RAMSEY THEORY

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Introduction

Ramsey Theory studies the appearance of some specific patterns in large structures. For example, Ben Green and Terence Tao proved that in the set of primes (the structure), there is an arbitrarily long arithmetic progression (the pattern) [9]. Another result in this area is Endre Szemerédi's theorem on the existence of an arbitrarily long arithmetic progression (the pattern) in subsets of the natural numbers that have positive upper density [17]. Franklin Ramsey classical theorem, reduced to the context of graphs, says that in any graph of large order (the structure), there is either a relatively large independent set, or a relatively large clique (the pattern) [16] (for the basic notions of graphs, see for example Diestel [4]). As a last example, let us mention a result in combinatorial geometry. Among many points in the plane that are in general position, there is a large subset which forms a convex polygon (see [7]). There are many more examples of Ramsey type theorems. For a good introduction on Ramsey Theory, see for example [8].

In this thesis, we focus on the question of finding graphs (more particularly trees) in larger graphs that satisfy certain conditions. We investigate the following conjecture by Martin Loebl, Jánós Komlós and Vera Sós.

Conjecture. *If a graph G has at least half of its vertices of degree at least k, then any tree with at most k edges embeds in G.*

The topic is also related to the Ramsey number of a tree. Indeed, for the special case of $k = \frac{n}{2}$, the Loebl-Komlós-Sós Conjecture implies that the Ramsey number $r(T, 2)$ of a tree T is $2|E(T)|$. This means that if we two-colour the edges of a complete graph on $2n$ vertices, we find any tree with at most n edges in one of the two colour classes.

There are two possible approaches how to investigate the Loebl-Komlós-Sós Conjecture. The first one is to reduce the problem only to certain classes of trees we intend to embed into G. For example, it is easy to see that the Loebl-Komlós-Sós Conjecture is true for the class of stars (a star of order $k+1$ consists of a central vertex of degree k connected to k vertices of degree 1).

The second approach to the question consists in strengthening the condition on the hosting graph, in particular to make it larger and denser. For example, if all vertices of the graph G have degree at least k, it is easy to embed any tree on $k+1$ vertices.

In this thesis, we develop both approaches. The first result is the solution of the Loebl-Komlós-Sós conjecture for the special class of trees with diameter at most 5. By diameter of a tree T, we understand the length of the longest path contained in T.

The second result is a solution of an approximate version of the Loebl-Komlós-Sós Conjecture. The strengthening on the graph G is the following.

- The size k of the tree is linear with respect to the order of the graph.
- The graph G is large.
- The size of the set of vertices with high degree must be slightly larger than $\frac{n}{2}$.
- The degree of the vertices of the mentioned set has to be slightly more than k.

This gives us the following theorem.

Theorem. *For every $\gamma, q > 0$ there is an $n_0 \in \mathbb{N}$ so that for all graphs G on $n \geq n_0$ vertices the following is true.*
If at least $(1+\gamma)\frac{n}{2}$ vertices of G have degree at least $(1+\gamma)qn$, then G contains, as subgraphs, all trees with at most qn edges.

This result is based on a paper of Miklós Ajtai, Janós Komlós and Endre Szemerédi [1], where they proved the above stated theorem for the special case when $q = \frac{1}{2}$. Their result can be easily generalised for $q \geq \frac{1}{2}$, but the case when $q < \frac{1}{2}$ needs new ideas.

The thesis is divided into two chapters. The first one, entitled *Regularity Lemma*, contains background information on this powerful tool in modern graph theory. It explains the notion of regularity, states the Regularity Lemma, gives a proof of it and describe several properties we need in the second part. All the facts of Chapter I are well-known. For a good survey on the topic, refer to [11].

The second chapter, entitled *Embedding of trees*, contains mostly, but not exclusively, new results. The solution of the Loebl-Komlós-Sós Conjecture for trees of diameter at most 5 can be found in Section 2.4. This section is based on [15]. For completeness, we quote a result of Cristina Bazgan, Hao Li and Mariusz Woźniak in Section 2.3 saying that the Loebl-Komlós-Sós Conjecture is true for paths. This comparison is especially interesting, as it envolves completely different classes of trees: one class contains trees with very small diameter, and the other one with very large diameter.

In Section 2.5, we give a solution to an approximative version of the Loebl-Komlós-Sos Conjecture, together with many tools for embedding trees using regularity. At the end of the section, we improve this result, extending it to a class of graphs more general than trees. A shorter but more dense proof of the approximate version of the Loebl–Komlós–Sós Conjecture can be found in [14].

Chapter 1

Regularity Lemma

After introducing the notion of regularity in Section 1.1, we state several equivalent formulations of the Regularity Lemma in Section 1.2 and we prove their equivalence in Section 1.3.

Then, we discuss the usual procedure used after the application of the Regularity Lemma in Section 1.4. The last section is devoted to some useful properties we can deduct from regularity.

Most of the material in this chapter is from the notes [11] written by the author during a course on Ramsey theory [13]. For a good survey on Regularity Lemma.

1.1 Notion of Regularity

For a graph $G = (V, E)$ and for two disjoint sets $X, Y \subseteq V$, denote by $e(X, Y)$ the number $|\{\{x, y\} \in G, x \in X, y \in Y\}|$. Then the *density* is defined by

$$\delta(X, Y) := \frac{e(X, Y)}{|X||Y|}.$$

Given an $\varepsilon > 0$, call a pair (A, B) *ε-regular*, if for any subsets $X \subseteq A$ and $Y \subseteq B$ with $|X| \geq \varepsilon|A|$ and $|Y| \geq \varepsilon|B|$, we have

$$|\delta(X, Y) - \delta(A, B)| < \varepsilon.$$

So, in a regular pair any the density between two significant subsets is about the same as in the whole pair. If a pair is ε-regular, it gives us extra information, because it can be approximated by a regular graph.

In the definition of ε-regularity, the number ε represents two different things: first, the portion of the set that forms a significant subset; and then, the difference of density allowed for significant sets. They have nothing in common, except that they are small. Therefore, we bound them by ε.

In the next definition we differentiate these two meaning of ε and use α for the portion of a set to be significant.

We say that a pair (A, B) is (ε, α)-*regular* if, for any subsets $X \subseteq A$ and $Y \subseteq B$ with $|X| \geq \alpha|A|$ and $|Y| \geq \alpha|B|$, it holds that $|\delta(X, Y) - \delta(A, B)| < \varepsilon$.

A partition $\mathcal{C} = \{V_0, V_1, \ldots, V_N\}$ of a vertex set $V(G)$ is called $(\varepsilon; N)$-*equitable* (or *equitable*), if

- $|V_i| = |V_j|$ for $i, j \geq 1$,
- $|V_0| \leq \varepsilon n$, where $n = |V(G)|$, and
- all but at most εN^2 pairs (V_i, V_j) with $i, j \geq 1$ are ε-regular.

Analogously, we we have the following definition.
A partition $\mathcal{C} = \{V_0, V_1, \ldots, V_N\}$ of a vertex set $V(G)$ is called $(\varepsilon, \alpha; N)$-*equitable*, if

- $|V_i| = |V_j|$ for $i, j \geq 1$,
- $|V_0| \leq \varepsilon n$, where $n = |V(G)|$, and
- all but at most εN^2 pairs (V_i, V_j) with $i, j \geq 1$ are (ε, α)-regular.

The sets $V_i \in \mathcal{C}$ are called *clusters*. The set V_0 is called the *exceptional set*. The only meaning of the exceptional set V_0 is to have the rest of the clusters of the same size. In this chapter, we formulate a version of Regularity Lemma that does not consider equitable partition.

We say that a partition $\mathcal{R}'$ refines a partition $\mathcal{R}$, if, for any choice of an element $R' \in \mathcal{R}'$, there exists an element $R \in \mathcal{R}$ such that $R' \subseteq R$. Then, we write $\mathcal{R}' \prec \mathcal{R}$.

1.2 Different versions of the Regularity Lemma

Here is the first formulation of Szemerédi's Regularity Lemma. We shall prove this version in Section 1.6.

Theorem 1.2.1 (The Regularity Lemma)**.** *For any $\varepsilon > 0$ and any $m \in \mathbb{N}$, there exist $M, n_0 \in \mathbb{N}$ such that every graph on $n \geq n_0$ vertices admits an $(\varepsilon; N)$-equitable partition of its vertex-set that is ε-regular, with $m \leq N \leq M$.*

The next formulation of the Regularity Lemma is the one we shall use in Section 2.5.

Theorem 1.2.2 (Second formulation of the Regularity Lemma)**.** *For every $\varepsilon, \alpha > 0$ and $m \in \mathbb{N}$, there exist $M, n_0 \in \mathbb{N}$ such that every graph G of order $n \geq n_0$ admits an equitable partition $\{V_0, V_1, \ldots, V_N\}$ of its vertex-set with $m \leq N \leq M$.*

In the following formulation of the Regularity Lemma, we do not require the partition to be equitable, but only semi-equitable.

Theorem 1.2.3 (Third formulation of the Regularity Lemma)**.** *For every $\varepsilon > 0$ and every $m \in \mathbb{N}$ there exist $M, n_0 \in \mathbb{N}$ such that every graph of order $n \geq n_0$ admits a partition $\{V_1, \ldots, V_N\}$ of its vertex set with $m \leq N \leq M$ and*

- $||V_i| - |V_j|| \leq 1$ *for all i, j, and*
- *all but at most εN^2 pairs $(V - i, V_j)$ are ε-regular.*

1.3 Different formulations are equivalent

In this section we prove the equivalence of the different formulations of the Regularity Lemma.

Proposition 1.3.1. *Theorems 1.2.1, 1.2.2 and 1.2.3 are equivalent.*

Proof. 1.2.1 $\Rightarrow$ 1.2.2:
Suppose $\varepsilon, \alpha > 0$ and $m \in \mathbb{N}$ are given. Set $\tilde{\varepsilon} := \min\{\varepsilon, \alpha\}$ and use Theorem 1.2.1 with values $\tilde{\varepsilon}$ and m. It gives us integers n_0 and M.

We claim that if (V_i, V_j) is $\tilde{\varepsilon}$-regular, then it is also (ε, α)-regular. Let $U_i \subseteq V_i$ with $|U_\ell| \geq \alpha |V_\ell| \geq \tilde{\varepsilon} |V_\ell|$ for $\ell = i, j$. Then,

$$\left| \frac{e(U_i, U_j)}{|U_i||U_j|} - \frac{e(V_i, V_j)}{|V_i||V_j|} \right| < \tilde{\varepsilon} \leq \varepsilon.$$

Also $|V_0| < \tilde{\varepsilon} n \leq \varepsilon n$ and at most $\tilde{\varepsilon} N^2 \leq \varepsilon N^2$ pairs (V_i, V_j) are not (ε, α)-regular.

1.2.2 $\Rightarrow$ 1.2.1:
Suppose an $\varepsilon > 0$ and a natural number $m \in \mathbb{N}$ are given. Then, set $\tilde{\varepsilon}, \tilde{\alpha} := \varepsilon$. Theorem 1.2.2 with $\tilde{\varepsilon}, \tilde{\alpha}$ and m implies Theorem 1.2.1 for ε and m.

1.2.1 $\Rightarrow$ 1.2.3:
Suppose an $m \in \mathbb{N}$ and an $\varepsilon > 0$ are given. Choose $\tilde{\varepsilon} := \frac{\varepsilon^2}{16}$ and use Theorem 1.2.1 with the values m and $\tilde{\varepsilon}$. Then, for any graph G with at least $n_0(\tilde{\varepsilon}, m)$ vertices, we get a vertex partition $V(G) := C_0 \cup C_1, \ldots, C_N$ such that $|C_0| < \tilde{\varepsilon} n$, $|C_i| = |C_j| =: s$ for $i, j = 1, \ldots N$, where $m \leq N \leq M$ and all but at most $\tilde{\varepsilon} N^2$ pairs (C_i, C_j) are $\tilde{\varepsilon}$-regular.
Distribute the vertices of C_0 between the sets $C_1, \ldots, C_N$ as evenly as possible. We get sets $V_1, \ldots, V_N$ with $|V_1| \leq |V_2| \leq \cdots \leq |V_N| \leq |V_1| + 1$ and no set V_0.
We claim that, if a pair (C_i, C_j) is $\tilde{\varepsilon}$-regular, the corresponding pair (V_i, V_j) is ε-regular.
Let $U_\ell \subseteq V_\ell$ with $|U_\ell| \geq \varepsilon |V_\ell|$ for $\ell = i, j$. Set $V_\ell^0 := V_\ell \cap C_0$, $U_\ell^0 := U_\ell \cap C_0$, and $S_\ell := U_\ell \setminus U_\ell^0$, i. e. $S_\ell = U_\ell \cap C_\ell$. Then, $V_\ell = V_\ell^0 \cup C_\ell$ and $U_\ell = U_\ell^0 \cup S_\ell$.
We want to prove that $|\delta(U_i, U_j) - \delta(V_i, V_j)| < \varepsilon$. For this, we show that

(i) $|\delta(S_i, S_j) - \delta(C_i, C_j)| < \tilde{\varepsilon}$,

(ii) $|\delta(V_i, V_j) - \delta(C_i, C_j)| < 5\tilde{\varepsilon}$,

(iii) $|\delta(S_i, S_j) - \delta(U_i, U_j)| < \frac{\varepsilon}{2}$.

To prove (i), observe that for $\ell = i, j$,

$$|S_\ell| \geq |U_\ell| - \tilde{\varepsilon}\frac{n}{N} \geq \varepsilon |V_\ell| - \tilde{\varepsilon}\frac{n}{N} \geq \tilde{\varepsilon}|C_\ell|.$$

As (C_i, C_j) is $\tilde{\varepsilon}$-regular, (i) holds.

We prove (ii) by contradiction. Assuming that (ii) does not hold, we get

$$\begin{aligned}
5\tilde{\varepsilon}\frac{n^2}{N^2} &\leq 5\tilde{\varepsilon}|V_i||V_j| \leq \big|\, e(V_i, V_j) - \delta(C_i, C_j)(|C_i \cup V_i^0||C_j \cup V_j^0|) \,\big| \\
&\leq |e(V_i, V_j) - e(C_i, C_j)| + \delta(C_i, C_j)\left(|C_i||V_j^0| + |C_j||V_i^0| + |V_i^0||V_j^0|\right) \\
&\leq (1 + \delta(C_i, C_j))(2\tilde{\varepsilon}\frac{n^2}{N^2} + \tilde{\varepsilon}^2\frac{n^2}{N^2}) \leq 2\tilde{\varepsilon}\frac{n^2}{N^2}(2 + \tilde{\varepsilon}) \\
&< 5\tilde{\varepsilon}\frac{n^2}{N^2},
\end{aligned}$$

a contradiction.

We prove (iii) also by contradiction and assuming, without loss of generality, that $|U_j| \geq |U_i|$.

$$\begin{aligned}
\frac{\varepsilon^2}{4}\frac{n}{N}|U_j| &< \frac{\varepsilon}{4}(\varepsilon - \tilde{\varepsilon})\frac{n}{N}|U_j| \\
&\leq \frac{\varepsilon}{2}|U_i||U_j| \leq \big|\ \delta(S_i, S_j) \cdot |U_i^0 \cup S_i||U_j^0 \cup S_j| - e(U_i, U_j)\ \big| \\
&\leq e(S_i, S_j) - e(U_i, U_j)| + \delta(S_i, S_j)\left(|S_i||U_j^0| + |U_j||U_i^0|\right) \\
&\leq (1 + \delta(S_i, S_j))\left(\tilde{\varepsilon}\frac{n}{N}(|S_i| + |U_j|)\right) \leq 4\tilde{\varepsilon}\frac{n}{N}|U_j| \\
&= \frac{\varepsilon^2}{4}\frac{n}{N}|U_j|,
\end{aligned}$$

a contradiction.

We have just shown that $|\delta(U_i, U_j) - \delta(V_i, V_j)| \leq |\delta(S_i, S_j) - \delta(C_i, C_j)| + |\delta(V_i, V_j) - \delta(C_i, C_j)| + |\delta(S_i, S_j) - \delta(U_i, U_j)| < 6\tilde{\varepsilon} + \frac{\varepsilon}{2} < \frac{\varepsilon^2}{2} + \frac{\varepsilon}{2} < \varepsilon$.

So, we have at most $\tilde{\varepsilon}N^2 < \varepsilon N^2$ pairs (V_i, V_j) that are not ε-regular.

1.2.3 $\Rightarrow$ 1.2.1:

Suppose an $\varepsilon > 0$ and an $m \in \mathbb{N}$ are given. Set $\tilde{\varepsilon} := \frac{\varepsilon}{2}$. Theorem 1.2.3, with $\tilde{\varepsilon}$ and m, gives us integers $\tilde{n}_0$ and M. Set $n_0 := \max\{\tilde{n}_0, \frac{M}{\varepsilon}\}$. Let G be a graph of order $n \geq n_0$, and $C_1, \ldots C_N$ its $(\tilde{\varepsilon}; N)$-equitable partition with $m \leq N \leq M$.

In every cluster C_i with $|C_i| > |C_1|$, choose some vertex $v_i \in C_i$. Set $V_0 := \{v_i; |C_i| > |C_1|\}$. Then $|V_0| < N \leq \varepsilon n$. For each cluster C_i, set $V_i := C_i \setminus V_0$. Then, $|V_1| = |V_2| = \cdots = |V_N|$.

We claim that, if the pair (C_i, C_j) is $\tilde{\varepsilon}$-regular, then the corresponding pair (V_i, V_j) is ε-regular. Let $U_\ell \subseteq V_\ell$ with $|U_\ell \geq \varepsilon|V_\ell|$ for $\ell = i, j$.

Observe that $|U_\ell| \geq \varepsilon|V_\ell| \geq \tilde{2}\varepsilon|V_\ell| \geq \tilde{\varepsilon}|C_\ell|$, for $\ell = i, j$. By regularity of the pair (C_i, C_j), we have

$$|\delta(U_i, U_j) - \delta(C_i, C_j)| < \tilde{\varepsilon}.$$

As $|V_\ell| \geq |C_\ell| - 1 \geq \tilde{\varepsilon}|C_\ell|$, we have

$$|\delta(V_i, V_j) - \delta(C_i, C_j)| < \tilde{\varepsilon},$$

implying the inequality

$$|\delta(U_i, U_j) - \delta(V_i, V_j)| < 2\tilde{\varepsilon} = \varepsilon.$$

We have at most $\tilde{\varepsilon}N^2 < \varepsilon N^2$ pairs (V_i, V_j) that are not ε-regular.
Hence the equivalence between Theorems 1.2.2, 1.2.1 and 1.2.3 is proved. □

1.4 Cleaning the graph

The ε-regularity (or (ε, α)-regularity) of a pair gives us extra information on the graph G. Indeed we can approximate this pair with a regular graph. We still have some pairs of clusters that are not regular, and therefore, we have no control on the degree of the vertices in this pair. We have also no information on the edges incident to the set V_0 and on the edges lying inside some cluster.

An other problem is when the density between a pair is very low (less than ε). Then, it may happen that two significant subsets in the pair have no edge between them. This does not suit us, as we want to take profit of the non-zero degree of the vertices to embed some graph in G.

Therefore, we will delete the undesirable edges to get a subgraph G_δ. We delete

- all edges incident to the set V_0,
- all edges lying in a cluster,
- all edges in irregular pairs, and
- all edges in pairs with low density ($\leq \delta$).

This procedure erases at most

$$\varepsilon n^2 + N\left(\frac{n}{N}\right)^2 + \varepsilon N^2\left(\frac{n}{N}\right)^2 + N^2\delta\left(\frac{n}{N}\right)^2 \leq (2\varepsilon + \frac{1}{m} + \delta)\, n^2 \qquad (1.1)$$

edges.

We shall call such a graph G_δ a *cleaned graph* (with minimal density δ).

1.5 Properties of cluster graphs

After the cleaning procedure of Section 1.4 we got a subgraph G_δ of a graph G. In the graph G_δ, all pairs are regular and have either density 0 or density at least δ. On such a subgraph G_δ, we define a *cluster graph* $H = H(G_\delta)$ as follows. The vertices of H are the clusters in G_δ and two vertices C, D in H are joined by an edge, if the density in the pair (C, D) is positive.

We use the same notation to denote the cluster (set of vertices in G_δ) and the vertex of the cluster graph.

In a cleaned graph, most of the vertices have a degree close to the average degree of the cluster in which it lies. Throughout this section, suppose that G_δ is a cleaned graph with minimal density δ and of order n with (α, ε)-regular pairs and cluster's size s.

Lemma 1.5.1. *Let (C, D) be a pair of clusters in G_δ. Then, all but at most αs vertices v in C have $deg_D(v) > deg_D(C) - \varepsilon s \geq (\delta - \varepsilon)s$.*

We call those $(1 - \alpha)s$ vertices *typical vertices with respect to D.*

Proof of Lemma 1.5.1. Suppose on the contrary that there is a set $C' \subseteq C$ of size $> \alpha s$ of vertices v with $deg_D(v) \leq deg_D(C) - \varepsilon s$. Then, $deg_D(C') \leq deg_D(C) - \varepsilon s$, implying

$$\frac{e(C, D)}{s^2} - \frac{e(C', D)}{|C'|s} \geq \varepsilon,$$

a contradiction with the regularity of the pair (C, D). □

In a similar way, we get the following lemma.

Lemma 1.5.2. *Let (C, D) be a pair of clusters in G_δ. Then, all but at most αs vertices v in C have $deg_D(v) < deg_D(C) + \varepsilon s := \frac{e(C,D)}{s} + \varepsilon s$.*

Proof of Lemma 1.5.2. Similarly as before, denote by C' the set of vertices v with $deg_D(v) \geq deg_D(C) + \varepsilon s$. If $|C'| \geq \alpha s$, then

$$\frac{e(C', D)}{|C'|s} - \frac{e(C, D)}{s^2} \geq \varepsilon.$$

This yields a contradiction. □

Corollary 1.5.3. *Let (C, D) be a pair of clusters in G_δ. Then all but at most $2\alpha s$ vertices v in C_1 have $deg_D(v) \in (deg_D(C) - \varepsilon s, deg_D(C) + \varepsilon s)$.*

Now, instead of looking for the degree in the whole cluster D, we are interested in the degree into a significant subset of D.

Lemma 1.5.4. *Let (C, D) be a pair of clusters in G_δ and let $D' \subseteq D$ with $|D'| \geq \alpha s$. Then, all but at most αs vertices v in C have $deg_{D'}(v') > deg_{D'}(C) - \varepsilon|D'| \geq (\delta - \varepsilon)|D'| \geq (\delta - 2\varepsilon)s$.*

We call those $(1-\alpha)s$ vertices *typical vertices with respect to D'*.

Proof of Lemma 1.5.4. First, observe that as D' is a significant set, we have

$$\frac{e(D', C)}{|D'|s} \leq \frac{e(D, C)}{s^2} + \varepsilon.$$

Denoting by C' the set of vertices v with $deg_{D'}(v') \leq deg_{D'}(C) - 2\varepsilon|D'|$, we have

$$\frac{e(D', C')}{|D'||C'|} \leq \frac{e(D', C)}{|D'|s} - 2\varepsilon \leq \frac{e(D, C)}{s^2} - \varepsilon.$$

This implies

$$\frac{e(D, C)}{s^2} - \frac{e(D', C')}{|D'||C'|} \geq \varepsilon,$$

a contradiction with the regularity of the pair (C, D). □

Lemma 1.5.5. *Let (C, D) be a pair of clusters in G_δ and let $D' \subseteq D$ with $|D'| \geq \alpha s$. Then, all but at most αs vertices v in C have $deg_{D'}(v') < deg_{D'}(C) + 2\varepsilon|D'| \leq deg_{D'}(C) + 2\varepsilon s$.*

Proof of Lemma 1.5.5. Similarly as before, if C' denotes the set of vertices v with $deg_{D'}(v') \geq deg_{D'}(C) + 2\varepsilon|D'|$, we get

$$\frac{e(D', C')}{|D'||C'|} \geq \frac{e(D', C)}{|D'|s} + 2\varepsilon \geq \frac{e(D, C)}{s^2} + \varepsilon,$$

again contradicting the regularity of the pair (C, D). □

After studying the degrees into cluster D or a significant subset of D, we turn our attention to the degree into a wider set. formulace

Lemma 1.5.6. *Let C be a cluster of graph G_δ. Then,*

1. *all but at most αs vertices v of cluster C has degree at least $deg_{G_\delta}(v) \geq deg_{G_\delta}(C) - \varepsilon n$ and*
2. *all but at most αs vertices v have $deg_{G_\delta}(v) \leq deg_{G_\delta}(C) + \varepsilon n$.*

Proof of Lemma 1.5.6. Let C' be the set of vertices v in cluster C that have $deg_{G_\delta}(v) < deg_{G_\delta}(C) - \varepsilon n$. Then, $deg_{G_\delta}(C') < deg_{G_\delta}(C) - \varepsilon n$. On the other hand, as $|C'| \geq \alpha s$, we have

$$deg_{G_\delta}(C') = \sum_{D \neq C} deg_D(C') \geq \sum_{D \neq C} deg_D(C) - \varepsilon s > deg_{G_\delta}(C) - \varepsilon n,$$

a contradiction. The second case is proved similarly. □

Lemma 1.5.7. *Let D be a cluster of graph G_δ. Let $\mathcal{C}$ be a set of clusters. Let $C' \subseteq C$ with $|C'| \geq \alpha s$, for each cluster $C \in \mathcal{C}$. Denote by $\mathcal{C}' = \bigcup_{C \in \mathcal{C}} C'$ the union of those subsets. Then*

1. *all but at most αs vertices v of cluster D has degree at least $deg_{\mathcal{C}'}(v) > deg_{\mathcal{C}'}(D) - 2\varepsilon s|\mathcal{C}|$ and*
2. *all but at most αs vertices v of cluster D have degree at most $deg_{\mathcal{D}'}(v) < deg_{\mathcal{C}'}(D) + 2\varepsilon s|\mathcal{C}|$.*

We call those $(1-\alpha)s$ vertices *typical vertices with respect to $\mathcal{C}'$* (We use only the first property).

Proof of Lemma 1.5.7. Let denote by D' the set of vertices v with $deg_{\mathcal{C}'}(v) \leq deg_{\mathcal{C}'}(D) - 2\varepsilon s|\mathcal{C}|$. If $|D'| \geq \alpha s$, then by regularity,

$$deg_{\mathcal{C}'}(D') = \sum_{C \in \mathcal{C}} deg_{C'}(D') > \sum_{C \in \mathcal{C}} deg_C(D) - 2\varepsilon s = deg_{\mathcal{C}'}(D) - 2\varepsilon s|\mathcal{C}|.$$

On the other hand, by the definition of D',

$$deg_{\mathcal{C}'}(D') \leq deg_{\mathcal{C}'}(D) - 2\varepsilon s|\mathcal{C}|,$$

a contradiction.

The second case is proved analogously. □

Lemma 1.5.8. *Let C be a cluster and let $C' \subseteq C$ be the set of vertices in C that has degree at least Δ in the graph G_δ. If $|C'| \geq \alpha s$, then $deg_{G_\delta}(C) \geq \Delta - \varepsilon n$.*

Proof of Lemma 1.5.8.

$$\begin{aligned} deg_{G_\delta}(C) &= \sum_{D \neq C} deg_D(C) = \sum_{D \neq C} \delta(C, D)s \\ &\geq \sum_{D \neq C} (\delta(C', D) - \varepsilon)s \geq deg_{G_\delta}(C') - \varepsilon n \\ &\geq \Delta - \varepsilon n \end{aligned}$$

□

Let us resume some of the properties we saw in the lemmas of this section. Let $C, D \in V(H)$ and $\mathcal{C} \subseteq V(H)$.

$$deg_{C'}(v) > deg_{C'}(D) - 2\varepsilon s \textit{ for all but at most } \alpha s \textit{ vertices } v \in D. \tag{1.2}$$

$$deg_{G_\delta}(v) > deg_{G_\delta}(D) - \varepsilon n \textit{ for all but at most } \alpha s \textit{ vertices } v \in D. \tag{1.3}$$

$$deg_{\mathcal{C}'}(v) > deg_{\mathcal{C}'}(D) - 2\varepsilon s|\mathcal{C}| \textit{ for all but at most } \alpha s \textit{ vertices } v \in D. \tag{1.4}$$

1.6 The proof of the Regularity Lemma

For any partition of a vertex-set, we define an index. This index is bounded by 1. In Lemma 1.6.1, we observe that refining the partition does not decrease its index. In Lemma 1.6.2, we show how to take profit of the irregularity of a pair to find a partition of the pair with higher index.

In the Index Pumping Lemma 1.6.3, on the base of those observations, we show how to refine a non-regular partition to get its refinement with higher index.

The proof of the Regularity Lemma 1.2.1 consists of iterating the use of the Index Pumping Lemma 1.6.3.

1.6.1 Index of a partition

For a graph $G = (V, E)$ and for disjoint $A, B \subseteq V$, we define the *index* $q(A, B)$ of a pair (A, B) as follows.

$$q(A, B) := \frac{|A||B|}{n^2} d^2(A, B) = \frac{e(A, B)^2}{|A||B|n^2}.$$

For a partition $\mathcal{A}$ of A and a partition $\mathcal{B}$ of B, let us define the index $q(\mathcal{A}, \mathcal{B})$ of these partitions as follows.

$$q(\mathcal{A}, \mathcal{B}) := \sum_{\substack{A' \in \mathcal{A} \\ B' \in \mathcal{B}}} q(A', B').$$

Note that for a partition $\mathcal{P} = \{C_1, \ldots, C_k\}$ of our vertex set V, we define

$$q(\mathcal{P}) = \sum_{i<j} q(C_i, C_j).$$

For the partition of the vertex set with an exceptional set C_0, we consider this set as a set of singletons, instead of taking it as a whole. We can do that, because we do not expect from the exceptional set nothing else than to be small, so we treat it in the proof as singletons and at the end put these singeltons together to form the exceptional set.

1.6.2 Refining the partition

To prove the Regularity Lemma, we need the following Cauchy-Schwartz inequality. For real numbers $m_1, \dots, m_k > 0$ and $e_1, \dots, e_k > 0$,

$$\sum_i \frac{e_i^2}{m_i} \geq \frac{(\sum_i e_i)^2}{\sum_i m_i}. \tag{1.5}$$

Lemma 1.6.1.

1. *Let $C, D \subseteq V$ be disjoint sets. If $\mathcal{C}$ is a partition of C and $\mathcal{D}$ is a partition of D, then $q(\mathcal{C}, \mathcal{D}) \geq q(C, D)$.*

2. *If $\mathcal{P}, \mathcal{P}'$ are partitions of V and $\mathcal{P}'$ refines $\mathcal{P}$, then $q(\mathcal{P}') \geq q(\mathcal{P})$.*

Proof of Lemma 1.6.1.
1) Let $\mathcal{C} = \{C_1, \dots, C_k\}$ and $\mathcal{D} = \{D_1, \dots, D_l\}$, then

$$q(\mathcal{C}, \mathcal{D}) = \sum_{i,j=1}^{k,l} q(C_i, D_j) = \sum_{i,j=1}^{k,l} \frac{|C_i||D_j|}{n^2} d^2(C_i, D_j) = \frac{1}{n^2} \sum_{i,j=1}^{k,l} \frac{e^2(C_i, D_j)}{|C_i||D_j|}$$
$$\underbrace{\geq}_{(1.5)} \frac{(\sum_{i,j=1}^{k,l} e(C_i, D_j))^2}{n^2 \sum_{i,j=1}^{k,l} |C_i||D_j|} = \frac{1}{n^2} \frac{e^2(C, D)}{|C||D|} = q(C, D)$$

2) Let $\mathcal{P} = \{C_1, \dots, C_k\}$ and for $i = 1, \dots, k$ let $\mathcal{C}_i$ be the partition of C_i induced by $\mathcal{P}'$, then

$$q(\mathcal{P}) = \sum_{i<j} q(C_i, C_j) \underbrace{\leq}_{1)} \sum_{i \leq j} q(\mathcal{C}_i, \mathcal{C}_j) \leq \sum_i q(\mathcal{C}_i) + \sum_{i<j} q(\mathcal{C}_i, \mathcal{C}_j) = q(\mathcal{P}')$$

□

Assuming that the pair (C, D) is not regular allows us to strengthen the previous lemma. Then, the irregularity allows us to increase the index.

