φ^* as

$$\Phi(t,s) = K^\theta(t,s) - \int_0^s \Phi(t,r)A^*(r)A(r)\Phi^*(s,r)\,\mathrm{d}r. \tag{2.1.29}$$

Indeed we have

$$\langle \left(\sum_{j=1}^n \int_0^s (\Phi(t,r)A_j(r)) \circ (\Phi(s,r)A_j(r))\ \mathrm{d}r\right) h, k\rangle_H$$
$$= \sum_{j=1}^n \int_0^s \langle \Phi(t,r)A_j(r)\langle \Phi(s,r)A_j(r), h\rangle_H, k\rangle_H \,\mathrm{d}r$$
$$= \sum_{j=1}^n \int_0^s \langle \Phi(t,r)A_j(r)\langle \Phi^*(s,r)h, A_j(r)\rangle_H, k\rangle_H \,\mathrm{d}r$$
$$= \langle \left(\int_0^s \Phi(t,r)A^*(r)A(r)\Phi^*(s,r)\,\mathrm{d}r\right) h, k\rangle_H$$

for all $h,k \in H$ and all $(t,s) \in \Lambda$.

Therefore, Theorem 2.1.1 simplifies to exact infinite-dimensional analogy of Theorem 1.1.2 when we set $Q(t) = I$, $t > 0$.

2.2 Uniqueness of the solution

In the present section we prove uniqueness of solutions to the integral equation (2.1.3) in the class of operators

$$\mathbb{M} = \{\Psi : \Lambda \to \mathcal{L}(V^*, V);\ \Psi \text{ strongly continuous};\ \Psi(t,t) \geq 0\ \forall t \in [0,T]\}.$$

More generally, we prove the following.

Theorem 2.2.1. *Let $K : \Lambda \to \mathcal{L}(V^*, V)$ be a strongly continuous mapping such that $K(t,t) \geq 0$ for each $t \in [0,T]$.*

Then the integral equation

$$\Psi(t,s) = K(t,s) - \sum_{j=1}^n \int_0^s (\Psi(t,r)A_j(r)) \circ (\Psi(s,r)A_j(r))\ \mathrm{d}r, \quad (t,s) \in \Lambda \tag{2.2.1}$$

has at most one solution in the class $\mathbb{M}$.

Proof. First, we show that every $\Psi \in \mathbb{M}$ satisfying the equation (2.2.1) is bounded on Λ.

Using Cauchy - Schwarz inequality we obtain for arbitrary $x, y \in V^*$

$$
\begin{aligned}
&|\langle \left(\sum_{j=1}^{n} \int_0^s (\Psi(t,r)A_j(r)) \circ (\Psi(s,r)A_j(r))\ \mathrm{d}r \right) x, y\rangle_{V,V^*}| \\
&\leq \sum_{j=1}^{n} \int_0^s |\langle [(\Psi(t,r)A_j(r)) \circ (\Psi(s,r)A_j(r))]\, x, y\rangle_{V,V^*}|\, \mathrm{d}r \\
&= \sum_{j=1}^{n} \int_0^s |\langle \Psi(t,r)A_j(r)\langle \Psi(s,r)A_j(r), x\rangle_{V,V^*}, y\rangle_{V,V^*}|\, \mathrm{d}r \\
&= \sum_{j=1}^{n} \int_0^s |\langle \Psi(t,r)A_j(r), y\rangle_{V,V^*}|\ |\langle \Psi(s,r)A_j(r), x\rangle_{V,V^*}|\, \mathrm{d}r \\
&\leq \sum_{j=1}^{n} \left(\int_0^s (\langle \Psi(t,r)A_j(r), y\rangle_{V,V^*})^2\ \mathrm{d}r \right)^{\frac{1}{2}} \left(\int_0^s (\langle \Psi(s,r)A_j(r), x\rangle_{V,V^*})^2 \right)^{\frac{1}{2}} \\
&= \sum_{j=1}^{n} \left(\int_0^s \langle [(\Psi(t,r)A_j(r)) \circ (\Psi(t,r)A_j(r))]\, y, y\rangle_{V,V^*}\ \mathrm{d}r \right)^{\frac{1}{2}} \\
&\qquad\qquad \left(\int_0^s \langle [(\Psi(s,r)A_j(r)) \circ (\Psi(s,r)A_j(r))]\, x, x\rangle_{V,V^*}\ \mathrm{d}r \right)^{\frac{1}{2}}. \qquad (2.2.2)
\end{aligned}
$$

In the last inequality we can increase the upper bound of the first integral from s to t because the integrand is nonnegative. Thus all we need is to estimate the term

$$\int_0^t \langle [(\Psi(t,r)A_j(r)) \circ (\Psi(t,r)A_j(r))]\, x, x\rangle_{V,V^*}\ \mathrm{d}r$$

for all $j = 1, \ldots, n$ and $x \in V^*$. Using (2.2.1) and the boundedness of the family $(K(t,s),\ t, s \in [0,T])$ in $\mathcal{L}(V^*, V)$ (which follows by the Resonance Theorem) we obtain

$$
\begin{aligned}
0 &\leq \int_0^t \langle [(\Psi(t,r)A_j(r)) \circ (\Psi(t,r)A_j(r))]\, x, x\rangle_{V,V^*}\ \mathrm{d}r \\
&\leq \sum_{j=1}^{n} \int_0^t \langle [(\Psi(t,r)A_j(r)) \circ (\Psi(t,r)A_j(r))]\, x, x\rangle_{V,V^*}\ \mathrm{d}r \\
&= \langle K(t,t)x, x\rangle_{V,V^*} - \langle \Psi(t,t)x, x\rangle_{V,V^*} \\
&\leq \langle K(t,t)x, x\rangle_{V,V^*} \leq C_1(T)\|x\|^2_{V^*}, \quad t \in [0,T]. \qquad (2.2.3)
\end{aligned}
$$

Now, by (2.2.1), (2.2.2), (2.2.3) and again by the boundedness of the family $(K(t,s),\ t, s \in [0,T])$ we have that

$$
\begin{aligned}
|\langle \Psi(t,s)x, y\rangle_{V,V^*}| &\leq \|K(t,s)\|_{\mathcal{L}(V^*,V)}\|x\|_{V^*}\|y\|_{V^*} \\
&\quad + \left| \langle \left(\sum_{j=1}^{n} \int_0^s (\Psi(t,r)A_j(r)) \circ (\Psi(s,r)A_j(r))\ \mathrm{d}r \right) x, y\rangle_{V,V^*} \right| \\
&\leq C_2(T)\|x\|_{V^*}\|y\|_{V^*}
\end{aligned}
$$

for all $x, y \in V^*$ and $(t,s) \in \Lambda$, which proves that the family of operators $(\Psi(t,s),\ (t,s) \in \Lambda)$ is bounded in $\mathcal{L}(V^*, V)$.

Assume that $\Psi_1, \Psi_2 \in \mathbb{M}$ are solutions to the equation (2.2.1) and set $\varphi : [0,T] \to [0,\infty)$, $\varphi(s) = \sup_{t\in[s,T]} \|\Psi_1(t,s) - \Psi_2(t,s)\|_{\mathcal{L}(V^*,V)}$. In virtue of (2.2.1) we have

$$\begin{aligned}
\varphi(s) &= \sup_{t\in[s,T]} \|\Psi_1(t,s) - \Psi_2(t,s)\|_{\mathcal{L}(V^*,V)} \\
&= \sup_{t\in[s,T]} \| \sum_{j=1}^{n} \int_0^s (\Psi_1(t,r)A_j(r)) \circ (\Psi_1(s,r)A_j(r)) \\
&\qquad - (\Psi_2(t,r)A_j(r)) \circ (\Psi_2(s,r)A_j(r)) \, \mathrm{d}r \|_{\mathcal{L}(V^*,V)} \\
&\leq \sum_{j=1}^{n} \int_0^s \sup_{t\in[s,T]} \| (\Psi_1(t,r)A_j(r)) \circ (\Psi_1(s,r)A_j(r)) \\
&\qquad - (\Psi_2(t,r)A_j(r)) \circ (\Psi_2(s,r)A_j(r)) \|_{\mathcal{L}(V^*,V)} \, \mathrm{d}r \\
&= \sum_{j=1}^{n} \int_0^s \sup_{t\in[s,T]} \| (\Psi_1(t,r)A_j(r)) \circ (\Psi_1(s,r)A_j(r)) \\
&\qquad - (\Psi_2(t,r)A_j(r)) \circ (\Psi_2(s,r)A_j(r)) \\
&\qquad \pm (\Psi_1(t,r)A_j(r)) \circ (\Psi_2(s,r)A_j(r)) \|_{\mathcal{L}(V^*,V)} \, \mathrm{d}r \\
&= \sum_{j=1}^{n} \int_0^s \sup_{t\in[s,T]} \| (\Psi_1(t,r)A_j(r)) \circ [(\Psi_1(s,r) - \Psi_2(s,r)) \, A_j(r)] \|_{\mathcal{L}(V^*,V)} \\
&\qquad + \sup_{t\in[s,T]} \| [(\Psi_1(t,r) - \Psi_2(t,r)) \, A_j(r)] \circ (\Psi_2(s,r)A_j(r)) \|_{\mathcal{L}(V^*,V)} \, \mathrm{d}r \\
&\leq \sum_{j=1}^{n} \int_0^s \sup_{t\in[s,T]} \|\Psi_1(t,r)\|_{\mathcal{L}(V^*,V)} \|A_j(r)\|_{V^*} \|\Psi_1(s,r) - \Psi_2(s,r)\|_{\mathcal{L}(V^*,V)} \|A_j(r)\|_{V^*} \\
&\quad + \sup_{t\in[s,T]} \|\Psi_1(t,r) - \Psi_2(t,r)\|_{\mathcal{L}(V^*,V)} \|A_j(r)\|_{\mathcal{L}(V^*,V)} \|\Psi_2(s,r)\|_{\mathcal{L}(V^*,V)} \|A_j(r)\|_{V^*} \, \mathrm{d}r \\
&\leq C(T) \int_0^s \sup_{t\in[s,T]} \| (\Psi_1(s,r) - \Psi_2(s,r)) \|_{\mathcal{L}(V^*,V)} \\
&\qquad + \sup_{t\in[s,T]} \| (\Psi_1(t,r) - \Psi_2(t,r)) \|_{\mathcal{L}(V^*,V)} \, \mathrm{d}r \\
&\leq 2C(T) \int_0^s \sup_{t\in[s,T]} \| (\Psi_1(t,r) - \Psi_2(t,r)) \|_{\mathcal{L}(V^*,V)} \, \mathrm{d}r \\
&\leq 2C(T) \int_0^s \sup_{t\in[r,T]} \| (\Psi_1(t,r) - \Psi_2(t,r)) \|_{\mathcal{L}(V^*,V)} \, \mathrm{d}r = 2C(T) \int_0^s \varphi(r) \, \mathrm{d}r.
\end{aligned}$$

Now, by the Gronwall lemma we obtain $\sup_{t\in[s,T]} \|\Psi_1(t,s) - \Psi_2(t,s)\|_{\mathcal{L}(V^*,V)} = 0$ for all $0 \leq s \leq T$ and the proof is complete.

□

2.3 Signal governed by stochastic evolution equation driven by Wiener process

In this section we will show that if the signal process is governed by linear evolution equation driven by a standard cylindrical Wiener process Theorem 2.1.1 reduces to Theorem 2.3.1 which is an infinite-dimensional analogue of the classical Kalman-Bucy Theorem (Theorem 1.1.1).

Let the signal $\theta = \{\theta_t, t \in [0,T]\}$ be an H - valued random process defined by stochastic evolution equation

$$d\theta_t = \mathcal{A}\theta_s \, ds + G \, d\mathcal{W}_t, \quad \theta_0 = 0, \tag{2.3.1}$$

where $\mathcal{A}$ is the infinitesimal generator of a strongly continuous semigroup $(S(t))_{t\in\mathbb{R}_+}$ in H, $G \in \mathcal{L}(H)$ and $\{\mathcal{W}_t, t \in [0,T]\}$ is an H - valued standard cylindrical Wiener process defined on a stochastic basis $(\Omega, F, P, (F_t))$.

Assume $\forall t > 0$, $S(t)G \in \mathcal{L}_2(H)$ and

$$\int_0^{T_0} | \, S(r)G \, |^2_{\mathcal{L}_2(H)} \; dr < \infty$$

for some $T_0 > 0$. Then, according to Section 1.2.3, the equation (2.3.1) has a unique mean-square continuous solution $\{\theta_t, t \in [0,T]\}$. The observation $\xi = \{\xi_t, t \in [0,T]\}$ is given by the equation (2.1.1) where $V = H$, $\{W_t, t \in [0,T]\}$ is a standard $\mathbb{R}^n$-valued Wiener process on $(\Omega, F, P, (F_t))$ independent of $\{\mathcal{W}_t, t \in [0,T]\}$ and $A : [0,T] \to \mathcal{L}(H, \mathbb{R}^n)$ is strongly measurable and bounded.

For all $0 \le r \le t \le T$ the process θ_t satisfies

$$\theta_t = S(t-r)\theta_r + \gamma_{t,r}, \tag{2.3.2}$$

where $\gamma_{t,r} = \int_r^t S(t-s)G \, d\mathcal{W}_s$ is a well defined centered Gaussian variable in H and is independent of θ_r (see Section 1.2.3 for more details).

In this case the equations (2.1.2) and (2.1.3) simplify to the infinte - dimensional analogue of standard Kalman - Bucy filter as shown in the following theorem.

Theorem 2.3.1. *The filter $\widehat{\theta}$ satisfies the stochastic differential equation*

$$d\widehat{\theta}_t = \mathcal{A}\widehat{\theta}_t \, dt + \Phi(t)A^*(t)\left(d\xi_t - A(t)\widehat{\theta}_t \, dt \right), \quad t \in [0,T], \tag{2.3.3}$$

where the solution to (2.3.3) is understood in the mild sense, i.e. $\widehat{\theta}_t$ solves the equation

$$\widehat{\theta}_t = \int_0^t S(t-s)\Phi(s)A^*(s) \, d\xi_s - \int_0^t S(t-s)\Phi(s)A^*(s)A(s)\widehat{\theta}_s \, ds. \tag{2.3.4}$$

The family of operators $(\Phi(t))_{t\in[0,T]}$: $H \to H$ defined as

$$\Phi(t) = \mathbb{E}\left[(\theta_t - \widehat{\theta}_t) \circ (\theta_t - \widehat{\theta}_t)\right]$$

for all $t \in [0,T]$ satisfies the differential equation

$$\dot{\Phi}(t) = \mathcal{A}\Phi(t) + \Phi(t)\mathcal{A}^* - \Phi(t)A^*(t)A(t)\Phi(t) + Q, \quad Q = GG^*$$

in the weak sense, that is

$$\frac{d}{dt}\langle \Phi(t)x, y\rangle = \langle \Phi(t)x, \mathcal{A}^*y\rangle + \langle \mathcal{A}^*x, \Phi(t)y\rangle - \langle A(t)\Phi(t)x, A(t)\Phi(t)y\rangle + \langle Qx, y\rangle \tag{2.3.5}$$

for all $x, y \in Dom(\mathcal{A}^*)$.

Proof. Note that the operator $\Phi(t)$ is in fact the operator $\Phi(t,s)$ defined in theorem (2.1.1) when $s = t$ and is selfadjoint. Set $\Phi(t) = \Phi(t,t)$ and $K(t) = K^{\theta}(t,t)$ on $t \in [0,T]$.

Using (2.1.2), (2.3.2), the independence of $\gamma_{t,s}$ and θ_s for all $0 \leq s \leq t$ and the definition of operator $\Phi(t,s)$ in theorem (2.1.1) we obtain

$$\begin{aligned}
\widehat{\theta}_t &= \int_0^t \mathbb{E}\left[(S(t-s)\theta_s + \gamma_{t,s}) \circ (\theta_s - \widehat{\theta}_s)\right] A^*(s)\,\mathrm{d}\xi_s \\
&\quad - \int_0^t \mathbb{E}\left[(S(t-s)\theta_s + \gamma_{t,s}) \circ (\theta_s - \widehat{\theta}_s)\right] A^*(s)A(s)\widehat{\theta}_s\,\mathrm{d}s \\
&= \int_0^t S(t-s)\mathbb{E}\left[\theta_s \circ (\theta_s - \widehat{\theta}_s)\right] A^*(s)\,\mathrm{d}\xi_s \\
&\quad - \int_0^t S(t-s)\mathbb{E}\left[\theta_s \circ (\theta_s - \widehat{\theta}_s)\right] A^*(s)A(s)\widehat{\theta}_s\,\mathrm{d}s \\
&= \int_0^t S(t-s)\Phi(s)A^*(s)\,\mathrm{d}\xi_s - \int_0^t S(t-s)\Phi(s)A^*(s)A(s)\widehat{\theta}_s\,\mathrm{d}s
\end{aligned}$$

for all $t \in [0,T]$ which is exactly (2.3.4).

