Doctoral thesis - supervisor's report

Analysis of unsteady flows of incompressible heat-conducting rate-type viscoelastic fluids with stress-diffusion

by *Michal Bathory*

Mathematical theory of viscous heat conducting fluid is very quickly developing field in the theory of partial differential equations and new recently developed mathematical tools allow us to study more and more complicated problems that describe flows of rather complicated fluids. One of the most appealing problems is to include the elastic effects with diffusion into the constitutive equations on one hand and to obtain a satisfactory existence theory for reasonable class of models on the other hand. The main goal of the thesis was to study incompressible heat conducting viscoelastic fluids with stress diffusion and to establish the *global in time* and *large data* existence of a weak solution for these problems.

Such goal was in fact very ambitious since all available results were either related to simpler problems or they treated systems that violated the basic physical principles. To be more specific, the existing theory can be roughly split into several cases. First, for the viscoelastic problems without stress diffusion, there are many results related to Oldroyd-B, Giesekus or similar models, however, in these models there does not appear the stress diffusion and techniques used for such problems cannot be applied (or are irrelevant) for problems with stress diffusion. Second, recently there was developed a theory for simplified models where the elastic effects are modeled through a scalar quantity. Such simplification directly leads to much better estimates which makes the whole analysis much easier. Next, there are many "ad-hoc" models for which the existence theory is known, but their physical interpretation is either unclear or they even do not fulfill the second law of thermodynamics. Last, there was no mathematical theory (related to global in time and large data existence of a weak solution) for models which would include viscous, thermal and elastic effect in a reasonable way.

I am happy to say that the main goal was fulfilled and that the obtained results enriches the theory for incompressible heat conducting fluids by the elastic effects with stress diffusion.

To be more specific. The thesis consists of two main parts and of introduction. While the last section, which corresponds to already published paper, deals with isothermal problem, the first part of the thesis is focused on the temperature dependent problem. The whole thesis is technically very complicated and the authors builds the existence of a solution by using a cascade of approximations and proper identification of the corresponding limiting objects. Honestly, I must admit, that there is nothing substantially new if one focus just on the limiting process and all methods and tools used in the paper belong to the standard and classical theory for partial differential equations. However, this is not the main critical issue in the analysis. The key novelties are the following:

1) The entropy estimates for the stress diffusion, i.e., the estimates based on the control of

$$\int \nabla \mathbb{B} : \nabla \mathbb{B}^{-1}.$$

While in the scalar setting the above term clearly gives the estimate for $\log \mathbb{B}$, in the tensorial setting, i.e., if $\mathbb{B}$ is a matrix, it was not clear what kind of information can one extract from the above term. In addition, the whole method was generalized to get the precise estimates based on the control of

$$\int \nabla \mathbb{B} : \nabla \mathbb{B}^{\alpha}.$$

for arbitrary $\alpha \in \mathbb{R}$.

2) Development of the proper approximative scheme and, in particular, the way how to obtain the positive definiteness of $\mathbb{B}$. Indeed, the positive definiteness of $\mathbb{B}$ must be shown at certain level of approximation in order to justify the testing of the equations by $\mathbb{B}^{-1}$, which is the essential point to get the proper entropy inequality.

3) Proper estimates for heat conducting viscoelastic fluids. It is remarkable that the estimates based just on the entropy inequality are not sufficient to define a notion of a weak solution and they must be improved. A nice way how to do it was developed in the thesis and the author finally obtained estimates, which are very similar to those used for incompressible Navier–Stokes–Fourier system.

To summarize, I think that the thesis contains very clever ideas how to deal with very complicated problems with stress diffusion and temperature dependent material coefficient. Although the author simplified the model by taking the special form of the Helmholtz free energy (to obtain an easier equation for the internal energy), it is the first global existence results for such fluids. In addition, the whole newly developed approach and method can be used also for much more complicated problems with more general constitutive equations.

The thesis is written clearly, all steps are carefully and rigorously justified, and the thesis is also completed by a physical overview of the problem and by several comments of the author. With some of theses comments I personally do not fully agree or I would write it differently, nevertheless, this also shows the way of Michal's own independent way of thinking, which I was really appreciating during last four years of his studies.

During his studies, Michal also spent a significant periods at universities abroad (Italy, Germany, Austria) and established some international connections. I believe that it was also the reason why he already got a postdoc position at University of Vienna in the group of Ulisse Stefanelli. Moreover, Michal already prepared two papers, which were not co-authored by myself (one published, one submitted), which also proves his ability for own independent research.

Therefore, I recommend Michal Bathory to be awarded by PhD degree.

Prague, August 27, 2020 Miroslav Bulíček