Lemma 1.6.2. *Let $\varepsilon > 0$ and let $C, D \subseteq V$ be disjoint sets. If (C, D) is not regular, then there are partitions $\mathcal{C} = (C_1, C_2)$ of C and $\mathcal{D} = (D_1, D_2)$ of D such that*

$$q(\mathcal{C}, \mathcal{D}) \geq q(C, D) + \varepsilon^4 \frac{|C||D|}{n^2}. \tag{1.6}$$

Proof of Lemma 1.6.2. If (C, D) is not a regular pair, then there exist $C_1 \subseteq C$ and $D_1 \subseteq D$ with $|C_1| \geq \varepsilon|C|$ and $|D_1| \geq \varepsilon|D|$ such that $|\eta| > \varepsilon$, where $\eta := d(C_1, D_1) - d(C, D)$. The partitions $\mathcal{C} = (C_1, C \setminus C_1)$ and $\mathcal{D} = (D_1, D \setminus D_1)$ satisfy (1.6).

To simplify the notation, set $e_{i,j} = e(C_i, D_j), e = e(C, D), c_i = |C_i|, d_j = |D_j|, c = |C|, d = |D|$. Now

$$\begin{aligned}
q(\mathcal{C}, \mathcal{D}) &= \frac{1}{n^2} \sum_{i,j=1}^{k,l} \frac{e_{i,j}^2}{c_i d_j} = \frac{1}{n^2} \frac{e_{1,1}^2}{c_1 d_1} + \frac{1}{n^2} \sum_{i+j>2} \frac{e_{i,j}^2}{c_i d_j} \\
&\underbrace{\geq}_{(1.5)} \frac{1}{n^2} \frac{e_{1,1}^2}{c_1 d_1} + \frac{1}{n^2} \frac{\left(\sum_{i+j>2} e_{i,j}\right)^2}{\sum_{i+j>2} c_i d_j} \\
&= \frac{1}{n^2} \left(\frac{e_{1,1}^2}{c_1 d_1} + \frac{(e - e_{1,1})^2}{cd - c_1 d_1} \right)
\end{aligned}$$

By definition, we have $e_{1,1} = c_1 d_1 \frac{e}{cd} + \eta c_1 d_1$, inserting this into the equation, we have

$$\begin{aligned}
n^2 q(\mathcal{C}, \mathcal{D}) &\geq \left(\frac{e_{1,1}^2}{c_1 d_1} + \frac{(e - e_{1,1})^2}{cd - c_1 d_1} \right) \\
&= \frac{1}{c_1 d_1} \left(c_1 d_1 \frac{e}{cd} + \eta c_1 d_1 \right)^2 + \frac{1}{cd - c_1 d_1} \left(e - c_1 d_1 \frac{e}{cd} - \eta c_1 d_1 \right)^2 \\
&= c_1 d_1 \left(\frac{e}{cd} + \eta \right)^2 + (cd - c_1 d_1) \left(\frac{e}{cd} - \frac{\eta c_1 d_1}{cd - c_1 d_1} \right)^2 \\
&\geq \frac{e^2}{cd} + c_1 d_1 \eta^2 \geq \frac{e^2}{cd} + \varepsilon^2 cd \cdot \varepsilon^2
\end{aligned}$$

So $q(\mathcal{C}, \mathcal{D}) \geq \frac{1}{n^2} \left(\frac{e^2}{cd} + \varepsilon^4 cd \right)$.

Hence, the partitions $\mathcal{C}$ and $\mathcal{D}$ satisfy (1.6) $\square$

1.6.3 The Index Pumping Lemma

The following Lemma, is the key tool in the proof of the Regularity Lemma.

Lemma 1.6.3 (Index Pumping Lemma)**.** *Let $0 < \varepsilon < \frac{1}{4}$ and $\mathcal{P} = \{C_0, C_1 \ldots, C_k\}$ be a partition of V with exceptional set C_0 and $|C_i| = |C_j|$ for $i, j \geq 1$. If $\mathcal{P}$ is not ε-regular, then there is a partition $\mathcal{P}' = \{C'_0, C'_1, \ldots, C'_l\}$ of V with exceptional set C'_0 such that*

1. $k \leq l \leq k \cdot 4^k$,
2. $|C'_0| \leq |C_0| + \frac{n}{2^k}$,
3. $|C'_1| = |C'_2| = \cdots = |C'_l|$,
4. $q(\mathcal{P}') \geq q(\mathcal{P}) + \frac{\varepsilon^5}{2}$.

Proof of Lemma 1.6.3. We have a non-regular partition $\mathcal{P} = \{C_0, C_1, \ldots, C_k\}$. For all $1 \leq i, j \leq k$, define a partition $\mathcal{C}_{ij}$ of C_i and a partition $\mathcal{C}_{ji}$ of C_j as follows.

If the pair (C_i, C_j) is ε-regular, then $\mathcal{C}_{ij} = \{C_i\}$ and $\mathcal{C}_{ji} = \{C_j\}$, but if the pair is not ε-regular, use the partition as in Lemma 1.6.2. Then $|\mathcal{C}_{ij}| = |\mathcal{C}_{ji}| = 2$ and $q(\mathcal{C}_{ij}, \mathcal{C}_{ji}) \geq q(C_i, C_j) + \frac{\varepsilon^4 |C_i||C_j|}{n^2}$. Let $\mathcal{C}_0 := \{\{v\}, v \in C_0\}$.

For each $i = 1, \ldots, k$, let $\mathcal{C}_i$ be the unique maximal partition (with respect to $\prec$) refining all $\mathcal{C}_{ij}$, with $j = 1, \ldots, k$. Then, $|\mathcal{C}_i| \leq 2^{k-1}$. Now, consider the partition $\mathcal{C} = \{\mathcal{C}_0\} \cup \bigcup_{i=1}^{k} \mathcal{C}_i$. We have $\mathcal{C} \prec \mathcal{P}$ and $k \leq |\mathcal{C}| \leq k \cdot 2^{k-1}$.

The index of the partition $\mathcal{C}$ satisfies

$$\begin{aligned}
q(\mathcal{C}) &= \sum_{1 \leq i < j \leq k} q(\mathcal{C}_i, \mathcal{C}_j) + \sum_{1 \leq i} q(\mathcal{C}_0, \mathcal{C}_i) + \sum_{0 \leq i} q(\mathcal{C}_i) \\
&\geq \sum_{1 \leq i < j \leq k} q(\mathcal{C}_i, \mathcal{C}_j) + \sum_{1 \leq i} q(\mathcal{C}_0, \mathcal{C}_i) + q(\mathcal{C}_0) \\
&\underbrace{\geq}_{1.6.2} \sum_{1 \leq i < j \leq k} q(C_i, C_j) + \varepsilon k^2 \frac{\varepsilon^4 c^2}{n^2} + \sum_{1 \leq i} q(\mathcal{C}_0, \mathcal{C}_i) + q(\mathcal{C}_0) \\
&= q(\mathcal{P}) + \varepsilon^5 \frac{k^2 c^2}{n^2} \\
&> q(\mathcal{P}) + \frac{\varepsilon^5}{2}.
\end{aligned}$$

The last inequality comes from $c \geq \frac{n(1-\varepsilon)}{k} \geq \frac{3}{4}\frac{n}{k}$. So, $\frac{k^2 c^2}{n^2} \underbrace{\geq}_{\varepsilon \leq \frac{1}{4}} \frac{3^2}{4^2} > \frac{1}{2}$.

Now, we modify the partition $\mathcal{C}$ to get a partition with all unexceptional clusters of the same size. As $|\mathcal{C}| \leq k \cdot 2^{k-1}$, the average size of a cluster is $\frac{n}{k \cdot 2^{k-1}}$. Set $d := \frac{c}{4^k}$. To construct the partition $\mathcal{P}'$, we divide each set into smaller sets of size d and put the rest-over into the set C_0'. For each cluster of $\mathcal{C}$ we put less then d elements into C_0'. We have

$$|C_0'| \leq |C_0| + (d-1)|\mathcal{C}| \leq |C_0| + \frac{n}{k}\frac{1}{4^k} \cdot k2^k = |C_0| + \frac{n}{2^k}.$$

Then the set C_0' is not too big, as we promised. What about the index of $\mathcal{P}'$? We did change the partition, but as we refined it we could only increase the index, so

$$q(\mathcal{P}') \geq q(\mathcal{C}') \geq q(\mathcal{P}) + \frac{\varepsilon^5}{2}.$$

This ends the proof of the Index Pumping Lemma. □

1.6.4 Proof of the Regularity Lemma

Now, with an iterate use of the Index Pumping Lemma, we prove the Regularity Lemma.

Proof of the Regularity Lemma 1.2.1. The main idea is to use the Index Pumping Lemma 1.6.3 $\frac{2}{\varepsilon^5}$ times. Let $\mathcal{P}_0$ be an initial partition with m clusters. If the given partition is regular, we are done. If not, then use inductively the Index Pumping Lemma 1.6.3. If a partition $\mathcal{P}_i$ is not regular, we find a partition $\mathcal{P}_{i+1}$ such that $q(\mathcal{P}_{i+1}) \geq q(\mathcal{P}_i) + \frac{\varepsilon^5}{2}$. As the index is bounded by 1, we can repeat this step at most $\frac{2}{\varepsilon^5}$ times. □

Chapter 2

Embedding of Trees

2.1 Introduction

Martin Loebl conjectured the following:

Conjecture 2.1.1 (Loebl Conjecture)**.** *If a graph G of order n has at least $\frac{n}{2}$ vertices of degree at least $\frac{n}{2}$, then any tree with at most $\frac{n}{2}$ edges embeds into G.*

Janós Komlós and Vera T. Sós generalised the Loebl Conjecture to the following [6]:

Conjecture 2.1.2 (Loebl-Komlós-Sós Conjecture)**.** *If a graph G has at least half of its vertices of degree k, then any tree with at most k edges embeds in the graph G.*

An other related conjecture comes from Paul Erdős and Vera T. Sós [5]. Instead of considering graphs with high median degree, they considered graphs with high average degree.

Conjecture 2.1.3 (Erdős-Sós Conjecture)**.** *Let G be a graph on n vertices with more than $\frac{n}{2}(k-1)$ edges. Then, any tree with at most k edges embeds into G.*

If true, the Erdős-Sós Conjecture would imply an immediate bound on the Ramsey number for trees. Indeed, if we colour a complete graph on ℓk vertices with ℓ colours, then in at least one colour, we have more than $(k-1)\frac{n}{2}$ edges. Then by the Erdős-Sós Conjecture, we would be able to embed any tree with at most k edges in this colour-class. Ajtai, Komlós, Simonovits and Szemerédi are working on a paper on the Erdős-Sós Conjecture for graphs on sufficiently many vertices (see [2]). Andrew McLennan proved that the Erdős-Sós Conjecture is true for trees of diameter at most four. The proof of this result can be found in [12].

It is trivial to see that both the Loebl-Komlós-Sós and the Erdős-Sós Conjecture are true for stars. Indeed, it is enough to find one vertex of degree k and embed the center of the star on this vertex. We prove the simple fact that the Loebl-Komlós-Sós Conjecture is true for dumbbells, (two stars with their centres joined by an edge) in the beginning of Section 2.4. Then, in the rest of the section, we prove that the conjecture is also true for any tree with diameter at most 5.

Cristina Bazgan, Hao Li and Mariusz Woźniak proved that the Loebl-Komlós-Sós Conjecture is true for the class of trees consisting of paths and also for the class of trees consisting of paths with one of its vertex identified with a centre of a star. We insert the proof of their theorem in Section 2.3, and find some other classes of trees for which the Loebl-Komlós-Sós Conjecture is true as an easy corollary of their theorem.

In Section 2.2, by extending an argument of Zhao [19], we prove that condition on the number of vertices with degree at least k can not be relaxed too much, for $k \leq \frac{n}{2}$. Indeed, we can not replace $\frac{n}{2}$ by $\frac{n}{2} - \sqrt{n} - 2$.

Section 2.5 presents an approximation of the Loebl-Komlós-Sós Conjecture for sufficiently large graphs; it is Theorem 2.5.1. This result is a joined work with Maya Stein. It is greatly inspired by a preprint of Ajtai, Komlós and Szemerédi [1]. In this thesis the proof of this result is divided in different independent lemmas, introducing many tools for the embedding of trees into sets of regular pairs. Also we present two different proofs, using different embedding techniques. At the end of the section, there is an easy generalisation of Theorem 2.5.1, extending the class of graphs that we can embed in G to a slightly wider class. This is Theorem 2.5.31. A shorter, but denser version of the proofs of Theorems 2.5.1 and 2.5.31 can be found in [14].

2.2 A graph not not containing all trees with at most k edges

In this section we show that, for $k \leq \frac{n}{2}$, there is a graph with $\frac{n}{2} - \sqrt{\frac{n}{q}} - \frac{n}{k}$ vertices with degree k that does not contain a certain tree of order $k+1$.

Construction 2.2.0.1. Let $k = qn$ with $q \leq 12$ and let G be a graph of order n with a vertex set $V_1 \cup \cdots \cup V_{\frac{1}{q}}$ such that $|V_i| = k$ for all $i = 1, \ldots, \frac{1}{q}$. Let $A_i \cup B_i = V_i$ be a partition of the vertex set V_i, for all $i \in [\frac{1}{q}]$ such that $|A_i| = \frac{qn}{2} - \sqrt{qn} - 1$.

Each vertex of the set A_i is adjacent to any vertex in V_i and sends exactly one edge in B_{i+1} (if $i = \frac{1}{q}$, then it sends to B_1) in the following way. Partition A_i in subsets $A_i^1, \ldots, A_i^m$, each of size at most $\sqrt{qn}$. We have that $m = \lceil |A_i|/\sqrt{qn} \rceil \leq \frac{\sqrt{qn}}{2} - 1$. Choose some vertices $v_1^{i+1}, \ldots, v_m^{i+1}$ in the set B_{i+1}. Now, each vertex from A_i^j is adjacent to the vertex v_j^{i+1}.

The vertices in $\bigcup A_i$ have degree k and $|\bigcup A_i| = \frac{1}{q}(\frac{qn}{2} - \sqrt{qn} - 1) = \frac{n}{2} - \sqrt{\frac{n}{q}} - \frac{1}{q}$.

Lemma 2.2.1. *Let T be a tree of order $k+1$ with one vertex v of degree $\frac{k}{2}$ and all vertices in $N(v)$ have degree 2. All other vertices are leaves. Then, G defined in Construction 2.2.0.1 does not contain T as a subgraph.*

Proof of Lemma 2.2.1. For contradiction, suppose that there is an embedding φ of the tree T in G.

A vertex v_j^i has at most $\sqrt{qn}$ neighbours in A_{i-1}. Therefore, the degree of the vertices in B_i is at most $\sqrt{qn} + |A_i| < q\frac{n}{2}$. This implies that the vertex $v \in V(T)$ cannot be embedded in any vertex of $\bigcup B_i$, as $deg(v) = \frac{qn}{2} > deg(u)$ for any $u \in \bigcup B_i$.

So $\varphi(v) \in \bigcup A_i$. By the symmetry of the construction of G, we may reduce ourself to the study of the case when $\varphi(v)$ is in one of the A_ℓ.

Denote by B' the set of vertices in $\bigcup B_i$ that have positive degree in $\bigcup A_i \setminus A_\ell$. The set B' consists of one vertex in $B_{\ell-1}$ and m vertices of B_ℓ. Therefore we have $|B'| \leq \frac{\sqrt{qn}}{2}$.

Now, the vertex $\varphi(v)$ has degree at most $\frac{\sqrt{qn}}{2}$ in B' and degree $\frac{qn}{2} - \sqrt{qn} - 2$ in A_ℓ. So we have that

$$deg_{B' \cup A_\ell}(\varphi(v)) \leq \frac{qn}{2} - 2.$$

The vertices in $B_\ell \setminus B'$ have neighbours only in A_ℓ. This is a contradiction with the fact that there is an embedding extending $\varphi \restriction v$.

□

2.3 The Loebl-Komlós-Sós Conjecture for paths

Next theorem says that the Loebl–Komlós–Sós Conjecture is true for paths.

Theorem 2.3.1 (Bazgan, Li, Woźniak)**.** *Any graph with at least half of its vertices having degree at least k contains any path of length k.*

Corollary 2.3.2. *The Ramsey number for paths $r(P, 2)$ is $2 \cdot |E(P)|$.*

Indeed, if we 2-colour the edges of a complete graph on $2n$ vertices, then, in one of the two colours, we have at least $\frac{n}{2}$ vertices of degree at least $\frac{n}{2}$. Theorem 2.3.1 implies then, that in this colour we can find any path of length at most n.

Proof of Theorem 2.3.1. The proof goes by contradiction. Let k be the smallest integer such that Theorem 2.3.1 does not hold. Then $k \geq 3$, as for $k = 2$ the theorem holds trivially. With this choice of k, let n be the smallest integer for which there a graph G of order n such tat G satisfy the hypothesis of Theorem 2.3.1, but not the conclusion. Also suppose that G is minimal, i. e. for each edge $e \in E(G)$ we have that $G - e$ does not satisfy the hypothesis of 2.3.1 anymore. Denote by L the set of vertices that have degree at least k. Observe that $S := V(G) \setminus L$ is an independent set.

By minimality of n, we have that G is connected, otherwise at least one of the components satisfies the hypothesis of 2.3.1 and is of smaller order. Therefore, we could embed the path in this component. Also we may assume that each vertex $v \in L$ has at most one neighbour of degree 1, otherwise if v_1, v_2 are neighbours of some $v \in L$ and $deg(v_1) = deg(v_2) = 1$, then the graph $G - \{v_1, v_2\}$ is of order $n-2$ and has at least $|L \setminus \{v\}| \geq \frac{n}{2} - 1 = \frac{|V(G) \setminus \{v_1, v_2\}|}{2}$ vertices of degree at least k. Therefore we can embed our path in $G - \{v_1, v_2\}$. Similarly we get the following lemma.

Lemma 2.3.3. *Let $X \subseteq V \setminus L$. Then $|X| < 2|N(X)|$.*

Proof of Lemma 2.3.3. For contradiction suppose that $|X| \geq 2|N(X)|$. Consider the graph $G' := G - X$. The order of G' is at most $n - 2|N(X)| =: n'$. The set $L(G')$ of vertices in G' with degree at least k contains $L \setminus N(X)$ and therefore $|L(G')| \geq \frac{n}{2} - |N(X)| = \frac{n'}{2}$. By our assumption, n is the smallest integer for which the Theorem 2.3.1 does not hold. Therefore we can embed a path of length k in $G' \subseteq G$, a contradiction. □

Lemma 2.3.4. *The graph G contains none of the following subgraphs*

1) a path P of length $k - 1$ with one extremity in the set L,

2) a path P' of length $k - 2$ with both extremities in the set L,

3) a cycle C of length k,

4) a cycle C' of length $k - 1$,

5) a cycle $\tilde{C}$ of length $k - 2$.

Proof of Lemma 2.3.4. 1) If v is the extremity of the path P that lies in L then it has at least 1 neighbour u that does not lie in the path P. Then the path $V(P) \cup \{u\}, E(P) \cup \{v, u\}$ is a path of length k in the graph G, a contradiction.

2) Let $v_1, v_2 \in L$ are the two extremities of the path P'. Then v_1 has at least one neighbour u that does not lie in the path P'. Then the path $V(P') \cup \{u\}, E(P') \cup \{v_1, u\}$ is a path in G of length $k-1$ with one extremity in L, a contradiction with 1).

3) We have that $|V(G)| \geq k+1$ and as G is connected there exists a vertex v in $V(G) \setminus V(C)$ that is adjacent to our cycle C. Then we can find a path of size k in the induced subgraph on $V(C) \cup \{v\}$, a contradiction.

4) First suppose that there is a vertex $v \in L \setminus V(C')$. As G is connected, we can fin a path connecting v with the cycle C' such that all vertices except one extremity lie in $V(G) \setminus V(C')$. Then we can find a path of length $k-1$ with one extremity in L in the graph G, a contradiction with 1). Therefore we may suppose that the cycle C' contains all vertices of the set L. If two vertices $u, v \in L$ are adjacent in C', then $C' - \{u, v\}$ is a path of length $k-2$ with both extremities in L, a contradiction with 2). So there are no consecutive vertices from L on C'. As $V(G) \setminus L$ is independent and $|L| \geq |V(G) \setminus L|$, we have that C' goes through all vertices from G and therefore is of length $|V(G)| \geq k+1$, a contradiction.

5) Suppose that there is a cycle of length $k-2$ and choose between all cycles of length $k-2$, the one with the most vertices in L.

First suppose that there exists a vertex in $L \setminus V(\tilde{C})$. As G is connected, there exists a path connecting this vertex with the cycle $\tilde{C}$. This path is of length at most 1, otherwise we can find a path of length $k-1$ with one extremity in L, a contradiction to 1). So we have an edge $\{v, u\} \in E(G)$, with $v \in L \setminus V(\tilde{C})$ and $u \in V(\tilde{C})$.

If the neighbour u_1 or u_2 of u in the cycle $\tilde{C}$ is in L, then the path with vertex set $V(P) \cup \{v\}$ and with edges $E(P) \cup \{u, v\} \setminus \{u, u_i\}$ is of length $k-2$ with both extremities u_i and v in L, a contradiction with 2). So any neighbour of v that lies on the cycle $\tilde{C}$ is from L and its neighbours on the cycle are in $V(G) \setminus V(\tilde{C})$.

Consider w, the neighbour of u_1 in the cycle $\tilde{C}$, different from u. If $\{v, w\} \in E(G)$, then the path $\{w, v, u\}$ together with $\tilde{C} - u_1$ forms a cycle of length $k-2$ with one more vertex from L than $\tilde{C}$, a contradiction. But as $u_1 \notin L$, we have that $w \in L$ and therefore between two neighbours of v in $\tilde{C}$ we have at least three vertices (two from $V(G) \setminus L$ and one from L between them). Hence $|N(v) \cap V(\tilde{C})| \leq (k-2)/4$ and thus $|N(v) \setminus V(\tilde{C})| \geq (3k+2)/4 \geq 2$ for $k \geq 2$. Observe that all neighbours

of v, not lying on $\tilde{C}$, are in $V(G) \setminus L$, otherwise we find a path of length $k-1$ with one extremity, the neighbour of v, lying in $L \setminus V(\tilde{C})$. This contradicts 1). So, by Lemma 2.3.3, at least one of these neighbours v_1 have degree greater than 1. All the neighbours of v_1 are in $V(\tilde{C}) \cup \{v\}$, otherwise there is a path of length k formed by the path of length 3 containing this neighbour, and the vertices v_1, v, u and by $\tilde{C} - \{u, u_1\}$. So let v_2 be its neighbour on the cycle $\tilde{C}$. Consider the path formed by the path of length 2 on vertices v, v_1, v_2 and by $\tilde{C} - e$, where e is one of the edges containing v_2. This path has length $k-1$ and has one extremity in L, a contradiction. Therefore we may assume that the cycle $\tilde{C}$ contains all vertices in L and thus $k-2 = |L| + |Y_C|$, where $Y_C := V(\tilde{C}) \setminus L$. Consider an orientation of our cycle $\tilde{C}$ and denote by S the set of vertices from L which has a successor in $\tilde{C}$ also in L. The cardinality of S is equal to the number of edges in $\tilde{C}$ with both extremities in L and thus is equal to

$$|S| = |E(\tilde{C})| - 2|Y_C| = k - 2 - 2|Y_C| = |L| - |Y_C|. \tag{2.1}$$

We claim that none of the vertices from S have a common neighbour in $V(G) \setminus V(\tilde{C})$. Indeed if $u \notin V(\tilde{C})$ is a common neighbour to $v_1, v_2 \in S$ then consider the path formed by the path of length 2 containing v_1, u, v_2 together with $\tilde{C} - \{e_1, e_2\}$, where e_i is an edge of $\tilde{C}$ with both extremities in L: v_i and its successor in the orientation of $\tilde{C}$. This path has length $k-2$ and has its both extremities in L, a contradiction with 2). Observe that each vertex from L has at least three neighbours in $V(G) \setminus V(\tilde{C})$. So

$$|N(S) \setminus V(\tilde{C})| = \sum_{v \in S} |N(v) \setminus V(\tilde{C})| \geq 3|S|.$$

As all vertices that are not in $\tilde{C}$ are not vertices from L, we have by (2.1) that

$$3|L| = 3|S| + 3|Y_C| \leq |V(G) \setminus L| + 2|Y_C|,$$

so $n \geq 4|L| - 2|Y_C| \geq 2n - 2|Y_C|$. This implies that $|Y_C| \geq \frac{n}{2}$, implying that the cycle $\tilde{C}$ is of length at least $n \geq k+1$, a contradiction. □

Let us turn back now to the proof of Theorem 2.3.1. By our assumption, the theorem holds for any $k' < k$. Therefore our graph G contains a path of length $k-1$. By 1), we know that both extremities of this path lie in $V(G) \setminus L$. We shall consider a subpath P, by deleting these two extremities. The path P has length $k-3$ and has its two extremities $v_1, v_2 \in L$.

The vertices v_1 and v_2 have each at least 3 neighbours that are not contained in $V(P)$ and any such neighbour is in L or we would get 2). Also remark that by 5), denoting by W_1 and W_2 the sets of neighbours in $V(G) \setminus V(P)$ of v_1 and v_2 respectively, we have $W_1 \cap W_2 = \emptyset$. Remark that the neighbours of W_1 and W_2 must lie on P, otherwise we get 1). Set $N_i := N(W_i) \setminus \{v_i\}$ and $N := N_1 \cup N_2 \subseteq L$.

Let $w_i \in W_i$ and let $u_i \in N_i$ be any of its neighbour. Now consider the neighbour w of u_i on the subpath of P from u_i to v_i. If $w \in L$, then the path formed by $P - \{w, u_i\}$ and by the path on vertices v_i, w_i, u_i is of length $k-2$ and has both extremities in L, a contradiction with 2).

Denote by S_i the set of such vertices w adjacent in P to some $u_i \in N_i$ and lying on the subpath with extremities u_i and v_i. Set $S := S_1 \cup S_2$. Any vertex in N generates one vertex that is in S. We want to show now that different vertices in N generates different vertices in S. So suppose that there exists $w \in S_1 \cap S_2$ with neighbours $u_i \in N_i$ on P and $u_i = N(w_i)$, for some $w_i \in W_i$. The cycle formed by $P - w$ and by the paths on vertices v_i, w_i, u_i is of length $k-1$, what contradicts 4). Thus we have

$$|S_1| + |S_2| = |S| = |N|. \tag{2.2}$$

Now consider an edge $e = \{u, v\} \in E(P)$ such that u is on the path from v to v_1. If u is connected to v_2 and at the same time v is connected to v_1, then $P - \{u, v\}$ together with the edges $\{u, v_2\}$ and v, v_1 from a cycle of length $k-2$, a contradiction with 5). For the same reason, we have that $\{v_1, v_2\} \notin E(P)$. This implies that

$$|N(v_1) \cap V(P)| + |N(v_2) \cap V(P)| \leq |V(P)| - 1 = k - 3.$$

As $v_1, v_2 \in L$, we have

$$|W_1| + |W_2| = |N(v_1)| - |N(v_1) \cap V(P)| + |N(v_2)| = |N(v_2) \cap V(P)| \geq k + 3.$$

Now we use Lemma 2.3.3 for $X := W_1 \cup W_2$ and get

$$2 + |N| \geq \frac{|W_1| + |W_2|}{2} \geq \frac{k+3}{2}.$$

Combining this result with (2.2), we get that $|N| + |S| \geq (k+3) - 4 = k - 1$. $N, S \subseteq V(P)$ and N and S are disjoint, as $N \subseteq L$ and $S \cap L = \emptyset$. Therefore we have that P is of length at least $k-2$, a contradiction.

□

Corollary 2.3.5 (Bazgan, Li, Woźniak)**.** *Any graph with at least half of its vertices having degree at least k contains any tree of order $k+1$ consisting of a path and a star with its centre on the path.*

Proof of Corollary 2.3.5. Let H be a graph of order $k+1$, consisting of a path P of length $k - s \leq k - 1$ with v, a vertex in P of degree $s+2$ in H. We want to embed H in G. First remember that we can assume that the set of vertices with degree less that k from an independent set. By Theorem 2.3.1 we find an embedding of a path of length at least k. There are two subpaths P_1 and P_2 of

length $k-1$ that are shifted by 1 one from the other. Consider the path P_1 embed P on P_1 and look at the vertex $u := \varphi(v)$. If u is a vertex of degree at least k, we can extend the embedding φ to $V(H)$, as $|N(u) \setminus \varphi(V(P))| \geq s$. So suppose that u is a vertex with degree $< k$. Then its neighbours in the path P_1 have degree at least k. Consider then the shift T by one of $\varphi(P)$ such that $T(\varphi(P))$ embeds on the path P_2. Then $T(\varphi(v))$ is a vertex with degree at least k and the embedding $T \circ \varphi$ can be extended to $V(H)$. □

Using the proof of Theorem 2.3.1 and pushing the argument in the proof of the corollary 2.3.5 a little bit further, we found with Maya Stein the following corollary.

Corollary 2.3.6. *Let G be a graph with at least half of its vertices having degree at least k. Let H be any tree of order $k+1$ consisting of a path of length $k-\ell$ and two stars with their centres anywhere on the path, but at even distance $\leq \ell$.*

Proof of Corollary 2.3.6. Let H be a graph of order $k+1$, consisting of a path P of length $k-\ell \leq k-2$ with v_1 and v_2, a vertices in P of degree s_1 and s_2 respectively in H such that the distance $d(v_1, v_2)$ between v_1 and v_2 in P is even and at most ℓ. We want to embed H in G.

Denote by L the set of vertices that have degree at least k and set $S := V(G) \setminus L$. Once again we assume that S is independent. By Theorem 2.3.1 we find an embedding of a path $\tilde{P}$ of length at least k.

Consider the subpath P_1 of length $k-\ell$, containing one of the extremities u of $\tilde{P}$. Embed P on P_1. Let $u_1 := \varphi(v_1)$ and $u_2 := \varphi(v_2)$. If both u_1 and u_2 are vertices in L, we can extend φ to $V(H)$.

Again we shall consider a shift T that shifts P_1 by one (i. e. $T(V(P_1))$ does not contain any extremities of $\tilde{P}$). If u_1 and u_2 are both in S, then $T \circ \varphi(\{u_1, u_2\}) \subseteq L$ and the embedding $T \circ \varphi$ can be extended to $V(H)$.

Therefore we may suppose that u_1 and u_2 lie in different sets L and S. But as the distance between u_1 and u_2 on the path is even and S is independent, this implies that on the path going from u_1 to u_2 there are two consecutive vertices from L. Denote this two consecutive vertices by w_1 and w_2 (with w_1 closer to u_1 than w_2).

Observe that $w_1 = T^m(u_1)$ for some $m < \ell$. If $T^m(u_2) \in L$, we are done, because $T^m \circ \varphi$ can be extended to $V(H)$. So, assume that $T^m(u_2) \in S$. Then, $T^{m+1}(u_2) \in L$ as well as $T^{m+1}(u_i) := w_2$, and the embedding $T^{m+1} \circ \varphi$ can be extended to $V(H)$.