Using the same arguments as above, equation (2.1.29) and self - adjointness of the operator $\Phi(t)$, $t \in [0,T]$ we have

$$\begin{aligned}
\Phi(t) &= K(t) - \int_0^t \mathbb{E}\left[(S(t-r)\theta_r + \gamma_{t,r}) \circ (\theta_r - \widehat{\theta}_r)\right] A^*(r) \\
&\qquad A(r)\left(\mathbb{E}\left[(S(t-r)\theta_r + \gamma_{t,r}) \circ (\theta_r - \widehat{\theta}_r)\right]\right)^*\,\mathrm{d}r \\
&= K(t) - \int_0^t S(t-r)\mathbb{E}\left[\theta_r \circ (\theta_r - \widehat{\theta}_r)\right] A^*(r)A(r)\left(S(t-r)\mathbb{E}\left[\theta_r \circ (\theta_r - \widehat{\theta}_r)\right]\right)^*\,\mathrm{d}r \\
&= K(t) - \int_0^t S(t-r)\Phi(r)A^*(r)A(r)\Phi(r)S^*(t-r)\,\mathrm{d}r.
\end{aligned}$$

Since the covariance operator $K(t),\ t \in [0,T]$ is a mild solution to the equation

$$\dot{K}(t) = \mathcal{A}K(t) + K(t)\mathcal{A}^* + Q, \qquad K(0) = 0,$$

which takes the form

$$K(t) = \int_0^t S(t-r)QS^*(t-r)\,\mathrm{d}r$$

we obtain

$$\Phi(t) = \int_0^t S(t-r)\left(Q - \Phi(r)A^*(r)A(r)\Phi(r)\right)S^*(t-r)\,\mathrm{d}r$$

which is known to be equivalent to the weak form of the equation (2.3.5) (cf. [15]). □

2.4 Signal goverened by stochastic evolution equation driven by fractional noise

In this section we will apply the general results to signal governed by stochastic evolution equations driven by fractional noise.

First, in Section 2.4.1 the general results are applied to $2m$-th order stochastic parabolic equation driven by Gauss-Voltera noise on a bounded domain. In Corollary 2.4.1 the main results are specified to the case of pointwise observation of the signal. Another application is stated in Section 2.4.2, where the general results are applied to stochastic heat equation driven by fBm on an unbounded domain with pointwise observation of the signal again.

2.4.1 Stochastic parabolic equation on a bounded domain driven by Gauss-Volterra process

Consider the signal given by the following parabolic equation

$$\partial_t u = L_{2m} u + \eta \tag{2.4.1}$$

on $[0,T] \times \mathcal{D}$ with initial condition $u(0,\cdot) = 0$ and with the Dirichlet boundary condition

$$\left.\frac{\partial^k u}{\partial \mathbf{x}^k}\right|_{[0,T] \times \partial\mathcal{D}} = 0, \quad k \in \{0, \cdots, m-1\},$$

where $\frac{\partial}{\partial \mathbf{x}^k}$ denotes the conormal derivative. The domain $\mathcal{D} \subset \mathbb{R}^d$ is open and bounded with smooth boundary and L_{2m} is a differential operator uniformly elliptic of order $2m$,

$$L_{2m} = \sum_{|k| \le 2m} a_k(\cdot)\partial^k \tag{2.4.2}$$

with $a_k \in C_b^\infty(\mathcal{D})$. The noise η is Gauss-Volterra in time and may be white or correlated in space. It is formally given as

$$\eta(t,\cdot) = G\frac{\mathrm{d}}{\mathrm{d}t}B_t,$$

where $\{B_t, t \in [0,T]\}$ is an α-regular Gauss-Volterra process on H and $G \in \mathcal{L}(H)$. Therefore, parabolic equation (2.4.1) can be reformulated as the stochastic evolution equation

$$\mathrm{d}\theta_t = \mathcal{A}\theta_s \,\mathrm{d}s + G \,\mathrm{d}B_t, \quad t \in [0,T], \quad \theta_0 = 0, \tag{2.4.3}$$

where $\mathcal{A} = L_{2m}\big|_{Dom(\mathcal{A})}$ and

$$Dom(\mathcal{A}) = \left\{ f \in W^{2m,2}(\mathcal{D}) : \frac{\partial^k f}{\partial \mathbf{x}^k} = 0 \text{ on } \partial\mathcal{D} \text{ for } k \in \{0, \cdots, m-1\} \right\}. \tag{2.4.4}$$

The operator $\mathcal{A}$ generates an analytic semigroup $(S(t), t \in [0,T])$ on H.

Signal $\{\theta_t, t \in [0,T]\}$ is assumed to be the mild solution to (2.4.3). If the noise term satisfies the assumptions of Theorem 1.2.18 (e.g. $G \in \mathcal{L}_2(H)$), the mild solution exists and is H-measurable mean-square continuous process. If the observation is given by the equation (2.1.1) where $V = H = L^2(\mathcal{D})$, $\{W_t, t \in [0,T]\}$ is independent of $\{B_t, t \in [0,T]\}$ and $A : [0,T] \to \mathcal{L}(L^2(\mathcal{D}), \mathbb{R}^n)$ is strongly measurable and bounded, Theorems 2.1.1, 2.2.1 and Remark 2.1.3 may be applied.

However, it may be interesting to consider the case when the only accessible information comes from observation of the signal at given points $z_1, \ldots, z_n \in \mathcal{D}$. Then the signal process has to be more regular.

By the analyticity of semigroup S there exists $\lambda \in \mathbb{R}$ such that the operator $(\lambda I - \mathcal{A})$ is strictly positive. Therefore, for $\delta \geq 0$, we can define the Hilbert space

$$V_\delta = Dom((\lambda I - \mathcal{A})^\delta)$$

equipped with the graph norm topology.

For simplicity, assume that $m = 1$, i.e. (2.4.1) is a stochastic heat equation. If the condition (1.2.15) of Theorem 1.2.18 is satisfied for some $\gamma > 0$ such that

$$0 \leq \delta < \alpha + \frac{1}{2} - \gamma$$

and, moreover,

$$\delta > \frac{d}{4},$$

then the signal θ has a continuous version in V_δ that is continuously embedded into $C(\mathcal{D})$ by the Sobolev theorem. This follows from the fact that $V_\delta \subset W^{2\delta,2}(\mathcal{D})$ (cf. [47]). Such a choice of δ is possible if

$$\alpha + \frac{1}{2} - \gamma > \frac{d}{4} \tag{2.4.5}$$

(note that if $G \in \mathcal{L}_2(H)$ we may put $\gamma = 0$ and, on the other hand, for $\gamma = d/4$ we may consider arbitrary $G \in \mathcal{L}(H)$).

Then in Theorems 2.1.1 and 2.2.1 we may put $V = V_\delta \hookrightarrow C(\mathcal{D})$ and as an example of observation operator we may take $A(t) = A$ defined as

$$A\theta_t = (\theta_t(z_1), \ldots, \theta_t(z_n)), \tag{2.4.6}$$

where $z_1, \ldots, z_n \in \mathcal{D}$, which corresponds to pointwise observation of the signal process θ.

In this case the equations (2.1.2) and (2.1.3) can be rewritten according to the following theorem.

Corollary 2.4.1. *Let the signal $\theta = \{\theta_t, t \in [0,T]\}$ satisfy the stochastic evolution equation*

$$\mathrm{d}\theta_t = \mathcal{A}\theta_s \,\mathrm{d}s + G \,\mathrm{d}B_t, \quad \theta_0 = 0,$$

where $\{B_t, t \in [0,T]\}$ is a Gauss - Volterra process on $L^2(\mathcal{D})$, $G \in \mathcal{L}(L^2(\mathcal{D}))$ and $\mathcal{A} = L_2\big|_{Dom(\mathcal{A})}$ is given by (2.4.2) and (2.4.4). Further assume that condition (2.4.5) holds. Consider the observation process $\xi = \{\xi_t, t \in [0,T]\}$ given by (2.1.1) with operator $A(t) = A$ defined by (2.4.6). Then the filter $\widehat{\theta}$ satisfies stochastic integral equation

$$\widehat{\theta}_t = \sum_{j=1}^{n} \int_0^t \Phi_{z_j}(t,s) \,\mathrm{d}\xi_s^j - \sum_{j=1}^{n} \int_0^t \Phi_{z_j}(t,s)\widehat{\theta}_s(z_j) \,\mathrm{d}s, \quad t \in [0,T], \tag{2.4.7}$$

where $\Phi_{z_i}\colon \Lambda \to C(\mathcal{D})$ is defined as $\Phi_{z_i}(t,s) = \mathbb{E}[(\theta_s - \widehat{\theta}_s)(z_i)\theta_t]$ for all $(t,s) \in \Lambda$, $i = 1, \ldots, n$ and integral equation

$$\Phi_{z_i}(t,s) = E[\theta_s(z_i)\theta_t] - \sum_{j=1}^{n} \int_0^s \Phi_{z_j}(s,r)(z_i)\Phi_{z_j}(t,r) \,\mathrm{d}r, \quad i = 1, \ldots, n \tag{2.4.8}$$

is satisfied.

Proof. From (2.1.2), the definition of operator Φ, the continuous embedding $V_\delta \hookrightarrow C(\mathcal{D})$ and (2.4.6) we have

$$\begin{aligned}
\widehat{\theta}_t &= \int_0^t \Phi(t,s)A^* \,\mathrm{d}\xi_s - \int_0^t \Phi(t,s)A^*A\widehat{\theta}_s \,\mathrm{d}s \\
&= \sum_{j=1}^n \int_0^t \mathsf{E}\left[\theta_t \langle \theta_s - \widehat{\theta}_s, A_j\rangle_{V,V^*}\right] \mathrm{d}\xi_s - \sum_{j=1}^n \int_0^t \mathsf{E}\left[\theta_t \langle \theta_s - \widehat{\theta}_s, A_j\rangle_{V,V^*}\right] \langle \widehat{\theta}_s, A_j\rangle_{V,V^*} \,\mathrm{d}s \\
&= \sum_{j=1}^n \int_0^t \mathsf{E}\left[(\theta_s - \widehat{\theta}_s)(z_j)\theta_t\right] \mathrm{d}\xi_s - \sum_{j=1}^n \int_0^t \mathsf{E}\left[(\theta_s - \widehat{\theta}_s)(z_j)\theta_t\right] \widehat{\theta}_s(z_j) \,\mathrm{d}s
\end{aligned}$$

which concludes the proof of (2.4.7).

Analogously, using (2.1.3) we obtain

$$\begin{aligned}
\Phi_{z_i}(t,s) &= \Phi(t,s)A_i \\
&= \mathsf{E}\left[\theta_t \langle \theta_s, A_i\rangle_{V,V^*}\right] \\
&\quad - \sum_{j=1}^n \int_0^s \mathsf{E}\left[\theta_t \langle \theta_r - \widehat{\theta}_r, A_j\rangle_{V,V^*}\right] \langle \mathsf{E}\left[\theta_s \langle \theta_r - \widehat{\theta}_r, A_j\rangle_{V,V^*}\right], A_i\rangle_{V,V^*} \,\mathrm{d}r \\
&= \mathsf{E}\left[\theta_s(z_i)\theta_t\right] - \sum_{j=1}^n \int_0^s \mathsf{E}\left[(\theta_r - \widehat{\theta}_r)(z_j)\theta_t\right] \left[\left(\mathsf{E}\left[(\theta_r - \widehat{\theta}_r)(z_j)\theta_s\right]\right)(z_i)\right] \mathrm{d}r \\
&= \mathsf{E}\left[\theta_s(z_i)\theta_t\right] - \sum_{j=1}^n \int_0^s \Phi_{z_j}(t,r) \left[\Phi_{z_j}(s,r)(z_i)\right] \mathrm{d}r
\end{aligned}$$

which concludes the proof of (2.4.8). □

Note that if the driving process $B = B^h$ is a fractional Brownian motion by Theorem 1.2.26 the condition (2.4.5) reads

$$h > \frac{d}{4} + \gamma,$$

and we can work even with the singular fBm, i.e. $h \in (0,1)$.

It may be interesting to specify the covariances $\mathsf{E}\left[\theta_s(z_i)\theta_t\right]$ that appear in the equation (2.4.8). Suppose, for simplicity, that the driving process $B_t = B_t^h$ is a fractional Brownian motion with the Hurst parameter $h > 1/2$, $n = 1$ (i.e. the process $\theta = \{\theta_t, t \in [0,T]\}$ is observed at a single point $z_1 \in \mathcal{D}$) and the noise term G is Hilbert-Schmidt, i.e. it may be expressed as

$$[G(f)](\xi) = \int_{\mathcal{D}} k(\xi,\eta) f(\eta) \,\mathrm{d}\eta, \quad f \in H = L^2(\mathcal{D}), \tag{2.4.9}$$

where $k \in L^2(\mathcal{D} \times \mathcal{D})$. It is also well known that the semigroup $(S(t), t \in \mathbb{R})$ may be represented by a Green function $g : [0,T] \times \mathcal{D} \times \mathcal{D} \to \mathbb{R}$, that is,

$$[S(t)(f)](\xi) = \int_{\mathcal{D}} g(t,\xi,\eta) f(\eta) \,\mathrm{d}\eta, \quad f \in H,\ t > 0. \tag{2.4.10}$$

Therefore, the composition $S(t)G$ may be written as

$$[S(t)G(f)](\xi) = \int_{\mathcal{D}} \widetilde{g}(t,\xi,\eta) f(\eta) \,\mathrm{d}\eta, \quad f \in H,\ t > 0, \tag{2.4.11}$$

where the composition kernel $\widetilde{g}$ is given by

$$\widetilde{g}(t,\xi,\eta) = \int_{\mathcal{D}} g(t,\xi,\lambda)k(\lambda,\eta)\,\mathrm{d}\lambda. \tag{2.4.12}$$

Now it is standard to compute the covariance

$$\mathsf{E}\,[\theta_s(z_1)\theta_t]\,(\eta) = \int_0^s\int_0^t \phi_h(\lambda,r)\int_{\mathcal{D}}\widetilde{g}(s-r,z_1,\xi)\widetilde{g}(t-\lambda,\eta,\xi)\,\mathrm{d}\xi\,\mathrm{d}\lambda\,\mathrm{d}r, \tag{2.4.13}$$

for $(s,t)\in\Lambda$, $\eta\in\mathcal{D}$, where $\phi_h(\lambda,r) = h(2h-1)\mid\lambda-r\mid^{2h-2}$ and $(\eta,\lambda)\in\Lambda$.