□

Remark 2.3.7. If we had to extend the embedding $T^\ell \circ \varphi$, then one of the extremities of the path P is mapped on the extremity of $\tilde{P}$ different from u. Therefore we cannot hope to have $d(v_1, v_2)$ larger than ℓ with this approach.

2.4 The Loebl-Komlós-Sós Conjecture for trees of small diameter

In this subsection, we prove that the Loebl-Komlós-Sós Conjecture 2.1.2 is true for the class of trees of diameter at most five.

First, we show the easy fact that the Loebl-Komlós-Sós 2.1.2 is true for the trees of diameter at most 3, i. e. for dumbbells.

Proposition 2.4.1. *Let G be a graph such that at least a half of its vertices have degree at least k, then any tree T of diameter at most 3 and with at most k edges embeds in G.*

Proof of Proposition 2.4.1. If there is an edge between two vertices of L, we can embed the center of the tree on those vertices and embed the leaves without any problem, as they are adjacent to vertices embedded in L. Suppose on the contrary that there is no edges with both end-vertices in L. Counting the number of edges between L and S we get

$$|L|\ k \leq e(L, V \setminus L) < |V \setminus L|\ k,$$

a contradiction.

□

Theorem 2.4.2. *Let G be a graph such that at least a half of its vertices have degree at least k, then any tree T of diameter at most 5 and with at most k edges embeds in G.*

Proof of Theorem 2.4.2. Let G be a graph such that at least a half of its vertices have degree at least k, and let T be a tree of order at most $k+1$ and of diameter at most 5.

We denote by L the set of vertices in G with degree at least k and we set $S := V(G) \setminus L$. We may assume that S is an independent set (we can delete any edge in $\binom{S}{2}$ without changing the sets L and S). Also we may assume that $|L| \leq |S| + 1$. Indeed, suppose this is not the case. Then, if there is an edge between L and S, choose such an edge and delete it. Either L and S keep their cardinality or the size of the set L decreases by one and the size of the set S increases by one. Continue to delete such edges as long as $|L| \leq |S| + 1$ or until there is no edge between L and S. If $e(L, S) = 0$, we have no problem to embed any tree of order

at most $k+1$ in the set L. Indeed, in the induced graph $G \restriction L$, all vertices have degree at least k.
We use the following notation.

$B := \{v \in L;\ deg_L(v) \geq \frac{k}{2}\}$,
$A := L \setminus B$,
$C := \{v \in S;\ deg(v) = deg_L(v) \geq \frac{k}{2}\}$,
$D := S \setminus C$,
$X := \{v \in L;\ deg_{L \cup C}(v) \geq \frac{k}{2}\} \supseteq B$ and
$Y := L \setminus X = \{v \in L;\ deg_{L \cup C}(v) < \frac{k}{2}\} \subseteq A$.

$e(M, K)$ shall denote the number of edges between two given sets M and K and $N_K(M)$ shall denote the set of all the neighbours of M lying in the set K.

For the tree T, we choose an edge containing the center (either the edge is the centre or it contains the center in one of its endvertices), and denote this edge by $\{r_1, r_2\}$. Let

$P := N(r_1) \setminus \{r_2\}$,
$Q := N(r_2) \setminus \{r_1\}$,
$R := N(P) \setminus \{r_1\}$,
$S := N(Q) \setminus \{r_2\}$,
$P' := \{v \in P;\ deg(v) \geq 2\}$,
$Q' := \{v \in Q;\ deg(v) \geq 2\}$.

Without loss of generality, we may assume that

$$|R \cup Q| < \frac{k}{2}.$$

Remark 2.4.3. Along the whole proof, we use many times the fact that both the degree of a vertex, as well as the cardinality of a set of vertices, are natural numbers. So, if for a set U of vertices and for a vertex u, we have $|U| < x + 1$ with $x \in \mathbb{R}$ and $deg(u) \geq x$, then $|U| \leq deg(u)$.

Lemma 2.4.4. *If there exists an edge $e = \{u, v\} \in E(G)$ such that $u \in X$ and $v \in C$, any tree T of order $k+1$ and diameter at most 5 embeds in G.*

Proof of Lemma 2.4.4. Let $\{u, v\} \in E(G)$ with $u \in X$ and $v \in C$. We define our embedding φ as follows.
Embed r_1 in u, r_2 in v and P' in $V(u) \cap (L \cup C)$. We can do so, as

$$|P'| \leq |R| < |R \cup Q| < \frac{k}{2}.$$

Set P_Q to be the set of vertices in P' that are embedded in the set C. Set

$$\begin{aligned} R_Q &:= N(P_Q) \cap R, \\ P_L &:= P' \setminus P_C, \\ R_L &:= R \setminus R_C. \end{aligned}$$

Embed R_Q in $N(\varphi(P_C)) \subseteq L$. Indeed, as each vertex in C has degree at least $\frac{k}{2}$ and

$$|R_C \cup (P_L)| + 1 \leq |R| + 1 < \frac{k}{2}.$$

Otherwise, $Q = \emptyset$, and by Proposition 2.4.1 we can embed T.
Embed Q in $N(v) \subseteq L$. Indeed, v has degree at least $\frac{k}{2}$ and

$$|Q \cup R_C \cup P_L| + 1 \leq |Q \cup R| + 1 < \frac{k}{2} + 1$$

and, therefore, $|Q \cup R_C \cup P_L| + 1 \leq deg(v)$.
Now, we can embed $P \setminus P'$, R_L and S without any problem as they are adjacent to vertices embedded in L. □

Remark 2.4.5. By Lemma 2.4.4, we can assume that $X = B$. Therefore, $A = Y$ and there is no B–C edges.

Lemma 2.4.6. *If there exists an edge $e = \{u, v\}$ such that $u, v \in B$, then any tree T of order $k + 1$ and diameter 5 embeds in G.*

Proof of Lemma 2.4.6. Let $\{u, v\} \in E(G)$ with $u, v \in B$. We define our embedding φ as follows.
Embed r_1 in u, r_2 in v and P' in $V(u) \cap L$. Indeed, $|P'| < \frac{k}{2}$. Now, embed Q in $N(v) \cap L$. We can do so, as $|P' \cup Q| + 1 \leq |R \cup Q| + 1 < \frac{k}{2} + 1$ and, therefore, $|P' \cup Q| + 1 \leq deg_L(v)$.
Now, we can embed $P \setminus P'$, R and S without any problem as they are adjacent to vertices embedded in L. □

Remark 2.4.7. By Lemma 2.4.6, we may assume that the set B is independent.

Lemma 2.4.8. *If there exists a vertex $v \in N(B) \cap L$ such that $deg_B(v) \geq \frac{k}{4}$, then any tree T of order $k + 1$ and diameter 5 embeds in G.*

Proof of Lemma 2.4.8. First, observe that $|P' \cup Q'| < \frac{k}{2}$ and, therefore, the smallest of $|P'|$ and $|Q'|$ is smaller than $\frac{k}{4}$. Suppose that the smallest is P'. The case when the smallest is Q' is done analogously.
We shall define our embedding φ as follows. Embed r_1 in the vertex v and $P' \cup \{r_2\}$ in $N(v) \cap B$. We can do so as $|P'| + 1 < \frac{k}{4} + 1$. Thus $|P'| \leq deg_B(v)$.

Now, embed the set Q' in $N(\varphi(r_2)) =: u$. Indeed, B is independent by 2.4.7, and therefore $N(u) \cap \varphi(P') = \emptyset$ and $|Q'| + 1 \leq \frac{k}{2} + 1$. Thus, $|Q'| + 1 \leq deg_L(u)$.
Now we can embed $P \setminus P'$, $Q \setminus Q'$, R and S without any problem as they are adjacent to vertices embedded in L. □

Lemma 2.4.9. *If there exists a vertex $v \in N(B) \cap L$ such that $deg_{L \cup C}(v) \geq \frac{k}{4}$, then any tree T of order $k+1$, diameter 5 (with $|R \cup Q| < \frac{k}{2}$) and*

$$|P'| < \frac{k}{4}$$

embeds in G.

Proof of Lemma 2.4.9. The proof goes along as the proof of Lemma 2.4.4, embedding r_1 in v, r_2 in $u \in N(v) \cap B$. Then, P' is embedded in $L \cup C$, $R_C := \varphi^{-1}(\varphi(P') \cap C)$ is embedded in L, Q in $N(u) \cap L$. At the end, we embed the leaves of T. □

Lemma 2.4.10. *If there exists a vertex $v \in N(B) \cap L$ such that $deg_{L \cup C}(v) \geq \frac{k}{4}$, then any tree T of order $k+1$, diameter 5 (with $|R \cup Q| < \frac{k}{2}$) and*

$$|P' \cup S| < \frac{k}{2}$$

embeds in G.

Proof of Lemma 2.4.10. By Lemma 2.4.9, we may assume that

$$\frac{k}{4} \leq |P'| \leq |R|.$$

This implies that $|Q| < \frac{k}{4}$. Embed r_2 in vertex v and r_1 in vertex $u \in N(v) \cap B$. Then, embed Q in $N(v) \cap (L \cup C)$. This is possible, as $|Q| + 1 < \frac{k}{4} + 1$. Thus, $|Q| + 1 \leq deg_{L \cup C}(v)$. Denote by Q_C the vertices of Q embedded in the set C, and set

$$\begin{aligned} Q_L &:= Q \setminus Q_C, \\ S_C &:= N(Q_C) \cap S, \\ S_L &:= N(Q_L) \cap S = S \setminus S_C. \end{aligned}$$

Now, we can embed the set S_C in $N(\varphi(Q_C))$. Indeed, $|S_C \cup Q_L| + 1 \leq |S| + 1 < \frac{k}{2} + 1$. Thus, $|S_C \cup G_L| + 1 \leq N(w)$, for any $w \in C$. Observe that by Remark 2.4.5, there is no $B - C$ edge, and therefore, $u \notin N(C)$.
Next, embed the set P' in $N(u) \cap L$. This is possible, as $|P' \cup Q_L \cup S_C| + 1 \leq |P' \cup S| + 1 < \frac{k}{2} + 1$. Thus, $|P' \cup Q_L \cup S_C| + 1 \leq deg_L(u)$.
Now, we can embed the vertices in $(P \setminus P') \cup S_L \cup R$ without any problem, as they are leaves adjacent to vertices embedded in L. □

Proposition 2.4.11. *Let G be a graph such that at least a half of its vertices have degree at least k, then any tree T of order at most $k+1$, of diameter 5 (with $|R \cup Q| < \frac{k}{2}$) and with*

$$|P'| < \frac{k}{4} \quad or \quad |P' \cup S| < \frac{k}{2}$$

embeds in G.

Proof of Proposition 2.4.11. Suppose this is not the case. Lemmas 2.4.9 and 2.4.10 imply that there is no vertex $v \in N(B) \cap L$ with $deg_{L \cup C}(v) \geq \frac{k}{4}$. We bring this fact to a contradiction.
Any vertex $v \in N := N(B) \cap L$ has $deg_D(v) \geq \frac{3k}{4}$. By a double edge-counting argument, we have

$$|A \setminus N|\frac{k}{2} + |N|\frac{3k}{4} \leq e(A, D) < |D|\frac{k}{2}. \tag{2.3}$$

Recall that by Remark 2.4.5, $A = Y$ and thus, $deg_D(v) \geq \frac{k}{2}$, for $v \in A$.
Dividing (2.3) by $\frac{k}{4}$, we get

$$2|A| + |N| < 2|D|. \tag{2.4}$$

Once more, by a double edge-counting and using Remark 2.4.7 and Lemma 2.4.8, we have

$$|B|\frac{k}{2} \leq e(N, B) < |N|\frac{k}{4}.$$

Then,

$$|N| > 2|B|. \tag{2.5}$$

Giving (2.4) and (2.5) together, we find

$$2|D| > 2|A| + 2|B| \geq n,$$

a contradiction with the fact that $|S| \leq \frac{n}{2}$.

□

Remark 2.4.12. We can assume now that $|P'| \geq \frac{k}{4}$ and $|P' \cup S| \geq \frac{k}{2}$. If $x \in (0, 1)$ such that $|P'| = x \cdot \frac{k}{2}$. Then,

$$|S| > (1 - x)\frac{k}{2}. \tag{2.6}$$

On the other hand, we have that $|R| \geq |P'| = x \cdot \frac{k}{2}$. This implies that

$$|Q \cup S \cup (P \setminus P')| < k - 2 \cdot x\frac{k}{2} = (1-x)k.$$

Using 2.6, we get $|Q \cup (P \setminus P')| < (1-x)\frac{k}{2}$. Then,

$$|Q \cup P| = |Q \cup (P \setminus P') \cup P'| < \frac{k}{2}.$$

The rest of the proof of Theorem 2.4.2 goes under this assumption.

Lemma 2.4.13. *If there exists a vertex $v \in N(B \cup C) \cap L$ such that $deg_L(v) \geq \frac{k}{4}$, then any tree T of order $k+1$ and diameter 5 (with $|R \cup Q| < \frac{k}{2}$, $|P'| \geq \frac{k}{4}$ and $|P \cup Q| < \frac{k}{2}$) embeds in G.*

Proof of Lemma 2.4.13. Embed r_2 in vertex v and r_1 in vertex $u \in N(v) \cap (B \cup C)$. Then, embed the set Q in $N(v) \cap L$. Indeed, $|Q| + 1 < \frac{k}{2} - |P'| \leq \frac{k}{4} + 1$, thus $|Q| + 1 \leq deg_L(v)$. Next, embed the set P in $N(u) \cap L$. We can do so, as $|Q \cup P| + 1 < \frac{k}{2} + 1$ and thus $|Q \cup P| + 1 \leq deg_L(u)$.
Now, we can embed the rest of the tree, as they are leaves adjacent to vertices embedded in the set L. □

Proposition 2.4.14. *Let G be a graph such that at least a half of its vertices have degree at least k. Then any tree T of order at most $k+1$ and with diameter 5 (with $|R \cup Q| < \frac{k}{2}$, $|P'| \geq \frac{k}{4}$ and $|P \cup Q| \geq \frac{k}{2}$) embeds in G.*

Proof of Proposition 2.4.14. By Lemma 2.4.13, we may assume that there is no vertex $v \in N := N(B \cup C) \cap L$ with $deg_L(v) \geq \frac{k}{4}$. By Remark 2.4.5 and a double edge-counting, we get

$$|A \setminus N|\frac{k}{2} + |N|\frac{3k}{4} - e(C, N) + |B|k - e(B, N) \leq e(L, D) < |D|\frac{k}{2}.$$

Using this fact in the next double edge-counting argument, we have

$$(|C|+|B|)\frac{k}{2}+|N|\frac{k}{4} \leq (|A|+|B|-|D|)\frac{k}{2}+|B|\frac{k}{2}+|N|\frac{k}{4} < (e(C \cup B), N) < |N|\frac{k}{2}.$$

Multiplying by $\frac{k}{4}$, we get

$$|N| > 2|C| + 2|B|. \tag{2.7}$$

A final double edge-counting gives us

$$(|A| + |B| + |C|)\frac{k}{2} < |A|\frac{k}{2} + |N|\frac{k}{4} \leq e(A, S) < |D|\frac{k}{2} + |C|\frac{k}{2} = |S|\frac{k}{2} + |C|\frac{k}{2}.$$

This implies

$$|S| > |L|,$$

a contradiction.

□

Propositions 2.4.11 and 2.4.14 imply Theorem 2.4.2.

□

2.5 The approximate version of the Loebl-Komlós-Sós Conjecture

In this section, we prove the following approximative version of the Loebl-Komlós-Sós Conjecture.

Theorem 2.5.1 (An approximate version of the L-K-S Conjecture)**.** *For every $\gamma, q > 0$ there is an $n_0 \in \mathbb{N}$ so that for all graphs G on $n \geq n_0$ vertices the following is true.*
If at least $(1+\gamma)\frac{n}{2}$ vertices of G have degree at least $(1+\gamma)qn$, then G contains, as subgraphs, all trees with at most qn edges.

The proof of this theorem is greatly inspired by a preprint of M. Ajtai, J. Komlós and E. Szemerédi [1], where they proved the Theorem 2.5.1 for $q := \frac{1}{2}$.

Theorem 2.5.2 (Ajtai, Komlós, Szemerédi)**.** *For every $\pi > 0$, there is an $n_0 \in \mathbb{N}$ such that for all graphs G on $n \geq n_0$ vertices the following holds.*
If at least $(1+\pi)\frac{n}{2}$ vertices of G have degree at least $(1+\pi)\frac{n}{2}$, then any tree with at most $\frac{n}{2}$ edges embeds in G.

Their approach can be generalised very easily for $q \geq \frac{1}{2}$ (see Remark 2.5.6), while the case $q < \frac{1}{2}$ needs a little different approach. We follow a hint given by Ajtai, Komlós and Szemerédi in [1], but give here a slightly stronger lemma then the original one. This is Lemma 2.5.5.

Before going into the details of the proof, we first state some useful tools as embedding of specific trees. In this way we get in touch with problems one at a time, making the reading easier. In Subsection 2.5.3, we show how to decompose trees into a small set of vertices and small rooted subtrees. The main idea again was taken in [1], and developed to other cases of configurations needed for the proofs.

Also, some lemmas in Subsection 2.5.5 are not directly used in the proof of Theorem 2.5.1, as Propositions 2.5.22, or 2.5.23. They follow the main lines

of the proof of Theorem 2.5.1, giving a good idea how the proof goes through, but letting aside some technical details making them not so general, but more readable.

On page 80, we have a discussion on the main differences between the proofs of Propositions 2.5.22, or 2.5.23 (i. e. the simplified versions) and Propositions 2.5.25 and 2.5.26 respectively (the non-simplified versions used in the proof of Theorem 2.5.1). We show what makes the proof non-simplified versions longer and more complicated.

We also give a variation of Poposition 2.5.25 and one of Proposition 2.5.26, that one can also use for the proof of Theorem 2.5.1.

At the end of this section, we apply the techniques developed so far to prove a stronger theorem. In a graph satisfying the hypothesis of Theorem 2.5.1, we embed some graphs obtained from trees of order at most $k+1$ by adding a few (well chosen) edges. This gives Theorem 2.5.31.

2.5.1 Tools for the proof of the approximative version

The first tool for the proof of Theorem 2.5.1 is Szemerédi's Regularity Lemma. We already stated it in Chapter **??**. We shall use the following version:

Theorem 1.2.2 (Szemerédi's regularity Lemma).
For every $\varepsilon, \alpha > 0$ and $m \in \mathbb{N}$, there exist $M, n_0 \in \mathbb{N}$ so that every graph G of order $n \geq n_0$ admits a partition of its vertex set $V(G) = V_0 \cup V_1 \cup \ldots \cup V_N$ such that

- $m \leq N \leq M$,
- $|V_0| \leq \varepsilon n$,
- $|V_1| = |V_2| = \cdots = |V_N|$,
- *all but at most εN^2 pairs (V_i, V_j) with $i \neq j$ are (ε, α)-regular.*

Next tool is a simplified version of Gallai-Edmonds Matching Theorem (see [4]). Before stating the theorem, we shall introduce the notion of k-factor and factor criticality.
A graph G has a *k-factor* if there exists a spanning subgraph $H \subseteq G$ that is k-regular. We shall say that a graph (or component) $G = (V, E)$ is *k-factor critical* if for any vertex $v \in V$ the graph $G - v$ has a k-factor. So G is 1-factor critical if for any vertex $v \in V(G)$, there exists a matching in E covering all vertices of $V \setminus v$.

Theorem 2.5.3 (Gallai-Edmonds Matching Theorem)**.** *Any graph H contains a set S of vertices such that all components of $H - S$ are 1-factor-critical and such that there is a matching that matches each vertex of S with a different component of $H - S$.*

For a proof of this theorem see [4].

The next two lemmas were taken from [1]. Lemma 2.5.5 is here a little stronger than the original one. For both lemmas we give a proof.

Lemma 2.5.4. *Let I be a finite set, and let $p, q, \zeta > 0$. For all $i \in I$, let $p_i, q_i \in (0, \zeta]$, so that*

$$\frac{p}{\sum_{i\in I} p_i} + \frac{q}{\sum_{i\in I} q_i} \leq 1. \tag{2.8}$$

Then there is a partition of I into I_p and I_q such that $\sum_{i\in I_p} p_i > p - \zeta$ and $\sum_{i\in I_q} q_i > q$.

Proof. For $i, j \in I$ set $i \preceq j$ if $\frac{p_i}{q_i} \leq \frac{p_j}{q_j}$. Let $\ell \in I$ be minimal in this (total) ordering of I with $p \geq \sum_{i\succ\ell} p_i$.
Set $I_p := \{i \in I : \ i \succ \ell\}$ and set $I_q := I \setminus I_p$. By the minimality of ℓ, we have that $p - \zeta < \sum_{i\in I_p} p_i$. So, all we have to show is that $q < \sum_{i\in I_q} q_i$.
Indeed, suppose otherwise. Then by (2.8), and by definition of ℓ, we have that

$$\frac{\sum_{i\in I_q} q_i}{\sum_{i\in I} q_i} < \frac{p - \sum_{i\in I_p} p_i}{\sum_{i\in I} p_i} + \frac{q}{\sum_{i\in I} q_i} \leq 1 - \frac{\sum_{i\in I_p} p_i}{\sum_{i\in I} p_i} = \frac{\sum_{i\in I_q} p_i}{\sum_{i\in I} p_i}.$$

Multiply with $\sum_{i\in I} p_i \cdot \sum_{i\in I} q_i$, subtract $\sum_{i\in I_q} p_i \cdot \sum_{i\in I_q} q_i$, and divide by $\sum_{i\in I_p} q_i \sum_{i\in I_q} q_i$ to obtain

$$\frac{p_\ell}{q_\ell} \leq \frac{\sum_{i\in I_p} p_i}{\sum_{i\in I_p} q_i} < \frac{\sum_{i\in I_q} p_i}{\sum_{i\in I_q} q_i} \leq \frac{p_\ell}{q_\ell},$$

a contradiction. □

Lemma 2.5.5. *Let $\bar{H}$ be a weighted graph on N vertices, with some weight-function ω. Let L be the set of all vertices $v \in V(H)$ with $\overline{deg}(v) \geq K$, for some $K \in \mathbb{N}$. Suppose that $|L| > \frac{N}{2}$.*
Then there are two adjacent vertices $A, B \in L$, and a matching M in $H - (A \cup B)$ such that one of the following holds.

(a) M covers all but at most one cluster in $N(A \cup B)$,

(b) M covers $N(A)$, and $\overline{deg}_{L\cup M}(B) \geq \frac{K}{2}$. Moreover, every edge in M has at most one endvertex in $N(A)$.

Proof of Lemma 2.5.5. We may suppose that $Y := V(H) - L$ is independent. Theorem 2.5.3 applied to the graph H (without considering the weights) yields a separator S and a matching M. Among all such matchings, assume M to be chosen so that it contains a maximal number of vertices of Y.

Clearly, if there is an edge AB with endvertices $A, B \in L' := L \setminus S$, then A and B lie in the same factor-critical component of $H - S$, thus, (a) holds. We may thus assume that L' is independent.

So, each edge that is not incident with S has one endvertex in L', and one in Y. Consider any component C of $H - S$. Since C is factor-critical, we have that $|(C - x) \cap Y| = |(C - x) \cap L'|$, for every $x \in V(C)$. Hence, C must be trivial. Thus, all components of $H - S$ are trivial.

Denote by X the set of those vertices of Y that are not covered by M. Set $\tilde{L} := N(L') \cap L$. Now, if there is a vertex $B \in \tilde{L}$ whose weighted degree into $H - X$ is at least $\frac{K}{2}$, then B, together with any of its neighbours A in L', satisfies (b). So we may assume that for each $B \in \tilde{L}$,

$$\overline{deg}_{V(H)-X}(B) < \frac{K}{2}, \tag{2.9}$$

and hence $\overline{deg}_X(B) \geq \frac{K}{2}$. Then, by double edge-counting of $E(X, \tilde{L})$, where we sum the weights of the edges,

$$|X| \geq \frac{|\tilde{L}|}{2}. \tag{2.10}$$

Furthermore, (2.9) implies that the weighted degree of $S' := \tilde{L} \cup (S - L)$ into L' is less than $|\tilde{L}|\frac{K}{2} + |S - L|K$, while each vertex of L' has weighted degree at least K into S'. Thus, again by double edge-counting, and by (2.10),

$$|X| + |S - L| \geq \frac{|\tilde{L}|}{2} + |S - L| > |L'|. \tag{2.11}$$

On the other hand, since Y is independent, M matches $S - L \subseteq Y$ to L'. Thus, $|L'| = |S - L| + |L - M|$, and so, by (2.11),

$$|X| > |L - M|.$$

Hence, since $|L| > \frac{N}{2}$, there is an edge $AB \in M$ with both clusters A, B in L, more precisely, with $A \in L'$, and $B \in \tilde{L}$. By (2.9), B has a neighbour D in X. then, the matching $M \cup \{BD\} - \{AB\}$ contains more vertices of Y than M does, a contradiction to the choice of M. □

Remark 2.5.6. Suppose that the weight function gives values $\omega(e) \leq 1$ for any edge e, and suppose that $K < \frac{N}{2}$. Then, we know that in lemma 2.5.5 case 1 occurs. This variant of the lemma is the one used in [1] to prove the approximation version of the Loebl Conjecture. In fact it can be used for the proof of Theorem 2.5.1, as long as $q \geq \frac{1}{2}$.

Lemma 2.5.7. *Let $\bar{H}$ be a weighted graph on N vertices, with some weight-function ω with $\omega(e) \leq 1$, for any $e \in E(H)$. Let L be the set of all vertices $v \in V(H)$ with $\overline{deg}(v) \geq K$, for some $K < \frac{N}{2}$. Suppose that $|L| > \frac{N}{2}$.*
Then there are two adjacent vertices $A, B \in L$, and a matching M in $H-(A \cup B)$ such that M covers all but at most one cluster in $N(A \cup B)$.

Proof of Lemma 2.5.7. As in Lemma 2.5.5, we use the Gallai-Edmonds Matching Theorem to get a separator S and a matching of H.
Observe that there is at least one component in $H - S$ that contains at least a vertex from L. Indeed, otherwise, we have $L \subseteq S$, so

$$\frac{N}{2} < |L| \leq |S| \leq \frac{N}{2},$$

a contradiction.

Now, observe that there is at most one component in $H - S$ that contains vertices from L. Each component, containing some vertex from L, has cardinality greater than $\frac{N}{2} - |S| + 1$. So, if there are at least two such components, we have

$$|V(H) > 2|S| - 2 + 2(\frac{N}{2} - |S| + 1) = N,$$

a contradiction.

therefore, there is a a unique component K in $H - S$ that contains vertices from L. Now, we show that there is an edge in $L \cap K$. If this is not the case, any vertex in $L \cap K$ has neighbours only in $S \cup (K \setminus L)$. So, we have that

$$|S| + |K \cap L| \geq \frac{N}{2} + 1.$$

Now, as vertices from L are contained in $S \cup K$ only, we have

$$|S| + |K \cap L| \geq |L| \geq \frac{N}{2} + 1.$$

Then,

$$|V(H)| \geq |K \cap L| + |K \setminus L| + 2|S| - 1 \geq N + 1,$$

again a contradiction. □

Lemma 2.5.8 (Partitioning the clusters)**.** *Let $0 < \alpha, \varepsilon, \sigma < \frac{1}{2}$ such that $\alpha\sigma < \frac{1}{5}$. Let H be a cluster graph with every edgeformed by an (ε, α)-regular pair of clusters. Lest each cluster have size s. Let $\mathcal{C} \subseteq V(H)$. Let $\mathcal{A}, \mathcal{B}$ be sets of clusters. For clusters $A \in \mathcal{A}$ let $N_A \subseteq \mathcal{C}$ such that*

$$deg_{N_A}(A) := \sum_{C \in N_A} \frac{e(A, C)}{s} \geq \frac{|V_A|}{y}(1 + 2\sigma) + |N_A|\varepsilon s, \tag{2.12}$$

and for clusters $B \in \mathcal{B}$ let $N_B \subseteq \mathcal{C}$such that

$$deg_{N_B}(B) := \sum_{C \in N_B} \frac{e(B,C)}{s} \geq \frac{|V_B|}{1-y}(1+2\sigma) + |N_B|\varepsilon s. \tag{2.13}$$

Then there exists a partition $C^{\mathcal{A}} \cup C^{\mathcal{B}}$ of each cluster C in $\mathcal{C}$ with the following properties.

If $N_A^{\mathcal{A}}$ denotes the set $\bigcup_{C \in N_A} C^{\mathcal{A}}$, then $deg_{N_A^{\mathcal{A}}}(A) \geq |V_A|(1+\frac{\sigma^2}{y})$, and

if $N^{\mathcal{B}_B}$ denotes the set $\bigcup_{C \in N_B} C^{\mathcal{B}}$, then $deg_{N_B^{\mathcal{B}}}(B) \geq |V_B|(1+\frac{\sigma^2}{1-y})$, and

$|C^{\mathcal{A}}|, |C^{\mathcal{B}}| \geq \alpha s$ for each cluster $C \in \mathcal{C}$.

Proof of Lemma 2.5.8. Set

$$x := \begin{cases} y+\sigma & \text{if } y < \sigma \\ y & \text{if } y \in [\sigma, 1-\sigma] \\ y-\sigma & \text{if } y > 1-\sigma \end{cases}$$

For each cluster $C \in \mathcal{C}$ we shall choose any subset $C^{\mathcal{A}}$ of size xs and set $C^{\mathcal{B}} := C \setminus C^{\mathcal{A}}$. We have that $|C^{\mathcal{A}}|, |C^{\mathcal{B}}| \geq \sigma s$. Then for clusters $A \in \mathcal{A}$ and $N_A^{\mathcal{A}}$, we have

$$\begin{aligned} deg_{N_A^{\mathcal{A}}}(A) &\geq \sum_{C \in N_A} (\delta(A,C) - \varepsilon)xs \\ &\geq x \cdot (deg_{N_A}(A) - |N_A|\varepsilon s) \\ &\geq x\frac{|V_A|}{y}(1+2\sigma). \end{aligned}$$

First suppose that $x = y \geq \sigma$, then

$$deg_{N_A^{\mathcal{A}}}(A) \geq |V_A|(1+2\sigma\frac{y}{y}) \geq |V_A|(1+\frac{\sigma^2}{y}).$$

The second possibility is when $x = y+\sigma$, if $y < \sigma$. Then

$$\begin{aligned} deg_{N_A^{\mathcal{A}}}(A) &\geq (y+\sigma)\frac{|V_A|}{y}(1+2\sigma) \\ &= |V_A|(1+2\sigma) + \frac{\sigma}{y}|V_A|(1+2\sigma) \\ &\geq |V_A|(1+\sigma^2). \end{aligned}$$

The last possibility is when $x = y - \sigma$, i. e. when $y \geq 1 - \sigma$. This gives us

$$\begin{aligned} deg_{N_A^{\mathcal{A}}}(A) &\geq (y-\sigma)\frac{|V_A|}{y}(1+2\sigma) \\ &\geq |V_A|(1+2\sigma - \frac{\sigma}{y} - \frac{2\sigma^2}{y}) \\ &\geq |V_A|(1+\frac{\sigma}{y}(2y-1-2\sigma)) \\ &\geq |V_A|(1+\frac{\sigma}{y}(2-2\sigma-1-2\sigma)) \\ &\geq |V_A|(1+\frac{\sigma^2}{y}). \end{aligned}$$

Now we shall check the degree for clusters $B \in \mathcal{B}$ into $N_B^{\mathcal{B}}$. We have

$$\begin{aligned} deg_{N_B^{\mathcal{B}}}(B) &\geq \sum_{C \in\ N)_B} (\delta(B,C) - \varepsilon)(1-x)s \\ &\geq (1-x)\cdot(deg_{N_B}(B) - |N_B|\varepsilon s) \\ &\geq (1-x)\frac{|V_B|}{(1-y)}(1+2\sigma). \end{aligned}$$

For $x = y$, we have $1 - y \geq \sigma$ and then

$$deg_{N_B^{\mathcal{B}}}(B) \geq |V_B|(1+2\sigma\frac{1-y}{1-y}) \geq |V_B|(1+\frac{\sigma^2}{1-y}).$$

For the case when $x = y - \sigma$, we have that

$$\begin{aligned} deg_{N_B^{\mathcal{B}}}(B) &\geq (1-y+\sigma)\frac{|V_B|}{1-y}(1+2\sigma) \\ &= |V_B|(1+2\sigma) + \frac{\sigma}{1-y}|V_B|(1+2\sigma) \\ &\geq |V_B|(1+\frac{\sigma^2}{1-y}). \end{aligned}$$

Now for $x = y + \sigma$ (i. e. $y < \sigma$), we have

$$
\begin{aligned}
deg_{N_B^B}(B) &\geq (1-y-\sigma)\frac{|V_B|}{1-y}(1+2\sigma) \\
&\geq |V_B|(1+2\sigma-\frac{\sigma}{1-y}(1+2\sigma) \\
&\geq |V_B|(1+\frac{\sigma}{1-y}(2-2\sigma-1-2\sigma) \\
&\geq |V_B|(1+\frac{\sigma^2}{1-y}).
\end{aligned}
$$

□

2.5.2 Some simple embeddings

In the next lemma we see how to use an edge, formed by a regular pair, to embed a tree of small size in it. This lemma will be widely used in the proof of other lemmas.

Lemma 2.5.9 (Embedding in an edge)**.** *Let $0 < \varepsilon, \alpha, \delta < 1$. Let t be a rooted tree and let (C, D) be an (ε, α)-regular pair with $s := |C| = |D|$ and density $\delta := \frac{e(C,D)}{s^2}$. If $\bar{C} \subseteq C$ and $\bar{D} \subseteq D$ such that $|\bar{C}|$, $|\bar{D}| \geq \frac{(\alpha s+|t|)}{(\delta-2\varepsilon)}$, then we can embed the tree t in $\bar{C} \cup \bar{D}$. Moreover we can choose in which of the two clusters we want to embed the root of the tree.*

Proof of Lemma 2.5.9. By (1.2) we have that all but at most αs vertices of $\bar{C}$ are typical with respect to $\bar{D}$. Choose any of those typical vertices to embed the root $r \in V(t)$ in. Because $\varphi(r)$ is typical with respect to $\bar{D}$, it has at least $(\delta - 2\varepsilon) \cdot |\bar{D}| \geq \alpha s + |t|$ neighbours in $\bar{D}$.
Between those neighbours of $\varphi(r)$ we want to choose typical vertices with respect to $\bar{C}$ to embed the neighbours of r in. From $\alpha s + |t|$ vertices in $\bar{D}$ (resp. in $\bar{C}$), at least $|t|$ of them are typical with respect to $\bar{C}$ (resp. to $\bar{D}$).
We shall continue to embed the tree t levelwise. At each step, we shall choose between the at least $|t|$ typical vertices, from a neighbourhood $N(v)$, some that is not already used by the embedding φ and embed the neighbours of $\varphi^{-1}(v)$ in. □

The following two lemmas show us how to embd larger tree into a matching and a cluster with high average degree into this matching.

First we bring some definitions. We say that (T, R) is a *rooted τ-tree*, if (T, R) is a rooted tree with root R and if $T - R$ decomposes into a forest of trees, each of cardinality at most τ.

A matching M is an $(\varepsilon, \alpha; \delta)$-*matching of* $\mathcal{C}$, if M covers $\mathcal{C}$ and if each edge of M is an (ε, α)-regular pair with density at least δ.

Lemma 2.5.10 (Embedding in a matching – I)**.** *Let $0 < \varepsilon, \alpha, \delta < 1$ with $2\varepsilon < \delta$. Let (T, R) be a rooted τ-tree. Let $\mathcal{C}$ be a set of clusters, each having size s. Let $\mathcal{N} \subseteq \mathcal{C}$ and $U \subseteq \bigcup_{C \in \mathcal{C}} C$. Let M be an $\varepsilon, \alpha; \delta)$-matching of $\mathcal{C}$ such that each matching edge has at most one end vertex in $\mathcal{N}$. Let $\alpha s \leq s' \leq s$. Then, for all $C \in \mathcal{C}$ let $C' \subseteq C$ be of size s' and denote $\mathcal{N}' := \cup_{C \in \mathcal{N}} C'$ and $\mathcal{C}' := \cup_{C \in \mathcal{C}} C'$. Let $A \notin \mathcal{C}$ be a cluster of size s and let $v \in A$ such that*

$$deg_{\mathcal{N}'}(v) \geq |T| + |U| + |\mathcal{N}| \cdot \Delta,$$

where $\Delta = \frac{(2\alpha s + \tau)}{(\delta - 2\varepsilon)}$. Then, there is an embedding of the tree T such that R embeds on v and $T \setminus \{R\}$ embeds in $\mathcal{C}' \setminus U$ and such that any vertex of T with odd distance to the root R is embedded on a vertex that has at least $(\delta - \varepsilon)s$ neighbours in cluster A.

Proof of Lemma 2.5.10. We embed the root R of the tree T on vertex $v \in A$. We shall then embed $T \setminus \{R\}$ in ℓ steps, where ℓ is the number of components in $T \setminus \{R\}$. In each step we embed the tree t^j forming the j-th component. We claim that for each step j, $1 \leq j \leq \ell$ we find an edge $e \in M$ such that

$$deg_{C'}(v) - |e \cap U_{j-1}| - |U| \geq \Delta, \tag{2.14}$$

for $C := e \cap \mathcal{N}$, where U_{j-1} denotes the set of vertices already used by the embedding before step j, i. e. $U_j := \varphi(R) \cup \bigcup_{i \leq j} \varphi(V(t^i))$. Indeed, suppose on the contrary, that there is no such edge. Then,

$$|T| + |\mathcal{N}| \cdot \Delta - |U_{j-1}| \leq \sum_{e \in M} deg_{e \cap \mathcal{N}'}(v) - |U| - |e \cap U_{j-1}| < |\mathcal{N}|\Delta,$$

a contradiction.
So let $e = \{C, D\}$, with $C := e \cap \mathcal{N}$, be a suitable edge with property (2.14). Inequality (2.14) implies that

$$\Delta \leq s' - |e \cap (U_{i-1} \cup U)| \leq |D' \setminus (U_{j-1} \cup U)| =: |\bar{D}|.$$

By (2.14), it is clear that $|N(v) \cap C' \setminus (U_{j-1} \cup U)| \geq \Delta$. Denote by $\bar{C}$ the set of vertices in $N(v) \cap C' \setminus (U_{j-1} \cup U)$ that are typical with respect to cluster A. Such vertices have each degree at least $(\delta - \varepsilon)s$ in cluster A. We have that $\bar{C} \geq \frac{(\alpha s + \tau)}{\delta - 2\varepsilon}$. Using Lemma 2.5.9, we embed the tree t^j in $\bar{C} \cup \bar{D}$ choosing to embed the root of t^j in cluster C.

□

Remark 2.5.11. In the proof of Theorem 2.5.1, the set U will be used for a set of vertices used by the embedding in some previous step of the embedding process. In the next lemma, we do not deal with such a set. Instead, we state and prove Lemma 2.5.13 to take care of the used vertices.

Lemma 2.5.12 (Embedding in a matching – II)**.** *Let $0 < \varepsilon, \alpha, \delta < 1$ with $\varepsilon < \delta$. Let (T, R) be a rooted τ-tree. Let $\mathcal{C}$ be a set of clusters, each having size s and $\mathcal{N} \subseteq \mathcal{C}$. Let M be an $(\varepsilon, \alpha; \delta)$-matching of $\mathcal{C}$. Let $\alpha s \leq s' \leq s$. Then, for each $C \in \mathcal{C}$ let $C' \subseteq C$ be of size s'. Denote by $\mathcal{C}' := \cup_{C \in \mathcal{C}} C'$ and $\mathcal{N}' := \cup_{C \in \mathcal{N}} C'$. Let $A \notin \mathcal{C}$ be a cluster of size s and let $v \in A$ such that*

$$deg_{\mathcal{N}'}(v) \geq |T| + |\mathcal{N}| \cdot \Delta + |M| \cdot \tau,$$

where $\Delta = \frac{(2\alpha s + \tau)}{(\delta - 2\varepsilon)}$. Then, there is an embedding of the tree T such that R embeds on v and $T \setminus \{R\}$ embeds in $\mathcal{C}'$ and such that any vertex of T with odd distance to the root R is embedded on a vertex that has at least $(\delta - \varepsilon)s$ neighbours in cluster A.

Proof of Lemma 2.5.12. We embed the root R of the tree T on vertex $v \in A$.
We embed $T \setminus \{R\}$ into ℓ steps, where ℓ is the number of components in $T \setminus \{R\}$. In each step, we embed the tree t^j, which is the j-th component of $T \setminus \{R\}$. Define U^{j-1} as the set of vertices in the clusters used so far by the embedding. We claim that there is an edge $e \in M$ such that

$$deg_{e'}(v) - |e \cap U^{j-1}| \geq |e \cap \mathcal{N}| \cdot \Delta + \tau. \tag{2.15}$$

Indeed, suppose for contradiction that this is not the case. Then,

$$|T| + |\mathcal{N}| \cdot \Delta + |M|\tau - |U^{j-1}| < |\mathcal{N}|\Delta + |M| \cdot \tau,$$

implying $|V(t^j)| < 0$, a contradiction.
Therefore we have found a suitable edge for the embedding of t^j. At each step $j \geq 1$, our embedding will satisfy the following conditions for edge $e = \{C, D\}$.

(a) If $\min\{deg_{C' \setminus U^{j-1}}(v), deg_{D' \setminus U^{j-1}}(v)\} \geq \Delta$, then $||C \cap U^j| - |D \cap U^j|| < \tau$

(b) If $deg_{D' \setminus U^{j-1}}(v) < \Delta$, then $deg_{D'}(v) < |C \cap U^{j-1}| + \Delta + \tau$ and
if $deg_{C' \setminus U^{j-1}}(v) < \Delta$, then $deg_{C'}(v) < |D \cap U^{j-1}| + \Delta + \tau$.

Without loss of generality assume that $deg_{C' \setminus U^j}(v) \geq deg_{D' \setminus U^j}(v)$. Then, $deg_{C' \setminus U^j}(v) \geq \Delta$. We claim that $|D' \setminus U^{j-1}| \geq \Delta$.
If $deg_{D' \setminus U^{j-1}}(v) \geq \Delta$, there is nothing to prove. So, assume that $deg_{D' \setminus U^{j-1}}(v) < \Delta$. Using Property (b), we have

$$
\begin{aligned}
|e \cap \mathcal{N}| \cdot \Delta + \tau &\leq deg_{e' \setminus U^{j-1}}(v) \\
&\leq |C \cap U^{j-1}| + s' - |e \cap U^{j-1}| \\
&\leq |D' \setminus U^{j-1}|.
\end{aligned}
$$

This implies the required inequality.

If $deg_{D' \setminus U^{j-1}}(v) < \Delta$, then we shall embed the root $rt(t^j)$ in cluster C. If $deg_{D' \setminus U^{j-1}}(v) \geq \Delta$, then we choose to embed the root $rt(t^j)$ of the tree t^j, depending on the following criteria.

Let $L_0(T^j)$ denote the set of vertices that are at even distance from the root $rt(t^j)$. Set $L_1(t^j) := V(t^j) \setminus L_0(t^j)$.. We want to embed the largest of $L_0(t^j)$ and $L_1(t^j)$ to the cluster with less used vertices, i. e. which has the smallest $|C \cap U^{j-1}|$,$|D \cap U^{j-1}|$. We choose to embed the root $rt(t^j)$ according to this criteria. To simplify the notation, we assume that the root embed in C' (Otherwise, just interchange the C's and the D's).

Denote by $\bar{C}$ the set of vertices in $N(v) \cap C' \setminus U^{j-1}$ that are typical with respect to cluster A and set $\bar{D} := D' \setminus U^{j-1}$. We have

$$
|\bar{C}|, |\bar{D}| \geq \Delta - \alpha s > \frac{\alpha s + \tau}{\delta - 2\varepsilon}.
$$

We embed the tree t^j in $\bar{C} \cup \bar{D}$ using Lemma 2.5.9 and choosing to embed the root of t^j in $\bar{C}$.
Now, we check that the define embedding of t^j fulfill Properties (a) and (b).

Property (a) follows from the facts that if $\min\{\} \geq \Delta$, then (a) for $i-1$ implies that $||C \cap U^{j-1}| - |D \cap U^{j-1}|| \leq tau$ and that we have embedded the largest part of t^j to the cluster with less used vertices. Therefore, the difference $||C \cap U^j| - |D \cap U^j||$ was kept under τ.
To see that Property (b) holds, first assume that $deg_{D' \setminus U^{j-2}}(v) < \Delta$. Then, (b) for $i-1$ implies (b) for i. So, assume now that $deg_{D' \setminus U^j}(v) \geq \Delta$. Then (a) for $i-1$ implies that $||C \cap U^{j-1}| - |D \cap U^{j-1}|| \leq \tau$. So,

$$
deg_{D'}(v) < |D \cap U^{j-1}| + \Delta \leq |C \cap U^{j-1}| + \Delta + \tau.
$$

After having embedded that last trees of $T - R$, we have defined φ for the whole tree T. This ends the proof of Lemma 2.5.12. □

In the proof of Theorem 2.5.1, we shall meet the situation when, for the embedding of a rooted subtree, we will have to use a pair of clusters that was already

used by an other rooted subtrees of the tree T. For this situation we shall use next lemma.

Lemma 2.5.13 (Embedding in used edges)**.** *Let $0 < \alpha, \varepsilon, \delta < 1$ with $2\varepsilon < \delta$. Let A, C, D be 3 clusters of size s, each. The pairs of clusters from A, C, D form (ε, α)-regular pairs. Let $\alpha s \leq s' \leq s$ and let $C' \subseteq C$ and $D' \subseteq D$ with $|C'|, |D'| = s'$. Moreover suppose that the density between the clusters C and D is at least δ. Let T be a rooted tree with root R such that $T - R$ decomposes into components of size at most τ, each. Let U be a subset of $C' \cup D'$ with the following properties:*
If $||C \cap U| - |D \cap U|| > \tau$, then

$$\min\{deg_{C'}(A) - |U \cap C|, deg_{D'}(A) - |U \cap D|\} < \Delta + 2\varepsilon s'. \tag{2.16}$$

Also

$$deg_{C'}(A) < |U \cap C| + \Delta + 2\varepsilon s' \Rightarrow deg_{C'}(A) < |U \cap D| + \Delta + 2\varepsilon s' + \tau \tag{2.17}$$
$$deg_{D'}(A) < |U \cap D| + \Delta + 2\varepsilon s' \Rightarrow deg_{D'}(A) < |U \cap C| + \Delta + 2\varepsilon s' + \tau \tag{2.18}$$

Let $A' \subset A$ be the set of vertices in A that are typical with respect to both C' and D'. If

$$deg_{C' \cup D'}(A) \geq |T| + |U| + 2(\Delta + 2\varepsilon s') + \tau, \tag{2.19}$$

where $\Delta = \frac{2\alpha s + \tau}{\delta - 2\varepsilon}$, then for any vertex $v \in A'$ we can embed the tree T into $A \cup C \cup D$ such that R embed onto v and $T - R$ embeds into $(C' \cup D') \setminus U$ such that any vertex of T with odd distance to the root R has at least $(\delta - \varepsilon)s$ neighbours in cluster A and that if we set $\tilde{U} := U \cup \varphi(V(T))$, then $\tilde{U}$ fulfill the same conditions as U, i. e.
If $||C \cap \tilde{U}| - |D \cap \tilde{U}|| > \tau$, then

$$\min\{deg_{C'}(A) - |\tilde{U} \cap C|, deg_{D'}(A) - |\tilde{U} \cap D|\} < \Delta + 2\varepsilon s'.$$

Also

$$deg_{C'}(A) < |\tilde{U} \cap C| + \Delta + 2\varepsilon s' \Rightarrow deg_{C'}(A) < |\tilde{U} \cap D| + \Delta + 2\varepsilon s' + \tau \tag{2.20}$$
$$deg_{D'}(A) < |\tilde{U} \cap D| + \Delta + 2\varepsilon s' \Rightarrow deg_{D'}(A) < |\tilde{U} \cap C| + \Delta + 2\varepsilon s' + \tau \tag{2.21}$$

Proof of Lemma 2.5.13. We are given a vertex $v \in A'$. Embed R in v, a typical with respect to the set C' and with respect to the set D'. By (1.4) we have

$$deg_{C' \cup D'}(v) \geq deg_{C' \cup D'}(A) - 4\varepsilon s' \geq |T| + |U| + 2\Delta. \tag{2.22}$$

For the components of $T - R$ we shall proceed inductively, embedding one component after the other. Suppose we are at step $i \geq 0$. We have embedded all the components t^j of $T - R$, for $j < i$, and want to embed component t^i. Set $V^{<i} := \bigcup_{j<i} V(t^j)$ and $U^{<i} := \varphi(V^{<i})$. Set $U_{i-1} := U \cup U^{<i}$. For each step $i \geq 1$ we have two possible cases.

(i) $\min\{deg_{C'\setminus U_{i-1}}(v), deg_{D'\setminus U_{i-1}}(v)\} \geq \Delta$

(ii) $\min\{deg_{C'\setminus U_{i-1}}(v), deg_{D'\setminus U_{i-1}}(v)\} < \Delta$

Then our embedding will satisfy the following.

(a) If case (i) holds, then either

$$||C \cap U_i| - |D \cap U_i|| \leq \tau,$$

or

$$\tau < ||C \cap U_i| - |D \cap U_i|| \leq ||C \cap U_{i-1}| - |D \cap U_{i-1}||\,.$$

Without loss of generality suppose that

$$deg_{C'}(A) - |C \cap U_{i-1}| \geq deg_{D'}(A) - |D \cap U_{i-1}|.$$

Then by (2.22)

$$deg_{C'}(v) - |C \cap U_{i-1}| \geq \Delta \tag{2.23}$$

We want to show that $|D' \setminus U_{i-1}| \geq \Delta$. If $deg_{D'\setminus U_{i-1}}(v) \geq \Delta$, we are done. So suppose that

$$deg_{D'\setminus U_{i-1}}(v) < \Delta \tag{2.24}$$

We claim then that

$$deg_{C'\cup D'}(A) - |U_{i-1}| \leq |D' \setminus U_{i-1}| + \Delta + 2\varepsilon s' + \tau \tag{2.25}$$

If $|D \cap U_{i-1}| \leq |C \cap U_{i-1}| + \tau$ then we get immediately (2.25). So suppose we have $|C \cap U_{i-1}| < |D \cap U_{i-1}| + \tau$.
Let $\ell \leq i-1$ be the minimal index with the property that $deg_{D'\setminus U_\ell}(v) < \Delta$. If $\ell = 0$, then

$$deg_{D'}(A) - |U \cap D| \leq deg_{D'\setminus U}(v) < \Delta$$

and by (2.18), we get

$$deg_{D'}(A) < |U \cap C| + \Delta + 2\varepsilon s' + \tau \leq |U_{i-1} \cap C| + \Delta + 2\varepsilon s' + \tau \tag{2.26}$$

implying at once (2.25). Now if $\ell > 0$, we have that $deg_{D'\setminus U_{i-2}}(v) \geq \Delta$. Together with (2.22), Property (a) we get

$$|C \cap U| < |D \cap U| + \tau.$$

Now if we have (2.18), we get directly (2.25). If Case (2.17) occurs, then

$$\begin{aligned} deg_{C'\cup D}(A) - |U_{i-1}| &< deg_{D'}(A) + |U \cap C| + \Delta + 2\varepsilon s' - |U_{i-1}| \\ &\leq s' + |U_{i-1} \cap C| - |U_{i-1}| + \Delta + 2\varepsilon s' \\ &\leq |D'| - |U_{i-1} \cap D| + \Delta + 2\varepsilon s', \end{aligned}$$

leading to (2.25).
Now by (2.19) and (2.25), we get

$$2(\Delta + 2\varepsilon s') + \tau \leq |D' \setminus U_{i-1}| + \Delta + 2\varepsilon s' + \tau,$$

which gives us finely that

$$|D' \setminus U_{i-1}| \geq \Delta + 2\varepsilon s' \geq \Delta. \tag{2.27}$$

Denote by $\bar{C}$ the set of vertices in $N_{C\setminus U_{i-1}}(v)$ that are typical with respect to cluster A and by $\bar{D}$ the set of vertices in $D' \setminus U_{i-1}$ that are typical with respect to cluster A. We have that $(\bar{C} \cup \bar{D}) \cap U = \emptyset$ and by (2.23) and (2.27) that $|\bar{C}|, |\bar{D}| \geq \frac{\alpha s+\tau}{\delta-2\varepsilon}$. We may use Lemma 2.5.9 to embed tree t^i.
Observe that if case (i), we may choose in which cluster we want to embed the root of the tree t^i. Therefore, we can guaranty that property (a) holds.
After having embedded all the components of $T - R$ we have define the required embedding.
We have now to check if properties (2.20) and (2.21) hold. So suppose that

$$deg_{C'}(A) < |\tilde{U} \cap C| + \Delta + 2\varepsilon s', \tag{2.28}$$

but

$$deg_{C'}(A) \geq |\tilde{U} \cap D| + \Delta + 2\varepsilon s' + \tau \tag{2.29}$$

The other case is proved analogously.
We have then $|\tilde{U}\cap D|+\tau < |\tilde{U}\cap C|$. If $||U\cap D|-|U\cap C|| > \tau$, then properties (2.20) and (2.21) hold from (2.17) and (2.18). So we may assume, by Property (a), that Case (ii) holds.
Set ℓ to be the largest index for which $\min\{deg_{C'\setminus U_{\ell-1}}(v), deg_{D'\setminus U_{\ell-1}}(v)\} \geq \Delta$. Then $||U_\ell \cap D| - |U_\ell \cap C|| < \tau$. By maximality of ℓ, we have that

$$\min\{deg_{C'\setminus U_\ell}(v), deg_{D'\setminus U_\ell}(v)\} < \Delta.$$

Suppose that $deg_{C'}(v) - |U_\ell \cap C| < \Delta$ then

$$\begin{aligned} deg_{c'}(A) \leq deg_{C'}(v) + 2\varepsilon s' &< \Delta + 2\varepsilon s' + |U_\ell \cap C| \\ &\leq \Delta + 2\varepsilon s' + |U_\ell \cap D| + \tau \\ &\leq \Delta + 2\varepsilon s' + |\tilde{U} \cap D| + \tau, \end{aligned}$$

a contradiction with (2.29).
Therefore, we know that

$$deg_{D'}(A) - 2\varepsilon s' - |U_\ell \cap D| \leq deg_{D' \setminus U_\ell}(v) < \Delta.$$

But then by (2.29)

$$\begin{aligned} deg_{C' \cup D'}(A) &< \Delta + 2\varepsilon s' + |U_\ell \cap D| + |\tilde{U} \cap C| + \Delta + 2\varepsilon s' \\ &\leq |U| + |T| + 2\Delta + 4\varepsilon s' + \tau, \end{aligned}$$

a contradiction with (2.19). This ends the proof of Lemma 2.5.13.
At least we have to check if

$$||C \cap \tilde{U}| - |D \cap \tilde{U}|| > \tau \Rightarrow \min\{deg_{C'}(A) - |\tilde{U} \cap C|, deg_{D'}(A) - |\tilde{U} \cap D|\} < \Delta + 2\varepsilon s'.$$

Either $||U \cap C| - |U \cap D|| > \tau$ and then

$$\begin{aligned} &min\{deg_{C'}(A) - |\tilde{U} \cap C|, deg_{D'}(A) - |\tilde{U} \cap D|\} \\ &\leq min\{deg_{C'}(A) - |U \cap C|, deg_{D'}(A) - |U \cap D|\} \\ &< \Delta + 2\varepsilon s', \end{aligned}$$

or at some step ℓ we had $min\{deg_{C'}(v) - |U_\ell \cap C|, deg_{D'}(v) - |U_\ell \cap D|\} < \Delta$, but by the typicality of vertex v, we have that

$$\begin{aligned} &min\{deg_{C'}(A) - |\tilde{U} \cap C|, deg_{D'}(A) - |\tilde{U} \cap D|\} \\ &\leq min\{deg_{C'}(A) - |U_\ell \cap C|, deg_{D'}(A) - |U_\ell \cap D|\} \\ &\leq min\{deg_{C'}(v) - |U_\ell \cap C|, deg_{D'}(v) - |U_\ell \cap D|\} + 2\varepsilon s' \\ &< \Delta + 2\varepsilon s', \end{aligned}$$

□

In the next lemma, we see how to take profit of clusters with high average degree.

Lemma 2.5.14 (Embedding using clusters with high average degree)**.** *Let $0 < \alpha, \varepsilon, \delta < 1$ with $2\varepsilon < \delta$. Let T be a rooted tree with root R, such that $T \setminus \{R\}$ decomposes into a forest for which each component is a tree of size at most τ. Let H be a cluster graph with clusters of size s each and in which each edge is (ε, α)-regular with density at least δ.*
Let $\alpha s \leq s' \leq s$ and for each $C \in V(H)$ let $C' \subseteq C$ be of size s'. Then denote by $V' := \cup_{C \in V(H)} C'$. Let $\mathcal{L}$ be a set of clusters of H with the property that if $C \in \mathcal{L}$ then

$$deg_{V'}(C) := \sum_{D \in V(H)} \frac{e(C, D')}{s} \geq |T| + |V(H)| \cdot (\Delta + 2\varepsilon s') + s',$$

where $\Delta = \frac{(2\alpha s+\tau)}{(\delta-2\varepsilon)}$. *If* $\mathcal{L}'$ *denotes the set* $\cup_{C\in\mathcal{L}} C'$ *and* $v \in A$ *be such that*

$$deg_{\mathcal{L}'}(v) := \sum_{C\in\mathcal{N}} \frac{e(A, C')}{s} \geq |T| + |\mathcal{L}| \cdot \Delta.$$

Then there is an embedding of the tree T *such that* R *embeds on* v *and* $T \setminus \{R\}$ *embeds in* $V' \setminus A$ *and such that any vertex of* T *with odd distance to the root* R *is embedded on a vertex that has at least* $(\delta - \varepsilon)s$ *neighbours in cluster* A.

Proof of Lemma 2.5.14. Embed the root R of the tree T in vertex $v \in A$. We shall embed $T\setminus\{R\}$ into ℓ steps, where ℓ is the number of components in $T\setminus\{R\}$. In each step we embed the tree t_j, which is the j-th component of $T\setminus\{R\}$. Define U_{j-1} as the set of vertices in the clusters used so far by the embedding.
Suppose we are in step $1 \leq j \leq \ell$ and we want to embed t_j. We claim that there is a cluster $C \in \mathcal{L}$ such that

$$deg_{C'}(v) - |C' \cap U_{j-1}| \geq \Delta \tag{2.30}$$

Suppose this is not the case. Then

$$|\mathcal{L}| \cdot \Delta < |T| - |U_{j-1}| + |\mathcal{L}| \cdot \Delta \leq \sum_{C\in\mathcal{L}} deg_{C'}(v) - |U_{j-1}| < |\mathcal{L}| \cdot \Delta,$$

a contradiction.
So assume that $C \in \mathcal{L}$ is a suitable cluster with property 2.30. Embed the root of t_j in a vertex $u \in C' \cap N(v)$ that is typical with respect to $V' \setminus A$. Then

$$deg_{V'\setminus A}(u) \geq |T| + |V(H)| \cdot \Delta. \tag{2.31}$$

We claim that there is a cluster $D \in V(H) \setminus A$ such that

$$deg_{D'}(u) \geq \Delta.$$

If not, then by (2.31) we have that

$$|V(H)| \cdot \Delta < |T| + |V(H)| \cdot \Delta - |U_{j-1}| \leq deg_{V'\setminus A}(v) < |V(H)| \cdot \Delta,$$

a contradiction.
Denote by $\bar{C} \subseteq C' \setminus U_{j-1}$ the set of vertices that are typical with respect to cluster A. We have that $|\bar{C} \geq \frac{\alpha s+\tau}{\delta-2\varepsilon}$. Set $\bar{D} := N(u) \cap D' \setminus U_{j-1}$. Then $|\bar{D}| \geq \frac{\alpha s+\tau}{\delta-2\varepsilon}$. The sets $\bar{C}$ and $\bar{D}$ contain no vertices used by the embedding. Lemma 2.5.9 ensures us the embedding of the rest of tree t_j.

□

2.5.3 Partitioning trees

On a rooted T with root R, we shall define a natural partial order $(V(T), \preceq)$ as follows: $u \preceq v$ if there exists a path from R to v containing vertex u. In this ordering R is the smallest vertex and any leaf ($\neq R$) is a maximal vertex. For any subtree T' of the tree T and a vertex $r \in V(T')$, we define $T'(r)$ as the subtree of T' induced by all vertices in $V(T')$ that are greater or equal to r in the partial order $\preceq$. Then r is the minimal vertex (or the root) of $T'(r)$. For any tree t forming one of the components of $T'(r) - r$, we shall denote by $v(t)$ the maximal vertex in T that is smaller than any vertex of the tree t, i. e. $v(t) = r$. We call this vertex $v(t)$ the *seed* of the tree t and say that t *grows* from $v(t)$.

Lemma 2.5.15. *Let $1 \leq \tau \leq \vartheta$. Then for any rooted tree (T, R) of order $\vartheta + 1$, there exists a set $\mathcal{R}$ of vertices of T of size at most $\frac{\vartheta}{\tau} + 1$ such that $T - \mathcal{R}$ decomposes into trees of order at most τ.*

Proof of Lemma 2.5.15. We shall define the set $\mathcal{R}$ inductively. For $i \geq 0$, $\mathcal{R}_i$ shall denote a set of vertices of T_i, where T_i shall denote the subtree containing the root R in $T - \mathcal{R}_i$ (if $R \in \mathcal{R}_{i-1}$, then $T_i = \emptyset$ and the process ends). Now set $T_0 := T$. In each step $i \geq 1$ define $\mathcal{R}_i$ as the set of vertices r which are minimal in T_{i-1} with the property that $|V(T_{i-1}(r))| > \tau$. Then, by minimality, $T_{i-1}(r) - r$ decomposes into a component containing the root (if $r \neq R$) and into components of size at most τ, containing vertices that are greater than r. If there is no such vertex r, i. e. $|V(T_{i-1})| \leq \tau$, then set $\mathcal{R}_i := R$. Observe that $T_{i-1} - \mathcal{R}_i$ decomposes into subtrees of size at most τ and in T_i. At the end of the process, set $\mathcal{R} := \bigcup_i \mathcal{R}_i$. We have to check that $|\mathcal{R}| \leq \frac{\vartheta}{\tau} + 1$. □

Lemma 2.5.16. *Let $1 \leq \tau \leq \vartheta$. Then for any rooted tree (T, R) of order $\vartheta + 1$, there exist sets $\mathcal{R}_A$ and $\mathcal{R}_B$ of size at most $\frac{\vartheta}{\tau} + 1$ each, such that $T - (\mathcal{R}_A \cup \mathcal{R}_B)$ decomposes into $\mathcal{T}_A \cup \mathcal{T}_B$, sets of subtrees of size at most τ and the following holds:*

- *For any tree $t_A \in \mathcal{T}_A$ we have that $v(t_A) \in \mathcal{R}_A$ and for any tree $t_B \in \mathcal{T}_B$ we have that $v(t_B) \in \mathcal{R}_B$.*
- *Any two vertices from the set $\mathcal{R}_A$ have even distance between them and any two vertices from the set $\mathcal{R}_B$ have even distance between them.*
- *No vertex from a tree $t \in \mathcal{T}_A$ is adjacent to a vertex from $\mathcal{R}_B$ and analogously no vertex from a tree $t \in \mathcal{T}_B$ is adjacent to a vertex from $\mathcal{R}_A$.*

Remark 2.5.17. This means that the subtrees in $\mathcal{T}_A$ grow all from vertices in $\mathcal{R}_A$ and analogously the subtrees in $\mathcal{T}_B$ grow from vertices in $\mathcal{R}_B$. The subtrees from $\mathcal{T}_A$ and $\mathcal{T}_B$ are connected by edges between vertices in $\mathcal{R}_A$ and vertices in $\mathcal{R}_B$.