2.4.2 Stochastic heat equation on an unbounded domain driven by fBm

Now, consider the signal governed by the following parabolic equation

$$\partial_t u = \Delta u + \eta$$

on $[0,T]\times\mathcal{D}$ with initial condition $u(0,\cdot)=0$ and with the Dirichlet boundary condition

$$u\Big|_{[0,T]\,\times\,\partial\mathcal{D}} = 0,$$

where Δ is a Laplace operator. The domain $\mathcal{D}\subset\mathbb{R}^d$ is open and has the C^m - extension property (cf. [23]). It is well known that this property is satisfied, for instance, if $\mathcal{D}=\mathbb{R}^d$, $\mathcal{D}=(\mathbb{R}^d)_+$ or if $\mathcal{D}$ is bounded with Lipschitz boundary. Unlike in the previous example, $\mathcal{D}$ is not necessarily bounded. The noise η is fractional in time and correlated in space. This system can be reformulated as the stochastic evolution equation

$$\mathrm{d}\theta_t = \Delta\theta_s\,\mathrm{d}s + G\,\mathrm{d}B_t^h, \quad \theta_0 = 0, \tag{2.4.14}$$

where $\{B_t^h, t\in[0,T]\}$ is a cylindrical fractional Brownian motion with Hurst parameter $h\in(0,1)$ on $\mathcal{D}$. The equation is considered in the Hilbert space $H=L^2(\mathcal{D})$. As in the previous example, the Dirichlet Laplacian generates an analytic semigroup $(S(t),t\geq 0)$ on H. The noise covariance G is supposed to be Hilbert-Schmidt on H. In virtue of Theorem 1.2.26 the equation (2.4.14) has a unique mild solution $\theta=\{\theta_t,t\in[0,T]\}$ which is continuous in time in the space $V_\delta = Dom\left((\beta-\Delta)^\delta\right)$ for a fixed β large enough and $0\leq\delta<h$. If, moreover,

$$h > \frac{d}{4}$$

then, due to Theorem 1.6.1 in [23], we can take $\delta\in(1/4,h)$ and the space V_δ is continuously embedded into the space of continuous functions $C(\mathcal{D})$ in the same way as in the previous example, i.e.

$$V_\delta \hookrightarrow C(\mathcal{D}) \hookrightarrow L^2(\mathcal{D}) = H.$$

Hence it make sense to consider observation operator defined as

$$A\theta_t = (\theta_t(z_1),\ldots,\theta_t(z_n)), \tag{2.4.15}$$

where $z_1, \dots, z_n \in \mathcal{D}$ which corresponds to pointwise observation of the signal process θ at these points. Therefore, to find the filter we can use again the Corollary 2.4.1.

Suppose, for simplicity, that $h > 1/2$ and $n = 1$ (i.e. the process $\theta = \{\theta_t, t \in [0,T]\}$ is observed at a single point $z_1 \in \mathcal{D}$). Since the noise term G is Hilbert-Schmidt the covariances $\mathsf{E}\,[\theta_s(z_i)\theta_t]$ that appear in the equation (2.4.8) may be expressed in the same way as in the previous example by (2.4.9), (2.4.10), (2.4.11), (2.4.12) and (2.4.13).

For example, if $\mathcal{D} = (0,\infty)$ (i.e. Δ is the Dirichlet Laplacian in $L^2(0,\infty)$), we have

$$g(t,\xi,\eta) = \frac{1}{\sqrt{4\pi t}}\left(e^{-\frac{(\xi-\eta)^2}{4t}} - e^{-\frac{(\xi+\eta)^2}{4t}}\right), \quad \xi,\eta \geq 0,$$

and if $\mathcal{D} = \mathbb{R}$ then g is the Gaussian kernel

$$g(t,\xi,\eta) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{(\xi-\eta)^2}{4t}}, \quad \xi,\eta \geq 0.$$

2.5 Signal governed by stochastic delayed evolution equation driven by Wiener process

In this final section of Chapter 2 the general results are applied to signal given by stochastic delayed evolution equation perturbed by Wiener process, therefore, the correlation between the increments of the signal is influenced by the delay in the drift instead of the time-correlated noise. Note that, because of the delay in the drift term, the infinite-dimensional Kalman-Bucy theorem (Theorem 2.3.1) can not be used directly. We will compute the mean value and covariances of the signal, which are necessary for application of Theorem 2.1.1 and are the main obstacle here. Then we will give an example of a delayed stochastic parabolic equation.

Let the signal $\theta = \{\theta_t, t \in [0,T]\}$ be an H - valued random process defined by delayed stochastic evolution equation

$$\begin{aligned} \mathrm{d}\theta_t &= (\mathcal{A}_0\theta_t + \mathcal{A}_1\theta_{t-r})\,\mathrm{d}t + G\,\mathrm{d}\mathcal{W}_t, \qquad t > 0, \\ \theta_t &= h_t, \qquad t \in [-r,0], \end{aligned} \tag{2.5.1}$$

where $r > 0$ denotes the delay, $\mathcal{A}_0$ is the infinitesimal generator of a strongly continuous semigroup $(S(t))_{t\in\mathbb{R}_+}$ in H, $\mathcal{A}_1 \in \mathcal{L}(H)$, $G \in \mathcal{L}(H)$, $\{\mathcal{W}_t, t \in [0,T]\}$ is an H-valued standard cylindrical Wiener process defined on a stochastic basis $(\Omega, F, P, (F_t))$ and $h \in \mathcal{C}([-r,0],H)$.

Assume $\forall t > 0$, $S(t)G \in \mathcal{L}_2(H)$ and

$$\int_0^{T_0} |\, S(r)G\,|^2_{\mathcal{L}_2(H)}\ \mathrm{d}r < \infty$$

for some $T_0 > 0$. Then for $t \in [0,r]$ the equation (2.5.1) has a unique mean-square continuous mild solution $\{\theta_t, t \in [0,r]\}$ given as

$$\theta_t = S(t)h_0 + \int_0^t S(t-v)\mathcal{A}_1 h_{v-r}\,\mathrm{d}v + \int_0^t S(t-v)G\,\mathrm{d}\mathcal{W}_v.$$

(cf. Section 1.2.3)

In fact, using mathematical induction the solution can be extended on the whole interval $[0,T]$. To be able to use results of the previous section we need also the mean and the covariance of this process. Let us denote

$$u_n(t) = \mathsf{E}\,[\theta_t], \qquad t \in [(n-1)r, nr], \tag{2.5.2}$$

$$\rho_{n,m}(t,s) = cov(\theta_t, \theta_s) = \mathsf{E}\,[(\theta_t - \mathsf{E}\,[\theta_t]) \circ (\theta_s - \mathsf{E}\,[\theta_s])], \qquad t \in [(n-1)r, nr], \; s \in [(m-1)r, mr], \tag{2.5.3}$$

where $n, m \in \mathbb{N}$, $\quad nr \leq T$, $mr \leq T$. Hereafter, if we use notation (2.5.2) and (2.5.3) we automatically assume that $nr \leq T$, $mr \leq T$, t, s lie in appropriate intervals, without loss of generality $m \leq n$ and if $m = n$ then $s \leq t$.

Theorem 2.5.1. *Assume $S(t)G \in \mathcal{L}_2(H)$ for $t > 0$ and*

$$\int_0^{T_0} \mid S(r)G \mid^2_{\mathcal{L}_2(H)} \,\mathrm{d}r < \infty$$

for some $T_0 > 0$. Then there exists a unique mean-square continuous solution $\{\theta_t, t \in [0,T]\}$ to the delayed stochastic evolution equation (2.5.1) and it satisfies

$$\theta_t = S(t)\theta_0 + \int_0^t S(t-v)\mathcal{A}_1\theta_{v-r}\,\mathrm{d}v + \int_0^t S(t-v)G\,\mathrm{d}\mathcal{W}_v, \qquad t \in [0,r], \tag{2.5.4}$$

$$\begin{aligned}\theta_t = {} & S(t-(n-1)r)\theta_{(n-1)r} + \int_{(n-1)r}^t S(t-v)\mathcal{A}_1\theta_{v-r}\,\mathrm{d}v \\ & + \int_{(n-1)r}^t S(t-v)G\,\mathrm{d}\mathcal{W}_v, \qquad t \in [(n-1)r, nr].\end{aligned} \tag{2.5.5}$$

Further we have

$$u_1(t) = S(t)h_0 + \int_0^t S(t-v)\mathcal{A}_1 h_{v-r}\,\mathrm{d}v, \tag{2.5.6}$$

$$u_n(t) = S(t-(n-1)r)u_{n-1}((n-1)r) + \int_{(n-1)r}^t S(t-v)\mathcal{A}_1 u_{n-1}(v-r)\,\mathrm{d}v \tag{2.5.7}$$

and for $m < n$

$$\rho_{1,1(t,s)} = \int_0^s S(t-v)QS^*(s-v)\,\mathrm{d}v, \qquad Q = GG^*, \tag{2.5.8}$$

$$\begin{aligned}\rho_{n,m}(t,s) = {} & S(t-(n-1)r)\rho_{n-1,m}((n-1)r, s) \\ & + \int_{(n-1)r}^t S(t-v)\mathcal{A}_1\rho_{n-1,m}(v-r,s)\,\mathrm{d}v,\end{aligned} \tag{2.5.9}$$

$$\begin{aligned}\rho_{n,n}(t,s) = {} & S(t-(n-1)r)\rho_{n-1,n-1}((n-1)r,(n-1)r)S^*(s-(n-1)r) \\ & + S(t-(n-1)r)\int_{(n-1)r}^s \rho_{n-1,n-1}((n-1)r, v-r)\mathcal{A}_1^* S^*(s-v)\,\mathrm{d}v \\ & + \int_{(n-1)r}^t S(t-v)\mathcal{A}_1\rho_{n-1,n-1}(v-r,(n-1)r)\,\mathrm{d}v S^*(s-(n-1)r) \\ & + \int_{(n-1)r}^t \int_{(n-1)r}^s S(t-v)\mathcal{A}_1\rho_{n-1,n-1}(v-r,w-r)\mathcal{A}_1^* S^*(s-w)\,\mathrm{d}w\,\mathrm{d}v \\ & + \int_{(n-1)r}^s S(t-v)QS^*(s-v)\,\mathrm{d}v.\end{aligned} \tag{2.5.10}$$

Proof. The formulas (2.5.4), (2.5.6) and (2.5.8) follow directly by the theory recalled in Section 1.2.3. Using the fact that θ_{v-r} is known at time $(n-1)r$ for all $v \in [(n-1)r, nr]$ formula (2.5.5) can be obtained in the same way as (2.5.4).

Formula (2.5.7) follows from (2.5.5). Indeed, we have

$$\begin{aligned} u_n(t) &= \mathsf{E}\left[S(t-(n-1)r)\theta_{(n-1)r} + \int_{(n-1)r}^{t} S(t-v)\mathcal{A}_1\theta_{v-r}\,\mathrm{d}\right] \\ &\quad + \mathsf{E}\left[\int_{(n-1)r}^{t} S(t-v)G\,\mathrm{d}\mathcal{W}_v\right] \\ &= S(t-(n-1)r)\,\mathsf{E}\left[\theta_{(n-1)r}\right] + \int_{(n-1)r}^{t} S(t-v)\mathcal{A}_1\,\mathsf{E}\left[\theta_{v-r}\right]\mathrm{d}v \\ &= S(t-(n-1)r)u_{n-1}((n-1)r) + \int_{(n-1)r}^{t} S(t-v)\mathcal{A}_1 u_{n-1}(v-r)\,\mathrm{d}v. \end{aligned}$$

Using the independence of increments of Wiener process we can obtain (2.5.9) as

$$\begin{aligned} \rho_{n,m}(t,s) &= cov(\theta_t, \theta_s) \\ &= cov\Bigg(S(t-(n-1)r)\theta_{(n-1)r} + \int_{(n-1)r}^{t} S(t-v)\mathcal{A}_1\theta_{v-r}\,\mathrm{d}v \\ &\qquad + \int_{(n-1)r}^{t} S(t-v)G\,\mathrm{d}\mathcal{W}_v\ ,\ \theta_s\Bigg) \\ &= S(t-(n-1)r)cov(\theta_{(n-1)r}, \theta_s) + \int_{(n-1)r}^{t} S(t-v)\mathcal{A}_1 cov(\theta_{v-r}, \theta_s)\,\mathrm{d}v \\ &= S(t-(n-1)r)\rho_{n-1,m}((n-1)r, s) + \int_{(n-1)r}^{t} S(t-v)\mathcal{A}_1\rho_{n-1,m}(v-r, s)\,\mathrm{d}v. \end{aligned}$$

Similarly, we can show (2.5.10). We have

$$\begin{aligned} \rho_{n,n}(t,s) &= cov(\theta_t, \theta_s) \\ &= cov\ \Bigg(S(t-(n-1)r)\theta_{(n-1)r} + \int_{(n-1)r}^{t} S(t-v)\mathcal{A}_1\theta_{v-r}\,\mathrm{d}v \\ &\qquad + \int_{(n-1)r}^{t} S(t-v)G\,\mathrm{d}\mathcal{W}_v, \\ &\qquad S(s-(n-1)r)\theta_{(n-1)r} + \int_{(n-1)r}^{s} S(s-v)\mathcal{A}_1\theta_{v-r}\,\mathrm{d}v \\ &\qquad + \int_{(n-1)r}^{s} S(s-v)G\,\mathrm{d}\mathcal{W}_v\Bigg), \end{aligned}$$

which is equal to

$$
\begin{aligned}
&S(t-(n-1)r)cov(\theta_{(n-1)r},\theta_{(n-1)r})S^*(s-(n-1)r)\\
&\quad+S(t-(n-1)r)\int_{(n-1)r}^{s} cov\Big(\theta_{(n-1)r},\theta_{v-r}\Big)\mathcal{A}_1^*S^*(s-v)\,\mathrm{d}v\\
&\quad+\int_{(n-1)r}^{t} S(t-v)\mathcal{A}_1 cov\Big(\theta_{v-r},\theta_{n-1)r}\Big)\,\mathrm{d}vS^*(s-(n-1)r)\\
&\quad+\int_{(n-1)r}^{t}\int_{(n-1)r}^{s} S(t-v)\mathcal{A}_1 cov(\theta_{v-r},\theta_{w-r})\mathcal{A}_1^*S^*(s-w)\,\mathrm{d}w\,\mathrm{d}v\\
&\quad+cov\Bigg(\int_{(n-1)r}^{t} S(t-v)G\,\mathrm{d}\mathcal{W}_v,\int_{(n-1)r}^{s} S(s-v)G\,\mathrm{d}\mathcal{W}_v\Bigg)\\
&=S(t-(n-1)r)\rho_{n-1,n-1}((n-1)r,(n-1)r)S^*(s-(n-1)r)\\
&\quad+S(t-(n-1)r)\int_{(n-1)r}^{s}\rho_{n-1,n-1}((n-1)r,v-r)\mathcal{A}_1^*S^*(s-v)\,\mathrm{d}v\\
&\quad+\int_{(n-1)r}^{t} S(t-v)\mathcal{A}_1\rho_{n-1,n-1}(v-r,(n-1)r)\,\mathrm{d}vS^*(s-(n-1)r)\\
&\quad+\int_{(n-1)r}^{t}\int_{(n-1)r}^{s} S(t-v)\mathcal{A}_1\rho_{n-1,n-1}(v-r,w-r)\mathcal{A}_1^*S^*(s-w)\,\mathrm{d}w\,\mathrm{d}v\\
&\quad+\int_{(n-1)r}^{s} S(t-v)QS^*(s-v)\,\mathrm{d}v.
\end{aligned}
$$

□

Consider a signal given by an analogous parabolic equation as in Section 2.4.1 but with certain delay $r>0$ and delayed drop rate $d>0$, i.e.

$$\partial_t u = L_{2m}u - du(t-r)+\eta \tag{2.5.11}$$

on $[0,T]\ \times\ \mathcal{D}$ with initial condition

$$u(t)=h,\quad t\in[-r,0],\quad h\in\mathcal{C}([-r,0],L^2(\mathcal{D}))$$

and with the Dirichlet boundary condition

$$\frac{\partial^k u}{\partial \mathbf{x}^k}\bigg|_{[0,T]\ \times\ \partial\mathcal{D}}=0,\quad k\in\{0,\cdots,m-1\},$$

where $\dfrac{\partial}{\partial \mathbf{x}^k}$ denotes the conormal derivative. The domain $\mathcal{D}\subset\mathbb{R}^d$ is open and bounded with smooth boundary and L_{2m} is a differential operator uniformly elliptic of order $2m$, given by (2.4.2).

In this example, the noise η is assumed to be given by cylindrical Wiener process. The system can be reformulated as the stochastic delayed evolution equation (2.5.1) in $H=L^2(\mathcal{D})$. The noise η is then formally given as

$$\eta(t,\cdot)=G\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{W}_t,$$

where $G\in\mathcal{L}(H)$ and $\{\mathcal{W}_t,t\in[0,T]\}$ is an H - valued standard cylindrical Wiener process defined on a stochastic basis $(\Omega,F,P,(F_t))$, $\mathcal{A}_0 = L_{2m}\Big|_{Dom(\mathcal{A})}$

with domain (2.4.4) and $\mathcal{A}_1 = -dI$. The operator $\mathcal{A}_0$ generates an analytic semigroup $(S(t), t \in [0,T])$ on H and $\mathcal{A}_1 \in \mathcal{L}(H)$.