Proof of Lemma 2.5.16. First we use Lemma 2.5.15 to get a set $\mathcal{R}$ of size at most $\frac{\vartheta}{\tau} + 1$ such that $T \setminus \mathcal{R}$ decomposes into subtrees of size at most τ. Let $\mathcal{R}'_A$ denote

the set of vertices from $\mathcal{R}$ that are at even distance from the root and by $\mathcal{R}'_B$ the set of vertices from $\mathcal{R}$ that have odd distance from the root R. Denote by S_A the set of vertices in some tree t from $\mathcal{T}$ with $v(t) \in \mathcal{R}'_A$ that is adjacent to some vertex in $\mathcal{R}'_B$. Observe that vertices in S_A have even distance to the root. Analogously define S_B. Vertices in the set S_B have odd distance to the root. Now define $\mathcal{R}_A := \mathcal{R}'_A \cup S_A$ and $\mathcal{R}_B := \mathcal{R}'_B \cup S_B$. Denote by $\mathcal{T}_A$ the set of components from $T - (\mathcal{R}_A \cup \mathcal{R}_B)$ that contain a vertex adjacent to a vertex from $\mathcal{R}_A$ and analogously denote by $\mathcal{T}_B$ the set of components from $T - (\mathcal{R}_A \cup \mathcal{R}_B)$ that contain a vertex adjacent to a vertex from $\mathcal{R}_B$. Observe that $\mathcal{T}_A \cap \mathcal{T}_B = \emptyset$. The components of $T - (\mathcal{R}_A \cup \mathcal{R}_B)$ are subsets of components of $T - \mathcal{R}$ and therefore are of size at most τ each. □

Lemma 2.5.18. *Let $1 \leq \tau \leq \vartheta$. Then for any rooted tree (T, R) of order $\vartheta + 1$, there exist sets $\mathcal{R}_A$ and $\mathcal{R}_B$ of size at most $3(\frac{\vartheta}{\tau}+1)$ each, such that $T-(\mathcal{R}_A \cup \mathcal{R}_B)$ decomposes into $\mathcal{T}_A \cup \mathcal{T}_B \cup \mathcal{T}_F$, sets of subtrees of size at most τ and the following holds:*

- *For any vertex v in a subtree $t \in \mathcal{T}_A \cup \mathcal{T}_B$, if v is adjacent to some vertex in $u \in \mathcal{R}_A \cup \mathcal{R}_B$, then $u = v(t)$.*
- *For any tree $t_A \in \mathcal{T}_A \cup \mathcal{T}_F$ we have that $v(t_A) \in \mathcal{R}_A$ and for any tree $t_B \in \mathcal{T}_B$ we have that $v(t_B) \in \mathcal{R}_B$.*
- $|\bigcup_{t \in \mathcal{T}_A} V(t)| \geq |\bigcup_{t \in \mathcal{T}_B} V(t)|$.

Remark 2.5.19. Again all subtrees from $\mathcal{T}_F$ and $\mathcal{T}_A$ grow from vertices in $\mathcal{R}_A$ and all subtrees from $\mathcal{T}_B$ grow from vertices in $\mathcal{R}_B$ and both parts are connected by edges between sets $\mathcal{R}_A$ and $\mathcal{R}_B$. Moreover subtrees from the sets $\mathcal{T}_A$ and $\mathcal{T}_B$ and "end-subtrees"; this means that there are no vertices from $\mathcal{R}_A \cup \mathcal{R}_B$ that are greater or equal to some vertex in a subtree from $\mathcal{T}_A \cup \mathcal{T}_B$.

The idea of the proof is the following. After partitioning the set of components into two sets $\mathcal{T}_A$ and $\mathcal{T}_B$, using Lemma 2.5.16, we consider "end-components", that is the ones that do not lie between two or more other components. Those shall define the set $\mathcal{T}_F$. We look which one of $\mathcal{T}_A \setminus \mathcal{T}_F$ and $\mathcal{T}_F \setminus \mathcal{T}_F$ is smaller. The smaller shall be denoted by T_B, the other one by $\mathcal{T}_A$ and we shall "switch" all $\mathcal{T}_F$ on the side of $\mathcal{T}_A$. This switching shall enlarge the set $\mathcal{R}$ only by few vertices.

Proof of Lemma 2.5.18. Use Lemma 2.5.16 to get a vertex-cut $\mathcal{R}'_A \cup \mathcal{R}'_B$ of size $\frac{\vartheta}{\tau} + 1$ each giving sets of components $\mathcal{T}'_A$ and $\mathcal{T}'_B$. Denote by $\mathcal{T}'_F$ the set of components of $T - (\mathcal{R}'_A \cup \mathcal{R}'_B)$ for which more than one vertex from $\mathcal{R}'_A \cup \mathcal{R}'_B$ is adjacent to some vertex of the given component. Without loss of generality suppose that $|\bigcup_{t \in \mathcal{T}'_B \setminus \mathcal{T}'_F} V(t)| \leq |\bigcup_{t \in \mathcal{T}'_A \setminus \mathcal{T}'_F} V(t)|$ (otherwise interchange A's and B's). Set $\mathcal{T}_B := \mathcal{T}'_B \setminus \mathcal{T}'_F$ and $\mathcal{T}_A := \mathcal{T}'_A \setminus \mathcal{T}'_F$. Observe that any tree t from $\mathcal{T}_A$ or $\mathcal{T}_B$ has only one vertex (the root of t) that is adjacent to some vertex $v(t) \in \mathcal{R}'_A \cup \mathcal{R}'_B$.

Now we want to "switch" $\mathcal{T}'_F$ on the A side. We do this as follows. For any tree t in $\mathcal{T}'_F \cap \mathcal{T}'_B$ denote by S_t the set of vertices in t that are adjacent to some vertex in $\mathcal{R}'_B$. By those vertices we shall enlarge the set $\mathcal{R}'_A$. Set $\mathcal{R}_A := \mathcal{R}'_A \cup \bigcup_{t \in \mathcal{T}'_F \cap \mathcal{T}'_B} S_t$. Observe that $|\bigcup_{t \in \mathcal{T}'_F \cap \mathcal{T}'_B} S_t| \leq 2|\mathcal{R}'_B|$ and therefore

$$|\mathcal{R}_A| \leq |\mathcal{R}'_A| + 2|\mathcal{R}'_B| \leq 3(\frac{\vartheta}{\tau} + 1).$$

This switching changes the set $\mathcal{T}'_F$ in a natural way, i. e.

$$\mathcal{T}_F := (\mathcal{T}'_F \cap \mathcal{T}'_A) \cup \bigcup_{t \in (\mathcal{T}'_F \cap \mathcal{T}'_B)} t - S_t,$$

where by $t - S_t$ here we understand the union of components of $t - S_t$.
From Lemma 2.5.16 it is clear that for $t \in \mathcal{T}_A$ we have $v(t) \in \mathcal{R}_A$ and for $t \in \mathcal{T}_B$ we have $v(t) \in \mathcal{R}_B$. Now any tree $t \in \mathcal{T}_F$ is adjacent to vertices from $\mathcal{R}_A$, therefore $v(t) \in \mathcal{R}_A$. At the end observe that any component in $T - (\mathcal{R}_A \cup \mathcal{R}_B)$ is a subset of some component from $T - (\mathcal{R}_A \cup \mathcal{R}_B)$ and therefore by Lemma 2.5.16 their size is at most τ. □

Lemma 2.5.20. *Let $1 \leq \tau \leq \vartheta$. Let (T, R) be a rooted tree of order $\vartheta + 1$. Let S_1 and S_2 be subsets of $V(T)$ each of size c such that all vertices of S_1 are at odd distance to R and all vertices of S_2 are of even distance from R. Then there exist sets $\mathcal{R}_A$ and $\mathcal{R}_B$ of size at most $\frac{\vartheta}{\tau} + 1 + 2c$ each, such that $T - (\mathcal{R}_A \cup \mathcal{R}_B)$ decomposes into $\mathcal{T}_A \cup \mathcal{T}_B$, sets of subtrees of size at most τ and the following holds:*

- *$S_1 \subseteq \mathcal{R}_B$ and $S_2 \subseteq \mathcal{R}_A$.*
- *For any tree $t_A \in \mathcal{T}_A$ we have that $v(t_A) \in \mathcal{R}_A$ and for any tree $t_B \in \mathcal{T}_B$ we have that $v(t_B) \in \mathcal{R}_B$.*
- *Any two vertices from the set $\mathcal{R}_A$ have even distance between them and any two vertices from the set $\mathcal{R}_B$ have even distance between them.*
- *No vertex from a tree $t \in \mathcal{T}_A$ is adjacent to a vertex from $\mathcal{R}_B$ and analogously no vertex from a tree $t \in \mathcal{T}_B$ is adjacent to a vertex from $\mathcal{R}_A$.*

Proof of Lemma 2.5.20. The proof is very similar to the proof of Lemma 2.5.16. We use Lemma 2.5.15 to get a set $\mathcal{R}$ of size at most $\frac{\vartheta}{\tau} + 1$ such that $T \setminus \mathcal{R}$ decomposes into subtrees of size at most τ. Let $\mathcal{R}'_A$ denote S_2 union the set of vertices from $\mathcal{R}$ that are at even distance from the root and $\mathcal{R}'_B$ denote S_1 union the set of vertices from $\mathcal{R}$ that have odd distance from the root R.
Denote by S_A the set of vertices in some tree t from $\mathcal{T}$ with $v(t) \in \mathcal{R}'_A$ that is adjacent to some vertex in $\mathcal{R}'_B$. Analogously define S_B. Now define $\mathcal{R}_A := \mathcal{R}'_A \cup S_A$ and $\mathcal{R}_B := \mathcal{R}'_B \cup S_B$. We have $S_1 \subseteq \mathcal{R}'_B \subseteq \mathcal{R}_B$ and $S_2 \subseteq \mathcal{R}'_A \subseteq \mathcal{R}_A$.

Denote by $\mathcal{T}_A$ the set of components from $T - (\mathcal{R}_A \cup \mathcal{R}_B)$ that contain a vertex adjacent to a vertex from $\mathcal{R}_A$ and analogously denote by $\mathcal{T}_B$ the set of components from $T - (\mathcal{R}_A \cup \mathcal{R}_B)$ that contain a vertex adjacent to a vertex from $\mathcal{R}_B$. Observe that $\mathcal{T}_A \cap \mathcal{T}_B = \emptyset$. The components of $T - (\mathcal{R}_A \cup \mathcal{R}_B)$ are subsets of components of $T - \mathcal{R}$ and therefore are of size at most τ each.

□

Lemma 2.5.21. *Let $1 \leq \tau \leq \vartheta$. Let (T, R) be a rooted tree of order $\vartheta + 1$. Let S_1 and S_2 be subsets of $V(T)$ each of size c such that all vertices of S_1 are at odd distance to R and all vertices of S_2 are of even distance from R. Then there exist sets $\mathcal{R}_A$ and $\mathcal{R}_B$ of size at most $3(\frac{\vartheta}{\tau} + 1 + 2c)$ each, such that $T - (\mathcal{R}_A \cup \mathcal{R}_B)$ decomposes into $\mathcal{T}_A \cup \mathcal{T}_B \cup \mathcal{T}_F$, sets of subtrees of size at most τ and the following holds:*

- *$S_1 \subseteq \mathcal{R}_B$ and $S_2 \subseteq \mathcal{R}_A$.*
- *For any vertex v in a subtree $t \in \mathcal{T}_A \cup \mathcal{T}_B$, if v is adjacent to some vertex in $u \in \mathcal{R}_A \cup \mathcal{R}_B$, then $u = v(t)$.*
- *For any tree $t_A \in \mathcal{T}_A \cup \mathcal{T}_F$ we have that $v(t_A) \in \mathcal{R}_A$ and for any tree $t_B \in \mathcal{T}_B$ we have that $v(t_B) \in \mathcal{R}_B$.*
- *$|\bigcup_{t \in \mathcal{T}_A} V(t)| \geq |\bigcup_{t \in \mathcal{T}_B} V(t)|$.*

Proof of Lemma 2.5.21. The proof goes analogously to proof 2.5.18. We use Lemma 2.5.20 to obtain vertex-sets $\mathcal{R}'_A$ and $\mathcal{R}'_B$ of size $\frac{\vartheta}{\tau} + 1 + 2c$ each and sets $\mathcal{T}_A$ and $\mathcal{T}_B \subseteq T - (\mathcal{R}'_A \cup \mathcal{R}'_B)$ of subtrees of size at most τ.
Denote by $\mathcal{T}'_F$ the set of components of $T - (\mathcal{R}'_A \cup \mathcal{R}'_B)$ for which more than one vertex from $\mathcal{R}'_A \cup \mathcal{R}'_B$ is adjacent to some vertex of the given component. Without loss of generality suppose that $|\bigcup_{t \in \mathcal{T}'_B \setminus \mathcal{T}'_F} V(t)| \leq |\bigcup_{t \in \mathcal{T}'_A \setminus \mathcal{T}'_F} V(t)|$. Set $\mathcal{T}_B := \mathcal{T}'_B \setminus \mathcal{T}'_F$ and $\mathcal{T}_A := \mathcal{T}'_A \setminus \mathcal{T}'_F$.
For any tree t in $\mathcal{T}'_F \cap \mathcal{T}'_B$ denote by S_t the set of vertices in t that are adjacent to some vertex in $\mathcal{R}'_B$. By those vertices we shall enlarge the set $\mathcal{R}'_A$. Set $\mathcal{R}_A := \mathcal{R}'_A \cup \bigcup_{t \in \mathcal{T}'_F \cap \mathcal{T}'_B} S_t$. Observe that $|\bigcup_{t \in \mathcal{T}'_F \cap \mathcal{T}'_B} S_t| \leq 2|\mathcal{R}'_B|$ and therefore

$$|\mathcal{R}_A| \leq |\mathcal{R}'_A| + 2|\mathcal{R}'_B| \leq 3(\frac{\vartheta}{\tau} + 1 + 2c).$$

This switching changes the set $\mathcal{T}'_F$ in a natural way, i. e.

$$\mathcal{T}_F := (\mathcal{T}'_F \cap \mathcal{T}'_A) \cup \bigcup_{t \in (\mathcal{T}'_F \cap \mathcal{T}'_B)} t - S_t,$$

where by $t - S_t$ here we understand the union of components of $t - S_t$.

By Lemma 2.5.20 we have that $S_1 \subseteq \mathcal{R}_B$, $S_2 \subseteq \mathcal{R}_A$, that any tree in $t \in \mathcal{T}_A$ has $v(T) \in \mathcal{R}_A$, that any tree $t \in \mathcal{T}_B$ has $v(t) \in \mathcal{R}_B$ and that the only vertices in $\mathcal{T}_A \cup \mathcal{T}_B$ adjacent to the set $\mathcal{R}_A \cup \mathcal{R}_B$ is the root of some subtree in $\mathcal{T}_A \cup \mathcal{T}_B$. By our construction we have that any tree $t \in \mathcal{T}_F$ has $v(t) \in \mathcal{R}_A$. By our assumption we have $|\bigcup_{t \in \mathcal{T}_A} V(t)| \geq |\bigcup_{t \in \mathcal{T}_B} V(t)|$ (in the opposite case we would have switched the A's and B's). $\square$

2.5.4 The proof of the approximate version

Now that we have formulated the tools we need in the proof of the approximative version of the Loebl-Komlós-Sós conjecture, we can prove the theorem. Before doing so, let us recall the statement of the theorem.

Theorem 2.5.1. *For every $\gamma, q > 0$ there is an $n_0 \in \mathbb{N}$ so that for all graphs G on $n \geq n_0$ vertices the following is true.*
If at least $(1+\gamma)\frac{n}{2}$ vertices of G have degree at least $(1+\gamma)qn$, then G contains, as subgraphs, all trees with at most qn edges.

The proof of Theorem 2.5.1 follows the main lines of the proof of Theorem 2.5.2 given by Ajtai, Komlós and Szemerédi (see [1]).

We first use the Regularity Lemma to partition the set of vertices of the graph into clusters, such that most pairs of clusters are regular. Then, we clean the graph deleting some edges and get a subgraph G_p in which we shall embed any tree with at most k edges. The cluster graph H has a similar property as G, i. e. that at least a little bit more than half of its clusters have average degree at least a little bit more than $k = q \cdot n$. We denote by L those clusters.

We use then the Gallai-Edmonds Matching Theorem to get a matching of the cluster graph H. Then, we find two suitable clusters A and B in L, joined by an edge in H, for which the neighbourhood is well covered by the matching and the set of clusters L.

Then, we partition the tree T into small subtrees such that there are few vertices connecting those subtrees.

Then, we embed the tree in the cluster graph. The vertices connecting the small trees embed into the clusters A and B. Because there are few of them, we can fit them in those two clusters. The small trees embed into the neighbourhood of those two clusters. Having partitioned the tree T into small pieces, we can easily embed each small subtree.

Proof of Theorem 2.5.1:
Suppose $0 < \gamma, q < 1$ are given. Then, set $\pi := \min\{\gamma, q\}$ and

$$\alpha := \frac{\pi^5 q}{25 \cdot 10^7}, \quad \varepsilon := \frac{\pi^4 q}{5 \cdot 10^5} \text{ and } m := \frac{500}{q\pi^2}.$$

Szemerédi's Regularity Lemma gives us two natural numbers N_0 and M_0 such that, for any graph G of order $n \geq N_0$, there exists an $(\varepsilon, \alpha; N)$-equitable partition with $m \leq N \leq M$. We set

$$\beta := \frac{\varepsilon}{M}, \text{ a density } \delta := \frac{\pi^2 q}{250}$$

and

$$n_0 := \max\{N_0, \frac{10^7 \cdot M^2}{\pi^4 q^2}\}.$$

We claim that Theorem 2.5.1 holds for this choice of $n_0 \in \mathbb{N}$.

So, assume G is a graph of order $n \geq n_0$ which has at least $(1+\gamma)\frac{n}{2} \geq (1+\pi)\frac{n}{2}$ vertices of degree at least $(1+\gamma)k \geq (1+\pi)k$ and we are given a tree T of order at most $k+1$. Regularity Lemma give us a partition of the vertices into clusters $C_0, C_1, \ldots C_N$ such that

1. $m \leq N \leq M$
2. $|C_0| \leq \varepsilon n$
3. $|C_i| = |C_j|$ for $i, j \in \{1, \ldots, N\}$
4. all but at most εN^2 pairs (C_i, C_j) with $i \neq j \in \{1, \ldots, N\}$ are (ε, α)-regular.

The density satisfies

$$4\varepsilon + \frac{1}{m} < \delta < \frac{\pi^2}{16}. \tag{2.32}$$

Now, we clean the graph G such that we delete all edges between irregular pairs, all edges between pairs with density smaller than δ, all edges that lies inside a cluster and all edges that are incident to C_0. Denote by G_δ the subgraph of G we get after having deleted all these edges. We shall embed the tree T into the subgraph G_δ. The subgraph G_δ has nearly as many edges as has graph G. Indeed, by (1.1) we have deleted at most

$$(2\varepsilon + \frac{1}{2m} + \frac{\delta}{2})\, n^2 < \delta n^2 < \frac{\pi^2 q}{8} n^2$$

edges. This gives us that

$$|E(G \setminus G_\delta)| \leq \frac{\pi^2 q}{8} n^2.$$

Therefore, for all but at most $\frac{\pi n}{4}$ vertices v, we have $deg_{G_\delta}(v) \geq deg_G(v) - \frac{\pi q n}{2}$. Hence,

$$G_\delta \textit{ has at least } (1+\tfrac{\pi}{2})\tfrac{n}{2} \textit{ vertices of degree at least } (1+\tfrac{\pi}{2})k. \tag{2.33}$$

Let us calculate now how many clusters of the graph G_δ has more than αs vertices with degree at least $(1+\frac{\pi}{2})k$. Suppose we have only $(1+\frac{\pi}{10})\frac{N}{2}$ such clusters. In all those clusters, we have at most $s \cdot (1+\frac{\pi}{10})\frac{N}{2}$ vertices with degree at least $(1+\frac{\pi}{2})k$. In the rest of the clusters, we have at most $\alpha s \cdot (1-\frac{\pi}{10})\frac{N}{2}$ vertices with degree at least $(1+\frac{\pi}{2})k$. All together, we have

$$\begin{aligned}\frac{n}{N}(1+\frac{\pi}{10})\frac{N}{2} + \alpha\frac{n}{N}(1-\frac{\pi}{10})\frac{N}{2} &\geq |\{v \in V(G_\delta);\ deg(v) \geq (1+\frac{\pi}{2})k\}| \\ &\geq (1+\frac{\pi}{2})\frac{n}{2}\end{aligned}$$

vertices with degree at least $(1+\frac{\pi}{2})k$. This implies that $\alpha \geq \frac{2}{5}\pi$, a contradiction with the choice of α. So, we know we have more than $(1+\frac{\pi}{10})\frac{N}{2}$ clusters with more than αs vertices with degree at least $(1+\frac{\pi}{2})k$. Denote by $\mathcal{L}$ the set of those clusters. By Proposition 1.5.8, we know that a cluster C that contains more than αs vertices with degree at least $(1+\frac{\pi}{2})k$ must itself have average degree at least

$$deg(C) > (1+\frac{\pi}{2})k - \varepsilon n > (1+\frac{\pi}{5})k.$$

We set $K := (1+\frac{\pi}{5})k$. If H denotes the cluster graph of G_δ, we apply Lemma 2.5.5 to H and K and obtain one of the following two cases:

1. H has a matching M', and an edge AB with $A, B \in \mathcal{L}$ such that M' covers all but at most one neighbour of $A \cup B$, or

2. H has a matching M', and an edge AB with $A, B \in \mathcal{L}$, so that each cluster in $N(A)$ meets a different edge of M', and so that the degree of B into $M' \cup \mathcal{L}$ is at least $(1+\frac{\pi}{5})\frac{k}{2}$.

In both cases, we slightly modify the matching M' to get a matching M such that $V(M) \cap (A \cup B) = \emptyset$. We delete any edges from the matching M' that are

incident to the vertex A or B. We have taken out at most 2 edges $\{e_A, e_B\}$ from M'. So, with the new matching $M := M' \setminus \{e_A, e_B\}$, we have for case 1

$$deg_M(A), deg_M(B) \geq (1+\frac{\pi}{5})k - \frac{3n}{N} \geq (1+\frac{\pi}{5}-\frac{3}{qm})k \geq (1+\frac{\pi}{10})k, \quad (2.34)$$

and for case 2

$$deg_M(A)(1+\frac{\pi}{10})k$$

and

$$deg_{M\cup\mathcal{L}\setminus A}(B) \geq (1+\frac{\pi}{5})\frac{k}{2} - \frac{3n}{N} \geq (1+\frac{\pi}{5}-\frac{3}{qm})\frac{k}{2} \geq (1+\frac{\pi}{10})\frac{k}{2}. \quad (2.35)$$

By the same argument, observe that for any cluster $C \in \mathcal{L}$,

$$deg_{V(H)\setminus(A\cup B)}(C) \geq (1+\frac{\pi}{10})k.$$

If case 1 occurs, we use Proposition 2.5.25. If case 2 occurs, we use Proposition 2.5.26.
We have to check if the conditions of the propositions are fulfilled. Set $\tau := \beta k$, $\vartheta := k$ and $\Delta := \frac{2\alpha s+\tau}{\delta-2\varepsilon}$.

First, observe that

$$\begin{aligned} N(\Delta + 2\varepsilon\tilde{s} + \tau) &\leq N\left(\frac{2\alpha s + \beta qn}{\delta - 2\varepsilon} + 2\varepsilon(1-2\alpha)s + \beta qn\right) \\ &\leq n\left(\frac{4(2\alpha+\varepsilon q)}{3\delta} + 2\varepsilon + \varepsilon q\right) \leq \frac{\pi^2 qn}{900}. \end{aligned}$$

For case 1, we have

$$2N(\Delta + \varepsilon\tilde{s} + \tau) + 2s \leq \frac{\pi^2 qn}{450} + \frac{\pi^2 qn}{250} < \frac{\pi qn}{10}.$$

Then

$$deg_M(A), deg_M(B) \geq \vartheta + 4|M|\bar{\Delta} + 2s,$$

satisfying the conditions of Proposition 2.5.25.

For case 2, we have

$$(\vartheta N(\Delta + 2\varepsilon\tilde{s} + \tau))^{\frac{1}{2}} + N\varepsilon s \leq (qn\frac{\pi^2 qn}{900})^{\frac{1}{2}} + \frac{\pi^4 q}{5\cdot 10^5}n < \frac{\pi qn}{20}.$$

Thus,

$$deg_M(A) \geq \vartheta + 2(\vartheta N\bar{\Delta})^{\frac{1}{2}} + N\varepsilon s$$

and

$$deg_{M\cup\mathcal{L}}(B) \geq \frac{\vartheta}{2} + (\vartheta N\bar{\Delta})^{\frac{1}{2}} + N\varepsilon s.$$

Similarly, for $C \in \mathcal{L}$, we have

$$deg_{V(H)\setminus(A\cup B)}(C) \geq \frac{\vartheta}{2} + (\vartheta N\bar{\Delta})^{\frac{1}{2}} + N\varepsilon s.$$

At least,

$$\vartheta = qn > \frac{\pi^2 qn}{100} > 8N(\Delta + \varepsilon s + \tau).$$

This satisfy the conditions of Proposition 2.5.26.

Before stating and proving Propositions 2.5.25 and 2.5.26, let us introduce their simplified versions. In this way, we can get in touch with the idea of the proof, without considering some details.

2.5.5 Simplified versions

We have found two possible configurations of a cluster graph H of our graph G and then using Lemma 2.5.25 and Lemma 2.5.26, each for a configuration of H, we shall embed our tree T into G. In the following two lemmas we assume a slightly stronger assumption on the possible two configurations of the cluster graph H to get the same result: the embedding of the tree T.

The proofs of those two propositions give a good idea of the proofs of Lemmas 2.5.25 and 2.5.26 without bothering with some technical details. For a discussion on why we cannot have similar simple proofs for Lemmas 2.5.25 and 2.5.26, see Remark 2.5.27.

Suppose that after using Regularity Lemma on our graph G, you clean it as in Section 1.4 to get a subgraph G_δ and let H be the cluster graph of G_δ. Denote by $\mathcal{L}$ the set of clusters in H that have average degree at least $(1+\frac{\pi}{5})k$, where $\pi := \min\{q, \gamma\}$. Suppose that there in an edge $\{A, B\} \in E(H)$ with $A, B \in \mathcal{L}$ and a matching M in H that do not cover A nor B, such that

(1') $N(A)$ and $N(B)$ is covered by M or

(2') $N(A)$ is covered by M and the degree of B into $M \cup \mathcal{L}$ is at least $(1+\frac{\pi}{5})\frac{k}{2}$,

and that each cluster in $N(A)$ meets a different edge of M and each cluster in $N(B)$ meets also a different edge of M.

Suppose that case (1') occurs. Then we use Lemma 2.5.22. If case (2') occurs, then we use Lemma 2.5.23.

Case 1' – (simplified version of case 1)

Proposition 2.5.22. *Let $0 < \alpha, \varepsilon, \delta < 1$ and $\tau, \vartheta, s, \in \mathbb{N}$ with $\tau \leq \vartheta$, $s(\delta - \varepsilon - 2\alpha) > 4\frac{\vartheta}{\tau}$ and $2\varepsilon < \delta$. Let H be the cluster-graph of a graph G with each cluster of size s and such that for each $\{C, D\} \in E(H)$ the pair (C, D) is an (ε, α)-regular pair in G with density at least δ. Let $\{A, B\}$ be an edge of the cluster graph H and M a matching in H such that $V(M) \cap (A \cup B) \neq \emptyset$ and each cluster neighbouring cluster A (resp. cluster B) meets a different edge of the matching M. If*

$$deg_M(A) \geq \vartheta + 4|M|\bar{\Delta} + 2s,$$

$$deg_M(B) \geq \vartheta + 4|M|\bar{\Delta} + 2s,$$

where $\bar{\Delta} := \Delta + 2\varepsilon s$ and $\Delta := \frac{2\alpha s + \tau}{\delta - 2\varepsilon}$, then any tree T of order at most $\vartheta + 1$ embeds in G.

Before entering the details of the proof, we give some sketch of it. First, we decomposes the tree T into a vertex-cut $\mathcal{R}$ of small size and small rooted subtrees of size at most τ. This set of small subtrees is partitioned into two sets $\mathcal{T}_A$ and $\mathcal{T}_B$.

Depending on the size of those sets $\mathcal{T}_A$ and $\mathcal{T}_B$, we partition our matching M into M_A and M_B, such that the degree of cluster A into M_A is large enough to be able to embed $\mathcal{T}_A$ and the degree of cluster B into M_B is large enough to embed the trees of $\mathcal{T}_B$.

Then, we define our embedding in $|\mathcal{R}|$ steps, where, in each of those steps, we embed a vertex v from $\mathcal{R}$ together with all the trees in $\mathcal{T}_A \cup \mathcal{T}_B$ that grow from v, i. e. trees with $v(t) = r$. For each of those trees, we choose a suitable edge in M_A or in M_B, that has still enough free space and embed in it the small tree levelwise.

Proof of Proposition 2.5.22. Choose any vertex $R \in V(T)$ as the root of the tree T. We first partition the rooted tree (T, R) using Lemma 2.5.16. We get sets of subtrees $\mathcal{T}_A$ and $\mathcal{T}_B$ and sets of vertices $\mathcal{R}_A$ and $\mathcal{R}_B$ with

$$|\mathcal{R}_A|, |\mathcal{R}_B| \leq \frac{\vartheta}{\tau} + 1 \leq 2\frac{\vartheta}{\tau}.$$

Set $\mathcal{R} := \mathcal{R}_A \cup \mathcal{R}_B$ and $\mathcal{T} := \mathcal{T}_A \cup \mathcal{T}_B$.

We partition the matching M into disjoint matchings M_A and M_B as follows. We set $p := |V_A| + 2|M|\bar{\Delta} + 2s$, $q := |V_B| + 2|M|\bar{\Delta}$, $p_i := deg_{e_i}(A)$, $q_i := deg_{e_i}(B)$, where e_i are the edges of the matching M and $\zeta = 2s$. We have

$$\frac{p}{\sum_{i=1}^{|M|} p_i} + \frac{q}{\sum_{i=1}^{|M|} q_i} \leq \frac{|V_A| + 2|M|\bar{\Delta} + \zeta}{deg_M(A)} + \frac{|V_B| + 2|M|\bar{\Delta}}{deg_M(B)} \leq 1.$$

Using Lemma 2.5.4, we get a partition $M_A = \{e_1, \dots e_{m_a}\}$ and $M_B := \{e_{m_a+1}, \dots, e_{|M|}\}$ of M with

$$deg_{M_A}(A) := \sum_{i=1}^{m_a} deg_{e_i}(A) > |V_A| + 2|M|\bar{\Delta}, \tag{2.36}$$

and

$$deg_{M_B}(B) := \sum_{i=m_a+1}^{|M|} deg_{e_i}(B) > |V_B| + 2|M|\bar{\Delta}.$$

We define our embedding φ in $|\mathcal{R}|$ steps. For step 1, set $R_0 := R$. At each step $i > 1$, choose a vertex $R_i \in \mathcal{R}$ that has a neighbour for which the embedding φ is already defined. In each step, we embed R_i (in cluster A, if $R_i \in \mathcal{R}_A$; and in cluster B ,otherwise) together with

$$T_i := \{t \in \mathcal{T}, v(t) = R_i\}$$

(in M_A and respectively in M_B). Set

$$V_i := \{R_i\} \cup \bigcup_{t \in T_i} V(t),$$

and

$$U_i := \bigcup_{\ell \leq i} \varphi(V_\ell).$$

So, the set U_i denotes the set of vertices used by the embedding φ after the step i.

At each step $i \geq 1$, our embedding will satisfy the following two conditions.

(a) Any vertex $v \in V_i$ that is a predecessor in our tree T of some vertex $R_k \in \mathcal{R}$ has at least $(\delta - \varepsilon)s$ neighbours in cluster A, if $R_k \in \mathcal{R}_A$; or in cluster B, if $R_k \in \mathcal{R}_B$.

(b) There are i vertices embedded in $A \cup B$.

Without loss of generality, suppose that $R_i \in \mathcal{R}_A$. The case when $R_i \in \mathcal{R}_B$ is analogue. Embed the vertex R_i in an unused vertex of cluster A that is typical with respect to cluster B and with respect to M_A (formally we mean here typical with respect to the union of all clusters contained in edges of M_A). In the first

step, we have $(1-2\alpha)s > 0$ vertices to choose from. At step $i > 1$, the property (a) ensures us that we have at least $(\delta - \varepsilon - 2\alpha)s - i > 0$ unused vertices to choose from. Set $v_i := \varphi(R_i)$.

Then, we have

$$deg_{M_A}(v_i) \geq deg_{M_A}(A) - |M_A|2\varepsilon s \geq |V_A| + |M_A|2\Delta.$$

Now, we use Lemma 2.5.10 to embed T_i in M_A by setting $\mathcal{C} := \bigcup_{\{C,D\}\in M_A} C \cup D$ and $\mathcal{N} := \mathcal{C} \cap N(A)$. Lemma 2.5.10 ensures us that R_i embeds on v_i and T_i embeds in $M_A \setminus U_{i-1}$. Also, we can embed in such a way that all vertices at odd distance from R_i have at least $(\delta - \varepsilon)s$ neighbours in cluster A. The latter property implies that the definition of φ, at step i, fulfills (a). Property (b) follows directly from the fact that at step i we embedded only the vertex R_i in cluster A.

After $|\mathcal{R}|$ steps, we have defined the embedding φ of all vertices in $V(T)$.

□

Case 2' – (simplified version of case 2)

Proposition 2.5.23. *Let $0 < \alpha, \varepsilon, \delta < 1$ and $\tau, \vartheta, s \in \mathbb{N}$ with $\tau \leq \vartheta$, $s(\delta - \varepsilon s - 7\alpha) > 6\frac{\vartheta}{\tau}$ and $2\varepsilon < \delta$. Let H be the cluster-graph of order N of a graph G with each cluster of size s and such that for each $\{C, D\} \in E(H)$ the pair (C, D) is an (ε, α)-regular pair in G with density at least δ. Let $\{A, B\}$ be an edge of the cluster graph H and M a matching in H such that $V(M) \cap (A \cup B) \neq \emptyset$ and each cluster neighbouring cluster A (resp. cluster B) meets a different edge of the matching M. Let $\mathcal{L}$ be the set of clusters C in $V(H) \setminus (V(M) \cup A \cup B)$ with high average degree in $V(H)$, i. e.*

$$deg_{V(H)\setminus(A\cup B)}(C) \geq \frac{\vartheta}{2} + \left(\vartheta N\bar{\Delta}\right)^{\frac{1}{2}} + N\varepsilon s. \tag{2.37}$$

Suppose that

$$deg_M(A) \geq \vartheta + 2\left(\vartheta N\bar{\Delta}\right)^{\frac{1}{2}} + N\varepsilon s,$$

and

$$deg_{M\cup\mathcal{L}}(B) \geq \frac{\vartheta}{2} + \left(\vartheta N\bar{\Delta}\right)^{\frac{1}{2}} + N\varepsilon s,$$

where $\bar{\Delta} := \Delta + 2\varepsilon s'$ and $\Delta := \frac{2\alpha s + \tau}{\delta - 2\varepsilon}$. Suppose that $\vartheta \geq 8N\bar{\Delta}$. Then, any tree T of order at most $\vartheta + 1$ embeds into G.

The idea of the proof is the following. First, we get some vertex-cut $\mathcal{R} = \mathcal{R}_A \cup \mathcal{R}_B$ of the tree T such that the components of $T - \mathcal{R}$ are subtrees of size at most τ (See Remark 2.5.24 for more discussion on this decomposition of the tree). The

set of components are partitioned into three sets $\mathcal{T}_F, \mathcal{T}_A$ and $\mathcal{T}_B$.

After our tree decomposition, we reserve in each cluster $C \neq A, B$ some part $C^{\mathcal{A}}$ for the embedding of $\mathcal{T}_F$, and some part $C^{\mathcal{B}}$, for the embedding of $\mathcal{T}_B$. The components of $\mathcal{T}_A$ will be embedded at the very end, after having finished to embed all components of $\mathcal{T}_F$ and $\mathcal{T}_B$. We can do so, as components in $\mathcal{T}_A$ are end-components, i. e. for any tree $t \in \mathcal{T}_A$ the subgraph $T - t$ is still connected.

Our embedding process will be defined in two phases. In the first phase, we shall define our embedding φ reduced to the vertices in $\mathcal{R}$ and in $\mathcal{T}_F \cup \mathcal{T}_B$, while, in the second phase, we shall define φ on the rest of the tree T, i. e. on the vertices of the components of $\mathcal{T}_A$. The first phase will be defined in $|\mathcal{R}|$ steps.

In each of these steps, we embed one vertex R_i of $\mathcal{R}$ together with all the components in $\mathcal{T}_F \cup \mathcal{T}_B$ that grow from the seed R_i. The vertex R_i will be embedded on a typical vertex of cluster A if $R_i \in \mathcal{R}_A$ and on a typical vertex of cluster B otherwise. Having embedded R_i, for each component $t \in \mathcal{T}_F \cup \mathcal{T}_B$ that grows from R_i, we shall find a suitable edge (either in the matching M or containing some cluster $C \in N(B)$ with large average degree) that is suitable for the embedding of the component t.

At the end, we embed the components of $\mathcal{T}_A$, without taking care of any reservation anymore.

Proof of Lemma 2.5.23. Choose any vertex $R \in V(T)$ as the root of the tree T. We first partition the rooted tree (T, R) using Lemma 2.5.18.
We get sets of subtrees $\mathcal{T}_A$, $\mathcal{T}_B$ and $\mathcal{T}_F$, with $|\bigcup_{t \in \mathcal{T}_B} V(t)| \leq |\bigcup_{t \in \mathcal{T}_A} V(t)|$, and sets of vertices $\mathcal{R}_A$ and $\mathcal{R}_B$ with $|\mathcal{R}_A|, |\mathcal{R}_B| \leq 3(\frac{\vartheta}{\tau} + 1) \leq 6\frac{\vartheta}{\tau}$ with the following properties.

- For each tree $t \in \mathcal{T}_A \cup \mathcal{T}_B$, there is only one vertex $v \in \mathcal{R} := \mathcal{R}_A \cup \mathcal{R}_B$ that is adjacent to some vertex of t.
- Trees from $\mathcal{T}_F \cup \mathcal{T}_F$ are adjacent only to seeds from $\mathcal{R}_A$.
- Trees from $\mathcal{T}_B$ are adjacent only to seeds from $\mathcal{R}_B$.

Set $\mathcal{R} := \mathcal{R}_A \cup \mathcal{R}_B$, $\mathcal{T}' := \mathcal{T}_F \cup \mathcal{T}_B$ are the components we shall embed first and $\mathcal{T} := \mathcal{T}' \cup \mathcal{T}_A$ is the set of all components of $T - \mathcal{R}$. Denote by V_F the set $\bigcup_{t \in \mathcal{T}_F} V(T)$. Analogously, we have $V_B := \bigcup_{T \in \mathcal{T}_B} V(t)$ and $V_A := \bigcup_{t \in \mathcal{T}_A} V(t)$.

Let $y \in \mathbb{R}$ be such that $|V_F| = y \cdot \vartheta$. Then, $|V_B| \leq \frac{(1-y)}{2}\vartheta$.

Set

$$\sigma := \left(\frac{N\bar{\Delta}}{\vartheta}\right)^{\frac{1}{2}}.$$

Set $\mathcal{A} := A$, $\mathcal{B} := \mathcal{L} \cup B$, $N_A := V(M)$, $N_B := V(M) \cup \mathcal{L}$ and $N_L := V(H) \setminus (A \cup B)$, for any $L \in \mathcal{L}$. By using Lemma 2.5.8, we get a σ-uniform partition $C^{\mathcal{A}} \cup C^{\mathcal{B}}$ of each cluster C in $V(H) \setminus (A \cup B)$ such that

$$deg_{N_A^{\mathcal{A}}}(A) \geq |V_F|(1 + \frac{\sigma^2}{y}) \geq |V_F| + \sigma^2\vartheta \geq |V_F| + N\bar{\Delta},$$

$$deg_{N_B^{\mathcal{B}}}(B) \geq |V_B|(1 + \frac{\sigma^2}{1-y}) \geq |V_B| + N\bar{\Delta},$$

and, for $L \in \mathcal{L}$,

$$deg_{N_L^{\mathcal{B}}}(L) \geq |V_B| + N\bar{\Delta},$$

where $N_D^{\mathcal{A}} := \bigcup_{C \in N_D} C^{\mathcal{A}}$ and $N_D^{\mathcal{B}} := \bigcup_{C \in N_D} C^{\mathcal{B}}$, for $D = A, B, L$. Similarly, we define $\mathcal{C}^{\mathcal{A}}$, $\mathcal{C}^{\mathcal{B}}$, $M^{\mathcal{A}}$, $M^{\mathcal{B}}$ and $\mathcal{L}^{\mathcal{B}}$.

Our embedding φ is defined in two phases. During the first one, we embed all vertices in $\mathcal{R}$ and all components of $\mathcal{T}'$. In the second phase, we embed the components of $\mathcal{T}_A$.

The first phase is defined in $|\mathcal{R}|$ steps. At each step $i > 1$, we choose a vertex $R_i \in \mathcal{R}$ that has a neighbour for which the embedding φ is already defined. If $i = 1$, set $R_0 := R$. In each step, we embed R_i in cluster A if $R_i \in \mathcal{R}_A$; and in cluster B, otherwise. Also, we embed

$$T_i := \{t \in \mathcal{T}', v(t) = R_i\}$$

in edges of the matching M or adjacent to some cluster $C \in \mathcal{L}$ neighbouring cluster B.
Set

$$V_i := R_i \cup \bigcup_{t \in T_i} V(t),$$

and

$$U^i := \bigcup_{\ell \leq i} \varphi(V_\ell).$$

In each step of the first phase of our embedding process, $\varphi(V_i)$ will satisfy the following properties.

(a) Any vertex $v \in V_i$ that is a predecessor of some vertex $R_k \in \mathcal{R}$ has at least $(\delta - \varepsilon)s$ neighbours in cluster A, if $R_k \in \mathcal{R}_A$; and $(\delta - \varepsilon)s$ neighbours in cluster B, if $R_k \in \mathcal{R}_B$.

(b) There are i vertices embedded in $A \cup B$.

Now, suppose that we are at step i and want to embed $R_i \in \mathcal{R}$ together with T_i. First, assume that $R_i \in \mathcal{R}_A$. We embed R_i in cluster A and the rest of V_i in $M^{\mathcal{A}}$. Let us choose a vertex that is

- typical to the cluster B,
- typical to the set $\mathcal{C}^{\mathcal{A}}$,
- typical to the set $\mathcal{C}$.

Properties (a) and (b) ensure that there are at least $(\delta - \varepsilon - 3\alpha)s - i > 0$ vertices to choose from the the neighbourhood of vertex $\varphi(v)$, where v is the predecessor of R_i.

Set $s' := |C^{\mathcal{A}}|$. Now, we use Lemma 2.5.10 to embed the components T_i in $G \setminus U^{i-1}$ such that R_i embeds on v_i, T_i embeds in $M^{\mathcal{A}} \setminus U^{i-1}$. We know that we can ensure that all vertices at odd distance from R_i have at least $(\delta - \varepsilon)s$ neighbours in cluster A. The latter property implies that the definition of φ, at step i, fulfills (a). Property (b) follows directly from the fact that we have embedded, at step i, only one vertex R_i in cluster A.

Now, suppose that $R_i \in \mathcal{R}_B$. We want to embed R_i in cluster A and the rest of V_i in M_B or some edge of H incident to some $C \in \mathcal{L}$. Choose a vertex v_i that is

- typical with respect to the cluster A,
- typical to $\mathcal{M}^{\mathcal{B}}$,
- typical to $\mathcal{L}^{\mathcal{B}}$,

By properties (a) and (b), we have at least $(\delta - \varepsilon - 3\alpha)s - i > 0$ vertices to choose from the neighbours of vertex v, where v is the predecessor of R_i. Set $\varphi(R_i) := v_i$.

Let $T^1 \subseteq T_i$ be maximal such that

$$deg_{M^{\mathcal{B}}}(B) - |N|\bar{\Delta} \geq \sum_{t \in T^1} |V(t)| + |U^{j-1} \cap M_B|.$$

Then,

$$deg_{M^{\mathcal{B}}}(v_i) \geq \sum_{t \in T^1} |V(t)| + |U^{j-1} \cap M_B| + 2|M|\Delta,$$

and

$$deg_{\mathcal{L}^{\mathcal{B}}}(v_i) \geq \sum_{t \in T_i \setminus T^1} |V(t)| + |U^{j-1} \cap M_B| + |\mathcal{L}|\Delta.$$

Embed T^1 using Lemma 2.5.10 and $T_i \setminus T^1$ using Lemma 2.5.14. We can embed V_i in $G \setminus U^{i-1}$ such that R_i embeds on v_i, T_i embeds in $\mathcal{C}^{\mathcal{B}} \setminus U^{i-1}$, and all vertices at odd distance from R_i have at least $(\delta - \varepsilon)s$ neighbours in cluster B. The latter property implies that the definition of φ, at step i, fulfills Property (a). Property (b) follows directly from the fact that we embedded at step i only one vertex R_i in cluster B.

The second phase defines the embedding φ for components of $\mathcal{T}_A$. We do not care about any reservation anymore. This phase is defined in $|\mathcal{R}_A|$ steps, where in each step $j \geq 1$ we embed all the trees in

$$T_j := \{t \in \mathcal{T}_A, v(t) = R_j\}$$

in edges of the matching M.
Set

$$V_j := \bigcup_{t \in T_j} V(t),$$

and

$$U^j := U \cup \bigcup_{\ell \leq j} \varphi(V_\ell),$$

where $U := \varphi(\mathcal{R}) \cup \bigcup_{t \in \mathcal{T}'} \varphi(V(t))$.
All vertices $R_j \in \mathcal{R}_A$ were embedded on typical vertices with respect to M. So, we can use Lemma 2.5.10 to define our embedding of V_j with $s' := s$, and $\mathcal{N} := \mathcal{C} \cap N(A)$. We embed V_j in $G \setminus U^{j-1}$ such that R_j embeds on v_j, and V_j embeds in $M \setminus U^{j-1}$.

We have defined φ for $V(T)$.This ends the proof of Proposition 2.5.23.

□

Remark 2.5.24. For case 2' (and analogously for case 2), we need a more complex tree decomposition than for case 1'.

If we use the same tree decomposition as in case 1', we could embed the smallest of V_A and V_B in the neighbourhood of cluster C, but we would need to embed

the whole set V_B, before to embed any vertex of V_A.

We would eventually embed a forest, that we would maybe not be able to glue together using $\mathcal{R}$ and $\mathcal{T}_A$.

In fact $\mathcal{T}_F$ denotes the components that have to be embedded as first. We can wait for the embedding of the components $\mathcal{T}_A \cup \mathcal{T}_B$ as long as we wish.

2.5.6 Case 1 (non-simplified)

Proposition 2.5.25 (Case 1)**.** *Let $0 < \alpha, \varepsilon, \delta < 1$ and $\tau, \vartheta, s \in \mathbb{N}$ with $\tau \leq \vartheta$, $s(\frac{\delta}{2} - 6\alpha) > 4\frac{\vartheta}{\tau}$. Let H be the cluster-graph of a graph G such that for each $\{C, D\} \in E(H)$ the pair (C, D) is an (ε, α)-regular pair in G with density at least δ. All clusters have size s each. Let $\{A, B\}$ be an edge of the cluster graph H and M a matching in H such that $(A \cup B) \cap V(M) \neq \emptyset$. If*

$$deg_M(A) \geq \vartheta + 4|M|\bar{\Delta} + 2s,$$

$$deg_M(B) \geq \vartheta + 4|M|\bar{\Delta} + 2s,$$

where $\bar{\Delta} := \Delta + 2\varepsilon s + \tau$ and $\Delta := \frac{2\alpha s + \tau}{\delta - 2\varepsilon}$, then any tree T of order at most $\vartheta + 1$ embeds into G.

Notation 2.5.25.1. Suppose that I is an ordered set of indices and let $S = \bigcup_{i \in I'} S_i$, where $I' \subseteq I$. We say that S has *shadow* I'. For a partition of I into subsets I_A and I_B, we denote by S_A and by S_B the subset of S that has its shadow in A and in B, respectively. Formally $S_A := \bigcup_{i \in I' \cap I_A} S_i$, and $S_B := \bigcup_{i \in I' \cap I_B} S_i$. We define $S^{\leq i} := \bigcup_{j \leq i} S_j$ and say that $S^{\leq i}$ is the subset of S with shadow at most i. Similarly define $S^{<i}$, $S^{\geq i}$, and $S^{>i}$. We can combine the just defined notations. Then $S_A^{\leq i} = \bigcup_{j \leq i;\ j \in I_A} S_j$.

Before entering the details, we give an idea of the proof.

First, we shall find a set $\mathcal{R} = \mathcal{R}_A \cup \mathcal{R}_B$ of special vertices in the tree T such that $T - \mathcal{R}$ decomposes into small components. Each component t is growing from a seed $v(t)$ that is a vertex in $\mathcal{R}$. If the seed of the small tree t is in $\mathcal{R}_A$, then t is adjacent to vertices only in $\mathcal{R}_A$. Similarly trees t with $v(t) \in \mathcal{R}_B$ are adjacent to vertices in $\mathcal{R}_B$ only. If $\mathcal{T}$ denotes the set of components in $T - \mathcal{R}$, it naturally partitions into $\mathcal{T}_A$ and $\mathcal{T}_B$. So, the decomposition of the tree remembers the shape of a dumbbell.

We shall partition the matching M into two submatchings, M_A for the set $\mathcal{T}_A$ and M_B for $\mathcal{T}_B$.

The next step is to order the seeds in $\mathcal{R}$ and define submatchings (not necessarily disjoint) of M_A and M_B, each submatching corresponding to the union of the components growing from the corresponding seed in $\mathcal{R}$. Then, we define the embedding φ inductively, and in each step, we embed one vertex $R_i \in \mathcal{R}$ and the components in $T - \mathcal{R}$ that grows from the seed R_i in the corresponding matching M_i.

Proof of Proposition 2.5.25. As the proof is relatively long, we decompose it into nine parts, each containing an underlined title, to make it more readable.

1. <u>Decomposition of the tree:</u>

Choose any vertex $R \in V(T)$ as the root of the tree T. We first partition the rooted tree (T, R) using Lemma 2.5.16. We get sets of subtrees $\mathcal{T}_A$ and $\mathcal{T}_B$ and sets of vertices $\mathcal{R}_A$ and $\mathcal{R}_B$ of vertices in $V(T)$ with $|\mathcal{R}_A|, |\mathcal{R}_B| \leq \frac{\vartheta}{\tau} + 1 \leq 2\frac{\vartheta}{\tau}$. Set $\mathcal{R} := \mathcal{R}_A \cup \mathcal{R}_B$ and $\mathcal{T} := \mathcal{T}_A \cup \mathcal{T}_B$. We shall define V_A as the set of vertices of all the components in $\mathcal{T}$ that has shadow in $\mathcal{T}_A$, i. e. $V_A := \bigcup_{t \in \mathcal{T}_A} V(t)$. Similarly define V_B.

2. <u>Partition of the matching</u>:

We partition the matching M into disjoint submatchings M_A and M_B, using Lemma 2.5.4 in the same way as we did in the proof of Proposition 2.5.22. We get $M_A = \{e_1, \dots e_{m_a}\}$ and $M_B := \{e_{m_a+1}, \dots, e_m\}$ of M with

$$deg_{M_A}(A) := \sum_{i=1}^{m_a} deg_{e_i}(A) > |V_A| + 2|M|\bar{\Delta},$$

and

$$deg_{M_B}(B) := \sum_{i=m_a+1}^{m} deg_{e_i}(B) > |V_B| + 2|M|\bar{\Delta}.$$

3. <u>Ordering of vertices in the vertex cut</u>:

Inductively, we order the vertices of $\mathcal{R}$ and define submatchings of M for each vertex of $\mathcal{R}$. Define $R_1 := R$. If R_i is defined, then choose R_{i+1} among any of the vertices of $\mathcal{R} \setminus \bigcup_{j \leq i} R_j$ that has a neighbour in a tree $t \in \mathcal{T}$ with $v(t) = R_j$, $j \leq i$.

This defines an index set $I := \{1, \ldots, |\mathcal{R}|\}$ with a natural partition $I_A := \{i \in I;\ R_i \in \mathcal{R}_A\}$ and $I_B := \{i \in I; R_i \in \mathcal{R}_B\}$. Set $V^i := \bigcup_{t \in \mathcal{T}; v(t)=R_i} V(t)$. Denote by $V^{\leq i}$ the subset of $V(T)$ with shadow at most $i \in I$, i. e. $V^{\leq i} := \bigcup_{j \leq i} V^j$ for $i \geq 0$. Set $V_A^{\leq i} := V_A \cap V^{\leq i}$. Analogously define $V_B^{\leq i}$.

4. Definition of submatchings:

We define the submatchings M_i in such a way so that we can embed the subtrees that grow from R_i in M_i. Set index $a_0 = 1$, $b_0 = m_a + 1$. Let a_i and b_i be minimal with the property that

$$\sum_{\ell=1}^{a_i} deg_{e_\ell}(A) \geq |V_A^{\leq i}| + a_i \bar{\Delta}, \tag{2.38}$$

and

$$\sum_{\ell=m_a+1}^{b_i} deg_{e_\ell}(B) \geq |V_B^{\leq i}| + (b_i - m_a)\bar{\Delta}.$$

For $i < j$, we have $a_i \leq a_j$ and $b_i \leq b_j$. Now, if $i \in I_A$, we define $M_i := \{e_{a_{i-1}}, \ldots, e_{a_i}\} \subseteq M_A$, and if $i \in I_B$, $M_i := \{e_{b_{i-1}}, \ldots, e_{b_i}\} \subseteq M_B$.

5. The inductive step:

Our embedding φ will be defined in $|\mathcal{R}|$ steps. In each step, we first embed R_i in cluster A, if $R_i \in \mathcal{R}_A$; and in cluster B, if $R_i \in \mathcal{R}_B$. Then, we embed all trees t with $v(t) = R_i$ (the trees that grow from the seed R_i) in the submatching M_i of M.

6. The properties of the embedding:

Let $U^i := \varphi(V^{\leq i} \cup \{R_1, \ldots, R_i\})$. For each step $i \geq 0$, our embedding φ will satisfy the following.

(a) $|(A \cup B) \cap U^i| \leq i$.

(b) If $v \in U^i$ and $\varphi^{-1}(v)$ precedes (in the ordering of the tree) some vertex in $\mathcal{R}_A$, resp. $\mathcal{R}_B$, then v has at least $\frac{\delta}{2}s$ neighbours in A, resp. B.

(c) For $CD = e_{a_i}$,
$||C \cap U^i| - |D \cap U^i|| > \tau \Rightarrow \min\{deg_C(A) - |U^i \cap C|, deg_D(A) - |U^i \cap D|\} < \Delta + 2\varepsilon$.
Also, $deg_C(A) < |U^i \cap C| + \Delta + 2\varepsilon s \Rightarrow deg_C(A) < |U^i \cap D| + \bar{\Delta}$,
and $deg_D(A) < |U^i \cap D| + \Delta + 2\varepsilon s \Rightarrow deg_D(A) < |U^i \cap C| + \bar{\Delta}$.

(d) For $CD = e_{b_i}$,
$||C \cap U^i| - |D \cap U^i|| > \tau \Rightarrow \min\{deg_C(B) - |U^i \cap C|, deg_D(B) - |U^i \cap D|\} < \Delta + 2\varepsilon s$.
Also, $deg_C(B) < |U^i \cap C| + \Delta + 2\varepsilon s \Rightarrow deg_C(B) < |U^i \cap D| + \bar{\Delta}$,
and $deg_D(B) < |U^i \cap D| + \Delta + 2\varepsilon s \Rightarrow deg_D(B) < |U^i \cap C| + \bar{\Delta}$.

(e) $U^i \cap e_j = \emptyset$, for $a_i < j \leq m_a$ or $b_i < j \leq m$.

(f) $|U^i \cap e_{a_i}| \leq |V_A^{\leq i}| - \Sigma_0^{a_i - 1}(A)$ and $|U^i \cap e_{b_i}| \leq |V_B^{\leq i}| - \Sigma_0^{b_i - 1}(B)$,

where, for $\ell \in \mathbb{N}$, we define

$$\Sigma_i^\ell(A) := \sum_{j=a_i}^{\ell} \big(deg_{e_j}(A)\big) - |e_{a_i} \cap U^i| - 2(\ell - a_i + 1)\bar{\Delta},$$

and

$$\Sigma_i^\ell(B) := \sum_{j=b_i}^{\ell} \big(deg_{e_j}(B)\big) - |e_{b_i} \cap U^i| - 2(\ell - b_i + 1)\bar{\Delta}.$$

The symbols $\Sigma_i^\ell(A)$ and $\Sigma_i^\ell(B)$ express the size of a subtree of T for which there is enough place in the edges $e_{a_i}, \ldots, e_\ell$ and in the edges $e_{b_i}, \ldots, e_\ell$, respectively. Remark that

$$\Sigma_0^{a_i - 1}(A) + \Sigma_i^{a_{i+1}}(A) = \Sigma_0^{a_{i+1}}(A) - |U^i \cap e_{a_i}|.$$

A similar equation holds for $\Sigma(B)$. Observe that properties (a)–(f) clearly hold for $i = 0$.

7. <u>The embedding of $\mathcal{R}$</u>:

For each step $i \geq 1$, we define the embedding as follows. Suppose that $R_i \in \mathcal{R}_A$ (The case when $R_i \in \mathcal{R}_B$ is define analogously).
Define a vertex in A to be i-typical if it is

- typical to cluster B,
- typical to both C and D, for $\{C, D\} = e_{a_{i-1}}$,
- typical to both C and D, for $\{C, D\} = e_{a_i}$,
- typical to $M_i \setminus (e_{a_{i-1}} \cup e_{a_i})$,

We embed R_i among the unused i-typical vertices of the neighbours of $\varphi(v)$, where v is the predecessor of R_i (if $R_i = R_1 = R$, choose a i-typical vertex in the cluster A without any other restrictions). Using the properties (a) and (b), we know that we have at least $(\frac{\delta}{2} - 6\alpha)s - i > 0$ unused vertices to choose from.

8. <u>The embedding of T_i</u>:

Let $T^1 \subseteq T_i$ be minimal such that

$$\Sigma_{i-1}^{a_{i-1}}(A) \leq \sum_{t \in T^1} |V(t)|.$$

Then

$$deg_{e_{a_{i-1}}}(A) \geq \sum_{t \in T^1} |V(t)| + |U^{i-1} \cap e_{a_{i-1}}| + 2\Delta + 4\varepsilon s + \tau.$$

Set $U := U^{i-1} \cap e_{a_{i-1}}$. Using Lemma 2.5.13, embed T^1 in the first edge $e_{a_{i-1}}$ of M_i. Properties (c) ensures that the conditions(2.16), (2.17) and (2.18) are fulfilled. (If only one of the clusters that form the matching edge is neighbouring A, we use Lemma 2.5.10 instead of Lemma 2.5.13.)

Now let $T^2 \subseteq T_i \setminus T^1$ be minimal such that

$$\Sigma_{i-1}^{a_i-1}(A) - \Sigma_{i-1}^{a_{i-1}}(A) \leq \sum_{t \in T^2} |V(t)|.$$

Then,

$$deg_{M_i \setminus (e_{a_{i-1}} \cup e_{a_i})}(v_i) \geq \sum_{t \in T^2} |V(t)| + (|M_i| - 2)(2\Delta + \tau).$$

Using Lemma 2.5.12, we embed T^2 in $M_i \setminus (e_{a_{i-1}} \cup e_{a_i})$, the "internal" edges of M_i.
Observe that

$$\Sigma_{i-1}^{a_i-1}(A) \leq \sum_{t \in T^1 \cup T^2} |V(t)| \tag{2.39}$$

Using Lemma 2.5.13 with $U = U^{i-1} \cap e_{a_i}$, we embed the trees of $T_i \setminus (T^1 \cup T^2)$ into e_{a_1}, the last edge of M_i ($U = \emptyset$ if $a_{i-1} \neq a_i$, and if $a_{i-1} = a_i$, then $T_i \setminus (T^1 \cup T^2) = \emptyset$). (If only one of the clusters that form the matching edge is neighbouring A, we use Lemma 2.5.10 instead of Lemma 2.5.13.)

We claim that

$$deg_{e_{a_i}}(A) \geq \sum_{t \in T_i \setminus (T^1 \cup T^2)} |V(t)| + 2\bar{\Delta} \tag{2.40}$$

Indeed, using the definition of $\Sigma(A)$, Property (f) for $i-1$, (2.38), and (2.39), we have

$$\begin{aligned}\sum_{t\in T_i\setminus(T^1\cup T^2)}|V(t)| &= |V_{\bar{A}}^{\leq i}| - |V_A^{<i}| - \sum_{t\in(T^1\cup T^2)}|V(t)| \\ &\leq \sum_{j-1}^{a_i} deg_{e_j}(A) - 2a_i\bar{\Delta} - |U^{i-1}\cap e_{a_{i-1}}| - (\Sigma_0^{a_{i-1}-1}(A) + \Sigma_{i-1}^{a_i-1}(A)) \\ &= deg_{e_{a_i}}(A) - 2\bar{\Delta}.\end{aligned}$$

9. <u>Checking the properties for step i</u>:

Property (a) follows immediately from the fact that the only vertices we embed in clusters A and B are vertices from $\mathcal{R}$. Therefore, at the end of step i, we have embedded at most i vertices in $A\cup B$.