Now, if the observation is given by the equation (2.1.1) where $V = H = L^2(\mathcal{D})$, $\{W_t, t \in [0,T]\}$ is independent of $\{\mathcal{W}_t, t \in [0,T]\}$ and $A : [0,T] \to \mathcal{L}(L^2(\mathcal{D}), \mathbb{R}^n)$ is strongly measurable and bounded then Theorems 2.1.1, 2.2.1 may be applied. Signal expectation $\mathsf{E}[\theta_t]$ and covariance $K^\theta(t,s)$ are given by Theorem 2.5.1.

3. Continuous dependence on a parameter

In this chapter we assume that the signal and the observation process depend continuously on a parameter. We show that under appropriate conditions the continuous dependence transfers to the covariance operator Φ and to the filter.

The chapter is divided into four sections. In the first part the basic setting is explained in detail and the main filtering result from the second chapter is applied.

Section 3.2 deals with the continuous dependence of the covariance of observation error which is proved in Theorem 3.2.1. As mentioned earlier, this mapping satisfies a nonlinear integral equation with non-Lipschitz right-hand side and, therefore, it does not seem to be possible to proceed in a standard way by means of the Gronwall lemma. Hence a method based on compactness of the family of solutions is used. The heart of the proof is based on Lemma 3.2.2 which is proved by Arzela-Ascoli theorem for mappings taking values in operator spaces, utilizing so-called collective compactness of solutions and their adjoints.

Section 3.3 contains the proof of continuous dependence of the filter (Theorem 3.3.1) which is the main result of the chapter.

In Section 3.4, these results are applied to the signal given by linear SPDE driven by cylindrical fractional Brownian motion. Two examples of signal are then studied in more detail: The heat equation perturbed either with distributed (in 3.4.1) or by pointwise fractional noise (in 3.4.2). Observations are finite-dimensional and the case of pointwise observation at some points in the domain, that my depend on the parameter, is also considered.

3.1 Preliminaries

Let $H = (H, \langle \cdot,\cdot \rangle_H, \|\cdot\|_H)$ be a separable Hilbert space. Consider a selfadjoint positive operator $\mathcal{B}$ on H with a compact resolvent. For $\alpha > 0$ consider a Hilbert space $V_\alpha = (V_\alpha, \langle \cdot,\cdot \rangle_{V_\alpha}, \|\cdot\|_{V_\alpha})$ defined by the fractional power of operator $\mathcal{B}$ as $V_\alpha = Dom(\mathcal{B}^\alpha)$ equipped with a graph norm $\|\cdot\|_{V_\alpha}$. Then (H, V_α) form together a rigged separable Hilbert space such that $V_\alpha \subset H$ and identifying H with the dual H^* the embeddings

$$V_\alpha \hookrightarrow H = H^* \hookrightarrow V_\alpha^*$$

are continuous and dense. The duality pairing between V_α and V_α^* is defined by the usual extension of the form $\langle u, v \rangle_{V,V^*} = \langle u, v \rangle_H$ for $u \in V_\alpha \subset H$ and $v \in H \subset V_\alpha^*$.

Consider a stochastic basis $(\Omega, F, P, (F_t))$, a compact set of parameters Λ and a family of signals $\{\theta_t^\lambda, t \in [0,T], \lambda \in \Lambda\}$ that are (F_t) - progressively measurable centered Gaussian processes with paths P-a.s. in $L^2([0,T], V_{\alpha+\vartheta})$ for some $\alpha > \vartheta > 0$ such that

$$\sup_{\lambda \in \Lambda, t \in [0,T]} \mathsf{E} \left\| \theta_t^\lambda \right\|_{V_{\alpha+\vartheta}}^2 < \infty \tag{3.1.1}$$

and for every $\epsilon > 0$ there exists $\delta > 0$ such that for all $\lambda \in \Lambda$ and for all

$s, t \in [0, T]$, $|t - s| < \delta$

$$\mathsf{E} \left|\left|\theta_t^\lambda - \theta_s^\lambda\right|\right|_{V_\alpha}^2 < \epsilon. \tag{3.1.2}$$

The condition (3.1.1) will be referred to as uniform boundedness of $\left\{\theta^\lambda\right\}$ and the property (3.1.2) as (mean-square) equicontinuity.

For every $\lambda \in \Lambda$ let $\xi^\lambda = \{\xi_t^\lambda, t \in [0, T]\}$ denote an $\mathbb{R}^n$ - valued observation process given as

$$\xi_t^\lambda = \int_0^t A^\lambda(s)\theta_s^\lambda \,\mathrm{d}s + W_t, \quad t \in [0, T], \tag{3.1.3}$$

where $\left(A^\lambda(s)\right)_{s\in[0,T]}$ is a family of linear operators $V_\alpha \to \mathbb{R}^n$ such that for every $\lambda \in \Lambda$ the mapping $t \to A^\lambda(t)$ is strongly measurable and uniformly bounded, that is,

$$\sup_{t\in[0,T],\lambda\in\Lambda} \left|\left|A^\lambda(t)\right|\right|_{\mathcal{L}(V_\alpha,\mathbb{R}^n)} < \infty.$$

Here $W = \{W_t, t \in [0, T]\}$ is a standard $\mathbb{R}^n$ - valued Wiener process independent of the family of signals $\{\theta^\lambda\}$.

Further, assume that for each $\lambda \in \Lambda$ and $t \in [0, T]$ the operator $A^\lambda(t)$ can be decomposed into functionals $A_1^\lambda(t), \ldots, A_n^\lambda(t) \in V_\alpha^*$ such that

$$A^\lambda(t)b = (\langle b, A_1^\lambda(t)\rangle_{V,V^*}, \ldots, \langle b, A_n^\lambda(t)\rangle_{V,V^*})^T$$

for all $b \in V_\alpha$. The dual operator $(A^\lambda)^*(t)$: $\mathbb{R}^n \to V_\alpha^*$ then satisfies

$$(A^\lambda)^*(t)z = \sum_{i=1}^n z_i A_i^\lambda(t) \tag{3.1.4}$$

for all $z \in \mathbb{R}^n$.

We will study the optimal filter $\widehat{\theta}_t^\lambda$, which is defined as

$$\widehat{\theta}_t^\lambda = \mathbb{E}[\theta_t^\lambda | F_t^{\xi_\lambda}],$$

where $(F_t^{\xi_\lambda})_{t\in[0,T]}$ is the filtration generated by the observation process ξ^λ.

Set $K^\lambda(t, s) = \mathbb{E}[\theta_t^\lambda \circ \theta_s^\lambda]$, $t, s \in [0, T]$, $\lambda \in \Lambda$. In virtue of the uniform boundedness of the processes $\{\theta_t^\lambda, t \in [0, T], \lambda \in \Lambda\}$ the family of mappings $K^\lambda : [0, T]^2 \to \mathcal{L}(V_\alpha^*, V_\alpha)$, $\lambda \in \Lambda$ is uniformly bounded in $\mathcal{C}([0, T]^2, \mathcal{L}(V_\alpha^*, V_\alpha))$, i.e.

$$\sup_{\lambda\in\Lambda} \left|\left|K^\lambda\right|\right|_{\mathcal{C}([0,T]^2,\mathcal{L}(V_\alpha^*,V_\alpha))} < \infty.$$

Then for a fixed value of the parameter λ we can apply Theorem 2.1.1.

Theorem 3.1.1. *Let $\Lambda = \{(t, s) \in [0, T]^2; 0 \le s \le t \le T\}$ and $\lambda \in \Lambda$. The filter $\widehat{\theta}^\lambda$ satisfies the stochastic integral equation*

$$\widehat{\theta}_t^\lambda = \int_0^t \Phi^\lambda(t, s)(A^\lambda)^*(s) \,\mathrm{d}\xi_s - \int_0^t \Phi^\lambda(t, s)(A^\lambda)^*(s)A^\lambda(s)\widehat{\theta}_s^\lambda \,\mathrm{d}s, \quad t \in [0, T], \tag{3.1.5}$$

where operator Φ^λ: $\Lambda \to \mathcal{L}(V_\alpha^*, V_\alpha)$ defined as $\Phi^\lambda(t, s) = \mathbb{E}[\theta_t^\lambda \circ (\theta_s^\lambda - \widehat{\theta}_s^\lambda)]$ for all $(t, s) \in \Lambda$ is strongly continuous and satisfies the integral equation

$$\Phi^\lambda(t, s) = K^\lambda(t, s) - \sum_{j=1}^n \int_0^s \left(\Phi^\lambda(t, r)A_j^\lambda(r)\right) \circ \left(\Phi^\lambda(s, r)A_j^\lambda(r)\right) \,\mathrm{d}r, \quad (t, s) \in \Lambda. \tag{3.1.6}$$

Moreover, for all $t \in [0,T]$, $\Phi^\lambda(t,t)$ is the covariance of the estimation error at time $t \in [0,T]$, that is,

$$\Phi^\lambda(t,t) = \mathbb{E}\left[(\theta_t^\lambda - \widehat{\theta}_t^\lambda) \circ (\theta_t^\lambda - \widehat{\theta}_t^\lambda)\right] \tag{3.1.7}$$

holds.

Remark 3.1.2. According to Lemma 2.2 in [27] process $\{\tilde{W}_t, t \in [0,T]\}$ defined as

$$\tilde{W}_t^\lambda = \xi_t^\lambda - \int_0^t \mathbb{E}[A^\lambda(r)\theta_r^\lambda \mid F_t^{\xi_\lambda}]\, \mathrm{d}r = \xi_t^\lambda - \int_0^t A^\lambda(r)\widehat{\theta}_r^\lambda\, \mathrm{d}r. \tag{3.1.8}$$

is $\mathbb{R}^n$ - valued $(F_t^{\xi_\lambda})$ - standard Wiener process called innovation process. The formula (3.1.5) can be rewritten as

$$\widehat{\theta}_t^\lambda = \int_0^t \Phi^\lambda(t,s)(A^\lambda)^*(s)\, \mathrm{d}\tilde{W}_s^\lambda, \quad t \in [0,T]. \tag{3.1.9}$$

3.2 Continuous dependence for the covariance

In the present Section, continuous dependence of the covariance operator Φ^λ on $\lambda \in \Lambda$ is shown. The main result of the section is stated below.

Theorem 3.2.1. *Under the assumptions in Section 3.1 if*

$$K^\lambda \to K^{\lambda_0}, \qquad \lambda \to \lambda_0, \quad \lambda \in \Lambda \tag{3.2.1}$$

in $\mathcal{C}\left([0,T]^2, \mathcal{L}(V_\alpha^*, V_\alpha)\right)$ and

$$A^\lambda \to A^{\lambda_0}, \qquad \lambda \to \lambda_0, \quad \lambda \in \Lambda \tag{3.2.2}$$

in $\mathcal{C}\left([0,T], \mathcal{L}(V_\alpha, \mathbb{R}^n)\right)$ then

$$\Phi^\lambda \to \Phi^{\lambda_0}, \qquad \lambda \to \lambda_0, \quad \lambda \in \Lambda \tag{3.2.3}$$

in $\mathcal{C}\left(\Lambda, \mathcal{L}(V_\alpha^*, V_\alpha)\right)$.

For the proof of Theorem 3.2.1 we need following lemma.

Lemma 3.2.2. *Under the assumptions in Section 3.1 the set of functions $\{\Phi^\lambda\}_{\lambda\in\Lambda}$ is relatively compact in $\mathcal{C}\left(\Lambda, \mathcal{L}(V_\alpha^*, V_\alpha)\right)$.*

Proof. In virtue of the infinite-dimensional version of the Arzela-Ascoli theorem the statement of Lemma 3.2.2 holds if and only if the family $\{\Phi^\lambda\}_{\lambda\in\Lambda}$ is uniformly bounded and equicontinuous in $\mathcal{C}\left(\Lambda, \mathcal{L}(V_\alpha^*, V_\alpha)\right)$ and there exists $(t,s) \in \Lambda$ such that $\{\Phi^\lambda(t,s)\}_{\lambda\in\Lambda}$ is relatively compact in $\mathcal{L}(V_\alpha^*, V_\alpha)$.

First we show that mappings $(t,s,\lambda) \to \Phi^\lambda(t,s)$, $\lambda \in \Lambda$ are uniformly bounded on $\Lambda \times \Lambda$. For arbitrary $x, y \in V_\alpha^*$, $(t,s) \in \Lambda$ and for all $\lambda \in \Lambda$ using Cauchy-Schwarz inequality we obtain

$$
\begin{aligned}
&|\langle \left(\sum_{j=1}^n \int_0^s \left(\Phi^\lambda(t,r)A_j^\lambda(r)\right) \circ \left(\Phi^\lambda(s,r)A_j^\lambda(r)\right) \mathrm{d}r \right) x,y\rangle_{V,V^*}| \\
&\le \sum_{j=1}^n \int_0^s |\langle \left[\left(\Phi^\lambda(t,r)A_j^\lambda(r)\right) \circ \left(\Phi^\lambda(s,r)A_j^\lambda(r)\right) \right] x,y\rangle_{V,V^*}| \,\mathrm{d}r \\
&= \sum_{j=1}^n \int_0^s |\langle \Phi^\lambda(t,r)A_j^\lambda(r)\langle \Phi^\lambda(s,r)A_j^\lambda(r), x\rangle_{V,V^*}, y\rangle_{V,V^*}| \,\mathrm{d}r \\
&= \sum_{j=1}^n \int_0^s |\langle \Phi^\lambda(t,r)A_j^\lambda(r), y\rangle_{V,V^*}| \; |\langle \Phi^\lambda(s,r)A_j^\lambda(r), x\rangle_{V,V^*}| \,\mathrm{d}r \\
&\le \sum_{j=1}^n \left(\int_0^s \left(\langle \Phi^\lambda(t,r)A_j^\lambda(r), y\rangle_{V,V^*} \right)^2 \mathrm{d}r \right)^{\frac{1}{2}} \left(\int_0^s \left(\langle \Phi^\lambda(s,r)A_j^\lambda(r), x\rangle_{V,V^*} \right)^2 \mathrm{d}r \right)^{\frac{1}{2}} \\
&= \sum_{j=1}^n \left(\int_0^s \langle \left[\left(\Phi^\lambda(t,r)A_j^\lambda(r)\right) \circ \left(\Phi^\lambda(t,r)A_j^\lambda(r)\right) \right] y,y\rangle_{V,V^*} \,\mathrm{d}r \right)^{\frac{1}{2}} \\
&\qquad\qquad \left(\int_0^s \langle \left[\left(\Phi^\lambda(s,r)A_j^\lambda(r)\right) \circ \left(\Phi^\lambda(s,r)A_j^\lambda(r)\right) \right] x,x\rangle_{V,V^*} \,\mathrm{d}r \right)^{\frac{1}{2}}. \qquad (3.2.4)
\end{aligned}
$$

In the last inequality we can increase the upper bound of the first integral from s to t because the integrand is nonnegative. Hence we only need to estimate the term

$$
\int_0^t \langle \left[\left(\Phi^\lambda(t,r)A_j^\lambda(r)\right) \circ \left(\Phi^\lambda(t,r)A_j^\lambda(r)\right) \right] x,x\rangle_{V,V^*} \,\mathrm{d}r
$$

for $j = 1, \ldots, n$ and $x \in V_\alpha^*$. By (3.1.6) and the uniform boundedness of the $\{K^\lambda\}_{\lambda\in\Lambda}$ we have

$$
\begin{aligned}
0 &\le \int_0^t \langle \left[\left(\Phi^\lambda(t,r)A_j^\lambda(r)\right) \circ \left(\Phi^\lambda(t,r)A_j^\lambda(r)\right) \right] x,x\rangle_{V,V^*} \,\mathrm{d}r \\
&\le \sum_{j=1}^n \int_0^t \langle \left[\left(\Phi^\lambda(t,r)A_j^\lambda(r)\right) \circ \left(\Phi^\lambda(t,r)A_j^\lambda(r)\right) \right] x,x\rangle_{V,V^*} \,\mathrm{d}r \\
&= \langle K^\lambda(t,t)x,x\rangle_{V,V^*} - \langle \Phi^\lambda(t,t)x,x\rangle_{V,V^*} \\
&\le \langle K^\lambda(t,t)x,x\rangle_{V,V^*} \le C_1(T)\|x\|_{V_\alpha^*}^2, \quad t \in [0,T], \quad C_1(T) < \infty. \qquad (3.2.5)
\end{aligned}
$$

Now, by (3.1.6), (3.2.4), (3.2.5) and again by the uniform boundedness of the family $\{K^\lambda\}_{\lambda\in\Lambda}$ we obtain

$$
\begin{aligned}
&|\langle \Phi^\lambda(t,s)x,y\rangle_{V,V^*}| \\
&\le \|K^\lambda(t,s)\|_{\mathcal{L}(V_\alpha^*,V)}\|x\|_{V_\alpha^*}\|y\|_{V_\alpha^*} \\
&\quad + |\langle \left(\sum_{j=1}^n \int_0^s \left(\Phi^\lambda(t,r)A_j^\lambda(r)\right) \circ \left(\Phi^\lambda(s,r)A_j^\lambda(r)\right) \mathrm{d}r \right) x,y\rangle_{V,V^*}| \\
&\le C_2(T)\|x\|_{V_\alpha^*}\|y\|_{V_\alpha^*}, \quad C_2(T) < \infty
\end{aligned}
$$

for all $\lambda \in \Lambda$, $x, y \in V_\alpha^*$ and $(t,s) \in \Lambda$, which proves that the family of operators $\{\Phi^\lambda\}_{\lambda\in\Lambda}$ is uniformly bounded in $\mathcal{C}\left(\Lambda, \mathcal{L}(V_\alpha^*, V_\alpha)\right)$.