Vertices $\mathcal{R}_A$ are embedded on vertices in cluster A that are typical with respect to cluster B, and vertices $\mathcal{R}_B$ on vertices in cluster B that are typical with respect to cluster A. So, vertices $v\in U^i\cap\varphi(\mathcal{R})$ satisfy Property (b). Now, by Lemmas 2.5.12 and 2.5.13, all vertices from $V_{\bar{A}}^{\leq i}$ that are from odd distance from $\mathcal{R}_A$ have at least $(\delta-\varepsilon)s \geq \frac{\delta}{2}s$ neighbours in cluster A. A similar argument holds for vertices in $V_{\bar{B}}^{\leq i}$.

Property (c) or (d) follows from Lemma 2.5.13.

Property (e) follows from the fact that we use only the edges from M_i to embed V^i.

If $e_{a_i}\neq e_{a_{i-1}}$, then $|U^i\cap e_{a_i}| = \sum_{t\in T_i\setminus(T^1\cup T^2)}|V(t)|$. Using Properties (e) and (f) for $i-1$, we have

$$\begin{aligned}\sum_{t\in T_i\setminus(T^1\cup T^2)}|V(t)| &= |V_{\bar{A}}^{\leq i}| - |V_A^{<i}| - \sum_{t\in(T^1\cup T^2)}|V(t)| \\ &\leq |V_{\bar{A}}^{\leq i}| - \Sigma_0^{a_{i-1}-1}(A) - |U^{i-1}\cap e_{a_{i-1}}| - \Sigma_{i-1}^{a_i-1}(A) \\ &\leq |V_{\bar{A}}^{\leq i}| - \Sigma_0^{a_i-1}(A),\end{aligned}$$

leading to Property (f).

Now, if $e_{a_i} = e_{a_{i-1}}$, then $|U^i\cap e_{a_i}| = \sum_{t\in T_i}|V(t)| + |U^{i-1}\cap e_{a_i}|$

$$\begin{aligned}\sum_{t\in T_i}|V(t)| &= |V_A^{\leq i}| - |V_A^{<i}| \\ &\leq |V_A^{\leq i}| - \Sigma_0^{a_{i-1}-1}(A) - |U^{i-1}\cap e_{a_{i-1}}| \\ &\leq |V_A^{\leq i}| - \Sigma_0^{a_i-1}(A) - |U^{i-1}\cap e_{a_i}|,\end{aligned}$$

leading to Property (f).
After the last step $|\mathcal{R}|$, we have embedded all the tree T. This ends the proof of Proposition 2.5.25. □

2.5.7 Case 2 (non-simplified)

Proposition 2.5.26 (Case 2)**.** *Let $0 < \alpha, \varepsilon, \delta < 1$ and $\tau, \vartheta, s \in \mathbb{N}$ with $\tau \leq \vartheta$, $s(\frac{\delta}{2} - 7\alpha) > 6\frac{\vartheta}{\tau}$. Let H be the cluster-graph of order N of a graph G with each cluster of size s and such that for each $\{C, D\} \in E(H)$ the pair (C, D) is an (ε, α)-regular pair in G with density at least δ. Let $\{A, B\}$ be an edge of the cluster graph H and M a matching in H such that $V(M) \cap (A \cup B) \neq \emptyset$. Let $\mathcal{L}$ be the set of clusters C in $V(H) \setminus (M \cup A \cup B)$ that has high average degree in $V(H)$, i. e.*

$$deg_{V(H)\setminus(A\cup B)}(C) \geq \frac{\vartheta}{2} + \left(\vartheta N\bar{\Delta}\right)^{\frac{1}{2}} + N\varepsilon s, \tag{2.41}$$

where $\bar{\Delta} := \Delta + 2\varepsilon s + \tau$ and $\Delta := \frac{2\alpha s+\tau}{\delta-2\varepsilon}$. Suppose that $\vartheta \geq 8N\bar{\Delta}$.

If

$$deg_M(A) \geq \vartheta + 2\left(\vartheta N\bar{\Delta}\right)^{\frac{1}{2}} + N\varepsilon s,$$

and

$$deg_{M\cup\mathcal{L}}(B) \geq \frac{\vartheta}{2} + \left(\vartheta N\bar{\Delta}\right)^{\frac{1}{2}} + N\varepsilon s,$$

then any tree T of order at most $\vartheta + 1$ embeds into G.

Before entering the details, we give an idea of the proof.

First, we find a vertex-cut $\mathcal{R} = \mathcal{R}_A \cup \mathcal{R}_B$ of the tree T that decomposes the tree into three parts: one middle part $\mathcal{T}_F$ that contains components lying between at least two vertices of the cut and two border parts $\mathcal{T}_A$ and $\mathcal{T}_B$.

After decomposing the tree, we reserve a part in each cluster for the middle part $\mathcal{T}_F$ and leave the rest for $\mathcal{T}_B$. We do this proportionally to the size of $\mathcal{T}_F$. So, if the size of $\mathcal{T}_F$ is a portion x of our tree T, we reserve a portion x of each cluster for $\mathcal{T}_F$,

letting a $(1-x)$ portion for $\mathcal{T}_B$. This leaves an average degree from cluster B of $(1-x)\frac{k}{2}$ to the the reserved part for $\mathcal{T}_B$, which is an upper bound for the size of $\mathcal{T}_B$.

We first embed $\mathcal{T}_F$ and $\mathcal{T}_B$, each in its respective reserved part, letting the embedding of $\mathcal{T}_A$ at the very end. The embedding of $\mathcal{T}_F$ is somehow easier, as it uses only matching edges and each cluster in the neighbourhood of A meets a different edge of the matching. For the embedding of $\mathcal{T}_B$, we define, for each set of components growing from a vertex $R_i \in \mathcal{R}$, a submatching M_i, similarly as in case 1.We embed $\mathcal{T}_B$ in the matching-edges, as long as the matching edges are not full. Then we use the clusters with large average degree to embed the rest of $\mathcal{T}_B$. Having embedded the sets $\mathcal{T}_F$ and $\mathcal{T}_B$, we embed $\mathcal{T}_A$. The embedding is defined as for $\mathcal{T}_F$, but not taking in account any reservation.

Proof of Proposition 2.5.26. We use the notations defined in 2.5.25.1. For faster orientation, the proof is divided into eleven parts, each depicted by an underlined title.

1. <u>Decomposition of the tree</u>.

We choose any vertex $R \in V(T)$ as the root of the tree T, and decompose the rooted tree (T, R) using Lemma 2.5.18. We get sets of subtrees $\mathcal{T}_A$, $\mathcal{T}_B$ and $\mathcal{T}_F$ such that $|\bigcup_{t\in\mathcal{T}_B} V(t)| \leq |\bigcup_{t\in\mathcal{T}_A} V(t)|$, and sets of vertices $\mathcal{R}_A$ and $\mathcal{R}_B$ such that $|\mathcal{R}_A|, |\mathcal{R}_B| \leq 3(\frac{\vartheta}{\tau}+1) \leq 6\frac{\vartheta}{\tau}$ with the following properties.

- For each tree $t \in \mathcal{T}_A \cup \mathcal{T}_B$, there is only one vertex $v \in \mathcal{R} := \mathcal{R}_A \cup \mathcal{R}_B$ that is adjacent to some vertex of t,
- trees from $\mathcal{T}_A \cup \mathcal{T}_F$ are adjacent only to vertices from $\mathcal{R}_A$ and
- trees from $\mathcal{T}_B$ are adjacent only to vertices from $\mathcal{R}_B$.

Set $\mathcal{T}' := \mathcal{T}_F \cup \mathcal{T}_B$. These are the components we shall embed first and $\mathcal{T} := \mathcal{T}' \cup \mathcal{T}_A$ is the set of all components of $T - \mathcal{R}$. Set $V_A := \bigcup_{t\in\mathcal{T}_A} V(t)$. Analogously define V_B and V_F.

2. <u>The reservation</u>.

Let $y \in \mathbb{R}$ be such that $|V_F| = y \cdot \vartheta$. We have then that $|V_B| \leq \frac{(1-y)}{2}\vartheta$.
Set

$$\sigma := \left(\frac{N\bar{\Delta}}{\vartheta}\right)^{\frac{1}{2}}.$$

We use Lemma 2.5.8, with $\mathcal{A} := A$, $\mathcal{B} := \mathcal{L} \cup B$, $N_A := M$, $N_B := M \cup \mathcal{L}$ and $N_L := V(H) \setminus (A \cup B)$ for any $L \in \mathcal{L}$; and we get a partition $C^{\mathcal{A}} \cup C^{\mathcal{B}}$ of each cluster C in $V(H) \setminus (A \cup B)$ such that

$$deg_{N_A^{\mathcal{A}}}(A) \geq |V_F|(1 + \frac{\sigma^2}{y}) \geq |V_F| + \sigma^2 \vartheta \geq |V_F| + N\bar{\Delta},$$

$$deg_{N_B^{\mathcal{B}}}(B) \geq |V_B|(1 + \frac{\sigma^2}{1-y}) \geq |V_B| + N\bar{\Delta},$$

and for $L \in \mathcal{L}$,

$$deg_{N_L^{\mathcal{B}}}(L) \geq |V_B| + N\bar{\Delta},$$

where $N_D^{\mathcal{A}} := \bigcup_{C \in N_D} C^{\mathcal{A}}$ and $N_D^{\mathcal{B}} := \bigcup_{C \in N_D} C^{\mathcal{B}}$, for $D = A, B, L$.
We define $\mathcal{C}^{\mathcal{A}}$, $\mathcal{C}^{\mathcal{B}}$, $M^{\mathcal{A}}$, $M^{\mathcal{B}}$, $M_i^{\mathcal{B}}$, and $\mathcal{L}^{\mathcal{B}}$ in a similar way.

3. <u>Ordering of the vertices of $\mathcal{R}$</u>.

Inductively, we order the vertices of $\mathcal{R}$ and define submatchings of M for each vertex of $\mathcal{R}$. Define $R_1 := R$. If R_i is defined, then choose R_{i+1} among any of vertices of $\mathcal{R} \setminus \bigcup_{j \leq i} R_j$ having a neighbour in a tree $t \in \mathcal{T}'$ with $v(t) = R_j$ for $j \leq i$. Observe that we order all vertices in $\mathcal{R}$. Indeed, the only vertex in $\mathcal{R}$ that is adjacent to some tree $t \in \mathcal{T}_A$ is the seed $v(t)$ of t.
This defines an index set $I := \{1, \ldots, |\mathcal{R}|\}$ with a natural partition $I_A := \{i \in I;\ R_i \in \mathcal{R}_A\}$ and $I_B := \{i \in I; R_i \in \mathcal{R}_B\}$. Set $T_i := \{t \in \mathcal{T}'; v(t) = R_i\}$. and $V^i := \bigcup_{t \in T_i} V(t)$. Denote by $V^{\leq i}$ the subset of $V(T)$ with shadow $\leq i \in I$, that is $V^{\leq i} := \bigcup_{j \leq i} V^j$ for $i \geq 0$. Then $V_F^{\leq i} := V^{\leq i} \cap V_F$. Analogously define $V_B^{\leq i}$.

4. <u>Partitioning $\mathcal{T}_B$</u>.

Now that we have ordered the vertices of $\mathcal{R}$, we partition the set $\mathcal{T}_B$ into subsets $\mathcal{T}_B'$ and $\mathcal{T}_B \setminus \mathcal{T}_B'$. The set $\mathcal{T}_B'$ will contain those subtrees that will be embedded using matching edges, and $\mathcal{T}_B \setminus \mathcal{T}_B'$ will contain the subtrees that will be embedded using clusters with high average degree.
Index the trees in $\mathcal{T}_B$ to satisfy the following condition. If $t^k \in T_i$ and $t^\ell \in T_j$ with $i < j$, then $k < \ell$. Then, let ℓ be the maximal index with

$$deg_{M^{\mathcal{B}}}(B) \geq \sum_{j=1}^{\ell} |V(t^j)| + 2|M|\bar{\Delta}.$$

Now, $\mathcal{T}_B' := \{t^j \in \mathcal{T}_B;\ j \leq \ell\}$. We denote by W_B the set of vertices in $\mathcal{T}_B'$ and

define $W_B^{\leq i}$ accordingly.

5. <u>The definition of submatchings</u>.

For each R_i, we define the submatchings M_i and embed the components of $\mathcal{T}'_B$ that grow from R_i (an empty set if $R_i \in \mathcal{R}_A$) in M_i. Set index $b_0 = 1$. Let $b_i \leq m$ be minimal with the following property.

$$\sum_{\ell=1}^{b_i} deg_{e_\ell^{\mathcal{B}}}(B) \geq |W_B^{\leq i}| + 2b_i\bar{\Delta}. \tag{2.42}$$

For $i < j$ we have $b_i \leq b_j$. Now define $M_i := \{e_{b_{i-1}}, \ldots, e_{b_i}\}$.

6. <u>The steps of the embedding</u>.

We define the embedding φ in two phases. During the first phase we embed all vertices in $\mathcal{R}$ and all components of $\mathcal{T}'$. In the second phase, we embed the components of $\mathcal{T}_A$. The first phase shall be defined in $|\mathcal{R}|$ steps. In each step we shall first embed R_i in cluster A, if $R_i \in \mathcal{R}_A$; and in cluster B, if $R_i \in \mathcal{R}_B$. Then we embed all trees $t \in \mathcal{T}'$ with $v(t) = R_i$ (the trees that grow from R_i, except for the components of $\mathcal{T}_A$) in edges of the submatching M_i of M or adjacent to some cluster $C \in \mathcal{L}$ that is neighbouring cluster B. The second phase shall define the embedding φ for components of $\mathcal{T}_A$. In the second phase we shall not care about any reservation anymore, nor the ordering of the vertices in $\mathcal{R}$.

7. <u>The properties of the embedding</u>.

Let $U^i := \varphi(V^{\leq i} \cup \{R_1, \ldots, R_i\})$. For each step $i \geq 0$ our embedding φ will satisfy the following:

(a) $|(A \cup B) \cap U^i| \leq i$.

(b) If $v \in U^i$ and $\varphi^{-1}(v)$ precedes some vertex in $\mathcal{R}_A$, resp. $\mathcal{R}_B$, then v has at least $\frac{\delta}{2}s$ neighbours in A, resp. B.

(c) For $CD = e_{b_i}$,
$||C^{\mathcal{B}} \cap U^i| - |D^{\mathcal{B}} \cap U^i|| > \tau \Rightarrow \min\{deg_{C^{\mathcal{B}}}(B) - |U^i \cap C^{\mathcal{B}}|, deg_{D^{\mathcal{B}}}(B) - |U^i \cap D^{\mathcal{B}}|\} < \Delta + 2\varepsilon s$.
Also $deg_{C^{\mathcal{B}}}(B) < |U^i \cap C^{\mathcal{B}}| + \Delta + 2\varepsilon s \Rightarrow deg_{C^{\mathcal{B}}}(B) < |U^i \cap D^{\mathcal{B}}| + \bar{\Delta}$.
and $deg_{D^{\mathcal{B}}}(B) < |U^i \cap D^{\mathcal{B}}| + \Delta + 2\varepsilon s \Rightarrow deg_{D^{\mathcal{B}}}(B) < |U^i \cap C| + \bar{\Delta}$.

(d) $U^i \cap e_j^{\mathcal{B}} = \emptyset$ for $b_i < j \leq m$, and $\varphi(W_B) \cap \mathcal{L} = \emptyset$.

(e) $|U^i \cap e^{\mathcal{B}}_{b_i}| \leq |W^{\leq i}_{\overline{B}}| - \Sigma_0^{b_i-1}(B)$,

where for $\ell \in \mathbb{N}$ we define

$$\Sigma_i^{\ell}(B) := \sum_{j=b_i}^{\ell} \Big(deg_{e^{\mathcal{B}}_j}(B) \Big) - |e^{\mathcal{B}}_{b_i} \cap U^i| - 2(\ell - b_i + 1)\bar{\Delta}.$$

The symbol $\Sigma_i^{\ell}(B)$ traduces the size of a subtree of T for which we have enough place in the edges $e_{b_i}, \ldots, e_{\ell}$, for its embedding. Remark that

$$\Sigma_0^{b_i-1}(B) + \Sigma_i^{b_\ell}(B) = \Sigma_0^{b_\ell}(B) - |U^i \cap e^{\mathcal{B}}_{b_i}|. \tag{2.43}$$

8. <u>The embedding of $\mathcal{T}_F$</u>.

First, suppose that at step i we have $R_i \in \mathcal{R}_A$. Then, we embed R_i in cluster A and V^i in $M^{\mathcal{A}}$. If v is the predecessor of R_i, choose from the the neighbourhood of vertex $\varphi(v)$ a vertex that is

- typical to the cluster B,
- typical to the set $M^{\mathcal{A}}$,
- typical to the set M.

Using properties (a) and (b), we know that we have at least $(\frac{\delta}{2} - 3\alpha)s - i > 0$ vertices to choose from. Now, we use Lemma 2.5.10 and embed the components of T_i.

9. <u>The embedding of $\mathcal{R}_B$</u>:

Suppose, now, that, at step i, we have $R_i \in \mathcal{R}_B$. Define a vertex in B to be *i-typical* if it is

- typical with respect to the cluster A,
- typical to the set $\mathcal{L}^{\mathcal{B}}$,
- typical to both $C^{\mathcal{B}}$ and $D^{\mathcal{B}}$, for $\{C, D\} = e_{b_{i-1}}$,
- typical to both $C^{\mathcal{B}}$ and $D^{\mathcal{B}}$, for $\{C, D\} = e_{b_i}$,
- typical to $M_i^{\mathcal{B}} \setminus (e_{b_{i-1}} \cup e_{b_i})$.

If v is the predecessor of vertex R_i, embed the vertex R_i in an unused neighbour of $\varphi(v)$ that is an i-typical vertex of cluster B. By properties (a) and (b), we have at least $(\frac{\delta}{2} - 7\alpha)s - i > 0$ unused vertices to choose from.

10. <u>Embedding $\mathcal{T}'_B$</u>.

If $T_i \cap \mathcal{T}'_B \neq \emptyset$, we define the embedding of T'_i as follows.
Let $T^1 \subseteq T'_i$ be a maximal subset satisfying

$$\Sigma_{i-1}^{b_{i-1}}(B) \geq \sum_{t \in T^1} |V(t)|. \tag{2.44}$$

Then

$$deg_{e^{\mathcal{B}}_{b_{i-1}}}(B) \geq \sum_{t \in T^1} |V(t)| + |U_{i-1} \cap e_{b_{i-1}}| + 2\bar{\Delta}.$$

Set $U := U_{i-1} \cap e_{b_{i-1}}$ and $s' := |C^{\mathcal{B}}|$. Use Lemma 2.5.13 to embed T^1 in $e^{\mathcal{B}}_{b_{i-1}}$— the reserved part of the first edge of the matching M_i. Conditions (2.16), (2.17) and (2.18) are satisfied by (c).
Now, let $T^2 \subseteq T'_i \setminus T^1$ be a maximal set satisfying

$$\Sigma_{i-1}^{b_i-1}(B) - \Sigma_{i-1}^{b_{i-1}}(B) \geq \sum_{t \in T^2} |V(t)|. \tag{2.45}$$

Then, by (1.4),

$$deg_{M_i^{\mathcal{B}} \setminus (e_{b_{i-1}} \cup e_{b_i})}(v_i) \geq \sum_{t \in T^2} |V(t)| + (|M_i| - 2)2\Delta + \tau.$$

Embed T^2 in the reserved part for $\mathcal{T}_B$ of the matching $M_i \setminus (e_{b_{i-1}} \cup e_{b_i})$ using Lemma 2.5.12.
Next, let $T^3 \subseteq T'_i \setminus (T^1 \cup T^2)$ be a maximal set satisfying

$$\Sigma_{i-1}^{b_i}(B) - \Sigma_{i-1}^{b_i-1}(B) \geq \sum_{t \in T^3} |V(t)|. \tag{2.46}$$

Then

$$deg_{e^{\mathcal{B}}_{b_i}}(B) \geq \sum_{t \in T^3} |V(t)| + |U_{i-1} \cap e_{b_i}| + 2\bar{\Delta}.$$

Set $U := U_{i-1} \cap e_{b_i}$ and $s' := |C^{\mathcal{B}}|$ (Observe that if $b_i > b_{i-1}$, then property (d) ensures that $U = \emptyset$). Now, use Lemma 2.5.13 to embed T^3 in $e^{\mathcal{B}}_{b_i}$.
Observe that $T^3 = T'_i \setminus (T^1 \cup T^2)$. Indeed, suppose on the contrary that

$$\sum_{t \in T'_i \setminus (T^1 \cup T^2)} |V(t)| > \Sigma_{i-1}^{b_i}(B) - \Sigma_{i-1}^{b_i-1}(B).$$

Then,

$$\sum_{t\in T_i'} |V(t)| > \Sigma_{i-1}^{b_i}(B).$$

On the other hand, using the definition of the index b_i and Property (e), we have

$$\begin{aligned}
\sum_{t\in T_i'} |V(t)| &= |W_B^{\leq i}| - |W_B^{<i}| \\
&\leq \sum_{j=1}^{b_i} deg_{e_j^{\mathcal{B}}}(B) - 2b_i\bar{\Delta} - |U^{i-1} \cap e_{b_{i-1}}| - \Sigma_0^{b_{i-1}-1}(B) \\
&= \sum_{j=1}^{b_i} deg_{e_j^{\mathcal{B}}}(B) - \sum_{j=1}^{b_{i-1}-1} deg_{e_j^{\mathcal{B}}}(B) + 2(b_{i-1}-1)\bar{\Delta} - 2b_i\bar{\Delta} - |U^{i-1} \cap e_{b_{i-1}}| \\
&= \Sigma_{i-1}^{b_i}(B),
\end{aligned}$$

a contradiction.

11. Embedding using clusters with large average degree.

If $T_i \setminus T_i' \neq \emptyset$, we define the embedding φ for the left-over trees in T_i. Observe that $\mathcal{L} \neq \emptyset$. Then,

$$\begin{aligned}
deg_{\mathcal{L}^{\mathcal{B}}}(B) &= deg_{\mathcal{L}^{\mathcal{B}} \cup M^{\mathcal{B}}}(B) - deg_{M^{\mathcal{B}}}(B) \\
&\geq |V_B| + |N|\bar{\Delta} - |W_B| - 2|M|\bar{\Delta} - \tau \\
&\geq |V_B \setminus W_B| + |\mathcal{L}|\bar{\Delta} - \tau \\
&\geq |V_B \setminus W_B| + |\mathcal{L}|(\Delta + 2\varepsilon s).
\end{aligned}$$

Then,

$$deg_{\mathcal{L}^{\mathcal{B}}}(v_i) \geq \sum_{t\in T_i\setminus T_i'} |V(t)| + |U^{i-1} \cap \mathcal{L}^{\mathcal{B}}| + |\mathcal{L}|\Delta. \tag{2.47}$$

We use Lemma 2.5.14 to embed $T_i \setminus T_i'$.

12. Controlling the properties of the embedding.

The definition of our embedding at step i satisfies the properties (a)–(e). Indeed, (a) is true because we have embedded only the vertices R_i in $A \cup B$.
Vertices $\mathcal{R}_A$ are embedded on vertices in cluster A that are typical with respect to cluster B and vertices $\mathcal{R}_B$ on vertices in cluster B that are typical with respect to cluster A. So vertices $v \in U_i \cap \varphi(\mathcal{R})$ satisfy Property (b), Now, by Lemma 2.5.12

and 2.5.13, all vertices from $V_{\bar{F}}^{\leq i}$ that are from odd distance from $\mathcal{R}_A$ have at least $(\delta-\varepsilon)s \geq \frac{\delta}{2}s$ neighbours in cluster A. A similar argument holds for vertices in $V_{\bar{B}}^{\leq i}$.

Property (c) follows from Lemma 2.5.13.

Now, observe that W_B is embedded in M only, and that $W_{\bar{B}}^{\leq i}$ is embedded in $\bigcup_{\ell\leq i} M_i$. If $b_i < j$, then $e_j \notin \bigcup_{\ell\leq i} M_\ell$, and thus, $U^i \cap e_j^{\mathcal{B}} = \varphi(W_{\bar{B}}^{\leq i}) \cap e_j = \emptyset$. Thus, Property (d) is satisfied.

For Property (e), if $e_{b_i} \neq e_{b_{i-1}}$, see that $|U^i \cap e_{b_i}^{\mathcal{B}}| = |U^{i-1} \cap e_{b_i}^{\mathcal{B}}| + \sum_{t\in T^3} |V(t)|$. Then,

$$\begin{aligned}\sum_{t\in T^3} |V(t)| &= |W_{\bar{B}}^{\leq i}| - |W_B^{<i}| - \sum_{t\in(T^1\cup T^2)} |V(t)| \\ &\leq |W_{\bar{B}}^{\leq i}| - |U^{i-1} \cap e_{b_i-1}^{\mathcal{B}}| - \Sigma_0^{b_{i-1}-1}(B) - \Sigma_{i-1}^{b_i-1}(B) \\ &= |W_B \leq i| - \Sigma_0^{b_i-1}(B).\end{aligned}$$

If $b_i = b_{i-1}$, see that $|U^i \cap e_{b_i}^{\mathcal{B}}| = |U^{i-1} \cap e_{b_{i-1}}^{\mathcal{B}}| + \sum_{t\in T_i'} |V(t)|$. Then

$$\begin{aligned}\sum_{t\in T_i'} |V(t)| + |U^{i-1} \cap e_{b_{i-1}}^{\mathcal{B}}| &= |W_{\bar{B}}^{\leq i}| - |W_B^{<i}| + |U^{i-1} \cap e_{b_{i-1}}^{\mathcal{B}}| \\ &\leq |W_{\bar{B}}^{\leq i}| - |U^{i-1} \cap e_{b_i-1}^{\mathcal{B}}| - \Sigma_0^{b_{i-1}-1}(B) + |U^{i-1} \cap e_{b_{i-1}}^{\mathcal{B}}| \\ &= |W_{\bar{B}}^{\leq i-1}| - \Sigma_0^{b_i-1}(B).\end{aligned}$$

Thus all properties are satisfied by the embedding defined at step i.

13. <u>Embedding the last part on the A-side</u>.

Now we begin the second phase of our embedding, i. e. the embedding of the tree of $\mathcal{T}_A$. We embed those trees one after the other. We embed the trees $t \in \mathcal{T}_A$ one after the other. If U denotes the set of vertices in G that were used so far for the embedding of $\mathcal{T}'$ and by the embedding of other components of $\mathcal{T}_A$, for $v_i = v(t)$,

$$\begin{aligned}deg_M(v_i) &\geq deg_M(A) - \varepsilon s \\ &\geq \vartheta + \left(8N\vartheta\bar{\Delta}\right)^{\frac{1}{2}} - \varepsilon s \\ &\geq |V(t)| + |U| + N\bar{\Delta} - \varepsilon s \\ &\geq |V(t)| + |U| + N\Delta.\end{aligned}$$

We embed the tree t using Lemma 2.5.10. When we have embedded all trees in $\mathcal{T}_A$, we have finished defining our embedding φ.

□

Remark 2.5.27. As the reader noticed, the proof of Proposition 2.5.25 and of Proposition 2.5.26 are longer and more complicated than the proof of their simplified versions, the Propositions 2.5.22 and 2.5.23, respectively.

We would like to outline here why some configuration of the matching raises some complications. The seed of the problem is the typicality of a vertex. With typicality, we always refer to a some set of vertices to which a given vertex has to be typical.

For each condition on typicality, we have some set of exceptional vertices in the cluster that do not satisfy the required condition. This set is small, but if we have many conditions on typicality, we cannot ensure that there is a vertex satisfying all the required conditions.

In particular, we cannot expect to have a vertex in some cluster C to be typical with respect to each cluster neighbouring cluster C. We can only choose a few sets of clusters, and find a vertex in cluster C that is typical to those sets.

Such a typical vertex has some expected degree into each of those sets of clusters, but we have no information on its degree to the clusters itself. This point is the source of the complication. To see why, imagine the following situation.

Suppose that the matching edge $\{C, D\}$ lies entirely in the neighbourhood of cluster A and suppose we have embedded a vertex R_i on a typical vertex $v_i \in A$. As long as the vertex v_i has enough unused neighbours in both clusters C and D, we can balance the embedding of the trees in T_i such that the used part in cluster C and the used part in cluster D is about the same. But eventually we use nearly all neighbours in one side and we have no choice where to embed the root of subtrees of T_i.

If the subtrees we embed next have a large difference between the size of the set of vertices with odd distance to the root and the set with even distance to the root, we may fill nearly completely one of the two clusters, say cluster C, letting cluster D nearly empty.

At the time we embed an other vertex R_j in cluster A, it can have still big degree in the edge $\{C, D\}$. We then believe that there is enough space to embed some subtrees from T_j in this edge, but have no place in cluster C. If all neighbours of $\varphi(R_j)$ are contained in such balanced edges, we have no suitable edge to embed T_j.

This is the reason to take extra care when we embed R_i and T_i, making the proof

longer and more complicated.

2.5.8 A second proof for case 1

In the proof of Theorem 2.5.1, we can use the following proposition, instead of Proposition 2.5.25, if case 1 occurs. Indeed,

$$deg_M(A), deg_M(B) \geq k(1+\frac{\pi}{10}) \geq k+\left(\frac{qn \cdot \pi^2 qn}{900}\right)^{\frac{1}{2}} + N\varepsilon s \geq \vartheta + 2(\vartheta N\bar{\Delta})^{\frac{1}{2}} + 2\varepsilon s.$$

Proposition 2.5.28. *Let $0 < \alpha, \varepsilon, \delta < 1$ and $\tau, \vartheta, s \in \mathbb{N}$ with $\tau \leq \vartheta$, $s(\frac{\delta}{2} - 6\alpha) > 4\frac{\vartheta}{\tau}$. Let H be the cluster-graph of a graph G with each cluster of size s and such that for each $\{C, D\} \in E(H)$ the pair (C, D) is an (ε, α)-regular pair in G with density at least δ. Let $\{A, B\}$ be an edge of the cluster graph H and M a matching in H such that $V(M) \cap (A \cup B) \neq \emptyset$. If*

$$deg_M(A) \geq \vartheta + 2\left(\vartheta N\bar{\Delta}\right)^{\frac{1}{2}} + |N|\varepsilon s,$$

and

$$deg_{M\cup\mathcal{L}}(B) \geq \vartheta + 2\left(\vartheta N\bar{\Delta}\right)^{\frac{1}{2}} + |N|\varepsilon s,$$

where $\bar{\Delta} := \Delta + 2\varepsilon s + \tau$ and $\Delta := \frac{2\alpha s + \tau}{\delta - 2\varepsilon}$, then any tree T of order at most $\vartheta + 1$ embeds into G.

Proof of Proposition 2.5.28. We use the same notation as in 2.5.25.1. The proof goes along the proof of Proposition 2.5.25, but instead of partitioning the matching M into two parts, we partition its clusters into two parts. For a faster orientation, we decompose the proof into nine parts, corresponding to parts in Proposition 2.5.25. As many of those parts are identical or very similar to their analogue in Proposition 2.5.25, we omit some details and just refer to the proof of Proposition 2.5.25, instead.

1. Decomposition of the tree.

We decompose the tree T as in Proposition 2.5.25.

2. Partitioning the clusters.

Let y be such that $|V_A| = y \cdot \vartheta$. Then $|V_B| \leq (1-y)\vartheta$.