Further, we show that the family $\{\Phi^\lambda\}_{\lambda\in\Lambda}$ is equicontinuous on Λ (as mappings with values in $\mathcal{L}(V_\alpha^*, V_\alpha)$). To this end, it is enough to show equicontinuity of $\{\Phi^\lambda(\cdot, s)\}_{\lambda\in\Lambda}$ on $[s,T]$ for all $s \in [0,T]$ and equicontinuity of $\{\Phi^\lambda(t,\cdot)\}_{\lambda\in\Lambda}$ on $[0,t]$ for all $t \in [0,T]$. By definition of Φ^λ and Cauchy - Schwarz inequality we have

$$\begin{aligned}
&\left|\langle\Big(\Phi^\lambda(t_1,s) - \Phi^\lambda(t_2,s)\Big)x,y\rangle_{V,V^*}\right| = \left|\mathsf{E}\,\Big(\langle\theta_s^\lambda - \widehat{\theta}_s^\lambda, x\rangle_{V,V^*}\langle\theta_{t_1}^\lambda - \theta_{t_2}^\lambda, y\rangle_{V,V^*}\Big)\right| \\
&\leq \sqrt{\mathsf{E}\,\left|\langle\theta_s^\lambda - \widehat{\theta}_s^\lambda, x\rangle_{V,V^*}\right|^2}\sqrt{\mathsf{E}\,\left|\langle\theta_{t_1}^\lambda - \theta_{t_2}^\lambda, y\rangle_{V,V^*}\right|^2} \\
&\leq \sqrt{\mathsf{E}\,||\theta_s^\lambda||_{V_\alpha}^2}\,||x||_{V_\alpha^*}\sqrt{\mathsf{E}\,\left|\langle\theta_{t_1}^\lambda - \theta_{t_2}^\lambda, y\rangle_{V,V^*}\right|^2}
\end{aligned}$$

for all $s \in [0,T]$, $t_1, t_2 \in [s,T]$, $\lambda \in \Lambda$ and all $x, y \in V_\alpha^*$. Therefore, using the mean - square equicontinuity and uniform boundedness of signals $\{\theta_t^\lambda, t \in [0,T], \lambda \in \Lambda\}$, for $\epsilon > 0$ we find $\delta > 0$ such that

$$\left|\langle\Big(\Phi^\lambda(t_1,s) - \Phi^\lambda(t_2,s)\Big)x,y\rangle_{V,V^*}\right| < \epsilon\,||x||_{V_\alpha^*}\,||y||_{V_\alpha^*} \tag{3.2.6}$$

for every $\lambda \in \Lambda$ and all $s \in [0,T]$, $t_1, t_2 \in [s,T]$, $|t_2 - t_1| < \delta$, which proves equicontinuity of $\{\Phi^\lambda(\cdot, s)\}_{\lambda\in\Lambda}$.

Furthermore, we have

$$\begin{aligned}
&\left|\left|\Big(\Phi^\lambda(t,s_1) - \Phi^\lambda(t,s_2)\Big)x\right|\right|_{V_\alpha} \\
&\leq \left|\left|\mathsf{E}\,\Big(\theta_t^\lambda\langle\theta_{s_1}^\lambda - \theta_{s_2}^\lambda, x\rangle_{V,V^*}\Big)\right|\right|_{V_\alpha} + \left|\left|\mathsf{E}\,\Big(\theta_t^\lambda\langle\widehat{\theta}_{s_2}^\lambda - \widehat{\theta}_{s_1}^\lambda, x\rangle_{V,V^*}\Big)\right|\right|_{V_\alpha} \\
&\leq \sqrt{\mathsf{E}\,||\theta_s^\lambda||_{V_\alpha}^2}\left(\sqrt{\mathsf{E}\,\left|\langle\theta_{s_1}^\lambda - \theta_{s_2}^\lambda, x\rangle_{V,V^*}\right|^2} + \sqrt{\mathsf{E}\,\left|\langle\widehat{\theta}_{s_2}^\lambda - \widehat{\theta}_{s_1}^\lambda, x\rangle_{V,V^*}\right|^2}\right)
\end{aligned} \tag{3.2.7}$$

for all $t \in [0,T]$, $s_1, s_2 \in [0,t]$, $\lambda \in \Lambda$ and all $x \in V_\alpha^*$. Using (3.1.9), (3.1.4), Itô isometry and the uniform boundednes of $\left(A^\lambda\right)_{\lambda\in\Lambda}$ and $\{\Phi^\lambda\}_{\lambda\in\Lambda}$ we have

$$\begin{aligned}
&\mathsf{E}\,\left|\langle\widehat{\theta}_{s_2}^\lambda - \widehat{\theta}_{s_1}^\lambda, x\rangle_{V,V^*}\right|^2 \\
&\leq 2\sum_{j=1}^n \mathsf{E}\,\left|\int_0^{s_1}\langle\Big(\Phi^\lambda(s_2,r) - \Phi^\lambda(s_1,r)\Big)A_j^\lambda(s), x\rangle_{V,V^*}\,\mathrm{d}\tilde{W}_{j,r}^\lambda\right|^2 \\
&\quad + 2\sum_{j=1}^n \mathsf{E}\,\left|\int_{s_1}^{s_2}\langle\Phi^\lambda(s_2,r)A_j^\lambda(s), x\rangle_{V,V^*}\,\mathrm{d}\tilde{W}_{j,r}^\lambda\right|^2 \\
&= 2\sum_{j=1}^n \int_0^{s_1}\left|\langle\Big(\Phi^\lambda(s_2,r) - \Phi^\lambda(s_1,r)\Big)A_j^\lambda(s), x\rangle_{V,V^*}\right|^2\,\mathrm{d}r \\
&\quad + 2\sum_{j=1}^n \int_{s_1}^{s_2}\left|\langle\Phi^\lambda(s_2,r)A_j^\lambda(s), x\rangle_{V,V^*}\right|^2\,\mathrm{d}r \\
&\leq C(T)\,||x||_{V_\alpha^*}^2\left(\int_0^{s_1}\left|\left|\Phi^\lambda(s_2,r) - \Phi^\lambda(s_1,r)\right|\right|_{\mathcal{C}([r,T],\mathcal{L}(V_\alpha^*,V_\alpha))}^2\,\mathrm{d}r + |s_2 - s_1|\right),
\end{aligned} \tag{3.2.8}$$

for $0 \leq s_1 < s_2 \leq T$ where $C(T) < \infty$.

Using (3.2.7), (3.2.8), the equicontinuity of $\{\Phi^\lambda(\cdot, r)\}_{\lambda\in\Lambda}$ on $[r,T]$ for all $r \in [0,T]$ shown in (3.2.6) and the mean-square equicontinuity and uniform boundedness of signals $\{\theta_t^\lambda, t \in [0,T], \lambda \in \Lambda\}$, for all $\epsilon > 0$ we can find $\delta > 0$ such that

$$\left|\left|\Big(\Phi^\lambda(t,s_1) - \Phi^\lambda(t,s_2)\Big)x\right|\right|_{V_\alpha} < \epsilon\, ||x||_{V_\alpha^*} \tag{3.2.9}$$

for every $\lambda \in \Lambda$ and all $t \in [0,T]$, $s_1, s_2 \in [0,t]$, $|s_2 - s_1| < \delta$ and $x \in V_\alpha^*$. This completes the proof of equicontinuity of $\{\Phi^\lambda\}_{\lambda\in\Lambda}$ on Λ.

It remains to show that there exists $(t,s) \in \Lambda$ such that $\{\Phi^\lambda(t,s)\}_{\lambda\in\Lambda}$ is relatively compact in $\mathcal{L}(V_\alpha^*, V_\alpha)$. This property is equivalent to collective compactness of the family $\{\big[\Phi^\lambda(t,s), (\Phi^\lambda)^*(t,s)\big]\}_{\lambda\in\Lambda}$ in $\mathcal{L}(V_\alpha^*, V_\alpha) \times \mathcal{L}(V_\alpha, V_\alpha^*)$, see Theorem B.2. Employing the compactness of embeddings

$$V_{\alpha+\vartheta} \hookrightarrow V_\alpha, \qquad V_{\alpha-\vartheta}^* \hookrightarrow V_\alpha^*$$

(cf. (3.1.1) for the definition of ϑ) it is enough to show $Range(\Phi^\lambda(t,s)) \subset V_{\alpha+\vartheta}$, $Range((\Phi^\lambda)^*(t,s)) \subset V_{\alpha-\vartheta}^*$ and that the uniform boundedness

$$\sup_{\lambda\in\Lambda} \left|\left|\Phi^\lambda(t,s)\right|\right|_{\mathcal{L}(V_\alpha^*, V_{\alpha+\vartheta})} < \infty \tag{3.2.10}$$

and

$$\sup_{\lambda\in\Lambda} \left|\left|(\Phi^\lambda)^*(t,s)\right|\right|_{\mathcal{L}(V_\alpha, V_{\alpha-\vartheta}^*)} < \infty \tag{3.2.11}$$

holds for arbitrarily chosen $(t,s) \in \Lambda$. We have that

$$\begin{aligned}
\left|\left|\Phi^\lambda(t,s)\right|\right|_{\mathcal{L}(V_\alpha^*, V_{\alpha+\vartheta})} &= \sup_{||x||_{V_\alpha^*}\le 1} \left|\left|\Phi^\lambda(t,s)x\right|\right|_{V_{\alpha+\vartheta}} \\
&\le \sup_{||x||_{V_\alpha^*}\le 1} \mathsf{E} \left|\left|\theta_t^\lambda\right|\right|_{V_{\alpha+\vartheta}} \left|\left|\theta_s^\lambda - \widehat{\theta}_s^\lambda\right|\right|_{V_\alpha} ||x||_{V_\alpha^*} \\
&\le \tilde{C}_1(T)\,\mathsf{E} \left|\left|\theta_t^\lambda\right|\right|_{V_{\alpha+\vartheta}} \left|\left|\theta_s^\lambda\right|\right|_{V_\alpha} \\
&\le C_1(T),
\end{aligned}$$

where the constant $C_1(T) < \infty$ does not depend on $\lambda \in \Lambda$ due to (3.1.1), which proves (3.2.10). Similarly, we have

$$\begin{aligned}
&\left|\left|(\Phi^\lambda(^*(t,s)\right|\right|_{\mathcal{L}(V_\alpha, V_{\alpha-\vartheta}^*)} = \sup_{||y||_{V_\alpha}\le 1} \left|\left|(\Phi^\lambda)^*(t,s)y\right|\right|_{V_{\alpha-\vartheta}^*} \\
&= \sup_{||y||_{V_\alpha}\le 1} \sup \left\{ \left|\langle \mathcal{B}^{2\vartheta}x, (\Phi^\lambda)^*(t,s)y\rangle_{V_\alpha^*}\right| ; \left|\left|\mathcal{B}^\vartheta x\right|\right|_{V_\alpha^*} \le 1,\ x \in V_{\alpha-2\vartheta}^* \right\} \\
&= \sup_{||y||_{V_\alpha}\le 1} \sup \left\{ \left|\langle \Phi^\lambda(t,s)\mathcal{B}^\vartheta z, y\rangle_{V_\alpha}\right| ; ||z||_{V_\alpha^*} \le 1,\ z \in V_{\alpha-\vartheta}^* \right\} \\
&\le \sup_{||y||_{V_\alpha}\le 1} \sup \left\{ \left|\langle \theta_t^\lambda, y\rangle_{V_\alpha} \langle \mathcal{B}^\alpha(\theta_s^\lambda - \widehat{\theta}_s^\lambda), \mathcal{B}^{-\alpha}\mathcal{B}^\vartheta z\rangle_H\right| ; ||z||_{V_\alpha^*} \le 1,\ z \in V_{\alpha-\vartheta}^* \right\} \\
&\le \sup_{||y||_{V_\alpha}\le 1} \sup \left\{ \left|\langle \theta_t^\lambda, y\rangle_{V_\alpha} \langle \mathcal{B}^{\alpha+\vartheta}(\theta_s^\lambda - \widehat{\theta}_s^\lambda), \mathcal{B}^{-\alpha} z\rangle_H\right| ; ||z||_{V_\alpha^*} \le 1,\ z \in V_{\alpha-\vartheta}^* \right\} \\
&\le \sup_{||y||_{V_\alpha}\le 1} \sup \left\{ ||y||_{V_\alpha} ||z||_{V_\alpha^*}\, \mathsf{E} \left|\left|\theta_t^\lambda\right|\right|_{V_\alpha} \left|\left|\theta_s^\lambda - \widehat{\theta}_s^\lambda\right|\right|_{V_{\alpha+\vartheta}} ; ||z||_{V_\alpha^*} \le 1,\ z \in V_{\alpha-\vartheta}^* \right\} \\
&\le C(T),
\end{aligned}$$

where the constant $C_2(T) < \infty$ does not depend on $\lambda \in \Lambda$. Therefore (3.2.11) holds true and the proof of Lemma 3.2.2 is complete. □

Now, we prove Theorem 3.2.1.

Proof. Set $\mathcal{C} = \mathcal{C}\left(\Lambda, \mathcal{L}(V_\alpha^*, V_\alpha)\right)$ and assume the converse, i.e. (3.2.1) and (3.2.2) hold and (3.2.3) does not. Then we can find $\epsilon_0 > 0$ and a sequence $\{\lambda_n\}_{n\in\mathbb{N}} \in \Lambda$ such that $\lambda_n \to \lambda_0$ and

$$\left\|\Phi^{\lambda_n} - \Phi^{\lambda_0}\right\|_{\mathcal{C}} > \epsilon_0, \quad n \in \mathbb{N}. \tag{3.2.12}$$

By the relative compactness of $\{\Phi^\lambda\}_{\lambda\in\Lambda}$ proved in Lemma 3.2.2 we can find a subsequence $\{\lambda_{n_k}\}_{k\in\mathbb{N}}$ and a limit $\Psi \in \mathcal{C}$ such that

$$\left\|\Phi^{\lambda_{n_k}} - \Psi\right\|_{\mathcal{C}} \to 0, \quad k \to 0. \tag{3.2.13}$$

We show that $\Psi = \Phi^{\lambda_0}$ and, therefore, (3.2.12) contradicts (3.2.13).

Using (3.1.6) we have

$$\begin{aligned}
&\left\|\Psi(t,s) - K^{\lambda_0}(t,s) + \sum_{j=1}^{n}\int_0^s \left(\Psi(t,r)A_j^{\lambda_0}(r)\right)\circ\left(\Psi(s,r)A_j^{\lambda_0}(r)\right)\,\mathrm{d}r\right\|_{\mathcal{L}(V_\alpha^*,V_\alpha)}\\
&\le \left\|\Psi(t,s) - \Phi^{\lambda_{n_k}}(t,s)\right\|_{\mathcal{L}(V_\alpha^*,V_\alpha)} + \left\|K^{\lambda_{n_k}}(t,s) - K^{\lambda_0}(t,s)\right\|_{\mathcal{L}(V_\alpha^*,V_\alpha)}\\
&\quad + \sum_{j=1}^{n}\bigg\|\int_0^s \left(\Psi(t,r)A_j^{\lambda_0}(r)\right)\circ\left(\Psi(s,r)A_j^{\lambda_0}(r)\right)\\
&\qquad\qquad - \left(\Phi^{\lambda_{n_k}}(t,r)A_j^{\lambda_{n_k}}(r)\right)\circ\left(\Phi^{\lambda_{n_k}}(s,r)A_j^{\lambda_{n_k}}(r)\right)\,\mathrm{d}r\bigg\|_{\mathcal{L}(V_\alpha^*,V_\alpha)}
\end{aligned}$$

for all $(t,s) \in \Lambda$. The first two terms tend to zero uniformly in Λ as $k \to \infty$ by (3.2.13) and (3.2.1). Using uniform boundedness of $\left\{\Phi^\lambda\right\}$ and $\left\{A^\lambda\right\}$ the third term can be estimated by

$$\begin{aligned}
&s\sum_{j=1}^{n}\left(\left\|\left(\Psi A_j^{\lambda_0}\right)\circ\left[\left(\Psi - \Phi^{\lambda_{n_k}}\right)A_j^{\lambda_0}\right]\right\|_{\mathcal{C}} + \left\|\left(\Psi A_j^{\lambda_0}\right)\circ\left[\Phi^{\lambda_{n_k}}\left(A_j^{\lambda_0} - A_j^{\lambda_{n_k}}\right)\right]\right\|_{\mathcal{C}}\right)\\
&\quad + s\sum_{j=1}^{n}\bigg(\left\|\left[\Psi\left(A_j^{\lambda_0} - A_j^{\lambda_{n_k}}\right)\right]\circ\left(\Phi^{\lambda_{n_k}}A_j^{\lambda_{n_k}}\right)\right\|_{\mathcal{C}}\\
&\qquad\qquad + \left\|\left[\left(\Psi - \Phi^{\lambda_{n_k}}\right)A_j^{\lambda_{n_k}}\right]\circ\left(\Phi^{\lambda_{n_k}}(s,r)A_j^{\lambda_{n_k}}(r)\right)\right\|_{\mathcal{C}}\bigg)\\
&\le C(T)\left(\left\|\Psi - \Phi^{\lambda_{n_k}}\right\|_{\mathcal{C}} + \left\|A^{\lambda_0} - A^{\lambda_{n_k}}\right\|_{\mathcal{C}([0,T],\mathcal{L}(V_\alpha,\mathbb{R}^n))}\right), \quad C(T) < \infty
\end{aligned}$$

which tends to zero uniformly in Λ as $k \to \infty$ by (3.2.1) and (3.2.2). It follows that

$$\Psi(t,s) = K^{\lambda_0}(t,s) - \sum_{j=1}^{n}\int_0^s \left(\Psi(t,r)A_j^{\lambda_0}(r)\right)\circ\left(\Psi(s,r)A_j^{\lambda_0}(r)\right)\,\mathrm{d}r$$

for all $(t,s) \in \Lambda$, hence Ψ solves (3.1.6) with $\lambda = \lambda_0$. In virtue of Theorem 2.2.1 on uniqueness of solutions to (3.1.6) we conclude that $\Psi = \Phi^{\lambda_0}$, which completes the proof of Theorem 3.2.1. □

3.3 Continuous dependence for the filter

In this section, continuous dependence of the filter $\hat{\theta}^\lambda$ on $\lambda \in \Lambda$ is proved, which is the main result of this chapter.

First, note that in virtue of (3.1.9), (3.1.4), Itô isometry and the uniform boundednes of $\left\{A^\lambda\right\}_{\lambda\in\Lambda}$ and $\{\Phi^\lambda\}_{\lambda\in\Lambda}$ we obtain

$$\begin{aligned}
\mathsf{E}\left\|\widehat{\theta}^\lambda_{t_2} - \widehat{\theta}^\lambda_{t_1}\right\|^2_{V_\alpha} &\le 2\,\mathsf{E}\left\|\int_0^{t_1}\Big(\Phi^\lambda(t_2,r) - \Phi^\lambda(t_1,r)\Big)(A^\lambda)^*(r)\,\mathrm{d}\tilde{W}^\lambda_r\right\|^2_{V_\alpha} \\
&\quad + 2\,\mathsf{E}\left\|\int_{t_1}^{t_2}\Phi^\lambda(t_2,r)(A^\lambda)^*(r)\,\mathrm{d}\tilde{W}^\lambda_r\right\|^2_{V_\alpha} \\
&= 2\int_0^{t_1}\left\|\Big(\Phi^\lambda(t_2,r) - \Phi^\lambda(t_1,r)\Big)A^\lambda_j(s)\right\|^2_{V_\alpha}\,\mathrm{d}r \\
&\quad + 2\int_{t_1}^{t_2}\left\|\Phi^\lambda(t_2,r)A^\lambda_j(s)\right\|^2_{V_\alpha}\,\mathrm{d}r \\
&\le C(T)\Big(\int_0^{t_1}\left\|\Phi^\lambda(t_2,r) - \Phi^\lambda(t_1,r)\right\|^2_{\mathcal{L}(V_\alpha^*,V_\alpha)}\,\mathrm{d}r + |t_2 - t_1|\Big)
\end{aligned}$$

for some $C(T) < \infty$ and all $t_1, t_2 \in [0,T]$, $t_1 < t_2$.

Therefore, using the equicontinuity of $\{\Phi^\lambda(\cdot,s)\}_{\lambda\in\Lambda}$ on $[r,T]$ for all $r \in [0,T]$ shown in (3.2.6) it follows that

$$\mathsf{E}\left\|\widehat{\theta}^\lambda_{t_2} - \widehat{\theta}^\lambda_{t_1}\right\|^2_{V_\alpha} \to 0, \quad |t_2 - t_1| \to 0$$

for every $\lambda \in \Lambda$, which yields

$$\widehat{\theta}^\lambda \in \mathcal{C}\left([0,T], L^2(\Omega, V_\alpha)\right).$$

Theorem 3.3.1. *Under the assumptions stated in Section 3.1 if*

$$\theta^\lambda \to \theta^{\lambda_0}, \qquad \lambda \to \lambda_0, \quad \lambda \in \Lambda, \tag{3.3.1}$$

in $\mathcal{C}([0,T], L^2(\Omega, V_\alpha))$ *and*

$$A^\lambda \to A^{\lambda_0}, \qquad \lambda \to \lambda_0, \quad \lambda \in \Lambda, \tag{3.3.2}$$

in $\mathcal{C}([0,T], \mathcal{L}(V_\alpha, \mathbb{R}^n))$ *then*

$$\widehat{\theta}^\lambda \to \widehat{\theta}^{\lambda_0}, \qquad \lambda \to \lambda_0, \quad \lambda \in \Lambda, \tag{3.3.3}$$

in $\mathcal{C}([0,T], L^2(\Omega, V_\alpha))$.

Proof. Given $\lambda \in \Lambda$ set

$$u(r) = \sup_{t\in[0,r]}\left\|\widehat{\theta}^\lambda_t - \widehat{\theta}^{\lambda_0}_t\right\|_{L^2(\Omega,V_\alpha)}, \quad r \in [0,T].$$

Note that u is nondecreasing and measurable on $[0,T]$ which follows from the continuity of the filter.

Using (3.1.5) and (3.1.3) we have

$$u(r) \le 2(I_1 + I_2),$$

where

$$I_1 = \sup_{t\in[0,r]} \mathsf{E} \left\| \int_0^t \Phi^\lambda(t,s)(A^\lambda)^*(s)A^\lambda(s)\Big(\theta_s^\lambda - \widehat{\theta}_s^\lambda\Big) \right.$$
$$\left. -\Phi^{\lambda_0}(t,s)(A^{\lambda_0})^*(s)A^{\lambda_0}(s)\Big(\theta_s^{\lambda_0} - \widehat{\theta}_s^{\lambda_0}\Big)\,\mathrm{d}s \right\|_{L^2(\Omega,V_\alpha)}^2,$$
$$I_2 = \sup_{t\in[0,r]} \mathsf{E} \left\| \int_0^t \Phi^\lambda(t,s)(A^\lambda)^*(s) - \Phi^{\lambda_0}(t,s)(A^{\lambda_0})^*(s)\,\mathrm{d}W_s \right\|_{L^2(\Omega,V_\alpha)}^2.$$

Furthermore, we can estimate

$$\begin{aligned}
I_1 &\leq \sup_{t\in[0,r]} 2\,\mathsf{E} \left\| \int_0^t \Big[\Phi^\lambda(t,s) - \Phi^{\lambda_0}(t,s)\Big](A^\lambda)^*(s)A^\lambda(s)\Big(\theta_s^\lambda - \widehat{\theta}_s^\lambda\Big)\,\mathrm{d}s \right\|_{L^2(\Omega,V_\alpha)}^2 \\
&+ \sup_{t\in[0,r]} 2\,\mathsf{E} \left\| \int_0^t \Phi^{\lambda_0}(t,s)\Big[(A^\lambda)^*(s) - (A^{\lambda_0})^*(s)\Big]A^\lambda(s)\Big(\theta_s^\lambda - \widehat{\theta}_s^\lambda\Big)\,\mathrm{d}s \right\|_{L^2(\Omega,V_\alpha)}^2 \\
&+ \sup_{t\in[0,r]} 2\,\mathsf{E} \left\| \int_0^t \Phi^{\lambda_0}(t,s)(A^{\lambda_0})^*(s)\Big[A^\lambda(s) - A^{\lambda_0}(s)\Big]\Big(\theta_s^\lambda - \widehat{\theta}_s^\lambda\Big)\,\mathrm{d}s \right\|_{L^2(\Omega,V_\alpha)}^2 \\
&+ \sup_{t\in[0,r]} 2\,\mathsf{E} \left\| \int_0^t \Phi^{\lambda_0}(t,s)(A^{\lambda_0})^*(s)A^{\lambda_0}(s)\Big(\theta_s^\lambda - \theta_s^{\lambda_0} + \widehat{\theta}_s^{\lambda_0} - \widehat{\theta}_s^\lambda\Big)\,\mathrm{d}s \right\|_{L^2(\Omega,V_\alpha)}^2 \\
&\leq C_1(T)\left\|\Phi^\lambda - \Phi^{\lambda_0}\right\|_{\mathcal{C}} + C_2(T)\left\|A^\lambda - A^{\lambda_0}\right\|_{\mathcal{C}([0,T],\mathcal{L}(V_\alpha,\mathbb{R}^n))} \\
&+ C_3(T)\left\|\theta^\lambda - \theta^{\lambda_0}\right\|_{\mathcal{C}([0,T],L^2(\Omega,V_\alpha))} + C_4(T)\int_0^r u(s)\,\mathrm{d}s
\end{aligned}$$

and

$$\begin{aligned}
I_2 &\leq \sup_{t\in[0,r]} \sum_{j=1}^n \mathsf{E} \left\| \int_0^t \Phi^\lambda(t,s)A_j^\lambda(s) - \Phi^{\lambda_0}(t,s)A_j^{\lambda_0}(s)\,\mathrm{d}W_s^j \right\|_{L^2(\Omega,V_\alpha)}^2 \\
&= \sup_{t\in[0,r]} \sum_{j=1}^n \int_0^t \left\|\Phi^\lambda(t,s)A_j^\lambda(s) - \Phi^{\lambda_0}(t,s)A_j^{\lambda_0}(s)\right\|_{L^2(\Omega,V_\alpha)}^2 \mathrm{d}s \\
&\leq C_5(T)\left\|A^\lambda - A^{\lambda_0}\right\|_{\mathcal{C}([0,T],\mathcal{L}(V_\alpha,\mathbb{R}^n))} + C_6(T)\left\|\Phi^\lambda - \Phi^{\lambda_0}\right\|_{\mathcal{C}},
\end{aligned}$$

where $C_1 - C_6$ are finite constants dependent only on T. We used boundedness of $\left\{\Phi^\lambda\right\}$ and $\left\{A^\lambda\right\}$ and Itô isometry. Therefore, we have

$$u(r) \leq \alpha(T) + \int_0^r C_4(T)u(s)\,\mathrm{d}s, \quad r \in [0,T],$$

where

$$\alpha(T) = \bar{C}(T)\Big(\left\|\Phi^\lambda - \Phi^{\lambda_0}\right\|_{\mathcal{C}} + \left\|A^\lambda - A^{\lambda_0}\right\|_{\mathcal{C}([0,T],\mathcal{L}(V_\alpha,\mathbb{R}^n))}$$
$$+ \left\|\theta^\lambda - \theta^{\lambda_0}\right\|_{\mathcal{C}([0,T],L^2(\Omega,V_\alpha))}\Big)$$

and $\bar{C}(T) < \infty$. Using Gronwall's inequality we obtain

$$\begin{aligned}
&u(T) \leq \alpha(T)\exp\{TC_4(T)\} \\
&\leq C(T)\Big(\left\|\Phi^\lambda - \Phi^{\lambda_0}\right\|_{\mathcal{C}} + \left\|A^\lambda - A^{\lambda_0}\right\|_{\mathcal{C}([0,T],\mathcal{L}(V_\alpha,\mathbb{R}^n))} + \left\|\theta^\lambda - \theta^{\lambda_0}\right\|_{\mathcal{C}([0,T],L^2(\Omega,V_\alpha))}\Big)
\end{aligned}$$

for a constant $C(T) < \infty$ independent of $\lambda \in \Lambda$.

Using assumptions (3.3.1), (3.3.2) and Theorem 3.2.1 we obtain

$$\sup_{t\in[0,r]} \left|\left|\widehat{\theta}_t^\lambda - \widehat{\theta}_t^{\lambda_0}\right|\right|_{L^2(\Omega,V_\alpha)} \to 0, \qquad \lambda \to \lambda_0, \quad \lambda \in \Lambda,$$

which completes the proof. □

3.4 Signal governed by stochastic evolution equation

In this section we apply the previous results to a signal given by stochastic evolution equation driven by fractional noise.

Let H be a separable Hilbert space and for any $\lambda \in \Lambda$, Λ being a compact metric space, let the H-valued signal θ^λ satisfy the equation

$$d\theta_t^\lambda = \mathcal{A}_\lambda \theta_t^\lambda dt + G_\lambda dB_t, \quad t \in [0,T], \tag{3.4.1}$$

where the linear operator $\mathcal{A}_\lambda : Dom(\mathcal{A}_\lambda) \subset H \to H$ is strictily negative, selfadjoint and has compact resolvent, hence it generates a compact strongly continuous analytic semigroup $\{S_\lambda(t), t \geq 0\}$ in H. The noise is given by two-sided cylindrical fBm $\{B_t, t \in \mathbb{R}\}$ with the Hurst parameter $h > \frac{1}{2}$ and, finally, G_λ is an operator

$$G_\lambda : U \to V_1',$$

where V_1' is the dual of V_1 with respect to topology of H.

By the analyticity of $S_\lambda, \lambda \in \Lambda$, the Hilbert spaces

$$V_\delta^\lambda = Dom((-\mathcal{A}_\lambda)^\delta), \quad \delta \geq 0 \tag{3.4.2}$$

equipped with the graph norm topology are well defined. We assume that V_δ^λ do not depend on $\lambda \in \Lambda$ for every $\delta \geq 0$ (the graph norms $\|\cdot\|_{V_\delta^\lambda}, \lambda \in \Lambda$ are equivalent) and we set $V_\delta = V_\delta^{\lambda_0}, \delta \geq 0$, for a fixed, arbitrarily chosen $\lambda_0 \in \Lambda$.

In order to use results from previous sections impose the following assumptions:

- *Uniform exponential stability*: For some $c_1 > 0$ and $\rho_1 > 0$ we have

$$||S_\lambda(t)||_{\mathcal{L}(H)} \leq c_1 e^{-\rho_1 t}, \quad t > 0 \tag{A1}$$

  for $\lambda \in \Lambda$.

- *Uniform singularity at time* $t = 0$: For some $c_2 > 0$ and $0 \leq \gamma < h$ we have

$$||S_\lambda(t) G_\lambda||_{\mathcal{L}_2(U,H)} \leq c_2 t^{-\gamma}, t > 0 \tag{A2}$$

  for $\lambda \in \Lambda$.