Set

$$\sigma := \left(\frac{N\bar{\Delta}}{\vartheta}\right)^{\frac{1}{2}}.$$

Set $\mathcal{A} := A$, $\mathcal{B} := B$ and $N_A, N_B := M$. Using Lemma 2.5.8, we get a partition $C^{\mathcal{A}} \cup C^{\mathcal{B}}$ of $\mathcal{C} = V(H) \setminus \{A, B\}$ such that

$$deg_{M^{\mathcal{A}}}(A) \geq |V_A|(1 + \frac{\sigma^2}{y}) \geq |V_A| + \sigma^2\vartheta \geq |V_A| + 2|M|\bar{\Delta},$$

$$deg_{M^{\mathcal{B}}}(B) \geq |V_B|(1 + \frac{\sigma^2}{1-y}) \geq |V_B| + 2|M|\bar{\Delta},$$

where $M^{\mathcal{A}} := \bigcup_{\{C,D\}\in M} C^{\mathcal{A}} \cup D^{\mathcal{A}}$. We define $M^{\mathcal{B}}, e^{\mathcal{A}}, e^{\mathcal{B}}$ analogously.

3. <u>Ordering $\mathcal{R}$</u>.

The inductive ordering of the vertices in $\mathcal{R}$ is done as in Proposition 2.5.25 and use the notation for $V_A^{\leq i}, V_B^{\leq i}$ and T_i.

4. <u>Defining the submatching</u>.

For each $i \in \{1, \ldots, |\mathcal{R}|\}$ we define a submatching M_i of the matching M as follows.
Let a_i and b_i be the minimal indices with the property

$$\sum_{\ell=1}^{a_i} deg_{e_\ell^{\mathcal{A}}}(A) \geq |V_A^{\leq i}| + 2a_i\bar{\Delta},$$

and

$$\sum_{\ell=1}^{b_i} deg_{e_\ell^{\mathcal{B}}}(B) \geq |V_B^{\leq i}| + 2b_i\bar{\Delta}.$$

For $i < j$, we have $a_i \leq a_j$ and $b_i \leq b_j$. Now, if $i \in I_A$, we set $M_i := \{e_{a_{i-1}}, \ldots, e_{a_i}\}$; and if $i \in I_B$, we set $M_i := \{e_{b_{i-1}}, \ldots, e_{b_i}\}$. If $i \in I_A$, then we embed T_i in $M_i^{\mathcal{A}}$; and if $i \in I_B$, we embed T_i in $M_i^{\mathcal{B}}$, where $M_i^{\mathcal{A}} := \bigcup_{e\in M_i} e^{\mathcal{A}}$ and analogously we define $M_i^{\mathcal{B}}$.

5. <u>The steps of the embedding</u>.

We define the embedding in $|\mathcal{R}|$ steps, where in each step i we embed vertex R_i in cluster A and T_i in $M_i^{\mathcal{A}}$, if $R_i \in \mathcal{R}_A$; and we embed R_i in cluster B and T_i in $M_i^{\mathcal{B}}$, if $R_i \in \mathcal{R}_B$. Let $U^i := \varphi(V^{\leq i} \cup \{R_1, \ldots, R_i\})$.

6. <u>The properties of the embedding</u>.

At each step $i \geq 1$, the embedding φ satisfies the following conditions.

(a) $|(A \cup B) \cap U^i| \leq i$.

(b) If $v \in U^i$ and $\varphi^{-1}(v)$ precedes some vertex in $\mathcal{R}_A$, or $\mathcal{R}_B$, then v has at least $\frac{\delta}{2}s$ neighbours in A and in B, respectively.

(c) For $CD = e_{a_i}$,
$||C^{\mathcal{A}} \cap U^i| - |D^{\mathcal{A}} \cap U^i|| > \tau \Rightarrow \min\{deg_{C^{\mathcal{A}}}(A) - |U^i \cap C^{\mathcal{A}}|, deg_{D^{\mathcal{A}}}(A) - |U^i \cap D^{\mathcal{A}}|\} < \Delta + 2\varepsilon s$.
Also $deg_{C^{\mathcal{A}}}(A) < |U^i \cap C^{\mathcal{A}}| + \Delta + 2\varepsilon s \Rightarrow deg_{C^{\mathcal{A}}}(A) < |U^i \cap D^{\mathcal{A}}| + \bar{\Delta}$
and $deg_{D^{\mathcal{A}}}(A) < |U^i \cap D^{\mathcal{A}}| + \Delta + 2\varepsilon s \Rightarrow deg_{D^{\mathcal{A}}}(A) < |U \cap C^{\mathcal{A}}| + \bar{\Delta}$.

(d) For $CD = e_{b_i}$,
$||C^{\mathcal{B}} \cap U^i| - |D^{\mathcal{B}} \cap U^i|| > \tau \Rightarrow \min\{deg_{C^{\mathcal{B}}}(B) - |U^i \cap C^{\mathcal{B}}|, deg_{D^{\mathcal{B}}}(B) - |U^i \cap D^{\mathcal{B}}|\} < \Delta + 2\varepsilon s$.
Also $deg_{C^{\mathcal{B}}}(B) < |U^i \cap C^{\mathcal{B}}| + \Delta + 2\varepsilon s \Rightarrow deg_{C^{\mathcal{B}}}(B) < |U^i \cap D^{\mathcal{B}}| + \bar{\Delta}$
and $deg_{D^{\mathcal{B}}}(B) < |U^i \cap D^{\mathcal{B}}| + \Delta + 2\varepsilon s \Rightarrow deg_{D^{\mathcal{B}}}(B) < |U^i \cap C^{\mathcal{B}}| + \bar{\Delta}$.

(e) $U^i \cap e_j^{\mathcal{A}} = \emptyset$ for $a_i < j \leq |M|$ and $U^i \cap e_j^{\mathcal{B}} = \emptyset$ for $b_i < j \leq |M|$,

(f) $|U^i \cap e_{a_i}^A| \leq |V_A^{\leq i}| - \Sigma_0^{a_i - 1}(A)$ and $|U^i \cap e_{b_i}^B| \leq |V_B^{\leq i}| - \Sigma_0^{b_i - 1}(B)$,

where where for $\ell \in \mathbb{N}$ we define

$$\Sigma_i^{\ell}(A) := \sum_{j=a_i}^{\ell} \left(deg_{e_j^{\mathcal{A}}}(A) \right) - |e_{a_i}^{\mathcal{A}} \cap U^i| - 2(\ell - a_i + 1)\bar{\Delta},$$

and

$$\Sigma_i^{\ell}(B) := \sum_{j=b_i}^{\ell} \left(deg_{e_j^{\mathcal{B}}}(B) \right) - |e_{b_i}^{\mathcal{B}} \cap U^i| - 2(\ell - b_i + 1)\bar{\Delta}.$$

7. <u>Embedding $\mathcal{R}$</u>.

Suppose we are at step i of our embedding process, and that $R_i \in \mathcal{R}_A$ (the other case is similar).
Define a vertex in A to be *i-typical* if it is

- typical to cluster B,
- typical to both $C^{\mathcal{A}}$ and $D^{\mathcal{A}}$, for $\{C, D\} = e_{a_{i-1}}$,
- typical to both $C^{\mathcal{A}}$ and $D^{\mathcal{A}}$, for $\{C, D\} = e_{a_i}$,

- typical to $M_i^{\mathcal{A}} \setminus (e_{a_{i-1}} \cup e_{a_i})$,

Properties (a) and (b) allows us to embed R_i among the unused i-typical vertices of the neighbours of $\varphi(v)$, where v is the predecessor of R_i .

8. Embedding T_i.

Similarly as in Proposition 2.5.25 we embed the subtrees from T_i in the matching edges of M, but using only the respective reserved parts of the clusters.

9. Checking the properties.

This is done similarly as in Proposition 2.5.25.
This ends the proof of Proposition 2.5.28. □

2.5.9 A second proof for case 2

In the proof of Theorem 2.5.1, we can use the following proposition, instead of Proposition 2.5.26, if case 2 occurs. Indeed, for a cluster $C \in \mathcal{L}$, we have

$$deg_{V(H)\setminus(A\cup B)}(C) \geq (1+\frac{\pi}{10})k \geq k + \frac{\pi^2 qn}{900} \leq \frac{\vartheta}{2} + N\bar{\Delta}.$$

Also,

$$(1+\frac{\pi}{10})k \geq k + \pi^2 qn(\frac{\pi^2 qn}{180} + 2\frac{n}{m}) \geq \vartheta + 5N\bar{\Delta} + 2s.$$

So,

$$deg_M(A) \geq \vartheta + 5N\bar{\Delta} + 2s \geq \vartheta + (6|M| + 2|\mathcal{L}|)\bar{\Delta} + 2s$$

and

$$deg_{M\cup\mathcal{L}}(B) \geq \frac{1}{2}(\vartheta + 5N\bar{\Delta} + 2s) \geq \frac{\vartheta}{2} + (3|M| + |\mathcal{L}|)\bar{\Delta} + 2.$$

Proposition 2.5.29. *Let $0 < \alpha, \varepsilon, \delta < 1$ and $\tau, \vartheta, s \in \mathbb{N}$ with $\tau \leq \vartheta$, $s(\frac{\delta}{2} - 7\alpha) > 6\frac{\vartheta}{\tau}$. Let H be the cluster-graph of order N of a graph G with each cluster of size s and such that for each $\{C, D\} \in E(H)$ the pair (C, D) is an (ε, α)-regular pair in G with density at least δ. Let $\{A, B\}$ be an edge of the cluster graph H and M a matching in H such that $V(M) \cap (A \cup B) \neq \emptyset$. Let $\mathcal{L}$ be the set of clusters C in $V(H) \setminus (V(M) \cup A \cup B)$ that has high average degree in $V(H)$, i. e.*

$$deg_{V(H)\setminus(A\cup B)}(C) \geq \frac{\vartheta}{2} + N\bar{\Delta}, \tag{2.48}$$

where $\bar{\Delta} := \Delta + 2\varepsilon s + \tau$ *and* $\Delta := \frac{2\alpha s+\tau}{\delta-2\varepsilon}$.

If

$$deg_M(A) \geq \vartheta + (6|M| + 2|\mathcal{L}|)\bar{\Delta} + 2s,$$

and

$$deg_{M\cup\mathcal{L}}(B) \geq \frac{\vartheta}{2} + (3|M| + |\mathcal{L}|)\bar{\Delta} + s,$$

then, any tree T *of order at most* $\vartheta + 1$ *embeds into* G.

Proof of Proposition 2.5.29. We use the notation defined in 2.5.25.1. For faster orientation, we decompose the proof in 14 parts, each depicted with an underlined title. As parts of the proof go along the proof of Proposition 2.5.26, we omit some details, and refer to the proof of Proposition 2.5.26, instead.

1. <u>Decomposition of the tree</u>.

We decompose the tree as in Proposition 2.5.26.

2. <u>Partition of $M \cup \mathcal{L}$</u>.

Set $p := |V_F| + 2|M|\bar{\Delta} + 2s$ and $q := |V_B| + (2|M| + |\mathcal{L}|)\bar{\Delta}$. Let $m := |M|$ and $\lambda = |\mathcal{L}|$. Set

$$p_i := \begin{cases} deg_{e_i}(A), & \text{for } i \leq m \\ deg_{C_i}(A) = 0, & \text{for } i > m, \end{cases}$$

where e_i are the edges of M and C_i are the clusters of $\mathcal{L}$. Similarly we define

$$q_i := \begin{cases} deg_{e_i}(B), & \text{for } i \leq m \\ deg_{C_i}(B), & \text{for } i > m. \end{cases}$$

Set $\zeta := 2s$. Observe that

$$\begin{aligned} \frac{p}{\sum_{i=1}^{m+\lambda} p_i} + \frac{q}{\sum_{i=1}^{m+\lambda} q_i} &\leq \frac{|V_F| + 2|M|\bar{\Delta} + \zeta}{deg_M(A)} + \frac{|V_B| + (2|M| + |\mathcal{L}|)\bar{\Delta}}{deg_{M\cup\mathcal{L}}(B)} \\ &\leq \frac{\vartheta + (6|M| + 2|\mathcal{L}|)\bar{\Delta} + 2s}{\vartheta + (6|M| + 2|\mathcal{L}|)\bar{\Delta} + 2s} \\ &= 1. \end{aligned}$$

Recall that $|V_F| = y \cdot \vartheta$ and $|V_B| \leq \frac{(1-y)}{2}\vartheta$.
Now, Lemma 2.5.4 gives us a partition $(M \cup \mathcal{L})_F$ and $(M \cup \mathcal{L})_B$ so that

$$deg_{M_F}(A) = deg_{(M\cup\mathcal{L})_F}(A) \geq |V_F| + 2|M|\bar{\Delta}, \tag{2.49}$$

and

$$deg_{(M\cup\mathcal{L})_B}(B) \geq |V_B| + (2|M| + |\mathcal{L}|)\bar{\Delta}. \tag{2.50}$$

3. Ordering $\mathcal{R}$.

We order the vertices in $\mathcal{R}$ as in Proposition 2.5.26.

4. Partitioning $\mathcal{T}_B$.

We partition the set $\mathcal{T}_B$ into the set $\mathcal{T}'_B$, the part that will be embedded in M_B; and into the set $\mathcal{T}_B \setminus \mathcal{T}'_B$, the part that will be embedded using clusters in $\mathcal{L}$. Index the trees in $\mathcal{T}_B$ to satisfy the following condition. If $t^k \in T_i$ and $t^\ell \in T_j$ with $i < j$, then $k < \ell$. Let ℓ be the maximal index with

$$deg_{M_B}(B) \geq \sum_{j=1}^{\ell} |V(t^j)| + 2|M|\bar{\Delta}. \tag{2.51}$$

Now, $\mathcal{T}'_B := \{t^j \in \mathcal{T}_B;\ j \leq \ell\}$. We denote by W_B the set of vertices in $\mathcal{T}'_B$ and define $W_B^{\leq i}$ accordingly.

5. Definition of the submatchings.

We need to define some submatchings M_i only for the embedding of $\mathcal{T}_B$, as each cluster in the neighbourhood of cluster A meets a different edge of the matching.

For each R_i, we define the submatchings M_i and embed the components of $\mathcal{T}'_B$ that grow from R_i (an empty set if $R_i \in \mathcal{R}_A$) in M_i. Let $M_B := \{e_1, \ldots, e_m\}$. Set an index $b_0 = 1$. Let $b_i \leq m$ be minimal with the following property.

$$\sum_{\ell=1}^{b_i} deg_{e_\ell}(B) \geq |W_B^{\leq i}| + 2b_i\bar{\Delta}. \tag{2.52}$$

For $i < j$ we have $b_i \leq b_j$. Now define $M_i := \{e_{b_{i-1}}, \ldots, e_{b_i}\}$. By (2.51), such an index b_i always exists.

6. The steps of the embedding.

The embedding is defined in two phases. In the first phase, we embed the trees of $\mathcal{T}_F$ in M_F; and the trees of $\mathcal{T}'_B$ in M_B; This phase is defined in $|\mathcal{R}|$ step, where at each step we embed the vertex $R_i \in \mathcal{R}$ in cluster A or B, together with the set $T'_i := \{t \in \mathcal{T}_F \cup \mathcal{T}'_B;\ v(t) = R_i\}$ in the matching M_F and M_B, respectively.

In the second phase, we embed the trees from $\mathcal{T}_B \setminus \mathcal{T}'_B$ and then, we embed the trees from $\mathcal{T}_A$. Denote by $\bar{T}_i := \{t \in \mathcal{T}_B \setminus \mathcal{T}'_B; v(t) = R_i\}$.

We recall that $V^i := \bigcup_{t \in T_i} V(t)$ and $U^i := \varphi(\bigcup_{j \leq i}\{R_j\} \cup V^j)$.

7. Properties of the embedding.

At each step $i \geq 1$ of the first phase, our embedding satisfies the following conditions.

(a) $|(A \cup B) \cap U^i| \leq i$.

(b) If $v \in U^i$ and $\varphi^{-1}(v)$ precedes (in the ordering of the tree) some vertex in $\mathcal{R}_A$ or $\mathcal{R}_B$, then v has at least $\frac{\delta}{2}s$ neighbours in A and in B, respectively.

(c) For $CD = e_{b_i}$,
$||C \cap U^i| - |D \cap U^i|| > \tau \Rightarrow \min\{deg_C(B) - |U^i \cap C|, deg_D(B) - |U^i \cap D|\} < \Delta + 2\varepsilon s$.
Also, $deg_C(B) < |U^i \cap C| + \Delta + 2\varepsilon s \Rightarrow deg_C(B) < |U^i \cap D| + \Delta + 2\varepsilon s + \tau$, and $deg_D(B) < |U^i \cap D| + \Delta + 2\varepsilon s \Rightarrow deg_D(B) < |U^i \cap C| + \Delta + 2\varepsilon s + \tau$.

(d) $U^i \cap e_j = \emptyset$ for $b_i < j \leq m$, and $\varphi(W_B) \cap \mathcal{L} = \emptyset$.

(e) $|U^i \cap e_{b_i}| \leq |W_{\bar{B}}^{\leq i}| - \Sigma_0^{b_i - 1}(B)$,

where

$$\Sigma_i^{\ell}(B) := \sum_{j=b_i}^{\ell} \big(deg_{e_j}(B)\big) - |e_{b_i} \cap U^i| - 2(\ell - b_i + 1)\bar{\Delta}.$$

8. Embedding $\mathcal{T}_F$.

First, suppose that we have $R_i \in \mathcal{R}_A$ at step i. Then, we embed R_i in cluster A and V^i in M_F. If v is the predecessor of R_i, choose from the the neighbourhood of vertex $\varphi(v)$ a vertex that is

- typical to the cluster B,
- typical to the set M_F,

- typical to the set M.

Using properties (a) and (b), we know that we have at least $(\frac{\delta}{2} - 3\alpha)s - i > 0$ vertices to choose from. Now, we use Lemma 2.5.10 and embed the components of T_i.

9. Embedding $\mathcal{R}_B$.

Suppose, now, that, at step i, we have $R_i \in \mathcal{R}_B$. Define a vertex in B to be *i-typical* if it is

- typical with respect to the cluster A,
- typical to the set $\mathcal{L}$,
- typical to both C and D, for $\{C, D\} = e_{b_{i-1}}$,
- typical to both C and D, for $\{C, D\} = e_{b_i}$,
- typical to $M \setminus (e_{b_{i-1}} \cup e_{b_i})$.

If v is the predecessor of vertex R_i, embed the vertex R_i in an unused neighbour of $\varphi(v)$ that is an i-typical vertex of cluster B. By properties (a) and (b), we have at least $(\frac{\delta}{2} - 7\alpha)s - i > 0$ unused vertices to choose from.

10. Embedding $\mathcal{T}'_B$.

The embedding of $T'_i := T_i \cap \mathcal{T}'_B$ is defined in a similar way as in the proof of Proposition 2.5.26, but using the whole clusters, instead of a reserved part only.

12. Checking the properties of the embedding after step i.

Checking the Properties (a) – (e) is done similarly as in Proposition 2.5.26.

13. Embedding the trees of $\mathcal{T}_B \setminus \mathcal{T}'_B$.

All the vertices of $\mathcal{R}_B$ are embedded in vertices of cluster B that are typical with respect to $\mathcal{L}$. For such a typical vertex v,

$$deg_{\mathcal{L} \setminus U}(v) \geq |V_B| - |W_B| - |U \cap \mathcal{L}| + |\mathcal{L}|\Delta,$$

where U denotes the vertices used so far. We embed then $\bar{T}_i$ in the graph using Lemma 2.5.14.

14. Embedding the trees of $\mathcal{T}_A$.

The embedding of the trees in $\mathcal{T}_A$ is done in the same way as in Proposition 2.5.26.

□

Remark 2.5.30. There is a variation to the proof of Proposition 2.5.29. It goes exactly along this proof, but at the time we have embedded all the vertices in W_B, we forget about the partition of $M = M_F \cup M_B$. We embed then $\mathcal{T}_B \setminus \mathcal{T}'_B$ using Lemma 2.5.14, without waiting until the end of the embedding of the trees of $\mathcal{T}_F$, and embed $\mathcal{T}_F$ and $\mathcal{T}_A$ in the whole matching M using Lemma 2.5.10.

2.5.10 A simple generalisation

Jan Foniok asked if the tools developed to prove Theorem 2.5.1 permits us to embed other graphs than trees. Indeed, we proved, with Maya Stein, that we can embed some graphs $\tilde{G}$, that we obtain from trees by adding some (carefully chosen) edges. Then, we embed the spanning tree T of $\tilde{G}$, using the technique of the proof of Theorem 2.5.1, with some extra precaution, to embed vertices that are adjacent in $\tilde{G} \setminus T$ in vertices of G that are also adjacent.

In the proof of Theorem 2.5.1, we use the fact that any tree is bipartite. Indeed, if we would like to embed a graph that is not bipartite, we surely would need to find suitable triangles in the cluster graph H. Therefore the graph $\tilde{G}$ has to be bipartite. Also, we require the circles of $\tilde{G}$ to be edge disjoint, so $\tilde{G}$ keeps a tree-like structure. Maybe, this condition may be slightly relaxed to get some stronger result. This would imply some changes in the values of $\delta, \varepsilon, \alpha$.

Theorem 2.5.31. *For every $\gamma, q > 0$ and for every $c \in \mathbb{N}$ there is an $n_0 \in \mathbb{N}$ so that for all graphs G on $n \geq n_0$ vertices the following is true.*
If at least $(1+\gamma)\frac{n}{2}$ vertices of G have degree at least $(1+\gamma)qn$, then any bipartite graph $\tilde{G}$ with at most $qn + c$ edges that contains c cycles, which are pairwise edge-disjoint, embeds in G.

Sketch of the proof of Theorem 2.5.31
The proof goes along the proof of Theorem 2.5.1. Set

$$\pi := \min\{\gamma, q\}, \quad \varepsilon := \frac{\pi^4 q}{5 \cdot 10^5}, \quad \alpha := \frac{\pi^5 q}{25 \cdot 10^7}, \quad \text{and} \quad m_0 := \max\{\frac{500}{\pi^2 q}, l\}.$$

Regularity Lemma applied to these values gives us two natural numbers N_0 and M_0. Set

$$\beta := \frac{\varepsilon}{M_0}, \quad \delta := \frac{\pi^2 q}{250}, \quad \text{and} \quad n_0 := \max\{N_0, \frac{3 \cdot 10^{12} M_0^2}{\pi^6 q^2}\}.$$

So, suppose G is a graph of order $n \geq n_0$ that satisfies the hypothesis of Theorem 2.5.31. As in the proof of Theorem 2.5.1, we find an $(\alpha, \varepsilon; N)$-equitable partition of the vertex set $V(G)$ with $m_0 \leq N \leq M_0$. We delete the undesirable edges in G and get a subgraph G_δ. We define a cluster graph H on G_δ, and using Lemma 2.5.5 on H, we find two adjacent clusters A, B and a matching M with $V(M) \cap (A \cup B) = \emptyset$ such that one of the following holds.

1. $deg_M(A), deg_M(B) \geq (1 + \frac{\pi}{20})qn$,
2. $deg_M(A) \geq (1 + \frac{\pi}{20})qn$, $deg_{M \cup \mathcal{L}}(B) \geq (1 + \frac{\pi}{20})\frac{qn}{2}$ and each cluster neighbouring A meets a different edge of the matching.

Now, we find inductively a spanning tree T in $\tilde{G}$ and a matching $\tilde{M}$ that is edge disjoint with T. Denote by Γ the set of all cycles in $\tilde{G}$. At each step $1 \leq \ell \leq c$, we delete one edge e_ℓ of some cycle σ_ℓ in such a way that the edges $\bigcup_k e_k$ forms a matching.

At step 1, choose any cycle in $\tilde{G}$ and denote it by σ_1. Choose any edge e_1 in σ_1 and delete it.

Now, suppose that we have deleted the edges $e_1, \ldots e_{\ell-1}$ in $\sigma_1, \ldots \sigma_{\ell-1}$, respectively, such that $\bigcup_{k<\ell} e_k$ from a matching. Denote by $\Gamma_{\ell-1}$ the set of cycles $\{\sigma_1, \ldots \sigma_{\ell-1}\}$. If there exists a cycle $\sigma \in \Gamma \setminus \Gamma_{\ell-1}$ that is adjacent to some of the cycles in $\Gamma_{\ell-1}$, denote this cycle by σ_ℓ. Otherwise, choose any cycle in $\Gamma \setminus \Gamma_{\ell-1}$ and denote it by σ_ℓ.

Observe that, by our construction and the fact that the cycles are in Γ pairwise edge-disjoint, $|V(\sigma_\ell) \cap \bigcup_{\sigma \in \Gamma_{\ell-1}} V(\sigma)| \leq 1$. As any cycle in Γ has length at least 4, there exists an edge $e_\ell \in E(\sigma_\ell)$ that is vertex disjoint with $\bigcup_{\sigma \in \Gamma_{\ell-1}} V(\sigma)$. Delete this edge e_ℓ.

After $|\Gamma|$ steps, we have found our spanning tree T and a matching $\tilde{M}$ that is edge disjoint with T.

Now, we apply Lemma 2.5.20, if case 1. occurs; and Lemma 2.5.21, if case 2. occurs. We get sets $\mathcal{R}_A$ and $\mathcal{R}_B \subseteq V(\tilde{G})$, each of size at most $3(\frac{2}{\beta} + 2c)$, and sets $\mathcal{T}_A$ and $\mathcal{T}_B$ (and $\mathcal{T}_F$ if we have case 2) of subtrees of T.

For an edge $e_\ell = \{u_\ell, v_\ell\} \in E(\tilde{M} \cap \sigma_\ell)$, $\ell \in \{1, \ldots, c\}$, we define the σ-*neighbourhood* of the edge e_ℓ as the set $\{x_\ell^u, x_\ell^v\}$, where

$$x_\ell^u := N(u_\ell) \cap V(\sigma_\ell) \setminus \{v_\ell\}$$
$$x_\ell^v := N(v_\ell) \cap V(\sigma_\ell) \setminus \{u_\ell\}$$

The vertices x_ℓ^u and x_ℓ^v are the neighbours of the edge e_ℓ in the cycle σ_ℓ.

As σ_ℓ has length at least 4, we know that $x_\ell^u \neq x_\ell^v$. Observe that in our tree ordering $\preceq$, it hods that $x_\ell^u \prec u_\ell$ and $x_\ell^v \prec v_\ell$.

The idea of the proof is to embed $T \subseteq \tilde{G}$ following the proof of Lemma 2.5.25, for case 1.; or the proof of Lemma 2.5.26, for case 2.; but with small changes for the vertices in $\tilde{M}$ and their σ-neighbourhood to make sure that vertices forming the edges e_ℓ are embedded in adjacent vertices in graph G.

So, suppose that at some point of our embedding process, we want to embed the first of the two vertices x_ℓ^u, x_ℓ^v, for some edge $e_\ell \in \tilde{M}$. Without loss of generality, suppose it is x_ℓ^u. As $x_\ell^u < u_\ell$ and $x_\ell^v < v_\ell$, the embedding φ is not defined for any of the vertices x_ℓ^v, u_ℓ and v_ℓ, yet. Embed x_ℓ^u as usual, but in the rest of the embedding process, we make sure that v_ℓ is embedded as the last of the vertices x_ℓ^v, u_ℓ, v_ℓ. This is possible, because $x_\ell^v < v_\ell$, as it is on the path from R to v_ℓ, and both vertices u_ℓ, v_ℓ are in $\mathcal{R}$. So, after the step when we embed x_ℓ^u, we can choose u_ℓ as some R_j, as soon as we need, i. e. before chosing v_ℓ.

Denote by w_1 the vertex from $\{u_\ell, x_\ell^v\}$ we embed first and set $w_2 := \{u_\ell, x_\ell^v\} \setminus w_1$. Without loss of generality, assume that $v_\ell \in \mathcal{R}_A$. Then, we shall embed w_2 on a vertex that is typical with respect to $N(w_1) \cap A$. This is possible, as $|N(w_1) \cap A| \geq (\delta - \varepsilon)s \geq \alpha s$. Then, at the time we want to embed v_ℓ, we have at least $((\delta-\varepsilon)^2 - 7\alpha)s - 3(\frac{2}{\beta} + 2c) > 0$ unused typical vertices to choose $\varphi(v_\ell)$ from.

After having embedded v_ℓ, we continue our embedding process as usual, but taking special care with the embedding of the other vertices vertices u_i, v_j, x_u^j and x_v^j. This ends the sketch for the proof of Theorem 2.5.31. $\square$

In this thesis, we investigated the Loebl-Komlós-Sós Conjecture. We brought three contributions to this topic.

1. We brought a solution for the class of trees with diameter at most 5 (Theorem 2.4.2).

2. We gave an approximate solution for large graphs strengthening the condition on the degree for the graph into which we want to embed the tree (Theorem 2.5.1).

3. Another result consists in embedding bipartite graphs containing few edge-dijoint cycles in a graph G satisfying the conditions of Theorem 2.5.1 (Theorem 2.5.31).

The first result is an exact one, but only for a very restricted class of trees. It seems that increasing the diameter of the trees rapidely increasses the number of cases one has to consider. This makes our approach difficult to use for trees with larger diameter.

The second result has no restriction on the structure of the tree, but, as any result using the Regularity Lemma, the result applies only for large and dense graphs. For sparse graphs, one may try to investigate the possibility of using the Sparse Regularity Lemma (see Kohayakawa [10]).

An interesting question is to which extend Theorem 2.5.31 can be generalised. We have showned that in a graph G satisfying the conditions of Theorem 2.5.31, we can embed any bipartite graph H satisfying the followings:

- the spanning tree of H has order at most $k+1$,
- any edge in $E(H)$ is contained in at most one cycle,
- the number of cycles is constant with respect to $n = |V(G)|$.

How much can we strengthen one of the last two conditions?
An other direction of investigation is to take out the approximation in Theorem 2.5.1. This would imply a solution of the Loebl-Komlós-Sós Conjecture for large graphs. Indeed, this was done by Zhao in the special case of the Loebl conjecture, i.e. when $k = \frac{n}{2}$ [19]. Ajtai, Komlós, Simonovits and Szemerédi are working on a paper that deals with the same question in the context of the Erdős-Sós conjecture [2].

Conclusion

In this thesis, we investigated the Loebl-Komlós-Sós Conjecture. We brought three contributions to this topic.

1. We brought a solution for the class of trees with diameter at most 5 (Theorem 2.4.2).
2. We gave an approximate solution for large graphs strengthening the condition on the degree for the graph into which we want to embed the tree (Theorem 2.5.1).
3. Another result consists in embedding bipartite graphs containing few edge-dijoint cycles in a graph G satisfying the conditions of Theorem 2.5.1 (Theorem 2.5.31).

The first result is an exact one, but only for a very restricted class of trees. It seems that increasing the diameter of the trees rapidely increasses the number of cases one has to consider. This makes our approach difficult to use for trees with larger diameter.

The second result has no restriction on the structure of the tree, but, as any result using the Regularity Lemma, the result applies only for large and dense graphs. For sparse graphs, one may try to investigate the possibility of using the Sparse Regularity Lemma (see Kohayakawa [10]).

An interesting question is to which extend Theorem 2.5.31 can be generalised. We have showned that in a graph G satisfying the conditions of Theorem 2.5.31, we can embed any bipartite graph H satisfying the followings:

- the spanning tree of H has order at most $k + 1$,
- any edge in $E(H)$ is contained in at most one cycle,
- the number of cycles is constant with respect to $n = |V(G)|$.

How much can we strengthen one of the last two conditions?

An other direction of investigation is to take out the approximation in Theorem 2.5.1. This would imply a solution of the Loebl-Komlós-Sós Conjecture for large graphs. Indeed, this was done by Zhao in the special case of the Loebl conjecture, i.e. when $k = \frac{n}{2}$ [19]. Ajtai, Komlós, Simonovits and Szemerédi are working on a paper that deals with the same question in the context of the Erdős-Sós conjecture [2].

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