- *Equicontinuity of the semigroups*: There exists $\alpha \geq 0$ such that $\gamma + \alpha < h$ and for any $x \in V_\alpha$ the mappings

$$S_\lambda(\cdot)x : [0,T] \to V_\alpha \text{ are continuous uniformly in } \lambda \in \Lambda. \tag{A3}$$

- *Continuous dependence*: For $t > 0$ and $\lambda_0 \in \Lambda$ we have

$$S_\lambda(t)G_\lambda \xrightarrow{\mathcal{L}_2(U,V_\alpha)} S_{\lambda_0}(t)G_{\lambda_0}, \quad \lambda \to \lambda_0. \tag{A4}$$

Note that the above conditions imply the uniform analyticity of the family of semigroups $(S_\lambda)_{\lambda\in\Lambda}$, hence by (A1) and (A2) we obtain

$$||S_\lambda(t)G_\lambda||_{\mathcal{L}_2(U,V_\delta)} \leq C_\delta e^{-\rho t} t^{-(\gamma+\delta)}, \quad t > 0 \tag{3.4.3}$$

for any $\delta \geq 0$ and a constant C_δ independent of $t > 0$ and $\lambda \in \Lambda$. Indeed, we may estimate

$$||S_\lambda(t)G_\lambda||_{\mathcal{L}_2(U,V_\delta)} \leq ||S_\lambda(t/2)G_\lambda||_{\mathcal{L}_2(U,H)} \, ||S_\lambda(t/2)||_{\mathcal{L}_2(H,V_\delta)}, \quad t > 0.$$

Also, notice that the inequality in (A2) may be verified only on a finite time interval $(0, T)$ for a $T > 0$ if we take into account (A1).

Using the above hypotheses we can apply Theorem 1.2.27 to obtain a strictly stationary solution to (3.4.1) with continuous paths in V_α, understood in the mild sense, which may be expressed as

$$\theta_t^\lambda = \int_{-\infty}^t S_\lambda(t-u)G_\lambda dB(u), \quad t \in [0, T]. \tag{3.4.4}$$

(Similar computations as in [19] yield a representation for the covariance $K^\lambda(t,s) = K^\lambda(t-s) = \mathbb{E}\Big[\theta_t^\lambda \circ \theta_0^\lambda\Big]$ for $(t,s) \in \Lambda$:

$$K^\lambda(t) = \int_{-\infty}^t \int_{-\infty}^0 S_\lambda(-u)G_\lambda G_\lambda^* S_\lambda(t-v)\gamma_h(u-v)dudv, \tag{3.4.5}$$

where $\gamma_h(u) = h(2h-1) \mid u \mid^{2h-2}, u \in \mathbb{R}$, (note that $S_\lambda(t)^* = S_\lambda(t)$). The integral (3.4.5) is correctly defined due to the estimate

$$\begin{aligned}
&\int_{-\infty}^t \int_{-\infty}^0 ||S_\lambda(-r)G_\lambda G_\lambda^* S_\lambda(t-v)||_{\mathcal{L}(H)} \, \gamma_h(r-v)drdv \\
&\leq \int_{-\infty}^t \int_{-\infty}^0 ||S_\lambda(-r)G_\lambda G_\lambda^* S_\lambda(t-v)||_{\mathcal{L}_1(H)} \, \gamma_h(r-v)drdv \\
&\leq \int_{-\infty}^t \int_{-\infty}^0 ||S_\lambda(-r)G_\lambda||_{\mathcal{L}_2(U,H} \, ||S_\lambda(t-v)G_\lambda||_{\mathcal{L}_2(U,H)} \, \gamma_h(r-v)drdv \\
&\leq c_0 \int_{-\infty}^t \int_{-\infty}^0 e^{\rho(r+v)}(-r)^{-\gamma}(t-v)^{-\gamma}\gamma_h(r-v)drdv
\end{aligned}$$

for $t \in [0, T]$, with some constant $c_0 < \infty$, which follows by (3.4.3) with $\delta = 0$ (so $V_\delta = H$). The right-hand side is finite since

$$\int_{-\infty}^t \int_{-\infty}^0 e^{\rho(r+v)}(-r)^{-\eta}(t-v)^{-\eta}\gamma_h(r-v)drdv < \infty, \quad t \in [0, T], \tag{3.4.6}$$

for any $\rho > 0$ and $0 \leq \eta < h$ which we will also use in the sequel.

We are now ready to verify the boundedness condition (3.1.1) and equicontinuity condition (3.1.2) from Section 3.1. Take $\vartheta \in (0, \alpha)$ such that $\gamma + \alpha + \vartheta < h$

and any $t \in [0,T]$, $\lambda \in \Lambda_0$. Using (3.4.6), (3.4.3) and the strict stationarity of (3.4.4) we have

$$
\begin{aligned}
\mathbb{E}\Big|\Big|\theta_t^\lambda\Big|\Big|_{V_{\alpha+\vartheta}}^2 &= \mathbb{E}\Big|\Big|\theta_0^\lambda\Big|\Big|_{V_{\alpha+\vartheta}}^2 \\
&= \Big|\Big|\int_{-\infty}^0\int_{-\infty}^0 S_\lambda(-r)G_\lambda G_\lambda^* S_\lambda(-v)\gamma_h(r-v)drdv\Big|\Big|_{\mathcal{L}_1(V_{\alpha+\vartheta})} \\
&\le \int_{-\infty}^0\int_{-\infty}^0 ||S_\lambda(-r)G_\lambda||_{\mathcal{L}_2(U,V_{\alpha+\vartheta})}||S_\lambda(-v)G_\lambda||_{\mathcal{L}_2(U,V_{\alpha+\vartheta})}\gamma_h(r-v)drdv \\
&\le C_{\alpha+\vartheta}^2\int_{-\infty}^0\int_{-\infty}^0 e^{\rho(r+v)}(-r)^{-(\gamma+\alpha+\vartheta)}(-v)^{-(\gamma+\alpha+\vartheta)}\gamma_h(r-v)drdv < \infty,
\end{aligned}
$$

with the last integral being independent of t, λ and finite due to (3.4.6) with $\eta = \gamma+\alpha+\theta$. This proves the boundedness of $\{\theta^\lambda\}_{\lambda\in\Lambda}$ in $\mathcal{C}([0,T], L^2(\Omega, V_{\alpha+\vartheta}))$.

By the strict stationarity it is enough to verify the equicontinuity at zero from the right. For $t \in [0,T]$ and $\lambda \in \Lambda$ we obtain

$$
\begin{aligned}
&\mathbb{E}\Big|\Big|\int_{-\infty}^t S_\lambda(t-u)G_\lambda dB_u - \int_{-\infty}^0 S_\lambda(-u)G_\lambda dB_u\Big|\Big|_{V_\alpha}^2 \\
&= \mathbb{E}\Big|\Big|\int_0^t S_\lambda(t-u)G_\lambda dB_u - \int_{-\infty}^0 (S_\lambda(t)-I)S_\lambda(-u)G_\lambda dB_u\Big|\Big|_{V_\alpha}^2 \\
&\le 2\mathbb{E}\Big|\Big|\int_0^t S_\lambda(t-u)G_\lambda dB_u\Big|\Big|_{V_\alpha}^2 + 2\mathbb{E}\Big|\Big|\int_{-\infty}^0 (S_\lambda(t)-I)S_\lambda(-u)G_\lambda dB_u\Big|\Big|_{V_\alpha}^2 \\
&\le 2\Big|\Big|\int_0^t\int_0^t S_\lambda(t-u)G_\lambda G_\lambda^* S_\lambda(t-v)\gamma_h(u-v)dudv\Big|\Big|_{\mathcal{L}_1(V_\alpha)} \\
&+ 2\Big|\Big|\int_{-\infty}^0\int_{-\infty}^0 (S_\lambda(t)-I)S_\lambda(-u)G_\lambda G_\lambda^* S_\lambda(-v)(S_\lambda(t)-I)\gamma_h(u-v)dudv\Big|\Big|_{\mathcal{L}_1(V_\alpha)} \\
&\le 2\int_0^t\int_0^t ||S_\lambda(t-u)G_\lambda||_{\mathcal{L}_2(U,V_\alpha)}\,||S_\lambda(t-v)G_\lambda||_{\mathcal{L}_2(U,V_\alpha)}\,\gamma_h(u-v)dudv \\
&\quad + 2\int_{-\infty}^0\int_{-\infty}^0 ||(S_\lambda(t)-I)S_\lambda(-u)G_\lambda||_{\mathcal{L}_2(U,V_\alpha)} \\
&\qquad\qquad ||(S_\lambda(t)-I)S_\lambda(-v)G_\lambda||_{\mathcal{L}_2(U,V_\alpha)}\,\gamma_h(u-v)dudv \\
&=: 2I_1(t,\lambda) + 2I_2(t,\lambda)
\end{aligned}
$$

Now $I_1(t,\lambda)$ can be estimated

$$
\begin{aligned}
I_1(t,\lambda) &= \int_0^t\int_0^t ||S_\lambda(u)G_\lambda||_{\mathcal{L}_2(U,V_\alpha)_{\mathcal{L}_2(U,V_\alpha)}}||S_\lambda(v)G_\lambda||_{\mathcal{L}_2(U,V_\alpha)}\gamma_h(u-v)dudv \\
&\le C_\alpha^2\int_0^t\int_0^t e^{-\rho(u+v)}(uv)^{-(\gamma+\alpha)}\gamma_h(u-v)dudv.
\end{aligned}
$$

The last term is independent of $\lambda \in \Lambda$ and tends to 0 as $t \to 0+$ when we take into account (3.4.6) with $\eta = \gamma+\alpha$.

For $I_2(t,\lambda)$ we construct an integrable majorant by (3.4.6) with $\eta = \gamma+\alpha$ and the equicontinuity of the semigroups in (A3) which implies that $|\ S_\lambda(\cdot) - I\ |_{\mathcal{L}(V_\alpha)}$ is bounded on $[0,T]$ by some $N_\alpha > 0$ depending only on α. Moreover,

$$
(S_\lambda(t)-I)S_\lambda(u)G_\lambda \xrightarrow{\mathcal{L}_2(U,V_\alpha)} 0, \quad t \to 0+ \tag{3.4.7}
$$

holds for any $u > 0$. The convergence (3.4.7) is obtained by the analyticity of S_λ and equicontinuity in (A3) again by Dominated Convergence Theorem. Indeed, if $\{f_n\}_{n\in\mathbb{N}}$ is an orthonormal basis in U we have

$$||(S_\lambda(t) - I)S_\lambda(u)G_\lambda f_n||_{V_\alpha} \to 0, \quad t \to 0+,$$

for any $n \in \mathbb{N}$ and $u > 0$. Moreover,

$$\begin{aligned}\sum_{n=0}^{\infty} ||(S_\lambda(t) - I)S_\lambda(u)G_\lambda f_n||_{V_\alpha}{}^2 &\leq N_\alpha^2 \sum_{n=0}^{\infty} ||S_\lambda(u)G_\lambda f_n||_{V_\alpha}{}^2 \\ &= N_\alpha^2 \, ||S_\lambda(u)G_\lambda||_{\mathcal{L}_2(U,V_\alpha)},\end{aligned}$$

which is finite by (3.4.3) with $\delta = \alpha$. Hence we have verified that the family $\{\theta^\lambda\}_{\lambda\in\Lambda} \subset \mathcal{C}([0,T], L^2(\Omega, V_\alpha))$ is equicontinuous.

Now, we verify the condition (3.3.1) on continuous dependence of θ^λ on λ. Assume $t \in [0,T]$ and $\lambda, \lambda_0 \in \Lambda$ then we have

$$\begin{aligned}&\mathbb{E}\Big|\Big|\theta_t^\lambda - \theta_t^{\lambda_0}\Big|\Big|_{V_\alpha}^2 = \mathbb{E}\Big|\Big|\int_{-\infty}^t S_\lambda(t-u)G_\lambda dB_u - \int_{-\infty}^t S_{\lambda_0}(t-u)G_{\lambda_0} dB_u\Big|\Big|_{V_\alpha}^2 \\ &= \mathbb{E}\Big|\Big|\int_{-\infty}^t (S_\lambda(t-u)G_\lambda - S_{\lambda_0}(t-u)G_{\lambda_0})dB_u\Big|\Big|_{V_\alpha}^2 \\ &= \Big|\Big|\int_{-\infty}^t \int_{-\infty}^t (S_\lambda(t-u)G_\lambda - S_{\lambda_0}(t-u)G_{\lambda_0}) \\ &\qquad\qquad (S_\lambda(t-v)G_\lambda - S_{\lambda_0}(t-v)G_{\lambda_0})^* \gamma_h(u-v)dudv\Big|\Big|_{\mathcal{L}_1(V_\alpha)} \\ &= \Big|\Big|\int_0^\infty \int_0^\infty (S_\lambda(u)G_\lambda - S_{\lambda_0}(u)G_{\lambda_0})(S_\lambda(v)G_\lambda - S_{\lambda_0}(v)G_{\lambda_0})^* \gamma_h(u-v)dudv\Big|\Big|_{\mathcal{L}_1(V_\alpha)} \\ &\leq \int_0^\infty \int_0^\infty ||S_\lambda(u)G_\lambda - S_{\lambda_0}(u)G_{\lambda_0}||_{\mathcal{L}_2(U,V_\alpha)} \\ &\qquad\qquad ||S_\lambda(v)G_\lambda - S_{\lambda_0}(v)G_{\lambda_0}||_{\mathcal{L}_2(U,V_\alpha)} \gamma_h(u-v)dudv.\end{aligned}$$

This upper bound does not depend on t and the integrand converges pointwise to zero as $\lambda \to \lambda_0$ by (A4). The Dominated Convergence Theorem (we use an upper bound constructed using (3.4.6) with $\eta = \gamma + \alpha$) yields desired convergence. We have verified that $\theta^\lambda \to \theta^{\lambda_0}$ in $\mathcal{C}([0,T], L^2(\Omega, V_\alpha))$ as $\lambda \to \lambda_0$.

3.4.1 Distributed fractional noise in heat equation

Consider the stationary solution of the equation

$$\begin{aligned}\frac{\partial u}{\partial t}(t,x) =& \lambda^1 \Delta u(t,x) + \eta_{\lambda^2}^h(t,x), \quad (t,x) \in [0,T] \times \mathcal{D}, \\ u(t,\cdot)\Big|_{\partial\mathcal{D}} =& 0, \quad t \in [0,T],\end{aligned} \tag{3.4.8}$$

where $\mathcal{D} \subset \mathbb{R}^d$ is a bounded domain with a smooth boundary $\partial\mathcal{D}$, Δ is the Laplace operator and the parameter $\lambda = (\lambda^1, \lambda^2)$ takes values in a compact metric space

$\Lambda = \Lambda^1 \times \Lambda^2, \Lambda^1 \subset (0,\infty)$. The noise $\eta^h_{\lambda^2}$ is viewed as a fractional noise with the Hurst parameter $h > 1/2$.

The equation (3.4.8) is treated rigorously as the Hilbert space-valued equation

$$d\theta^\lambda_t = \mathcal{A}_\lambda \theta^\lambda_t dt + G_\lambda dB^h_t, \quad t \in [0,T], \tag{3.4.9}$$

for $\lambda \in \Lambda$ as in (3.4.1), where we set

$$U = H = L^2(\mathcal{D}), \quad \mathcal{A}_\lambda = \lambda^1 \Delta, \quad Dom\mathcal{A}_\lambda = W^{2,2}(\mathcal{D}) \cap W^{1,2}_0(\mathcal{D}),$$

B^h is the cylindrical fractional Brownian motion in U and $G_\lambda = G_{\lambda^2} : \Lambda^2 \to \mathcal{L}_2(U,H)$ is continuous.

It is well known that $\mathcal{A}_1$ is strictly negative and generates strongly continuous compact semigroup on H which we denote by S (here we formally assume that $1 \in \Lambda^1$). For the semigroups S_λ generated by $\mathcal{A}_\lambda, \lambda \in \Lambda$ we have

$$S_\lambda(t) = S(\lambda^1 t), \quad t > 0, \lambda \in \Lambda. \tag{3.4.10}$$

To establish continuous dependence of the filter we verify (A1), (A2), (A3) and (A4). Firstly, S is is exponentially stable so the condition (A1) is satisfied.

Furthermore,

$$||S_\lambda(t)G_\lambda||_{\mathcal{L}_2(U,H)} \leq \left|\left|S(\lambda^1 t)\right|\right|_{\mathcal{L}(H)} ||G_{\lambda^2}||_{\mathcal{L}_2(U,H)} \leq K$$

for some $K < \infty$ by the Resonance Theorem, compactness of Λ^1 and continuous dependence of G_{λ^2} on $\lambda^2 \in \Lambda^2$. Therefore, (A2) is verified with $\gamma = 0$.

Taking arbitrary $\alpha > 0$ so that

$$\gamma + \alpha = \alpha < h \tag{3.4.11}$$

we obtain (A3) by analyticity of S and (3.4.10).

Finally, the continuous dependence in (A4) is verified as follows: First, we observe that in our case (A4) follows from the weaker condition

$$S_\lambda(t)G_\lambda \xrightarrow{\mathcal{L}_2(U,H)} S_{\lambda_0}(t)G_{\lambda_0}, \quad \lambda \to \lambda_0,$$

for $t > 0$. For $t > 0$ fixed taking $c > 0$ such that $\lambda^1 t > 0$ for every $\lambda^1 \in \Lambda^1$ we may write

$$\begin{aligned} &||S_\lambda(t)G_\lambda - S_{\tilde{\lambda}}(t)G_{\tilde{\lambda}}||_{\mathcal{L}_2(U,V_\alpha)} \\ &\qquad \leq ||S(c)||_{\mathcal{L}(H,V_\alpha)} \left|\left|S(\lambda^1 t - c)G_\lambda - S(\tilde{\lambda}^1 t - c)G_{\tilde{\lambda}}\right|\right|_{\mathcal{L}_2(U,H)} \end{aligned}$$

and we use analyticity of the semigroup S.

Now let $\{f_n\}_{n\in\mathbb{N}}$ be an orthonormal basis in U, then we have

$$\begin{aligned} ||S_\lambda(t)G_\lambda - S(t)_{\tilde{\lambda}}G_{\tilde{\lambda}}||_{\mathcal{L}_2(U,H)} &= \sum_{n=0}^{\infty} ||(S_\lambda(t)G_{\lambda^2} - S_{\tilde{\lambda}}(t)G_{\tilde{\lambda}^2})f_n||^2_H \\ &= \sum_{n=0}^{\infty} \left|\left|(S(\lambda^1 t)G_{\lambda^2} - S(\tilde{\lambda}^1 t)G_{\tilde{\lambda}^2})f_n\right|\right|^2_H \\ &\leq 2\sum_{n=0}^{\infty} \left|\left|S(\lambda^1 t)(G_{\tilde{\lambda}^2} - G_{\lambda^2})f_n\right|\right|^2_H \\ &+ 2\sum_{n=0}^{\infty} \left|\left|(S(\lambda^1 t) - S(\tilde{\lambda}^1 t))G_{\tilde{\lambda}^2}f_n\right|\right|^2_H \end{aligned}$$

for $t > 0, \lambda = (\lambda^1, \lambda^2) \in \Lambda$ and $\tilde{\lambda} = (\tilde{\lambda}^1, \tilde{\lambda}^2) \in \Lambda$. The right-hand side converges to 0 if $\lambda \to \tilde{\lambda}$ by the Resonance Theorem, compactness of Λ and continuous dependence of G_{λ^2} on $\lambda^2 \in \Lambda^2$.

We have verified the conditions for continuous dependence of the filter $\hat{\theta}^\lambda$ in Theorem 3.3.1 for arbitrary $0 < \alpha < h$.

Assume moreover, that the condition

$$\alpha > \frac{d}{4} \tag{3.4.12}$$

is additionally satisfied. Then by the Sobolev embedding theorem and [47] we have

$$V_\alpha \hookrightarrow W^{2\alpha,2}(\mathcal{D}) \hookrightarrow \mathcal{C}^{0,\beta}(\mathcal{D}), \tag{3.4.13}$$

where $W^{2\alpha,2}(\mathcal{D})$ is the Sobolev space and $\mathcal{C}^{0,\beta}(\mathcal{D})$ is the space of uniformly β-Hölder continuous functions on $\mathcal{D}$, $\beta = (4\alpha - d)/2$. Hence, for arbitrary chosen set of points $z_1^\lambda, \dots, z_n^\lambda \in \mathcal{D}$ (possibly depending on $\lambda \in \Lambda$) the evaluation map

$$A^\lambda \varphi = (\varphi(z_1^\lambda), \dots, \varphi(z_n^\lambda)), \quad \varphi \in V_\alpha \tag{3.4.14}$$

is well defined for $\lambda \in \Lambda$. Suppose that the mapping $\lambda \mapsto (z_1^\lambda, \dots z_n^\lambda)$ is continuous. Then $A^\lambda : \Lambda \to \mathcal{L}(V_\alpha, \mathbb{R}^n)$ is continuous as well, since by (3.4.13) for a constant $c_0 > 0$ we have

$$\sup_{\varphi \in V_\alpha} \{ \left|\varphi(z_i^\lambda) - \varphi(z_i^{\tilde{\lambda}})\right|, ||\varphi||_{V_\alpha} \leq 1\} \leq c_0 \sup_{\varphi \in \mathcal{C}^{0,\beta}} \{ \left|\varphi(z_i^\lambda) - \varphi(z_i^{\tilde{\lambda}})\right|, ||\varphi||_{\mathcal{C}^{0,\beta}} \leq 1\} \to 0$$

whenever $\lambda \to \tilde{\lambda}$ for $i = 1, \dots, n$. This verifies the condition (3.3.2) and we may conclude that Theorem 3.3.1 with V_α defined as above holds for signal defined by (3.4.9) and the observation process

$$\begin{aligned} d\xi_t^\lambda &= A^\lambda \theta_t^\lambda dt + dW_t, \quad t \in [0, T], \\ \xi_0^\lambda &= 0, \end{aligned}$$

with arbitrary $\mathbb{R}^n$-valued Wiener process W which is independent of B^h and pointwise observation A^λ given by (3.4.14). In this case the equations for the filter (3.1.5) and (3.1.6) in Theorem 3.1.1 can be simplified in the same way as in the Corollary 2.4.1.

Note that since we assume $h > 1/2$, both (3.4.11) and (3.4.12) are satisfied if either $d = 1, 2$ or $d = 3$ and $h > 3/4$.

3.4.2 Pointwise fractional noise in heat equation

Consider the signal given as a stationary solution to the parabolic equation

$$\begin{aligned} \frac{\partial u}{\partial t}(t, x) &= \lambda^1 \Delta u(t, x) + \delta_{\lambda^2} \eta^h(t), \quad (t, x) \in [0, T] \times \mathcal{D}, \\ u(t, \cdot)\Big|_{\partial \mathcal{D}} &= 0, \quad t \in [0, T]. \end{aligned} \tag{3.4.15}$$

The setup is similar to the previous example except that the noise η^h is not distributed on the whole domain $\mathcal{D}$, but is scalar and acting at the point $\lambda^2 \in \mathcal{D}$. Here, $\mathcal{D} \subset \mathbb{R}^d$ is again a bounded domain with smooth boundary $\partial\mathcal{D}$, Δ is the Laplace operator, δ_y stands for the Dirac distribution at $y \in \mathcal{D}$ and the parameter $\lambda = (\lambda^1, \lambda^2)$ takes values in a compact metric space $\Lambda = \Lambda^1 \times \Lambda^2$, where $\Lambda^1 \subset (0, \infty)$ and $\Lambda^2 \subset \mathcal{D}$. The noise η^h is an one-dimensional fractional Brownian motion with the Hurst parameter $h > 1/2$.

We treat (3.4.15) as the equation

$$d\theta_t^\lambda = \mathcal{A}_\lambda \theta_t^\lambda dt + G_\lambda dB_t, \quad t \in [0, T],$$

for $\lambda \in \Lambda$ as in (3.4.1), where

$$U = \mathbb{R}, \quad H = L^2(\mathcal{D}), \quad \mathcal{A}_\lambda = \lambda^1 \Delta, \quad Dom\mathcal{A}_\lambda = W^{2,2}(\mathcal{D}) \cap W_0^{1,2}(\mathcal{D}),$$
$$G_\lambda = \delta_{\lambda^2}$$

with a scalar fractional Brownian motion B^h. The semigroups S and S_λ are the same as in the previous example, it is therefore sufficient to verify (A2), (A3) and (A4). Note that as $U = \mathbb{R}$, the Hilbert-Schmidt and operator norms are equal for operators defined on U.

To verify (A2) we estimate

$$\begin{aligned} ||S_\lambda(t)G_\lambda||_{\mathcal{L}_2(U,H)} &= \left|\left|S(\lambda^1 t)\delta_{\lambda^2}\right|\right|_{\mathcal{L}(U,H)} \\ &\le \left|\left|S(\lambda^1 t)\right|\right|_{\mathcal{L}(V_\gamma^*,H)} ||\delta_{\lambda^2}||_{\mathcal{L}(U,V_\gamma^*)} \\ &\le c_0 t^{-\gamma}, \quad t > 0, \end{aligned} \tag{3.4.16}$$

for some $c_0 > 0$ whenever $d/4 < \gamma < 1$. We used the analyticity of S, isomorphism $V_\gamma^* \cong Dom(-\mathcal{A})^{-\gamma}$, compactness of Λ and continuous dependence of δ_{λ^2} on $\lambda^2 \in \Lambda^2$ in $(\mathcal{C}^{0,\beta})^* \hookrightarrow V_\gamma^*$, where $\beta = (4\gamma - d)/2$. Assuming that

$$\frac{d}{4} < h, \tag{3.4.17}$$

we have verified (A2). Fix γ such that $d/4 < \gamma < h$. Then (A3) is satisfied for any $\alpha \ge 0$ with

$$\gamma + \alpha < h$$

by the analyticity of S. Finally, to verify (A4) we examine the norm in $\mathcal{L}_2(U, H)$ as in the previous example and estimate

$$\begin{aligned} ||S_\lambda(t)G_\lambda - S_{\tilde\lambda}(t)G_{\tilde\lambda}||_{\mathcal{L}_2(U,H)} &= ||S_\lambda(t)G_\lambda - S_{\tilde\lambda}(t)G_{\tilde\lambda}||_{\mathcal{L}(U,H)} \\ &\le ||S_\lambda(t)(G_\lambda - G_{\tilde\lambda})||_{\mathcal{L}(U,H)} && =: H_1 \\ &+ ||(S_\lambda(t) - S_{\tilde\lambda}(t))G_{\tilde\lambda}||_{\mathcal{L}(U,H)} && =: H_2 \end{aligned}$$

for $\lambda = (\lambda^1, \lambda^2) \in \Lambda$ and $\tilde{\lambda} = (\tilde{\lambda}^1, \tilde{\lambda}^2) \in \Lambda$. The term H_1 is estimated similarly as in (3.4.16) as

$$H_1 \le c_0 t^{-\gamma} \, ||\delta_{\lambda^2} - \delta_{\tilde{\lambda}^2}||_{\mathcal{L}(U,V_\gamma^*)}$$

by analyticity of S. We see that H_1 tends to 0 as $\lambda \to \tilde{\lambda}$ by continuous dependence of δ_{λ^2} on λ^2 in $(\mathcal{C}^{0,\beta})^* \hookrightarrow V_\gamma^*$. For H_2 we have

$$\begin{aligned} H_2 &= \left|\left|(S(\lambda^1 t) - S(\tilde{\lambda}^1 t))\delta_{\tilde{\lambda}^2}\right|\right|_{\mathcal{L}(U,H)} \\ &= \left|\left|(S(\lambda^1 t) - S(\tilde{\lambda}^1 t))(-A)^\gamma (-A)^{-\gamma} \delta_{\tilde{\lambda}^2}\right|\right|_{\mathcal{L}(U,H)} . \end{aligned}$$

Now $\delta_{\tilde{\lambda}^2} \in \mathcal{L}(U, V_\gamma^*)$ and it easily follows that $H_2 \to 0$ as $\lambda \to \tilde{\lambda}$, which verifies (A4). As in the previous example we may also examine the conditions under which we shall consider pointwise observation of the signal as defined in (3.4.14). Similarly, we obtain the condition $d/4 < \alpha$ which can be satisfied only when $d = 1$.

List of Figures

Glossaries

Abbreviations

KB	Kalman-Bucy
SPDE	stochastic partial differential equation
fBm	fractional Brownian motion
lfBm	Liouville fractional Brownian motion

Spaces of functions and operators

$C([0,T],X)$	space of X-valued continuous functions on $[0,T]$
$C^k([0,T],X)$	space of X-valued k times continuously differentiable functions on $[0,T]$
$\mathcal{C}^{0,\beta}(X)$	space of uniformly β-Hölder continuous functions on X
$\mathcal{L}(X,Y)$	space of bounded linear operators mapping a Banach space X to a Banach space Y
$\mathcal{L}(X)$	space of bounded linear operators on a Banach space X
$\mathcal{L}_2(X,Y)$	space of Hilbert-Schmidt operators mapping a Hilbert space X to a Hilbert space Y
$\mathcal{L}_2(X)$	space of Hilbert-Schmidt operators on a Hilbert space X
$\mathcal{L}_1(X)$	space of trace class operators on a Hilbert space X
$L^p(X,Y)$	Lebesgue-Bochner space of Y-valued functions on X
$W^{s,p}(X)$	Sobolev space on X
$H^s(X)$	$W^{s,2}(X)$

Miscellaneous

Γ	gamma function
I	identity operator
Δ	Laplace operator
$\partial\mathcal{D}$	boundary of $\mathcal{D} \subset \mathbb{R}^d$
$\stackrel{\text{law}}{=}$	equality in distribution
TrA	trace of operator A
A^*	adjoint operator of operator A
A^T	a transposition of matrix A
Λ	set $\{(t,s) \in [0,T]^2; 0 \leq s \leq t \leq T\}$

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Appendices

A. Fractional calculus

In this section we recall some basic definitions from the theory of fractional integration which is a useful tool when dealing with singular fractional Brownian motion.

Consider a separable Hilbert space $(V, \langle\cdot,\cdot\rangle_V)$, a parameter $\alpha \in (0,1)$ and a function $\varphi \in L^1([0,T], V)$.

Definition A.1. *Left-sided and right-sided fractional (Rieman-Liouville) integrals of function φ are defined as*

$$\left(I^{\alpha}_{0+}\varphi\right)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\varphi(s)\,\mathrm{d}s$$

and

$$\left(I^{\alpha}_{T-}\varphi\right)(t) = \frac{1}{\Gamma(\alpha)} \int_t^T (s-t)^{\alpha-1}\varphi(s)\,\mathrm{d}s$$

for almost all $t \in [0,T]$ where $\Gamma(\cdot)$ is the gamma function.

The inverse operators are called fractional derivatives and are given by the Weyl representations.

Definition A.2. *Left-sided fractional derivative of a function $\psi \in I^{\alpha}_{0+}(L^1([0,T],V))$ is defined as*

$$\left(D^{\alpha}_{0+}\psi\right)(t) = \frac{1}{\Gamma(1-\alpha)}\left(\frac{\psi(t)}{t^{\alpha}} + \alpha\int_0^t \frac{\psi(t)-\psi(s)}{(t-s)^{\alpha+1}}\,\mathrm{d}s\right)$$

and right-sided fractional derivative of a function $\psi \in I^{\alpha}_{0-}(L^1([0,T],V))$ is defined as

$$\left(D^{\alpha}_{T-}\psi\right)(t) = \frac{1}{\Gamma(1-\alpha)}\left(\frac{\psi(t)}{(t-T)^{\alpha}} + \alpha\int_t^T \frac{\psi(s)-\psi(t)}{(s-t)^{\alpha+1}}\,\mathrm{d}s\right).$$

For more information on fractional calculus see, for example, [46].

B. Collective compactness of a family of operators

In this section we will recall a characterization of totally bounded sets of linear operators which is based on collectively compactness and is studied by Palmer [43]. Note that in complete metric spaces totally boundedness is equivalent to relative compactness.

Let X, Y be real or complex Banach spaces and let the unit ball in X be denoted by X_1.

Definition B.1. *A subset $\mathcal{R} \subset \mathcal{L}(X, Y)$ is called collectively compact if and only if*

$$\mathcal{R}X_1 = \{Kx :, K \in \mathcal{R}, x \in X_1\}$$

has compact closure.

Theorem B.2. *A collectively compact subset $\mathcal{R}$ of $\mathcal{L}(X, Y)$ is totally bounded if and only if $\mathcal{R}^* = \{K^*, K \in \mathcal{R}\}$ is collectively compact.*

Proof. See Theorem 3.1 in [43]. □